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ABSTRACT

An influential proposal about aquiring mathematical knowledge is that it entails linking instruction-based concepts to intuitions derived from informal activities. In the case of non-positive numbers, informal knowledge is unlikely to emanate from observing physical objects, because non-positive objects or sets of objects do not exist. However, it is hypothesized that such intuitions could derive from experience with actions that undo other actions, such as decrementing a collection or returning to a starting point. Game-like activities involving positive and non-positive actions are used in an exploratory study of 4- to 7-year-old children to examine this hypothesis. The results suggest that children do develop action-based intuitions of non-positive quantities prior to formal instruction, although this knowledge must be described as qualitative and precursory. Similar informal activities could be adapted for use in the early grades to promote an intuitive basis for formal concepts that will be presented later. Activities emphasizing ordinal concepts may be especially useful, as these appear less likely than cardinal concepts to be acquired through casual experience. (Author)

Precursors of Non-positive Integer Concepts

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Running head:

NON-POSITIVE CONCEPTS

Abstract

An influential proposal about acquiring mathematical knowledge is that it entails linking instruction-based concepts to intuitions derived from informal activities. In the case of non-positive numbers, informal knowledge is unlikely to emanate from observing physical objects, because non-positive objects or sets of objects do not exist. However, it is hypothesized that such intuitions could derive from experience with actions that undo other actions, such as decrementing a collection or returning to a starting point. Game-like activities involving positive and non-positive actions are used in an exploratory study of 4- to 7-year old children to examine this hypothesis. The results suggest that children do develop action-based intuitions of non-positive quantities prior to formal instruction, although this knowledge must be described as qualitative and precursory. Similar informal activities could be adapted for use in the early grades to promote an intuitive basis for formal concepts that will be presented later. Activities emphasizing ordinal concepts may be especially useful, as these appear less likely than cardinal concepts to be acquired through casual experience.

Precursors of Non-positive Integer Concepts

Historically, adopting a distinct numeral for zero, thus recognizing it as a quantity like other numbers, was a major conceptual breakthrough that made possible our place value computational system. The legitimation of negative numbers was equally important for advancing the understanding of mathematical structure. For today's students, the quantification of non-positive integers continues to be a difficult and important conceptual construction.

Young children readily acquire certain notions about the qualities of zero which make it a "special number" through informal mathematical activities or rote learning of formulas. For example, children as young as three or four years display some implicit knowledge of number facts involving zero, such as $\underline{n} - \underline{n} = 0$ (Starkey & Gelman, 1982). Five- to seven-year olds can recite properties of zero ("zero is nothing," "when you take zero from a number you have that number," etc.), and even perform better on standard arithmetic problems involving zero (e.g., $\underline{n} + 0 = ?$; $\underline{n} - \underline{n} = ?$) than on corresponding non-zero problems (Evans, 1983; Wellman & Miller, 1986).

However, the acquisition of such facts about zero does not necessarily imply that children readily conceptualize zero as a legitimate member of a number system. It is possible that young children initially represent zero qualitatively--as an <u>absence</u> of number--rather than as a number per se. An example of qualitative reasoning comes from a procedure in which 5- to 7-year old children were asked to compare the number of marbles in two empty containers: "Does one of them have more marbles, or do they have the same number?" (Davidson, 1984). Significantly more of the younger children refused to make this quantitative comparison, either arguing that the comparison was not possible because no numbers were visible, or treating the question as nonsensical. For instance, one 5 1/2-year old boy replied, "There's no number to it...cause there's none in here; no marbles in them." Most children who solved the problem were momentarily puzzled, and then answered "it's the same number" with a wide grin--apparently amused at applying a numerical interpretation to this novel case. Wellman and Miller (1986) also found anomalies in younger children's knowledge about zero. Many children who knew that zero denotes nothing nevertheless judged that one is the smallest number, for example. Presumably, children who represent zero as a member of a number system would not hesitate to make numerical comparisons involving zero, nor be inconsistent about whether it is the smallest number they know.

