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ABSTRACT

Designed as an avenue of communication for mathematics educators concerned with the views, ideas, and experiences of two-year college students and teachers, this journal contains articles on mathematics exposition and education, as well as regular features presenting book and software reviews and math problems. The first of two issues of volume 14 contains the following major articles: "Technology in the Mathematics Classroom," by Mike Davidson; "Reflections on Arithmetic-Progression Factorials," by William E. Rosenthal; "The Investigation of Tangent Polynomials with a Computer Algebra System," by John H. Mathews and Russell W. Howell; "On Finding the General Term of a Sequence Using Lagrange Interpolation," by Sheldon Gordon and Robert Decker; "The Floating Leaf Problem," by Richard L. Francis; "Approximations to the Hypergeometric Distribution," by Chitra Gunawardena and K. L. D. Gunawardena; and "Generating 'JE(3)' with Some Elementary Applications," by John J. Edgell, Jr. The second issue contains: "Strategies for Making Mathematics Work for Minorities," by Beverly J. Anderson; "Two-Year Mathematics Pioneers," an interview with Allyn J. Washington; "Using Linear Programming To Obtain a Minimum Cost Balanced Organic Fertilizer Mix," by Stephen J. Turner; "Problems Whose Solutions Lie on a Hyperbola," by Steven Schwartzman; and "The Shape of a Projectile's Path: Explorations with a Computer Algebra System," by John H. Mathews and Robert Lopez. (BCY)

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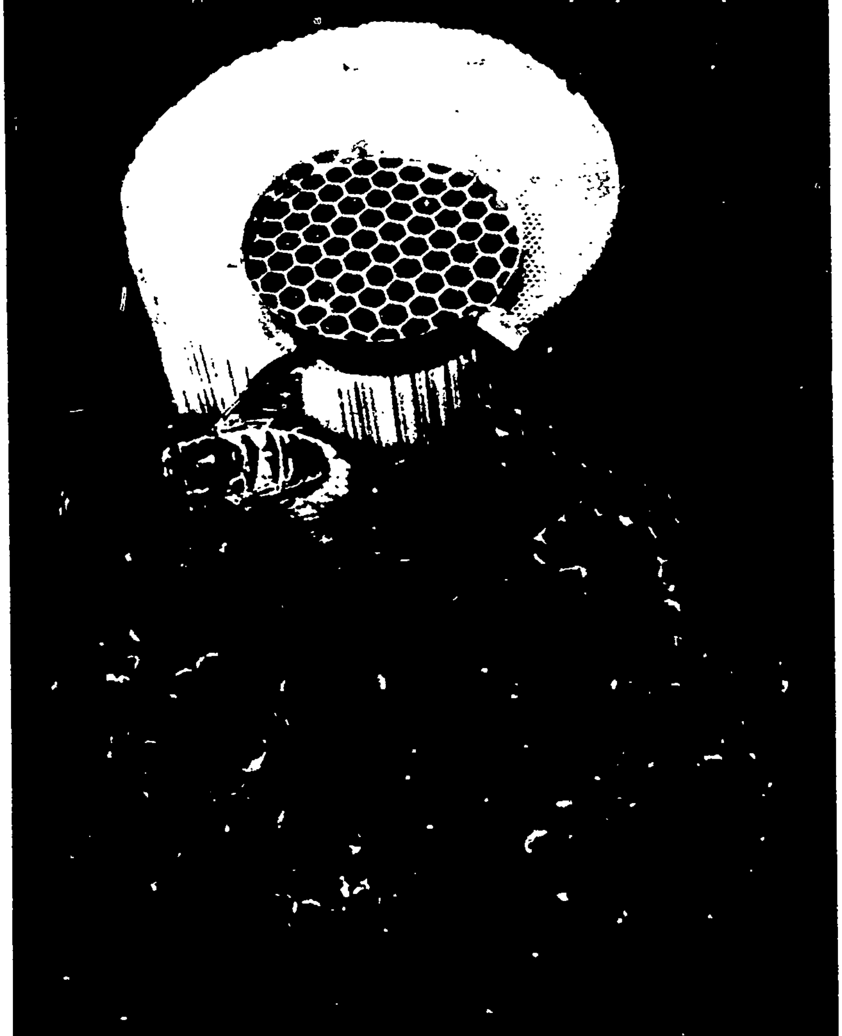
A honeycomb and a ceramic substrate for a catalytic converter share a common geometric structure. Sam McInroy explains why in



Honeybees, Hexagons, and the Industrial Workplace.

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PURPOSE: AMATYC remains committed to its original purposes:

- To provide a national forum for the exchange of ideas
- To further develop and improve the mathematics education of students of two-year colleges
- To coordinate activities of affiliated organizations on a national level
- To promote the professional development and welfare of its members

The AMATYC Review provides an avenue of communication for all mathematics educators concerned with the views, ideas and experiences pertinent to two-year college teachers and students.

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Editor's Comments

This is the last issue of *The AMATYC Review* that I will edit and that Roy Cameron will produce. Six years have passed very quickly for me. And, I thank AMATYC for giving me the opportunity to serve the two-year college mathematics community. I am very proud of the efforts of the journal staff and authors. The staff has included **advertising managers** Eleanor Young and Larry Gilligan; **production managers** Paul Dudenhefer and Roy Cameron; **artist** Robert Burghardt; **editorial panelists** Agnes Azzolino, Ione Boodt, Mike Davidson, Joyce Friske, Brian Hickey, Gordon Hoagland, Nelson Rich, Martha Wood, Carla Thompson, Dennis Reissig, Travis Thompson, Max Cisneros, Marvin Johnson, Charlotte Newsom, Jacqueline Thornberry, and August Zarcone; **columnists** Greg Foley, Les Tanner, John Edgell, Michael Ecker, Shoa Mah, Ralph Selig, Robert Stong, Judy Cain, Joe Browne, Deborah Crocker, Igor Malyshev, and Joanne Becker; and the **numerous reviewers** who have helped to evaluate manuscripts. To these individuals and to the authors who have either successfully or unsuccessfully endured through a sometimes lengthy and painful publication process, I offer a sincere thank you.

I am especially indebted to my friend and colleague, Roy Cameron, for his work as the production manager. Roy's first issue appeared in March, 1988. Throughout our combined tenure with *The Review* we formed a cohesive team. I supplied Roy with the content and he took care of everything else that had to be done in order to get the journal to you in a timely manner. Roy worked with the computer publisher, corresponded with the authors to correct errors, pasted-up the pages, and coordinated the efforts of the printer and mailer. He took great pride in producing a journal that did not have any blank spaces, was free of errors, and was visually pleasing.

Roy and I attempted to build upon the work done by the previous editor, Jay Huber, and his production manager, Paul Dudenhefer, to produce a better journal. Similarly, I expect that *The Review* will continue to improve under the leadership of the new editor, Joe Browne, and his production manager, Jane Covillion (both of Onondaga Community College in Syracuse, NY). However, they will need your help. Send them your suggestions. Send them your articles. Write "Letters to the Editor." Support the columnists. In sum, make it a personal responsibility to help the staff produce a valuable journal for the AMATYC membership.

In closing, I would just like to comment on the content of this issue of *The Review*. While I have always tried to include at least two education articles in each issue, this issue contains only one article in that category. The main reason that this has occurred is that I wanted to clear a backlog of articles that were accepted prior to January 1, 1992 and allow the new editor to have a fresh start. This situation should offer a hint to potential authors. There is a need for more mathematics education articles.

Don Cohen, Editor

Letter to the Editor

Dear Editor:

I read *A Time for Change in Remedial Mathematics* by Edward D. Laughbaum in your Spring 1992 issue and I certainly agree with the fact that developmental mathematics is in need of reform. Ed mentions the use of graphing calculators as the direction developmental mathematics should take. While it is clear that calculator use as well as the use of computers and other technologies must be a part of mathematics instruction at any level, I believe that they must be integrated into instruction to produce a holistic methodology.

Ed also mentions that less time should be spent teaching algebraic manipulations. I wholeheartedly agree. However, developmental mathematics courses tend to be prerequisite courses rather than terminal courses. Therefore, instructors must identify the skills that are needed to help to insure the success of their students in subsequent courses. Also, business and industry are turning more and more to developmental mathematics to help to upgrade workers' skills. Again, it is incumbent upon us to identify and teach those skills necessary to make the workers as productive as possible.

These and other complex issues have led the Mathematics Special Professional Interest Network, a subgroup of the National Association for Developmental Education (NADE) to form a task force of two and four year college mathematics educators. Their project, entitled *Restructuring the Mathematical Bridge: Mathematics Reform for Underprepared College Students*, will use input from both developmental and nondevelopmental instructors to produce and disseminate a reformed developmental mathematics curriculum. The task force plans to include models for instruction and assessment.

The project will have four phases: surveying the current status and extensively assessing the needs of developmental mathematics, developing a curriculum and instruction model based on extensive input from concerned parties, piloting and refining the model, and disseminating the results. The task force is presently organizing committees to implement various aspects of the project and plans to work with representatives from professional mathematics organizations as well as other interested groups. Project funding is currently being sought.

It is anticipated that the curriculum model that will be developed will complement the curricular changes currently occurring at secondary and college levels. Possible outcomes include a reasoned inclusion of new technology and recent pedagogical research, significant modifications to the traditional topics taught in developmental courses, more connections between mathematics and its applications, and an increased focus on nontraditional problem solving. Plans call for pilot testing the materials at 10 institutions of varying types.

What does the *Bridge* project want with you? We are looking for people who are willing to give of their time and energy to serve on four project committees: Curriculum, Instruction, Aassessment, and Dissemination. In addition, we need to people to lead brainstorming sessions at their schools. The linchpin of reforming the structure of developmental mathematics is a massive influx of information from its practitioners. If you are interested in becoming involved in the *Bridge* project, please contact me. Ed's article has indicated a need that must be addressed. Hopefully, we can work together to meet that need.

Sincerely,

William Thomas, Jr.
University of Toledo/ComTech, Scott Park Campus
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ABOUT THE COVER

Honeybees, Hexagons, and the Industrial Workplace

by Samuel F. McInroy
Corning Community College, Corning, NY 14830

Sam has taught all levels of mathematics at Corning Community College since 1964. He is past President of NYSMATYC, a past Northeast Regional Vice President of AMATYC, and a past Chairperson of the Faculty Association at Corning Community College. As this article appears, he is sequestered somewhere in the high plains of Wyoming while enjoying a semester's sabbatical leave in order to devote full attention to completing a book on mathematical puzzles.

Mathematicians have known from ancient times that there are but three possible tessellations of the plane by regular polygons all of the same kind: triangles, squares, and hexagons. If every cell in this mosaic is to have a constant area, then the hexagonal tiling creates this with the least perimeter. Mother Nature has known from time immemorial that this efficient honeycomb pattern—translated into perfectly packed three-dimensional hexagonal prisms—would best satisfy the specialized needs for certain structures, for example within the hive or nest of bees, wasps, and similar insects.

More recently, industries, such as Corning Incorporated, the leading substrate producer for catalytic converters installed in cars built by American automobile manufacturers, have incorporated tessellated compon nts into their products. Inasmuch as industrial substrates are coated by precious metal catalysts that convert exhaust gasses into harmless compounds, the preferred option utilized would balance the need for economizing on the amount of a very expensive coating while providing a maximum of coated surface area and strength with a minimum of back pressure as the exhaust gasses pass through the catalytic chamber. Since the substrates are manufactured via an extrusion from a metal die, additional considerations in the selection of a preferred geometry involve die cost, usable die life, and the success rate in bringing the substrate from its animated green state to an intermediate dry state and finally through a high heat firing state.

Even though there is a great deal to recommend the hexagonal design, certain quality control difficulties encountered in manufacture have, to date, biased industry's choices towards triangular and squared cross sections. The scientist or technician who solves this manufacturing hobgoblin will usher in an era in which this industrial application will be as efficient as the one employed naturally by the honeybee.

Credits: Special thanks to Dr. Paul A. Tick, Research Associate, Richard Bernard, Senior Mechanical Engineer and Kevin R. Brundage, Senior Glass Scientist, all of Corning Incorporated and to Mr. Michael Gilmartin, Professor of English at Corning Community College.

VIEWPOINT

Technology in the Mathematics Classroom

by Mike Davidson
Cabrillo College, Aptos, CA 95003

Mike Davidson graduated with a BS in Mathematics from Tarleton State University in Stephenville, Texas, in 1969, and with an MS in Algebraic Topology from Texas Christian University, in Fort Worth, Texas, in 1973. He has taught mathematics for 15 years, in Texas, Colorado and California, from sixth grade through college.

Mathematics is really two things. First, it is a huge library of time-tested and proven techniques, procedures, and formulas. Second, mathematics is an attitude, a mind-set, with its own special logic, motives, and goals. The former are our "tools of the trade," while the latter makes us artists and craftsmen. Skill in the use of tools can exist in the absence of the creative spark that raises them out of the purely mechanical. We in mathematics education have tacitly believed that learning the manipulations would somehow kindle that fire in the minds of our students, that the logic and the rationality would somehow "rub off" on our students. We have assumed that if a student mastered mathematical mechanics, the mathematical mind would follow. Sometimes that was the case . . . but never, it seems, without the student's natural fascination with structure and pattern.

Over the decades, since the axiomatic approach to teaching mathematics supplanted the scientific (experimental) approach, mathematics teachers have succeeded in making the subject dull enough that the number of people majoring in mathematics has dwindled. The impetus for the mathematical mind-set to arise is that attraction of seeing order come out of chaos, the drive to see some new pattern, ". . . to boldly go where no one has gone before." Our axiomatic approach has eliminated that thrill of exploration and reduced all of introductory mathematics to a mechanical application of dusty principles. Still, a few students do manage to find their way to the beauty, elegance and fire of mathematics.

Mathematics educators have relied on exposure to axiom systems and mechanical manipulations to spark the students interest, to set them on the path to gaining that mathematical high ground, that point-of-view, that mathematical mind, that has brought such phenomenal success in the sciences. However, that approach is succeeding less and less often. Technology has caught up with mathematics as it has with much of industry. In the Industrial Revolution, technology freed workers from most purely mechanical tasks, greatly increasing the speed, reliability and quality of production. We stand at the dawning of a similar revolution in mathematics.

Many of our colleagues argue that if we allow students access to technology, they will come to depend on the technology and will never come to understand what they are doing. They are exactly right on both points. We do come to depend on our technology – how many people still know how to render down animal fat into tallow to make candles? And, if we continue to rely on exposure to the manipulations in mathematics to teach the mathematical mind-set, while at the same time reducing the students' contact time with them, truly, the students will *never* understand mathematics.

Much of the mathematics software currently available is designed to provide students with a variety of similar problems organized by topics. Does drill improve understanding? It gives the students access to more manipulation problems, instant feedback for correctly entered answers and even hints if the student is unsuccessful. But the purpose of drill is to reduce the need for students to "think" about problems by "conditioning" the students' responses to certain situations. Having to think about each problem slows down the manipulations. In theory, if students do enough problems of a specific type, they need not think at all. They become trained to make correct responses: they see a problem they recognize and do the right thing automatically. While Pavlovian conditioning may make the students faster and more accurate mechanically, it robs them of flexibility. They may know *what* to do in certain situations without knowing *why* they are doing it. They may be mechanically perfect and understand nothing. But perhaps the worst side effect of drilling is that students never really learn to conquer new problems, problems unlike those to which they have been conditioned to respond. When confronted with a genuinely original problem, they have no idea how to begin. Drill has its uses, as I pointed out earlier, but conditioned responses are *not* creativity. If the only problems truly worth solving are those that haven't been solved before, increasing the amount of drill required of the students is counterproductive.

As a member of the editorial panel of the *Review,* I see an increasing number of articles on the "use of technology" in the classroom. I would like to share with you my criteria for evaluating the content of those articles for what it is worth. If the technology applied in the article does nothing more than speed up the process (or improve its reliability or quality), I classify it as "sterile." For example, using a graphing calculator to approximate the real roots of a polynomial is sterile. It offers only an increase in efficiency over the old process. It poses no new questions, opens no new doors. It deals with the same problem set as before. It merely filters old techniques through new technology.

On the other hand, a "creative" application of technology can raise more questions in the students' minds than it answers. It can give the students something to think about that the old process didn't address for lack of time or lack of breadth. An example of a creative application of technology is the use of a graphing calculator to help students explore the relationship between the coefficients of a polynomial and the shape of its graph. Push the "leading term test" to the limit, to include all the terms not just the first one.

I tell my beginning precalculus students that I expect them by the end of the course to be able to tell me how each term in the polynomial function $f(x)=x^4-5x^2+x$ contributes to the overall shape of its graph. Since each student has access to a graphing calculator, the calculator becomes a tool to answer questions that we normally don't pose for lack of time or the lousy graphing ability of the students. The students can then be asked to speculate on how the terms of a function, in general, relate to the shape of its graph, and to compare that to how the factors of a function, in general, relate to the shape of its graph . . . .

In the classroom, the distinction between sterile and creative technological applications is a relatively easy one to make if you ask the right questions:

1) Are the students asking more questions or less than usual?
2) Are the questions about the technology or the mathematics?
3) Are the mathematical questions of higher or lower order than usual?
4) Do some of the questions extend the discussion into new but related areas?

In summary, technology can save us or sink us in the classroom. Creative applications of technology can restore much of the thrill of exploration by giving even our less skillful students tools to take them where they could not have easily gone before. But we must learn how to pass on the "mathematical mind" to our students without the drill and manipulations. We must reinfect them with the excitement of discovery, with the dramatic power of analytical reasoning. We have a lot to learn, but we stand at the door to a new era in mathematics education (and with heartfelt thanks to Bill Shakespeare and Gene Roddenberry) that is our "undiscovered country."

MATHEMATICAL EXPOSITION

Reflections on Arithmetic-progression Factorials

by William E. Rosenthal
Ursinus College, Collegeville, PA 19426

Bill received his PhD in mathematics from the State University of New York at Stony Brook in 1983. He has devoted his professional and personal lives to an ongoing quest for ways to reinvent humane, humanistic, and life-affirming mathematics pedagogies. In August 1990, Ursinus acknowledged his creation of the course "Humanistic Calculus" with the inaugural Sears Roebuck Award for Teaching Excellence and Campus Leadership.

In a short communication appearing in the Fall, 1990 issue of *The Review*, David L. Farnsworth (1990) obtains closed-form expressions for the "even and odd factorials"

$$2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2)(2n) \qquad \text{(Even)}$$

and

$$1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)(2n-1)\,. \qquad \text{(Odd)}$$

His ideas are condensed as follows: factoring out a 2 from each component of (Even) yields $2^n \cdot n!$ as a closed form of an "even factorial"; since the product of (Even) and (Odd) is $(2n)!$, an "odd factorial" closes to

$$\frac{(2n)\,!}{2^n \cdot n\,!}\,.$$

Professor Farnsworth's piece moved me to recall some long-lost work of my own and led me to consider a generalization of his results – then more. Here is a synopsis of my investigations.

I looked first at the "natural" extensions of (Even) and (Odd) to products in which the factors are in (finite) arithmetic-progression, being either multiples of a fixed positive integer k, as with

$$k \cdot 2k \cdot 3k \cdot \ldots \cdot [(n-1)k]\,(nk), \qquad (0_k)$$

or taking the form $(ik + 1)$, where i ranges from 0 to $n-1$, as in the product

$$(1)(k+1)(2k+1) \ldots [(n-2)k+1]\,[(n-1)k+1]\,. \qquad (1_k)$$

Both expressions comprise n factors; I will soon explain their strange-looking labels. Please note that (0_k) and (1_k) reduce respectively to (Even) and (Odd) when $k = 2$ and that each takes the value of $n!$ when $k = 1$.

Being amenable to the identical manipulation used to evaluate its particular case (Even), the product (0_k) is receptive to "closure":

$$k \cdot 2k \cdot 3k \cdot \ldots \cdot [(n-1)k](nk) = [k \cdot k \cdot k \cdot \ldots \cdot k \cdot k][1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)n] \qquad (n \text{ factors of } k)$$

$$= k^n \cdot n! \, .$$

Expression (1_k) is less closure-friendly since, when $k > 2$, multiplication by its companion (0_k) leads to neither a factorial nor another recognizable quantity. When $k = 3$, these expressions are

$$3 \cdot 6 \cdot 9 \cdot \ldots \cdot (3n-3)(3n) \qquad (0_3)$$

and

$$1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3n-5)(3n-2), \qquad (1_3)$$

whose product is

$$[1 \cdot 3][4 \cdot 6][7 \cdot 9] \ldots [(3n-5)(3n-3)][(3n-2)(3n)] \, .$$

And, $(3n)!$ is among the many things this product is not. A third arithmetic progression product is needed to fill the gaps and fashion $(3n)!$. This is

$$2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n-4)(3n-1) \, , \qquad (2_3)$$

each of whose factors leaves a remainder of 2 upon division by 3. Here lies a faithful analogue to the origin: odd-even situation: in the $k = 2$ case, interlacing the factors in two products (the even and odd factorials) fills out $(2n)!$. This creates a closed form for the odd product, each of whose factors has a value of 1 mod 2, in terms of a product already found for its even partner, which comprises factors that are 0 mod 2. (Hence, the unusual labelling scheme.) When $k = 3$, three arithmetic-progression products must be employed to compose $(3n)!$. And so it goes.

Thus, a revised generalization of the odd-even factorial question: the general case of "modulus" k calls for not only the two products (0_k) and (1_k) but a family of k products, whose members are

$$(0k+1)(1k+1)(2k+1) \ldots [(n-2)k+1][(n-1)k+1] \qquad (1_k)$$

$$(0k+2)(1k+2)(2k+2) \ldots [(n-2)k+2][(n-1)k+2] \qquad (2_k)$$

$$\ldots$$

$$(0k+i)(1k+i)(2k+i) \ldots [(n-2)k+i][(n-1)k+i] \qquad (i_k)$$

$$\ldots$$

$$(0k+(k-1))(1k+(k-1))(2k+(k-1)) \ldots [(n-2)k+(k-1)][(n-1)k+(k-1)] \qquad ((k-1)_k)$$

$$(0k+k)(1k+k)(2k+k) \ldots [(n-2)k+k][(n-1)k+k].^1 \qquad (0_k)$$

I'll choose the name *arithmetic-progression factorial* for any product of this type, whose generic form is (i_k). For each pair of positive integers n and k, the members of this family multiply to a grand product of $(kn)!$, affording some hope of finding a closed form for the arithmetic-progression factorial (1_k) — and, perhaps, its siblings as well.

Once again, I'll illustrate by specifying to $k=3$. Here, $(3n)!$ is the product of the "big factors" (1_3), (2_3), and (0_3):

$$[1\cdot 4\cdot 7\cdot \ldots \cdot(3n-5)(3n-2)][2\cdot 5\cdot 8\cdot \ldots \cdot(3n-4)(3n-1)][3\cdot 6\cdot 9\cdot \ldots \cdot(3n-3)(3n)]=(3n)!. \quad (A_3)$$

Since the third factor has value $3^n\cdot n!$, this identity "solves" the problem of evaluating the arithmetic progression factorial (1_3):

$$1\cdot 4\cdot 7\cdot \ldots \cdot(3n-5)(3n-2) = \frac{(3n)!}{[3^n\cdot n!]\,[2\cdot 5\cdot 8\cdot \ldots \cdot(3n-4)(3n-1)]}. \quad (B_3)$$

Of course, the answer on the right is worse than the question on the left, as this pseudo-solution is even more complicated and itself contains an open-form factor! Realizing this, I mourned an hour of apparently misbegotten effort and continued working with a consolation prize – the product of the **two** unknown expressions (1_3) and (2_3). Returning to (A_3) and dividing by only the third big factor results in

$$[1\cdot 4\cdot 7\cdot \ldots \cdot(3n-5)(3n-2)][2\cdot 5\cdot 8\cdot \ldots \cdot(3n-4)(3n-1)] = \frac{(3n)!}{3^n\cdot n!},$$

from which something tangible can be salvaged. First associate pairs of consecutive integers to obtain

$$[1\cdot 2][4\cdot 5][7\cdot 8]\ldots[(3n-5)(3n-4)][(3n-2)(3n-1)] = \frac{(3n)!}{3^n\cdot n!}.$$

Next observe that since $\binom{m}{2} = \frac{m(m-1)}{2}$, each pair is twice a binomial coefficient. This observation inspires rewriting $(3i-2)(3i-1)$ as $2!\binom{3i-1}{2}$. Since the left-hand side of the previous identity consists of the product of $(3i-2)(3i-1)$ for i ranging from 1 to n, pushing the n factors of 2! to the denominator of the right-hand side and using $(2!)^n\cdot 3^n = (3!)^n$ leads to a product of binomial coefficients:

$$\binom{2}{2}\binom{5}{2}\binom{8}{2}\cdots\binom{3n-4}{2}\binom{3n-1}{2} = \frac{(3n)!}{(3!)^n\cdot n!}.$$

Similarly, for $k=4$, the decomposition

$$[1\cdot 5\cdot 9\cdot \ldots \cdot(4n-7)(4n-3)][2\cdot 6\cdot 10\cdot \ldots \cdot(4n-6)(4n-2)]$$
$$[3\cdot 7\cdot 11\cdot \ldots \cdot(4n-5)(4n-1)][4\cdot 8\cdot 12\cdot \ldots \cdot(4n-4)(4n)] = (4n)!$$

hints at dividing by the fourth big factor (in the form $4^n\cdot n!$) and regrouping the surviving pieces of the left-hand side into triads of consecutive integers. The result is

$$[1\cdot 2\cdot 3][5\cdot 6\cdot 7][9\cdot 10\cdot 11]\ldots[(4n-7)(4n-6)(4n-5)][(4n-3)(4n-2)(4n-1)] = \frac{(4n)!}{4^n\cdot n!}.$$

Here, as a consequence of

$$(4i-3)(4i-2)(4i-1) = 3!\binom{4i-1}{3},$$

each three-factor grouping on the left is the product of 3! with a "choose-three" binomial coefficient. The same arithmetic that was successful for $k=3$ leads to

$$\binom{3}{3}\binom{7}{3}\binom{11}{3}\cdots\binom{4n-5}{3}\binom{4n-1}{3} = \frac{(4n)!}{(4!)^n \cdot n!}.$$

A general pattern has now been revealed. It appears as if the identity

$$\binom{k-1}{k-1}\binom{2k-1}{k-1}\binom{3k-1}{k-1}\cdots\binom{(n-1)k-1}{k-1}\binom{nk-1}{k-1} = \frac{(kn)!}{(k!)^n \cdot n!}$$

holds for all $k>2$ (and $n \geq 1$), and I invite you to demonstrate this discovery by considering the entire family $\{(i_k)\}$. In the underused "$\prod$-notation" for products, this identity reads as

$$\prod_{i=1}^{n}\binom{ik-1}{k-1} = \frac{(kn)!}{(k!)^n \cdot n!}.$$

This relation, inductively discovered by starting with $k=3$, also holds when $k=2$, as it then reduces to (Odd) and its closed-form equivalent. Since both sides have value 1 when $k=1$, the identity extends "down" an additional case.

Of course, this discovery doesn't speak to the original question of finding a closed-form expression for the arithmetic-progression factorial

$$(0k+1)(1k+1)(2k+1)\ldots[(n-2)k+1][(n-1)k+1] \qquad (1_k)$$

In the spirit of the professoriate, I've answered a question other than the one asked. Rather than an answer to a generalized question, this discovery describes a "Jeopardy"-like question for a generalized answer. Professor Farnsworth's (specific) question "What is $1\cdot3\cdot5\cdot\ldots\cdot(2n-3)(2n-1)$?" led to his (specific) answer

$$\frac{(2n)!}{(2!)^n \cdot n!},$$

which I have extended to

$$\frac{(kn)!}{(k!)^n \cdot n!}.$$

Were this expression to be revealed under the "Factorials" category of some futuristic episode of "Jeopardy," the generalized question

$$\text{"What is } \prod_{i=1}^{n}\binom{ik-1}{k-1}\text{ ?"}$$

would be an unimpeachable response!

Still, what about the abandoned question of evaluating the product (1_k)? Well, you may recall that an answer has been manufactured for the $k=3$ case: the unsatisfying right-hand side of (B_3). By reopening $3^n \cdot n!$ into $3 \cdot 6 \cdot 9 \cdot \ldots \cdot (3n-3)(3n)$, grouping each factor with the adjacent integer in $2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n-4)(3n-1)$, and identifying the ensuing pairs as binomial coefficients, this answer assumes the somewhat more glamorous form

$$\frac{(3n)!}{2^n \cdot \prod_{i=1}^{n} \binom{3i}{2}}$$

—which, however, remains an order of magnitude more complex than the question. Similar unniceties can be similarly obtained for other values of k.[2]

Aside from its complexity, the above "answer" corrupts the spirit of Professor Farnsworth's quest, which is to find **closed-form** expressions for arithmetic-progression factorials. This brings me to my final point, which lies within the purview of both the philosophy and the linguistics of mathematics. In this paper, I've applauded

$$\frac{(kn)!}{(k!)^n \cdot n!}$$

for being in closed-form and scorned (among other expressions)

$$1 \cdot 4 \cdot 7 \cdot \ldots \cdot (3n-5)(3n-2) \quad \text{and} \quad \binom{3}{2}\binom{6}{2}\binom{9}{2} \cdots \binom{3n-3}{2}\binom{3n}{2}$$

for their indeterminacy. Yet are closed- and open-form expressions necessarily as distinct in character as we make them out to be? Please consider your answer—then my claim that the **only** distinction between

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)n \qquad \text{(in the dreaded open-form)}$$

and

$$n! \qquad \text{(in the celebrated closed-form)}$$

is that the latter is a familiar, universally accepted **nickname** for the former. Yet I don't perceive the nickname $n!$ as intrinsically superior to $1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)n$ —since, being a humanly created abbreviation, $n!$ isn't intrinsically anything! It is closed-form by dictate, by fiat, by definition, and by contrivance: some folks wearied of writing out the product of the first n positive integers and, having failed to express this product in terms of quantities known to them (as had long before been done with the sum of these selfsame integers), they chose to create a name ("n factorial") and a nickname ("$n!$") for it. A nickname and a name—no more and no less.[3]

As a community, mathematicians have chosen to name ubiquitous open-form expressions such as $1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)n$, $x \cdot x \cdot \ldots \cdot x$, and

$$\frac{n(n-1)\ldots(n-r+1)}{1 \cdot 2 \cdot \ldots \cdot r},$$

and to promulgate their names within both the classroom and the ambient culture, while eschewing names for the likes of

$$\binom{3}{2}\binom{6}{2}\binom{9}{2}\cdots\binom{3n-3}{2}\binom{3n}{2}.^{4}$$

This product could be honored with its own appellation – but why bother to christen a quantity or object likely never to be heard from again? Still, I think we should acknowledge that many of the "simple" expressions we work with derive their "simplicity" from the fact that they are mere chosen nicknames for frequently occurring quantities. I recall reading a comment by Neil Postman, a brilliant critic of the good, the bad, and the ugly in human communication, to the effect that language and meaning are made not only by the definitions we give to words but also by our choices to create definitions and names for certain objects and not for others. (Unfortunately, I cannot locate the reference.) The activities of mathematics and mathematics education, each of which is tantamount to no more and no less than meaning making mediated by language, are no exceptions.

Notes

1 Please note that I've purposefully complicated the writing of the "eldest" members (0_k) and (1_k) so as to make them fit in with their relatives. Also, I've kept the denotation of (0_k) for the last-written expression, although this breaks the convention of tagging each product by its first factor. For consistency, think of a product as named by the residue mod k of each of its factors.

2 If you are despairing of ever seeing a pithy equivalent to (1_k), fear not. The magic wand of the gamma function can be waved to evaluate this product and also the other members of the family $\{(i_k)\}$ (Abramowitz & Stegun, 1965, p. 255; see also footnote 4 below). I find this connection to be spiritually pleasing, inasmuch as this family is in a sense a generalization of the factorial function (for the degenerate case $k=1$, the "only child" (0_k) has value $n!$), as is the gamma function.

3 I think it's important to distinguish between contrived closed-form expressions (e.g., $n!$) and more "naturally" arising ones such as $\frac{n(n+1)}{2}$ – although mathematical extremists might protest that "n" is just as humanly created as $n!$, being itself a nickname for the open-form expression $1+1+\ldots+1$ with n summands.

4 An expression such as $1\cdot 4\cdot 7\cdot\ldots\cdot(3n-5)(3n-2)$ falls somewhere between these extremes, with its nickname $\frac{\Gamma(n+(1/3))\,3^n}{\Gamma(1/3)}$ known to a proper subset of all mathematicians and scientists but very few other humans. It's also interesting to note that (Even) and (Odd) themselves respectively enjoy the quasi-common "double-factorial" nicknames of $(2n)!!$ and $(2n-1)!!$ (Abramowitz & Stegun, 1965, p. 258).

References

Abramowitz, M. & Stegun, I. [Eds.] (1965). *Handbook of mathematical functions.* New York: Dover.

Farnsworth, D. (1990). Odd and even factorials. *The AMATYC Review. 12*(1), 46-47.

The Investigation of Tangent Polynomials with a Computer Algebra System

by

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Computer algebra systems (CAS) such as Derive, Maple and *Mathematica* are influencing the way we teach mathematics. With the assistance of CAS, we can examine new avenues for exploring old problems, and perhaps gain new insights. For this paper, we take a close look at the fact that the limit of "the secant line" is "the tangent line." We recast this situation in the notation of polynomial approximation and view the secant line as the Newton polynomial $P_1(x)$ of degree one passing through the two points $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$. The tangent line is the Taylor polynomial

$$T_1(x) = f(x_0) + f'(x_0)(x-x_0).$$

It is well known that $T_1(x)$ is the limit of $P_1(x)$ as $h \to 0$. Figure 1 shows $f(x) = e^{x/4}\cos(x)$ and the linear approximations $T_1(x)$ and $P_1(x)$ based on $x_0 = 0$ and $h = \frac{1}{4}$.

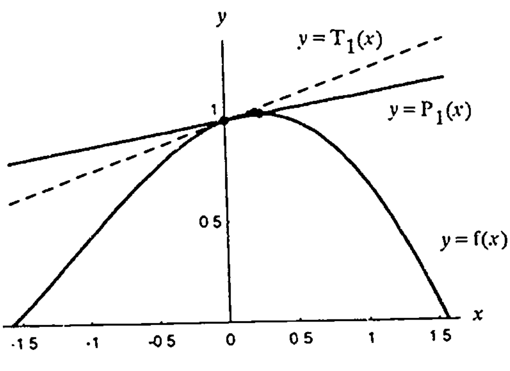

Figure 1. Graph of $f(x)$, $P_1(x)$, and $T_1(x)$ (dashed line).

This article shows how the Newton polynomial $P_n(x)$ is derived on a given interval. The computer algebra system *Mathematica* is used to assist with the construction of $P_n(x)$ and to demonstrate that the limit of a certain sequence $\{P_n(x)\}$ of Newton polynomials is the Taylor polynomial $T_n(x)$ of degree n. Hence, the Taylor polynomial is visualized as "the limit of approximating polynomials" and is imagined to be "the tangent *polynomial*" on the given interval.

