DOCUMENT RESUME

ED 361 350 TM 020 403

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TITLE Conceptual Understanding of the Arithmetic Mean. Draft.

PUB DATE Apr 93

NOTE 13p.; Paper presented at the Annual Meeting of the American Educational Research Association (Atlanta, GA, April 12-16, 1993).

PUB TYPE Reports - Research/Technical (143) -- Speeches/Conference Papers (150)

EDRS PRICE MF01/PC01 Plus Postage.

DESCRIPTORS *Age Differences; Child Development; *College Students; Comparative Testing; *Comprehension; Concept Formation; Difficulty Level; *Elementary School Students; Fundamental Concepts; Grade 4; Grade 8; Higher Education; Intermediate Grades; Junior High Schools; *Junior High School Students; Knowledge Level; *Mathematical Concepts; Scores; Test Items

IDENTIFIERS *Arithmetic Mean; Oral Examinations; Paper and Pencil Tests

ABSTRACT

The understanding of children and adults regarding four component properties of the arithmetic mean was studied. The component properties were: (1) the mean is a data point between extreme values of a score distribution; (2) the sum of deviations about the mean equals zero; (3) when the mean is calculated, any value of zero must be taken into account; and (4) the average value represents the values that were averaged. Also studied were the relative difficulties of these properties, the potential of varied problem formats to facilitate understanding of these concepts, and differences between methods of investigating understanding (individualized oral tests and large-group paper-and-pencil tests). A 16-item test that presented the properties in story and numerical form was completed by 41 fourth graders, 60 eighth graders, and 40 college students. Data were analyzed quantitatively and qualitatively. Mastery of all properties increased with age. Story format was superior to numerical format, and the first two properties were significantly easier to master than the latter two, possibly because of an artifact of item format. Differences between individualized oral tests and paper-and-pencil methods can cause varying conclusions to be drawn about subjects' understanding of the arithmetic mean. (SLD)

Conceptual Understanding of the Arithmetic Mean

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Paper presented at the American Educational Research Association Annual Meeting, Atlanta, Georgia, April 12-16.

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Abstract

The present study had four purposes: (a) to investigate children and adults' understanding of four component properties of the arithmetic mean, (i.e., (1) the mean is a data point located between the extreme values of a score distribution (2) the sum of the deviations about the mean equals zero (3) when the mean is calculated, any value of zero must be taken into account, and (4) the average value is representative of the values that were averaged) (b) to determine the relative difficulty of these four component properties of the mean (c) to determine the differential potential of varied problem formats (story vs. numerical) to facilitate understanding of the arithmetic mean, and (d) to discuss differences between different types of methods (i.e., individualized oral tests and large group paper-and-pencil tests) for contributing to research about, and investigating understanding of the arithmetic mean.

41 fourth graders, 60 eighth graders, and 40 college students completed a sixteen-item test that presented the four properties in story and numerical formats. Data was analyzed using quantitative methods (accuracy scores were calculated using repeated measures analysis of variance) and qualitative methods (written justifications for answers were classified into six discrete categories).

Results indicated that (a) mastery of all properties increased with age (b) story format was superior to numerical format, and (c) properties 1 and 2 were significantly easier to master than were properties 3 and 4, possibly because of an artifact of item format. Differences between individualized oral tests and large group paper-and-pencil methods can cause varying conclusions to be drawn about peoples' understanding of the arithmetic mean.

Conceptual Understanding of the Arithmetic Mean

Introduction

The arithmetic mean, or average, is a fundamental descriptive statistic that forms the basis for many inferential statistics, and also appears frequently in a variety of everyday contexts. Early research into persons' understanding of the arithmetic mean suggested that the mean may frequently be taught and/or learned primarily as a computational algorithm rather than as a conceptual construct. For instance, Carpenter et al. (1981) reported that adolescents had difficulty on problems that required going beyond a straightforward application of the computational algorithm for the mean. A majority of students could not find the value of a missing numerical entry, given the mean and the values of all the other numerical entries. This illustrates that students' conceptual knowledge of the mean was limited, as they were unable to modify the familiar algorithm for computing the mean in ways that allowed them to adapt it for use in a non-standard situation. Similarly, Pollatsek et al. (1981) interviewed college students, who exhibited difficulty with weighted mean problems. As a result of this investigation, they concluded that college students treated the mean solely as a computational formula. Extending Pollatsek's work, Mevarech (1983) concluded that a computational definition or understanding of the mean was not modified by exposure to formal instruction in statistics.

The focus on student performance in this early research suggested that some aspects of conceptual knowledge about the mean were absent. This led to subsequent research that began the process of identifying specific aspects of conceptual knowing (e.g., properties of the mean and varying interpretations of the mean), and of determining the extent to which people have acquired knowledge about these different conceptual aspects. Assessing the nature of students' conceptual understanding of the mean has the potential not only to inform educators' understanding of students' knowledge of the mean, but additionally holds promise for suggesting directions for classroom practice that will incorporate both computational and conceptual aspects of the mean.

