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ABSTRACT

The goal of this paper is to renew interest in the structural option to algebra instruction. Concern for the usual secondary school algebra curriculum related to simplifying expressions, solving equations, and rationalizing numerators and denominators is viewed from three pedagogical approaches: (1) structural approach, (2) empirical approach, and (3) drill approach. The arguments for the structuralist approach are presented in four sections. The first section presents an analysis of algebraic symbol manipulation into six components of knowledge: (1) morphological, (2) graphological, (3) parsing, (4) transformational, (5) pragmatic, and (6) semantic components. The second section discusses the deductive structure of algebraic symbol manipulation. The third section discusses the visual structure of algebra and summarizes two empirical studies involving visual parsing and visual transformations that implicate the visual structure of algebra in students' unconscious assimilation of algebraic knowledge. The fourth section presents a structural algebra curriculum. The paper concludes by warning against abandoning entirely the formalist aspect of algebra in implementing a strictly empirical approach to algebra. Contains 72 references. (LDR)

The Structural Algebra Option:

A Discussion Paper

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The Structural Algebra Option: A Discussion Paper

Introduction

I wish in a short space, to offer new interpretations and explanations for some familiar events and phenomena connected with symbol skills education in elementary algebra; also to point to some new pedagogical possibilities that ensue from these perspectives, and to discuss over-arching themes of secondary school algebra curriculum. This wide scope precludes definitive treatment of any one topic. But I hope in a panoramic sweep to raise new options for consideration in the algebra curriculum debate.

Firstly, to define the terrain more clearly, I am concerned here with the usual secondary school curriculum related to simplifying algebraic expressions, solving equations, rationalizing numerators and denominators, etc. Three standard pedagogical approaches to this topic can be found in the literature:

> The *structural approach* views algebraic symbol manipulation as an exercise in explicit reasoning and logical deduction. Standard textbooks of the 'new math' legacy of the 1960s and 1970s take this approach (e.g., Brown, Smith, & Dolciani, 1986; Dolciani, Wooton, & Beckenbach, 1983), as do more recent efforts to provide computer tutors to instruct students, and to diagnose and remediate their errors (e.g., McArthur, 1987; McArthur, Stasz, Hotta, Peter, & Burdorf, 1988). Researchers concerned with structural models of expressions/equations and solution processes address this aspect of algebra (e.g., Bundy, & Welham, 1981; Carry, Lewis, & Bernard, 1980; Matz, 1980).
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> The *empirical approach* sees the key to competence in ready access to referential domains like real world situations (Fey, 1989; Nemirovsky, & Rubin, 1991; Usiskin, & Senk, 1990), graphs and tables (Confrey, 1991, 1992; Dugdale, 1990; Goldenberg, 1991; Kaput, 1987, 1989; Yerushalmy, 1991a, 1991b; Yerushalmy, & Gafni, 1992), and arithmetic domains (Hyde; Kilpatrick, Mason et al) all of which can serve to anchor otherwise arbitrary and forgettable rules, and to facilitate reasonableness checks on erroneous procedures (Booth, 1989; Resnick, Cauzinille-Marmeche, & Mathieu, 1987).
>
> The *drill approach*, now regaining ascendancy in school practice, trains students to non-reflective competence through repetitive, incremental practice (e.g., Kumon Math; Saxon, 1990, 1991).

A confluence of factors impinge upon the pedagogical debate in algebra. I selectively review several of these factors and highlight aspects of their influence on the pedagogical choices.

1. <u>Technology</u>: Due to the automation of algebraic manipulation on readily available calculators and computers, the scientific utility imperative that has driven the symbol skills algebra curriculum through decades and centuries is

no longer operative, or at least much weakened (Fey, 1984; NCTM, 1989; NCTM, 1991; NRC, 1989). This fact has raised doubts for many educational theorists and psychologists about the usefulness of the drill approach which, by design, makes little direct contribution to the learner other than behavioral facility (Conference Board of the Mathematical Sciences, 1983; Thorpe, 1989).