Preliminary Results

Isaac Newton developed and refined methods for fitting polynomials to curves in the second half of the fifteenth century (Whiteside, 1976, Vol. IV, pp. 14-73; Vol VIII, pp. 236-257). In the early 1690's he used these methods to calculate the apparent path of a comeut from individual sightings of its orbit (Whiteside, 1976, Vol. VII, p. 672). Today Newton polynomials are a familiar topic in numerical analysis (cf. Mathews, 1987, p. 186), and take the following form:

The Newton polynomial of degree $n = 1$ is

$$P_1(x) = a_0 + a_1(x - x_0). \tag{1}$$

The coefficients in equation (1) are determined by forcing $P_1(x)$ to pass through the two points $(x_0, f(x_0))$ and $(x_0+h, f(x_0 + h))$. This leads to the linear system:

$$\begin{aligned} a_0 + a_1(x_0 - x_0) &= f(x_0) \\ a_0 + a_1(x_0+h-x_0) &= f(x_0 + h), \end{aligned} \tag{2}$$

the solution of which is $a_0 = f(x_0)$ and $a_1 = \dfrac{f(x_0+h)-f(x_0)}{h}$. Hence, the polynomial $P_1(x)$ can be expressed in the form:

$$P_1(x) = f(x_0) + \frac{f(x_0 + h) - f(x_0)}{h}(x - x_0). \tag{3}$$

Assuming that $f(x)$ is differentiable, and letting h approach zero in equation (3), we find that the limit of the Newton polynomial $P_1(x)$ is the Taylor polynomial $T_1(x)$, i.e.

$$\begin{aligned} \lim_{h\to 0} P_1(x) &= f(x_0) + \lim_{h\to 0}\frac{f(x_0 + h) - f(x_0)}{h}(x - x_0) \\ &= f(x_0) + f'(x_0)(x - x_0) = T_1(x). \end{aligned} \tag{4}$$

A similar phenomenon occurs for quadratic polynomials. We begin with the Newton polynomial

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1). \tag{5}$$

The coefficients in equation (5) are determined by forcing $P_2(x)$ to pass through the three points $(x_k, f(x_k))$ for $x_k = x_0 + hk$ and $k=0,1,2$. This leads to the lower-triangular linear system

$$\begin{aligned} a_0 &= f(x_0) \\ a_0 + h\,a_1 &= f(x_0 + h) \\ a_0 + 2h\,a_1 + 2h^2 a_2 &= f(x_0 + 2h). \end{aligned} \tag{6}$$

This system can be solved by forward substitution to obtain:

$$a_0 = f(x_0),\ a_1 = \frac{f(x_0 + h) - f(x_0)}{h} \ \text{ and } \ a_2 = \frac{f(x_0) - 2f(h + x_0) + f(x_0 + 2h)}{2!\,h^2}.$$

Thus the polynomial $P_2(x)$ takes on the form:

$$P_2(x)=f(x_0)+\frac{f(x_0 + h)-f(x_0)}{h}(x-x_0) + \frac{f(x_0)-2\,f(h + x_0)+f(x_0 + 2h)}{2!\,h^2}(x-x_0)(x-x_1)\,. \tag{7}$$

Using the notation $h = \Delta x$, the coefficient a_2 can be viewed as one half of the second order Newton difference quotient $\frac{\Delta^2 f}{\Delta x^2}$. It is well known that $\frac{\Delta^2 f}{\Delta x^2}$ is the forward difference approximation for $f''(x_0)$, and tends to this quantity as $h \to 0$. Thus, we can conclude from equation (7) that the limit of the Newton polynomial $P_2(x)$ as $h \to 0$ is the Taylor polynomial $T_2(x)$:

$$\lim_{h\to 0} P_2(x) = f(x_0) + \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h}(x-x_0)$$

$$+ \lim_{h\to 0}\frac{f(x_0)-2f(h+x_0) + f(x_0+2h)}{2!\,h^2}(x-x_0)(x-x_0 - h)$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 = T_2(x)\,. \tag{8}$$

The software *Mathematica* can solve systems, differentiate and find limits. Additionally, it can symbolically manipulate quantities involving an arbitrary function $f(x)$. The dialogue for establishing (4) starts by defining the function P1[x] in equation (3):

```
P1[x_,x0_,h_] := f[x0] + (f[x0+h]-f[x0])(x-x0)/h
```

$$f[x0] + \frac{(x - x0)\ (-f[x0] + f[h + x0])}{h}$$

We then invoke *Mathematica's* **Limit** procedure and let h → 0. As anticipated, the result is the Taylor polynomial of degree n = 1:

```
T1[x_] = Limit[P1[x,x0,h],h->0,Analytic->True]
```

```
f[x0] + (x - x0) f'[x0]
```

CAS Investigation of Polynomials of Higher Degree

A natural question to ask now is: "What about polynomial approximation of higher degrees?" Exploration of the Newton polynomials involves complicated symbolic manipulations and is prone to error when carried out with hand computations. These derivations can become instructive and enjoyable when they are performed with computer algebra software.

Let $P_3(x)$ be the Newton polynomial that passes through the four points $(x_k, f(x_k))$ for $x_k = x_0 + hk$ and $k = 0, \ldots, 3$. It may be shown that the Taylor polynomial $T_3(x)$ is the limit of $P_3(x)$.

We shall use the power of *Mathematica* to assist us with this derivation. We begin by setting y(x) equal to the general form of a Newton polynomial by issuing the following *Mathematica* commands:

```
n = 3;
Vars =Table[a[k],{k,0,n}];
Eqns =Table[0,{4}];
y[x_] =Sum[a[j] Product[x-x0-h i,{i,0,j-1}],{j,0,n}]
```

The output generated by the computer is:

```
a[0] + (x - x0) a[1] + (x - x0) (-h + x - x0) a[2] +
(x - x0) (-2h + x - x0) (-h + x - x0) a[3]
```

Now we form the set of four equations that force the polynomial to pass through the four equally-spaced points:

```
n = 3;
Do[Eqns[[k+1]]=y[x0+k*h]==f[x0+k*h],{k,0,n}];
TableForm[Eqns]
```

$$a[0] == f[x0]$$

$$a[0] + h\,a[1] == f[h + x0]$$

$$a[0] + 2h\,a[1] + 2h^2\,a[2] == f[2h + x0]$$

$$a[0] + 3h\,a[1] + 6h^2\,a[2] + 6h^3\,a[3] == f[3h + x0]$$

Then we solve this linear system, construct the polynomial $P_3(x)$, and store it as the function **P[x,x0,h]** (since it involves the additional parameters x_0 and h).

```
Solutions = Solve[Eqns,Vars];
Solutions = First[MapAll[Together,Solutions]];
Y = y[x]/.Solutions;
P[x_,x0_,h_] = Y
```

$$f[x0] + \frac{(x-x0)\,(-f[x0]+f[h+x0])}{h} + \frac{(x-x0)\,(-h+x-x0)\,(f[x0]-2f[h+x0]+f[2h+x0])}{2\,h^2} + \frac{(x-x0)\,(-2h+x-x0)\,(-h+x-x0)\,(-f[x0]+3f[h+x0]-3f[2h+x0]+f[3h+x0])}{6\,h^3}$$

Finally, we compute the limit to verify that our conjecture was correct:

```
T[x_] = Limit[P[x,x0,h],h->0,Analytic->True]
```

$$f[x0] + (x-x0)\,f'[x0] + \frac{(x-x0)^2\,f''[x0]}{2} + \frac{(x-x0)^3\,f^{(3)}[x0]}{6}$$

Eureka! The limiting case of $P_3(x)$ as $h \to 0$ is the Taylor polynomial $T_3(x)$. Observe that the option **Analytic->True** must be used in *Mathematica's* limit procedure. This is a mathematicians way to tell the computer that f(x) is "sufficiently differentiable."

An Example

It is instructive to visually see how the Newton polynomials converge to the Taylor polynomial. For illustration we use the function $f(x) = e^{x/4}\cos(x)$ and the point $x_0 = 0$. We then draw the graphs of Newton polynomials with $h = 1/4$ and $h = 1/16$ and compare them with the Taylor polynomial. First, we enter f into the session by typing **f[x_] = Exp[x/4]Cos[x]** . Then we use the built in *Mathematica* command **Series** to generate the Taylor polynomial for f(x) centered at $x_0 = 0$ of degree 3:

```
T[x_] = Normal[Series[f[x],{x,0,3}]]
```

$$1 + \frac{x}{4} - \frac{15\,x^2}{32} - \frac{47\,x^3}{384}$$

Figure 2 shows a comparison of the Taylor polynomial $T_3(x)$ and f(x):

```
gr[0]=Plot[T[x],{x,-4.7,3.5},PlotRange->{-3.0,1.2},
          PlotStyle->Dashing[{0.02,0.02}],
          Ticks->{Range[-4,3,1],Range[-3,3,1]}];
grf=Plot[f[x],{x,-4.7,3.5},PlotRange->{-3.0,1.2},
          PlotStyle->Thickness[0.006],
          Ticks->{Range[-4,3,1],Range[-3,3,1]}];
Show[grf,gr[0]];
```

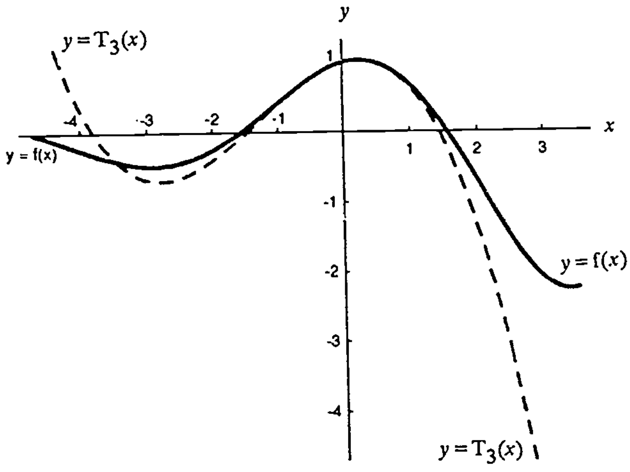

Figure 2. f(x) and $T_3(x)$ (the dashed curve).

Figure 3 compares $P_3(x,0,1/4)$, the Newton polynomial with $x_0 = 0$ and $h = 1/4$, and the four equally-spaced points on which it is based. This gr· ɲh was obtained by issuing the subroutine call **graph[1/4]**. The syntax for the subroutine **graph[h]** is listed in the appendix.

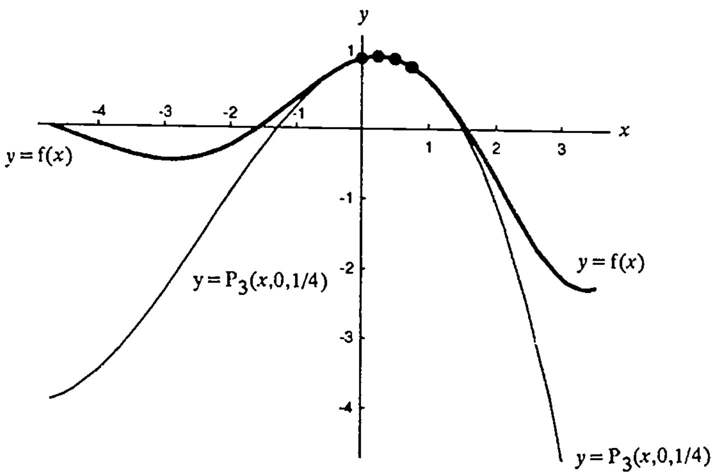

Figure 3. $P_3(x,0,1)$ and f(x) (the "thick" curve)

We can easily obtain a comparison of f(x) and $P_3(x)$ for smaller and smaller values of h. The command **graph[1/16]** constructs $P_3(x)$ for h =1/16 and stores it in **gr[1/16]**. Then the following command graphs the Newton polynomials for h =1/4 and h =1/16, along with the original function f(x) and its Taylor polynomial $T_3(x)$:

```
Show[{grf,gr[0],gr[1/16],gr[1/4]}];
```

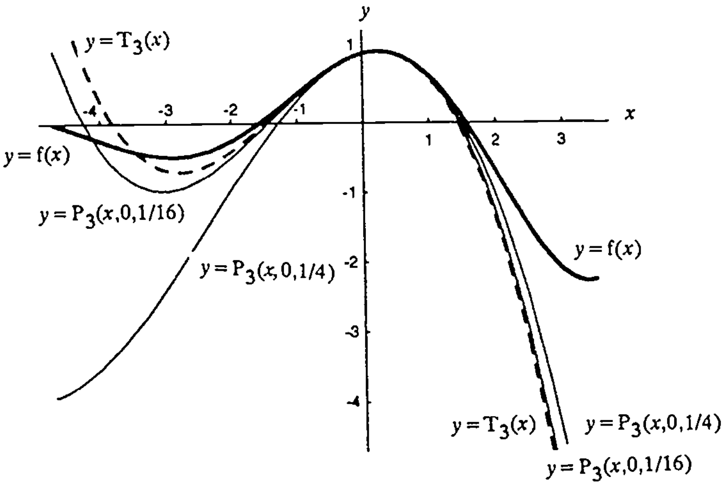

Figure 4. f(x) (the "thick" curve), $P_3(x,0,1/4)$, $P_3(x,0,1/16)$ (closest to the dashed curve) and $T_3(x)$ (dashed)

The General Case

The command to form an arbitrary Newton polynomial is:

```
Sum[a[j] Product[x-x0-h i,{i,0,j-1}],{j,0,n}]
```

and was used in the above derivations. Proceeding to an arbitrary case is accomplished by merely changing the value of n to some value other than $n=3$. Experimentation has revealed that *Mathematica* can quickly solve the general case for degrees up to $n=9$.

Concluding Remarks

We have shown how a CAS such as *Mathematica* can be used in a meaningful way to make mathematical exploration enjoyable, even when computations become laborious. One should note that the CAS is limited in that it is unable to solve our problem for a general case (n arbitrary), but rather requires specific values of n. The interested reader is encouraged to formulate a proof for such a general case.

Appendix

The following is the listing of the subroutine for plotting the Newton polynomial, the points which it is to pass through, and the Taylor polynomial. The programming language is *Mathematica*.

```
graph[h_] := Block[{},
        xys  =Table[{h(k-1),f[h(k-1)]},{k,4}];
        dots =ListPlot[xys,PlotStyle->{PointSize[0.02]}];
```

```
        grf   = Plot[f[x],{x,-4.7,3.5},
        PlotRange->{-3.0,1.2},
        Ticks->{Range[-4,3,1],Range[-3,1,1]}];
gr[h] = Plot[P[x,0,h],{x,-4.7,3.5},
        PlotRange->{-3.0,1.2},
        PlotStyle->Thickness[0.002],
        Ticks->{Range[-4,3,1],Range[-3,1,1]}];
Show[{grf,gr[h],dots}]; ]
```

References

Mathews, J. H. (1987). *Numerical methods.* Englewood Cliffs, NJ: Prentice-Hall.

Whiteside, D. T. (Ed.) (1976). *The mathematical papers of Isaac Newton.* New York: Cambridge University Press.


TEXAS INSTRUMENTS
X=1.6315789 Y=-3.677778
RANGE
ZOOM
TRACE
GRAPH
2nd
INS
DEL
ALPHA
MODE
MATH
MATRX
PRGM
VARS
CLEAR
SIN
COS
TAN
EE
ENTER
ON

On Finding the General Term of a Sequence Using Lagrange Interpolation[1]

by

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Bob is also interested in using technology to enhance the teaching of mathematics at all levels in the undergraduate curriculum. He gratefully acknowledges the National Science Foundation for grant number USE-91-53298 to support his work in calculus reform.

We are all familiar with the story of Gauss' teacher who assigned the class the makework chore of summing one through 100 only to be pestered in moments by Gauss who presented not only the correct answer, 5050, but also the general formula for the sum of the first n integers. Unfortunately, few of us are blessed with such insight (let alone with students of such a caliber). Therefore, whenever we encounter a similar problem involving finding the general term for a sequence, we are far more likely to apply Edison's prescription of 2% inspiration mixed in with 98% perspiration.

In the present article, we will consider an effective method for determining the general terms of certain sequences. This technique involves little, if any, effort, especially if appropriate computer software is available. The sole limitation on the method is that it requires the general term to be of purely polynomial form with rational coefficients.

We will illustrate the approaches with two examples. The first involves finding a formula for

$$\sum_{k=0}^{n} k^5 \tag{1}$$

The terms of this sequence are $S_0 = 0^5 = 0, S_1 = 0^5 + 1^5 = 1, S_2 = 33, S_3 = 276, \ldots$ The second example involves finding an expression for the general term in the sequence

$$1, 8, 35, 112, 294, 672, 1386, 2640, 4719, \ldots \tag{2}$$

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for $n = 6, 8, 10, \ldots$ which is adapted from a problem which arose in a separate investigation (Gordon, 1991) involving the patterns in the coefficients when $\cos n\theta$ is expanded. We should anticipate that the formula for the summation (1) will involve a polynomial of degree 6 in n. (In general, the formula for Σk^p is always a polynomial of degree $p+1$; see Paul, 1985 and its references.) Alternatively, we might notice that the sequence of partial sums satisfies the difference equation

$$S_{n+1} = S_n + (n+1)^5 .$$

Therefore, the rules for applying the discrete formulation of the method of under-determined coefficients would also indicate using a sixth degree polynomial. As for the sequence (2), we would probably have little feel, if any, for what the pattern should turn out to be.

Lagrange Interpolating Polynomials

Our suggested approach is based on the use of the Lagrange interpolating polynomial of degree n,

$$P_n(x) = \sum_{k=0}^{n} f(x_k)\, L_k(x) ,$$

where

$$L_k(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})\,(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\,(x_k-x_1)\ldots(x_k-x_{k-1})\,(x_k-x_{k+1})\ldots(x_k-x_n)} = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x-x_j}{x_k-x_j} .$$

Notice that $L_k(x_k) = 1$ for any $k = 0, 1, \ldots, n$ since all factors cancel out. Further, if $j \neq k$, then $L_k(x_j) = 0$ since $L_k(x)$ contains the factor $(x - x_j)$. Therefore, $P_n(x_k) = f(x_k)\, L_k(x_k) = f(x_k)$ for each k. Thus, $P_n(x)$ is a linear combination of $n+1$ polynomials of degree n which passes through each of the $n+1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

To apply this to the problem of finding the general term of a sequence, we simply use the indices $0, 1, 2, 3, \ldots, n$ as the x-values and we use either the partial sums

$$0, 1,\; 1 + 2^5 = 33,\; 1 + 2^5 + 3^5 = 276, \ldots$$

of the series (1) or the actual terms of the sequence in (2) as the y-values. Working the first example by hand using the first 7 terms,

x	0	1	2	3	...	6
y	0	1	33	276	...	12201

we obtain as the Lagrange interpolating polynomial

$$P_6(x) = 0\,\frac{(x-1)\,(x-2)\,(x-3)\ldots(x-6)}{(0-1)(0-2)(0-3)\ldots(0-6)} + 1\,\frac{(x-0)\,(x-2)\,(x-3)\ldots(x-6)}{(1-0)\,(1-2)\,(1-3)\ldots(1-6)}$$

$$+ 33\,\frac{(x-0)\,(x-1)\,(x-3)\ldots(x-6)}{(2-0)\,(2-1)\,(2-3)\ldots(2-6)} + \ldots + 12201\,\frac{(x-0)\,(x-1)\,(x-3)\ldots(x-5)}{(6-0)\,(6-1)\,(6-5)\ldots(6-5)} ,$$

which is a polynomial of degree 6.

Using a computer program (written by Sheldon Gordon) which calculates, graphs and displays the formula for the Lagrange interpolating polynomial through any desired set of points (a similar function is available through the MathCad package), we find (within seconds) that the corresponding polynomial is:

$$0.166666\,x^6 + 0.5\,x^5 + 0.416666\,x^4 - 0.083333\,x^2$$

for the partial sums. We first consider how to deal with this expression. To begin, we observe that the lead coefficient 0.166666 is essentially 1/6 and we therefore factor it out of the polynomial to obtain, after some judicious rounding,

$$\frac{x^6 + 3x^5 + 2.5x^4 - 0.5x^2}{6} = \frac{1}{6}x^6 + \frac{1}{2}x^5 + \frac{5}{12}x^4 - \frac{1}{12}x^2 = \frac{x^2(2x^4 + 6x^3 + 5x^2 - 1)}{12}$$

In terms of n, this gives

$$\sum_{k=0}^{n} k^5 = \frac{n^2\,(2n^4 + 6n^3 + 5n^2 - 1)}{12} = \frac{n^2\,(n+1)^2\,(2n^2 + n - 1)}{12},$$

which is the general formula for the sum of the fifth powers. Once such a formula is produced, we can prove it using the standard type of induction argument.

In a similar way, we consider the sequence of numbers in (2) as input to the same program in the form (6,1), (8,8), (10,35), ... and so obtain the polynomial

$$0.000043403\,x^6 - 0.00086806\,x^4 + 0.00277777\,x^2 .$$

We might be tempted to expect that the polynomial has rational coefficients. As such, based on our experience above, we might consider the multiplicative inverse, 23039.88, of the lead coefficient, so that the lead coefficient is essentially 1/23040. Using this, the polynomial factors as

$$\frac{x^6 - 20x^4 + 64x^2}{23040} = \frac{x^2\,(x^2-4)\,(x^2-16)}{23040} \qquad (3)$$

after some judicious rounding to undo the effects of the computer's initial roundings. Thus, in terms of n, we find that the general term for the sequence is

$$\frac{n^2\,(n^2-4)\,(n^2-16)}{23040},\ n = 6, 8, 10, \ldots .$$

It is simple to verify that this is indeed correct for the data values given.

We note that if less than nine decimal accuracy is used for the coefficients in example 2, we might well miss the fact that the lead coefficient is 1/23040. In turn, this would distort the judiciously rounded values for the other coefficients when the lead term was factored out. We can circumvent this potential problem by factoring out the lead decimal coefficient. Then, assuming that

$$p(x) = \left(\frac{A}{B}\right)(x^6 - 20.0000000\,x^4 + 63.99965440\,x^2),$$

we can substitute in any of the points, say (6,1), to determine algebraically the ratio A/B as a fraction.

We note that one characteristic of the Lagrange interpolating polynomial is that if a polynomial of degree less than m fits the given $m+1$ points, then the formula automatically "collapses" to produce that lower degree polynomial. This is precisely what occurred in the second example above where we used nine points which we would expect to lead to an eighth degree polynomial. Thus, in trying to determine a polynomial relationship that fits a set of data values, it might seem that the best strategy is to use *all* the values available. However, that approach could be misleading. That is, if we have seven data points available and use all of them, then the Lagrange method will produce a polynomial of degree six (or less) that passes through all of the points. It is conceivable, though, that the actual pattern might be described by, say, a tenth degree polynomial (or no polynomial at all) and we would have no way of checking if the result is correct.

Consequently, a more intelligent approach might be to use one less than the maximum number of known points, find the corresponding interpolating polynomial, and then check that the remaining point fits it as well. If it does, it provides a good indication that the result is correct. If the last point does not fit, then we can use all of the points, obtain the maximum interpolating polynomial based on the data, and hope that it is correct, pending obtaining additional data values.

To illustrate the potential pitfalls in making too definitive a decision, consider the sequence:

$$0, 0, 0, ..., 0, {}^{10}P_0, {}^{11}P_1, {}^{12}P_2, ...$$

for $n \geq 1$ whose first 10 terms are zero and whose subsequent terms are given by ${}^{m}P_r$, the number of permutations of m objects taken r at a time. Based on the first six or eight terms only, we would likely conclude that the sequence is identically zero. However, if the later terms are also noticed, then the pattern is radically different and, in fact, is given by

$$(n-1)(n-2)\ldots(n-10) \quad \text{for all } n.$$

Using Derive to Determine the Polynomial

Alternatively, it is possible to apply some of the common computer algebra systems to perform some of this work in rational form. To illustrate this, suppose we apply Derive. We first write the Lagrange interpolating polynomial as

$$\sum_{k=1}^{n} y_k \prod_{j=1}^{k-1} \frac{x-v_j}{v_k-v_j} \prod_{j=k+1}^{n} \frac{x-v_j}{v_k-v_j}$$

where $v_k = x_k$ and $y_k = f(x_k)$, for each k. We construct this formula in Derive in several steps using the Author command. First, we must declare V and Y to be vectors of dimension N. Then Author

ELEMENT (Y, K)

which becomes Expression #1. Next, Author

PRODUCT ((X−ELEMENT(V,J))/(ELEMENT(V,K) -ELEMENT (V,J)),J,1,K-1)

which becomes Expression #2. Then Author

PRODUCT ((X-ELEMENT(V,J))/(ELEMENT (V,K)-ELEMENT (V,J)),J,K+1,N)

which becomes Expression #3. Finally, Author

SUM(#1 #2 #3, K, 1, N)

which becomes Expression #4 and is identical to the above formula for the Lagrange interpolating polynomial.

This provides a simple, one-line formula for the interpolating polynomial in Derive. To find the sum of the fifth powers of the integers, we use the Manage-Substitute option to make the following substitutions:

7 FOR N
[0,1,2,3,4,5,6] FOR V
[0,1,33,276,1300,4425,12201] FOR Y

Upon using the Simplify command, we get precisely the expression obtained previously for the general term of the sum in exact form.

Advantages of Using Interpolating Polynomials

Finally, we note that if we assume the general term of a sequence is a polynomial of degree n in x

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n ,$$

we can substitute pairs of data values into the polynomial to obtain a set of $n+1$ equations in the $n+1$ unknowns to determine the coefficients. The resulting system can be solved using standard linear algebra methods, such as Gaussian elimination, using appropriate software, preferably a package which performs the operations using rational arithmetic to avoid the rounding problems inherent in floating point arithmetic.

Despite the wider availability of linear algebra software packages, the authors' preference is the interpolation approach. It involves considerably less typing. (Compare the need to input a 7 × 8 matrix instead of just seven pairs of points.) Second, it is faster, particularly if we want to experiment with different sets of points. Last, and perhaps most important, it seems less sensitive to successive rounding errors caused by the large range of values used by the data in such a problem when using programs based on floating point arithmetic. In fact, this is pointed out in many texts on numerical analysis. See, for example, Gerald and Wheatley (1988, p. 183).

Interested readers should contact Sheldon Gordon for information on the availability of the Lagrange Interpolating Polynomial program used above.

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The Floating Leaf Problem

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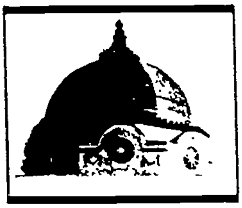

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Hasty answers to deceptively simple questions often prove in error. As is evident in classical problems, wrong results can hinge on strategies of doubling or averaging. Well known from the pages of history is the Delian Problem of antiquity. This long-ago attempt at doubling the volume of a cube failed dramatically and tragically when each of the cube's edges was doubled. Nor did the 20 mile per hour trip from Athens to Delos and the 30 mile per hour return on the same Grecian road yield an average speed of 25 miles per hour for the entire journey (as the unsuspecting traveler might have thought)!

Problems of such a kind, namely, those suggesting a careless doubling or averaging disposition, often conform on second thought to a good analytical procedure that leads to correct answers. One needs to look no farther than the farmer's peaceful pond in springtime to conjure up another puzzling predicament of doubling and averaging. It begins with a water plant of a single leaf. All generated leaves in this initial version are assumed to be of the same size when full grown.

Problem 1. Suppose that the number of full-grown leaves on a water plant doubles daily so that an entire pond is covered, without overlapping, in ten days. If two plants are used, in how many days will the pond be covered?

Is the answer five days? This is the impulsive response. Careful thought, however, reveals that the correct number of days is nine. By the ninth day, each of the two original plants will have given rise to a half covering of the pond. Thus, on the ninth day, the entire pond is covered. Beyond the ninth day, the nonoverlapping feature is lost if the generated leaves are restricted to the surface of the pond.

What About Three Plants?

Now suppose that three plants are used. How many days would then be required? The mere manipulation of a few whole numbers no longer suffices.

Let us return to the two plant problem for a moment. At the end of x days, a single plant results in 2^x leaves. [In the beginning (when $x = 0$), there is but one leaf.] It is assumed that this growth pattern prevails for all real number values of x in the appropriate time interval. Hence, two plants result in $2(2^x)$ leaves. Moreover, 2^{10} leaves cover the pond. Very simply,

$$2(2^x) = 2^{10} \Rightarrow 2^{x+1} = 2^{10} \Rightarrow x + 1 = 10 \Rightarrow x = 9.$$

Using this strategy, the three plant problem will fall nicely in place. Accordingly,

$$3(2^x) = 2^{10}.$$

As is quickly seen, whole numbers no longer suffice. More precisely,

$$2^x = \frac{2^{10}}{3}$$

$$x \log 2 = \log 2^{10} - \log 3$$

$$x = \frac{10\ (\log\ 2) - \log\ 3}{\log\ 2}.$$

Decimally, x is roughly 8.41 days. It is obvious that the answer lies between 8 and 9 if we consider that two plants cover the pond in nine days and four plants cover the pond in eight days.

A Generalized Look

The generalization to n original plants is fairly straightforward. In such a case,

$$n(2^x) = 2^{10}$$

$$2^x = \frac{2^{10}}{n}$$

$$x \log 2 = \log 2^{10} - \log n$$

$$x = \frac{10 \log 2 - \log n}{\log 2}.$$

Note also that $\log 2^{10}$ must exceed $\log n$ if a positive number of days is required. As $\log 2^{10} > \log n$, then $n < 2^{10}$. Common sense also dictates this restriction on n.

Leaves of Different Sizes

The plants above produced leaves of a same full-grown size. How might the analysis unfold in the event that the doubling pattern continues and all leaves produced by a plant are of the same full-grown size but leaves of different plants have different areas?

Problem 2. Suppose one plant's leaves can cover the pond in ten days and the other in twenty days. Working together, in how many days will the pond be covered?

At this point, there is likely some skepticism of an answer based on averaging; that is, 15 days just can't be right. Moreover, if the answer is x days, neither does the following equation work:

$$\frac{x}{10} + \frac{x}{20} = 1.$$

This latter approach assumes that one day's contribution to covering is the same as that for any other day. Caution must also be exercised in comparing leaf sizes. It does not follow that a large leaf is twice the size of a small leaf. Instead,

$$2^{10} \text{ large leaves} = 2^{20} \text{ small leaves, or}$$

$$1 \text{ large leaf} = \frac{2^{20}}{2^{10}} = 2^{10} = 1024 \text{ small leaves.}$$

In x days, the two plants together cover the pond. Hence,

$$2^x \text{ large leaves} + 2^x \text{ small leaves} = 2^{20} \text{ small leaves}$$

$$2^{10}(2^x) \text{ small leaves} + 2^x \text{ small leaves} = 2^{20} \text{ small leaves}$$

Therefore, we have

$$2^x(2^{10} + 1) = 2^{20}$$

$$2^x = \frac{2^{20}}{2^{10} + 1}$$

$$x = \frac{\log 2^{20} - \log (2^{10} + 1)}{\log 2}.$$

Roughly, x becomes 9.999 days. Interestingly, the plant with the small leaves makes a minor, virtually negligible, contribution to the covering of the pond. Note again that the plant with large leaves could provide a covering in only ten days.

The generalization follows the same basic strategy as above. If one plant can cover the pond in a days and another plant in b days (where a is less than b), then

$$2^a \text{ large leaves} = 2^b \text{ small leaves}$$

$$1 \text{ large leaf} = 2^{b-a} \text{ small leaves}$$

$$2^x \text{ large leaves} + 2^x \text{ small leaves} = 2^b \text{ small leaves}$$

$$2^{b-a}(2^x) + 2^x = 2^b$$

$$2^x(2^{b-a} + 1) = 2^b$$

$$x \log 2 = \log 2^b - \log (2^{b-a} + 1)$$

$$x = \frac{\log 2^b - \log (2^{b-a} + 1)}{\log 2}.$$

If $b = 2a$, as happened above, then

$$x = \frac{2 \log 2^a - \log (2^a + 1)}{\log 2}.$$

This is approximately $\frac{\log 2^a}{\log 2}$ or a. As before, the larger leaf plant provides virtually all of the covering.

Conclusion

There are other variations on this problem of the floating leaves. Perhaps the doubling process could be changed to one of tripling, or, generally, to one of multiplication by a number k. A greater number of plants than just two but having variable leaf sizes comes to mind. Likewise, there are different starting time possibilities. And more, much more! Such are the many facets of the problem. All are characterized by a subtlety or complexity which misses the eye at first glance.

A mathematical aside concerns the shape of the leaves, be they of fixed or varying sizes. It is theoretically possible to find shapes and sizes fitting the number choices of this overall problem so as to cover select ponds without overlapping. Whether or not Mother Nature will distribute and properly arrange these leaves is of course another matter!

Approximations to the Hypergeometric Distribution

by

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When sampling from a finite population is done without replacement, the resulting discrete probability distribution is known as the hypergeometric distribution. Applications for the hypergeometric distribution can be found in quality control (e.g., acceptance sampling), ecology (e.g., capture-recapture method), and games (e.g., bridge). The paper by Henry, Smith and Trapp (1992) presents applications involving card games. In this paper we describe two methods that can be used to approximate hypergeometric probabilities: the binomial and normal approximation. They are appropriate for classroom enrichment, when the hypergeometric distribution is taught.

To derive the density function for the hypergeometric distribution, suppose a population consists of a finite number of N elements ($N \geq 1$), and there are M elements of type I ($0 \leq M \leq N$) and the remaining $N-M$ elements are of type II. A random sample of size n ($1 \leq n \leq N$) is selected without replacement. Let X denote the number of elements in the sample that are of type I. Then $X \geq 0$ and $X \geq n-N+M$ [since the number of type II elements in the sample $(n-X)$, cannot exceed the number of type II elements in the population $(N-M)$]; thus $X \geq \max(0, n-N+M)$. Since $X \leq n$ and $X \leq M$, $X \leq \min(n,M)$. The distribution of the random variable X is known as *hypergeometric distribution*, and its probabilities are given by

$$P(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} \tag{1}$$

where k is an integer subject to the restrictions $\max(0, n-N+M) \leq k \leq \min(n,M)$ (see Bain & Engelhardt, 1992).

Using the definition of $\binom{n}{r}$, a recurrence relation for hypergeometric probabilities can be obtained. Since

$$(n-r+1)! = (n-r+1)(n-r)! \quad \text{and} \quad r! = r(r-1)!\,,$$

we have $\binom{n}{r} = \dfrac{n!}{r!(n-r)!} = \dfrac{n!(n-r+1)}{r(r-1)!(n-r+1)!} = \dfrac{n-r+1}{r}\dbinom{n}{r-1}$. Thus,

$$\binom{M}{k} = \frac{M-k+1}{k}\binom{M}{k-1} \quad \text{and}$$

$$\binom{N-M}{n-k+1} = \frac{N-M-(n-k+1)+1}{n-k+1}\binom{N-M}{n-k} = \frac{N-M-n+k}{n-k+1}\binom{N-M}{n-k}\,.$$

Substituting for $\binom{M}{k}$ and $\binom{N-M}{n-k}$ in (1), we have

$$\mathrm{P}(X=k) = \frac{\dfrac{M-k+1}{k}\dbinom{M}{k-1}\dfrac{n-k+1}{N-M-n+k}\dbinom{N-M}{n-(k-1)}}{\dbinom{N}{n}}$$

which is equivalent to

$$\mathrm{P}(X{=}k) = \frac{(M-k+1)\,(n-k+1)}{k(N-M-n+k)}\,P(X{=}k-1) \quad \text{for } k \geq 1. \qquad (2)$$

As shown in Example 1 below, recursion formula (2) offers an efficient method for calculating hypergeometric probabilities. However, it should also be noted that with the availability of the nCr key on most calculators, the required probabilities can be calculated directly by using (1).

Example 1. The set of letters in the game of Scrabble consists of 56 consonants and 42 vowels. What is the probability that your initial draw of the seven letters will contain at least three consonants [assume that the two blanks are removed before the draws]?

Solution. Let X denote the number of consonants in your initial draw of the seven letters. Then X has a hypergeometric distribution with $N = 98, M = 56, n = 7$, and

$$\mathrm{P}(X=k) = \frac{\dbinom{56}{k}\dbinom{42}{7-k}}{\dbinom{98}{7}}, \quad 0 \leq k \leq 7 \qquad (3)$$

$$\mathrm{P}(\text{at least 3 consonants}) = \mathrm{P}\,(X \geq 3) = \sum_{k=3}^{7} P(X=k)\,.$$

From (3), we have

$$P(X=3) = \frac{\binom{56}{3}\binom{42}{4}}{\binom{98}{7}} = .2243.$$

Using (2), we have the following:

$$P(X=4) = \frac{(56-4+1)(7-4+1)}{4(98-56-7+4)}P(X=3) = \frac{(53)(4)}{4(39)}.2243 = .3048$$

$$P(X=5) = \frac{(56-5+1)(7-5+1)}{5(98-56-7+5)}P(X=4) = \frac{(52)(3)}{5(40)}.3048 = .2377$$

$$P(X=6) = \frac{(56-6+1)(7-6+1)}{6(98-56-7+6)}P(X=5) = \frac{(51)(2)}{6(41)}.2377 = .0986$$

$$P(X=7) = \frac{(56-7+1)(7-7+1)}{7(98-56-7+7)}P(X=6) = \frac{(50)(1)}{7(42)}.0986 = .0168$$

Therefore, $P(X \geq 3) = .2243 + .3048 + .2377 + .0986 + .0168 = .8822$.

Another way to obtain the required probability is to use the fact that $P(X \geq 3) = 1-P(X \leq 2)$, find $P(X = 0)$ using (2), and then use (3) to find $P(X = 1)$ and $P(X = 2)$.

In both instances the individual probabilities can be obtained directly. The aim of the example is to illustrate the ease of use of the recurrence relation (3).

However, if N is large we may not be able to use a calculator to evaluate the combinations in (1). For example, some calculators will not compute nCr for $n = 150$ and $r = 50$. In these cases we need to approximate the hypergeometric distribution.

The Binomial Approximation for the Hypergeometric Distribution

Under certain conditions, hypergeometric probabilities can be approximated by binomial probabilities. The mathematical basis for the binomial approximation is that as M and N become infinite with $p = M/N$, a fixed positive constant, hypergeometric probabilities converge to binomial probabilities. Thus, for each possible value k in $\{0,1,2,..., n\}$,

$$\lim_{N\to\infty} \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} = \binom{n}{k} p^k(1-p)^{n-k}, \tag{4}$$

where $p = M/N$ is a positive constant.