A quantitative understanding of number involves dissociating number from physical or spatial qualities a collection of objects may have. Further, it involves recognizing the role of units in the organization of the whole numbers: any adjacent pair of whole numbers differ by a unit. Notice that this also entails an implicit coordination of ordinal and cardinal conceptions of number: numbers that differ ordinally (i.e., in terms of position in the number sequence) are understood to differ cardinally (i.e., in terms of a certain amount). Children's quantification of non-positive integers must include an appreciation of these same properties, but must also be complicated by a lack of perceptual cues, because non-positive numbers are not representable in the environment as collections of physical objects.

In school mathematics, a quantitative conception of zero is perhaps not critical until children encounter multidigit addition and subtraction problems involving place value concepts. In such problems, zero must be interpreted as a definite number of units, tens, hundreds, and so on (rather than as something qualitative, such as a place holder to keep the columns straight), or difficulties with carrying and borrowing will ensue. This is illustrated by VanLehn (1983), who cataloged 16 types of errors children exhibit when performing multidigit subtraction--12 of which involve difficulties with zero. An example is the "borrow across zero" procedure: Faced with a problem such as 405 minus 327, unable to borrow directly from the zero, children borrow instead from the four hundred (i.e., decrementing the 4 to 3, without incrementing the 0 to 9). As Kamii (1985) points out, many of children's difficulties in school mathematics can be traced to poor comprehension of place value. Error analyses such as VanLehn's suggest that part of the confusion about place value may be connected with the problem of quantifying zero. Analogously, it is possible that some of students' difficulties with algebra, somewhat later in the curriculum, may stem from the problem of quantifying negative numbers, or conceptualizing them as legitimate members of the number system.

An assumption underlying the present work is that autonomous mathematical understanding depends on constructing links between formal concepts received through instruction and intuitions derived from informal activities. If children experience difficulty with non-positive concepts in school mathematics, as was just suggested, it is important to understand something about the informal activities that are related to these concepts. An exploratory study was designed to examine the development of intuitions about non-positive quantities in the context of informal game-like activities. The study focuses on the age period of 4 to 7 years, before children have received much if any formal instruction on these topics. Thus, the method necessarily involves looking "beyond what children say explicitly for knowledge that is implicit in what they do" (Resnick, 1989).

Two hypotheses guided the study. The first is that intuitive representations of zero and negative numbers are based on actions, rather than objects. If intuitive number knowledge derived only from experience with perceptible objects, non-positive intuitions would be impossible. However, whereas non-positive collections of material objects do not exist, non-positive actions using material objects do exist and are familiar to children from infancy (Langer, 1980; 1986). These non-positive actions include negations--such as decrementing a collection, or dividing a collection into parts; and they include null actions--such as restoring a collection to its previous state by doing the inverse of a previous action.[1] Systematic relations between positive and negative actions mean that both positive and negative quantities can be represented in a concrete way within a single system of actions. For example, positive and negative integers are representable as the <u>number of times</u> objects are added to or removed from a collection. Another possibility is representation in terms of <u>numbers of movements</u> in forward or backward directions along a line.[2]

A second hypothesis concerns the conditions for quantification of non-positive quantities. Several researchers have remarked on the fact that each integer has at once two meanings: a cardinal meaning, by which it expresses the size of a collection of elements; and an ordinal meaning, by which it expresses a certain position relative to other numbers (Gelman & Gallistel, 1978; Piaget 1941/1965; Resnick, 1983). The distinction between the two connotations is not trivial: cardinal conceptions emphasize similarities between units (since elements must be construed as similar in order to treat them as a collection); whereas ordinal conceptions emphasize differences (since each number occupies a unique position in the ordered series). The number concept is thus a synthesis of cardinal and ordinal meanings. The present hypothesis is that this analysis applies as well to constructing non-positive numerical concepts. By this interpretation, children's "qualitative" conceptions may consist of uncoordinated, one-sided cardinal or ordinal meanings, and the precondition for quantification would be the coexistence of both ordinal and cardinal meanings. Accordingly, the exploratory tasks used in this work were designed to include both cardinal and ordinal components.

Method

<u>Subjects</u>

Forty-five children were recruited from preschool, kindergarten, and first grade classrooms in an urban area of mid-Michigan. The children ranged in age from 4 to 7 years, and included approximately equal numbers of boys and girls. They were questioned individually in a familiar room at their schools, using the game-like activities described below. Due to scheduling problems, not all children completed all tasks; in particular, many of the youngest did not finish the last task (n's for each task are reported in the results section). Responses were recorded on audiotape and then transcribed.