2. Equity: Skill in algebra eludes most students in most schools, but minority students and females are proportionately over-represented among the algebra-injured (Fennema, 1980; Fennema, 1985; Mullis, Dossey, Owen, & Phillips, 1991). Because algebra serves a gatekeeper function for exclusion from a variety of technical and scientific fields, there is a moral/social imperative that curricular modifications improve all students' learning benefits, but especially students from underachieving groups (NCTM, 1989, 1991; NRC, 1989). Indirectly, this imperative weighs in against curriculum innovations that may be too cost intensive in terms of hardware or teacher training requirements. Also, this imperative weighs in favor of drill curricula which have been more equitably successful in training students as symbol manipulators.

3. History: Structural algebra was a major focus of the 'new math' curricula of the 1960's and 1970's (e.g., Haag, 1961). Indeed, part of the new math agenda was to transfer to algebra a share of the concern for deductive reasoning that previously resided in the geometry curriculum (College Entrance Examination Board, 1959). The reasons usually cited for the failure of the new math initiative include the inherent difficulty and abstractness of the deductive approach (it mainly was intended for college-bound students), and the lack of preparation and training of teachers (NACOME, 1975). Because of equity issues, and because the previous sustained and determined effort of the new math era was not successful, educators may be reluctant to revisit this territory of structural algebra.

4. Technology (again): Apart from its influence in diminishing the scientific utility imperative for algebra symbol skills, technology also contributes positively to educational potential through linked representation microworlds (Confrey, 1991, 1992; Dugdale, 1990; Goldenberg, 1991; Kaput, 1992; McArthur, 1990; Thompson, 1989). The major educational intention of such microworlds, at least in their current trajectory of development, is to help students develop multirepresentational capabilities and perspectives (Kaput, 1987; Thompson, 1989). This technological front supports empirical approaches to algebra which, by definition, are multirepresentational.

I believe that the score from assessing and integrating these various factors favors the ascendancy of empirical algebra. The skills curriculum, despite a certain appeal for its equity performance, simply is obviated by the availability of computer symbol manipulators. The structural approach suffers from the inherent difficulty of deduction in an abstract domain (it scores low on equity) and from the fact that it already has been the subject of intensive reform efforts where it failed (NACOME, 1975). Empirical approaches traditionally have been caricatured in curricula by routine word problems that present only a very limited range of contextual association (Brown, Collins, & Duguid, 1989; Caldwell, & Goldin, 1987; Rosnick, & Clement, 1980). But the evolving initiatives in linked representation microworlds are poised to breath new life into this crucially important area of algebra education.

I seek to stimulate a renewed interest in the structuralist option by arguing the following points:

1. In its logical deductive structure, algebraic symbol manipulation is a relatively simple domain; much simpler, for example, than geometric proof.

2. The difficulties encountered with the structural approach do not stem from its inherent abstractness or deductive complexity, but from particular visual characteristics of algebraic language that facilitate superficial assimilation at the expense of reflective engagement.

3. A modified structural curriculum can be designed to counter the dependence on visual features, and to successfully engage students in structural thinking rather than in non-reflective, incremental skill acquisition.

Building such a case requires a detailed model of algebraic symbol skills that is presented in the next section. Before moving to this, I conclude the introduction with a statement of the overall intentions of this line of investigation. The objective is not to defuse interest in the exciting developments in empirical approaches to algebra, but to work towards a balanced curriculum of excellence in which the complementary aspects of structural and empirical algebra both are viable.