This approximation is intuitively reasonable since the binomial distribution is applicable when the sample is chosen with replacement, while the hypergeometric is applicable when the sample is chosen without replacement. If the population is large in comparison to the sample size, then the probability of selecting a type I item is not changing much from trial to trial and for all practical purposes it can be viewed as being constant.

The above result provides an approximation for the hypergeometric distribution when the number of items selected n, is small in relation to the population size N. A rule of thumb is that the approximation is satisfactory if $n/N \leq .05$.

Example 2. A manufacturing plant receives a shipment of 200 engines to be used in the production of lawn mowers. The lot must be checked to ensure the quality of the engines. The lot will be rejected if it appears that more than 5% of the engines are defective. The quality control inspector selects five engines at random for testing. Assuming that the lot contains ten defectives, compare the hypergeometric and the approximate binomial probabilities for obtaining exactly two defectives in the sample.

Solution. Let X denote the number of defectives in the sample. Then X has a hypergeometric distribution with $N = 200, M = 10, n = 5$. Since $\frac{n}{N} = \frac{5}{200} = .025 < .05$ binomial approximation to the hypergeometric distribution is appropriate with $p = \frac{M}{N} = \frac{10}{200} = .05$

The exact hypergeometric probability: $P(X = 2) = \frac{\binom{10}{2}\binom{190}{3}}{\binom{200}{5}} = .0200.$

The approximate binomial probability: $P(X = 2) = \binom{5}{2} (.05)^2(.95)^3 = .0214$

How good is the binomial approximation? Generally, the larger the population size N is in relation to the sample size n, the better the approximation. Table 1 shows the exact hypergeometric probabilities and the approximate binomial probabilities for three hypergeometric distributions with (a) $N = 100$, $M = 5$, $n = 5$ (b) $N = 200$, $M = 10, n = 5$ and (c) $N = 500, M = 25, n = 5$. In each case $\frac{n}{N} \leq .05$ and $p = .05$.

	Exact Hypergeometric Probability			Approximate Binomial Probability
X	(a)	(b)	(c)	
0	.7696	.7717	.7730	.7734
1	.2114	.2075	.2052	.2036
2	.0184	.0200	.0209	.0214
3	.0006	.0008	.0010	.0011
4	6.3091×10^{-6}	1.5736×10^{-5}	2.3541×10^{-5}	2.9688×10^{-5}
5	1.3282×10^{-8}	9.9382×10^{-8}	2.0185×10^{-7}	3.125×10^{-7}

Table 1. Comparison of Hypergeometric and Binomial Probabilities.

Notice that the binomial probabilities are considerably closer to the exact probabilities as N becomes large.

The expected value and the variance for the hypergeometric distribution are

$$E(X) = np, \text{ and } \operatorname{Var}(X) = np(1-p)\frac{N-n}{N-1}, \text{ where } p = \frac{M}{N}.$$

The expected value is identical to that for the binomial distribution, and the variance differs from that of the binomial distribution by the factor $\frac{N-n}{N-1}$. The quantity $\frac{N-n}{N-1}$ is called the *finite population correction factor*. If $\frac{N-n}{N-1}$ is close to 1 then the approximation is fairly good. For the three hypergeometric distributions considered in Table 1, $\frac{N-n}{N-1}$ take values .956, .980 and .992 respectively.

Normal Approximation to the Hypergeometric Distribution

When N is large in relation to the sample size n, but n itself is large, the binomial approximation to hypergeometric probabilities may be awkward to calculate. In such cases, the normal distribution can be used to approximate the probabilities.

If the random variable X has a hypergeometric distribution, then the distribution of the random variable

$$Z = \frac{X - np}{\sqrt{np(1-p)\frac{N-n}{N-1}}},$$

where $p = \frac{M}{N}$, is approximately standard normal for large N. Thus,

$$P(X \le k) = \Phi\left(\frac{k + .5 - np}{\sqrt{np(1-p)\frac{N-n}{N-1}}}\right),$$

where Φ denotes the standard normal probability distribution function. The normal approximation is appropriate whenever $np \ge 5$ and $n(1-p) \ge 5$.

Example 3. In a certain city of 100,000 voters 40,000 are Democrats. Five hundred voters are selected at random without replacement. What is the probability that the sample contains at least 220 Democrats?

Solution. Let X denote the number of Democrats in the sample. Then X has a hypergeometric distribution with $N = 100{,}000$, $M = 40{,}000$ and $n = 500$.

Also, $np = 500(.4) = 200$ and $n(1-p) = 500(.6) = 300$, hence the distribution of

$$Z = \frac{X - 500(.4)}{\sqrt{500(.4)(.6)\frac{100{,}000-500}{100{,}000-1}}} = \frac{X-200}{10.927}$$

4.)

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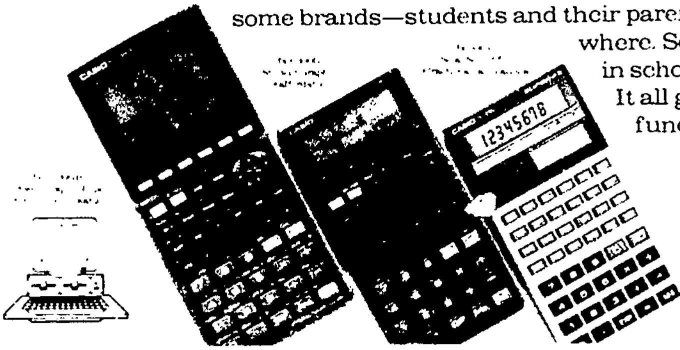

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is approximately standard normal. Thus,

$$P(X \geq 220) = 1 - P(X \leq 219) = 1 - \Phi\left(\frac{219.5 - 200}{10.927}\right)$$

$$= 1 - \Phi(1.7846) = 1 - .9628 = .0372.$$

The HP 21S Stat/Math calculator has a built in program to calculate the binomial probabilities, and it took about 5 minutes to compute the cumulative binomial probability $P(X \leq 219) = 0.9620$. Thus, $P(X \geq 220) = 0.0380$. It should be noted that MINITAB will not compute, but EXECUSTAT will compute the above binomial probabilities.

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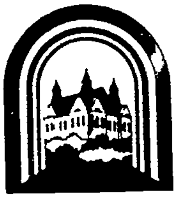
Generating *JE*(3) With Some Elementary Applications

by John J. Edgell, Jr.
Southwest Texas State University, San Marcos, TX 78666



John has long been interested in extensions and applications of the ancient Chinese trigonal array of numbers traditionally associated with Pascal. He recently rode in a fifth Multiple Sclerosis 150 bike tour from Houston to Austin with his oldest son on tandem and he plays tournament handball.

The ideas and applications in this article are an extension of the ideas and applications suggested in the short cover article, "*JE*(3)*M*2," that appeared in the Spring 1992 issue of *The AMATYC Review* (Edgell, 1992). This paper offers an interesting approach on how to introduce such material to students.

Almost every mathematician or student of mathematics has been enthralled with the ancient trigonal[1] array of binomial coefficients traditionally known by Pascal's name. Actually, it is well documented (Burton, 1991) that the array was known as an ancient method to the Chinese. It appeared in a work of Chia Hsien about 1050 A.D. A similar arrangement was known to the Arabs about the same time as recorded by Omar Khayyam. Later the trigonal array appeared in print in a European book on the title page of the *Rechnung* (1527) by Peter Apain (1495-1552). The array is easily generated using symmetric trigonal binary addition, $_a a + {}_b b$, (see Figure 1).

$$\begin{array}{ccccccccccc}
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0+1 & & 1+0 & & 0 & & \\
 & 0 & & 0+1 & & 1+1 & & 1+0 & & 0 & \\
0 & & 0+1 & & 1+2 & & 2+1 & & 1+0 & & 0 \\
 & 0+1 & & 1+3 & & 3+3 & & 3+1 & & 1+0 & \\
 & & & & & \vdots & & & & &
\end{array}$$

Figure 1. The trigonal array in terms of symmetric trigonal binary sums.

The Swiss mathematician Leonhard Euler (1707-1783) (Burton, 1991) later expressed the coefficients in a combinatorial form in 1781, but published in 1784. A combinatorial form, $\begin{smallmatrix} a \\ b \quad c \end{smallmatrix} = \frac{a!}{b!c!}$, not only generates the coefficient on the expansion of $(x^1 + x^0)^{m-1}$, but also keeps track of the exponent $\binom{a}{bc} x^{b(1)+c(0)}$. For example,

[1]Arrays in the given shape are more traditionally called "triangular" or "trilateral." While those labels might be seen as emphasizing angles and sides, respectively, the "trigonal" label should be seen as placing an emphasis upon points (trigonal, quadragonal, pentagonal, hexagonal, ...).

$$(x^1+x^0)^{5-1} = \binom{4}{40}x^{4(1)+0(0)} + \binom{4}{31}x^{3(1)+1(0)} + \binom{4}{22}x^{2(1)+2(0)} + \binom{4}{13}x^{1(1)+3(0)} + \binom{4}{04}x^{0(1)+4(0)} .$$

Quite often the array is enumerated in the combinatorial form, such as the fifth row depicted in Figure 2.

$$\begin{matrix} 4 & 4 & 4 & 4 & 4 \\ 40 & 31 & 22 & 13 & 04 \end{matrix}$$

Figure 2. The fifth row of the array in combinatorial form.

Since the sum of each row of the array is two to a power, dividing each term of a row of the array by the corresponding sum of the row, gives a row of terms whose sum is one. The row can be interpreted as a fair, discrete probability space (Edgell, 1991) (see Figure 3).

$$\frac{1}{2^4} \quad \frac{4}{2^4} \quad \frac{6}{2^4} \quad \frac{4}{2^4} \quad \frac{1}{2^4}$$

Figure 3. The fifth row of the array as a fair, discrete probability space.

One of the properties of the array is that it serves as a transformation from the arithmetic sequence, $A(1, 1) = (1, 2, 3, ..., n, n+1, ...)$ to the geometric sequence $G(1, 2) = (2^0, 2^1, 2^2, ..., 2^{n-1}, (2^{n-1})2, ...)$ (see Figure 4).

Value of terms in sequence	The transformation is the sum of the row entries	Value of terms in sequence
$A(1,1)$	transforms to	$G(1,2)$
1 term	1	is 2^0
2 terms	1 + 1	is 2^1
3 terms	1 + 2 + 1	is 2^2
4 terms	1 + 3 + 3 + 1	is 2^3
5 terms	1 + 4 + 6 + 4 + 1	is 2^4
⋮	⋮	⋮

Figure 4. The array as a transformation from $A(1, 1)$ to $G(1, 2)$.

The idea of developing a trigonal array from trinomial coefficients can be extended to the multinomial expansion of

$$(x^{n-1} + x^{n-2} + ... + x^1 + x^0)^{m-1} . \quad (1)$$

Such an array serves as a transformation from $A(1, n-1)$ to $G(1, n)$. For example, the transformation from $A(1, 2)$ to $G(1, 3)$ is shown in Figure 5. The method for generating the array along with some applications are explained later in the article.

Number of nodes $A(1,2)$	Transformation $JE(3)$	Sum of entries $G(1,3)$
1	1	3^0
3	1 + 1 + 1	is 3^1
5	1 + 2 + 3 + 2 + 1	is 3^2
7	1+3+6+7+6+3+1	is 3^3
⋮	⋮	⋮

Figure 5. $JE(3)$ as a transformation from $A(1, 2)$ to G(1, 3).

Edgell (1984) labeled the array formed by the multinomial expansion of line (1) "$JE(n)$." For example, the binomial expansion (Figure 1) is $JE(2)$ (the integral, J, extension, E, where $n = 2$). The array in Figure 5 is $JE(3)$. Each of these trigonal arrays, $JE(n)$, have some important applications in such areas as arithmetic, algebra, combinatorics, probability, and discrete geometry.

A situation which can be used to introduce students to the idea of generating the transformations from $A(1, n-1)$ to $G(1, n)$, $JE(n)$, is a network problem stated in a spider-prey format. For instance, let $A(1, 1)$ be the nodes of a spider's web (see Figures 6, 7).

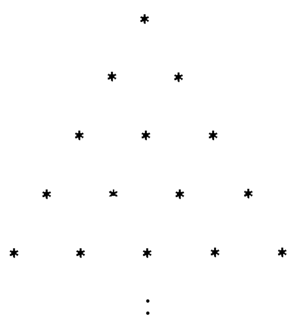


Figure 6. The nodes of the web, $A(1, 1)$

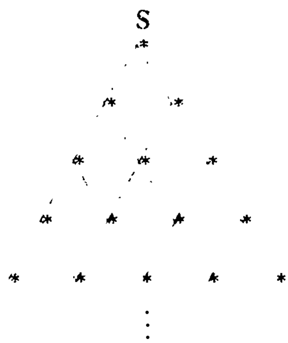


Figure 7. The web with the spider, S.

The movement rule of the problem is that the spider always travels downward, not retracing a path, stays on the web and can change direction only at the nodes. How many different paths can the spider take to get to a prey at any node?

```
                S
                *
                1
             *     *
             1     1
          *     *     *
          1     2     1
       *     *     *     *
       1     3     3     1
    *     *     *     *     *
    1     4     6     4     1
                .
                .
                .
```

Figure 8. Results of the problem.

Extract the numerical parts to get $JE(2)$, the transformation from $A(1, 1)$ to $G(1, 2)$.

The trigonal array of numerals, $JE(3)$, given earlier in Figure 5 can also be generated intuitively by using a similar spider-fly problem. The nodes of the web are laid out so that the arithmetic sequence $A(1, 2)$ is apparent (Figure 9).

To construct the web, connect each node to each of three nodes on the next level of the sequence located symmetrically below (Figure 10). Place the spider at the apex of the array.

```
                  *
            *     *     *
      *     *     *     *     *
*     *     *     *     *     *     *
                  .
                  .
                  .
```

Figure 9. Nodes of the web, $A(1, 2)$

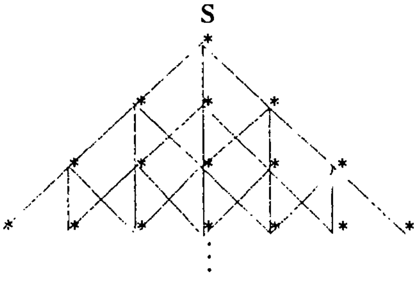


Figure 10. The web connecting each node to each of three nodes symmetrically below.

The number of possible paths to each node is given in Figure 11.

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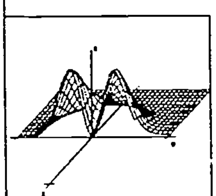

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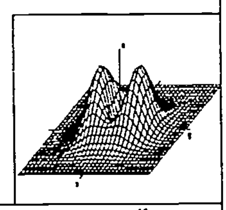

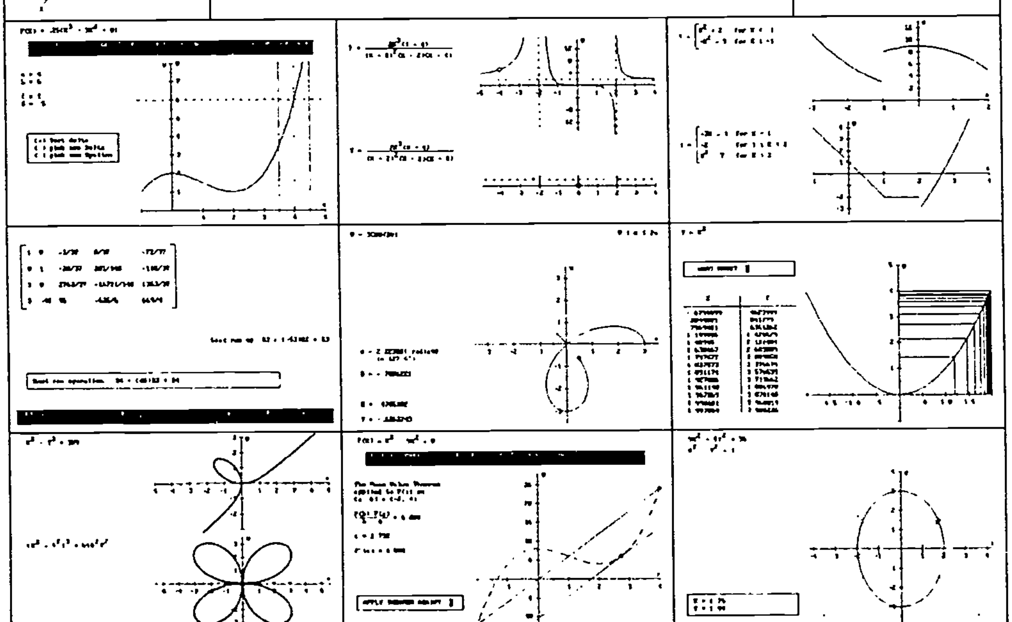

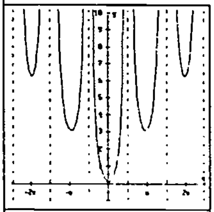
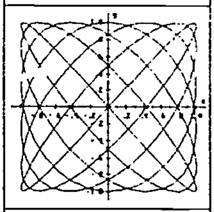
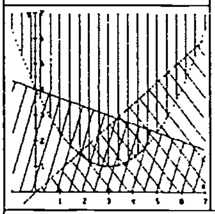
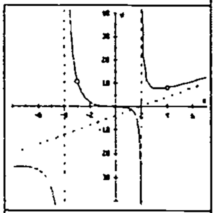
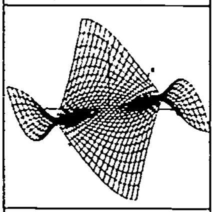
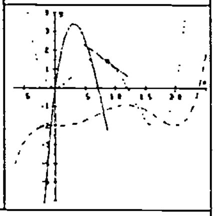

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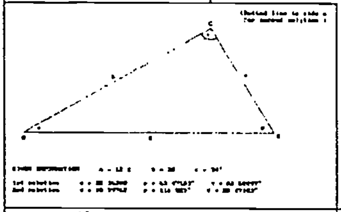
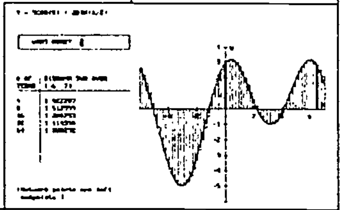
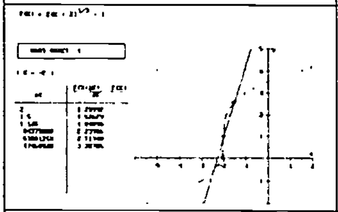
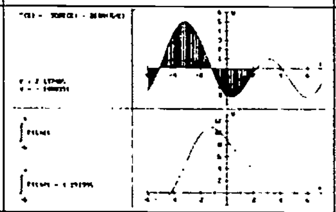
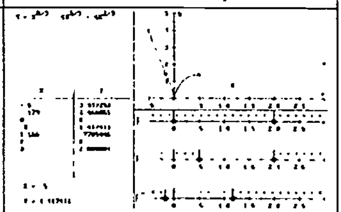
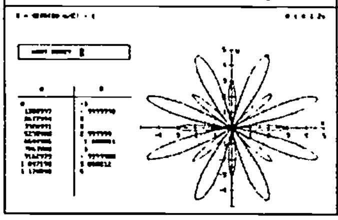
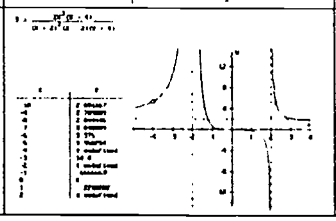
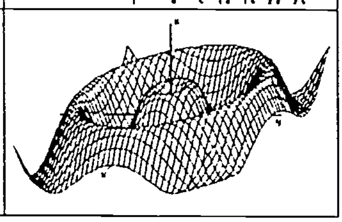

```
            S
            *
            1
      *     *     *
      1     1     1
   *     *     *     *     *
   1     2     3     2     1
*     *     *     *     *     *     *
1     3     6     7     6     3     1
```

Figure 11. The number of paths to each of the nodes.

Extract the numerical part to get $JE(3)$. $JE(3)$ can also be generated by the use of the technique of symmetric trigonal trinary addition,

$$\begin{matrix} a & & b & & c \\ & & a + b + c & & \end{matrix}$$

One of the first activities involving $JE(3)$ might be to show the transformation from $A(1,2)$ to the geometric sequence $G(1,3)$ by taking the sum of the entries of each row, similar to $JE(2)$ (see Figure 5). In addition to the transformation from an arithmetic sequence to a geometric sequence the algebraic applications of $JE(2)$ extend to $JE(3)$. Each of the entries of a row of $JE(3)$ is a coefficient of the expansion of

$$(x^2 + x^1 + x^0)^{m-1},$$

where m is the row number.

$$(x^1 + x^1 + x^0)^0 = 1x^0.$$
$$(x^1 + x^1 + x^0)^1 = 1x^2 + 1x^1 + 1x^0 .$$
$$(x^1 + x^1 + x^0)^2 = 1x^4 + 2x^3 + 3x^2 + 2x^1 + 1x^0 .$$
$$(x^1 + x^1 + x^0)^3 = 1x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x^1 + 1x^0.$$

Similarly, the combinatorics associated with $JE(2)$ extend to $JE(3)$. Let $\begin{smallmatrix} a \\ b\,c\,d \end{smallmatrix}$ represent $\frac{a!}{b!c!d!}$. Similar to the role of combinatorial expressions for entries of $JE(2)$, the combinatorial entries associated with $JE(3)$ (Figure 12) not only determine the coefficients of $(x^2 + x^1 + x^0)^{m-1}$, but also help to account for the exponent of the variable, $\binom{a}{bcd}x^{b(2)+c(1)+d(0)}$. For instance,

$$(x^2+x^1+x^0)^{3-1} = \binom{2}{200}x^{2(2)+0(1)+0(0)} + \binom{2}{110}x^{1(2)+1(1)+0(0)} + \left[\binom{2}{101}x^{1(2)+0(1)+1(0)} + \binom{2}{020}x^{0(2)+2(1)+0(0)}\right] + \binom{2}{011}x^{0(2)+1(1)+1(0)} + \binom{2}{002}x^{0(2)+0(1)+2(0)} .$$

0
000

1 1 1
100 010 001

2 2 2 2 2 2
200 110 101 + 020 011 022

3 3 3 3 3 3 3 3 3 3
300 210 201 + 120 111 + 030 102 + 021 012 003

Figure 12. $JE(3)$ in a combinatorial form.

Also, similar to $JE(2)$, expressed as a sequence of fair, discrete probability spaces, $P(m(JE(2)))$, $JE(3)$ can be expressed as a sequence of fair discrete probability spaces, $P(m(JE(3)))$, by dividing each of the entries of $JE(3)$ by the sum of the row entries for each row.

$$P(m(JE(3)))$$

$$\frac{1}{3^0}$$

$$\frac{1}{3^1} \quad \frac{1}{3^1} \quad \frac{1}{3^1}$$

$$\frac{1}{3^2} \quad \frac{2}{3^2} \quad \frac{3}{3^2} \quad \frac{2}{3^2} \quad \frac{1}{3^2}$$

$$\frac{1}{3^3} \quad \frac{3}{3^3} \quad \frac{6}{3^3} \quad \frac{7}{3^3} \quad \frac{6}{3^3} \quad \frac{3}{3^3} \quad \frac{1}{3^3}$$

$$\vdots$$

Figure 13. $JE(3)$ expressed as a sequence of fair, discrete probability spaces, $P(m(JE(3)))$.

Clearly, $JE(2)$ and $JE(3)$ have many similar applications across the domains of arithmetic, algebra, combinatorics, probability and statistics. But, the patterns of the discrete geometry represented by the cover design $JE(3)\,M2$ (Edgell, 1992) and similar discrete geometry configurations generated by $JE(2)$, $JE(3)$, $JE(4)$,..., $JE(n)$,... may provide insight to new conjectures and applications.

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SHORT COMMUNICATIONS

A General Test for Divisibility by an Odd Integer Not a Multiple of Five

by Matt Ogren (student) and Travis Thompson
Harding University, Searcy, AR 72143

At Harding University, Math 275 (Introduction to Mathematical Thought) was implemented for mathematics majors in order to help bridge the gap between lower level courses and higher level "theory-proof" courses. Travis Thompson, the instructor for this course, decided to sample areas of mathematics that require creativity, but yet, allow the student to get a firm understanding of the material. Elementary number theory was one of the areas chosen for exploration. The theorem that is stated and proved in this paper was an idea conceived by Matt Ogren, one of the students in Math 275. The authors were unable to find this theorem in print and believe it to be original.

Typically, survey courses in mathematics for liberal arts and elementary education majors include divisibility rules in the work on number theory. For example, an integer is divisible by 3 if and only if the sum of the digits of the integer is divisible by 3. The strange rule for divisibility by 7 is sometimes presented and sometimes omitted: A positive integer is divisible by 7 if and only if when 1) the last (right) digit is removed and doubled; and 2) the doubled last digit is then subtracted from the number represented by the remaining digits; 3) then this difference is divisible by 7. For example, consider 179,326. Double the last digit (6) to 12; consider $17{,}932-12 = 17{,}920$. Since 7 divides 17,920, 7 divides the original number of 179,326.

This divisibility theorem for 7 was generalized and proved, in part, by Matt Ogren. It goes without saying that the following test lacks practicality in most situations. But the creative process so vital to mathematical discovery was nurtured and allowed to take root . . . "large trees from little acorns grow."

Theorem. Let Q be an odd positive integer that is not a multiple of 5 and let A be any positive integer. Consider the set

$$K = \{m \mid m \text{ is a positive integer and } Q \text{ divides } (10m+1)\}.$$

Let M be arbitrarily chosen from set K. Then A is divisible by Q if and only if when 1) the last (right) digit is removed and multiplied by M; and 2) the M-multiple of the last digit is then subtracted from the number represented by the remaining digits; 3) then Q divides this difference.

Example. Suppose we wish to test 14,229 for divisibility by 17. Since 17 divides $10(5)+1$, M can be 5. Therefore, since 17 divides $1{,}422-5(9) = 1377$, we know that 17 divides 14,229. Using the same number 14,229, we note that 17 divides $10(22)+1$. Therefore M can also be 22. Since 17 divides $1{,}422-22(9)$, the theorem implies that 17 divides 14,229.

Proof. Let $A = \sum_{k=0}^{n} 10^k a_k$ be given and Q an odd positive integer that is not divisible by 5. Let

$$K = \{m \mid Q \text{ divides } (10m+1), m \text{ a positive integer}\}.$$

Note that K cannot be the empty set since Q is not a multiple of 5. Let M be arbitrarily chosen from the set K. Then, there exists an integer P_1 such that

$$QP_1 = 10M + 1.$$

Now according to hypothesis, suppose

$$Q \text{ divides } 10^{n-1} a_n + 10^{n-2} a_{n-1} + \ldots + 10a_2 + a_1 - Ma_0 .$$

Then there exists an integer P_2, such that

$$QP_2 = 10^{n-1} a_n + 10^{n-2} a_{n-1} + \ldots + 10a_2 + a_1 - Ma_0 ,$$

or

$$10(QP_2 + Ma_0) = 10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10^2 a_2 + 10a_1 .$$

Since $A = \sum_{k=0}^{n} 10^k a_k$, we have $10(QP_2 + Ma_0) + a_0 = A$. It follows that

$$10QP_2 + \underbrace{(10M + 1)}a_0 = 10QP_2 + \underbrace{QP_1}a_0 = Q(10P_2 + P_1 a_0) = A .$$

Therefore, Q divides A.

Conversely, assume that Q divides A. Then, there exists an integer P_3 such that $QP_3 = A$. Let

$$Z = 10^{n-1} a_n + 10^{n-2} a_{n-1} + \ldots + 10a_2 + a_1 - Ma_0 .$$

Therefore,

$$10Z = 10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10^2 a_2 + 10a_1 - 10Ma_0$$

$$= \underbrace{10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10^2 a_2 + 10a_1 + a_0} - a_0 - 10Ma_0$$

$$= A - a_0 - 10Ma_0$$

Therefore,

$$10Z = A - a_0(1 + 10M) = A - a_0(QP_1) = QP_3 - a_0 QP_1 = Q(P_3 - a_0 P_1)$$

and we have that Q divides $10Z$. By hypothesis, Q does not have a factor of 2 and does not have a factor of 5. Since Q and 10 are relatively prime and Q divides $10Z$, we know that Q must divide Z and the theorem is complete.

The Numerical Solution of a Simple Geometric Question

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As mathematics educators now rethink the teaching of calculus, there is cogent argument that a greater role be given to the use of calculation. Indeed, calculus evolved as the practice of exact and approximate calculation and measurement. It is important that students be provided with meaningful examples that can be solved only with numerical methods. We would like to give an example of one such problem, which, although not truly a real-world application, is simply stated and geometrically interesting.

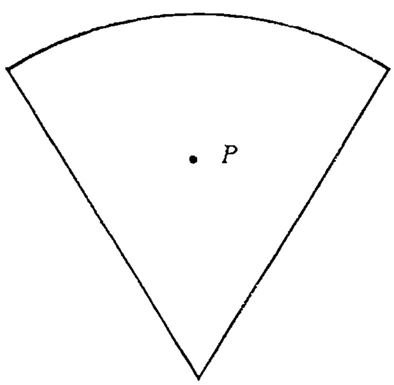

Figure 1

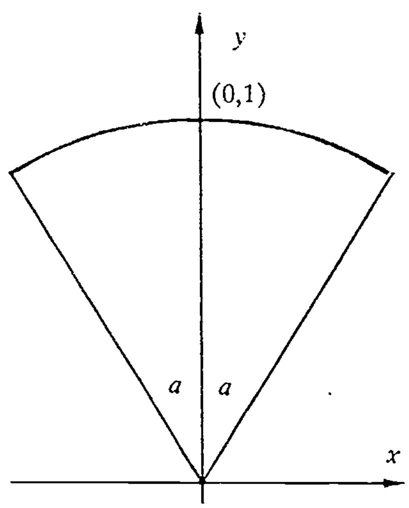

Figure 2

Consider a circular sector having centroid at point P (Figure 1). If the central angle is small, the centroid will be closer to the arc than to the vertex, while, if the central angle is large (for example, π radians), the centroid will be closer to the vertex than to the arc. Continuity suggests that, for some central angle, the centroid will be equidistant from the vertex and the arc. Let us try to determine the measure of this central angle.

In Figure 2, we have designated the radius to be 1, the measure of the central angle to be $2a$ and the angle bisector to be the y-axis. The centroid of the circular sector lies on this angle bisector. The moment M_x about the x-axis through the vertex is given by (see Figure 3)

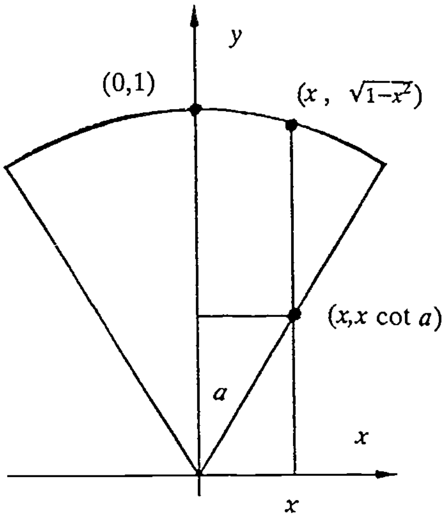

Figure 3

$$M_x = \int_{-\sin a}^{\sin a} \int_{x \cot a}^{\sqrt{1-x^2}} y \; dy \; dx$$

$$= \frac{1}{2} \int_{-\sin a}^{\sin a} \left[(1-x^2) - (x^2 \cot^2 a) \right] \; dx$$

$$= \frac{2}{3} \sin a \; .$$

Since the area of the sector is a, the distance of the centroid above the vertex is

$$\bar{y} = \frac{2 \sin a}{3a} .$$

The value of a for which the centroid is at the midpoint of the angle bisector segment is the solution of the equation

$$\frac{2 \sin a}{3a} = \frac{1}{2},$$

which can be written as

$$a = \frac{4}{3} \sin a \; . \qquad (1)$$

Equation (1) is called a **fixed-point** problem since substituting the solution a into the function on the right-hand side must yield the same value a. A graphing calculator with trace and zoom features can help determine an approximate solution. A more accurate solution can often be found using fixed-point iteration. Specifically, if we wish to solve the equation

$$x = g(x), \qquad (2)$$

where g is continuous, we begin with an initial guess x_0 and define the sequence of *iterates* $\{x_n\}$ by the formula $x_{n+1} = g(x_n)$ for $n = 1, 2, \ldots$. From the graph shown in Figure 4, it is seen that for any initial estimate a_0 in the interval $(0,\pi)$, the iterates converge to the unigue solution of (1). The reader is invited to experiment graphically with other initial estimates in the interval $(0,\pi)$.

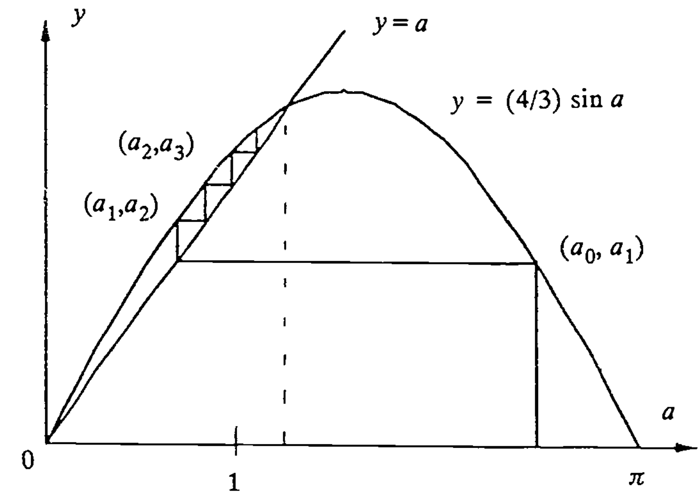

Figure 4

A scientific calculator can greatly ease the computation necessary to determine the iterates. Begin by keying in an initial guess a_0 and then key in [sin x] [×] 4 [÷] 3 [=], yielding a_1. Repeating these five keystrokes will give us a_2, and so on. If we begin with an initial guess of $a_0 = 1$, fixed point iteration applied to equation (1) yields the following:

n	a_n
0	1
1	1.1220
2	1.2013
3	1.2433
⋮	
9	1.2756
10	1.2757
11	1.2757

Thus, the approximate solution is $a = 1.2757$ or about 73 degrees. Therefore, the centroid of a circular sector will be equidistant from the vertex and the arc when the central angle is approximately 146°.

We note that there are other methods to approximate the solution of (1). For example, the Newton-Raphson method, which is somewhat more sophisticated, provides a more rapid rate of convergence.

Patterns Created by the Continued Fraction Representations of Two Sets of Transcendental Numbers

by Michael Schapira
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Decimals is a method of approximating real numbers by power series. When the number being approximated is rational, the decimal expansion will end in a repeating sequence (which also includes repeating zeros), and conversely, the presence of a repeating sequence at the end of a decimal indicates that the number being represented is rational. However, decimal expansions in general provide no clue as to whether a number is algebraic or transcendental. Algebraic numbers are solutions to polynomial equations with rational coefficients; transcendental numbers are real numbers which are not algebraic.

Continued fractions is another method of representing numbers. In a decimal, the digits comprising the representation of the number are coefficients of a power series. In continued fractions there is similarly a sequence of integers. This sequence of integers is used to generate a sequence of fractions which in turn converges to the number being represented. Let us call the sequence of integers "generators" and the sequence of fractions "continued fractions."

It is well known from a theorem by LaGrange that if the generators are periodic (repeating), then the number represented is algebraic (Khinchin, 1964). This paper presents two sets of transcendental numbers whose generators apparently display surprising nonperiodic regularity.

The Construction of Continued Fractions

Let us begin by considering $e^{1/4} = 1.284025\cdots$. First, note that

$$e^{1/4} = 1.284025\cdots = 1 + .284025\cdots .$$

Since the reciprocal of .284025 ⋯ is 3.520811⋯, we can write

$$1.284025\cdots = 1 + \frac{1}{3.520811\cdots} .$$

Similarly,

$$3.520811\cdots = 3 + .520811\cdots ,$$

and since the reciprocal of $.520811\cdots$ is $1.920079\cdots$, we can write

$$3.520811\cdots = 3 + \frac{1}{1.920079\cdots} .$$

So far we have

$$e^{1/4} = 1.284025\cdots = 1 + \cfrac{1}{3 + \cfrac{1}{1.920079\cdots}} .$$

Continuing in a similar fashion

$$e^{1/4} = 1.284025\cdots = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1.086862\cdots}}}$$

and

$$e^{1/4} = 1.284025\cdots = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{11.512495\cdots}}}}$$

In this example the integers 1,3,1,1,11 form the first five terms of the sequence of generators for the continued fraction representation of $e^{1/4}$ (assuming that rounding errors have not yet affected the results).

