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ABSTRACT

Constructivist epistemologies have been instrumental in recent research on mathematics learning and have provided a basis for recent mathematics education reform efforts. Although constructivism has the potential to inform changes in mathematics teaching, it offers no particular vision of how mathematics should be taught; models of teaching based on constructivism are needed. Data are presented from a whole-class teaching experiment in which constructivist perspectives encountered problems of teaching practice. The analysis of the data led to the development of a model of teacher decision making with respect to mathematical tasks. Central to this model is the creative tension between the teacher's goals for student learning and the responsibility to be sensitive and responsive to the mathematical thinking of the students. Themes represented include: (1) Students' thinking/understanding is taken seriously and given a central place in the design and implementation of instruction; (2) The teacher's knowledge evolves simultaneously with the growth in the students' knowledge; (3) Planning for instruction is seen as including the generation of a hypothetical learning trajectory; (4) The continually changing knowledge of the teacher creates continual change in the teacher's hypothetical learning trajectory. Contains 47 references. (Author/GW)

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Reconstructing Mathematics Pedagogy from a Constructivist Perspective

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Running Head: RECONSTRUCTING MATHEMATICS PEDAGOGY

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Reconstructing Mathematics Pedagogy from a Constructivist Perspective

Abstract

Constructivist epistemologies have been instrumental in recent research on mathematics learning and have provided a basis for recent mathematics education reform efforts. Although constructivism has the potential to inform changes in mathematics teaching, it offers no particular vision of how mathematics should be taught; models of teaching based on constructivism are needed. Data are presented from a whole-class, constructivist teaching experiment in which constructivist perspectives encountered problems of teaching practice. The analysis of the data led to the development of a model of teacher decision making with respect to mathematical tasks. Central to this model is the creative tension between the teacher's goals for student learning and his responsibility to be sensitive and responsive to the mathematical thinking of his students.

Constructivist perspectives on learning have been central to much of the recent empirical and theoretical work in mathematics education (Steffe & Gale, in press; von Glasersfeld, 1991) and as a result, have contributed to shaping mathematics reform efforts (National Council of Teachers of Mathematics, 1989, 1991). Although constructivism has provided mathematics educators with useful ways to understand learning and learners, the task of reconstructing mathematics pedagogy based on a constructivist view of learning is a considerable challenge, one that the mathematics education community has only begun to tackle. Although constructivism provides a useful bases for thinking about mathematics learning in classrooms, it does not tell us how to teach mathematics. Constructivism can contribute in important ways to the effort to reform classroom mathematics teaching, however it does not stipulate a particular model.

The use of the word "pedagogy," above, is meant to signify all contributions to the mathematical education of students in mathematics classrooms. As such, I include not only the multi-faceted work of the teacher, but also the contributions to classroom learning of curriculum designers, educational materials developers, and educational researchers . Mathematics pedagogy might be operationally defined using the following thought experiment. Picture twenty-five learners in an otherwise empty classroom. That which would appropriately be added to this picture to increase the potential for mathematics learning is pedagogy.

This paper describes data from a classroom teaching experiment, in which the researcher served as mathematics teacher, the analysis of that data, and an emerging theoretical framework for mathematics pedagogy which derives from that analysis. The paper contributes to a dialogue on *what teaching might be like if it is built on a constructivist perspective on knowledge development*. The specific focus of this paper is on decision making with respect to mathematics content and mathematical tasks for classroom learning. The paper begins with an articulation of the constructivist perspective which under girds the research and teaching and a review of pedagogical theory development based on constructivism which preceded this study and contributed to its theoretical foundation. This study examines the pedagogical decisions which result from the accommodation of the researcher's theoretical perspectives to the problems of teaching practice.

A Constructivist Perspective

The widespread interest in constructivism among mathematics education theorists, researchers, and practitioners has led to a plethora of different meanings for "constructivism." While terms such as "radical constructivism" and "social constructivism" provide some orientation, there is a diversity of epistemological perspectives even within these categories (cf. Steffe & Gale, in press). Therefore, it seems important to describe briefly the constructivist perspective on which our research is based.

Constructivism derives from a philosophical position that we as human beings have no access to an objective reality, i.e. a reality independent of our way of knowing it. Rather, we construct our knowledge of our world from our perceptions and experiences which are themselves mediated through our previous knowledge. Learning is the process by which human beings adapt to their experiential world.

From a constructivist perspective, we have no way of knowing whether a concept matches an objective reality. Our concern is whether it works (fits with our experiential world). Von Glasersfeld (in press, 1987) refers to this as "viability," in keeping with the biological model of learning as adaptation developed by Piaget (1970). To clarify, a concept "works" or is viable to the extent that it does what we need it to do: to make sense of our perceptions or data, to make an accurate prediction, to solve a problem, or to accomplish a personal goal. Confrey (in press) points out that a corollary to the radical constructivist epistemology is its "recursive fidelity - constructivism is subject to its own claims about the limits of knowledge. Thus, it is only true to the extent that it is shown useful in allowing us to make sense of our experience." (p. 10) When what we experience differs from the expected or intended, disequilibrium results and our adaptive (learning) process is triggered. Reflection on successful adaptive operations (reflective abstraction) leads to new or modified concepts.

Perhaps the most divisive issue in recent epistemological debates (Steffe & Gale, in press) is whether knowledge development (particularly relational knowledge) is seen as fundamentally a social process or a cognitive process. The difference in the two positions seems to be whether the social or the cognitive is viewed as figure or ground. The radical constructivist position focuses on the individual's construction, thus taking a cognitive or psychological perspective. Although social interaction is seen as an important context for learning, the focus is on the resulting reorganization of individual cognition.

> For Piaget, just as for the contemporary radical constructivist, the "others" with whom social interaction takes place, are part of the *environment*, no more but also no less than any of the relatively "permanent" objects the child constructs within the range of its lived experience. (von Glasersfeld, in press)

On the other hand, epistemologists with a socio-cultural orientation see higher mental processes as socially determined; individual knowledge derives from the social dimension. "Sociocultural processes are given analytical priority when understanding individual mental functioning rather than the other way around." (Wertsch and Toma, in press) From a social perspective, knowledge resides in the culture, which is a system that is greater than the sum of its parts.

Our position eschews either extreme, and builds on the theoretical work of Cobb, Yackel, and Wood(in press; Wood, Cobb, and Yackel; in press; Cobb, 1989) and Bauersfeld (1988) whose theories are grounded in both radical constructivism (von Glasersfeld, 1991) and symbolic interactionism (Blumer, 1969). Cobb (1989) points out that the coordination of the two perspectives is necessary to understand learning in the classroom. The issue is not whether the social or cognitive dimension is primary, but rather what can be learned from combining analyses from these two perspectives. I draw an analogy with physicists' theories of light. Neither a particle theory nor a wave theory of light is sufficient to characterize physicists' data. However, each theoretical construct has made a significant contribution to theoretically based research; considering light to be a particle and considering light to be a wave have each been useful. Coordinating the findings that derive from each perspective has led to advancements in the field. Likewise, it seems useful to coordinate analyses based on psychological (cognitive) and sociological perspectives in order to understand knowledge development in classrooms.

Psychological analysis of mathematics classroom learning focuses on the individual's knowledge of and about mathematics[1], her understanding of the mathematics of the others, and her sense of the functioning of the mathematics class. Sociological analysis focuses on taken-as-shared knowledge and classroom social norms(Cobb, Yackel, & Wood, 1989). "Taken-as-shared" (Cobb, Yackel, & Wood, 1992; Streeck, 1979) emphasizes that members of the classroom community, having no direct access to each other's understanding, achieve a sense that some aspects of knowledge are shared, while having no way of knowing whether the ideas are in fact shared. "Social norms" refer to that which is understood by the community as constituting effective participation in the mathematics classroom community. The social norms include the expectations that community members have of the teacher and students, conceptions of what it means to do mathematics in that community, and the ways that mathematical validity is established.

> ...it is useful to see mathematics as both cognitive activity constrained by social and cultural processes, and as a social and cultural phenomenon that is constituted by a community of actively cognizing individuals (Wood, Cobb, & Yackel, in press).

Constructivism and Mathematics Pedagogy

Understanding learning as a process of individual and social construction provides teachers with a conceptual framework with which to understand the learning of their students. While the development of such understandings is extremely valuable, this paper focuses on the question of how constructivism might contribute to a reconstruction of mathematics pedagogy. How might it inform the development of a framework for fostering and supporting learners' constructions of powerful ideas? Wood, Cobb, and Yackel (in press) assert that

...teachers must... construct a form of practice that fits with their students' ways of learning mathematics. This is the fundamental challenge that faces mathematics teacher educators. We have to reconstruct what it means to know and do mathematics in school and thus what it means to teach mathematics.

As I stated above, constructivism, as an epistemological theory, does not define a particular way of teaching. It describes knowledge development whether or not there is a teacher present or teaching going on. Konold (in press) argues, "not that a teacher's epistemology has *no* effect on how he or she teaches, rather that its effects are neither straightforward nor deterministic." There is no simple function that maps teaching methodology onto constructivist principles. A constructivist epistemology does not determine the appropriateness or inappropriateness of teaching strategies. Bauersfeld (in press) states,

The fundamentally constructive nature of human cognition and the processual emergence of themes, regularities, and norms for mathematizing across social interaction, to bring the [psychological] and the social together, make it impossible to end up with a simple prescriptive summary for teaching. There is no way towards an operationalization of the social constructivist perspective without destroying the perspective.

The commonly used misnomer, "constructivist teaching," however, suggests to the contrary that constructivism offers one set notion of how to teach. The question of whether teaching is "constructivist" is not a useful one and diverts attention from the more important question of how effective it is. From a theoretical perspective, the question that needs attention is, "In what ways can constructivism contribute to the development of useful theoretical frameworks for mathematics pedagogy?"