A Model of Algebraic Symbol Skills

This section provides an analysis of algebraic symbol manipulation into components of knowledge that can be independently characterized (though they may interact complexly in actual cognitive processing of algebra). The purpose here is to provide a vocabulary and to make distinctions that can be built upon or challenged in subsequent sections. The basic model itself is consistent with most of the work that is already in the literature (e.g., Carry, Lewis, & Bernard, 1980; Ernest, 1987; Kieran, 1992; Wagner, Rachlin, & Jensen, 1980) --though few authors have found it necessary to describe more than one or two components in a given study. A linguistics lexicon is employed here by loose (rather than firm or principled) analogy to levels of language structure.

1. Morphological Component:
The morphological component specifies the fundamental elements of meaning in an algebraic expression or equation. These elements include numbers, operations (addition, subtraction, multiplication, division, exponentiation, radical, and negation), variables, etc. Individual morphological elements have received sustained analysis (e.g., Herscovics, & Kieran, 1980; Küchemann, 1978; Usiskin, 1988). This component is distinguished from the graphological

forms by which these meanings may be signified.

2. Graphological Component:
The graphological component specifies the written forms for presentation of the morphemes described above, as well as the physical indicators that serve parsing functions. For instance, the operation of addition is symbolized by ~+~ (where ~ represents a blank space of a certain width), but multiplication (usually) is represented only by horizontal juxtaposition; exponentiation by diagonal juxtaposition, etc. Parsing functions are signaled by such graphological elements as parentheses, brackets, and braces, vincula (compare $\sqrt{3xy}$ and $\sqrt{3x}y$), and raising (compare 2^{xy} and $2^x y$).

Presumably competence in algebra entails a detailed knowledge of the morphological and graphological components, and of the correlation between them. Most psychological analyses of algebra skill simply have assumed these components and begun with higher level elements. For instance, Carry, Lewis and Bernard (1980) note the necessity of accounting for algebraic performance expressed as strings of symbols, but they restrict their analysis to structural tree diagrams which display certain aspects of representation more conveniently.

3. Parsing Component:
The parsing component specifies the conventions for grouping together morphs as they occur in symbol strings. In addition to delineating the functioning of the graphological parsing elements described above, the parsing components delineates the conventional hierarchy of operations (e.g., multiplication has precedence over addition or subtraction, etc.). The crucial cognitive function of parsing knowledge has been noted by several authors (Ernest, 1987; Thompson, & Thompson, 1987; Larkin, 1989; Norman, 1986).

4. Transformational Component:
The components discussed thus far permit the parsed graphological representation of morphs in expressions and equations. But the activity of algebraic symbol manipulation consists in the transforming of expressions and equations into new expression and equations. For instance the expression $3x^2 - 2^2$ can be transformed to $(3x - 2)(3x + 2)$ by application of a difference of squares rule. The transformational component specifies all such rules that normally would be available to an expert algebra symbol manipulator.

5. Pragmatic Component:
The manipulation of algebraic symbols is not just an arbitrary application of transformational rules to expressions and equations. For instance $3x^2 + 2x + 4 = x^2 + 2x^2 + 2x + 4 = x^2 - 4 + 4 + 2(x^2 + x) + 4 = (x - 2)(x + 2) + 2[2 + (x^2 + x) + 2]$ is a formally correct symbolic derivation, but it accomplishes none of the tasks that experts normally find it useful to accomplish in algebra. The pragmatic component governs the selection and sequencing of transformation to accomplish standard tasks (like simplifying

fractional expressions, rationalizing denominators, etc.). It has been extensively analyzed as *strategic knowledge* by Carry, Lewis, and Bernard, (1980), and for the special case of linear equation solving as *meta-level inferencing* by Bundy and Welham (1981).

6. Semantic Component:
The semantic component specifies the referential domains for the graphological elements. For instance, the link between ~+~ and some experiential notion of combining is located here. Similarly, the reference of a variable to a number (in a numerical domain) or to a quantity (in some 'real world' situation) resides in the semantic component. This aspect of algebraic knowledge is fundamental to algebraic applications and more generally to empirical algebra, as discussed in the introduction. But working within a formal system explicitly denies referential extension. Thus a formal analysis of structural algebra makes no reference to the semantic aspect.