Procedure

1. Card game. The initial task was a card game designed to elicit spontaneous definitions and comments about the number zero. Children were shown two stacks of nine cards each, marked with numerals ranging from 0 to 9 (0 to 5 for children younger than 6 years) and a few cards marked with the letter X. A stuffed animal, described as not understanding the game very well, served as their opponent. Children and the opponent (with help from the interviewer) turned over cards from their respective stacks, the high card being the winner on each turn. Cards marked X could be designated any numerical value by the player. Cards were prearranged so that both players drew X against numbered cards, and children drew a high card against the opponent's zero on two turns. The opponent designated his X cards as zero, whereupon children were asked to explain to him whether this was a good idea. Probe questions were asked about (a) what zero means, and (b) whether zero is a number.

2. Bees and Flowers. To explore understanding the cardinality of zero, subjects were asked to line up a set of small wooden bees and wooden flowers so that "every bee gets a flower." Questions were then asked to bring zero into the conversation: "How many bees don't have a flower?" "How many flowers do we have to add?" The two questions of interest were then asked: (a) "I want you to add zero bees. Can you do that? Why (or

why not)?" (b) "I want you to take away zero flowers. Can you do that? Why (or why not)?"

3. Mailman game. To explore ordinal notions, children played a game that required moving a toy van to deliver mail along a cardboard display representing a street of houses. The nine residences along the street were numbered in the manner of a number line from -4 to +4. The display deviated from a standard number line in that the zero space was unnumbered, and minus signs were absent (plus and minus segments of the line were signified by a large blue circle at one end and a large red circle at the other). Children moved the van according to symbols on cards: blue arrows represented moving a certain number of spaces in a positive direction, red arrows represented movements in the opposite direction, and cards with blue or red zeros indicated no movement. The van's starting point for each problem was the middle (zero) house. After verifying the children's understanding of these rules, six problems were administered, requiring the following combinations of movements: <-3, +2>, <-2, +2>, <-1, 0>, <-2, +1>, <-2, +3>, <0, 0>. Of interest were children's reactions to parts of the task involving zero: (a) whether they believed the different colored zero cards to mean the same; (b) whether they identified the middle house with an address of zero; and (c) whether they treated this as a valid address when making moves.

4. Hippo game. In a task involving both ordinal and cardinal aspects of negative and positive integers, children played a zookeeper who puts food pellets into a hippo's bowl—or takes some out—according to colored marks on a spinner. Quantities of food pellets were represented by colored marks along the spinner's edge in the following order: blue

(= -4 pellets), purple (-3), red (-2), pink (-1), black (0), yellow (+1), pale green (+2), green (+3), dark green (+4). A white space (90 degree arc) separated the -4 and +4 endpoints of this color-coded number line.

The first part of the task assessed whether children would discover the numerical ordering of the color codes and use this to infer the meaning of each code. Children were shown a food dish, a bag of food pellets, a picture of a hippo, and the spinner. They were told that there was a rule whereby each color meant to add or take away a certain number of the hippo's pellets, and that their job was to figure out this rule. Two initial hints were provided: a cue card with a yellow mark and a picture of one pellet (indicating +1), and a card with a purple mark and a picture of three missing pellets (drawn with dotted lines, indicating -3). After the cue cards were explained, children spun the spinner seven times (if the white separator space was spun, they spun again). After each trial they were asked for the numerical meaning of the color spun. The interviewer recorded the color and the response on a score sheet, then turned over another cue card revealing the numerical value of the color spun, and reminded children to use all hints provided.

The second part of the task involved simple arithmetic with positive and negative integers. The idea of owing pellets was introduced by asking what would happen if the hippo had three pellets and the zookeeper spun blue, meaning four pellets should be taken away. After agreeing that the hippo would owe (give back) one pellet at the next feeding, children were shown debit cards--cards with one, two, or three dots--to "help the hippo remember how many he has to give back next time." Three problems were then given: (a) "The spinner says to give the hippo three, and he owes one. How many will he have left?" (b) "The spinner says to take away four, and he has two. How many will he have left?" (c) "The spinner says to give the hippo two, and he owes two. How many will he have left?"