It is overly simplistic and not useful to connect constructivism to teaching with the romantic notion, "Leave students alone and they will construct mathematical understandings." Likewise, "Put students in groups and let them communicate as they solve problems," is not much more helpful. History provides unsolicited empirical evidence with respect to these approaches. Generations of outstanding mathematicians who were engaged in mathematical problems, who communicated with their colleagues about their work, required thousands of years to develop mathematics that we expect our average elementary school students to construct (Richards, 1991). Thus, while it is useful to have students work problems and communicate about their ideas, it does not seem to be adequate as a prescription for mathematics teaching. The challenge is: *How can mathematics teachers foster students' construction of powerful mathematical ideas which took the community of mathematicians thousands of years to develop?*

Richards asserts,

> It is necessary [for the mathematics teacher] to provide a structure and a set of plans that support the development of informed exploration and reflective inquiry without taking initiative or control away from the student. The teacher must design tasks and projects that stimulate students to ask questions, pose problems, and set goals. Students will not become active learners by accident,[2] but *by design*, through the use of the plans that we structure to guide exploration and inquiry.

Through empirical data and model building, this study attempts to examine the process of constituting pedagogical designs.

Recent Theoretical Work on Pedagogical Frameworks

Little recent work in mathematics education has provided theoretical frameworks (consistent with constructivism) to guide mathematics instruction. This seems to be the result of several factors:

(1) It is only recently that empirically based models for studying mathematics learning in classrooms have been articulated (cf. Wood, Cobb, Yackel, and Dillon, in press). Earlier empirical work which derived from and contributed to epistemological theory was focused on the cognitive development of individual learners (cf. Steffe, von Glasersfeld, Richards, & Cobb, 1983).

(2) Traditional views of mathematics, learning, and teaching have been so widespread that researchers studying teachers' thinking, beliefs, and decision making have had little access to

teachers who had well-developed constructivist perspectives and who understood and were implementing current reform ideas. As a result there has been a lack of connection between research on learning (which has focused on constructivism) and research on teaching[3] (which has focused for the most part on traditional instruction).

(3) The need for pedagogical frameworks is sometimes obscured by the tendency to assume that constructivism defines an approach to teaching.

Recent work which has contributed to developing pedagogical frameworks has focused primarily on the roles of the teacher. Laborde (1989) identifies three basic types of decisions that are made by mathematics teachers[4]: the choice of content, the planning of how the learners might engage the content, and the interventions that the teacher makes.

Recent work has focused more on the second and third of these categories of teacher responsibility. Brousseau (1987) asserts that part of the role of the teacher is to take the non-contextualized mathematical ideas which are to be taught and embed them in a context for student investigation. Such a context should be personally meaningful to the students, allowing them to solve problems in that context, the solution of which might be a specific instantiation of the idea to be learned. The teacher's job is to propose a learning situation within which students seek a response to the *milieu* not a response that is solely intended to please the teacher. For the problem to foster the learning of powerful mathematical ideas, the students must accept the problem as their problem[5], they must accept the responsibility for truth (Balacheff, 1987). Brousseau calls this the *devolution* of the problem.

The planning of instruction based on a constructivist view of learning faces an inherent tension. Brousseau emphasizes that students must have freedom to make a response to the situation based on their past knowledge of the context and their developing mathematical understandings. If the situation leads the students to a particular response, no real learning takes place. However, "if the teacher has no intention, no plan, no problem or well-developed situation, the child will not do and will not learn anything" (Brousseau, 1987, p. 8 - my translation). In the last two sentences, the learning which does not take place is the development of the mathematical ideas which have been identified as the goal of instruction. Students under these conditions learn other things such as how to respond appropriately to the teacher's leading questions.

The creation of appropriate problem contexts (*situations a-didactiques*) is not sufficient for learning. Brousseau points out that situations must be created for the decontextualizing and depersonalizing of the ideas (*situations didactiques*). Learning involves being able to use the ideas beyond the narrow context of the original problem situation. "The teaching process should allow this shift of pupils' interest from being a *practitioner* to becoming a *theoretician*" (Balacheff, 1987).

Also necessary is what the French researchers call "situations for institutionalization" (Brousseau, 1987, Douady, 1985) in which ideas constructed or modified during problem solving attain the status of knowledge in the classroom community. This is consistent with the notion of mathematical knowledge as social knowledge, as knowledge which is taken-as-shared by the classroom community.

This last point brings us to the responsibilities of the teacher in the class situation. The teacher has the dual role of fostering the development of conceptual knowledge among her students and of facilitating the constitution of shared knowledge in the classroom community. Cobb, Wood, and Yackel (in press) have demonstrated that classroom conversations about mathematics, facilitated by the teacher, result in taken-as-shared mathematical knowledge. They have also described a second type of conversation which focuses on what constitutes appropriate and effective mathematical activity in the classroom. Such discussion contributes to the constitution and modification of social norms for mathematical activity, the *contrat didactique*[6] (Brousseau, 1981).

The *Professional Standards for School Mathematics* (National Council of Teachers of Mathematics, 1991) is consonant with these perspectives. Its authors envision teachers' responsibilities in four key areas:

• Setting goals and selecting or creating mathematical *tasks* to help students achieve these goals;

• Stimulating and managing classroom *discourse* so that both the students and the teacher are clearer about what is being learned;

• Creating a classroom *environment* to support teaching and learning mathematics;

• *Analyzing* student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions. (p. 5)

In practice these aspects of teaching, which respond to the social and cognitive aspects of learning, are inextricably interwoven. However, for the purpose of analysis, we can productively focus on particular features of this "cloth." The analysis of data reported on in this paper focuses on the constitution of what the *Standards* calls "tasks"[7] or what Brousseau (1987) calls "situations" for learning. As such, the paper emphasizes the cognitive aspects of the learning although these aspects cannot be focused on in isolation from the social interactions of the mathematics classroom.

As cited above, Brousseau (1987) asserts that teaching must be guided by intentions for learning and plans for situations for learning. These decisions as to the nature and sequence of the mathematics to be taught are made, according to Laborde (1989), based on hypotheses about epistemology and learning. The *Standards* lists three areas of concern which underlay the selection and generation of tasks, "the mathematical content, the students, and the ways in which students learn mathematics" (p. 26). Steffe (1991) stresses that the teachers' plans must be informed by the "mathematics *of* students." "The most basic responsibility of constructivist teachers is to learn the mathematical knowledge of their students and how to harmonize their teaching methods with the nature of that mathematical knowledge" (Steffe & Wiegel, 1992). Articulating what it might mean to teach mathematics based on a constructivist view of learning involves answering the following question, "*How might a balance be developed between the teacher's goals and direction for learning and the teacher's valuing of and responsiveness to the mathematics of his students?*"

Theory Meets Practice in the Context of a Teaching Experiment

This section focuses on data from a classroom teaching experiment, in order to analyze situations in which a constructivist theoretical perspective came up against the realities of real students in a real classroom. The non-routine problems of teaching require an elaboration and modification of theories of learning and teaching. When the researcher/theorist assumes the role of teacher in a research project, he is uniquely positioned to study in a direct way the interaction of his theory and practice. Particularly, this report focuses on the teacher/researcher's on-going decision making with respect to the mathematical content of the course and the tasks and questions which provided a context for study of that content. This section begins with some brief background on the teaching experiment.

The teaching experiment was part of the Construction of Elementary Mathematics (CEM) Project a three-year study of the mathematical and pedagogical development of prospective elementary teachers. The project studied the prospective teachers in the context of an experimental teacher preparation program designed to increase their mathematical knowledge and to foster their development of views of mathematics, learning, and teaching that were consistent with the views espoused in recent reform documents (e.g., National Council of Teachers of Mathematics 1989; 1991). Data collection with 26 prospective elementary teachers (20 of whom finished the program) proceeded throughout a mathematics course, a course on mathematics learning and teaching, a five-week pre-student-teaching practicum, and a fifteen-week student-teaching practicum.

The research on the mathematics course and the course on mathematics learning and teaching employed a constructivist teaching experiment methodology, as described by Cobb and Steffe (1983) for research with individual subjects. We adapted that methodology to research on classroom mathematics (in the manner of Cobb, Yackel, and Wood, in press). The author taught all classes. Classes were videotaped and field notes were taken by project researchers. Videotapes of classes were transcribed for analysis. The author kept a reflective notebook in which he recorded his thinking immediately following teaching and planning sessions. Following each class the author met with a second project researcher to discuss what he and his colleague inferred to be the conceptualizations of the students[8] at that point and to plan for the next instructional intervention. These meetings were audio taped.

The teaching experiment methodology involves "hypothesizing what the [learner] might learn and finding ways of fostering this learning" (Steffe, 1991, p.177). This research report represents an extension of the teaching experiment methodology. Whereas the teaching experiment was created to learn about students' developing conceptions (our primary emphasis), analysis of the decision making of the teacher/researcher in posing problems is potentially a rich source for learning about teaching (Cobb, 1992). This paper is based on such an analysis.

Class lessons generally consisted of small group problem solving and teacher-led whole-class discussions. No lectures were given. The primary mathematical goal of the course was for students to learn to identify multiplicative relationships. Previous research on a variety of populations (Hart, 1981; Inhelder & Piaget, 1958; Karplus, Karplus, Formisano, & Paulson, 1979) and our pretest data with this population of students had shown that identifying ratio relationships tends to be difficult and that additive comparisons are often used where multiplicative comparisons (ratios) are more appropriate. The mathematical content of the course began with exploration of the multiplicative relationship involved in evaluating the area of rectangles.