The Deductive Structure of Algebraic Symbol Manipulation

In this section, I argue that in its logical operations algebraic symbol manipulation is an inherently simple domain --far simpler for example than geometric proof. In this regard I distinguish algebraic symbol manipulation, which *assumes* the existence and uniqueness of identities, inverses, etc., and which may include certain other rules (e.g., some exponent laws) as axioms, from abstract algebraic approaches which start with minimal (or nearly minimal) axiom sets and proceed through existence and uniqueness proofs (e.g. uniqueness of the additive identity) to build up properties of the number systems.[1]

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[1]This is a gloss of a fundamental and possibly controversial point. Elementary algebra evolved as a tool for scientific and commercial application long before it was formalized in abstract analyses of number systems (Kline, 1980). Thus there is a history of usage that may diverge from frameworks of explanation. For instance, the operations of addition and multiplication are defined within abstract algebraic treatments of the rational numbers, whereas exponentiation is addressed within analytic treatments of the real numbers. These are even different branches of mathematics! But rules like $(a^b)^c = a^{bc}$, $a^{b+c} = a^b a^c$, etc., participate together with rules like (ab)c = a(bc), a(b + c) = ab + ac, etc., as part of a completely integrated domain of practice. What I am suggesting is that structural treatments of elementary algebra can specify exponent rules (or indeed, any rules of convenience) as foundational for the purposes of developing rigorous derivations within elementary (school) algebra. Whether there is pedagogical merit in engaging students in rigorous processes of mathematical derivation outside of the content of established mathematical theory is a matter for discussion. It might be noted that attempts to formalize exponentiation within algebraic treatments of real numbers are unresolved. For instance, Macintyre (1979) concludes one such initiative as follows:

> The most interesting problem provoked by the above is that of showing that there are no "exotic" laws [of real numbers], i.e. that every law is a consequence of the laws of +, •, -, $^{-1}$, 0, 1 together with $x^1 = x$, $x^{y+z} = x^y x^z$, $x^{yz} = (x^y)^z$, $(xy)^z = x^z y^z$.
> It seems difficult to prove such a theorem by the methods of real algebra used

To begin this analysis, I review some aspects of logical deduction and analyze their difficulty for learners. Logical deduction often is based on conditional inferences which begin with a conditional statement consisting of an antecedent and a consequent, one of which is asserted or denied in a subsequent statement. For instance the logical principle modus ponens asserts the conditional, *if p then q*, and the antecedent, *p*, from which one may deduce the truth of the consequent, *q*. There are four inferential possibilities, but only modus ponens and its contrapositive modus tolens are logically sound (Table 1).

Table 1
Conditional Inferences (Evans, 1982, p. 121)

Modus ponens		If p then q
(MP)		p
	Therefore,	q
Denial of the antecedent		If p then q
(DA)		not p
	Therefore,	not q
Affirmation of the consequent		If p then q
(AC)		q
	Therefore,	p
Modus tolens		If p then q
(MT)		not q
	Therefore,	not p

DA and AC are common inferential errors. Table 2 reports percentages of errors in logic by experimental subjects in three studies. Importantly, the percentages of DA and AC errors are very similar, in keeping with the interpretation that conditionals are being mistaken for equivalences (bi-conditionals) (Evans, 1977; Griggs, 1976; Taplin, 1971; Taplin & Staudenmayer, 1973). Reasoning from equivalences is less error prone because asserting the bi-conditional and the truth or falsity of either the antecedent or consequent assures the truth or falsity, respectively, of the other.

above. (p. 97)

Table 2
Percentage of Adult Subjects Endorsing Conditional Inferences in Several Studies, for an Affirmative Rule, If p then q (Evans, 1982, p. 129)

Study	MP	DA	AC	MT
Taplin (1971)	92	52	57	63
Taplin and Staudenmayer (1973)	99	82	84	87
Evans (1977)	100	69	75	75

I argue now that the logical structure of algebraic symbol manipulation involves, almost exclusively, bi-conditional reasoning; almost never, conditional reasoning. A contrast with geometric proof will help establish this point.