5. Counting task. To explore the coordination of ordinal and cardinal aspects of zero, children were asked to count a set of 6 plastic flowers, starting with zero (i.e., 0, 1, 2,...). They then counted the same objects starting with one. This resulted in two different totals, and they were asked to explain the discrepancy. Credit was given for any reference to the mistake of counting from zero. This is considered evidence of cardinal-ordinal coordination because it is zero's cardinal value which implies that zero is not useful for enumerating objects. The following additional questions were also asked: (a) "Which is bigger, zero or one?" (b) "What is the smallest number?" (c) "Do you think there are any numbers smaller than zero? Why or why not?"

All subjects received the card game first and the counting task last; the other tasks were administered in counterbalanced order. Incorrect answers were followed by probe questions until the interviewer judged that an accurate indication of ability had been obtained. Throughout the interview, the term "zero" was supplemented by "nothing" or "none" as necessary for clarity, and these terms were considered acceptable in scoring responses if the substance of the response was correct.

Subjects received dichotomous scores (correct vs. incorrect) on 16 items coded from interview transcripts; the coding for these items was based both on subjects' statements and the interviewer's commentary describing their nonverbal actions with the materials. A randomly selected half of the sample was recoded by a second rater; the mean interrater agreement on the 16 categorical items was 85.25%. In addition, subjects received scale scores (number of correct responses) on 4 items derived from the score sheets for the spinner trials of the hippo task, making a total of 20 items per child.[3]

Results

The 16 dichotomous interview questions, and percentages of children answering each correctly, are listed as items 1-7 and 12-20 in Table 1. Items 8-11 of the table are the following numerically-scored variables, obtained from the spinner trials: (a) Percent of positive numbers correctly identified (e.g., a child spins green and correctly infers that it means "add 3"); (b) Percent of negative numbers correctly identified (e.g., a child spins red and correctly infers that it means "take away 2"); (c) Percent of positive numbers identified with the correct numerical sign (e.g., a child spins green and states that it means "add 1"); (d) Percent of negative numbers identified with the correct sign (e.g., a child spins red and states that it means "take away 1").

Insert Table 1 about here

Presentation of the results will be organized as follows: (a) items reflecting verbal knowledge about number, (b) items focusing primarily on cardinal properties, (c) items focusing primarily on ordinal properties, and (d) items requiring coordiration of cardinal and ordinal properties.

Verbal knowledge

Children display a range of verbal knowledge, and generally speaking there are only slight age differences on these items (please refer to Table 1). First, all children are able to count accurately collections of the size used in this study (item 18). When asked to count from zero instead of one, 85% are successful (item 17). At the beginning of the interview, prior to any discussion about numbers, 78% give an adequate definition of zero as "nothing" or "none" (item 1). However, only 37% affirm that zero is a number, when the question is posed at the outset of the interview (item 2). In the mailman game, 81% of the children correctly believe that the color in which the numeral zero is written has no effect on its value, although other colored symbols are used in this task to indicate moves in opposite directions (item 6). Three quarters of the children assert that one is bigger than zero (item 15), and 61% state that zero is the smallest number (item 16). At the end of the interview, 39% believe that there might be numbers smaller than zero (item 20). These latter responses are about evenly split between those that mention getting the idea from one of the tasks, such as the debit cards used in the hippo game, and those with more fanciful speculations—for instance, that half of zero, or several zeros together, might be smaller than zero.

As would be expected, there is some clustering among these verbal knowledge items: item 1 (definition of zero) is correlated with item 15 (stating one is greater than zero), $\underline{phi} = .70$, $\underline{p} < .001$, and with item 16 (stating zero is the smallest number), $\underline{phi} = .47$, $\underline{p} < .01$. Items 15 and 16 are also interrelated, $\underline{phi} = .45$, $\underline{p} < .01$, although 23% of the children state both that one is the smallest number and that one is greater than zero, consistent with Wellman and Miller's (1986) finding.