This process can be generalized as follows: Given an irrational number N, let $r_0 = N$. Define $g_i = [r_i]$ where $[x]$ represents the integer portion of x and define

$$r_{i+1} = \frac{1}{r_i - g_i}, \; i \geq 0 .$$

Relating these definitions to our example, we see that

$$r_0 = N = 1.284025\cdots ,$$

$$g_0 = [r_0] = 1,$$

$$r_1 = \frac{1}{r_0 - g_0} = \frac{1}{.284025\cdots} = 3.520811\cdots ,$$

$$g_1 = [r_1] = 3,$$

$$r_2 = \frac{1}{r_1 - g_1} = \frac{1}{.520811\cdots} = 1.920079\ldots ,$$

and so forth.

Having derived the generators, the next major step is to use them to generate a sequence of fractions $\frac{a_i}{b_i}$ that converge to $e^{\frac{1}{4}}$. To do this, define the ordered pairs

$$(a_{-2}, b_{-2}) = (0, 1) \text{ and } (a_{-1}, b_{-1}) = (1, 0).$$

Then, all subsequent ordered pairs are defined recursively as follows

$$(a_i, b_i) = g_i (a_{i-1}, b_{i-1}) + (a_{i-2}, b_{i-2}),\ i \geq 0,$$

where $h(c, d) = (hc, hd)$ and $(c, d) + (e, f) = (c + e, d + f)$.

Applying this to our example, we obtain the following table.

i	g_i	a_i	b_i	$\frac{a_i}{b_i}$
−2		0	1	
−1		1	0	
0	1	1	1	1
1	3	4	3	1.333333···
2	1	5	4	1.25
3	1	9	7	1.285714···
4	11	104	81	1.283950···

The last column shows how the ratios $\frac{a_i}{b_i}$ apparently converge towards $e^{1/4} = 1.284025\cdots$. Careful examination suggests that the even terms and the odd terms converge monotonically to $N = e^{1/4} = 1.284025\cdots$ from opposite directions.

Two Sets of Transcendental Numbers

Now let us consider the generators for the continued fraction representations of two sets of transcendental numbers. In the tables that follow only those generators that occur prior to roundoff error are presented. Table 1 shows the beginning of the generator sequences for the first of these sets, namely $e^{1/k}$, where $k = 1,\cdots,5$. Examination of the table appears to reveal that in each column from some point on, two out of every three consecutive entries are 1 and that the sequence of third entries forms an arithmetic progression with a common difference of $2k$. In addition, it appears that from some point on in the table, the entries in all the rows also form arithmetic progressions.

Table 2 shows the beginning of the generator sequences for the second of these sets, namely $e^{\frac{1}{k+.5}}$, where $k = 1,\cdots,5$. In this table there is no simple relationship of the terms within the columns. However the i^{th} terms of the expansions form arithmetic progressions for some i's. In these cases it is not obvious whether or not the differences for the various arithmetic progressions are related, nor for which i's we might expect the progressions to occur. The rows, however, clearly form arithmetic progressions.

Table 1. Generators for $e^{1/k}$, $k = 1,\cdots,5$

i	$e^{1/1} = 2.7182\cdots$	$e^{1/2} = 1.64870\cdots$	$e^{1/3} = 1.3956\cdots$	$e^{1/4} = 1.2840\cdots$	$e^{1/5} = 1.2214\cdots$
1	2	1	1	1	1
2	1	1	2	3	4
3	2	1	1	1	1
4	1	1	1	1	1
5	1	5	8	11	14
6	4	1	1	1	1
7	1	1	1	1	1
8	1	9	14	19	24
9	6	1	1	1	1
10	1	1	1	1	1
11	1	13	20	27	34
12	8	1	1	1	
13	1	1	1	1	
14	1	17			
15	10				

Table 2: Generators for $e^{\frac{1}{k+.5}}$, $k = 1,\cdots,5$

i	$e^{\frac{1}{1.5}} = 1.94772\cdots$	$e^{\frac{1}{2.5}} = 1.4918\cdots$	$e^{\frac{1}{3.5}} = 1.3307\cdots$	$e^{\frac{1}{4.5}} = 1.2488\cdots$	$e^{\frac{1}{5.5}} = 1.1993\cdots$
1	1	1	1	1	1
2	1	2	3	4	5
3	18	30	42	54	66
4	7	12	17	22	27
5	1	1	1	1	1
6	1	1	1	1	1
7	10	17	24		
8	53	98			
9	1				
10	11				

Conclusion

It is important to note that the patterns presented in this paper were developed experimentally and the observations are not being stated as general rules. However, the patterns do offer clues as to how one might recognize the decimal representations of powers of e.

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A Case for In-Context Placement Testing

by

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Introduction

Until 1988, placement testing for entry level mathematics courses at both Arizona State University (ASU) and the Prescott, Arizona campus of Embry-Riddle Aeronautical University (ERAU) was mandatory and followed a model which is widely used in post-secondary institutions. A multiple choice test was given to all incoming students to assess their background in mathematics. It was furnished by an outside agency which requested that questions not be released to the public and was not written to correspond directly to any particular college course. This type of placement testing will be referred to as "out-of-context" placement, since it is designed to test prerequisites entirely outside of the course which the student wishes to take. In 1988, the authors independently became dissatisfied with the results of their placement programs and instituted new placement programs which turned out to have essentially the same structure.

In the new placement programs, students are advised to select courses on the basis of prior mathematics coursework at either pre-college or college levels. This advice is intended to place each student in the highest level mathematics course in which he or she should be successful while at the same time advancing the student's academic program. All entry level courses (from algebra through calculus) begin with a review of important prerequisite material. Students are tested over this material at the end of the first few days of class. Students scoring poorly are advised to drop to prerequisite courses during a special drop-back period. This is an "in-context" placement system.

In independent trials with this in-context placement structure at each institution, an improvement in student attitudes about the placement process has been observed. There has been an increase in the number of students electing to start at higher levels with no decline in success rates. In fact, most courses have improved success rates. The relative merits of in-context and out-of-context placement will be discussed in detail in this paper.

Problems Arising From Out-of-Context Placement Testing

Theoretical Problems. Research in statistics and educational psychology has shown that the correlation coefficient between a placement test and a course grade should be .80 or higher if use of the test is to have a significant effect on success rates. When the correlation is lower, use of the test is not likely to result in a significant improvement in the success rate. In fact, an unacceptable number of students may be misadvised by their placement scores. Hassett & Smith (1983) summarized this research and applied it to mathematics placement.

Other studies have also concluded that many placement tests do not reach the desired correlation of .80 with course grade. Noble & Sawyer (1988) state:

> Numerous studies have examined the relationships between admissions and placement test scores and specific course grade.... The mathematics validity studies comprised a larger portion of the research on predicting specific course grades. A variety of predictors were used, including ACT subtests and composite scores; SAT-V, SAT-M, and SAT-Total scores; high school rank; and scores on specifically developed mathematics placement tests. The correlation coefficients ranged from .04 to .75. (p. 3)

The inaccuracy of out-of-context placement for many students is also apparent in a study done at Bucks County Community College in 1986 and 1987. Hoelzle (1987) showed that 47% of the students who placed themselves above the course predicted by the placement test were successful.

In 1987, both ERAU and ASU used the MAA placement tests. The correlations with course grades ranged from .30 to .50. Correlations using other available standard instruments (ACT math, SAT-M) were in the same range or lower. The placement instruments available to us did not correlate well enough with course grades to be used as the main predictors of success.

Practical Problems. Advisors believed that placement test scores were extremely valid. Thus they were hesitant to permit students to register for courses other than the ones indicated by the placement test. This tended to make advising superficial. Since some advisors were unfamiliar with the course content of mathematic courses, they accepted the results of the placement test as correct even when the student protests were supported by sufficient preparatory coursework.

Most of the students who took the out-of-context mathematics placement tests were not enrolled in a mathematics course when they took the test. Moreover, the importance of the test was often not made clear to the students, so that they made no effort to review any material. As a consequence, their mathematics knowledge and skills were at an ebb and they tended to score lower than they would have scored during the school year.

The combination of advisor ignorance and student rustiness forced many students to register for courses they had already passed elsewhere. When those courses were remedial, the students were delayed a semester or more in starting required courses. They also adopted poor work habits that usually led to low or failing course grades. Instructors in remedial classes complained of poor attendance and poor attitudes by students who said they already knew the course material.

Many students became hostile. They had good reason. They had passed the course at another institution, but were being required to repeat it. The placement test was perceived as a barrier to registration, not an aid. A great amount of office staff time was spent with students complaining about the system and filling out petitions to override placement scores.

The placement test was acting as a filter instead of a pump in the mathematics pipeline. The university system was placing students who had taken four or more years of college preparatory mathematics in high school into a precalculus curriculum. Consequently, many competent high school students were being turned into mediocre college students in one short semester. Furthermore, the placement process was not meeting its goal of improving success rates in the targeted classes.

There was also an adverse financial effect. University funds that could have been used on instructional assistance were being spent on the mandated placement testing program. Of course, out-of-context testing does make an advisor's job easier. Advisors can process a large number of students in a relatively short amount of time because they have a template to follow. Such "placement procedures tend to be used for their practical advantages, such as their greater ease of operation, or their greater ease of explanation to staff and students" (Noble & Sawyer, 1988, p. 2).

Advantages of In-Context Placement Testing

In 1988, both ASU and ERAU eliminated compulsory use of out-of-context placement testing and began to use in-context testing. Data analysis of the effects of their change in approach will be discussed in the next section, although additional analysis remains to be done. Analysis not withstanding, the practical advantages of this new system were immediately obvious.

Advising Became More Thorough. At Arizona State advisors received new materials on how to look at a students' record in mathematics. A self-advisement flow chart was created that students and advisors could use to guide the choice of a mathematics course based upon a student's background and past performance in mathematics courses. Seminars for advisors stressed the importance of focusing on each student's mathematics preparation rather than on a single placement test score. Without a trivial algorithm advisors were forced to worry about the quality of their advice. The University Testing Center still provided the MAA placement tests as an optional support vehicle and campus computer sites made available the self-advising computer program *Are You Ready for Calculus*? This is a program developed by Dr. David Lovelock of the University of Arizona.[1]

At Embry-Riddle, advisors were given more training in mathematics placement. The time usually spent during student orientation for placement testing was used for mathematics orientation. This was directed by a faculty member from the mathematics department. Students were allowed to ask questions and they were given outlines for the various courses open to them. They were encouraged to sign up for the highest level course for which they felt they had a chance to be successful. Accordingly, students who had done well in a year of high school calculus, but who

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[1]This software, along with similar "readiness" software is in the public domain. It can be obtained for nominal charge to cover costs of the computer disk, handling and mailing. Address requests for information to Mathematics Department, University of Arizona, Tucson, AZ 85721.

had not taken the AP Calculus Examination, were allowed to register for second-semester calculus.

On Both Campuses Student Hostility Disappeared. Teachers and office staff no longer have to contend with hostile students who have covered the material before. Students are no longer being told that they cannot enroll in a mathematics course because they have "failed" the placement test. Instead they are told that the Mathematics Department will help them decide if the course they have enrolled in appears to be too difficult, but will leave the actual decision up to each student.

At ASU students are given two weeks to make their decision. The first week's assignments are based on a review of essential prerequisite material, and a test on this material is given on Thursday or Friday. Students who score below 70% are advised to drop to a lower level course. This can be accomplished through the end of the second week of the semester by special arrangement with the registration office.

At ERAU students have the first three weeks to decide if they wish to transfer. This is an extension beyond the normal drop/add period, but because results have shown significant positive benefits the administration has allowed this arrangement to continue.

Enrollment in Remedial Courses Decreased at Both Schools. In general, students enrolling in remedial courses do so because of a rational self-assessment of their needs. The experience of the course instructors through several semesters bears out the validity of this statement. The teachers of these courses have indicated that attendance has improved and that student attitudes are better. As was hoped, success rates in introductory classes have not deteriorated. With the absence of compulsory placement testing students elect to start out in a higher level course, and they do so with less risk of failure.

Results of Preliminary Data Analysis

The goal of in-context placement is to put each student into the course most likely to lead to success. As the new system began to operate, trends in enrollment and success rates were observed and data were collected. This section contains analysis in three areas: predictive validity, enrollment trends, and success rates.

Predictive Validity. An earlier study (1983) at ASU had indicated a correlation of .80 between course grades and grades on the first week review test in College Algebra. Similarly, a correlation of .85 between course grades and grades on the first week review test in Calculus I was observed at ERAU in 1987. These correlations fell within the desired range and were used as justification for using the first test as a placement criterion. No revalidation was planned under the current structure since the policy of encouraging students with low scores to leave the course makes it impossible to see how they would do if they stayed.

However, one new instructor at ASU, teaching 490 students in three large sections, failed to inform students with low grades that they could drop to a prerequisite course. The grade results from these sections provide data which allow us to study the effectiveness of the review test as a predictor. This data indicate that the first test was quite useful as a placement advisory tool:

Table 1

Intermediate Algebra: Fall 1989
Course Grade vs Score on First Week Test in Numbers of Students at ASU

		Course Grade		
		ABC	DEW	Totals
1st	ABC	180	108	288
WEEK	DE	32	170	202
	Totals	212	278	490

a) It is clear that students with low first scores were at risk. Of the students who were unsuccessful, i.e., D or E, on the first test, 84% (170/202) failed to achieve a satisfactory course grade. In contrast, 63% (180/288) of those students who received a satisfactory grade on the first test passed the course with an A, B or C.

b) By using the first test score as a predictor, it is possible to identify an at-risk group whose actual success rate in the course was only 16% (32/202). If this entire group, i.e., 202 students, were excluded, the success rate would be raised from 43% (212/490) to 63% (180/288).

c) If the first test had been used for placement, only 15% (32/212) of the students who passed the course would have been excluded while 61% (170/278) of those who eventually failed the course would have been excluded. The key objective of a placement test is to identify students who are at risk of not succeeding in the course. The major error would have been the exclusion of students who in fact could satisfactorily pass the course. In identifying 61% of the students destined to fail and only 15% of the students appearing to be at risk but who actually achieved success in the course, the value of this in-context procedure is apparent.

Enrollment Trends and Success Rates. Both ASU and ERAU have observed decreased enrollment in remedial courses and increased enrollment in standard required courses. In looking at the engineering and computer science freshmen at ERAU over a four year period, we note that enrollment in Calculus I has continued to rise. These students take Calculus I as their first mathematics course that counts toward the degree. In 1987, 51% of these students registered for College Algebra based on the results of an out-of-context placement test. In 1988, the in-context test was first used as a mandatory placement vehicle and all students who failed this test were automatically dropped back to College Algebra (Jenkins, 1989). In 1989 and 1990 the drop-back after the first test became advisory rather than mandatory. Table 2 shows the placement and success rates for these four years.

Table 2

Placement and Success Rates for Freshmen
Engineering and computer Science Students at ERAU

	College Algebra	Calculus I	Calculus II
	Place (ABC)	Place (ABC)	Place (ABC)
1987	51% (88%)	42% (60%)	7% (75%)
1988	27% (65%)	69% (67%)	4% (67%)
1989	17% (52%)	79% (77%)	4% (86%)
1990	14% (71%)	80% (60%)	6% (75%)

Has the change in placement policies favorably affected the success rates at ASU and ERAU? It certainly has not lowered success rates at either school. At Embry-Riddle the success rate, i.e., A, B, or C, in Calculus I fluctuated somewhat, but the enrollment increased dramatically. What is significantly more important is the rise in the percent of students who achieved success in their first required math course, i.e., Calculus I, in their degree program during their first semester in school. This can be seen from Table 2. Consider all of the freshmen in engineering or computer science at ERAU to be the "calculus pool." In 1987, 30% [(.42)(.60) + (.07)(.75)] of this pool was successful in a calculus course in the first semester. The remaining 70% of the pool either took College Algebra or received a D, E or W in calculus. In 1988, the first year of in-context placement, 49% of the calculus pool students were successful in calculus in their first semester. In 1989 the success rate took another jump to 64% before tumbling back to 53% in 1990. Even this last figure represents a remarkable improvement over the 1987 rate of 30%.

Similarly at Arizona State University the percent of College Algebra students earning A, B, or C has increased from 50% to 65% since the change in placement testing. The A, B, C rate in beginning calculus remained at the same level of 48% until the Fall '90 semester, when it rose to 57%. While there have been changes in personnel, and, consequently, instruction, in these math courses over the time in question, the dire predictions by some faculty of large scale failures without out-of-context placement testing have not occurred.

It seems clear that in-context placement pumps students into the major in accordance with their original career choices, whereas the mandatory out-of-context placement filters them out, lowers self confidence, and causes unnecessary changes in their career choices. A memorable, and tragic, example of a freshman student at ERAU illustrates this point. In fact, it was primarily this case history that provided impetus to the author to instigate an in-context placement program.

The student had had five years of college preparatory mathematics in high school. However, she did poorly on the mandatory out-of-context placement test and was forced to take College Algebra. Part of the reason for her low score could have been that some of the questions on the placement test covered material she had not seen in over two years. Realizing that she knew the mathematics of College Algebra, bored and lacking challenge, she rarely studied and skipped several classes. Consequently she earned a course grade of B, instead of the A she should have had. The next semester the same pattern with the same result was repeated in Calculus I. The following fall semester she did poorly in Calculus II. She had finally run into some new material and had forgotten how to study mathematics. What had the system done? Through the use of an inept placement process it had taken a highly competent high school student and turned her into a mediocre college student.

Summary

An effective placement system is important to helping students and faculty achieve educational goals. Our experience at ASU and at ERAU has led us to believe that reliance on a single out-of-context placement examination is not conducive to achieving the desired degree of excellence, and may even become a harmful filter that keeps students out of the course most beneficial to them. By contrast, in-context testing shows that more students with prior success in mathematics prove to be well prepared if they are given some course time for serious review. We believe that an in-context system involving continuing positive interaction between students and faculty in the classroom is the more effective placement process.

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REGULAR FEATURES

Educational Reflections

Cooperative Learning

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Currently, there is increased attention to cooperative learning as a methodology, structure, or teaching technique in the mathematics classroom. This paper presents three perspectives on cooperative learning. The three parts of this paper are: (a) a clarification of the meaning of cooperative learning, (b) some views of cooperative learning by professional groups and organizations in mathematics education and individuals, and (c) a summary of some of the research findings regarding cooperative learning.

The Meaning of Cooperative Learning

What is cooperative learning? The term cooperative learning probably triggers the idea "group" in the minds of most educators. This is an accurate, but rather superficial idea. Cooperative learning is more complex than simply getting students to work in small groups. The key word is "cooperative" or "cooperation." Cooperative learning does involve groups of students. They approach problems, assignments, or projects as a team. The groups may vary in size. Slavin and Karweit (1981) define cooperative learning as "the instructional strategies in which students work in small cooperative groups or teams to master academic materials and are rewarded for doing well as a group" (p. 30). The primary idea is working and being rewarded as a group, not as a group of individuals.

Artzt and Newman (1990) point out that cooperative learning is more than the mere grouping of students. They state that cooperative learning is not: (a) sitting in a small group to work problems; (b) sitting in a small group, but working problems individually; or (c) sitting in small groups while only one or two students do the work. Cooperative learning should involve all students, should increase interaction among students, and should result in a team effort or collaboration on the particular task assigned. There are many structures, reasons, and situations appropriate for cooperative learning in the mathematics classroom. Some of these are voiced by individuals or professional groups and organizations.

Views of Individuals and Professional Groups and Organizations

Why should we attempt to incorporate cooperative learning into the mathematics classroom? Individuals and major professional groups and organizations in mathematics education include cooperative learning as one strategy to promote the much needed changes called for in the teaching and learning of mathematics.

The National Council of Teachers of Mathematics (NCTM), in its *Curriculum and Evaluation Standards for School Mathematics* (Commission on Standards for School Mathematics, 1989), states that an "active view of the learning process must be reflected in the way much of mathematics is taught" (p. 10) and includes in a list of instructional strategies to meet this goal, "group and individual assignments...[and] discussion between teacher and students and among students" (p. 10). In other words, the learning of mathematics is an active process and cooperative learning is an instructional strategy that promotes that process. NCTM (Commission on Teaching Standards for School Mathematics, 1991) also states, in the *Professional Standards for Teaching Mathematics*, that one of the "major shifts in the environment of mathematics classrooms...needed to move from current practice to mathematics teaching for the empowerment of students...[are shifts] toward classrooms as mathematical communities – away from classrooms as simply a collection of individuals" (p. 3). Further, NCTM (Commission on Teaching Standards for School Mathematics, 1991) states:

> Students' learning of mathematics is enhanced in a learning environment that is built as a community of people collaborating to make sense of mathematical ideas.... Classroom structures that can encourage and support this collaboration are varied: students may at times work independently, in pairs or in small groups. (p. 58)

If you agree that students construct and improve their understanding of mathematics in learning environments that operate as communities, cooperative learning is one structure you should attempt to incorporate into the classroom.

Also, concerning learning or construction of knowledge and effective teaching, the National Research Council (1989) offers the following viewpoint:

> In reality, no one can *teach* mathematics. Effective teachers are those who can stimulate students to *learn* mathematics. Educational research offers compelling evidence that students learn mathematics well only when they construct their own mathematical understanding.... This happens most readily when students work in groups, engage in discussion, make presentations, and in other ways take charge of their own learning. (pp. 58-59)

This is a strong statement in support of using cooperative learning as one type of classroom organization to promote learning and understanding. Professional groups and organizations support and encourage the use of cooperative learning.

One goal of education is to prepare students for jobs or careers. It is interesting to note that Henry Pollak (cited in Commission on Standards for School Mathematics, 1989), a well known industrial mathematician, lists as one of the expectations for new employees by industry, "The ability to work with others on problems" (p. 4). In reality, the workplace is a community and each person must become a contributing, functioning part of that community. It only seems reasonable that cooperative learning will improve students' abilities to work together. It will encourage students to value the opinions of all members of the group, to compromise, to listen, and to discuss problems and problem situations.

Davidson (1990b) makes the following claims regarding the reasons and situations appropriate for cooperative learning:

> Small-group cooperative learning can be used to foster effective mathematical communication, problem solving, logical reasoning, and the making of mathematical connections.... Small-group cooperative learning methods can be applied with all age levels of students, all levels of the mathematics curriculum from elementary school through graduate school, and all major topics areas in mathematics. Moreover, small groups working cooperatively can be used for many different instructional purposes: in the discussion of concepts, inquiry/discovery, problem solving, problem posing, proofs of theorems, mathematical modeling, the practice of skills, review, brainstorming, sharing data from different groups, and the use of technology. (p. 52)

Considering these claims and the above views, it would be foolish to fail to investigate, informally and formally, this approach to teaching. So, formally, what information does research offer concerning cooperative learning?

Research on Cooperative Learning

Cooperative learning techniques have been a part of education for many years. At the end of the 19th century, Parker was one of the most successful practitioners of cooperative learning, followed by another supporter in the 1930's, Dewey (Johnson, Johnson, Holubec, & Roy, 1984). As early as the 1920's, research on cooperative situations was carried out. The theory of Lewin, expanded upon by Deutsch, concerning cooperative and competitive situations is the foundation for most of the more recent cooperative learning research (Johnson et al., 1984). Starting in the early 1970's (Totten, Sills, Digby, & Russ, 1991) research on cooperation in classroom situations and its effects on achievement, attitude, social development, and other factors began. Slavin (1989/1990) states that, "cooperative learning is one of the most thoroughly researched of all instructional methods" (p. 52).

Research on cooperative learning, not necessarily specific to mathematics, has been abundant and the results have been favorable. Several reviews and meta-analyses of research on general cooperative learning (Johnson, 1989; Johnson, Maruyama, Johnson, Nelson, & Skon, 1981; Pepitone, 1980; Slavin, 1981, 1989/1990) are available. Davidson (1990a) states that the results summarized in these reviews indicate positive effects of cooperative learning in the areas of: (a) achievement, (b) learner's self-confidence, and (c) social skills and relations. Totten et al. (1991) provide a comprehensive annotated bibliography of research and applications of cooperative learning in a variety of content areas, including mathematics. The majority of research studies offer supporting evidence of cooperative learning.

Research on cooperative learning in mathematics has increased. Summaries of research on cooperative learning in mathematics (Davidson, 1985; Webb, 1985) show that, in at least 40% of the cases, students in cooperative learning situations demonstrate higher achievement. Cooperative learning had positive effects on students' attitudes and confidence. Webb (1991) analyzed research on cooperative learning in mathematics for indicators of effectiveness based on student interactions. She concluded that the most effective small groups are those where students are free to talk about what they understand and don't understand, give each other detailed responses on problems, and give each other a chance to discuss. Some particular results on cooperative learning in mathematics, at various grade levels are mentioned here as examples.

Phelps and Damon (1989) studied mathematics learning and development of spatial concepts using cooperative learning with students in Grade 4. Cooperative learning was effective when reasoning was required, but not with rote learning indicating that collaboration may deepen understanding of concepts and improve problem solving. Gender differences lessened in the cooperative learning classes. Good, Reys, Grouws, and Mulryan (1989/1990) gathered data in an observational study of elementary classrooms and concluded that students participating in cooperative learning were more actively involved in learning, more motivated, and more enthusiastic about mathematics. They also observed that some students still tended to work alone and that better materials for teachers using cooperative learning strategies were needed. Yackel, Cobb, and Wood (1991) researched small-group problem solving at Grade 2. They found that learning opportunities and dialogue not found in traditional classrooms occurred. The teacher's role and the use of appropriate activities are identified as crucial features of this cooperative learning study.

The studies detailed above, and many others, have been carried out at the elementary school level. Some research results at higher grade levels exists. In particular, cooperative learning at the college level has been receiving more attention recently. For example, Dees (1991) gathered data in a college remedial mathematics course to find whether cooperative learning helped students increase their problem solving abilities. The results showed significant differences in favor of cooperative learning. Students using cooperative learning performed as well or better on every measure in the study.

Sheets and Heid (1990) discuss cooperative learning aspects that emerged during the implementation of computer tools in the mathematics classroom at the beginning algebra level and at the beginning calculus level. The evolving new roles of students and teachers are identified along with characteristics of the collaborative teams. This combination of technology and cooperative learning is emerging as one of the more favorable strategies in the calculus reform movement. Beers (1991) describes a combination of microlabs and cooperative learning labs used in calculus at Simmons College. She points out that the cooperative learning labs promote conversations about calculus and increases self-confidence. Dubinsky and Schwingendorf (1991) use cooperative learning groups in a computer laboratory environment at Purdue University. They have found this approach to be more successful than any other methods they have tried, but are continuing to refine the cooperative learning aspects of the laboratory to ensure involvement by all students. Research and interest in cooperative learning at the college level is growing and seems promising.

Conclusions

Gill (1990) completely summarized the merits of cooperative learning with the following:

1. In the real world, skills and knowledge are used cooperatively.
2. Cooperative learning promotes social growth.

3. Cooperative learning harnesses the greatest single motivator, peer pressure, and makes it beneficial.
4. Cooperative learning promotes life skills.
5. Studies have shown an increase in learning, achievement, and performance for all members.
6. Research tells us that cooperative learning allows for a freer exchange of thoughts and ideas without fear of failure.
7. Individual achievement results from the support and assistance given from each group member.
8. Cooperative learning increases problem solving skills and develops higher level thinking skills.
9. Cooperative learning increases student self-esteem.
10. Research shows that cooperative learning should be used in the classroom when we want students to learn more, like school better, like each other better, and make students responsible for their learning. (p. 4)

Can we, in mathematics education, afford to ignore a classroom strategy that helps students develop in ways anticipated by business and industry; decreases peer pressure, fear of failing, and lack of self-esteem; and improves achievement, attitude toward mathematics, problem solving, and higher order thinking? I think not. We must investigate the potential of cooperative learning at all levels of mathematics education and take advantage of its power to improve the teaching and learning of mathematics.

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SUBMISSION OF ARTICLES

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Mathematics: An International View

Edited by Igor Malyshev and Joanne Rossi Becker
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This column is as an avenue for discussion of the first two years of college mathematics with a global perspective. Because mathematics is an international culture, it is important to follow the development of mathematics education throughout the world. We feel that by sharing information from other countries with systems of education different from ours, we will gain an enriched understanding of, and a better perspective on, our own system. A completely different viewpoint on a topic can serve to stimulate our imaginations and help us find new solutions to a problem which will work in our context. As we move toward dramatic changes in both the curriculum and instruction in mathematics called for in many reports, reflection on practices used in other countries can only help our efforts.

For this column, we would like to publish two types of items: material with an international origin or context that you could use directly with your students; or information about the content and structure of coursework in other countries. We welcome submissions of 3-4 pages or suggestions of mathematicians from other countries from whom we could solicit relevant material.

Send contributions for this column to Igor G. Malyshev at the address above.

Integral and Combinatorial Identities

by G.I. Prizva
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Professor Georgii Prizva has been teaching mathematics at Kiev University, Ukraine (former Soviet "republic"), for 25 years. His research interests are in Probability and Statistics, but for about as long as he has been teaching he has been involved in running miscellaneous mathematics competitions (Olympiads) at Kiev City and all-Ukrainian levels. He composes many original olympiad problems and trains teams of high school and college students for those competitions. He also has published extensively in the field of popularization of mathematics.

In this article we show an alternative method of proving some combinatorial identities. The proofs are based on a minimal knowledge of integration (and as a result available to second year students), but are powerful enough to handle rather complicated cases. The first three examples are chosen from the material offered at different mathematical competitions, and the last example belongs to the author.

In intermediate algebra or discrete mathematics courses our students learn the Newton's binomial expansion formula

$$(a+x)^n = a^n + n\,a^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots + x^n = \sum_{i=0}^{n} C(n,i)\,a^{n-i}x^i. \qquad (1)$$

Given that the case for $n=1$ has been verified, and, the validity of (1) for $n=k$ presumed, proof by induction requires that we establish formula (1) for $n=k+1$. Let's illustrate how integration can be applied here. Using (1) for $n=k$ we find that

$$\int_0^x (k+1)(a+y)^k dy = (k+1)\int_0^x (a^k + k\,a^{k-1}y + \ldots + y^k)\,dy$$

$$= (k+1)\,a^k x + \frac{(k+1)k}{2!}a^{k-1}x^2 + \ldots + x^{k+1}.$$

On the other hand,

$$\int_0^x (k+1)(a+y)^k dy = (a+x)^{k+1} - a^{k+1},$$

which concludes the proof.

Using integration, as in the illustration above, we can establish other combinatorial identities.

Example 1. To prove that

$$C(n,0) - \frac{1}{2}C(n,1) + \frac{1}{3}C(n,2) - \ldots + \frac{(-1)^n}{n+1}C(n,n) = \frac{1}{n+1}, \qquad (2)$$

where $C(n, k)$ is the number of "k-combinations out of n," we introduce the function

$$f(x) = x - \frac{1}{2}C(n,1)x^2 + \frac{1}{3}C(n,2)x^3 - \ldots + \frac{(-1)^n}{n+1}C(n,n)x^{n+1}.$$

Obviously, for $x = 1, f(1)$ gives us the left side of (2). Then, since

$$f'(x) = C(n,0) - C(n,1)x + C(n,2)x^2 - \ldots + (-1)^n C(n,n)x^n,$$

using (1) we find that $f'(x) = (1-x)^n$, and its antiderivative

$$f(x) = \frac{-(1-x)^{n+1}}{n+1} + C.$$

The constant C can be found from the fact that $f(0) = 0$ and $C = f(0) + 1/(n+1) = 1/(n+1)$. That is,

$$f(x) = -\frac{(1-x)^{n+1}}{n+1} + \frac{1}{n+1}, \quad \text{and} \quad f(1) = \frac{1}{n+1} \qquad \text{QED}$$

Example 2. As in example 1, to prove the identity

$$C(n,0) + 2\cdot 3\, C(n,1) + 3\cdot 3^2\, C(n,2) + \ldots + (n+1)\cdot 3^n\, C(n,n) = 4^{n-1}(3n+4),$$

we reduce the problem to finding the value $f(1)$ of the function

$$f(x) = C(n,0) + 2\cdot 3\, C(n,1)x + 3\cdot 3^2\, C(n,2)x^2 + \ldots + (n+1)\cdot 3^n\, C(n,n)x^n.$$

Its antiderivative (to the extent of unknown constant) has the form:

$$F(x) = C(n,0)x + 3\,C(n,1)x^2 + 3^2\,C(n,2)x^3 + \ldots + 3^n\,C(n,n)x^{n+1}$$
$$= x[C(n,0) + C(n,1)(3x) + C(n,2)(3x)^2 + \ldots + C(n,n)(3x)^n].$$

Again, from (1) we find $F(x) = x(1+3x)^n$, and therefore

$$f(x) = F'(x) = (1 + 3x)^n + 3nx(1 + 3x)^{n-1},$$

and, finally, $f(1) = 4n + 3n\,4^{n-1} = 4^{n-1}(3n + 4)$. QED

The following two cases are less elementary.

Example 3. The identity we shall prove now is

$$\sum_{m=0}^{n} \frac{(-1)^m}{C(n,m)} = \frac{n+1}{n+2}\left(1+(-1)^n\right).$$

As a supporting integral relation we consider the formula

$$\int_0^1 x^m (1-x)^n\,dx = \frac{m!\,n!}{(m+n-1)!},$$

which can be either borrowed from a reference source or proved using repeated integration by parts. Then leaving some details to the reader we find:

$$\sum_{m=0}^{n} \frac{(-1)^m}{C(n,m)} = (n+1)\sum_{m=0}^{n} \frac{(-1)^m\, m!\,(n-m)!}{(n+1)!} = (n+1)\sum_{m=0}^{n}(-1)^m \int_0^1 x^{n-m}(1-x)^m\,dx$$

$$= (n+1)\int_0^1 x^n \sum_{m=0}^{n}\left(1-\frac{1}{x}\right)^m dx \text{ (as a sum of a geometrical progression)}$$

$$= (n+1)\left(\int_0^1 x^{n+1}\,dx - \int_0^1 (x-1)^{n+1}\,dx\right) = \frac{n+1}{n+2}\left(1+(-1)^n\right). \quad \text{QED}$$

Example 4. Let's prove the identity

$$\sum_{m=0}^{n} \frac{(-1)^m\, C(n,m)}{a\,m+1} = \frac{n!\,a^n}{(a+1)(2a+1)\ldots(n\,a+1)},$$

which is a significant generalization of (2) and is reducible to it when $a = 1$. As before, we start with an integral relation, in this case

$$\int_0^1 (1-x^a)^n\,dx = \frac{n!\,a^n}{(a+1)\,(2a+1)\ldots(n\,a+1)},$$

which can be obtained with a substitution of variables $y = x^a$ and then multiple integration by parts. As a result, using (1) and the relation above, we find:

$$\sum_{m=0}^{n} \frac{(-1)^m\, C(n,m)}{a\,m+1} = \sum_{m=0}^{n}(-1)^m\, C(n,m)\int_0^1 x^{am}\,dx = \int_0^1 \sum_{m=0}^{n}(-1)^m\, C(n,m)\,x^{am}\,dx$$

$$= \int_0^1 (1-x^a)^n\,dx. \quad \text{QED}$$

THE CHALKBOARD

Edited by
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This column is intended as an idea exchange. We hope to facilitate an open exchange of ideas on classroom management, teaching techniques, tips for helping students get past the usual stumbling blocks, techniques for improving student participation, etc. We know there are lots of good ideas out there, and this is your chance to share them. Our backlog is almost exhausted! Please send your contributions to either editor.

Getting Help

I'm not sure how many times I have, in one way or another, told a student to get help on some topic, but I am sure that it's at least an order of magnitude greater than the number of times that the student actually did go get that help. Some of the time, the reason may be that the student doesn't even know what to ask about. I may have said "Be sure to memorize the quadratic formula," or "Get some help with rational exponents," but they may not have even understood what I was talking about. Consequently, they ignore the suggestion and never get the help they need.

At my college a major supplier of help for mathematics students is the Math Lab. Tutors are available there to help students on either a drop-in or appointment basis. In hopes of making it easier for students to know what to ask for, I've started using the referral form, below. It can be given to the student or clipped to a paper I am returning. The form reminds the student where help is available as well as specifying the topics to work on. This last feature helps the laboratory staff as well. (JB)

Mathematics Laboratory Referral

Student ______________________ Class______________

Please report to the Mathematics Laboratory (Room A108) to obtain assistance in the following topic(s) which seem(s) to need help. It would assist the staff if you took along this form and some recent papers showing the difficulty.