A school geometry problem starts with a theorem to be proved. Usually the theorem is expressed as an implication: If certain given conditions X hold, then conclusion Y follows. Now the theorem could be expressed as a bi-conditional: X iff (X and Y). But it is the deductive structure of the solution processes rather than the theorem statement that is at issue. For instance to prove that X implies Y, a typical strategy is proof by contradiction; assume not Y and prove not X. As we've seen, proof by contradiction is based upon modus tolens, one of the rules of conditional inference. In this and many other instances, geometry proof depends upon inferential reasoning.

We turn now to the case of algebraic symbol manipulation as illustrated by a typical derivation: $3x^2 - 27y^2 = 3(x^2 - 9y^2) = 3(x - 3y)(x + 3y)$. (For the sake of completeness we include the tacit steps: $3x^2 - 27 = 3x^2 - 3 \cdot 9 = 3(x^2 - 9) = 3(x^2 - 3^2) = 3(x - 3)(x + 3)$.) In its logical structure, this derivation constitutes a proof of the equivalence of $3x^2 - 27y^2$ with $3(x - 3y)(x + 3y)$. But again, it is the deductive structure of the solution process rather than the theorem statement that is at issue. In such derivations each step is derived from the previous one by substitution: 27 iff $3 \cdot 9$ yields the substitution $x^2 - 27$ iff $x^2 - 3 \cdot 9$, and so on. The solution may require considerable procedural skill in comparing the structure of a given expression (e.g., $x^2 - (3y)^2$) to the condition of a rule $a^2 - b^2$, noting the correspondence of structural components and rigorously applying substitutions of component parts. This may be technically demanding, but it is not logically obscure. Substitution of equivalences is an aspect of bi-conditional, not conditional, reasoning.

There are other differences between the logic of algebraic symbol manipulation and geometric proof. In algebra the conclusion to be reached is not given. Rather, one starts with an initial expression (or equation), and an instruction such as "factor" (or "solve"). Knowing the characteristics of the desired end-state, and deciding what transformational rules to apply to get there, are part of the pragmatic component (Kirshner, 1987). Thus the pragmatic component in algebra necessarily is based upon a knowledge of routine tasks. Geometry theorems, which do provide the conclusion sought, need not have a routine character (Anderson, 1983b). It is this difference that justifies the term pragmatic knowledge for algebra, but not for geometry, where creative strategies may need to be devised and implemented. This is another factor that may serve to make algebraic derivation a more tractable study than geometry proof (though of course educational goals and values need not favor more tractable problems).

The Visual Structure of Algebra

If, as argued above, the deductive structure of algebra symbol manipulation does not depend upon difficult inferential reasoning, then some other sources of difficulty need to be identified to explain (1) students' legendary difficulties with the subject, (2) the failure of the new math movement which involved a massive mobilization of resources to support structural algebra in schools, and (3) the predilection of many teachers and students for incrementalist skill-drill curricula like Saxon Algebra and Kumon Algebra in which opportunities to grapple with the (relatively simple) logical structure of the domain are minimized.

A comparison to computer programming can highlight this last point. Programming education is characterized by an explicit curriculum in which the components of knowledge to be acquired are carefully categorized and fully explicated for students. And care is taken to assure that declarative representations are integrated into students' initial construction of the domain. Anderson, Conrad, and Corbett (1989) describe their design of a LISP tutor as follows:

> Students must encode such information [explicit descriptions about the rules of the programming language] in a declarative representation ... and use it to guide their programming.... There are clear pedagogical implications of this initial stage of using declarative knowledge. One is that one should carefully fashion it so that the target productions will be compiled. (p. 475)

Furthermore successful students generally are able to describe the structure of the language that they have learned, as a result of which "explanation helps students immediately correct their code" (Anderson, Conrad, & Corbett, 1989, p. 501). But algebra students, even those who are relatively successful, generally are unable to provide coherent accounts of their own knowledge base (Davis, 1984, Kirshner, 1989), and instruction-resistant algebra errors are commonplace. Algebra just seems to have a mind of its own.