Cardinality

Portions of the interview that deal primarily with cardinal values include items 3 and 4 (adding and subtracting zero), and items 12, 13, and 14 (combining positive and negative amounts). Children receive credit on items 3 and 4 if they show in some way that adding or subtracting zero leaves the number of objects unchanged. Almost all who accomplish this do so by going through a motion of adding or subtracting with an empty hand. These children are also amused by the questions, as if enjoying their insight in finding an obvious solution to a cryptic problem. On both problems, most failures (86% of addition and 95% of subtraction failures) involve removing objects from the display--indicating an effort to produce or show zero rather than to add or subtract zero. Success on items 12, 13, and 14 requires grasping that the debit cards used in the hippo game symbolize negative quantities, and determining the result of combining negative with positive quantities.

Increasing age trends are found for four of these five items (Table 1). Performance on item 3 is also correlated with item 4, $\underline{phi} = .55$, $\underline{p} < .001$, as is item 12 with item 14, $\underline{phi} = .32$, $\underline{p} < .05$, and item 13 with item 14, $\underline{phi} = .50$, $\underline{p} < .005$.

Ordinality.

Aspects of ordinal understanding are assessed through items 5 and 7 (mailman game), and items 8, 9, 10, and 11 (spinner trials). Recall that in the mailman game children must demonstrate understanding the use of blue and red arrows to signify forward or backward movements before proceeding with the task, and that the middle, or zero, space is left unlabelled in order to observe whether children treat it as part of the number sequence. Although children otherwise successfully follow the game rules, only 39% do so when dealing with the middle space (item 5). The others skip over it and resume moving the given number of spaces on the other side, which results in delivery of "mail" to the wrong address. At the close of the task, when asked whether there is a space that "goes with" the zero cards, fewer than half associate the middle space with zero (item 7). The findings suggest that most children in this age range do not infer the ordinal location of zero, or expect that zero occupies a definite position just before one in an ordered series. Correctly using the middle space (item 5) is positively related to stating that one is greater than zero (item 15), *phi* = .42, $p < .05$, and to stating that zero is the smallest number (item 16), *phi* = .38, $p < .05$.

Similarly, the spinner trials assess whether children use available information (i.e., the cue cards) to infer the numerical value of color-coded marks on a spinner. Across seven trials, children average just over a quarter correct, indicating that they do not immediately grasp that the colors code an ordered series (items 8 & 9). (However, 92% of subjects get one or both of the last two trials correct, suggesting that when enough cue cards have been furnished they do take note of the series.) When credit is given for just ascertaining whether color codes represent positive or negative values, success rates more than double (items 10 & 11). This shows that the incorrect responses are not random, and suggests that although children may not readily construct a linear order ranging from negative to positive values, they nonetheless may intuit a "global" order consisting of <negative, positive>. Consistent with the mailman task results, most children seem not to infer that a zero occupies the midpoint: overall, responses are correct on only 5% of trials in which zero is spun. Because ignoring the existence of a zero code would throw off children's estimates of the other codes, this may partly explain the low percentage of exactly correct responses as well as the higher percentage of responses that are in the right ball park.

Responses to negative spins reveal U-shaped age trends (items 9 & 11). Such patterns are often indicative of shifts in strategy use (Strauss, 1982). For instance, it is possible that the youngest children achieve some of their success on negative spins through a preference for taking rather than giving pellets; whereas, the oldest children succeed through deductive strategies. By this interpretation, the middle group rejects irrelevant solutions, but falls short of deductive solutions. The strategy shift interpretation is supported by the second part of the same task (items 12, 13 & 14) which demonstrates greater conceptual competence in the oldest group than in the two younger groups.

Cardinal-ordinal coordination

The problem that most explicitly addresses coordination of cardinality and ordinality is that of reconciling two counts in the counting task. Having counted a set of objects from zero to a total of five, then from one to a total of six, children are asked to explain why the counts are different (item 19). Although all children identify the correct count, only 35% explain the difference by referring to the inappropriateness of counting from zero. The others either have no explanation, or refer to perceptual evidence for the correct count (e.g., "you can see there are six"). This item is positively related to item 1 (definition of zero), phi = .38; item 3 (adding zero), phi = .47; item 4 (subtracting zero), phi = .45; item 12 (adding minus to plus), phi = .39; and item 15 (stating one greater than zero), phi = .41 (all significant at $p < .05$ or greater).