Data and Analysis

The data presented describes situations of practice in which the ideas about mathematics teaching and the constructivist ideas about learning discussed earlier required elaboration. In particular, this section of the paper focuses on three teaching situations, examining the relationship among the teacher's decision making and the classroom activities. The data presented are from the first 5 weeks of the fifteen-week mathematics course and is taken from class transcripts, the teacher/researcher's notes, field notes from other researchers, and student journals. The data presented are representative of the first instructional unit (8 ninety-minute classes). The organization of the data into three instructional situations reflects the activity around subtopics which seemed to characterize this unit of instruction. For each situation, a description is provided of the challenge that faced the teacher as construed by the teacher, the decision that he made to respond to that challenge, and the subsequent classroom interaction that was constituted by the students and teacher.

The Rectangles Problem initiated the classroom interactions for this unit of instruction.

The Rectangles Problem

As the instructor, I chose to begin the exploration of multiplicative relationships in the evaluation of area of rectangles. My purpose was to focus on the multiplicative relationships involved, not to teach about area. The lesson that I chose was one that I had used several times before with similar groups of students. The lesson is based on the observation that although many prospective elementary teachers respond to area of rectangle problems by multiplying, their choice of multiplication is often the result of having learned a procedure or formula rather than the result of a solid conceptual link between their understandings of multiplication and their understandings of measuring area. This lesson, which was designed to foster the development of that link, was planned to be completed in one day, although I anticipated that it might continue into the next class.

The lesson began with a small cardboard rectangle being given to each of the small groups of students seated at the classroom's six large rectangular tables. The problem was to determine how many rectangles, of the size and shape of the rectangle that they were given, could fit on the top surface of their table. Rectangles could not be overlapped, could not be cut, nor could they overlap the edges of the table. Students were instructed to be prepared to describe to the whole class how they solved this problem. (This task will be referred to subsequently as "Problem 1.")

The students worked collaboratively and generally had little trouble solving this initial problem. The universal strategy was to use the given rectangle as a measure to count the number of rectangles along the length of the table, and the number of rectangles along the width of the table and to multiply these two numbers. However, a question arose in a few of the groups as to whether the rectangle should be kept in the same orientation (see Figure 1a) as it is moved along the length and along the width or whether the rectangle should be turned so that the measuring was always done using the same edge of the rectangle (see Figure 1b). The groups eventually agreed that the former, holding the rectangle in the same orientation, was appropriate.

[Insert Figure 1 about here]

When the small groups came back together for whole class discussion, I asked the students how they solved the problem. They described their method. I then asked, "Why did you multiply those numbers together?" This proved to be difficult for the students to answer. Some asserted that "it seemed like the easiest way," or "in previous math classes you learned the formula for areas." Some responded that they did it because it works; they had seen examples of how the product was the same as the counting up of all the rectangles. I pressed them to consider whether there was any reason to expect that it would <u>always</u> work. Most of the students seemed puzzled by this question. However, Molly offered the following:

> Molly: Well, it would work because, um, multiplying and adding are related in that multiplying is, is like adding groups and so it would always work because you add them up to see how many is in the square and to multiply the groups that go like that, that'll always work. You would get the same number, I'm saying if you added them or if you multiplied that side times that side. Because you're adding, I mean, you're multiplying the number of groups by the number in the groups which is the same as adding them all up. (2,125)[9]

Molly went on to demonstrate on the chalkboard how in a rectangular array [my language] of rectangles each row was a group (see Figure 2) and the number of rectangles in a row was the number in each group. She observed that summing the rectangles in each row (repeated addition) was equivalent to multiplying the number in a row by the number of rows.

[Insert Figure 2 about here]

The other students' subsequent comments suggested that only a few of them perceived that Molly's explanation had advanced the discussion in any significant way.

Situation One. What instructional situation might afford other students the opportunity to construct understandings similar to Molly's? It wasn't that the other students were puzzled by Molly's explanation; they seemed unaffected by it. They continued to respond to the question, "Why multiply?" in ways that indicated that the question did not demand for them this type of justification. Responses included "Cause that's the way we've been taught." and "... it's a mathematical law." Asking the students for explanations and justifications was not sufficient. Our classroom community had not established what counts for mathematical justification (Simon & Blume, 1993). It did not seem that continuing the already lengthy discussion would be fruitful.

I had engaged them in a problem solving activity using a hands-on activity and fostered communication in small and large groups. Yet only a few of my students showed evidence of learning the mathematics that I had intended that they learn. The challenge that I faced was a product of both cognitive and social factors. Cognitively, the majority of my students were employing a well-practiced procedure which was not well-examined conceptually. Socially, they did not have a view of mathematical activity in general and of appropriate activity for our classroom in particular that included the type of relational thinking and development of justification in which I was attempting to engage them.

I realized that the development of norms for classroom mathematics activity would take some time. Such norms would result from the activities in which we engaged as a mathematical community and the discussions that we had about that activity. Their competence in providing justification would grow as they engaged in discussions in which the demand for justification was consistently present (Simon & Blume, 1993). Thus, from a social perspective, I needed to continue the process that I had begun with them. However, this process could not happen in the abstract. Particular content and tasks were needed as the context for the constituting of appropriate mathematical activity. Thus, I returned to my role of problem poser, but "Which problems?" The traditional approach, assigning practice problems similar to the original one, seemed inappropriate. After all, the students were already able to generate correct answers, the real problem was why multiplication was appropriate. I needed to find problems which necessitated an understanding of the link between the solution strategy, counting the number of rectangles along the length and the width and multiplying those quantities, and the goal of determining the total number of rectangles that could be laid out on the table.

To generate such problems, I made use of conceptual difficulties that I had previously observed among students working on the Rectangles Problem (Problem 1). For example:

> **Problem 2**: Bill said, "If the table is 13 rectangles long and 9 rectangles wide, and if I count 1, 2, 3...,13 and then again 1, 2, 3..,9, and then I multiply, 13 x 9, then I have counted the corner rectangle twice." Respond to Bill's comment.

Problem 2 seemed to engage students in making a conceptual link between the goal of counting all of the rectangles and the prevalent solution strategy of counting rectangles along the two sides and multiplying. The following excerpts from the class transcript show the development of these connections. (Note: "Simon" refers to the teacher.)

> Lois: Well, you have to count twice cause if he didn't it wouldn't be 13 by 9, it would be more like 13 by 8 or 12 by 9.. Because there are 13 across and there's 9 down, you can't say there's twelve across because there are 13 and you can't say there are 8 down because there are 9 down.
>
> Simon: Candy?
>
> Candy: If they were walking steps and he started walking and counting each step as he walked and when he got to the corner, he'd have to take 2 steps in the same block.
>
> Simon: So you're saying he is counting them twice. Lilly?

Lilly: I thought, yes he's counting them twice because each corner represents two different rows, represents a horizontal row and a vertical row... I made a diagram with arrows. If you look at the direction of the arrow, they each represent one row. But the corner is the only one that represents two.

[At this point Lilly goes to the board to demonstrate and explain how she used her diagram]

Ellen: Well, I can see you are counting the corner blocks twice but you're not counting the same thing twice because, which is what she's saying, because, that block represents both the length and then the width, so you're not. The question to me sounds like this: Bill thinks that he's going to have to subtract one from his answer just because he would have counted that little square two times, but he wouldn't because he's counting two different things when he is counting that block twice. (3,119-129)

Up to this point, the students are still talking about the proper use of the formula, length-times-width. Karen shifts the discussion to the conceptual underpinnings of this procedure.

Karen: When we're multiplying thirteen times nine we're trying to see ...how much nine thirteens are. If we're looking at it thirteen times nine... or excuse me, how many nines there are.... So if I'm looking at one nine, two nines, three nines, four nines, I could find out how much it would be with those numbers but if I'm looking for thirteen nines I'd would want to see how many of those I would have if I would add them up or if I would multiply them, thirteen times. How many, how much would that be if I had thirteen nines or nine thirteens? I'm looking for the amount, total amount that that would be if I was multiplying how many groups of those or how many sets of those would I have, if I would add each one of them up to get the total amount?

Karen has made some progress in justifying the use of multiplication. However, Toni goes back to how one uses the formula appropriately. Her explanation is based on her identification of the problem as an area problem and her knowledge of how to measure length times width. Once again I attempt to refocus Toni (and, I suspect, other students) on the underlying conceptual issue.

Toni: When we're trying to find how many rectangles would fill that rectangle, we're looking for the area. And when we find area we multiply length times width and the columns would probably represent the length and the rows would represent the width. So by multiplying the number of columns times the number of rows you get total area or the total number of rectangles that would fill the rectangle.

Simon: Why does that work, that when we multiply the number of columns times the number of rows we get the area?

Toni: Cause the columns represent length and the rows represent width.

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Molly: Well, I thought again it referred back to when you're using a row to represent the units in a group, and the columns to represent the number of groups, and since multiplication is the same as repeated addition that when you multiplied the number of units in a group by the number of groups you would get the total number of parts in the whole.

Simon: And how is that connected to this issue about the corner?

Molly: Because it, the corner not only represents a one, it's just one, numbering of a group or it's also numbering a part of that unit, a unit in that group so it's not, it's two different things, just like when they were saying it's a row and a column, well, it's two different things, it's a unit and also representing a group.

Candy: You cannot treat it, I mean, it makes it confusing to try to look at the length times the width....You should really treat it as so many sets or so many groups like nine groups, nine sets, not, having thirteen groups of nine. That way you're not even going to deal with the corner and you won't even have that problem.

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Karen: Have we responded to Bill's problem about him thinking that he has double counted?

Simon: It sounds like you're not totally satisfied that we have. Why don't you--

At this point, Karen brings us back to the original problem. For her, it is not enough to decide whether one would be double counting the corner rectangle; it is also important to understand Bill's thinking which led to his confusion.