In this section, I argue that algebra, which has evolved over a millennium (see Cajori, 1928), is different in kind from wholly artificial languages like computer languages, whose functioning is contrived in advance by its developers at a given point in time --a slight exaggeration, but nearly true relative to the evolution of algebraic language. In the case of algebra, notational attributes which facilitate cognitive functioning and acquisition may have evolved in ways that were not explicitly intended, and which consequently, may not be fully accommodated for in instructional practices. But my claim for unintended cognitive attributes goes beyond mere notational device (the graphological component) to include aspects of the very construction of algebra content (the transformational component). In this section I briefly summarize two empirical studies that implicate the visual structure of algebra in students' unconscious assimilation of algebraic knowledge. In the next section I sketch a curricular model that accommodates for these implicit characteristics in developing students explicit structural understanding of the topic.

Visual Parsing

One component of parsing knowledge involves a well-known hierarchy of operations. A succinct (propositional) account of this hierarchy involves a system of *operation levels* (Schwartzman, 1977):

Operation Levels

Level 1 operations	addition, subtraction
Level 2 operations	multiplication, division
Level 3 operations	exponentiation, radical

This system groups together inverse operations in a natural way. Parsing rules for algebra expressions follow this hierarchy:

Parsing Rule

1. Precedence is assigned to the highest level operation.
2. For operation of equal level, precedence is assigned to the left-most operation.

For example $1 + 3x^2$ is parsed as $1 + [3(x^2)]$ because exponentiation (Level 3) has precedence over multiplication (Level 2) which has precedence over addition (Level 1). $3 - x + y$ is parsed as $(3 - x) + y$ (rather than $3 - (x + y)$) because the left-most Level 1 operation has precedence over the Level 1 operation on the right.

It is instructive to note that such propositionally accurate accounts of operation hierarchy are *not* provided in standard curricula. Rather, incomplete mnemonic devices like My Dear Aunt Sally for Multiply Divide Add Subtract (Keedy, 1986) provide an informal guide to operation precedence for some operations. Most textbooks devote only three or four pages to developing parsing skills (Kirshner, 1989).

Kirshner (1989) identified visual correlates to the propositional structure of operation level:

Operation Levels (Visual)

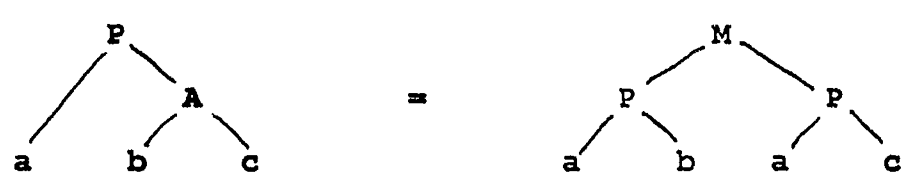
Level 1 operations	wide spacing	$(a + b;\ a - b)$
Level 2 operations	horizontal/vertical juxtaposition	$(ab;\ \frac{a}{b})$
Level 3 operations	diagonal juxtaposition	$(a^b;\ \sqrt[a]{b})$

With operation level defined in these visual terms the character of the hierarchy of operations rule is altered: The propositional construct that exponentiation has precedence over multiplication which has precedence over addition becomes the implicit knowledge that diagonal juxtaposition 'ties tighter than' horizontal juxtaposition which 'ties tighter than' wide spacing. Kirshner (1989) verified that many students who are fully competent in parsing algebraic expressions in ordinary notation are unable to transfer this ability to a contrived notation in which propositional information about operations is present but visual/spatial characteristics are distorted; and they are unable to give coherent accounts of their parsing knowledge. These data suggest that parsing knowledge may not be conveyed through the explicit curriculum (which anyway is inaccurate and incomplete) but induced directly through immersion in symbol skill activities.