The spinning trials also involve coordinating cardinal and ordinal properties, because cardinal information (marks on cue cards) is used to infer ordinal positions (colors on the spinner). Results, already described, suggest a twofold interpretation. On the one hand, children have little trouble with the purely symbolic aspect of the task--comprehending that the drawings on the cue cards stand both for positive or negative amounts of pellets and for a position on the spinner. On the other hand, they have difficulty with the quantitative aspect--inferring the precise relation of the cards' cardinal values to specific positions on the spinner.

Discussion

It should be recognized that the present findings are a conservative estimate of children's competencies, because the procedures do not include direct instruction in non-positive concepts nor assessment of children's ability to grasp these concepts with the benefit of instruction. Rather, the purpose is to explore whether any existing intuitions about non-positive quantities can be elicited in the context of informal activities.

The results support several proposals about the organization of children's intuitive knowledge:

1. Children manifest considerable potential for using familiar actions to represent non-positive quantities. This includes representing a negative cardinal as a number of withdrawals from a collection, representing a negative ordinal as a number of movements made in a direction opposite to positive movements, and representing zero as a null movement or a null change in a collection. Almost all children understand the procedures well enough to play the games, and in most cases these action-based representations provide sufficient support for adopting more conventional forms of symbolic representation (such as colored arrows, or drawings made with dotted lines) for negative quantities.

2. Throughout the age range studied, knowledge of negative numbers as well as the more familiar zero can be described as qualitative. This is evinced by the first interview questions, in which the majority of children define zero as "nothing," but do not define it as a number. Further, as just noted, children generally do well on symbolic or verbal aspects of tasks, such as counting from zero, not confusing the meaning of zero with the color in which it is written, and using symbols such as arrows, colored marks, or drawings on cue cards to stand for negative numbers. Children do less well on strictly quantitative aspects: in the hippo task, they produce many approximate but few exactly accurate solutions; in the mailman and hippo tasks, many children do not expect that a zero point separates negative and positive values in the numerical series; in the counting task, most do not point out that counting from zero produces an inaccurate result.

3. There is substantial development during this age period in the understanding of non-positive cardinality. Only about 10% of the youngest group demonstrate that adding or subtracting zero means to leave a cardinal value unchanged, compared to about 50% of the oldest. (Manifesting this understanding in a practical situation thus apparently trails behind verbal knowledge of the formula "adding [or taking away] zero leaves the same number," as reported by Starkey & Gelman, 1982, and Wellman & Miller, 1986.) Competence in combining positive and negative amounts also shows increasing age trends.

4. There is some indication that intuitions about ordinality may lag behind those about cardinal quantity. For instance,the oldest children's accuracy on ordinality problems (items 5, 7, 8 & 9) ranges from 31 to 53% (M = 39.75), whereas accuracy on cardinality problems (items 3, 4, 12, 13, & 14) ranges from 38 to 82% (M = 56.80).

5. The results are consistent with the hypothesis that a qualitative conception of non-positive integers is associated with a poor coordination of the cardinal and ordinal meanings of non-positive numbers. A possible explanation of the difficulty of this coordination would be the lag of ordinal intuitions, just noted.

Educational Implications

If, as has been suggested, there is a lag in ordinal intuitions about non-positive quantities, this could be accounted for by a scarcity of ordinal evidence about zero and negative values in everyday experience, as compared with cardinal evidence. The potential for cardinal evidence exists, for instance, when children decrement a collection and end up with zero objects, while attending to the negating actions themselves. The potential for ordinal evidence exists, for instance, when children perform movements oriented in opposite directions, while attending to how these movements bring them back to, and beyond, the starting point. However, abstracting ordinal numerical properties from the general forms of ordered actions may be cognitively more demanding than abstracting cardinal properties from the general forms of grouping (or separating) actions.

One reason why this would be the case is that many common grouping or separating actions are directed toward discrete units (e.g., marbles, blocks, pellets, etc.), which tends to facilitate attending to the positive, negative, or neutral properties of the actions themselves. In contrast, many common actions embodying ordinal properties take place in a spatial medium. This would necessitate the additional conceptual demands of imagining ordered units, or equidistant positions, along the given spatial path, and attributing a zero or "starting point" to a particular location among these (unless the path is somehow already artificially marked, as on a Monopoly game board, etc). Accordingly, construing specific actions as having positive, negative, or neutral ordinal properties may be a more abstract ability than the corresponding interpretations vis-a-vis cardinal properties.