Karen: It appears to me that Bill's thinking about addition, counting in his mind or at least in his thoughts as far as counting by ones, I'm not sure if it was Candy or who, said, maybe Lilly, said that that one represents two different things but in his mind, at least from what he said, it appears that he only sees one as representing one thing and that is a counting number. He thinks he's already counted it.

Candy and Karen's comments seem to demonstrate an understanding of how the counting (vertically and horizontally) and the multiplication are related. I push for further verbalization of the ideas involved to ascertain whether others in the class have constructed similar meanings.

Simon: So Bill thinks we're counting one, two, three, four, five, six, seven, eight, nine boxes, right? And then we're counting one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen boxes and if that's true, we definitely counted this as a box twice, right? Now what are you telling? What are you counting when we count, one, two, three, four, five, six, seven, eight, nine, what did I just count?

Class: Rows.

Simon: ...This is our number of rows? OK. What did I count here? One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen.

Class: Columns.

My spontaneous analysis is that many of the students have latched onto the notion of rows and columns in an unexamined way. The shift from counting boxes to counting rows and columns does not in itself lead to a connection between counting the total number and multiplying the two counts. (It may be that the creation of multiple units, rows and columns, is progress towards this goal) I refocus my questions on this connection.

Simon: So I'm not counting boxes at all?

Class: No.

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Simon: OK. Isn't it a little mysterious that we're never counting boxes here and we wind up with a number of boxes? Does that bother anybody? We didn't count boxes here, we didn't count boxes there and we wind up with boxes at the end. Tammy?

Tammy: If each box represents a portion of the row so we're really counting boxes but we're just putting them in a set instead of individually.

Simon: OK, so which way are you thinking about the sets going? This way or that way? Choose one.

Tammy: Vertically?

Simon: So this is a set? OK. So you are saying this is nine what?

Tammy: Nine separate units inside of a set.

Simon: ...OK. So here I counted nine boxes in a set and then here I'm counting what?

Tammy: Thirteen.

Simon: Thirteen what?

Tammy: Boxes in a set.

[Ellen is shaking her head]

Simon: Ellen, you don't like that.

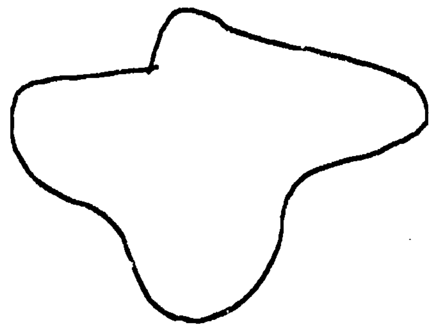
Ellen: If you're going to do it that way, I think you have to say that you're going to take the column as a set, nine boxes in one set...then the thirteen in the row is the number of sets that you have so it's not actually boxes, it's the number of that same type of set that you have. (3, 136-187)

Ellen's final comment evoked many nods of agreement from her classmates. I considered, however, that for a student to follow an explanation might not require the same level of understanding as needed to generate an explanation. Still it seemed clear to me from students' verbalizations that the number of students who were seeing a connection between multiplication and counting the total number of rectangles had increased. Perhaps Problem 2 shifted the discussion from my problem, justifying the method that the students believed to be valid, to a community problem, how to account for the "double counting."[10] Problem 2 seemed to provide a puzzlement, at least initially for most of the students.

Situation Two. As we proceeded to explore the multiplicative relationship involved in evaluating the area of a rectangle, I came to believe that the context in which we were working, area, was not well understood by many of the students. They seemed to think about area as generated by multiplying length times width. Although my primary focus was on multiplicative relationships, not on area, it seemed clear that an understanding of area was necessary in order for students to think about constituting the quantity (area) and evaluating that quantity. What action could I take as a teacher?

It seemed that if indeed these students were unclear about what is meant by area, that the traditional response of "reviewing" the idea would be inadequate. Surely by this time, junior year of college, they had been present at many such reviews and had some ideas which could be built upon. I chose instead to pose a problem that would push them to extend their understanding of area. I posed the following:

Problem 3: How can you find the area of this figure?



Students generated ideas in small groups. When we reconvened as a whole class, they shared their ideas. Two methods stimulated a great deal of discussion. The first was a suggestion that a string be put along the outline of the "blob" and then, without changing the length of the string, reshape it into a rectangle, a figure whose area we know how to determine. This method was based on an assumption, that was eventually rejected by the class, that figures that have the same perimeter also have the same area. The second method was to cut out cookie dough of a constant thickness so that it exactly covered the blob. Then, cut the cookie dough into squares of a given size. Roll up the remaining dough and roll it out to the original thickness, again cutting out squares, repeating this procedure until there is not enough dough remaining to make an additional square. This method was accepted by the class as theoretically sound; however they predicted that, in practice, it would be difficult to carry out accurately.

The problem generated more than discussion on the validity of these methods. Because of the contrast in the methods proposed by the students, we were able to consider the issue of conservation of area, that is, under what changes in shape is the original amount of area preserved. This was a challenging context for the students to think about the meaning of area. Considerable dialogue ensued in which students seemed to be using the notion of area appropriately, comparing area among different geometric shapes, and distinguishing it from the notion of perimeter which was brought up by the string strategy.

Situation Three. After providing students the opportunity to articulate why they multiplied to determine the total number of rectangles, and after our work with the "blob problem" (#3), I raised again the issue that they had brought up in solving the original problem, whether to turn the rectangle (Figure 1b) or to maintain its orientation (Figure 1a). I demonstrated the former, holding the rectangle one way to measure the length and turning it 90 degrees to measure the width (as in Figure 1b). Most of the students responded, saying that this method was not correct, that it did not give the number of rectangles that could fit on the table. After some discussion, I asked an extension of that question, does the number obtained when we measure this way and multiply the two numbers have any meaning? "Does it tell us anything about this particular table?" (This question is referred to as Problem 4. It was actually posed before the blob problem. Its discussion was interrupted based on my perception that students needed an opportunity to focus on the meaning of area. we then returned to Problem 4 following Problem 3.)

Students were not only convinced that the number generated was not useful, they also developed consensus that when you turn the rectangle to measure the other dimension you generate a set of "overlapping rectangles." They reasoned that since the rectangles are overlapping, the number that is obtained is nonsense.

I was surprised by their response and concerned about what it indicated about their understanding of the multiplicative (area) unit. I had posed this extension to encourage an understanding of the multiplicative relationship between linear measures of the rectangle and area measures. I anticipated that students might connect a visual sense of area (perceived/measured amount of surface) with an abstract concept of multiplication. After all, this measurement strategy and subsequent computation was identical to what they previously had learned to do with a ruler. In prior courses, when I posed this question, several students understood that the quantity resulting from multiplying the two linear measures represented the number of square units of area on the table. The square unit had sides equal to the length of the rectangle used for measuring. (See Figure 3.) The discussion that ensued generally led to a consensus with respect to this point.

[Insert Figure 3 about here]

However, unlike my previous experiences, no one in this class seemed to see this method as generating square units of sides equal to the length of the rectangle. Rather they were confident in their view that the number generated by this method was nonsense because it resulted in overlapping rectangles. I tried in different ways to promote disequilibrium so the students would reconsider the issue. One spontaneous question that I asked was the following.

Simon: Out in the hall I have two [rectangular] tables of different sizes. I used this method ... where I measure across one way, turn the [rectangle], measure down the other way, and multiply....When I multiplied using [this] method, on table A I got 32 as my answer and [when I measured] table B [using the same rectangle and the same method], I got 22. Now what I want to know is, [having used] the method of turning the rectangle, is table A bigger, is table B bigger, or don't you have enough information from my method to tell? (3,220)

Students decided that table A was bigger than table B, because 32 is greater than 22. I then asked, "32 what and 22 what?" They remained unshaken in their resolve that the 32 and 22 did not count anything that was meaningful and that this method generated overlapping rectangles (as in the upper left-hand corner of Figure 1b).

How could I understand the thinking of these students and how could I work with them so that they might develop more powerful understandings? I had never been confronted with these questions in this context before. I had always posed the problem (of turning the rectangle) within the whole class discussion and there had always been some students who explained about the square units to their classmates. I had assumed that the other students understood. My attempts at creating disequilibrium with my current students, a key part of my theory and practice, had been ineffectual.

One advantage of our teaching experiment design was that time was structured into the project to reflect, in collaboration with a colleague, on the understandings of the students. Our reflection led to the following hypotheses.

As a result of the initial problem (Problem 1) and because the measurement was still being done using the cardboard rectangle, the students anticipated incorrectly that the unit of area would be rectangles of the size of the cardboard one. They considered that when they lay the rectangle along one edge of the table that they were making a row of rectangles, a set, an iterable unit. (The discussion of Problems 1 and 2 had encouraged students to think of a row as an iterable unit; Molly's view of a unit of units now seemed to be taken-as-shared.) When they moved the rectangle down the other side, they were counting the number of iterations. This way of viewing the situation was adequate for the original problems, but inadequate for solving to the extension question (Problem 4) where a unit other than the rectangle itself was being created. The students were not "seeing" that measuring with the rectangle was a process of subdividing the length and width of the table into smaller linear units and that these units together implied a rectangular array of units, the size of these units determined by the size of the linear units. (A detailed analysis of the understandings involved is presented in Simon and Blume, 1992.)

Having constructed an understanding of the students' thinking[11], I still needed to generate an appropriate instructional intervention. I reasoned that if students had misanticipated the unit of area, assuming that the cardboard rectangle was the appropriate measure, then providing them with a context that did not invite misanticipation might provide them the opportunity to determine an appropriate unit of area based on linear units. Eventually, they would still need to sort out the problem involving the turned rectangles. This thinking led me to generate Problems 5 and 6.