- ___ Polynomial subtraction
- ___ Factoring: trinomials
- ___ Factoring: diff. of squares
- ___ Negative exponents
- ___ Fractional exponents
- ___ Radicals
- ___ Simplifying algebraic fractions
- ___ Writing equation of a line
- ___ Exponential functions: basics
- ___ Logarithmic functions: basics
- ___ Solving exp and log equations
- ___ Quadratic formula
- ___ Solving linear inequalities
- ___ Solving quadratic inequalities
- ___ Composition of functions
- ___ Inverse functions

Other (or additional instructions): ______________________

____

Instructor______________________

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The Problem Section

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Greetings, and welcome to still another Problem Section! Before getting down to business, I'd like just to say how wonderful it was to see so many of you in Seattle (Nov. 1991 AMATYC Convention)! Not having been to an AMATYC convention for three years, I found it wonderfully refreshing and invigorating. We had a great time at the Pacific Science Center—even if I did chicken out of riding that bicycle on a thin circle some 50 feet up in the air.

One thing that disappoints is how few AMATYC members are even aware that we have a problem section. Here it is more than a decade of doing this, and I don't mind so much not being known personally, but so many have never really paid attention to this section. W. .id a Jimmy Connors not notice that there are open, unused tennis courts across the street?

This mathematical court is definitely open and awaits your input to make it better. So, please be reminded once more that the *AMAT ı C Review* Problem Section seeks not mundane exercises but lively and interesting problems and their solutions from all areas of mathematics. Particularly favored are teasers and challenges of an elementary or intermediate level that have some applicability to the lives of two-year college math faculty and their students. We welcome computer-related submissions, but bear in mind that programs should supplement, not supplant, the mathematics.

To submit material for this department, send your new problem proposals only, preferably typed or printed neatly with separate items on separate pages, to the Problem Editor at the home address shown below. If you have a solution to your proposal, please include it along with any relevant comments, history, generalizations, special cases, observations, and/or improvements. Include a mailing label or self-addressed envelope if you'd like to be assured a reply.

All solutions to others' proposals should be sent directly to the Solutions Editor at his home address. Addresses are:

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If you see your own problem printed later, please send Prof. Stong a copy of your solution, regardless of whether you sent me one originally with your proposal or not. This is not mandatory as policy, and you will automatically be listed among solvers in any case. However, this is the safest way to assure that your approach will be considered for publication as a featured solution. *Thanks*!

Old Problems and Solutions Via Incubation

I and many others have long noted the value of letting the subconscious work on problems, where they may incubate, thriving on the admixture of a stream of assorted ideas. Perhaps it was the combination of teaching infinite series again and seeing the following AMATYC problem cited in Stan Rabinowitz's compilation of problems from several dozen journals (including this one) that brought this one to the fore. In retrospect, this one really seems easy now. (Recall the joke about the professor who works on a problem he calls obvious for a whole week and then tells his class that he was right: it was obvious after all!)

Solution to E-3. (Fall/Winter 1983 *AMATYC Review*, page 55). Proposed by Michael W. Ecker (then at the Worthington Scranton Campus of Penn State U):

a) Let $F(x) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^x}$ where the numerator uses the Euler phi function (which counts the number of positive integers under n that are relatively prime to n). Determine the domain of F.

The answer is $(2, \infty)$. As with a p-series in intermediate calculus, the domain is a connected set of x-values: If a is in the domain of F and $b > a$, then b is in the domain of F as well by a simple term-by-term comparison.

It suffices to show that $x > 2$ implies x is in the domain of F, and that 2 is not in the domain of F. For the first part, note that $\phi(n) < n$, so each term is less than $1/n^{x-1}$, for which latter series convergence is assured for $x-1 > 1$ (p-series result), or $x > 2$.

For $x = 2$, consider terms $n = p$ (a prime) only. Then $(p) = p-1$, and our subseries' typical term is $(p-1)/p^2$. By a limit comparison, this subseries converges if and only if the series of terms $1/p$ converges; however, $\sum_{p:\ \text{prime}} \frac{1}{p}$ is a known divergent series. This concludes the argument.

Note: I still do not have a full answer for E-3's part b), the alternating series version of part a.

Teaser Solutions

Teasers are math problems that typically take just a few minutes to at most a half-hour to an hour. Solutions follow the next issue. All correspondence to this department should go to the Problem Editor.

Teaser #3. Two numbers have a sum of 8 and a product of 4. What is the sum of the reciprocals of the two numbers?

Solution and Comment: The two numbers x and y satisfy $x+y = 8$ and $xy = 4$, and we want the value of $1/x + 1/y$. This last expression equals $(x+y)/(xy) = 8/4 = 2$. If one wishes a moral to such a simple story, it is dual: a) one need not always solve for the individual variables; b) appropriate form changes are at the heart of solving many problems—even ones much more involved than this.

Teaser #4. What is the smallest possible value for the sum of a positive real number and its reciprocal? Simple proof, please.

Solution and Comment: One could define $f(x) = x + 1/x$ for $x > 0$, get $f'(x)$ and set it to 0 to find $x = 1$ (remember: $x > 0$), and test to confirm an absolute minimum at $x = 1$. Thus, the smallest possible value is $f(1) = 2$.

However, a calculus proof is not needed. If $x \geq 2$ *or* $x \leq .5$, $f(x) \geq 2 = f(1)$ *because* x or $1/x$ (respectively) alone exceeds or equals 2. We need consider only elements of $(.5, 2)$. Moreover, since $f(1/x) = f(x)$, we may restrict attention to $[1,2]$, on which closed interval f will have a minimum no larger than 2.

Suppose $f(x) < 2$ for one or more elements x satisfying $1 < x < 2$. Then

$$[f(x)]^2 = (x + 1/x)^2 = x^2 + 1/x^2 + 2 = f(x^2) + 2 < 4 \text{ (since } 0 < f(x) < 2).$$

Thus, $f(x^2) < 2$ as well, though $x^2 > x$ need not belong to $(1,2)$. If we repeat this argument with x^2, we are led to an increasing, unbounded infinite sequence of numbers $x, x^2, x^4, x^8, \ldots$ with corresponding function values all under 2. But $f(x) > 2$ for all $x > 2$, at least. This contradiction shows that there cannot be such an x with $f(x) < 2$, so $2 = f(1)$ is indeed the minimum value.

Alternative: An even easier argument uses the quadratic formula: Set $x + 1/x = v$, v a possible value. This is equivalent, for positive x, to $x^2 - vx + 1 = 0$, and the latter has a real solution if and only if the discriminant $v^2 - 4$ is nonnegative. Thus, $v \geq 2$, confirming the minimality of $f(1) = 2$.

New Teasers

Teaser #5. It is obvious that $a + 1/a = b + 1/b$ if and only if $a = b$ or $a = 1/b$. Prove it – without calculus.

Teaser #6. Prove that there exist irrational numbers a and b such that a^b is a rational number. (This is an oldie-goldie, but thanks to Charles Ashbacher of Cedar Rapids, Iowa for the reminder, who in turn thanks Prof. Valery Marenich, Institute of Mathematics in Novosibirsk, Russia.)

Teaser #7. For k and n whole numbers, demonstrate that $(kn)!/(n!^k)$ is integral, too. (This was proposed by Greg Foley to me in person years ago at the Atlanta AMATYC convention, where Greg handed this to me on a cocktail napkin.)

Teaser #8. Proposed by Stephen Plett, Placentia, CA.

R^2, with the operations indicated below, is almost a vector space. What goes wrong? What subset(s) of $\mathbf{R}^2$ would be a vector space with these operations?:

$$\text{abstract addition: } (a, b) + (c, d) = (a+c, bd)$$

$$\text{abstract scalar multiplication: } k(a, b) = (ka, b^k).$$

New Problems

Set W Problems are due for ordinary consideration May 1, 1993. However, regardless of deadline, no problem is ever closed permanently, and new insights to old problems – even Teasers – are always welcome. An asterisk on a problem indicates that the proposer did not supply a solution with the proposal.

Problem W-1. Proposed by Jim Africh, College of DuPage, Glen Ellyn, IL.

Solve for x: $\sqrt[3]{1+\sqrt{x}} + \sqrt[3]{1-\sqrt{x}} = \sqrt[3]{5}$

Proposer's Comment: "This is a non-esoteric problem that can be solved by any mathematics student with a solid background in algebra."

Problem W-2. Proposed by Juan Bosco Romero Marquez, Avilia, Spain.

Prove that, if $w \geq 1$ and $0 < b < c$, then

$$\frac{wb+c}{w+1} \leq \frac{b+c}{2} \leq \frac{b+wc}{w+1}$$

Problem Editor's Comment: There is an obvious interpretation, so this one could actually be a teaser for some people.

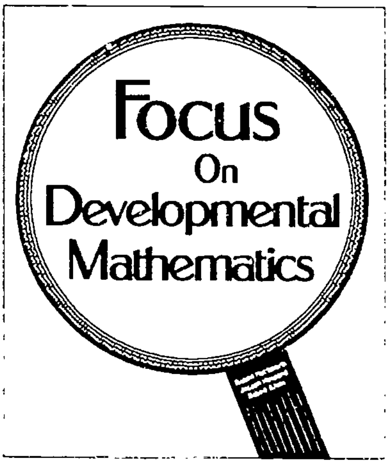
Problem W-3. Proposed by the Problem Editor, Penn State U, Lehman, PA.

Look at the first few Pythagorean triples: (3,4,5); (5,12,13); (8,15,17); (7,24,25); (9,40,41). Now prove: a) Every primitive Pythagorean triple contains one and only one element divisible by 3. b) Same, but divisible by 5.

Problem Editor's Comment: It is well-known that one element has the form 2uv, with one of the generators u, v even, and this element is divisible by 4. The (3,4,5) example shows that this problem completes the question of a divisibility generalization. For another look at triples, however, see the next problem.

Problem W-4. Proposed by Louis I. Alpert, Bronx Community College (part of the City University of New York), Bronx, NY.

If primitives u, v generate Pythagorean triples u^2-v^2, $2uv$, u^2+v^2, find the primitives m, n in terms of u, v that generate the triple twice that generated by u and v.



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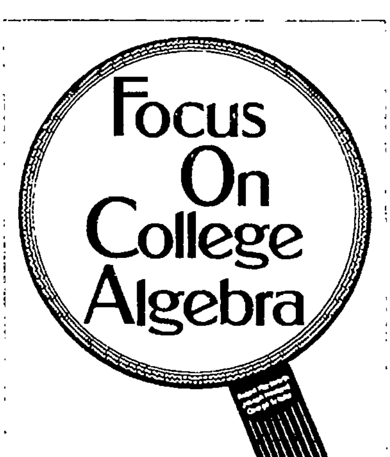

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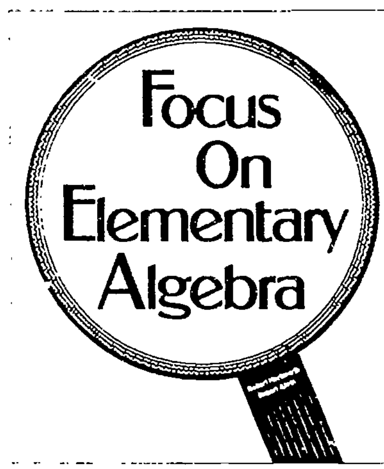

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Problem Editor's Comment: Note that the new triple generated cannot be primitive, by definition. Dr. Alpert, who actually offered this as a teaser, comments that there is an interesting reversal of positions in the new triple. Since readers may care to explore this and this question might fit in nicer here (e.g., see W-3), I've included this with the regular problems.

Problem W-5. Proposed by the Problem Editor.

Find all functions f defined on the set of reals for which $f(a+b) = f(a) + f(b)$.

Problem W-6. Proposed by Stephen Plett, Fullerton College, Fullerton, CA.

Upon looking at my digital wristwatch, I noted that all of the digits were different. Was this unusual? What's the exact probability that a digital timepiece will display distinct digits?

Problem Editor's Comment: Most readers should be alert to make appropriate assumptions and interpretations prior to attempting to solve the problem. Until then, you may be solving a different problem than others have in mind. (E.g., do we assume 12-hour display? ... HH:MM:SS format? Are lead zeros displayed?)

Thanks to a few activists such as Steve Plett and Bob Stong, I now have an actual backlog of problems. However, I still welcome more very elementary ones, of which I have virtually none left with which to attract new solvers.

Set U Solutions

N-secting the Angle

U-1. Proposed by Stephen Plett, Fullerton College, Fullerton, CA.

The repetitive nature of the polar graph $r = \cos(kt)$ is well known for k an integer. Generalize by finding the smallest period of the graph for k rational.

Solution by the proposer only.

In polar coordinates, the points identified with (r,t) are those of the form $((-1)^j r, t + j\pi)$ with j an integer, when r is not zero. Writing $k = m/n$ with m and n relatively prime, one seeks the smallest s for which $(\cos(m(t+s)/n), t+s)$ is the same point as $(\cos(mt/n), t)$ for all t. Taking $t=0$, the point $(1,0)$ must be the same as $(\cos(ms/n), s)$, so $s = j\pi$ where j is an integer and mj/n is an integer with the same parity as j. Taking $j = hn$ to make mj/n integral, one must have $hn \equiv hm$ modulo 2. Thus, the smallest s is n when m and n are both odd and is $2n$ if either m or n is even.

Note. Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA observes that this problem is fully discussed in the paper "Investigating The Petals of Hybrid Roses" by Kenneth S. Gordon in *Mathematics and Computer Education*, vol. 26, no. 1.

The Count of Monte Triangle

U-2. Proposed by Stephen Plett, Fullerton College, Fullerton, CA.

Suppose the usual two-dimensional x,y coordinate system is modified so that the positive y-axis is formed at a 60-degree angle to the positive x-axis. Consider any lattice point (a,b) in this system's first quadrant (i.e., a point (a,b) with integral coordinates and $a > 0$, $b > 0$). Each such point (a,b) determines a unique parallelogram formed by the two axes ($y=0, x=0$), the line $x=a$, and the line $y=b$.

How many equilateral triangles are contained within or on this parallelogram if the triangles have lattice points for vertices and sides parallel to the axes and the lines $x+y = \text{constant}$?

Solutions by Robert Bernstein, Mohawk Valley Community College, Utica, NY and the proposer.

The triangles with an edge of length k occur in pairs forming parallelograms with a pair of opposite vertices (x,y) and $(x+k,\ y+k)$ where $0 \le x,\ x+k \le a,\ 0 \le y$, and $y+k \le b$. Hence, the number of such triangles is $2(a-k+1)(b-k+1)$. For convenience, one may suppose $a \le b$. The total number of triangles is then

$$2\,[\,ab + (a-1)\,(b-1) + \ldots + 1(b-a+1)] = a(a+1)(3b-a+1)/3.$$

Attack of the Killer Denominators

U-3. Proposed by Ken Boback, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA.

Rationalize the denominator of F: $F = 1/(1 + \sqrt[3]{2} + \sqrt[3]{4}\,)$

Follow-up challenge: Change "4" to "5" in the expression.

Complete solutions including the challenge by N.D. Aggarwal and S.K. Aggarwal, Embry-Riddle Aeronautical University, Daytona Beach, FL; Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Jim Culliver and Davis Finley, Community College of Southern Nevada, North Las Vegas, NV; Scott Higinbotham, Middlesex Community College, Burlington, MA; Stephen Plett, Fullerton College, Fullerton, CA; Bella Wiener, University of Texas - Pan American, Edinburg, TX; and the proposer. **Solutions without the challenge** by Jim Africh, College of DuPage, Glen Ellyn, IL; Joseph Browne, Onondaga Community College, Syracuse, NY; William Radulovich, Florida Community College at Jacksonville, Jacksonville, FL; G. Russell, Brevard Community College, Melbourne, FL; J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, WI, and Grant Stallard, Manatee Community College, Bradenton, FL.

Since $1/(1 + x + x^2) = (x-1)/(x^3-1)$, $F = \sqrt[3]{2} - 1$. For $1/(1 + u + v)$ with $u^3 = a$ and $v^3 = b$, multiplying numerator and denominator by $1 - u - v + u^2 - uv + v^2$ gives a new denominator of $(1 + a + b) - 3uv$. Then further multiplication by $(1 + a + b)^2 + 3uv(1 + a + b) + 9u^2v^2$ completely rationalizes the denominator to $(1 + a + b)^3 - 27ab$. Applying this with $a = 2$ and $b = 5$ gives

$$(1 - \sqrt[3]{2} - \sqrt[3]{5} + \sqrt[3]{4} - \sqrt[3]{10} + \sqrt[3]{25})\ (64 + 24\sqrt[3]{10} + 9\sqrt[3]{100}\,)/242.$$

A Short Series

U-4. Proposed by the Solutions Editor.

When the Cincinnati Reds won the 1990 World Series in four straight games, CBS may have lost as much as 150 million dollars in ad revenue. It was asserted that the average length of the series is six games.

a) Assuming the two teams are evenly matched in a best-of-seven series, what is the exact expected number of games played to obtain a winner?

b) Can the probabilities be chosen so that the expected length of the series is exactly six games?

Solutions by N.D. Aggarwal and S.K. Aggarwal, Embry-Riddle Aeronautical University, Daytona Beach, FL; Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Nicholas Belloit, Florida Community College at Jacksonville, Jacksonville, FL; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Joseph Browne, Onondaga Community College, Syracuse, NY; Mike Dellens, Austin Community College, Austin, TX; Lawrence Gilligan, University of Cincinnati, Cincinnati, OH; Stephen Plett, Fullerton College, Fullerton, CA; G. Russell, Brevard Community College, Melbourne, FL; Frank Soler, De Anza College, Cupertino, CA; and Grant Stallard, Manatee Community College, Bradenton, FL.

Letting p and $q=1-p$ be the probabilities of winning for the two teams, the probability that the series ends with $k+4$ games is $C(k+3, k)(p^4q^k + p^kq^4)$, where $C(k+3, k)$ is the number of ways of choosing the k games won by the loser of the series from among the first $k+3$ games. The expected number of games in the series is then

$$E = 4 + 4p + 4p^2 + 4p^3 - 52p^4 + 60p^5 - 20p^6 .$$

and at $p = 1/2$ the value is $93/16 = 5.8125$. To find the maximum value, write E in terms of $x = p-1/2$ to obtain

$$E = 5.8125 - x^2 [11.75 - 23x^2 + 20x^4].$$

Since the quadratic $11.75- 23y + 20y^2$ has no real roots, it is always positive, and the maximum value of E occurs at $x=0$. Alternatively, the derivative of E is

$$E' = -8(p-1/2)[15p^2(p-1)^2 - (2p - 1)^2 + 2].$$

For p between zero and one, $(2p - 1)^2$ is at most 1, and the last factor is always positive. Thus E' is zero only at $p=1/2$, is positive for smaller values, and negative for larger values, giving a maximum at $p=1/2$.

Note. Lawrence Gilligan observes that part a) is an exercise in "Introduction to Finite Mathematics" by Kemeny, Snell, and Thompson. A generalization of part b) appears as problem E3386 in the *American Mathematical Monthly* with the solution in the March 1992 issue.

Two-Part Harmony

U-5. Proposed by Jayanthi Ganapathy, University of Wisconsin-Oshkosh.

Find all harmonic functions $u(x,y)$ (i.e. those satisfying $u_{xx} + u_{yy} = 0$) for which $u_{xx}u_{yy}-u_{xy}^2$ is identically zero.

Solutions by N.D. Aggarwal and S.K. Aggarwal, Embry-Riddle Aeronautical University, Daytona Beach, FL; Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Stephen Plett, Fullerton College, Fullerton, CA; Joseph Weiner, University of Texas-Pan American, Edinburg, TX; and the proposer.

Substituting $u_{yy} = -u_{xx}$ into the second condition gives $-(u_{xx}^2 + u_{xy}^2) = 0$, so u_{xx}, u_{yy}, and u_{xy} are all zero. Because $u_{yx} = u_{xy}$ in the presence of continuity, it follows that u_x and u_y do not depend on x and y, so are constants. Hence $u(x,y) = a + bx + cy$, where a, b, and c are constants.

Book Reviews

Edited by John J. Edgell, Jr.

TRIGONOMETRY FOR COLLEGE STUDENTS, 5th edition by Karl J. Smith, Brooks/Cole Publishing Co., Monterrey, CA, 1991, 410 pages, ISBN 0-534-13728-8.

With the extraordinary proliferation in college level trigonometry texts of varying standards in recent years, it is heartening to encounter a truly excellent, exemplary, and definitive edition. Karl J. Smith's book is first class in all respects.

The author adopts the unit circle approach to developing trigonometric functions which immediately places the emphasis on conceptual understanding. The quality of exposition is first rate. There are all the usual Karl Smith trademarks: the tantalizingly interesting historical notes, which are sure to educate and inform the instructor as well as the students; the ample problem sets at the end of each section offering a variety of applications from the fields of geography, physics, surveying, astronomy, space science, engineering, business, oceanography, etc. Other features include: suggestions for further study of the end of each chapter; a cumulative review with accompanying sample examinations every two chapters; and, an appendix reviewing basic geometry to assist in the transition from high school algebra for those ill-advised students who arrive in college without any knowledge of geometry.

Any instructor who has grappled with the problem of how to teach students the methods of graphing trigonometric functions when translations, additions and changes in amplitude and period are involved, will find the novel ideas of framing, used in Chapter 2, particularly illuminating.

I very strongly recommend this superb book as a required text for a college trigonometry course. It is written in simple terms to accommodate weaker students, but the wide variety of problems and applications should also satisfy and inspire the more capable student.

Reviewed by Stewart C. Welsh, Southwest Texas State University, San Marcos, TX 78666.

ON THE SHOULDERS OF GIANTS: NEW APPROACHES TO NUMERACY, Edited by Lynn Arthur Steen, National Academy Press, Washington D.C., 1990, 216 pages. ISBN 0-309-04234-8.

In an era of budget cutbacks in community colleges, we are all being called upon to search for ways to capture and challenge the minds of our students in both present day applications and classical philosophy. However, we must do so under severe financial restrictions. Occasionally we discover new ways, but often we revise tried and true methodologies. As professionals, we are all well aware of the sequence of building upon the ideas of those who preceded us, and then, in turn, contributing to the pool of knowledge. Unfortunately, it is frequently difficult to trace some important ideas back to their problematic roots, and to cite true historically or culturally important personalities. **ON THE SHOULDERS OF GIANTS: NEW APPROACHES TO NUMERACY,** endeavors to supplement the standard tools across a variety of traditional topics. The authors of the essays not only justify applications of modern technology, but they help students to make independent mathematical generalizations as well.

One of the particularly nice features is the emphasis upon hands-on, concrete experiences for learning ideas. For example, one of the essays develops the concept of conic sections by describing a demonstration that involves the placing of a physical model of a cone in water and observing the water line. Most of these suggestions are inexpensive and use ordinary equipment.

For those interested in some innovative, concrete, hands-on techniques for teaching important mathematical applications and ideas which do not require expensive laboratory equipment and yet retains the essence of classical mathematics, one should browse this book for ideas.

Reviewed by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA 52406.

THE HISTORY OF MATHEMATICS-AN INTRODUCTION, 2nd edition by David M. Burton, Wm. C. Brown Publishers, Dubuque, IA, 1991, 678 pages, ISBN 0-697-11196-2.

In years past, standard references for a history of mathematics course designed for secondary teachers of mathematics have included, (Ball, 1968), (Bell, 1965), (Eves, 1983), and (Newman, 1956). More recently Burton's book, **THE HISTORY OF MATHEMATICS - AN INTRODUCTION,** (Burton, 1985), has become a primary resource for this course. The second edition continues the same emphasis upon developing a historical perspective based primarily upon the mathematical problems relative to the times and the notable work of those mathematicians upon whose shoulders we build. A particularly attractive facet of the text is that there are many exercises for the student to gain experience in arriving at a historical appreciation of the tools and skills and problems of the times. Problems are presented to be the focus about which mathematics has evolved. The associated personalities and issues of the items are included which give color and dynamics to the mathematical evolvement. A particularly nice addition to the second edition, which is a major concern of **AMATYC,** is an increase in attention to the influence of women mathematicians. Students usually find enough problems to be accessible and seem to appreciate the role that challenging problems of the times have had upon the growth and maturing of mathematics as a dynamic, ever-growing body of knowledge.

Reviewed by John J. Edgell, Jr., Southwest Texas State University, San Marcos, Texas.

References

1. *A Short Account of the History of Mathematics*, by W.W. Rouse Ball, Dover Publications, Inc., New York, 1960, 522 pages, ISBN 0-486-20630.
2. *Men of Mathematics*, by E.T. Bell, Simon and Schuster Publications, New York, 1965, 590 pages.
3. *The History of Mathematics - An Introduction*, by David M. Burton, Allyn and Bacon, Inc., Boston, 1985, 678 pages, ISBN 0-205-08095-2.
4. *An Introduction to the History of Mathematics,* 5th edition, by Howard Eves, Saunders College Publishing, Philadelphia, 1983, 593 pages, ISBN 0-03-062064-3.
5. *The World of Mathematics,* Four volumes, by James R. Newman, Simon and Schuster, New York, 1956, 2469 pages.

Send Reviews to: Dr. John J. Edgell, Jr., Editor, Book Reviews, *The AMATYC Review*, Mathematics Dept., Southwest Texas State University, San Marcos, TX 78666

Software Reviews

Edited by Shao Mah

Title: Mathematical Plotting Package, version 3.50

Authors: Howard Lewis Penn, Jim Buchanan and Frank Pittilli

Address: Mathematics Department
U.S. Naval Academy
Annapolis, MD 21402

Price: No charge; software available by sending two formatted 5-1/4" diskettes to Prof. Howard Lewis Penn

Hardware Requirements: IBM compatible computer with at least 512 k of memory and a CGA, EGA, VGA or Hercules board but a color monitor with EGA or VGA graphics board is recommended.

The Mathematics Plotting Package was created to be used in conjunction with learning calculus at the US Naval Academy. It was intended to provide a calculus course with computer graphics. The package contains two disks.

Disk one, MPP, has eight modules:

1. MPP (Mathematics Plotting Program)
2. Root (Root finding by Newton, bisection or secant)
3. Integral (Evaluation with trapezoidal, Simpson or Riemann sums)
4. Slope (Definition of derivative illustration)
5. Contour (Plots up to 15 contour lines)
6. Vector Fields (plots vector fields)
7. Double Integrals (Rectangular or polar coordinates)
8. Triple Integrals (Rectangular, cylindrical or spherical coordinates)

The package is menu driven. The choice of any one of the eight modules is available from the menu. Each module provides a brief explanation in how to use it at the beginning of the module on the screen. A help window is also provided for many modules. In each window, the program lists the available options. Therefore, a user will find that the Mathematics Plotting Package is very simple to use.

The plotting program, MPP, can plot y as a function of x, x as a function of y, polar equations or parametric equations. It can also draw up to six different graphs to be shown on one screen. The graphs are very well constructed with a high degree of accuracy on the VGA monitor. The program does not provide the zooming feature but a user can change the maximum and minimum values of the variables to make the graphs bigger or smaller. The colors of the graphs can also be changed by simply pressing the <F5> key. If the user desires, text can also be inserted on screen along with graphs.

The root module allows the user to find roots of an equation using Newton's method, the bisection method or the secant method. For the Newton's method, a user must provide the initial value by estimating from the graph of the equation on the screen. The program will then show how each successive estimate is obtained. When two estimate values are too close, the program rescales the graph and continues. The program stops when two estimated values agree to six digits. The bisection method or the secant method must be provided with the initial values of left and right endpoints.

The integral module allows a user to evaluate a definite integral by using Riemann sums, Trapezoidal rule or Simpson's rule.

The slope module illustrates the approximation of the tangent line. Similar to the Root module, the program will automatically stop its calculations when a desired accuracy is reached. It is very helpful for a student to understand the convergent process of a tangent line.

The contour module can plot up to 15 contour curves of function of two variables and it can be arranged to graph an equation of two variables. The plottings are considerably slower when variables have rational exponents. However, the contour curves are nicely drawn out in different colors if one is using a color monitor.

The vector field module draws a set of two dimensional vectors for a vector function. These vectors are drawn to their appropriate lengths and the vector field is illustrated quite nicely.

The double and triple integrals are the last two modules of disk one. The double integral module accepts either rectangular or polar coordinates. The program also provides the graph of the integrating region for the integral. The triple integral module performs triple integral in rectangular, cylindrical, or spherical coordinates, and the triple integral module is the only module which has no graphs.

Disk two, MPP3D, is a three-dimensional plotting program. The program has no explanations in the manual but it is mainly self-explanatory. The graph of the solid (after the program draws) appears in a box without axes. The graph can be rotated to be viewed from different view-points. MPP3D is a very well constructed software.

In conclusion, although the Mathematics Plotting Package does not have the capacities of symbolic differentiations and integrations, its programs have provided many practical concepts in calculus for student learning. Compared to much available commercial software, the graphs are better constructed, and the speed of graph drawing is faster. The reviewer has found MPP very handy to use for a calculus course.

Send Reviews to: Shao Mah, Editor, Software Reviews

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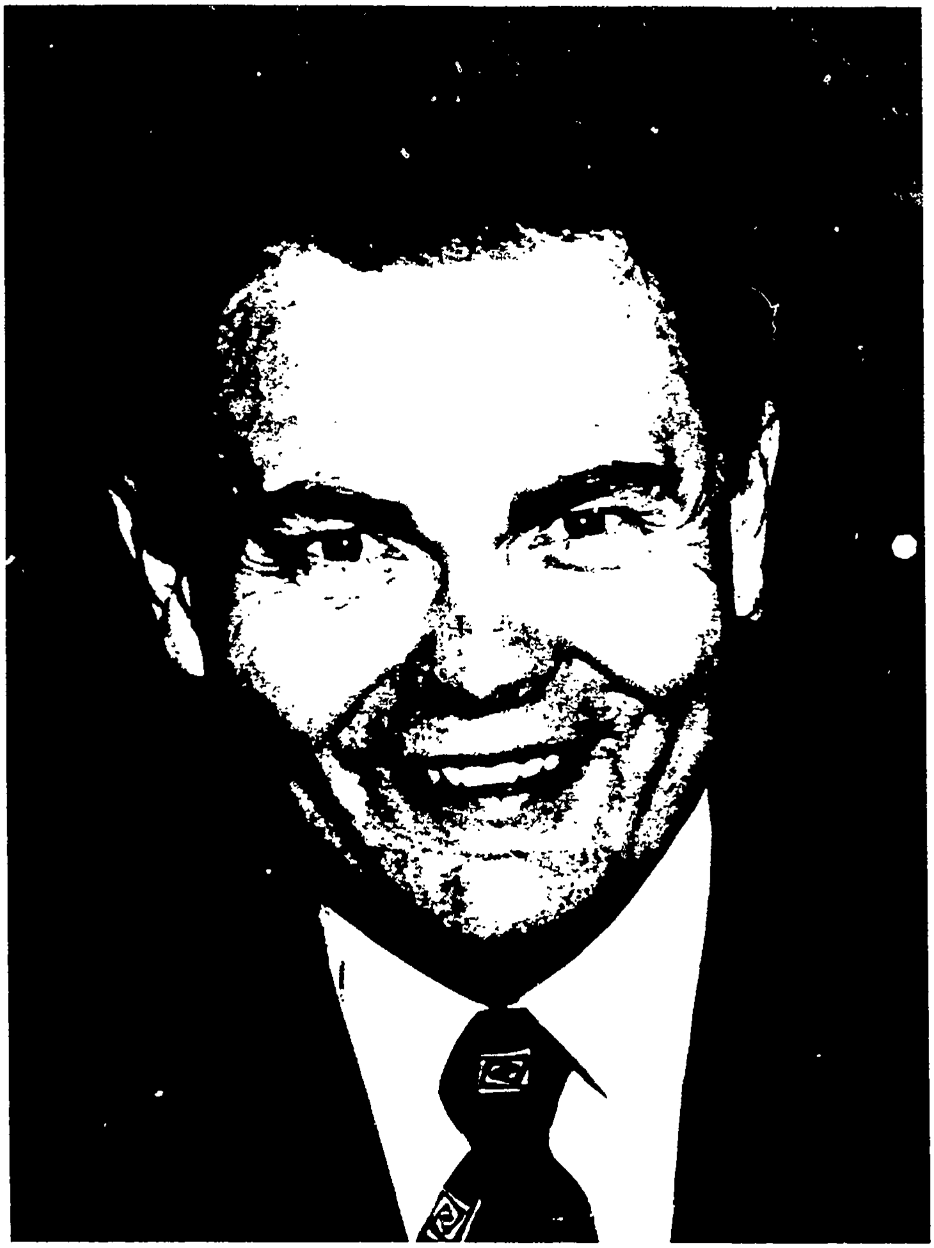

ALLYN J. WASHINGTON

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American Mathematical Association of Two-Year Colleges

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PURPOSE: AMATYC remains committed to its original purposes:

- To provide a national forum for the exchange of ideas
- To further develop and improve the mathematics education of students of two-year colleges
- To coordinate activities of affiliated organizations on a national level
- To promote the professional development and welfare of its members

The AMATYC Review provides an avenue of communication for all mathematics educators concerned with the views, ideas and experiences pertinent to two-year college teachers and students.

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Editor's Comments

New Editorial Staff

Greetings from your new editorial staff for *The AMATYC Review*: Jane Covillion, Production Editor, and Joe Browne, Editor. We have been handling manuscripts for about a year now, but, as I write this, we still are going through the apprehensive experience of actually producing our first issue. We ask your patience with the inevitable slips we will make at first. On the other hand, please feel free to make suggestions on how you think this could be made a better journal.

We want to thank the former editors, Don Cohen and Roy Cameron, for six very productive years and the excellent job they did. We also appreciate all their help and suggestions when we needed them.

Policy Change

Comments from referees and Editorial Panel members over the past couple of years have questioned the inclusion of specific calculator or computer commands and programs. The feeling frequently expressed was that most people did not actually read them or want them included. This has led to the policy below which is included in the newest *Guidelines for Authors*. Implementation will not be instantaneous, but this is the direction we are headed.

> Technology-oriented articles may be grouped into two, not necessarily distinct, categories: technology used as a teaching aid and technology used as a mathematical tool. In either case, the major intent of an article should be to help teachers and students to learn about mathematics, not about the machine or software. References to technology should be as generic as possible (e.g., "using a computer algebra system we find..." rather than "using Derive's [specific command] on an IBM PS/2-55 yields [specific output]"). Program listings, specific commands, and sequences of button pushes are usually inappropriate, though short segments may be included when they are essential to understanding the (mathematical or pedagogical) point being made.

About the cover

In this issue we begin a new occasional feature: interviews with people who have made significant impact on the two-year college mathematics scene. Readers are invited to make suggestions on who should be included in the future. We begin with Allyn J. Washington. Over a million students have used Al's books which have essentially defined technical mathematics for the past thirty years. He has received the NYSMATYC Award for Outstanding Contributions to Mathematics Education. After over twenty years of teaching at Dutchess Commu· college, Al "retired" to the beautiful mountains of California. He continues to produce revisions of the technical mathematics books and attends several mathematical meetings each year.

VIEWPOINT

Strategies for Making Mathematics Work for Minorities

by Beverly J. Anderson
Director, Office of Minority Affairs
The Mathematical Sciences Education Board of the National Research Council
and Professor of Mathematics, University of the District of Columbia

(Editor's note: These remarks were the concluding part of Dr. Anderson's keynote address at the 1991 AMATYC Annual Conference in Seattle. Much of the earlier portion of the address was devoted to history and statistics concerning minority students and the two-year college. The interested reader should consult the various reports listed under "References.")

By the year 2000, minorities will constitute one in every three American students. It has also been projected that from 1985 to the year 2000, over 21 million new jobs will be created (U.S. Department of Labor, 1988). These new jobs, even those not requiring a college education, will require basic skills in mathematics and the ability to reason. More than half of these jobs will require some education beyond high school and almost a third will require a college education. Thus, over the next ten years, Americans must take significant steps to keep minorities in school and focused on the appropriate academic areas those jobs will demand. Hence, it is no longer just an educational issue but, indeed, it has become an economic one as we consider who ultimately will be supporting American systems, such as social security.

A Vision

In the year 2000, as we take that flight into the future, I see the two-year colleges having strong articulation programs with four-year institutions, and those four-year institutions will include Historically Black Colleges and Uni-versities (HBCU), Hispanic Serving Institutions (HSI), and four-year tribal colleges. I see two-year institutions setting world-class standards, especially in the area of mathematics, that will guarantee unchallenged acceptance of two-year college students for the continuation of higher education in any university. I see four-year institutions accepting graduates from two-year colleges with marked enthusiasm, knowing that an influx of these graduates will not devalue their institutions. I see remedial courses designed to ensure student success, i.e. small classes staffed with experienced, well-motivated and talented teachers, as well as with student mentors. I see two-year colleges in partnerships with HBCUs, HSIs, and majority institutions to produce more teachers of mathematics prepared to teach in urban school systems, and in heavily minority populated schools. I see two-year colleges identifying potential teachers of mathematics, chemistry, and physics, as well as potential engineers, and scientists, and working hard at strengthening these students, especially in mathematics. I see these students, perhaps during their second year, having joint enrollment in both the two and four-year institutions that are in joint partnership. I see two-year institutions having numerical targets for transferability, and numerical targets for minorities to transfer into teacher education programs in the mathematical sciences, as well as in engineering, and other mathematics-based programs. I see mathematics faculty at two-year colleges asking themselves the question posed by Dr. Tilden Lemelle, the new president of the University of the District of Columbia, in his first address to the faculty: How does what we do prepare our students for living and for making a living? I see mathematics faculty in two-year institutions serving as mentors for students, especially minority students to help them see what is and can be for them – to show them a future in the mathematical sciences. I see mathematics faculty in two-year institutions working closely with school teachers and faculty in four-year institutions to strengthen programs and facilitate student transition. I see mathematics faculty providing good educational advising, serving to create and sustain mathematics clubs, and serving as the core change agents at the two-year institutions.