To encourage linkages between students' own intuitions and the formal non-positive concepts presented in class, educators could develop activities that emphasize ordinal properties and their relations to cardinal properties. Examples of such activities might be the following: (a) progressing in a series of ordered movements from a starting point, and retracing these steps back to and beyond the starting point; (b) establishing units along the direction of travel in order to facilitate the tracing and retracing of movements; (c) verifying the relation between (ordinal) positions relative to a starting point and the (cardinal) quantity of steps required to arrive at that position; (d) using the ordinal notion of "starting point" as an intuitive basis for the necessity of a number zero that is neither positive nor negative. It would also be useful to explore informal terminologies that exploit the equivalence of action sequences and constants (see footnote 2). For instance, "if you have 0 and you take-away 4, then you have 4 take-aways left;" a similar approach would employ the notion of debt and actions of "giving back."

A second point concerns negative numbers specifically. As formal concepts, these usually are not introduced before middle school, and this is well from both pedagogical and cognitive developmental viewpoints. However, this often means introducing ideas that are in conflict with students' own intuitions that have been well founded and deeply held for years--such as the impossibility of subtracting a larger number from a smaller number (Davidson, 1987b). One might suspect that instruction requiring students to overturn long standing convictions could contribute to the alienation some develop toward mathematics, or the belief that mathematics consists of conventions established by authorities (Erlwanger, 1973). Adopting informal activities and terminologies such as those suggested above during the early grades should help establish intuitive foundations for the formal concepts that will be presented later.

References

Davidson, P. (1984). The development of numerical reasoning in children: the role of classes, relations and functions. Dissertation Abstracts International, 45, 1036B. (University Microfilms No. 84-13356).

Davidson, P. (1987a). Early function concepts: Their development and relation to certain mathematical and logical abilities. Child development, 58, 1542-1555.

Davidson, P. (1987b). How should non-positive integers be introduced in elementary mathematics? Proceedings of the International Group for the Psychology of Mathematics Education, 11, 430-436.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. Journal of Children's Mathematical Behavior, 1, 7-26.

Evans, D. (1983). Understanding zero and infinity in the early school years. Dissertation Abstracts International. (University Microfilms No. 83-26285).

Fuson, K.C. (1982). An analysis of the counting-on solution procedure in addition. In T.P. Carpenter, J.M. Moser, & T.A. Romberg (Eds.), Addition and subtraction: A cognitive perspective (67-81). Hillsdale, NJ: Lawrence Erlbaum.

Gelman, R. & Gallistel, C. (1978). The child's understanding of number. Cambridge, MA: Harvard University Press.

Kamii, C. (1985). Young children reinvent arithmetic. New York: Teachers College Press.

Langer, J. (1980). The origins of logic: Six to twelve months. New York: Academic Press.

Langer, J. (1986). The origins of logic: One to two years. New York: Academic Press.

Piaget, J. (1965). The child's conception of number. New York: Norton. (Originally published 1941.)

Resnick, L. (1983). A developmental theory of number understanding. In H. Ginsburg (Ed.), The development of mathematical thinking (pp. 109-151). New York: Academic.

Resnick, L. (1989). Developing mathematical knowledge. American Psychologist, 44, 162-169.

SPSS, Inc. (1986). SPSSX User's guide (2nd Ed). Chicago: SPSS Inc.

Starkey, P. & Gelman, R. (1982). The development of addition and subtraction abilities prior to formal schooling in arithmetic. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), Addition and subtraction: a cognitive perspective (pp. 99-116). Hillsdale, NJ: Erlbaum.

Strauss, S. (Ed.), (1982). U-shaped behavioral growth. New York: Academic.

VanLehn, K. (1983). On the representation of procedures in repair theory. In H. Ginsburg (Ed.), The development of mathematical thinking (pp. 197-252). New York: Academic.