Examples of this more recent trend in research include Goodchild's 1988 study, in which students' ability to interpret the mean as a representative number, as a measure of location, and as an expected value was assessed, and Mokros & Russell's 1992 study, in which children's natural conceptualizations of the mean as a representative value that summarizes a data set were examined. Examples of inquiries into component properties of the mean include Mevarech's work (1983), which found that process errors occurred that were related to students' erroneous equation of the properties of the arithmetic mean with the properties of simple numbers, and the work of Strauss and Bichler (1988), who identified and assessed comprehension of seven distinct properties of the arithmetic mean by children and adolescents.

The present study continues the work of researchers like Strauss and Bichler by focusing specifically on providing a better understanding of how selected properties of the mean are understood from childhood through adulthood.

Research Goals

Four major research purposes formed the basis of this study:

(1) Investigation of fourth grade, eighth grade, and college students' understanding of four component properties of the arithmetic mean that had not been mastered by 14-year-olds (Strauss & Bichler, 1988), as follows:

   (a) Property 1: the mean is a data point located between the extreme values of a score distribution

   (b) Property 2: the sum of the deviations about the mean equals zero

   (c) Property 3: when the mean is calculated, any value of zero must be taken into account

   (d) Property 4: the average value is representative of the values that were averaged

(2) Examination of the relative difficulty of these four component properties of the mean

(3) Determination of the differential potential of two problem formats, story format and numerical format, to facilitate persons' understanding of the mean, and

(4) Discussion of differences between different types of methods (i.e., individualized oral tests and large group paper-and-pencil tests) for contributing to research about, and investigating understanding of the arithmetic mean.

Methods

Subjects

Forty-one fourth graders, sixty eighth graders, and forty college students served as experimental subjects. All subjects were voluntary participants in the experiment.

Materials

A sixteen-item questionnaire was constructed. For each of the four properties of the arithmetic mean, there were four questionnaire items (two in story form, and two in numerical form) which could be answered by a single yes/no answer or a single numeral. This information formed the quantitative data base. Students also provided written justifications for a portion of their responses, so that the reasoning processes used to arrive at a particular answer could be tracked. This latter information formed the qualitative data base. Throughout this study, the quantitative and qualitative data are presumed to inform each other, rather than functioning to provide alternative interpretations of a single phenomenon.

Procedure

Questionnaires were administered to intact classroom groups of students by either the experimenters or students' classroom teachers. Subjects were told that questionnaire items were being pilot tested for age-appropriateness in an attempt to minimize anxiety-induced performance decrements. Performance was untimed, and most students completed the test in approximately twenty minutes. Groups of subjects were debriefed following completion of the experimental task.

Results and Discussion

Quantitative data were analyzed by a series of repeated measures analysis of variance, followed by post-hoc comparisons (t-tests for independent samples). Qualitative data were categorized according to a six-category classification scheme that was synthesized by the researchers after close examination of the data. Major results were obtained as follows:

(1) **Older students' understanding of the four component properties of the arithmetic mean significantly surpassed younger students' understanding of these properties.** Means for fourth graders, eighth graders, and college students were respectively .67, .96, and 1.29 (corresponding to 33%, 48%, and 64% of the test items being answered with a response consistent with the arithmetic mean). A repeated measures analysis of variance revealed significant differences amongst these three means ($F(2, 138)=21.35$, $MSe=1.46$, $p<.00001$). Two-tailed t-tests for independent samples revealed that all possible pairwise comparisons differed significantly. This was an expected finding, and replicated results obtained by other experimenters (e.g., Strauss and Bichler, 1988).

(2) **The present subject sample gave answers consistent with the arithmetic mean somewhat less frequently than did other subject samples.** When comparing the results of the present study to the Strauss and Bichler study (which used Israeli students, and which investigated student knowledge about all four of the component properties of the mean examined in the present study), it was found that 4th graders gave answers consistent with the arithmetic mean on average 33% of the time (as compared to 38% of the time with Strauss and Bichler's 10-year-

olds), and 8th graders gave answers consistent with the arithmetic mean on average 48% of the time (as compared to 62% of the time with Strauss and Bichler's 14-year-olds). Three plausible explanations exist for this finding. First, differences could be due to varying amounts of prompting and use of informational probe questions employed by the two studies. Strauss and Bichler used a significant amount of prompting and probing of subject responses, while the present study employed no prompting or use of probe questions. Second, differences between samples could be due to differences in testing and response formats. Strauss and Bichler used an oral testing and response format, which could have been more sensitive to any conceptual understanding of the mean that may have present. Conversely, our study used a paper-and-pencil test, which more closely approximated standard classroom practice, but which may have been less able to detect the presence of conceptual understanding. Support for this contention comes from the work of Carraher et al. (1987), who found that students had greater accuracy rates and used different thinking strategies when they gave oral as opposed to written responses to verbal and numerical problems. Third, different teaching strategies could have been used with each subject group to foster conceptual understanding of the arithmetic mean, with some strategies being more effective than others, or, alternatively, a single set of instructional strategies could have been implemented with both groups, but with implemented with differential effectiveness for each subject group.