> **Problem 5**: Two people work together to measure the size of a rectangular region, one measures the length and the other the width. They each use a stick to measure with. The sticks, however, are of different lengths. Louisa says, "The length is four of my sticks."

Ruiz says, "The width is five of my sticks." What have they found out about the area of the rectangular region?

Students worked Problem 5 in groups of three. Some of the students were able to see how the use of different size sticks could be thought of as determining an array of non-square rectangles. In the class discussion that followed, Toni explained,

Toni: ...the way we looked at it, it would pretty much be the same thing as a rectangle because if you had a rectangle, it doesn't matter what size your stick is because in a rectangle, the length and width of the rectangle, like the width of the rectangle is smaller than the length of the rectangle so it doesn't really matter, I mean, the area ... would be 20 [of these small rectangles]. (6,593)

Toni's explanation came at the end of class. Students came to the next class having thought further about her explanation.

Ellen: I didn't really understand until Toni drew that diagram on the board, that I found something out about area and when she put that on the board, I realized that I had started thinking that you were starting with the unit, and you had to start with the unit to figure out the area. But what she did was she sort of went, in my mind she sort of went backwards, she ended up with a unit of measurement by um making the rectangle,

Simon: (inaudible)

Ellen: Like by making the rectangle out of the lengths and the widths of the sticks, okay. Because the unit you're using is a rectangle that has a length the size of the one stick and the width the size of the other stick, so you're sort of going backwards and ending up with a unit that means something....(6,61-63)

Through the discussion of this problem and an extension question which asked them to express the area of Ruiz and Louisa's rectangular area in terms of other units, I was persuaded that most of the students understood the relationship between the linear measures and the area measures in this context. The next step was to see if they could use this understanding to revisit the problem involving turning the rectangle. Would the understandings that they had developed in Problem 5 allow them to question their assumption of the cardboard rectangle as the appropriate unit of measure. Problem 6 revised Problem 4 in an attempt to make the problem more concrete by having them actually measure out particular rectangular regions using the method in question.

> **Problem 6**: I used your [cardboard] rectangle and my method (rotating the rectangle) to measure two rectangular regions; one was 3x4 and the other was 5x2. Draw these regions (real size). Record all that you can determine about their areas.

Half of the small groups determined that squares were useful units for describing the areas of the rectangular regions and could explain their thinking.

> Eve: First when we started drawing it, we drew like all the rectangles, OK? So it showed the overlapping up in the corner, but then we thought just take away the overlapping, ... and just think of it as a side, like this is a side and this is a side, like the sticks that Louisa and Ruiz used, just to get this little edge right here as a stick and this little edge right here as a stick, OK? So, we came out with ours was, well, they're all the same [refers to units used to measure the length and the width], three, three sticks by four sticks, OK? And since, like this one here is the same length and this length here, does everybody understand that? OK, so what it actually makes is squares because they're the same length ... so actually you could talk about your area in squares, like you wouldn't talk about your rectangles cause it has no relevance. (8,553)

Some of the students who had not previously seen the usefulness of square units in Problem 6, did so in the course of the class discussion. However, a number of students, who saw that measuring in squares worked for the problem, were unclear how one would know "when to use rectangles and when to use squares."

Discussion

The discussion of the teaching situations is divided into two parts. The first part examines the role of teacher that emerges in terms of the decision making about content and task.

This discussion, which focuses on the composite picture of teaching seen across the three situations presented, leads to the articulation of a model of teacher decision making called "The Mathematics Teaching Cycle." The second part of the discussion highlights particular aspects of each of the situations (considered separately), in an attempt to elaborate the role of teacher as described by the model.

As teacher/researcher I began the mathematics course with particular theoretical perspectives on teaching and learning, some of which were articulated in the "Background" section of this paper. However, not surprisingly, these perspectives did not "tell" me what to do as challenges arose. My responses to these challenges often were not the result of well-articulated models. Rather they were emerging patterns of operation in problem situations. It is only through a posteriori analysis that such patterns are characterized and used to enhance, modify, or develop theory. Thus, the reader should keep in mind that the theoretical aspects of this section grew out of the data analysis and were not part of my explicit thinking as I was planning and teaching. Also, these patterns of operation were established over repeated encounters with similar problem situations. I had taught essentially this way for many years. This particular teaching experience led to further elaboration of my teaching; the teaching experiment design led to a new level of analysis of the teaching.

In this section, I use the first person singular to refer to my actions and thinking as the teacher. I use the third person, often referring to "the teacher" to designate ideas which I am lifting from the particular context in which I was the teacher.[12]

Unpacking the Teaching Episodes: Developing Theory

This section analyzes the teacher's role as decision maker as it emerges across the three teaching situations.

The rectangles lesson was shaped by my understanding of the multiplicative relationship between the area of a rectangle and its linear measures. (The focus on my personal knowledge does not discount that this knowledge can be viewed as socially constituted and taken-as-shared in the mathematics education community. Rather it is expedient here to focus on my particular interpretation of these socially accepted ideas.) My previous experience with prospective elementary teachers[13] led me to hypothesize that my students would not share this knowledge. Rather, I expected that their knowledge would be rule-bound and that the concepts underlying the formula for the area of a rectangle would be unexplored. The disparity between my understanding, which I judged to be useful, and my sense of their understanding defined my learning goal for the first segment.

Having established my initial goal as an understanding of the relationship of multiplying length x width to the evaluation of the area of a rectangle, I considered possible learning activities and the types of thinking and learning that they might provoke. Following is a partial reconstruction of my thought process.

> I suspect that for many of my students A = l x w is a formula that has no conceptual roots. Concrete experience with area might be helpful. I need to keep in mind that they come to the task already knowing the rote formula. A learning situation that does not look like their previous experiences with area might preempt their resorting immediately to rote procedures.
>
> Tiling a rectangular region would provide the concrete experience. If I can envision a situation in which they form multiple units, sets of tiles, they may see the appropriateness of multiplication, in essence "deriving" the formula. I will not mention area, but ask them to find out how many tiles. However, if they are to make connections with l x w, they must do more than count each tile.
>
> If I give them only one small tile, they will need to look for an efficient way of determining the number of tiles - which will encourage them to go beyond counting all of the tiles. If I use rectangular tiles, they will not be able to measure mindlessly; they will need to consider how their measuring relates to the placement of the tiles on the table. Measuring with a non-square rectangle to determine the area encourages a level of visualization that is not required when one uses a ruler to determine square units, that is,

they will have to take into account what they are counting, the unit of measure, which is based on how they are laying the tiles on the table.

The preceding thought process provides an example of the reflexive relationship between the teacher's design of activities and consideration of the thinking that students might engage in as they participate in those activities. The consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage make up the hypothetical learning trajectory, a key part of the Mathematical Learning Cycle described below. Besides the teacher's knowledge of mathematics and his hypotheses about the students' understandings, several areas of teacher knowledge come into play including the teacher's theories about mathematics teaching and learning; knowledge of learning with respect to the particular mathematical content area (deriving from the research literature or the teacher's own experience with learners); and knowledge of mathematical representations, materials, and activities. The Mathematical Learning Cycle, described in the next section, portrays the relationship of these areas of knowledge to the design of instruction.

The only thing that is predictable in teaching is that classroom activities will not go as predicted. While the teacher creates an initial goal and plan for instruction, it generally must be modified many times (perhaps continually) during the study of a particular conceptual area. As students begin to engage in the planned activities, the teacher communicates with and observes the students which leads the teacher to new understandings of the students' conceptions. The learning environment evolves as a result of interaction among the teacher and students as they engage in the mathematical content. Steffe (1990) points out that, "A particular modification of a mathematical concept cannot be caused by a teacher any more than nutriments can cause plants to grow." (p. 392) A teacher may pose a task. However it is what the students make of that task and their experience with it that determines the potential for learning.

Student responses to the rectangle problem led me to believe that students did not adequately understand what is meant by area. As a result, I generated a new learning goal, understanding area. This goal temporarily superseded but did not replace the original learning goal. Towards this end I posed the "blob" problem anticipating that the students would brainstorm some ways to find area, discuss those ways, and in so doing strengthen their understanding of area. However, the specifics of what happened resulted in additional, unanticipated learning. First, students proposed the string strategy. Based on the understanding that I had developed of the students' conceptual difficulties in considering the string strategy, and based on the interesting contrast that I saw between the two strategies (as a result of my own mathematical understandings), I revised my goal for instruction once again. I now saw as the (local) goal to facilitate students' understanding of conservation of area (not limited to Piaget's assessment of the concept with children). I intended for my students to engage the question, "Under what types of change in shape does the area of a region remain invariant?"

My interest in them constructing answers to this question were based on three factors. (1) I believed that it would further their understanding of area, my motivation for posing the blob problem, (2) I saw an opportunity for learning based on the juxtaposition of the two strategies - an opportunity which I had neither planned for nor anticipated, and (3) I already believed that the concept of invariance, which I had thought about previously in relation to arithmetic concepts but not area, was an important one. This final point also points out how my own understanding of the mathematical connections involved is enhanced as I attend to the mathematical thinking of my students. This evolution of the teacher's mathematical knowledge is also revealed in the analysis of the third episode, the data involving measuring with only the long side of the rectangle (Problem 4).

My original goal that motivated the rectangles lesson was for my students to understand the evaluation of the area of a rectangle as a multiplicative relationship between the linear measures of the sides. For me, as I began instruction, such an understanding involved connecting an understanding of multiplication as repeated addition with the notion of identical rows of units of area and understanding the relationship between linear units and area units. The latter concept was represented by the issue of turning the rectangle to measure- I had not unpacked it further.