So then, WHAT MORE CAN TWO-YEAR COLLEGES DO to make mathematics work for minorities? I will recommend ten strategies on what we can do to make mathematics work for minorities:

Strategy I: Shift our paradigm, if necessary, to one which allows us to behave under the belief that all students can and must learn mathematics, and that minorities can succeed in mathematics and mathematics-based fields. Set high expectations for all students and, most of all, make sure that students know these expectations.

Strategy II: Set up articulation and collaborative programs with HBCUs, HSIs, as well as majority institutions, to facilitate smooth transferability. Also, work closely with faculty in schools and four-year institutions to strengthen programs and facilitate student transition. You may want to examine the program, Exploring Transfer, directed by Dr. Janet Lieberman of Fiorello H. LaGuardia Community College, and Dr. Colton Johnson of Vassar College. A collaborative, voluntary program, this program emphasizes experiential education, collaborative structures, and the power of the site.

Strategy III: Make the teaching profession a glamorous and rewarding one – one worthy, of the best students! Identify the best potential teachers among your ranks, provide incentives for them to go into the teaching profession, and provide them with the strongest possible program to prepare them for the four-year institutions. Have your pre-teacher education program designed like those in the connecting universities, and with appropriate support for student success. Develop a mechanism for joint enrollment in both the two-year and the four-year institutions. Bear in mind that today, only 9% of the public senior high school teachers of mathematics are minorities, which is evidence of a wide disparity between the supply of minority mathematics teachers and the proportion of minority students in virtually all states. Also, less than one-half of all public senior high school teachers of mathematics (47% with primary assignment) actually have a college major in mathematics.

Strategy IV: Intensify minority recruitment showing students the advantages of beginning their college work in a two-year institution, and of the ties that the two-year institutions have with four-year institutions. Be sure to let them know about ties to HBCUs and HSIs and majority institutions without steering them into specific schools. HBCUs may be especially receptive to developing strong ties in teacher education in mathematics and science.

Strategy V: Promote and communicate with appropriate minority communities' research on accomplishments of minority mathematicians and scientists who began their college training in two-year institutions. Also, follow your own students, and have successful ones come back to the college and talk to the students.

Strategy VI: Restructure remedial courses for success, incorporating co-operative learning, peer tutoring and computer assisted instruction as supplements to traditional teaching methods. Have experienced, well-motivated, and talented teachers work with students in small classes, and promote group study inside and outside of the classroom setting.

Strategy VII: Set numerical targets for transferability of minority students. Also, prepare this group of students with the capability of meaningful choice for immediate employment in a technological society, bearing in mind that all workers should be prepared to adapt to emerging technology, or prepare them for the continuation of higher education. You might want to examine closely the success of Austin Community College in transferability. Approximately 9 out of every 10 students who transfer from Austin Community College to public colleges and universities are still enrolled a year later (survivability is nearly 40% higher than the statewide average). They claim that transfer success comes from paying attention to curriculum; courses at Austin Community College are similar in content, emphasis, and difficulty to those in neighboring universities. Two-year college faculty have frequent contact with faculty at nearby universities. A program in place for nearly a decade brings Austin Community College onto the University of Texas campus two nights a week to university students. For those students, the university counts hours of enrollment at Austin Community College toward a student's minimum full-time enrollment obligation at the university.

Strategy VIII: Establish partnerships with industry, and give all students an opportunity to benefit from programs developed under these partnerships. In response to an industrial need, Seattle Central Community College developed a two-year biotech program. Together, industry leaders and the college faculty planned the course of study. Industry leaders also sat on the community college advisory boards and gave instruments and equipment to the college. College President, Dr. Charles Mitchell, said that it is hard for two-year colleges to keep up with advancing technology in isolation; hence, it is necessary to form partnerships with industry.

Strategy IX: Seek financial and human resources from government, especially agencies that are in need of mathematicians and scientists and those charged with improving mathematics and science education. Develop programs for increasing minority participation in science and mathematics, such as mentoring and career awareness programs with agencies such as The National Science Foundation, Department of Education, and National Security Agency.

Strategy X: Promote the teaching and learning function in mathematics. According to the Mathematical Association of America's document, A Call for Change, "to adequately prepare students for the 21st Century, the nation's mathematics educators must create classrooms that recognize students and teachers as thinkers, doers, investigators, and problem solvers." Walter Massey tells us, "The most important factor listed by minority students at successful institutions was a supportive environment; the presence of mentors, study groups, science and mathematics clubs, good advising and remedial courses when needed."

Conclusion

The role of two-year institutions to increase minority participation in the mathematical sciences cannot be overstated. This group of institutions is serving, and will continue to serve, a majority of the minority student population in college. You are uniquely positioned to determine, to a large degree, the success that we as a nation will have in increasing minority representation in the mathematical sciences, and for preparing minorities to make a productive life for themselves in the 21st Century. Two-year colleges have a wonderful opportunity to provide our country with a second wave of students who will become prepared in the mathematical sciences and in education to take on the many jobs that will require mathematics and/or their ability to use current and emerging technology. It certainly is clear that any effort to recruit more majors in mathematics-based fields, to strengthen the undergraduate major in mathematics-based fields, and to prepare students for life, and for making a living, must be carried out in a manner that includes two-year colleges as a full partner. It is no longer a matter of good-will to merely allow for the emergence of talents from the growing population of minorities, but rather, it is in our best interest for two-year institutions to actively encourage and sustain such an emergence for the survivability of our country, and of our way of life. The future rests with you. Thank you.

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President George Bush, "the Education President," set a goal of leading the world in mathematics and science education by the year 2000. Incredibly, we are closer to that goal now than we were a few years ago. It is not, however, due to any improvement in this country, but rather to the deterioration of education in many others.

Peter Hilton

(The remarks by Peter Hilton found at various places in this issue are taken from his keynote address at the Region II conference of NYSMATYC, October 17, 1992, in Binghamton NY.)

SPECIAL FEATURE

Two-Year College Mathematics Pioneers:
Allyn Washington

In what we hope will be a continuing series, we will be publishing interviews with people who have had a major impact on two-year college mathematics. Readers are invited to make suggestions on who might be included in the future. We start the series with Allyn Washington, author of the very successful technical mathematics books used all over North America and in some other parts of the world. After a full career, Al "retired" to the mountains of California. This interview took place at the 25th Anniversary Conference of NYSMATYC on April 4, 1992.

The AMATYC Review: Let's start with a few basics – where you were born, where you grew up, and where some mathematical interests showed up in your early years.

Allyn Washington: I am a Connecticut Yankee, born and brought up in Connecticut and lived there until I was about 27. My first interest in math started in high school with my mathematics teacher, who was also the principal. He did all kinds of things, and he taught the math.

AR: Not a very big school?

High school graduation, 1948.

AW: Well, we had a graduating class of 59 people in 1948. Then I went to Trinity College in Hartford, which was nearby. I could afford it because I commuted. There was a real great guy there in the Mathematics Department, too. I also had a physics professor who became my advisor. They didn't have very many physics majors, so

I became an advisee of this physics professor. He pushed me into physics, so I graduated with a dual major in math and physics.

AR: What was it about those two professors that made them stand out?

AW: They were both just so good at how they explained, at how they presented the material. I don't know, it just got to me. I had an interest in math by this time, anyway. I happened to get this one fellow for most of my math courses. What he did, it just fit me, that's all I can tell you. And the physics, well, I really had not intended to major in physics. But it turned out I got my masters degree in physics. It's not in math. I took several math courses in it because of the relationship to the field.

AR: Where did you get that masters degree?

AW: At Brown University.

AR: You didn't go very far from home, then.

AW: No. Well, I was away from home then; that was over a hundred miles from Hartford to Providence. There was a slight interruption for the Korean War. I was Federalized in the National Guards, so I lost a year between my sophomore and junior years.

AR: Did you go into college teaching straight out of grad school?

AW: I finished up my masters degree and Trinity College, where I graduated from college, had an opening. In the mid-fifties math and physics teachers were few and far between. You could just about name a spot and go to it. I had intended to go on for a Ph.D., but they said "Hey, would you like to try teaching for a year or two to see how you like it." They came after me; I didn't even seek after a teaching job.

AR: Times have certainly changed. You didn't know at that point that you wanted to go into teaching?

AW: Well, I considered it, but I hadn't made that decision. I thought this would be a great chance to find out about teaching, so I went for 2 years. I taught both math and physics at Trinity College in Hartford. Then I met another guy, we were both bachelors, and he taught chemistry. He came up to me one day and said "Lets try teaching out West." This was absolutely on a lark. So he and I wrote to various schools seeing if he could get a job teaching chemistry and I math and/or physics. We both got jobs in Boise, Idaho. So the following year that's where I was, in Boise, Idaho.

I met a gal teaching chemistry there who is now my wife. They didn't pay too well in Idaho back then; I guess they thought they were paying you with scenery. Based on the salary there I could not even cover the basics if I were to marry her, and she wouldn't be able to continue to work there because of nepotism rules. By the way, it was made perfectly clear that *she* would be the one who would not be able to stay at the college. That's an area in which things have changed for the better. So I wrote a bunch of letters around, mostly to California, and one to a college that was just getting started the following year in New York. For various reasons, mostly because I could start with the college, I thought that would be a great place to go. That's how I got to Dutchess [Community College in Poughkeepsie, NY].

AR: So you were there to help create a new college?

AW: I was there, Day One, at Dutchess in 1958. It was quite an experience, a very useful experience.

AR: Were you explicitly seeking a two-year college position?

AW: Well, the Boise, Idaho, Junior College where we happened to find a job together, that's how I got into two-year college. And I liked that. I liked the emphasis on teaching. Trinity was a little of both, emphasis on teaching and advancement and research. What I liked best was the teaching aspect of it and that's why we got the job at Boise together. That's when I decided on being a teacher. I did intend to finish my Ph.D. in math rather than in physics, and in 1961 I filled out the applications for some schools. I also started writing a book.

The Book

AR: Why write a book?

AW: I just thought I could possibly serve the students I had better than the books available for the tech math courses I was teaching.

AR: Were there tech math books at that time?

AW: Yes, there were. There was Rice and Knight, which was probably the best one, although there were others.

AR: What sorts of things did you think you could do better?

AW: Organization. Fit the curriculum we had at Dutchess better. The electronics people needed topics at specific times. I was teaching tech math as they had set it up, and in the second year we covered complex numbers. The students in the second half of the first year were studying AC circuits in their electronics courses. They were getting the math after their need for it.

AR: So they were having their electronics instructor teach some of their math.

AW: Right, to fill in the gaps. Electronics had the greatest need for timing. But we were also teaching vectors in the second semester while the physics teacher had been using vectors all year. What I did was organize the book so the course flowed, rather than taking a book and hop, skip and jumping around in it.

AR: So it was not a great, magnificent thing you were trying to do; it was more mundane.

AW: It was just a pragmatic thing. I was going to create something my students could use. After I wrote it I sent it to a publisher and said "Do you like it?" They said it looked pretty good and it went from there.

AR: It wasn't hard to sell, then?

AW: Not really. I wrote to Addison-Wesley because they didn't have a book of that kind. They said they had been thinking about it (whether they really had been, I don't know). They reviewed it, and the reviewers seemed to think there would be a possible market for it. I figured it would be easier for them to publish it than for me to have to write up and print all the notes. I was using notes at this particular point. They got the book done and it took off.

AR: Was it used mostly at two-year schools?

AW: Well, the basic tech math I wrote initially was primarily used in two-year schools. There was some use in high schools, but not too much. I wrote a calculus book off the calculus version that's used mostly in four-year schools, in the 4 year technical programs.

From book jacket, 1964

AR: My own first experience with one of your books was while teaching technical calculus at Oklahoma State.

AW: That's the kind of school that picked up the calculus book, but the basic tech math with calculus is mostly an algebra-trigonometry-geometry book. It's primarily used in two-year schools. I'd say that over half the students using it are in electronics, or at least something related to electronics. Also, just as a sidelight, Allyn and Bacon reviewed it in the middle 70's and found it to be the fourth leading book used for regular algebra-trig courses at that time.

AR : What do you think led to much of the rise to the two-year college? I believe that, time-wise, it coincided with much of what you did.

AW: That's hard for me to see. It's true I got into two-year college teaching just about when the two-year c · ege seemed to take off. But one emphasis hit in 1957 while I was in Boise, Sputnik! The Russians were all of a sudden ahead of us in technology. Come on now, no way should that happen. That was one huge influence on American education.

AR: But if I think of tech math, and then I think of the math which came as a result of Sputnik, the "new math," I don't believe I'm thinking of the same thing at all.

AW: I never did incorporate the new math into my books, as such. I kept to the more traditional developments of the material. We used the new approach in some of our other courses at Dutchess, but not in the tech math.

AR: How about the presentation of the calculus? The tech math treatment is a little different, especially in the sense of rigor. How much of that did you pioneer?

AW: Well, all I did there was to put calculus in my book the way I remembered *learning* it. I'm not sure what was in the textbook, but I don't remember learning epsilondelta proofs, so I didn't put them in. What I remember learning about the Fundamental Theorem of Calculus takes half of a page, so zing-zing-bing, there it is.

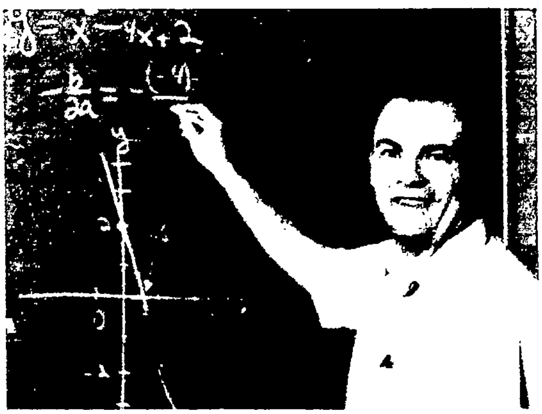
AR: A lot of trusting?

AW: Some trusting, but there was logic involved. There just wasn't formal proof. A few verbal statements with a little bit of algebra, and all of a sudden you've got the Fundamental Theorem. What bothered me was one of the calculus books of the time which took about 100 pages to get it. The calculus book I did took about a half of page. I think the way to do it is not necessarily rigorously, but hopefully to say things understandably rather than do it all symbolically. I don't think students, when they first see these ideas, follow the symbolism that well. I think they can sometimes just follow statements, a simple presentation. You might say its more the calculus of the forties than of the fifties and sixties, far more intuitive.

AR: In your own opinion, what do you think you added that made a difference, that is, made your book so successful ?

AW: I think the fact that it did *fit* technical programs made it take off in technical institutions. My first big adoptions were in the Carolinas, a whole series of institutes there. Also, the midwest had a lot of technical colleges. These were the leading adoptions, initially. It also fit the kind of background that a lot of people teaching it had. They hadn't always gone through a rigorous mathematics background themselves. They could read my book and understand it. I was just trying to keep it down to earth the best I could.

AR: How big did it get?



From Benjamin/Cummings brochure, 1982.

AW: Well, the tech math market today is down considerably from what it was. Ten years ago it was at its peak, in the early eighties. From what I have been led to believe through the publisher, a book that has ten percent of the market is doing pretty well, twenty percent is doing very well. For many years, during most of the eighties, I had anywhere from 50 to 80 percent of the market, something in that order. I don't know what it is now.

AR: How many different versions have you had to prepare?

AW: There is a metric version which is used a lot in Canada; all units are metric in that. The ones used here have a mix of metric units plus standard British units. Britain technically is on the metric system, only a lot of people still use English measure. A fellow from Peru translated the 3rd edition into Spanish in the middle eighties and it sold a fairly good amount, 10,000 plus, in South America. Total sales of the tech math books is over a million.

AR: Did work on the book ever become overwhelming?

AW: Yes. In the late seventies I was teaching full time, and the books were more than a full time job themselves at that time. Things kind of got out of hand. I took two leaves of absence from the college, without pay, just to finish off production of the books. I had a medical problem, also, a heart attack in 1978. At that point I said "Something's got to go, and I don't want it to be me!" Books are far more flexible as far as what you do, how you do it, when you do it, that kind of thing. That's why I gave up teaching. I loved teaching, I always did. But books become kind of dominant. Unless you've done it, unless you've gone through it, you can't know how much. Some of these [production schedules] can be horrendous. You can spend all day, I'm talking about from 8 o'clock in the morning to 10 o'clock at night, I mean every day of the year, on these things. The proof reading, the checking, it just gets completely out of hand, time-wise.

AR: Some authors basically start up a small business getting other people to do portions of the work.

AW: Oh, I've got people I call on, I count on, who assist me. I go through this routine, though I'm kind of demanding on myself. I've got to have a good solid check. I want them to check it and I want it double checked, but I still do things myself. I'd hate to have a math book out there with many errors in it. They do creep in, through, despite everything.

Professional Organizations

AR: Let's shift a little bit. With respect to professional organizations, I know you got into NYSMATYC right at the beginning [1967]. What were you in before that, and what spurred you to get involved in a two-year college organization?

AW: Well, my professional organization prior to NYSMATYC was MAA. I've been a member since 1958. But, as people have alluded to frequently, they didn't fit particularly well with what I was doing. They were into high-powered math with an occasional remark regarding how it was taught, but the emphasis was almost completely on topics that very few people except the presenters understood. They had no feel, at the time, for the problem of teaching at a four-year school, let alone a two-year school. There just wasn't any teaching in it at all. And my thoughts were on the teaching ideas. Herb Gross was the instigator of NYSMATYC. We kept trying joint meetings with AMTNYS [Association of Mathematics Teachers of New York State], but AMTNYS was also not geared for the two-year school.

So, some us got together to see if we could find others with problems like our own. Then we could ask each other how you handle this and how you deal with that—get some discussion going among ourselves. I thought NYSMATYC was a great idea. Herb Gross, who was also instrumental in starting AMATYC, was always saying "Let's go national, let's go national," and we started to go national in 1973.

AR: What was your role in that?

AW: The very first meeting of AMATYC was organized in 1973. The people from *The MATYC Journal* [now *Mathematics and Computer Education*] announced a meeting of those interested in supporting a national group in the Fall of 1973. We had a NYSMATYC meeting scheduled in the Spring and we felt it would be great for two-year people who want to go to NYSMATYC to also go to AMATYC. But we didn't like the idea that they would be having a meeting, we would be having a meeting, and we would probably take away from each other and the attendance would not be there. So we tried to come up with a joint meeting. That's what finally happened in the Spring of 1974. AMATYC had its organizational group get together, the NYSMATYC conference followed, and then we had a banquet on Saturday night. The first part, Thursday and Friday I think, was the beginnings of AMATYC.

AR: What happened at the initial meeting of AMATYC?

AW: Well, really it was kind of a foregone conclusion that we would form an organization. There were probably 25 or 30 people there for the organizational meeting itself. At that point we pretty well concluded it would happen and we got ourselves together and did it. But there had been a lot of talk about it and some preliminary work done before that to even get a meeting together. It had been discussed for several years before that happened.

AR: You were the president-elect for NYSMATYC at the time, which put you in charge of its conference. Did you have to organize the AMATYC conference, too?

AW: We didn't have to plan the AMATYC meeting, the people at *The MATYC Journal* were involved for the AMATYC part of it, what became AMATYC. We had to coordinate the thing, make the arrangements, that was our part of the job. Mike Steuer, who was at Nassau [Community College], was very helpful. He was kind of a liaison between the groups and the hotel. He ran into New York City making arrangements and he did a great job; he was a savior.

Prior to that you could go out and talk about forming a national organization, but it was *The MATYC Journal* and Herb Gross, first president of NYSMATYC, who made it happen. In fact, *The MATYC Journal* had been *The NYSMATYC Journal* back about 1970. It grew out of the original NYSMATYC newsletter into a national journal. It was attached to AMATYC for a while, and then became independent.

Life After Mathematics

AR: What do you do now besides mathematics?

AW: Play bridge. My wife and I are duplicate bridge players. It was a very successful year in 1991. We go to tournaments of various types around the country. We actually had what is called a regional win. We were first overall in a regional event, which is the second highest rated event in the country. Not too many people do that.

AR: That's great!

AW: Of course we also have lousy games, but we had a good game those days. That would probably be my primary activity other than traveling. We've been to Alaska and Europe. I recommend both very highly.

Al with wife, Millie, 1991.

AR: And the business part of your life remains the book; you're still working on it?

AW: Oh, yes, that's still it. I mean I'm only between editions, now. I can see signs that things are picking up pretty fast.

AR: Do you ever do an adjunct semester?

AW: I have. A few years back at Lake Tahoe I taught summer school and a fall semester. I haven't done that for a few years now. I like teaching; I wish I could do more of it, but right now the nearest college is about 45 miles away. Also, it ties one down. Even if you teach one course, you are tied up for the entire semester. I like to travel and would hate to commit myself to a course and then say "Hey, I'm not going to be here for the next two weeks."

AR: Goals for the future?

AW: Goals for the future? I haven't given too much thought to the future. I'm certainly not going to start any new books! hope to continue on with my old book writing for the forseeable future. I have no idea what comes up after that.

AR: Can you get to the organizational meetings in California?

AW: I normally plan to go in northern California. I think I've attended all but one of those since I've been out there, and I've attended the majority of AMATYC conventions. And NYSMATYC, I get to about half of them. I kind of combine those with trips to see my grandson if it works.

AR: I imagine that helps sell the idea to the family.

AW: Well, they get a business trip to endure, as well. I just wish we could get away for the heck of it. I wish we could be not so budget-oriented. I know that's a necessity of life, but it does hinder. I have a daughter who teaches music, and she's worried every day she's going to lose her job. They keep cutting programs, such things as music, and I think they are very important in an education as well. There is a lot more to education than math. In college I took those courses I had to, that were forced on me, but what did I end up doing? Writing. You just never know. Education is so incomplete if you don't know enough about geography or history. I'd like to see people come out of college with a better rounded education.

AR: Now this sounds a little strange from a person whose livelihood is technology. You're talking about a liberal arts education.

AW: That's right! In my own college career I missed getting a minor in social science by one course. I had taken psychology, philosophy, government courses, as well as a couple years of Spanish. There's more to life than math, no two ways about it.

MATHEMATICAL EXPOSITION

Using Linear Programming to Obtain a Minimum Cost Balanced Organic Fertilizer Mix

by

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Introduction

The demand for organic produce is well known, and an increasing number of home gardeners are turning to organic fertilizers to deliver the required nutrients to their plants. Until recently, the home gardener had to create her own organic fertilizers by combining manures (green and otherwise) with sand, clay, loam, table scraps and mineral sources such as greensand, black rock phosphate, and colloidal phosphate. Such mixtures build on three key elements which are essential to plant health and growth. The first of these is nitrogen (N) which promotes plant growth and resistance to insects. The second element is phosphorus (P) which is essential for proper root growth. The third element is potassium (K) which also aids in plant growth and resistance to disease. These three elements are so important that the percent of each present in a fertilizer is always listed in the order N-P-K. Thus, a 3-2-2 fertilizer is 3% nitrogen, 2% phosphorus, and 2% potassium, whereas a 5-2-4 fertilizer is 5% nitrogen, 2% phosphorus, and 4% potassium.

Recently, organic gardeners have been able to obtain pre-mixed organic fertilizers at many garden centers and from most seed catalogues. However, if a gardener wants to combine different fertilizers in order to produce a mixture with a specific N-P-K combination at minimum cost, then she must solve a particular type of optimization problem. One of the most popular mixtures used by nonorganic farmers is 10-10-10, which is called a "balanced" fertilizer since it contains equal percents of the three nutrients. When no such balanced organic fertilizer is offered by a supplier, then the buyer who wants to use one is faced with the problem of whether or not a balanced fertilizer can be created by combining a subset of the set of available unbalanced fertilizers. The 1992 Fedco Seeds catalogue[1], provides the following relevant data:

Blood Meal	(11 - 0 - 0)	@ \$6.00 per five pound bag;
Bone Meal	(6 -12 - 0)	@ \$5.00 per five pound bag;
Greensand	(0 - 1 - 7)	@ \$3.00 per five pound bag;
Sul-Po-Mag	(0 - 0 -22)	@ \$2.80 per five pound bag;
Sustane	(5 - 2 - 4)	@ \$4.25 per five pound bag.

In this case the organic gardener who desires a balanced fertilizer must use a combination of the above choices. Clearly, she would like to obtain that combination that satisfies the requirements at a minimum per pound cost.

Linear programming can be used to model the general problem. Assume there are n fertilizers to choose from and let

x_i represent the number of pounds of fertilizer i used in the mix;
N_i represent the percent of nitrogen in fertilizer i as a decimal;
P_i represent the percent of phosphorus in fertilizer i as a decimal;
K_i represent the percent of potassium in fertilizer i as a decimal;
c_i represent the cost per pound of fertilizer i,

for $i = 1, 2, 3, ..., n$.

The problem is to minimize the linear cost function

$$\sum_{i=1}^{n} c_i x_i$$

subject to the condition that the percents of N, P, and K in the mixture are equal. Letting the weight of the mixture be denoted by W, this condition is equivalent to

$$\sum_{i=1}^{n} [(N_i)(x_i)/W] = \sum_{i=1}^{n} [(P_i)(x_i)/W] = \sum_{i=1}^{n} [(K_i)(x_i)/W].$$

Thus, we must have,

$$\sum_{i=1}^{n} [(N_i)(x_i)] = \sum_{i=1}^{n} [(P_i)(x_i)] = \sum_{i=1}^{n} [(K_i)(x_i)].$$

The last condition can be forced by creating three linear equations as shown below, wherein all right-hand-side values are the same positive parameter, a.

$$\sum_{i=1}^{n} [(N_i)(x_i)] = a,$$

$$\sum_{i=1}^{n} [(P_i)(x_i)] = a,$$

$$\sum_{i=1}^{n} [(K_i)(x_i)] = a.$$

If a balanced fertilizer can be created from the available set of unbalanced fertilizers, then the above linear programming problem will have a solution, and the solution will represent the proportion of each fertilizer to be used in the least cost balanced fertilizer mix.

Let us return to the Fedco data as an example. Let

Blood	represent the number of pounds of blood meal;
Bone	represent the number of pounds of bone meal;
Green	represent the number of pounds of greensand;
SPM	represent the number of pounds of sul-po-mag;
Sustane	represent the number of pounds of sustane

to be used in the mix.

The problem is to minimize the objective function

$$\$1.20\ \text{Blood} + \$1.00\ \text{Bone} + \$0.60\ \text{Green} + \$0.56\ \text{SPM} + \$0.85\ \text{Sustane}$$

subject to the linear constraints

$$.11\ \text{Blood} + .06\ \text{Bone} + .00\ \text{Green} + .00\ \text{SPM} + .05\ \text{Sustane} = 1,$$
$$.00\ \text{Blood} + .12\ \text{Bone} + .01\ \text{Green} + .00\ \text{SPM} + .02\ \text{Sustane} = 1,$$
$$.00\ \text{Blood} + .00\ \text{Bone} + .07\ \text{Green} + .22\ \text{SPM} + .04\ \text{Sustane} = 1.$$

As we will see later, 1 is used as the common right hand side value, without loss in generality, in order to make subsequent calculations easier. Finally, we need to add the non-negativity constraints

$$\text{Blood} >= 0,\ \text{Bone} >= 0,\ \text{Green} >= 0,\ \text{SPM} >= 0, \text{and Sustane} >= 0,$$

in order to rule out a nonsensical solution.

The following solution to the Fedco problem is easy to obtain using a standard linear programming package:

Blood = 4.5454,
Bone = 8.3333,
Green = 0.0000,
SPM = 4.5454,
and Sustane = 0.0000.

This mixture would cost \$16.33 (rounded to the nearest cent), and since the mixture weighs 17.4242 pounds, the minimum cost per pound is 94 cents (rounded to the nearest cent). Thus, a minimum cost balanced fertilizer can be obtained by mixing 6 parts blood meal with 11 parts bone meal and 6 parts sul-po-mag.

There is one remaining issue to be resolved. The application rate is the recommended amount of fertilizer to be applied per square unit of garden. The application rate depends on the percent of each nutrient present in the fertilizer. Note that the above procedure allows for the determination of the proportion of each fertilizer to be used to obtain a balanced fertilizer, but the resulting percent of each nutrient present in the mix must be calculated. That is, one of the expressions,

$$\sum_{i=1}^{n} [(N_i)(x_i)/W],$$

$$\sum_{i=1}^{n} [(P_i)(x_i)/W],$$

$$\sum_{i=1}^{n} [(K_i)(x_i)/W],$$

must be evaluated. The choice of the constant 1 as the common right-hand-side value allows this evaluation to be accomplished by obtaining the reciprocal of W where

$$W = \sum_{i=1}^{n} x_i .$$

From the Fedco solution, we have $W = 17.4242$ (rounded to four places) giving the recriprocal value of .0574. Therefore, 6 parts blood meal mixed with 11 parts bone meal and 6 parts sul-po-mag results in a balanced 5.74-5.74-5.74 organic fertilizer that costs 94 cents per pound.

Reference

1. *Fedco Seeds 1992*, Fedco Co-op Seed Packers, Waterville, Maine, page 48, 1992.

There is an unfortunate confusion between training and education in the United States. While i·: traffic I observe the widespread need for training in the operation of an automobile, but we offer "Driver Education." In our schools we have dire need for a better educated faculty, but we offer "teacher training."

It is impossible to educate without training, but all too easy to train without educating.

Peter Hilton

Problems Whose Solutions Lie on a Hyperbola

by
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Steven Schwartzman has taught mathematics for 25 years. He recently finished an etymological dictionary of mathematical terms, a project which combined his interests in mathematics and language.

Textbooks that deal with ellipses, parabolas, and hyperbolas typically mention situations or objects that exemplify the three conic sections. For example, non-circular ellipses are found in the orbits of planets, in the shapes of certain gears, in "whispering galleries," and in the appearance of circles seen at an angle. Parabolas are found in the trajectories of many objects launched in a gravitational field, including the orbits of some, but not all, comets; in the reflectors behind headlight and flashlight bulbs, satellite antennas, and certain microphones; and in the main support cables of suspension bridges. When it comes to hyperbolas, however, the only example ordinarily given is LORAN, a navigational system used by ships and aircraft. In this article I will fill that gap by considering some interesting yet simple problems whose solutions are partial or complete hyperbolas. In each case the solution will have an equation of the type

(1) $y = f(x) = \dfrac{kx}{x-c}$, where neither k nor c is 0.

Let us begin, therefore, with a brief discussion of (1), whose graph is given in Figure 1. The graph is drawn with c and k both positive, but either constant may be positive or negative.

The domain of the function is $x \neq c$, because when $x = c$ the denominator of the fraction is 0. When x is very close to c, the absolute value of the function becomes quite large, so the line $x = c$ is a vertical asymptote. As for the horizontal dimension, when x has a large absolute value, $\lim\limits_{x\to\infty} \dfrac{kx}{x-c} = k$, and $\lim\limits_{x\to-\infty} \dfrac{kx}{x-c} = k$ too, so the line $y = k$ is a horizontal asymptote. Since the numerator of $f(x)$ is 0 if and only if $x = 0$, the origin is the only x- or y-intercept. The graph is a hyperbola whose center is located at (c, k), the point where the two asymptotes cross. The transverse axis of the hyperbola, which is inclined at 45° to the x-axis, lies on the line $y = x + (k-c)$.

If c and k happen to be equal, equation (1) becomes

(2) $y = \dfrac{kx}{x-k}$, where $k \neq 0$.

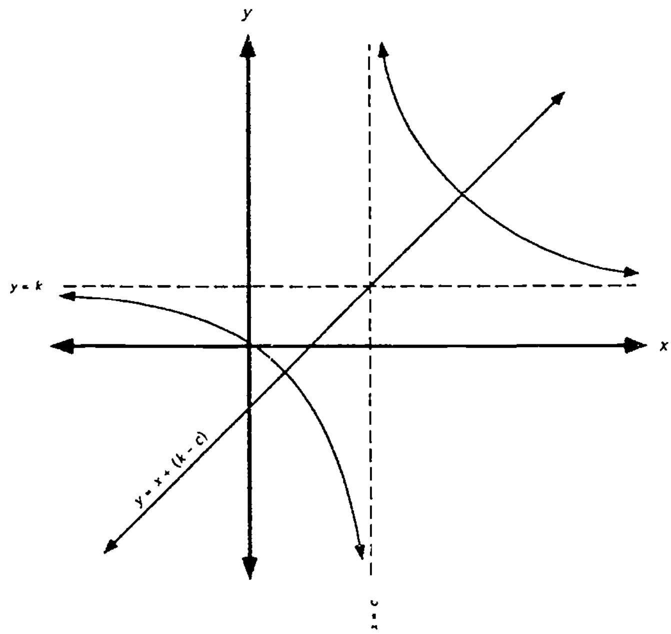

Figure 1. $y = \dfrac{kx}{x-c}$

This second graph is shown in Figure 2. The transverse axis of the hyperbola now lies on the line $y = x$. As a consequence of the curve's symmetry about that line, the function in (2) is its own inverse. That symmetry is manifested algebraically in the fact that

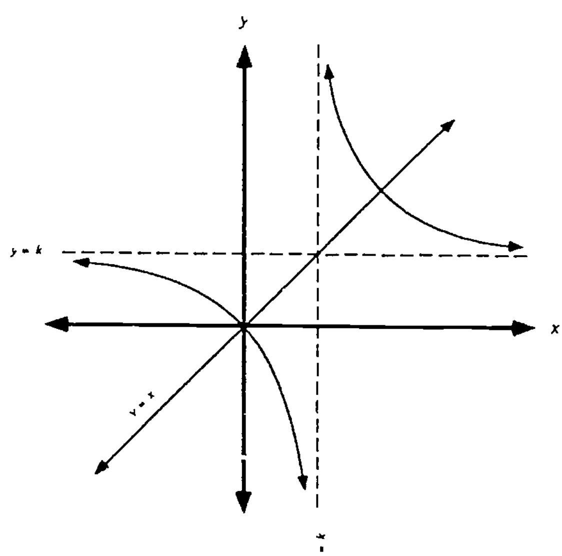

Figure 2. $y = \dfrac{kx}{x-k}$

interchanging x and y in (2) leads to the new equation $x = \frac{ky}{y-k}$, which, when solved for y, is identical to (2).

Now that we are familiar with some properties of graphs of hyperbolas, let us look at a few specific problems.

Percent of Increase and Decrease

This first question has fooled many people over the years, and may always continue to do so:

(3) If the price of an item increases by a certain percent, and then decreases by the *same* percent, how will the final price compare to the original price?

Most people believe that the final price will be the same as the original price, but a simple example disproves that popular misconception. Suppose the price of the item starts out at \$1 and increases 50%: the new price will now be \$1.50. If this new price of \$1.50 then decreases 50%, the item will end up selling for \$.75, which doesn't equal the original price. The reason for the disparity is that although the *percents* of increase and decrease are identical, they apply to different *amounts:* \$1 the first time, but \$1.50 the second time. The only way the final price can end up the same as the original price is if the percent of increase and decrease is 0, because only then will the percent of change be applied to the same amount both times.

Now let's recast the problem slightly:

(4) Suppose an item originally selling for a price p undergoes a change of $x\%$ (where the change might be up or down); what second change of $y\%$ will result in the item's returning to its original price?

If we treat y as a function of x, the domain of the function is $-100 < x < \infty$. The most the original price could decrease is 100%, corresponding to $x = -100$, but then the price would be 0, and no percent of increase could bring it back up. For that reason, we exclude $x = -100$ from the domain. At the other end of the scale, the original price could increase by any percent, no matter how large, and there would always be some subsequent percent of decrease to bring it back down to its original value.

Now let's solve the problem. If the original price p changes by $x\%$, the first amount of change will be $\frac{x}{100}p$ and new price will be $p + \frac{x}{100}p$. That new price in turn will change by $y\%$, so the second amount of change will be $\frac{y}{100}\left(p + \frac{x}{100}p\right)$, and the final price will be $\left(p + \frac{x}{100}p\right) + \frac{y}{100}\left(p + \frac{x}{100}p\right)$. Since the final price should equal the original price,

$$(5) \quad \left(p + \frac{x}{100}p\right) + \frac{y}{100}\left(p + \frac{x}{100}p\right) = p$$

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Notice that p is a factor of every term in the equation. Therefore the original price of the item is irrelevant; only the percents of change are significant. Solving equation (5) for y gives:

$$(6) \quad y = \frac{-100x}{x + 100} .$$

The graph, with some typical solutions labeled, appears in Figure 3. It is one branch of a hyperbola.