Wellman, H. & Miller, K. (1986). Thinking about nothing: development of concepts of zero. British Journal of Developmental Psychology, 4, 31-42.

Footnotes

[1] It might be supposed that if children gain intuitions about negative quantities from activities such as removing items from a collection, as is suggested here, these intuitions could in fact arise perceptually by simply attending to the items that have been removed. However, as perceptually given, the collection of items that have been removed and the collection of items that still remain are both positive quantities. It is only by attending first of all to the action by which the objects have been removed that one might hit on the idea of interpreting these removed objects as tokens or as symbols which designate the negation of objects in the original collection.

[2] In formal terms, such representations amount to identifying each integer n with a unary operation $fx = x + n$ (thus $fx = x - n$ when n is negative). In such a system, addition and subtraction are defined as binary compositions of the form gfx. For example, if $fx = x + 5$ and $gx = x - 5$, then $gfx = x$ (cf. Cayley's theorem that every group is isomorphic to a transformation group). This formulation corresponds quite closely to actual early addition and subtraction strategies, in which children first find the result of fx, and then sequentially perform gx on this result to obtain the solution (Davidson, 1987a; Fuson, 1982). From a pedagogical standpoint, this formulation identifies negative quantities with the action of subtracting, and thus suggests ways of talking about unfamiliar quantities in terms of familiar actions such as "taking away," "going back," and so forth.

[3] Some further, incidental information about certain of these items (such as types of errors or successes) was also recorded and is mentioned in the results section where appropriate.

Table 1

Percent of correct responses by age level

Task and description of item	Age level: Youngest	Middle	Oldest	Age Trends: Linear	Quadratic
Card game (n = 45)					
1. Accurate spontaneous definition of zero	71	69	93	2.11	1.14
2. States that zero is a number	42	19	53	.57	3.93**
Bees and Flowers (n = 39)					
3. Shows how to add zero	15	36	58	5.04**	.00
4. Shows how to subtract zero	8	14	42	4.59**	.54
Mailman game (n = 36)					
5. Treats zero house as a valid position	38	39	40	.15	.02
6. States blue zero means same as red	75	92	73	.12	2.02
7. Identifies zero cards with zero position	38	48	53	.55	.02
Hippo game: spinner trials (n = 37)					
8. Exactly correct on positive spins	23	22	31	.66	.29
9. Exactly correct on negative spins	35	17	35	.00	4.36**
10. Correct sign on positive spins	57	65	69	1.37	.06
11. Correct sign on negative spins	74	45	72	.04	8.49***
Hippo game: arithmetic problems (n = 33)					
12. Solves minus 1 plus 3	11	9	38	3.00*	1.62
13. Solves plus 2 minus 4	37	60	82	4.01**	.03
14. Solves minus 2 plus 2	50	50	64	.40	.18

(table continues)

	Age level			Age Trends	
Task and description of item	Youngest	Middle	Oldest	Linear	Quadratic
Counting task (n = 27)[a]					
15. States one is greater than zero		69	82	.60	-
16. States the smallest number is zero		50	75	1.84	-
17. Counts accurately from zero		87	82	.11	-
18. Counts accurately from one		100	100	.00	-
19. Explains the counting discrepancy		33	38	.04	-
20. Might be numbers smaller than zero		42	36	.07	-

Note. Age groups were divided so as to obtain approximately equal n's per group. For the youngest third of the sample (n = 14) the average age is 4 years 10 months; for the middle third (n = 16) 5 years 10 months, and for the oldest third (n = 15) 7 years 5 months. As explained in the text, not all children completed all tasks.

For spinner trials (items 8-11), scores are group means of percent of successful trials per subject; for all other variables, scores indicate percentage of the group giving correct responses. Age trends for spinner trials refer to F tests for linear and quadratic trends (df = [1,35]). All other trends refer to the likelihood-ratio chi-square, using the log-linear method of evaluating linear and quadratic models (df = 1). Thus, the values shown are the chi-squares of the difference between the model of independence and the linear model, and between the model of independence and the quadratic model, respectively (SPSS, Inc., 1986, p. 581).

[a]For the counting task, the two younger groups are combined because too few of the youngest group completed the task.

$^{*}p < .10$; $^{**}p < .05$; $^{***}p < .01$.