(3) **Presenting the component properties of the mean in story format resulted in significantly greater understanding than did presentation of the**

properties in numerical format. Means for story format and numerical format were 1.07 and .87, respectively, and differed significantly ($F(1, 138)=20.82$, $MSe=.48$, $p<.00001$). These scores represent (respectively) 53% and 43% of test items being answered with a value corresponding to the arithmetic mean. This finding supports recent research that indicates that many peoples' mathematical knowledge that they use in daily tasks tends to be situated in familiar everyday contexts rather than being organized around abstract formalisms and notation systems of pure mathematics as typically taught in school (Harris, 1991; Lave, 1988). It is hypothesized that the story settings were more likely than numerical examples to elicit rich, vivid scenarios that encouraged subjects to imagine quantitative relationships and reason concretely about the questions posed to them.

(4) **Properties 1 (the mean is a data point located between the extreme values of a score distribution) and 2 (the sum of the deviations about the mean equals zero) were accompanied by higher accuracy rates than were properties 3 (when the mean is calculated, any value of zero must be taken into account) and 4 (the average value is representative of the values that were averaged).** The means for Property 1, Property 2, Property 3 and Property 4 were respectively, 1.32, 1.19, .67, and .70. Again, these scores translate into 66%, 59%, 33%, and 35% of test items being answered with a value corresponding to the arithmetic mean. A main effect was obtained for the variable of component property ($F(3, 414)=58.00$, $MSe=.51$. $p<.00001$). with the number of questionnaire items being answered with a value corresponding to the arithmetic mean differing significantly between property 1 and properties 3 (F(3,

137)=21.00, MSe=.51, p<.01) and 4 (F(3, 137)=19.00, MSe=.51, p<.01), and between property 2 and properties 3 (F(3, 137)=13.50, MSe=.51, p<.01) and 4 (F(3, 137)=12.00, MSe=.51, p<.01), but with no differences being obtained between properties 1 and 2 or between properties 3 and 4. This is very likely due to an artifact of item design, as properties 1 and 2 contained two response alternatives (yes or no), and properties 3 and 4 contained a theoretically infinite number of response alternatives (i.e., a single numerical value selected from the set of rational numbers), making properties 1 and 2 more vulnerable to chance guessing. In fact, analysis of the qualitative data revealed that significantly greater numbers of students accompanied a "correct" answer with an incorrect justification for the former two properties than they did for the latter two properties. Specifically, an incorrect justification for a response was produced by 18% of subjects for property 1, 38% of subjects for property 2, 0.7% of subjects for Property F, and 0.7% of subjects for Property G, supporting the notion that chance may be primarily responsible for the relative difficulty of the different properties. Incorrect justifications included: No explanation supplied; states "I don't know"; states "I am guessing"; insists the answer is conditional upon certain characteristics of the number set used to illustrate the statistical property being measured, but the explanation given is inconsistent with a value corresponding to the arithmetic mean; invokes reasoning based on a notion of social justice; incorrectly interprets the intention of the problem; invokes an irrelevant reason; or simply repeats some portion of the original wording of the problem.

(5) Different testing formats, such as individualized, oral testing and large group, paper-and-pencil tests may have implications for the ways in which results about persons' understanding of the arithmetic mean are interpreted. Typical classroom paper-and-pencil tests are relatively more cost-and time-efficient, but may either underestimate knowledge (because oral response formats may reveal more information about students' thinking than written response formats), or overestimate knowledge (as in the case where guessing occurs, and qualitative data are not collected to reveal the extent to which students are getting the right answer for the wrong reason). Conversely, individualized oral tests provide a wealth of information about subtle nuances of student learning, but are relatively less time- and cost-efficient, and may overestimate student knowledge (especially if they rely excessively on probing and prompting of student responses). Trade-offs between instrument sensitivity, efficiency, and practicality seem inevitable, and warrant consideration when choosing an instrument for research or classroom practice.

Future Directions

Future research efforts might profitably focus on three issues: (a) refining the theoretical framework that represents current knowledge of student understanding about the arithmetic mean (b) further developing knowledge about persons' understanding of the mean by continuing investigation into properties, interpretations, and other conceptual aspects of the mean, and (c) investigating alternative forms of assessment methods of student knowledge of the mean that can be used to inform instructional decision making in constantly changing classroom environments.

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