<u>**Visual Transformation**</u>

Whereas this first study implicates visual structure in the conventional system of notational parsing, I want further to argue that visual pattern reaches into the heart of algebraic structure --the transformational component. Transformational rules seem to vary with respect to their degree of *visual salience*. Some rules like $a(b + c) = ab + ac$, $(a^b)^c = a^{bc}$, $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$, etc., have a certain visual coherence that other rules like $a^2 - b^2 = (a - b)(a + b)$, $(a + b)^2 = a^2 + 2ab + b^2$, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ lack.

In a recent study (Awtry, in preparation) two groups of grade 7 students, complete novices in algebra, were asked to memorize a set of eight rules, four of which were of the visually salient variety, four of which were not. One group was taught the rules in standard algebraic notation. The other group was introduced to a tree notation in which the parse of the expression is encoded in a hierarchy of operation nodes. For instance the rule $a^{b+c} = a^b a^c$ is encoded in tree notation as follows (where P, M, and A represent power, multiplication, and addition, respectively):



It must be noted that tree notation also is a visual medium. Indeed it is strongly visual. But the visual characteristics that contribute to the salience of ordinary notation rules seem to be related to visual parsing features (diagonal juxtaposition, horizontal juxtaposition, and spacing) that are not preserved in tree notation. Thus the visual character of tree notation does not insure that transformational rules will have differential visual salience.[2]

Two kinds of tasks were given to the students in the study: recognition and rejection tasks. In recognition tasks students were presented with an initial expression and asked to select from among six choices (including "none," but never as the correct response) the expression that could be derived from the given expression by legal application of one of the given rules. In the rejection tasks, no rule exactly applied to the given expression, though there was one rule that nearly (but not exactly) applied. Thus recognition tasks check the student's ability to recognize a routine application of a rule; whereas, rejection items invite the student to overgeneralize the context of application of algebra rules. The correct response for these latter items is "none".

The results were that in ordinary notation, visually salient rules were significantly easier to recognize than non visually-salient rules, but for tree notation items visual and non-visual rules were equally difficult. This pattern of results did not hold true for the rejection items. Indeed, in ordinary notation students more often overgeneralized visual items by applying them to inappropriate expressions than non-visual items (though the differences in this direction were not significant).

These data raise the possibility that students schooled in standard curricula may not be developing equally propositional representations for all of the transformational rules encountered. Some rules, based on their high degree of visual salience, may be slipping into usage more easily. But these visual rules appear to be just the ones that students most often overgeneralize, as illustrated by students' common errors like $(a + b)^2 = a^2 + b^2$ (an overgeneralization of $(ab)^2 = a^2b^2$), $\frac{a}{c} + \frac{b}{d} = \frac{a + b}{c + d}$ (an overgeneralization of $\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}$), etc. (Davis, Jockusch, & McKnight, 1978; Laursen, 1978; Matz, 1980; Schwartzman, 1977).

A Structural Algebra Curriculum

The foregoing provides a framework for explaining anomalous features of current algebra teaching and learning, including: (1) the non-rigorous presentation of parsing rules in textbooks; and (2) the apparent predilection of students to learn from

[2]My fuller speculation is that transformational rules in algebra evolved in dialectical relation to parsing features. But this historical analysis has yet to be undertaken.

examples and practice rather than from explanation and principles. Because students generally are successful in acquiring the ability to determine the correct parse of algebraic expressions, curriculum materials that present parsing rules rigorously may seem unnecessary. Unaware of the visual structure of notation, curriculum authors may make the reasonable assumption that students' parsing skill reflects a propositional foundation for parsing rules. Indeed, current theories of acquisition of intellectual skills often start with acquisition of declarative representations (Anderson, 1983).