The classroom discussion, however, pushed me to reexamine these understandings and to further elaborate my map of the conceptual terrain. (The use of the term "map" in this context is meant to emphasize that the teacher's understandings serve as a map as he engages in making sense of students' understandings and identifies potential learnings.) The students misanticipation of the area unit (assumption that the area would necessarily be measured in terms of cardboard rectangles) led me to explore the importance of anticipating an appropriate unit. Anticipating the area unit seemed to involve both an anticipation of the organization of the units, a rectangular array, and an understanding that the linear units define the size and shape of the units within that array. (For a fuller discussion, see Simon and Blume, 1992.) The multiplicative relationship, therefore, involved the coordination of the linear units to determine an area unit within an anticipated rectangular array. What I had observed in my students had changed both my perspective on my students' knowledge and my perspective on the mathematical concepts involved (my internal map). This reorganization of my perspectives led to a modification in my goals, plans for learning activities, and the learning/thinking that I anticipated.

The Mathematics Teaching Cycle

The analysis of these teaching episodes presented has led to the development of the Mathematics Teaching Cycle (Figure 4) as a model of the cyclical interrelationship of aspects of teacher knowledge, thinking, decision making, and activity that seems to be demonstrated by the data.

The three episodes create a picture of a teacher whose teaching is directed by his conceptual goals for his students, goals which are constantly being modified. The original lesson involving the rectangles on the table was not a random choice nor was it chapter one in someone's textbook. The goal for and design of the lesson were based on relating two factors, the teacher's mathematical understanding and the teacher's hypotheses about the students' knowledge. I refer to "hypotheses" about students knowledge to emphasize that the teacher has no direct access to students' knowledge. He must infer the nature of the students' understandings from his interpretations of the students' behaviors, based on his own schemata with respect to mathematics, learning, students, etc. Implied is that the teacher can only compare his understanding of a particular concept to his construction of the students' understandings, not to the students' "actual" understandings.

As the teacher, my perception of students' mathematical understanding is structured by my understandings of the mathematics in question. Conversely, what I observe in the students' mathematical thinking affects my understanding of the mathematical ideas involved and their interconnections. These two factors are interactive spheres of teacher thought.

Steffe (1990) states,

> ...using their own mathematical knowledge, mathematics teachers must interpret the language and actions of their students and then make decisions about possible mathematical knowledge their students might learn. (p. 395)

The teacher's learning goal provides a direction for a hypothetical learning trajectory. I use the term "hypothetical learning trajectory" to refer to the teacher's prediction as to the path by which learning might proceed. It is hypothetical because the actual learning trajectory is not knowable in advance. It characterizes an expected tendency. Individual students' learning proceeds along idiosyncratic, although often similar paths. This assumes that individuals' learning has some regularity to it (cf. Steffe, Von Glasersfeld, Richards, & Cobb, 1983), that the classroom community constrains mathematical activity often in predictable ways, and that many of the students in the same class can benefit from the same mathematical task. A hypothetical learning trajectory provides the teacher with a rationale for choosing a particular instructional design; I make my design decisions based on my best guess of how learning might proceed. This can be seen in the thinking and planning that preceded my instructional interventions in each of the teaching situations described.

The hypothetical learning trajectory is made up of three components: the learning goal which defines the direction, the learning activities, and the hypothetical learning process - a prediction of how the students' thinking/understanding will evolve in the context of the learning activities. The creation of and on-going modification of the hypothetical learning trajectory is the central piece of the model which is diagrammed in Figure 4. The notion of a hypothetical learning trajectory is not meant to suggest that the teacher always pursues one goal at a time or that only one trajectory is considered. Rather, it is meant to underscore the importance of having a goal and rationale for teaching decisions and the hypothetical nature of such thinking. Note the development of a hypothetical learning process and the development of the learning activities have a symbiotic relationship; the generation of ideas for learning activities is dependent on the teacher's hypotheses about the development of students' thinking and learning; further generation of hypotheses of student conceptual development depends on the nature of anticipated activities.

[Insert Figure 4 about here]

The choice of the word "trajectory" is meant to refer to a path, the nature of which can perhaps be clarified by the following analogy. Consider that you have decided to sail around the world in order to visit places that you have never seen. One does not do this randomly (e.g. go to France, then Hawaii, then England) nor is there one set itinerary to follow. Rather, you acquire as much knowledge relevant to planning your journey as possible. You then make a plan. You may initially plan the whole trip or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing and about the areas that you wish to visit. You change your plans with respect to the order your destinations. You modify the length and nature of your visits as a result of the interactions with people along the way. You add destinations which prior to your trip were unknown to you. The path that you travel is your "trajectory." The path that you anticipate at any point in time is your "hypothetical trajectory."

The generation of a hypothetical learning trajectory prior to classroom instruction is the process by which (according to this model) the teacher develops his plan for classroom activity. However, as the teacher interacts with and observes the students, the teacher and students collectively constitute an experience. This experience by the nature of its social constitution is different from the one anticipated by the teacher. Simultaneous with and in interaction with the social constitution of classroom activity is a modification in the teacher's ideas and knowledge as he makes sense of what is happening and what has happened in the classroom. The diagram in Figure 4 indicates that the assessment of student thinking (which goes on continually in the teaching model presented) can bring about adaptations in the teacher's knowledge which, in turn, lead to a new or modified hypothetical learning trajectory.

Figure 5 describes the relationship among various domains of teacher knowledge, the hypothetical learning trajectory, and the interactions with students. Beginning at the top of the diagram, the teacher's knowledge of mathematics in interaction with the teacher's hypotheses about the students' mathematical knowledge contribute to the identification of a learning goal. These domains of knowledge, the learning goal, and the teacher's knowledge of mathematical activities and representation, his knowledge of students' learning of particular content, as well as the teacher's conceptions of learning and teaching (both within mathematics and in general) contribute to the development of learning activities and a hypothetical learning process.

[Insert Figure 5 about here]

The modification of the hypothetical learning trajectory is not something that only occurs during planning between classes. The teacher is continually engaged in adjusting the learning trajectory that he has hypothesized to better reflect his enhanced knowledge. Some times fine tuning is in order, while at other times the whole thrust of the lesson must be discarded in favor of a more appropriate one. Regardless of the extent of modification, changes may be made at any or all of the three components of the hypothetical learning trajectory: the goal, the activities, and/or the hypothetical learning process.

Aspects of Teaching

Each of the three teaching situations portrays particular aspects of what teaching, which embodies reform principles, might be like. I discuss a few of these in this section.

The original rectangles problem was planned for one or two classes; instead eight classes were spent on the mathematics that was generated. Experienced teachers might affirm that it is difficult to determine in advance exactly how long it will take to teach a particular concept. However, the discrepancy between the amount of time anticipated and the amount of time spent in this case is well beyond the imprecision of planning. This discrepancy points at the experimental nature of mathematics teaching. "Experimental" denotes the ongoing cycle of hypothesis generation (or modification) and data collection that characterizes the teaching portrayed.

In the first situation, Problem 1 involving the tiling of the tables, I, as the teacher, perceived a lack of understanding among a majority of the students of the relationship between length-times-width and the counting of all the rectangles on the table. My response was to pose additional problems based on students' thinking that I had witnessed in the past. I selected thought processes that I thought students could determine as not viable, but which would likely be problematic initially for them to invalidate. (Problem 2 is an example.) My rationale was that previous students' conceptual difficulties (from the teacher/researcher's perspective) are potential difficulties for my current students and represent useful hurdles for them to encounter in the development of more powerful ideas.

This approach represents a sharp contrast from the approach to instruction characteristic of traditional mathematics instruction and represented by mathematics textbooks. Traditional instruction tends to focus on one skill or idea at a time and then provide considerable routine practice to "reinforce" that learning. The mathematics is subdivided into small fragments for instruction so that students can experience success on a regular basis. In contrast, situation one demonstrates a view of learning as one involving a complex network of connections. Learning is likely to be fostered by challenging the learner's conceptions using a variety of contexts. The teacher can be compared to an athletic coach who employs a variety of practice activities which challenge the athletes' strength and skill often beyond what is required of the athlete in competition (dribbling two basketballs while blindfolded, playing a soccer game where each player may not touch the ball two consecutive times, performing a figure skating program three times in a row with only a 1 minute rest in between). These activities are not aimed at constant success, but rather at increased competence. Growth is a result of challenge to body and mind. Conceptual difficulties that I have previously observed in students are not to be avoided; rather they provide particular challenges, which if surmounted by the students, result in conceptual growth. This fits with the French researchers' notion of "*obstacles epistemologiques*" (Bachelard, 1938 cited in Brousseau, 1983), that overcoming certain obstacles is a natural and essential part of conceptual development. These obstacles are a result of prior concepts which, although adaptive in earlier contexts, are maladaptive given the demands of the current problem situation.

A second feature of the approach seen in situation one is subtler. As a teacher, I often do not have a well developed map of the mathematical conceptual area in which I am engaging my students; that is, I may not have fully articulated for myself (or found in the literature) the specific connections that constitute understanding or the nature of development of understanding in that area. Rather, as was the case when I started the rectangles instructional unit, my knowledge of what it means to understand the particular concept may be carried in part by particular problem situations. The kinds of difficulties that students encounter provide me with key pieces of what it means to understand. Thus, in such cases, my operational definition of understanding is the ability to overcome these particular difficulties; I may not have unpacked the difficulties in order to understand the conceptual issues that are implicated. Thus, even if I do not have a thorough knowledge of what constitutes mathematical understanding in a particular domain, having a rich set of problem situations which challenge students and having knowledge of conceptual difficulties which they typically encounter provide me with an approximation that lets me be reasonably effective in promoting learning in the absence of more elaborated knowledge. (This is not to suggest that the more elaborated understanding would not be more powerful.) Indeed engaging students in these problem situations and with these conceptual difficulties gives me an opportunity to learn more about what it means to understand the concepts involved.