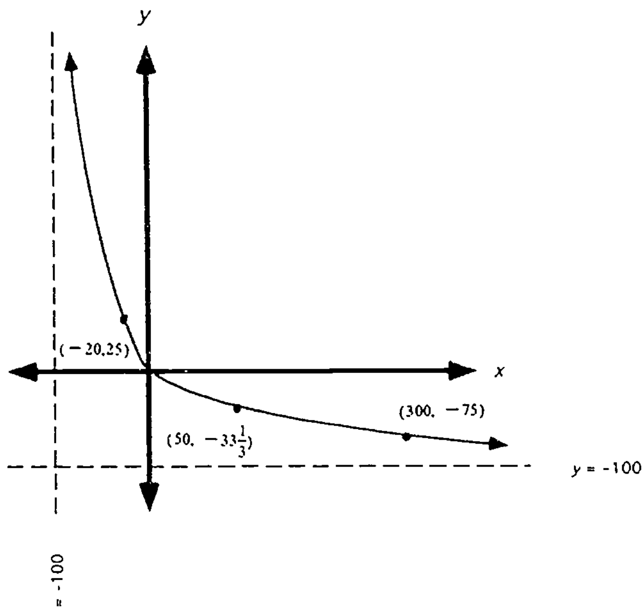

Figure 3. $y = \dfrac{-100x}{x + 100}$

As mentioned earlier, the point (0, 0) represents the only case in which $x = y$. All other points on the graph necessarily lie in the 2nd or 4th quadrants, since if the original price first decreases ($x < 0$), it must then increase ($y > 0$) to return to its original value, and vice versa. As x approaches -100, the first change drops the price of the item close to 0, and a relatively huge percent of increase is required in the second change to bring the price back to its original value. That observation accounts for the vertical asymptote at $x = -100$. On the other hand, if the first change is large ($x \to \infty$), the second change must be close to -100 to reduce the price to its original amount. That observation accounts for the horizontal asymptote at $y = -100$.

Comparing (6) to (2), we see that $k = -100$, and the function in (6) is its own inverse. For example, if a first change of -50% must be followed by a second change of $+100\%$ to get back to the original price, then it is also true that a first change of $+100\%$ must be followed by a second change of -50% to get back to the original price. This relationship in the values of x and y is no more intuitive than is the answer to (3), but it follows from the curve's symmetry about the line $y = x$.

Averages of Averages

A problem that people miss even more than (3) is the following:

(7) If you travel from A to B at an average speed of 20 m.p.h. and return from B to A along the same route at an average speed of 60 m.p.h., what is your average speed for the round trip?

Almost everyone thinks the average speed for the round-trip is 40 m.p.h., but it isn't. To see why the popular answer is wrong, assume the distance between A and B is 120 miles. The trip from A to B would take 6 hours, and the trip back would take 2 hours. The total distance covered would be 240 miles, and the total time spent traveling would be 8 hours, so the average speed for the round trip comes out to 30 m.p.h. If the distance between A and B had been, say, twice as great, all related distances and times would also have been twice as great, and the average speed would have remained the same as before. Consequently, the average speed is independent of the distance between A and B.

In general, let d be the distance between A and B, let r be the average speed from A to B, and let s be the average speed from B back to A. Then $\frac{d}{r}$ is the time spent going to B, and $\frac{d}{s}$ is the time spent returning to A. Since the total distance for the round trip is $2d$, the average speed for the round trip is total distance divided by total time, or $\frac{2d}{\frac{d}{r}+\frac{d}{s}}$. Dividing the numerator and denominator by d, we have:

(8) $$\text{Average speed} = \frac{2}{\frac{1}{r}+\frac{1}{s}}.$$

In an article on rate problems, Lawrence S. Braden (1991) pointed out that the average speed for the round trip is the *harmonic* mean of the two speeds involved, whereas the commonly offered answer is the *arithmetic* mean of the two speeds involved.

Now, as we did with (3), let's rephrase the problem presented in (7):

(9) If you travel from A to B at an average speed of r m.p.h. and return from B to A along the same route at an average speed which is x times as great as the first speed, what is your average speed y for the round trip?

Assuming y is a function of x, the domain of the function is $x > 0$, since the distance traveled is treated as an unsigned number; x can't be 0, because then you would never return to A. Substituting in (8), average speed $= y = \frac{2}{\frac{1}{r}+\frac{1}{xr}}$, which yields

(10) $$y = \left(\frac{2x}{x+1}\right) r.$$

The graph appears in Figure 4. Because the domain is limited to nonnegative numbers, the graph is only one part of one branch of a hyperbola.

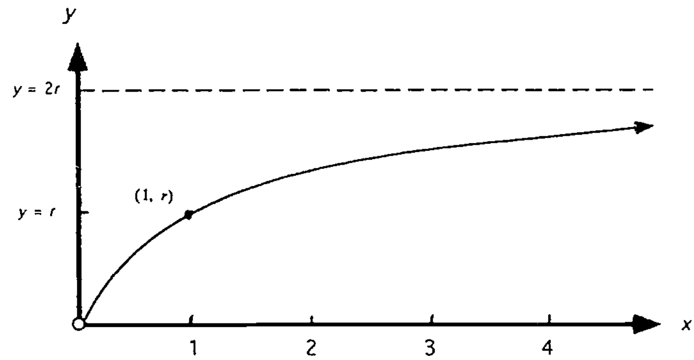

Figure 4. $y = \left(\frac{2x}{x+1}\right)r$

Applying the data from (7) to (10), the average speed returning is 3 times the average speed going, so $x = 3$, and the average speed for the round trip is $\frac{6}{4}$ as great as the average speed going: $\frac{6}{4}$(20 m.p.h.) = 30 m.p.h.

In general, as the average return speed becomes greater and greater relative to a fixed average speed for the first part of the trip, we have:

$$\lim_{x \to \infty} \left(\frac{2x}{x+1}\right) r = 2r .$$

This limit appears geometrically in Figure 4 as the horizontal asymptote $y = 2r$. In other words, no matter how fast your speed on the return trip, the best average speed you can expect for the round trip is not quite twice your speed for the first part of the trip. In light of this fact, another problem posed by Braden (1991) – in which Einstein was given an average speed going of 15 m.p.h. and a round-trip average speed of 30 m.p.h., and was asked to find the average speed returning – clearly has no solution.

The Child is Father to the Man

Daryll Keeling (1992) wrote a letter to the daily Austin *American-Statesman* mentioning that he was born when his father was 20 years old and spuriously claiming that he would catch up to his father by age 35. Nelson C. Haldane (1992) replied with an equally spurious letter saying that the son would actually catch up to his father at age 40. A variant of Haldane's "logic" goes something like this. When the son is 10, the father is 30, or 3 times his age. When the son is 20, the father is 40, or 2 times his age. Every time the son's age increases by 10 years, the ratio of the father's age to the son's age decreases by 1 unit. Therefore, when the son's age increases from 20 to 30, the ratio drops to 1, and the father and son are the same age.

Now let's recast the problem slightly.

(11) A father and son are born on the same date but 20 years apart. When the son is x years old, what fraction y of his father's age will he be?

Treating y as a function of x, the domain may be taken to be $0 \leq x \leq 120$, x an integer. Ages can't be negative, and presumably someone's age at birth is 0. Biologically speaking, there must be an upper limit on the father's age, but it's hard to say what that upper limit is. Let's arbitrarily let the father reach age 140, which means the son will reach age 120. Although we express infants' ages in weeks or months, we rarely use anything but whole numbers of years for adults' ages; so for simplicity let's restrict the domain to integers here.

Since the son's age at any time is x, the father's age at the same time is $x + 20$. The ratio of their ages is given by

(12) $$y = \frac{x}{x + 20}$$

The graph is shown in Figure 5. All the points lie on one branch of a hyperbola. At birth the son is "no fraction" of his father's age. At age 10, the son is $\frac{1}{3}$ of his father's age. At age 20, the son is $\frac{1}{2}$ of his father's age. By age 40, the son has advanced to $\frac{2}{3}$ of his father's age, and by age 100, to $\frac{5}{6}$ of his father's age. Faced with the limits of biology (though not the limits of mathematics), the son won't ever be more than $\frac{6}{7}$ of his father's age. If we could travel to some sort of fantasy world where the son and his father might age forever, the ratio of their ages would approach a limit of 1. That limit is represented geometrically by the horizontal asymptote $y = 1$.

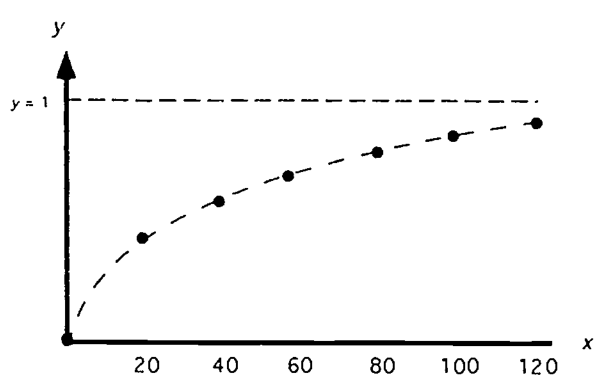

Figure 5. $= \frac{x}{x + 20}$

14.

When Adding is the Same as Multiplying

Some years ago, I wrote an article (Schwartzman, 1987) about solutions of the general equation $[f(x)] \, \Delta \, [g(x)] = [f(x)] * [g(x)]$, where Δ and $*$ are any two different operations chosen from the six basic operations of arithmetic (adding, subtracting, multiplying, dividing, raising to powers, extracting roots). If we restrict ourselves to numbers rather than functions, two cases are within the scope of this article. The first of the two cases is:

(13) Find two numbers x and y such that adding the numbers gives the same result as multiplying them.

The statement of the problem in (13) leads to the equation $x + y = xy$, from which

(14) $$y = \frac{x}{x-1}.$$

Based solely on the wording in (13), there seems to be no reason to exclude any number from the problem, and many people would assume that the domain should be $x \in \Re$. As is obvious from (14), however, the value $x = 1$ is not in the domain of the function. In other words, if the sum and product of two numbers are equal, then neither number can be 1.

The graph appears in Figure 6, and the solutions to (14) form a complete hyperbola. Comparing to (2), we see that $k = 1$ and that this function is its own inverse. The line

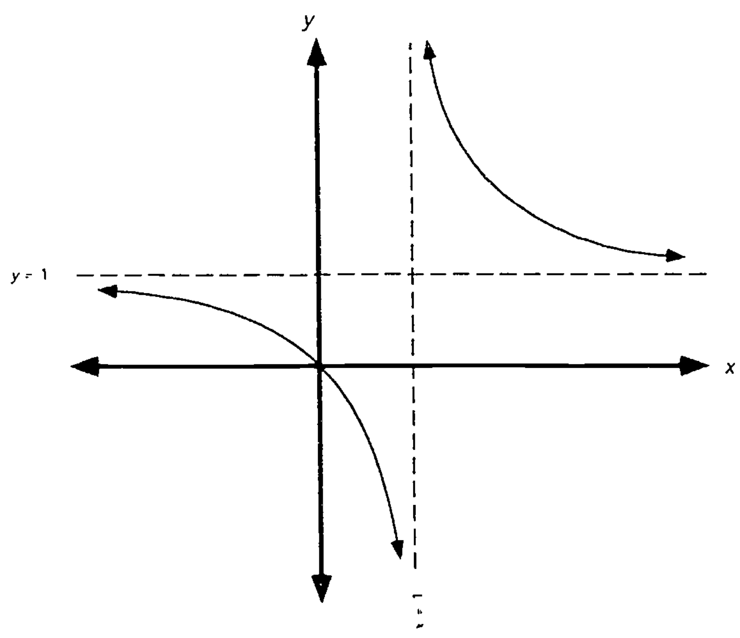

Figure 6. $y = \dfrac{x}{x-1}$

of symmetry $y = x$ intersects the hyperbola at (0,0) and (2, 2), showing that 0 and 2 are the only two instances of a number which can be added to itself or multiplied by itself with the same result. The symmetry of the equation $x + y = xy$ can also be explained by the commutativity of addition and of multiplication.

Making one change in (13), we can pose the following problem:

(15) Find two numbers x and y such that subtracting y from x gives the same result as multiplying y by x.

The statement of the problem leads to the equation $x - y = xy$, from which

(16) $$y = \frac{x}{x+1}.$$

The graph appears in Figure 7, and is once again a complete hyperbola. As with (13), it might at first appear that no number should be excluded from consideration in (15), but equation (16) shows that $x = -1$ must be excluded from the domain of the function.

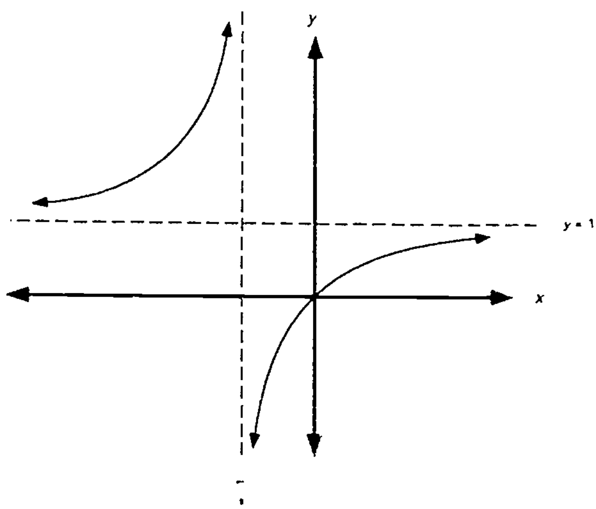

Figure 7. $y = \dfrac{x}{x+1}$

It is interesting to note that the function in (16) can be derived from the function in (14) by replacing every x in (14) with $-x$. That corresponds to the replacement of the word *adding* in (13) by the word *subtracting* in (15), given that subtraction is the opposite of addition.

Conclusion

Although most students probably conceive of hyperbolas as geometric entities, we have seen that hyperbolas can model various algebraic, real-world relationships. The asymptotes to a given hyperbola correspond to limiting or "forbidden" values in the matching real-world situation.

Many people may be further surprised to find out that a single hyperbola can model two very different situations. For instance, if (10) is slightly rewritten, we have

$$\text{(10a)} \quad y = \left(\frac{x}{x+1}\right)(2r).$$

Except for the constant multiplier $2r$, the function in (10a) is the same one that appears in (16), provided that we restrict ourselves to a common domain. Yet who would have expected that a problem involving average speeds has essentially the same solution as a problem in which multiplication is equivalent to subtraction?

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Plato and Spiro Agnew had one (and probably only one) idea in common, that education is a threat to the establishment. The difference is that Plato thought this was good.

Peter Hilton

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The Shape of a Projectile's Path: Explorations with a Computer Algebra System

by

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Brooklyn born and raised, Dr. Robert J. Lopez has a 1970 Purdue Ph.D. in Relativistic Cosmology. At Rose-Hulman Institute of Technology since 1985, he has pioneered the school's foray into the use of computer algebra systems in the classroom, co-authored (with Mel Maron) ***Numerical Analysis: A Practical Approach****, and won the Dean's Outstanding Teacher Award.*

When a projectile is shot into the air, we want to know how far it will go, how high it will rise, and the time until impact. See [1,2,3,6,7]. The formulas involved in the model where air resistance is neglected can be found in almost every calculus textbook. Answers are readily obtained from the direction and magnitude of the initial velocity. With a computer algebra system (CAS), it is easy for students to obtain and explore explicit formulas for the model which assumes air resistance is proportional to the velocity of the projectile. Exploration of these models constitute a computer laboratory exercise and can be implemented with CAS software such as Mathematica or Maple. Alternately, the mathematical analysis can be done with pencil and paper in conjunction with graphics software or a graphics calculator.

First Model: Zero air resistance

First, we assume that the only force is gravity, the starting point is the origin $(x_0, y_0) = (0,0)$, and the initial velocity is $V_0 = (v_x, v_y)$. Applying Newton's law $f = ma$, the differential equations describing the horizontal and vertical motion of the projectile are $m x''(t) = 0$ and $m y''(t) = -mg$, respectively, with the initial conditions $x(0) = 0, y(0) = 0, x'(0) = v_x$ and $y'(0) = v_y$. These differential equations are easy to solve, and their solution is:

(1) $x(t) = v_x t$ and $y(t) = v_y t - \frac{g t^2}{2}$.

Eliminate the parameter t in equations (1) to obtain the relationship between x and y:

(2) $$y = \frac{v_y x}{v_x} - \frac{g x^2}{2 v_x^2}.$$

The range, r, is obtained by setting $y = 0$ in (2) and solving for the non-zero solution x:

(3) $$r = \frac{2 v_x v_y}{g}.$$

If v_0 is the magnitude of V_0, and θ is the angle between the horizontal and V_0, then $v_x = v_0 \cos(\theta)$ and $v_y = v_0 \sin(\theta)$ so that the range is given by:

(4) $$r(v_0, \theta, g) = \frac{2 v_0^2 \sin(\theta) \cos(\theta)}{g} = \frac{v_0^2 \sin(2\theta)}{g}.$$

The angle which gives the maximum range is found by solving $\frac{\partial r}{\partial \theta} = \frac{2 v_0^2 \cos(2\theta)}{g} = 0$, which produces $\theta = \frac{\pi}{4}$; substituting this value into the range formula yields:

(5) $$r_{\max} = r(v_0, \frac{\pi}{4}, g) = \frac{v_0^2}{g}.$$

The maximum altitude occurs where $\frac{dy}{dx} = \frac{v_y}{v_x} - \frac{g x}{v_x^2} = 0$, which yields the point:

(6) $$(x_c, y_c) = \left(\frac{v_x v_y}{g}, \frac{v_y^2}{2g} \right).$$

Second Model: Air resistance proportional to velocity

We again assume that the starting point is the origin $(x_0, y_0) = (0,0)$ and that the initial horizontal and vertical components of velocity are v_x and v_y, respectively. Newton's law states that the sum of all forces acting on the projectile is zero. The vertical components produce the equation $f_1 + f_2 + f_3 = 0$, where $f_1 = ma = m\,y''(t)$, $f_2 = K\,y'(t)$ and $f_3 = mg$ are the forces due to acceleration, friction which is proportional to velocity, and gravity, respectively. The direction of f_2 depends on the sign of $y'(t)$ which is negative when $y(t)$ is increasing and positive when $y(t)$ is decreasing. Hence, the differential equations describing the motion of the projectile are:

15.

(7) $\quad m\,x''(t) + K\,x'(t) = 0$ and $m\,y''(t) + K\,y'(t) + m\,g = 0$,

with initial conditions $x(0) = 0, y(0) = 0, x'(0) = v_x$ and $y'(0) = v_y$. These two second order differential equations have solutions:

(8) $$x(t) = \frac{m\,v_x}{K}\left(1 - e^{-Kt/m}\right)$$

(9) $$y(t) = \frac{m}{K^2}\left(K\,v_y + m\,g\right)\left(1 - e^{-Kt/m}\right) - \frac{m\,g\,t}{K}.$$

From equation (8), $t = \frac{-m}{K}\ln\left|1 - \frac{Kx}{m\,v_x}\right|$ which can be substituted into equation (9) to obtain the relationship between x and y:

(10) $$y = \frac{v_y\,x}{v_x} + \frac{m}{K}\frac{g\,x}{v_x} + \left(\frac{m}{K}\right)^2 g\ln\left|1 - \frac{K}{m}\frac{x}{v_x}\right|.$$

Equation (10) describes the projectile trajectory. Substituting $k = \frac{K}{m}$ in (10) gives

(11) $$y = \frac{v_y\,x}{v_x} + \frac{g\,x}{k\,v_x} + \frac{g\ln\left|1 - \frac{k\,x}{v_x}\right|}{k^2}.$$

Enter this last formula into Mathematica with the command:

y[x_] = Vy x/Vx + g x/ (k Vx) + g Log[1 - k x/Vx] /k^2

The computer response is:

$$\frac{Vy\,x}{Vx} + \frac{g\,x}{Vx\,k} + \frac{g\,\mathrm{Log}\left[1 - \frac{k\,x}{V_x}\right]}{k^2}.$$

To have Mathematica find the limit as $k \to 0$, enter the command:

Limit [y [x] , k -> 0] / / Expand which gives:

(12) $$\frac{Vy\,x}{Vx} - \frac{g\,x^2}{2\,Vx^2}.$$

Notice that expression (12) agrees with equation (2). This shows, not surprisingly, that the model without air resistance is a limiting case of the model with air resistance.

The two models can be further investigated by expanding formula (11) in a series about $x = 0$, the first six terms are:

S[x_] = Series [y [x], {x, 0, 6}] //Normal

$$(13)\quad \frac{Vy\,x}{Vx} - \frac{g x^2}{2\,Vx^2} - \frac{g k x^3}{3\,Vx^3} - \frac{g k^2 x^4}{4\,Vx^4} - \frac{g k^3 x^5}{5\,Vx^5} - \frac{g k^4 x^6}{6\,Vx^6} - \ldots$$

Letting $k \to 0$ in (13), all but the first two terms go to zero, and the result is the parabolic model (2). The range is obtained by setting $y = 0$ in (13), and solving for x. Truncating this series and applying a root finding technique for polynomials gives only a coarse approximation for the range. In Example 1 we use the root finding procedure in Mathematica to solve $y = 0$ directly from equation (11) and obtain an accurate numerical approximation for the range.

Next, consider the maximum height of the curve. This is determined by solving $y'(x) = 0$ for x_c. The derivative is obtained simply by typing **y′** like **[x]** , and the result is:

$$(14)\quad \frac{Vy}{Vx} + \frac{g}{Vx\,k} - \frac{g}{Vx\,k\left(1 - \frac{k\,x}{Vx}\right)}.$$

Then the point $(x_c, y_c) = (x_c, y(x_c))$ is calculated symbolically by the commands:

```
xc = x/. Simplify[Solve[y' [x] = =0, x] [[1]]];
yc = y[xc];
{xc,yc}
```

The result is:

$$(15)\quad \left\{\frac{Vx\,Vy}{g + Vy\,k}, \frac{Vy^2}{g + Vy\,k} + \frac{Vy\,g}{k\,(g + Vy\,k)} + \frac{g\,\mathrm{Log}\left[1 - \frac{Vy\,k}{g + Vy\,k}\right]}{k^2}\right\}.$$

Again, we find the limit $k \to 0$ as in this model and obtain:

```
Limit [{1xc,yc},k->0]
```

$$(16)\quad \left\{\frac{Vx\,Vy}{g}, \frac{Vy^2}{2g}\right\}.$$

The point in (16) is the same as the one in equation (6) for zero air resistance. Again, the limiting case for the model with air resistance is the zero air resistance case.

The range is defined by setting $y = 0$ in equation (11) and solving for x. This is a nonlinear equation that is best investigated numerically, using Mathe-matica's root finding procedure. We examine several representative cases of the behaviors of the two models to get insight into the result of solving (11) for x when $y = 0$. The choice of parameter $k = 0.12$ produced a graph that is representative of the behavior of the model.

Example 1. Compare the two models using $g = 32$, $v_0 = 160$, $\theta = \frac{\pi}{4}$ and $k = 0.12$ where the components of $\mathbf{V}_0$ are $v_x = 160\cos\left(\frac{\pi}{4}\right) = 80\sqrt{2}$ and $v_y = 160\sin\left(\frac{\pi}{4}\right) = 80\sqrt{2}$.

Solution: The Mathematica function for (11) that involves the parameters v_x, v_y, g, and k is formed by typing Y [x _, Vx _, Vy _, g _, k _] = y[x] and for the given values of the parameters, we obtain $y = f_1(x)$, where:

f1[x_] = Y[x, 160Cos[Pi/4], 160Sin[Pi/4], 32, 12/100]

$$(16)\quad x + \frac{5\,\mathrm{Sqrt}[2]x}{3} + \frac{20000\,\mathrm{Log}\left[1 - \frac{3x}{125\,2^{9/2}}\right]}{9}.$$

If $v_x = 80\sqrt{2}$, $v_y = 80\sqrt{2}$, and $g = 32$ are fixed and $k \to 0$ in equation (11), the limiting case is obtained by typing:

f0[x_] = Limit[Y[x,160Cos[Pi/4],160Sin[Pi/4],32,k],k->0]

This gives

$$(17)\quad x - \frac{x^2}{800}.$$

This result is not surprising, because it was anticipated by the limit in (12). We issue the command s[x _] = **Series[f1[x],{x,0,5}]//Normal** and obtain the first five terms of the Taylor series for (16):

$$(18)\quad x - \frac{x^2}{800} - \frac{x^3}{3125\,2^{17/2}} - \frac{9x^4}{12800000000} - \frac{27x^5}{244140625\,2^{35/2}}$$

Since the coefficients of x^n for $n \geq 3$ in this series are negative, the graph $y = f_1(x)$ will lie below the graph $y = f_0(x)$.

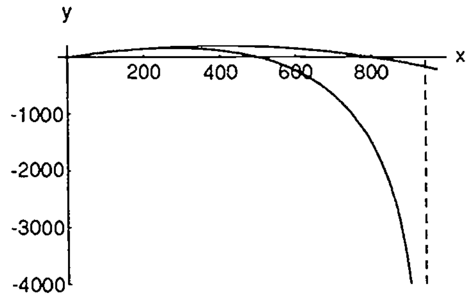
Further analysis of $f_1(x)$ reveals that the domain is $x \leq \frac{v_x}{k} = \frac{125\,2^{9/2}}{3}$ because the term $\ln\left|1 - \frac{kx}{v_x}\right| = \ln\left|1 - \frac{3x}{125\,2^{9/2}}\right|$ has a vertical asymptote at $x = \frac{125\,2^{9/2}}{3} \approx$ 942.809 which forces $f_1(x) \to -\infty$ as $x \to \frac{125\,2^{9/2}}{3}$. This should be contrasted with the parabola $y = f_0(x) = x - \frac{x^2}{800}$ which does not descend as fast as the curve $y = f_1(x)$. The two curves are shown in Figure 1. In Figure 2 the restrictions $0 \leq f_0(x)$ and $0 \leq f_1(x)$ are imposed on the curves of Figure 1 because they are trajectories of projectiles.

The numerical approximation for the range for the path $y = f_1(x)$ can be found by issuing the command:

FindRoot[Y[x,160Cos[Pi/4],160Sin[Pi/4],32,0.12] = = 0,{x,500}]

{ x -> 499.471 }

This value 499.471 is considerably less than in the zero air resistance case where the range is 800. Next, we substitute into the formulas for the point (x_c, y_c) in (15), and find the point on this curve $y = f_1(x)$ where the altitude is maximum:



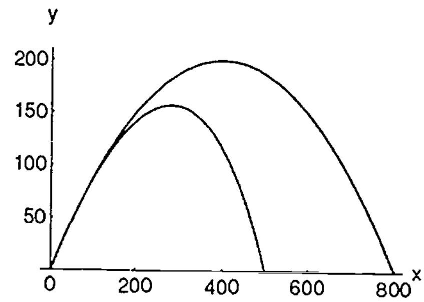


Figure 1. $y = f_0(x)$ and $y = f_1(x)$.

Figure 2. $y = f_0(x)$ and $y = f_1(x)$.

{xc,yc}/.{Vx->160Cos[Pi/4],Vy->160Sin[Pi/4],g->32,k->0.12}
{ 280.847, 156.909}

This point (280.847, 156.909) is lower than the point (400, 200) for the zero air resistance case. Another interesting fact concerns the slopes at impact. For the zero air resistance model, the slope at impact can be shown to be -1 but for $y = f_1(x)$, $f_1'(499.471) = -1.65545$. Thus, the graph with air resistance does not rise as high, its range is less, and the slope at impact is steeper than the graph for no air resistance. This helps explain why a "high pop fly ball" in baseball comes down at a steeper angle than the angle at which the ball was hit.

Example 2. By using equation (11) and $g = 32$, $v_0 = 160$, $\theta = \frac{\pi}{4}$, $v_x = 80\sqrt{2}$ and $v_y = 80\sqrt{2}$, determine how the range in the second model depends on the parameter k.

Solution: Consider the set of values $\{k_j\} = \{2.0, 1.0, 0.5, 0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$, and for each value $k = k_j$ graph the trajectory defined by equation (11). These trajectories are shown in Figure 3. In addition, the dashed trajectory which is computed from (2), and which corresponds to $k = 0$, is also shown. It is the path for the zero air resistance model.

Fig 3

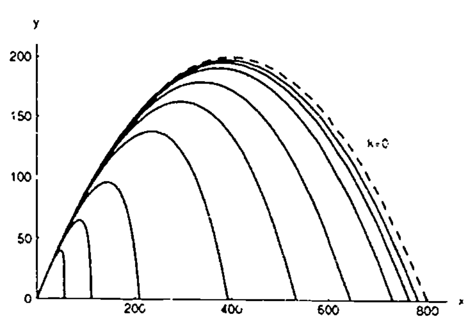


Figure 3. Trajectories for various values of k in Example 2.

153

The values of x where $y = 0$ can either be read from the graphs in Figure 3 or a root finding method can be used with equation (11) to solve $y = 0$. Nine solutions were found with Mathematica's root finding procedure and are summarized in Table 1. The graph of the curve $x = x(k)$ is given in Figure 4.

k_j	2.0	1.0	0.5	0.2	0.1	0.05	0.02	0.01	0.005
$x(k_j)$	56.55	111.86	208.64	392.79	534.46	643.95	730.35	763.78	781.52

Table 1. The range $x = x(k_j)$ for the parameter $k = k_j$ in Example 2.

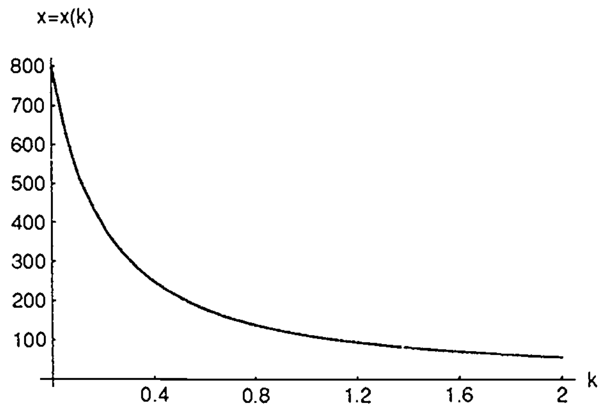


Figure 4. Range $x = x(k)$ for the parameter k in Example 2.

Example 3. By using equation (11) and $v_0 = 160, g = 32$, and $k = 0.12$, determine how the range in the second model depends on the launch angle $\theta = a$.

Solution: The components of the initial velocity are $v_x = 160\cos(a)$ and $v_y = 160\sin(a)$. If the launch angle a is measured in degrees, then equation (11) becomes:

$$(19) \quad y = \frac{5x\sec(a\pi/180)}{3} + \frac{20000}{9}\ln\left(1 - \frac{3x\sec(a\pi/180)}{4000}\right) + x\tan(a\pi/180).$$

The Mathematica function for y that involves this parameter a is:

Y[x,160Cos[a Pi/180],160Sin[a Pi/180],32,12/100]

The output is

$$\frac{5x\,\mathrm{Sec}\left[\frac{a\,\mathrm{Pi}}{180}\right]}{3} + \frac{20000\,\mathrm{Log}\left[1 - \frac{3x\,\mathrm{Sec}\left[\frac{a\,\mathrm{Pi}}{180}\right]}{4000}\right]}{9} + x\,\mathrm{Tan}\left[\frac{a\,\mathrm{Pi}}{180}\right].$$

Numerical methods can be employed to determine that the maximum range, $x_{max} \approx 510.7$, is obtained when the initial launch angle is approximately $a_{max} \approx 39°$. Figure 5(a) shows the graphs of (19) when the launch angle is taken as $a = 6.5°$, 13°, 19.5°, 26°, 32.5°, and 39°, and Figure 5(b) shows the graphs when $a = 39°$, 45.5°, 52°, 58.5°, 65°, 71.5°, 78°, and 84.5°. The range $x = x(a)$, as a function of a, can be read from the graph (the x-intercept to the right of the origin) or a table of values can be computed numerically from (19) after setting $y = 0$. See Table 2.

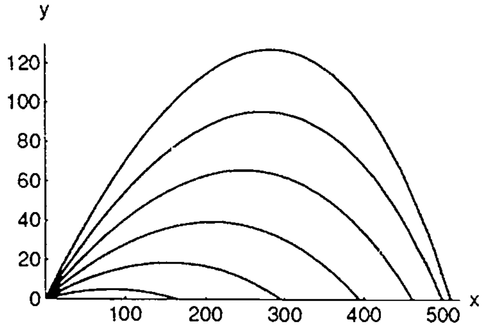


Figure 5(a). Trajectories for the initial angle $0 < a \leq a_{max}$ for Example 3.

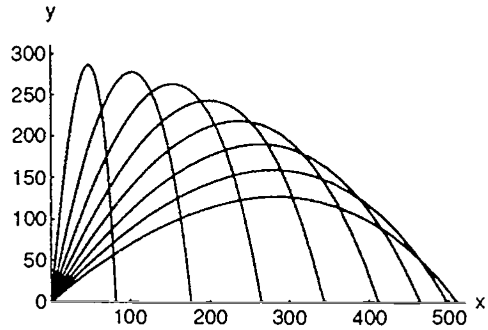


Figure 5(b). Trajectories for the initial angle $a_{max} \leq a < 90°$ for Example 3.

a_j	13.0°	19.5°	26.0°	32.5°	39.0°	45.5°	52.0°	58.5°	65.0°	71.5°	78.0°
$x(a_j)$	296.2	394.7	461.8	499.6	510.7	497.7	463.5	411.2	343.8	264.3	176.0

Table 2. The range $x = x(a_j)$ for the launch angles $a = a_j$ in Example 3.

Figure 6 shows a smooth curve through the data points generated by solving (19) numerically with $y = 0$. It exhibits the dependence $x = x(a)$ sought in Example 3. Figure 6 also shows graphically that the maximum range is approximately 510 when the initial launch angle is approximately 39°. This can be done visually by setting the cursor at the apparent maximum and letting Mathematica digitize the coordinates. A numerical optimization method can be employed to produce the more accurate result $a_{max} = 38.609°$, and $x_{max} = 510.728$ as the coordinates of the maximum point in Figure 6. Note that $a_{max} = 38.609°$ is considerably less than the 45° angle which maximizes the range when there is no friction.

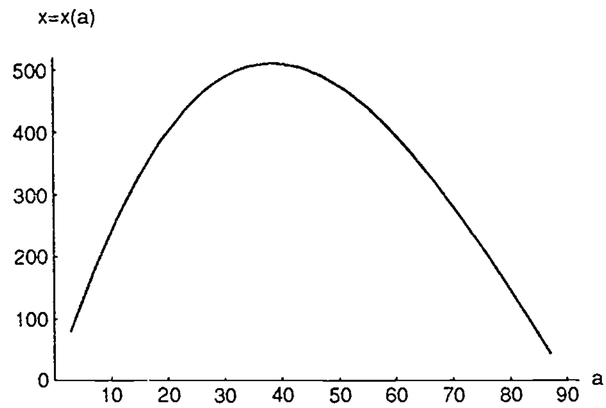


Figure 6. Range as a function of the initial angle a for Example 3.

Conclusion

We believe that elementary examples of projectile motion should be included in calculus and physics courses for three reasons. First, the ideas intrinsic to the models are, in themselves, useful and interesting. In fact, there is even an element of beauty in the discovery that simple formulas successfully describe the complexities of the physical world. Second, mathematical analysis, including the use of limits and series, can be used to compare and analyze mathematical models. Third, examples like we have discussed are stepping stones along the path to "mathematical maturity." For those students with no inherent interest in the physical contents of the models of projectile motion, there is still the need to develop modes of thinking about mathematical objects and expressions.

The notions of implicit functions and dependence on parameters are, by far, the most important constructs the student can meet in a discussion such as presented in this paper. The power of computer algebra systems to present and manipulate such fundamental concepts in very concrete ways is demonstrated in the discussions. We look forward to the development of more pedagogical materials based on experimentation and exploration via modern computer tools.

Acknowledgment

Ideas, inspiration, and reinforcement for pursuing this research were acquired at the NSF Summer workshop: "Integrating Calculus and Physics for Freshmen," held at the Colorado School of Mines, Golden, Co., July 26-28, 1990. The authors would like to thank Joan R. Hundhausen and F. Richard Yeatts who were enthusiastic in their encouragement of developing course materials for calculus.

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SHORT COMMUNICATIONS

More Poetry in Mathematics

by
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Richard L. Francis is a professor of mathematics at Southeast Missouri State University, where he has taught since 1965. He received a B.S. degree from Southeast Missouri State University and master's and doctorate degrees (including post-doctoral work) from the University of Missouri (Columbia). His major scholarly interests are number theory and the history of mathematics.