But rigorous comprehension and application of transformational rules must be founded upon an explicit model of parsing hierarchies. For instance to successfully apply the rule $\frac{ab}{ac} = \frac{b}{c}$ requires knowing that it specifies multiplication as the main operation in numerator and denominator, and then being able to identify such conditions in a given expression. Now if parsing skills are detached (by visual properties of notation) from explicit knowledge of operations, little sense can be made of teachers' rule explanations, and the student is well-launched to a superficial, apropositional acquisition of visually salient rules. Students' proclivity for learning through examples and practice (Sweller, & Cooper, 1985) without attending to explanations, is a natural consequent, as is the subsequent proliferation of curricula that promote non-reflective incremental practice (Kumon Math; Saxon, 1991; Saxon, 1992).

A successful structural algebra curriculum must take active measures to counteract the passive seductions of visual structure. The method that I advocate uses a verbal support system (VSS) to instantiate propositional aspects of parsing structure. The VSS provides a lexicon for structural elements of algebraic expressions. For instance each expression has a *dominant* or *principal* operation defined as the least precedent operation according to the rules for parsing. Becoming conversant with this term requires an explicit knowledge of parsing rules, as well as practice in creating explicit structural maps of expressions. But a further structural lexicon is needed. The subexpressions joined by the dominant operation might be called the *principal subexpressions*. From here, standard vocabulary items like *term* (or *factor*) can rigorously be defined as "the principal subexpressions of an expression whose dominant operation is addition (or multiplication)". Mastering this vocabulary constitutes a propositional grounding for the parsing structure of expressions and equations. Such mastery must precede introduction to transformational rules which alter structure.

Following mastery of the structural vocabulary, it becomes possible to instantiate algebra rules propositionally. For instance the description of $\frac{ab}{ac} = \frac{b}{c}$ as a rule for canceling common factors can take on a precise meaning traced back to the parsing structure of expressions. Errors like $\frac{x+3}{x+5} = \frac{3}{5}$ now can be analyzed as violating the scope of the cancellation rule, because "factor" has a precise meaning. The propositional basis for parsing structure can constrain overgeneralization of transformations. Ultimately it is possible to link the pragmatic component to an explicit analysis of structural states. For instance standard tasks like *simplify the fractional expression* can be related to an end state of applying the cancellation law for fractions, which requires achieving a preceding state of having the numerator and denominator in *factored* form. Explicating the pragmatic structure of various tasks is something that good teachers of structural algebra might try and do anyway. But the VSS moves one past the very natural assumption that students who can parse expressions and perform simple transformations successfully must have a propositional base for their knowledge of (Anderson, 1983; Anderson, Conrad, & Corbett, 1989). It provides for a language of shared meanings between student and teacher.

Conclusions

Internal structure and external reference are complementary and equally vital aspects of algebraic knowledge. The twin circumstances of computer/calculator symbol manipulators that remove the practical imperative for mastery of symbol skills, and a history of failure at implementing structural algebra (including the current retreat to non-reflective, incremental practice) have pushed many educators to an exclusively empirical agenda: **If it has no referential basis, it has no pedagogical value!** But to deny the value of learning to reason within a closed (referentially truncated) system is to abandon entirely the logicist/formalist aspect of mathematics. Such a position needs to be carefully considered.

I have suggested in this paper that structural algebra uses a relatively limited repertoire of logical operations (that may be tractable to students) and that a reasoning, discursive structural curriculum can replace the mindless incrementalist approaches gaining ascendancy today. Ironically this entails subverting (for a time) the visual structures that allow students a certain surface facility with expressions and transformations. But the visual aesthetic woven into the form and content of algebraic language that supports easy symbol fluency for the mathematically mature can overwhelm the propositional aspect of the language for the novice.

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