Underlying Situation Two is an idea that highlights a difference between teaching based on a traditional view of learning and teaching based on a constructivist perspective. Rather than "review" what is meant by area or assign "practice problems," my approach was to challenge the students in a way that might push them to extend their conceptions of area. The review and practice approach is based on learning as improving storage and retrieval of received information. (While I am not negating the importance of memory or of information, I contend that it is not what is most important, most interesting, and most problematic for educators in the domain of mathematics.) My approach in Situation Two reflected a view of understanding as a network of connected ideas which is further elaborated as the knowledge is used to solve novel problems.

Situation Three was the most difficult to analyze. I brought to the teaching situations a view that learning is triggered by disequilibrium. When the students were convinced that the rotated rectangle method of measuring and calculating provided no useful information about the table, I tried in every way I could to provoke disequilibrium, but to no avail. In-depth analysis of the data suggests that my interpretations of the students' thinking which led to their conclusion was not adequate. Whereas I had thought that they saw my method as counting the number of overlapping rectangles, I now believe that they were saying that the method involving turning the rectangle counted **nothing** because in the process I was overlapping the rectangles. This apparent difference in thinking may account for my inability to foster disequilibrium.

Having failed to promote disequilibrium, I embarked intuitively on another strategy. I backed away from the particular problem to try to focus on a part of the understanding demanded by the problem. Posteriori analysis suggests that what I was doing was fostering the development of knowledge which, when the students returned to that problem, would then contribute to the students' experience of a cognitive conflict. In this case, if I could help students build an understanding of the relationship between linear and area measures of a rectangle, they would then experience a conflict between those understandings and the expectation that measuring with the cardboard rectangle resulted in a measurement where that rectangle was the unit of area. This teaching episode seems to emphasize that disequilibrium is not created by the teacher. He can try to promote disequilibrium. However, the success of such efforts are in part determined by the adequacy of his model of students understanding. It also seems to support the notion that learning does not proceed linearly. Rather, there seem to be multiple sites in one's web of understandings on which learning can build.

In Summary

Constructivist views of learning have provided a theoretical foundation for mathematics education research and a framework within which teachers can understand their students. However, constructivism also poses a challenge to the mathematics education community to develop models of teaching which build on and are consistent with this theoretical perspective. Small group interaction, non-routine problems, and manipulative materials can be valuable tools in the hands of mathematics teachers. Yet these tools in themselves are not sufficient to allow teachers to be the architects of productive learning situations resulting in conceptual growth.

By what means can a teacher help students to develop new, more powerful mathematical concepts? Novice teachers, who want their students to 'construct" ideas, often ask for the knowledge from their students, consciously or unconsciously hoping that at least one student will be able to explain the idea to the others (Simon, 1991). Such an approach does not deal with the key question of "How does a teacher work with a group of students who do not have a particular concept to foster their development of that concept?"

The principal currencies of the mathematics teacher (if lecturing is rejected as an effective means of promoting concept development) are the posing of problems or tasks and the encouragement of reflection. The data analysis described in this paper and resulting Mathematics Teaching Cycle address the issue of the process by which a teacher can make decisions as to the content, design, and sequence of mathematical tasks. The model emphasizes the important interplay between the teacher's plans and the teacher/students' constitution of classroom activities. The former involves creation of instructional goals and hypotheses about how students might move towards those goals as a result of their collective engagement in particular mathematical tasks. However, the teacher's goals, hypotheses about learning, and design of activities change continually as the teacher's own knowledge changes, a result of his involvement in the culture of the mathematics classroom.

> A goal structure for mathematics education such as the one elaborated by Treffers (1987) is needed in specifying possible learning environments by teachers. But this element of possible learning environments is just as dependent on the experiential fields that constitute learning environments as the latter are dependent on the former. Mathematics educators should not take their goals for mathematics education as fixed ideals that stand uninfluenced by their teaching experiences. Goal structures that are established prior to experience are only starting points and must undergo experiential transformation in actual learning and teaching episodes. (Steffe, 1991, p. 192)

Steffe's comments seem to underscore the cyclical nature of this teaching process.

The Mathematics Teaching Cycle portrays a view of teacher decision making with respect to content and tasks which has been shaped by the meeting of a social constructivist perspective with the challenges of the mathematics classroom. Several themes are particularly important in the approach to decision-making represented by this model.

1. Students' thinking / understanding is taken seriously and given a central place in the design and implementation of instruction. Understanding students' thinking is a continual process of data collection and hypothesis generation.

2. The teacher's knowledge evolves simultaneously with the growth in the students' knowledge. As the students are learning mathematics, the teacher is learning about mathematics, learning, teaching, and about the mathematical thinking of his students.

3. Planning for instruction is seen as including the generation of a hypothetical learning trajectory. This view acknowledges and values the goals of the teacher for instruction and the

importance of hypotheses about students' learning processes (ideas which I hope I have demonstrated are not in conflict with constructivism).

4. The continually changing knowledge of the teacher (see #2) creates continual change in the teacher's hypothetical learning trajectory.

These last two points address directly the question raised earlier in the paper of balance between direction (some may call this "structure") and responsiveness to students, a creative tension which shapes mathematics teaching. The model suggests that as mathematics teachers we strive to be purposeful in our planning and actions, yet unattached to (flexible in) our goals and expectations.

The mathematics education literature is strong on the importance of listening to students and assessing their understanding. However, the emphasis on anticipating students' learning processes is not developed by most current descriptions of reform in mathematics teaching. Research on development of particular mathematical knowledge (cf. Steffe, Von Glasersfeld, Richards, & Cobb, 1983; Thompson, in press) informs such anticipation. Perhaps one explanation for the success of Cognitively Guided Instruction (Carpenter, Fennema, Peterson, & Carey, 1988), which was based on research on children's thinking (Carpenter & Moser, 1983), is that it increased teachers' ability to anticipate children's learning processes.

The data from this study must be seen in its particular context. The teaching practice was embedded in a teacher education program; the mathematics students were prospective elementary students. As the teacher I felt no pressure to teach from a preset curriculum nor to "cover" particular mathematical content, a condition that is probably the exception rather than the rule for mathematics teachers. Mathematics teaching with other populations involves a set of different constraints. Research in other contexts will inform us as to the degree of context dependence of the observed phenomena.

A possible contribution that can be made by the analysis of data and resulting model reported on in this paper is to encourage other researchers to examine teachers' "theorems-in-action" and to make explicit assumptions, beliefs, and emerging theory about teaching. At the minimum, the paper should serve to emphasize the need for models of mathematics teaching which are consistent with and built upon emerging theories of learning.

A well-developed conception of mathematics teaching is as vital to mathematics teacher educators as well-developed conceptions of mathematics are to mathematics teachers. Informed decisions in each case are dependent upon a clear sense of the nature of the content Considering the Mathematics Teaching Cycle as a way to think about mathematics teaching means that teachers would need to develop abilities beyond those already currently focused on in the mathematics education reform, particularly the ability to generate hypotheses about students' understandings (which goes beyond soliciting and attending to students' thinking), the ability to generate hypothetical learning trajectories, and the ability to engage in conceptual analysis related to the mathematics that they teach. This last point supports proposed reforms of mathematics for teachers (cf. Cipra, 1992; Committee on the Mathematical Education of Teachers, 1991).

Finally, it should be noted that the role of the mathematics teacher as portrayed in this paper is a very demanding one. The pedagogical responsibility that I took on in this project may be inappropriate to expect classroom teachers to accept. Teachers will need relevant research on children's mathematical thinking, innovative curriculum materials, and professional support in order to meet the demands of this role.

Footnotes

[1]Ball (1988) defines knowledge of mathematics as conceptual and procedural knowledge of the subject and knowledge about mathematics as:...understandings about the nature of the discipline--where it comes from, how it changes, and how truth is established. Knowledge about mathematics also includes what it means to "know" and "do" mathematics, the relative centrality of different issues, as well as what is arbitrary or conventional versus what is necessary or logical....

[2]I interpret Richards statement, "Students will not become active learners..." to reflect his interest in fostering more independent and reflective mathematical investigations and discussions among students. From a constructivist perspective, students are always active learners, however the nature of what is constructed in different classroom contexts may vary greatly.

[3]A mission of the National Center for Research in Mathematical Sciences Education at the University of Wisconsin is to bring these two domains of inquiry together.

[4]Laborde attributes these three points to Tom Romberg, University of Wisconsin.

[5]Although each problem solver may construct a somewhat different understanding of the problem, negotiation commonly takes place in the classroom to arrive at a taken-as-shared interpretation of the problem.

[6]The contrat didactique is also established by classroom routines which are not explicitly discussed.

[7]The Standards defines "tasks" as "the projects, questions, problems, constructions, applications, and exercises in which students engage. They provide the intellectual contexts for students' mathematical development" (p.20).

[8]In this section, "students" refers to the prospective elementary teachers participating in the teaching experiment.

[9]The numbers denote transcript number, paragraph number.

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[10] We cannot assume that the issue of whose problem was it is clear cut. In contrast to the majority, a few of the students took on the problem of justifying their method in Problem 1 as their problem. On the other hand, not all of the students took on Problem 2 as their problem. Molly's understanding of Problem 1 probably left her unchallenged by Problem Two. I suspect that some students did not see any puzzlement in Problem 2; for them knowing how to execute the procedure correctly was all that was necessary.

[11] At this point I am generalizing. This seemed to be useful even though undoubtedly individual differences existed in students' understandings.

[12] Because I was the teacher in the data on which the discussion is based, I use male pronouns in this section when referring to the teacher generically.

[13] Hypotheses of students' understandings may be based on information from a variety of sources: experience with the students in the conceptual area, experience with them in a related area, pretesting, experience with a similar group, and research data. Initial hypotheses often lack data that are available as work with the students proceeds. Thus, the hypotheses are expected to improve (i.e., become more useful).

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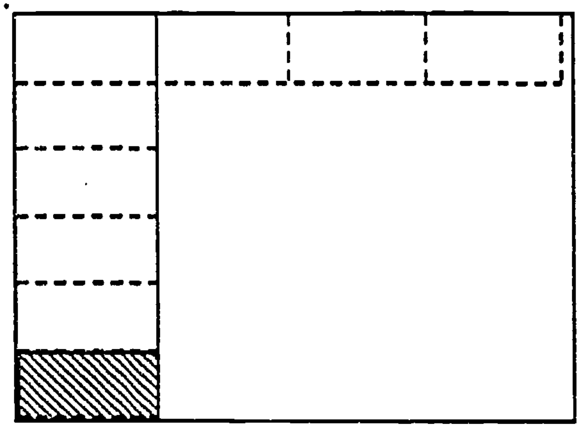


Fig. 1a

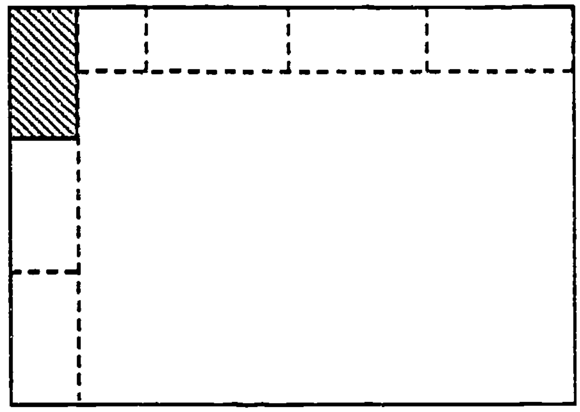
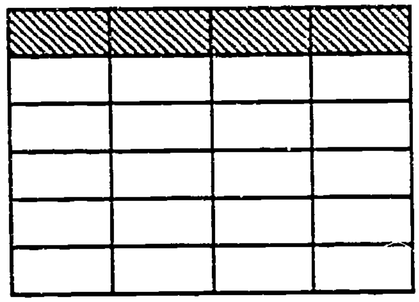


Fig. 1b



One group of four rectangles

Fig. 2

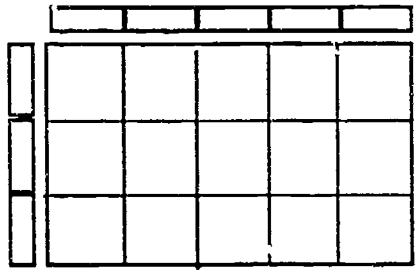


Fig. 3

Mathematics Teaching Cycle

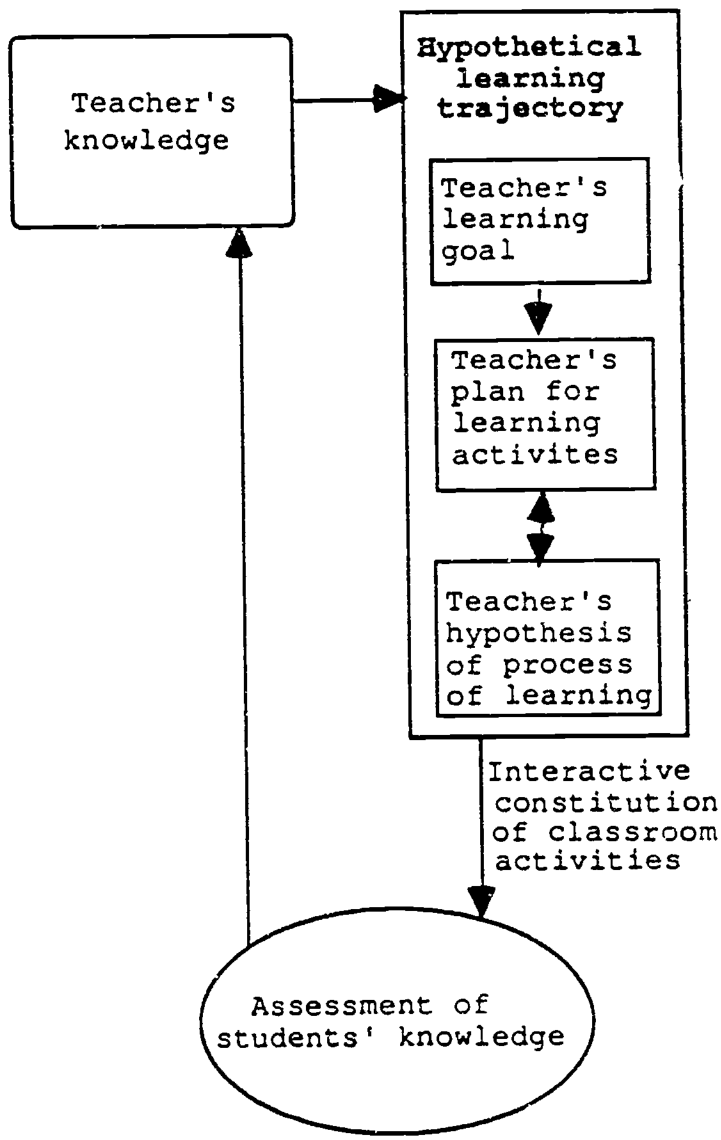
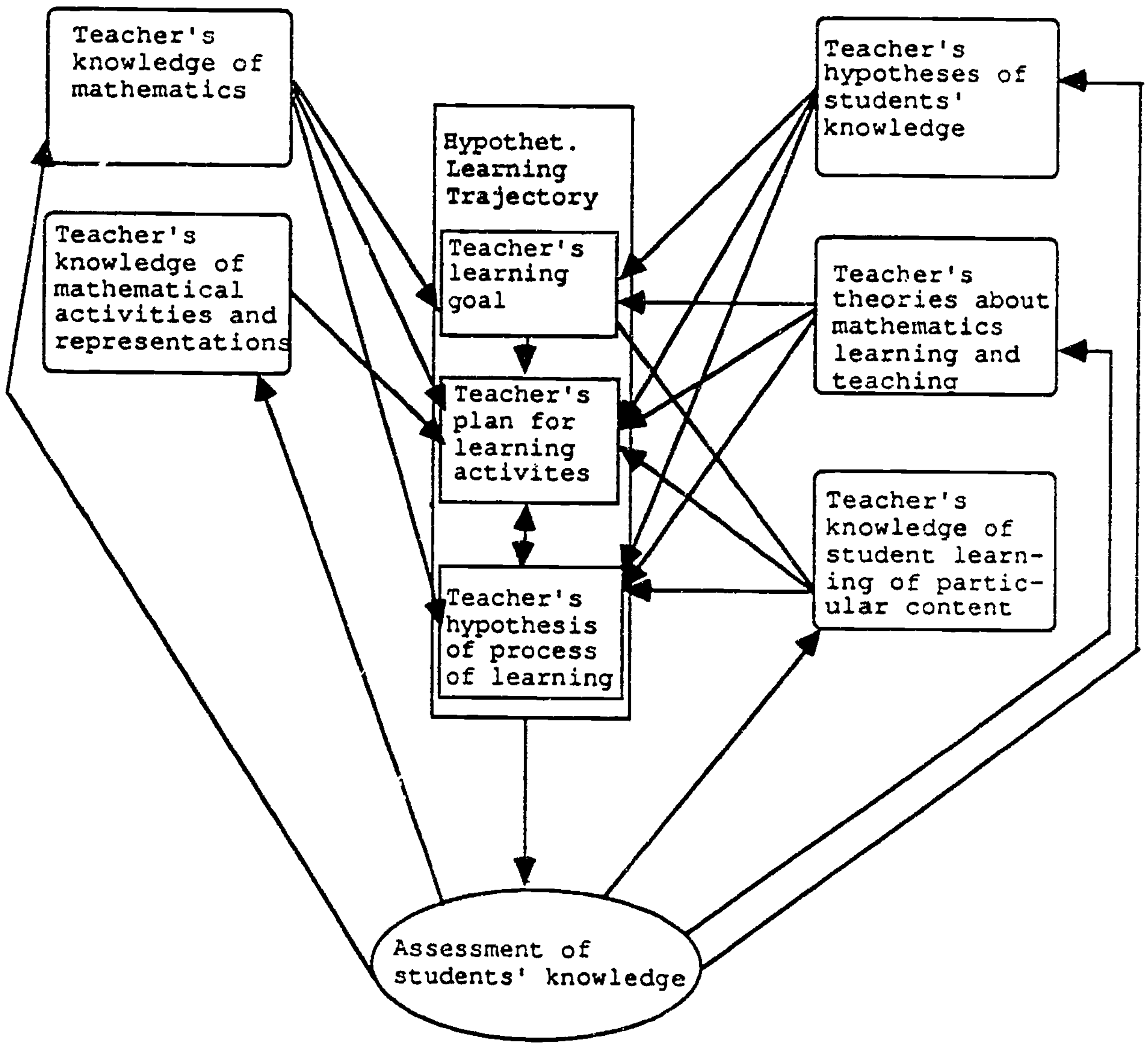


Fig. 4

Mathematics Teaching Cycle



Note: The domains of teacher knowledge also inform "assessment of students' knowledge" directly. However, because this was not emphasis of the model, and in order to simplify the diagram, those arrows are not included.

Fig. 5