Pie-eyed "Math"

Now Euler decided to try,
Finding *e* to the power xi,
"Add a one, take the sum,
The result will become,
A zero, if *x* equals pi."

Eureka!

Archimedes, a man of respect,
Drew an angle he wished to trisect,
Though a ruler was used,
He left students confused,
For the proof at first glance seemed correct.

Guess

It appears to work out every time,
This conjecture of Goldbach sublime.
Numbers starting with four,
Even so, evermore,
Decompose to a prime plus a prime.

The Derivative

Hear now Bishop Berkeley exclaim,
As Newton he questions by name,
"Departed almost,
Each quantity's ghost?"
"Your fluxion's a logical shame."

Delos

With compass and straightedge in hand,
A cube Euclid sought to expand,
And thus duplicate,
Yet he always got eight,
As the plague grew worse through the land.

The Dead Mathematicians Society Crossword Puzzle

by Jay Abramson and Robert Forsythe, student
Texas State Technical College/Amarillo
Amarillo, TX 79111

Jay Abramson is currently a mathematics instructor at Amarillo College, Amarillo, Texas. He has taught mathematics on both the secondary and post-secondary levels. Jay earned his Master of Science degree in mathematics at the University of New Hampshire in 1987. He also earned a Bachelor of Science degree in geology from the University of New Mexico in 1981. His current interests are researching math history and writing textbooks and suspense novels.

Robert Forsythe is currently working on an Associate in Applied Science in Metrology with the Endorsement in Technical Writing at Texas State Technical College. He earned an Associate in Applied Science in Laser Electro-Optics from Texas State Technical College in 1992. He was selected as a member of Who's Who Among Students in American Junior Colleges, 1992-1993.

ACROSS

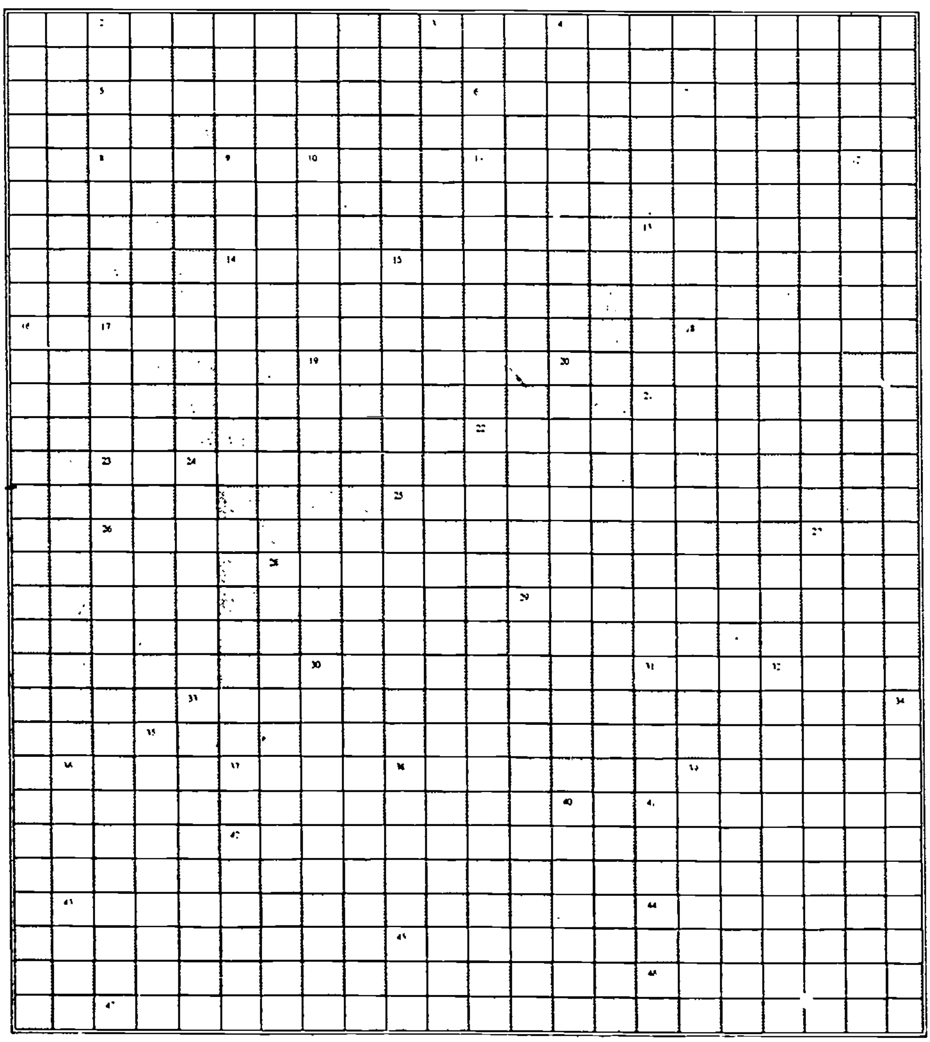
1. $E = mc^2$
3. Eureka!
5. Wrote a classic word problem book "How To Solve It." Said that "mathematics was the cheapest science."
6. Solved equations by use of ______Groups.
8. His elements influenced geometry for 2000 years.
11. Norwegian who cataloged huge collections of groups.
13. Logician who developed set diagrams.
14. His sum is the common definition of the integral.
16. Principia
18. Blind mathematician after whom "e" is named.
19. A woman with many rings and ideals.
22. Examined the square roots of -1.
23. Although a theorem bears his name, he was an early antagonist of calculus.
25. Rabbit sequence
26. Though known for his work with quarternions, once wrote "Cayley is forging the weapons of a future generation of physi-cists."
28. Solved the cubic equation $x^3 + px = q$, where p and q are positive.
29. Examined infinite sets using one-to-one correspondence
31. Lucasian professor best known for uniform convergence and the theorem that bears his name, though some say incorrectly.
33. His paradoxes included that motion was impossible.
37. Studied algebraic equations, one of the more interesting involved $c/4$, where $c = 2\pi$.
41. Though not the founder, the first real giant of probability theory.
42. Cantor's teacher, who then denied Cantor's transfinite numbers.
43. A spring stretched to its limits.
44. Proved the Fundamental Theorem of Algebra, "Few, but ripe."
45. French lawyer who used letters for numbers. (alt sp)
46. Single sided surface. (alt sp)
47. Theorem used in Boolean algebra.

DOWN

1. Developed a sieve to catch primes.
2. Studied logarithms prior to miscalculating the end of the world.
4. Greatest mathematician of the 20th century. Posed 23 problems in 1900.
6. His key statistical work set forth his ideas on regression and correlation.
7. He was determined to understand determi-nants.
9. His solutions to Euler's problems represent-ed a transition from the heavily geometrical presentation to an analytical one.
10. His cuts formed the real numbers.

DOWN (continued)

12. His series represents a function by a power series in $(x-a)$.
15. This witch loved curves.
17. A mostly ignored secondary school teacher who lectured on a continuous nowhere differentiable function; it was subsequently published by a student.
20. From India, through Hardy, to England and number theory.
21. The cart before the coordinate system.
24. A little late in inventing calculus. (alt sp)
25. His theorems, written in the margins of texts, have gone centuries without proofs.
27. His algebra is used to design computers.
30. Logician who wrote Alice in Wonderland.
32. A differential kind of woman. (alt sp)
34. Number theorist whose symbol is used to speed computations with quadratic residues.
35. First to systemize logic, developed the three Primary Laws of Thought.
36. This "mad, bigoted Catholic" formed the basis for modern complex variable theory.
38. His groups have the commutative property.
39. His debauchery and scared personal life did not keep him from a cubic solution. (alt sp)
40. Developed a method to solve a system with the use of determinants.
45. "____" Neuman



See p. 84 for answers.

Lucky Larry

by Philip G. Hogg
Punxsutawney, PA

Editor's Note: Lucky Larry is the infamous student who, in spite of making numerous errors, manages to get right answers. Is he in your class? Send samples of his work to the editor for inclusion in future issues.

Lucky Larry #1

In an algebra class I gave the following logarithmic equation. Lucky Larry "canceled" the logs, but still got the right answer.

$$2 \ln x = \ln x + \ln 4$$

$$2 \cancel{\ln} x = \cancel{\ln} x + \cancel{\ln} 4$$

$$2x = x + 4$$

$$x = 4$$

Lucky Larry #2

In calculus, Lucky Larry's form recognition isn't very good and his algebra is weak, but the answer is right.

$$\int_4^9 \frac{dx}{x-\sqrt{x}} = \ln \left| x-\sqrt{x} \right|_4^9$$

$$= \ln \left| (9-3) - (4-2) \right|$$

$$= \ln 4$$

Lucky Larry #3

I tried to count how many errors Lucky Larry made in this one, but I'm still not sure.

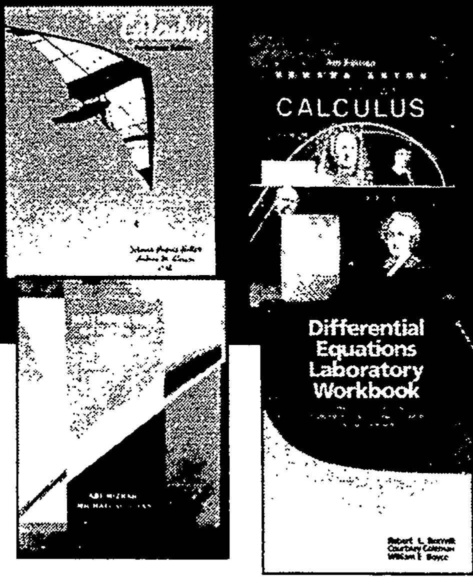
$$\frac{1}{2x+3} \geq 0$$

$$\frac{1}{2x} + \frac{1}{3} \geq 0$$

$$\frac{1}{2x} \geq -\frac{1}{3}$$

$$3 \geq -2x$$

$$-\frac{2}{3} \geq x$$



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Lucky Larry #4

I asked for a proof of the Remainder Theorem. Here is Lucky Larry's.

$$\begin{array}{r|l} & \text{f} \\ \hline x - a & \text{f}(x) \\ & \underline{\text{f}(x) - \text{f}(a)} \\ & \quad\quad \text{f}(a) = \text{remainder} \end{array}$$

Lucky Larry #5

I asked the class to solve $x + 3 = \sqrt{3x + 7}$. Larry brought me this.

$$x + 3 = \sqrt{3x + 7}$$

$$x + 3 = 3x + 7$$

$$-4 = 2x$$

$$x = -2 \text{ (which checks!)}$$

I pointed out the error of not squaring both sides of the equation, and told him to expect two roots. He came back with this.

$$x + 3 = \sqrt{3x + 7}$$

$$x^2 + 3x + 9 = 3x + 7$$

$$x^2 + 2 = 0$$

$$(x + 2)(x + 1) = 0$$

$$x = -2 \qquad x = -1 \text{ (Both check!!)}$$

> The effect of the computer on college teaching has been retarded by the tenure system. The non-tenured *dare* not change and the tenured *need* not change.
>
> Peter Hilton

MATHEMATICS EDUCATION

A Survey of Two-Year College Mathematics Programs: The Boom Continues

by
Anne E. Watkins
California State University, Northridge
Donald J. Albers
Mathematical Association of America
Don O. Loftsgaarden
University of Montana
Data Analyst

Ann Watkins is Professor of Mathematics at Cal State Northridge. She taught at Los Angeles Pierce College for 15 years where she became interested in statistics education. Ann and her husband, Bill, are the editors of the ***College Mathematics Journal*** *and the parents of two teenaged daughters.*

Don Albers is on leave from Menlo College while serving as Associate Director of the Mathematical Association of America. He is the coauthor of ***Mathematical People****. Don is wild about baseball, especially the Dodgers, and hiking in the Southwest.*

Don Loftsgaarden is chair of the Department of Mathematical Sciences where he has taught since 1967. He has been involved in data gathering efforts for the mathematics profession for 13 years. He is an avid sports fan and enjoys camping during the summer.

AMATYC is one of the 14 organizations that comprise the Conference Board of the Mathematical Sciences (CBMS). Every five years since 1965 CBMS has conducted a study of undergraduate mathematical sciences programs in the United States. These surveys provide a detailed portrait of enrollment, instruction, and faculty in universities, four-year colleges, and two-year colleges. This article presents an overview of the data collected in the 1990-1991 survey of two-year college mathematics programs, which are reported in full in Albers, Loftsgaarden, Rung, & Watkins (1992). (The term "mathematics program" refers to the mathematics, computer science, and statistics taught by the group of all mathematics and computer science faculty in a two-year college.)

The findings of the 1990-1991 CBMS survey underscore the importance in undergraduate mathematics education of the 1018 two-year college mathematics programs. In these programs, large numbers of students reap the advantages of small class size, extensive opportunity for remediation, and an educated, experienced, and diverse faculty. Meanwhile, the faculty struggles with the problems of teaching remedial mathematics, excessive dependence on part-time faculty, student motivation, and filling advanced classes.

The survey was mailed in the fall of 1990 to the mathematics program heads at a stratified random sample of 212 of the 1018 two-year college mathematics programs in the United States. Forty-eight percent responded. Projections were made using standard procedures for stratified random samples. The data from this survey are in good agreement with similar data when available from other surveys.

Enrollment

Total Enrollment in Mathematics Programs

The Fall 1990 enrollment in courses in two-year college mathematics programs totaled almost 1,400,000, which is about 38% of all post-secondary mathematics, statistics, and computer science enrollment. Enrollment has resumed its steep climb, increasing by 35% between 1985 and 1990, after declining slightly between 1980 and 1985. See Figure 1. (In contrast, the total number of two-year college students increased by 24%.) Fewer than 1% of two-year college students are mathematics majors.

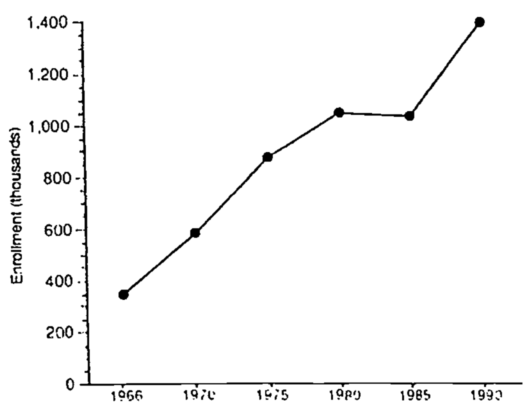


Figure 1. Enrollment (in thousands) in mathematics programs at two-year colleges: Fall 1966, 1970, 1975, 1980, 1985, 1990.

Enrollment in Specific Courses

The growth in mathematics program enrollment can be attributed largely to growth in remediation, which accounts for 67% of the enrollment increase from 1985 to 1990. Enrollment in remedial courses has climbed from a third of the total mathematics program enrollment in 1970 to more than half in 1990 (see Table 1). Fifty-eight percent of the enrollment in mathematics (excluding computer science) courses is at the level of intermediate algebra or below. Not surprisingly, remediation was classified as a major problem by 65% of department heads.

Courses showing large percentage increases in enrollment over 1985 include elementary algebra (45%), intermediate algebra (73%), college algebra (70%), math for liberal arts (218%), non-mainstream calculus (162%) and elementary statistics (62%). See Table 2. Pre-algebra, listed for the first time on the 1990 survey, debuts with an enrollment of about 45,000. The decrease in college algebra/trig enrollment appears to be a result of restructuring these courses as precalculus/elementary functions, which showed roughly an equivalent increase.

Table 1. Enrollment (in thousands) in mathematical sciences and computer science courses by level of courses in mathematics programs at two-year colleges: Fall 1966, 1970, 1975, 1980, 1985, 1990.

Level	1966	1970	1975	1980	1985	1990
Remedial (Courses 1-6)	109 (32%)	191 (33%)	346 (40%)	441 (42%)	482 (47%)	724 (52%)
Precalculus (7-11)	96 (28%)	134 (23%)	152 (17%)	180 (17%)	188 (18%)	245 (18%)
Calculus (12-17)	42 (12%)	59 (10%)	73 (8%)	86 (8%)	97 (9%)	128 (9%)
Computing (29 36)	5 (1%)	13 (2%)	10 (1%)	95 (9%)	98 (10%)	98 (7%)
Statistics (24 25)	5 (1%)	16 (3%)	27 (3%)	28 (3%)	36 (3%)	54 (4%)
Other (18 28,37)	91 (26%)	171 (29%)	266 (31%)	218 (21%)	133 (13%)	144 (10%)
TOTAL	348	584	874	1048	1034	1393

Note: This table was constructed using Table 2. Course numbers used in the groupings are also found in Table 2.

Two-year college mathematics departments have traditionally had difficulty offering the full range of lower division mathematics courses. Although there was an encouraging improvement in the availability of baccalaureate-level courses between 1970 and 1990, many students will still be unable to complete the first two years of baccalaureate-level mathematics. Linear algebra, discrete mathematics, finite mathematics, mathematics for liberal arts, mathematics for elementary school teachers, elementary programming, and many other computer science courses are offered at fewer than half of all two-year colleges.

Enrollment in Math Courses Taught by Other Departments

Many associate of arts degree programs and technical/occupational programs in two-year colleges teach their own mathematics. As part of this survey, the head of the mathematics program estimated the enrollment in such courses at his or her college.

The growth in enrollment in these mathematics courses has traditionally outstripped the growth in enrollment in mathematics programs. From 1970 to 1985, these courses increased in enrollment by 292%, while mathematics program enrollment

Table 2. Enrollment (in thousands) in mathematical sciences and computer science courses in mathematics programs at two-year colleges: Fall 1966, 1970, 1975, 1980, 1985, 1990.

	1966	1970	1975	1980	1985	1990
Remedial level						
1. Arithmetic	15	36	67	121	77	79
2. General mathematics	17	21	33	25	65	68
3. Pre-algebra	na	na	na	na	na	45
4. Elementary algebra	35	65	132	161	181	262
5. Intermediate algebra	37	60	105	122	151	261
6. High school geometry	5	9	9	12	8	9
Precalculus level						
7. College algebra	52	52	73	87	90	153
8. Trigonometry	18	25	30	33	33	39
9. Coll alg & trig(comb)	15	36	30	41	46	18
10. Precalc/elem fns	7	11	16	14	13	33
11. Analytic geometry	4	10	3	5	6	2
Calculus level						
12. Mainstream calc I						53
13. Mainstream calc II	40	58	62	73	80	23
14. Mainstream calc III						14
15. Non-mainstream calc I	na	na	8	9	13	31
16. Non-mainstream calc II	na	na				3
17. Differential equations	2	1	3	4	4	4
Services courses						
18. Linear algebra	1	1	2	1	3	3
19. Discrete mathematics	na	na	na	na	L	1
20. Finite mathematics	3	12	12	19	21	29
21. Math for liberal arts	22	57	72	19	11	35
22. Business math	17	28	70	57	33	26
23. Math for elem teachers	16	25	12	8	9	9
24. Elementary statistics	4	11	23	20	29	47
25. Probability & statistics	1	5	4	8	7	7
26. Technical mathematics	19	26	46	66	31	17
27. Tech math (calc level)	1	3	7	14	4	1
28. Use of hand calculators	na	na	4	3	6	L
Computing						
29. Computers & society	na	na	na	na	na	10
30. Data proc (elem or adv)	na	na	na	na	36	21
31. Elem prog (languages)	3	10	6	58	37	32
32. Advanced programming	na	na	na	na	5	8
33. Database management	na	na	na	na	na	4
34. Assembly lang prog	na	na	na	na	4	2
35. Data structures	na	na	na	na	2	1
36. Other comp. sci courses	2	3	4	37	14	20
37. Other math courses	8	14	32	27	14	23
TOTAL	348	584	874	1048	1034	1393

na means not available and L means some but fewer than 500
Mainstream calc is for math, physics, sci & engr, non mainstream for bio, soc & mgmt sci
Prior to 1990 aggregate sums for Main Calc I, II & III were reported
Prior to 1990, aggregate sums for Non Main Calc I & II were reported

increased by 77%. However, from 1985 to 1990, enrollment in mathematics programs increased by 35%, while enrollment in courses outside mathematics programs increased by only 12%. Enrollment in these courses is now about 29% as large as enrollment in mathematics programs.

Most students who take business math, data processing, and computer science and programming do so outside of mathematics programs. Table 2 shows a decrease in enrollment in business mathematics, technical mathematics, and data processing taught within mathematics programs. These courses showed a similar decrease in enrollment when they were taught outside of mathematics programs.

Instruction

Instructional Formats

In 94% of two-year college mathematics programs, most faculty use the standard lecture-recitation system with classes of 40 or fewer. In 5% of two-year college mathematics programs, large lecture sections are the norm.

The instructional innovations of the 1970s that allow students to pace their learning – personalized system of instruction, audio-tutorial, modules, computer-assisted instruction, programmed instruction – continue to decline in popularity. Only a small percentage of the two-year college mathematics faculty use them today.

Innovations in Calculus Courses

Innovations in calculus instruction of the late 1980s had not gained much of a toehold in Fall 1990. Fewer than 5% of the sections of Mainstream Calculus I, II, or III or Non-Mainstream Calculus I or II assign group projects or have a writing component.

Average Number of Students Per Section

In Fall 1990, the average number of students per section for mathematics and statistics courses was 28. The average number of students per section in computer science courses was 18. Table 3 shows that class sizes are smaller than in four-year colleges and universities.

Table 3. Average section size by level of course in two-year colleges and four-year colleges and universities: Fall 1990.

	Two-Year Colleges	Four-Year Colleges and Universities
Remedial (Courses 1-6)	29	31
Precalculus (Courses 7-11)	27	35
Calculus (Courses 12-17)	24	35
Computer science (Courses 29-36)	18	29
Statistics (Courses 24-25)	29	37

Course numbers are for two-year college courses See Table 2

Use of Computers and Calculators

The computer has arrived in two-year college mathematics classes, especially in advanced classes. Department heads report that in a typical week 23% of the faculty assign homework requiring use of the computer. Computer assignments are regularly given in 9% of all sections of mathematics (excluding computer science). Linear algebra, with 40%, is the course that has the largest percentage of sections in which computer assignments are regularly given. Statistics follows with 29%.

Mathematics program heads estimate that there is substantial use of computers by faculty for constructing tests or assignments, but only 10% of the faculty use a computer algebra system in a typical week.

Computers are available in moderate numbers for use by mathematics students and mathematics faculty, but the percentage of two-year colleges with no computers for use in mathematics classrooms is still quite large. In fact, "computer facilities for classroom use" is listed as a major problem by 28% of department heads.

Excluding computer science courses, calculators are recommended for use in 48% of all sections, up from 29% in 1980. As was the case with the computer, the more advanced the course, the more likely it is that calculators are recommended. The percentage varies from 88% of the sections of differential equations to only 12% of the sections of arithmetic, which apparently continues to be a pencil-and-paper algorithm course.

Math Labs and Other Student Services

Over 86% of two-year colleges operate a math lab or tutorial center, most of which employ students and paraprofessionals. Placement examination, available in 60% of two-year colleges, is the only other student service typically offered. See Table 4.

Table 4. Percent of two-year colleges offering various services to students: Fall 1990.

Service	% of two-year colleges offering
Math lab or tutorial center	86%
Advisory placement examinations	60%
Mandatory placement examinations	58%
Honors sections	17%
Regular participation in math contests	17%
Lectures/colloquia for students	15%
Active math club	12%
Social activities for majors and faculty	7%

The Faculty

Use of Part-Time Faculty

The part-time faculty now numbers 13,700, nearly twice the size of the full-time faculty, which numbers about 7200 (see Figure 2). From 1985 to 1990, the full-time faculty increased by 15%, the part-time faculty by 84%, and student enrollment by 35%.

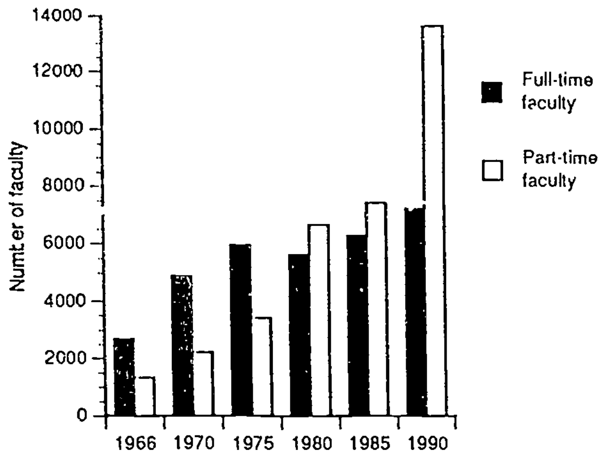


Figure 2. Number of full-time and part-time faculty in mathematics programs at two-year colleges: Fall 1966, 1970, 1975, 1980, 1985, 1990.

Part-time instructors make up 65% of the two-year college mathematics program faculty and teach 42% of all sections and 51% of the sections of remedial mathematics. About half of all sections are taught either by part-time instructors or full-time instructors teaching extra hours for extra pay. It's no wonder that 42% of mathematics program heads classify "the need to use temporary faculty for instruction" as a major problem.

Other Employment of Mathematics Program Faculty

Supplementing the part-time faculty, 44% of the full-time faculty teach extra hours for extra pay, averaging 4.7 additional hours for these faculty members.

Seventy-three percent of part-time instructors either have full-time employment elsewhere or are graduate students (see Table 5).

Table 5. Other employment of part-time faculty in two-year college mathematics programs: Fall 1990.

Other employment of part-time faculty	Percent of part-time faculty
Employed full-time in:	
a high school	30%
a two-year college	9%
a four-year college	3%
industry or other	26%
Graduate student	5%
No full-time employment	27%

Teaching Load

The average required teaching load of a full-time mathematics program faculty member is 14.7 contact hours a week, down from 16.1 hours in 1985. See Table 6.

Table 6. Teaching load for full-time faculty members in mathematics programs at two-year colleges: Fall 1990.

Teaching load-contact hours	9	10-12	13-15	16-18	19-21	22
Percent of two-year schools	0.4%	25.2%	57.3%	11.3%	5.4%	0 4%

- Full-time average contact hours: 14.7
- Percent of the full-time faculty who teach extra hours for extra pay: 44%
- Average number of extra hours for extra pay: 4.7

Part-time faculty members teach an average of 6.1 hours a week, up from 5.7 hours a week in 1985. In 19% of mathematics programs, "part-time" instructors teach an average of 9 hours or more.

Education of the Full-Time Faculty

The percentage of the full-time two-year college mathematics program faculty with a doctorate has risen to 16.5%, although fewer than 2% of new full-time hires in 1989-1990 had doctorates. Many faculty members apparently earn doctorates while on the job. The percentage of full-time faculty members whose highest degree is a bachelor's degree is down to 4% (compared to 27% of the part-time faculty). Table 7 gives the highest degree and field of degree for full-timers.

Table 7. Highest degree of full-time faculty in mathematics programs at two-year colleges by field and level of highest degree: Fall 1990.

	Highest degree				
Field	Doctorate	Masters+1	Masters	Bachelors	TOTAL
Mathematics	8%	26%	31%	3%	68%
Mathematics Education	6%	5%	6%	L	17%
Statistics	L	1%	1%	0%	2%
Computer Science	L	1%	2%	1%	4%
Other fields	2%	1%	5%	L	9%
TOTAL	17%	34%	45%	4%	100%

L: Fewer than half of 1%.

Education of the Part-Time Faculty

The percentage of part-time two-year college mathematics program faculty with either a doctorate or a master's degree plus one year has dropped since 1970 and the percentage with a bachelor's degree has increased. Table 8 gives the highest degree and field of degree for part-timers.

Table 8. Highest degree of part-time faculty in mathematics programs at two-year colleges by field: Fall 1990.

	Highest degree				
Field	Doctorate	Masters+1	Masters	Bachelors	TOTAL
Mathematics	1%	8%	27%	11%	47%
Mathematics Education	1%	3%	8%	5%	17%
Statistics	L	L	1%	L	2%
Computer Science	L	L	2%	4%	7%
Other fields	4%	4%	12%	7%	27%
TOTAL	8%	15%	50%	27%	100%

L: Fewer than half of 1%

Sex of the Full-Time Faculty

Women comprise 34% of the full-time faculty in mathematics programs, up from 21% in 1975 (see Figure 3). Women make up 49% of the full-time mathematics program faculty under the age of 40, a remarkable percentage given that in each of the years from 1970 to 1986, 35% or fewer of the master's degrees awarded in the mathematical sciences went to women (National Research Council, 1990, p. 102).

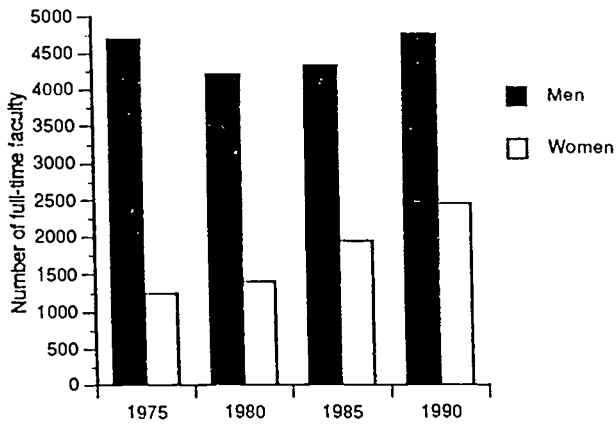


Figure 3. Number of male and female full-time faculty members in mathematics programs at two-year colleges: Fall 1975, 1980, 1985, 1990.

Ethnicity of the Full-Time Faculty

Ethnic minorities comprise 16% of the full-time mathematics program faculty (up from 7% in 1975) and 26% of the full-time mathematics program faculty members under the age of 40. See Tables 9 and 10.

Table 9. Number of ethnic minority full-time faculty members in mathematics programs at two-year colleges: Fall 1975, 1980, 1985, 1990.

	1975	1980	1985	1990
Number of full-time ethnic minority faculty members	416	450	753	1155
% ethnic minorities among full-time faculty members	7%	8%	12%	16%

Table 10. Ethnic group distribution of full-time faculty and of full-time faculty under age 40 in mathematics programs at two-year colleges (Fall 1990) and percent of master's degrees in mathematical sciences awarded (1985).

Ethnic Group	Percent of faculty	Percent of faculty under age 40	Percent of U.S. master's degrees *
Non-Hispanic white	84%	74%	87%
Asian/Pacific Islander	4%	6%	8%
Hispanic	7%	12%	2%
Black	4%	8%	2%
Native American	1%	L	L

L: Fewer than half of 1%

* Includes U.S. citizens only. [Source: National Research Council, A Challenge of Numbers: People in the Mathematical Sciences, National Academy Press, Washington, DC, 1990 p.47. Their source: National Center for Education Statistics of the U.S. Department of Education, unpublished data.]

Age of the Full-Time Faculty

The average age of the mathematics program faculty has risen to 45 years, about the same as the faculty in four-year college and university mathematics and statistics departments. The percentage under age 40 slid from 47% in 1975 to 23% in 1990.

Sources and Sinks of Full-Time Faculty

Over 700 people were newly hired for full-time teaching (both permanent and temporary) in mathematics programs at two-year colleges in 1990. Forty-seven percent of them had taught previously in the program into which they were hired.

The year before they were hired full-time (1989-1990), 62% of the new hires were teaching and 29% were in graduate school. In the first CBMS surveys, the majority of new hires were secondary school teachers. In 1990, fewer than 10% of the new hires were secondary school teachers.

In 1989-1990, only 33% of the full-time faculty who left two-year college teaching did so because of death or retirement.

Administration of Two-Year College Mathematics Programs

Given the opportunity to complain, department heads didn't. Remediation was the only problem classified as major by an overwhelming percentage of department heads (65%), followed by salary levels/patterns (47%), the need to use temporary faculty for instruction (42%), and student motivation (38%).

During the 1980s, many two-year colleges reorganized so that the mathematics program was administered by a division head rather than by a mathematics department chair. The percentage of two-year college mathematics programs administered under various structures in 1990 can be found in Table 11.

Table 11. Administrative structure of two-year college mathematics programs: Fall 1990.

Administrative structure	Percent of two-year college mathematics programs
Mathematics department	36%
Mathematics and computer science department	8%
Mathematics and science division or department	40%
No department structure	3%
Other (mostly department or division with mathematics and other disciplines)	13%

References

Albers, D. J., Loftsgaarden, D. O., Rung, D. C. & Watkins, A. E. (1992). *Statistical abstract of undergraduate programs in the mathematical sciences and computer science in the United States: 1990-91 CBMS survey.* Washington, DC: Mathematical Association of America.

National Research Council. (1990). *A challenge of numbers: People in the mathematical sciences. Washington, DC: National Academy Press.* Their source: National Center for Education Statistics of the U.S. Department of Education. (1988). *Digest of education statistics, 1988,* p. 102.

> The number of math majors is down; test scores are down; all numbers are down except the number of administrators. Administrators should be self-effacing facilitators, but too many are self-regarding complicators.
>
> Peter Hilton

REGULAR FEATURES

Mathematics: An International View

Edited by Igor Malyshev and Joanne Rossi Becker

San Jose State University, One Washington Square, San Jose, CA 95192

This column is an avenue for discussion of the first two years of college mathematics with a global perspective. Because mathematics is an international culture, it is important to follow the development of mathematics education throughout the world. We feel that by sharing information from other countries with systems of education different from ours, we will gain an enriched understanding of, and a better perspective on, our own system. A completely different viewpoint on a topic can serve to stimulate our imaginations and help us find new solutions to a problem which will work in our context. As we move toward dramatic changes in both the curriculum and instruction in mathematics called for in many reports, reflection on practices used in other countries can only help our efforts.

For this column, we would like to publish two types of items: material with an international origin or context that you could use directly with your students; or information about the content and structure of coursework in other countries. We welcome submissions of 3-4 pages or suggestions of mathematicians from other countries from whom we could solicit relevant material.

Send contributions for this column to Igor G. Malyshev at the address above.

Teaching International Awareness Through Mathematics

by Richard H. Schwartz

College of Staten Island, Staten Island, NY

Richard H. Schwartz earned his doctorate in civil engineering at Rutgers University and is an associate professor of mathematics at the College of Staten Island. He frequently speaks and contributes articles on mathematical connections to current issues.

The world faces a wide variety of interrelated problems today. Population is growing extremely rapidly and is projected to double in approximately 40 years. Environmental threats such as the depletion of the ozone layer, potential global warming, acid rain, and the destruction of tropical rain forests have global implications. Hunger stalks many areas and an estimated 20 million people die annually due to insufficient food. People in many countries lack clean water, adequate shelter, employment, and other basic needs. In view of these many global problems, it is not surprising that in 1989, contrary to its general practice of choosing a "Person of the Year," *Time* magazine chose instead "Our Endangered Planet," Planet Earth, as "Planet of the Year."

Yet college students are generally unaware of these critical issues that threaten their futures and those of every person on earth. Based on the results of a recent survey of geographic knowledge of students in the United States and the (former) Soviet Union, the National Geographic Society concluded that American citizens show "an astonishing lack of awareness of the world around them." (*New York Times*, 1989)

It is essential that students become aware of the nature of current threats and of possible solutions. Mathematics teachers can play a role by providing problems that relate to current critical issues. For the past 17 years I have been giving a course, "Mathematics and the Environment" at the College of Staten Island. By relating the material to today's major problems, it has helped to motivate students in their mathematical studies. It provides answers to questions frequently heard from students in general mathematics classes such as, "Why do I have to learn mathematics? What is it good for? How will I use it when I get out of school?"

The following problem illustrates the suggested approach. Creative mathematics teachers can create many similar problems from daily news articles, the many recent books on environmental and other global issues, and the resources listed at the end of this article.

> Using data from the 1992 World Population Data Sheet (discussed later), indicate several ways of comparing the wealthy countries (and regions) with poor countries.

This is an open-ended question that enables students to work (possibly in groups) in gathering and analyzing data. They could find the range in life expectancies (82 years - 42 years for females, and 76 years - 40 years for males, with Japan having the highest value and Afghanistan and Guinea-Bissau, respectively, having the lowest value in each case). They could find the country (Switzerland) having the highest per capita GNP ($32,790) and the lowest ($80 for Mozambique) and find the ratio. They could compare birth rates, death rates, infant mortality rates, and population doubling times for various countries and regions. They could draw scatterplots for birth rates vs. per capita GNP, for example, to see how the wealth of a country is related to other important variables.

. Other kinds of mathematics problems which can be presented using information on the Population Data Sheet involve computing percent increases in population and the percent of the world's population living in the United States or some other country or region, and drawing bar charts and histograms for such things as birth rates and physical quality of life indices. Different coun-tries such as China and India, and regions, can be compared using a wide variety of data.

To increase connections to international issues, news and magazine articles related to the course can be discussed; the course can be related to events such as Earth Day and U.N. conferences related to population, hunger, environment, water resources, habitat, desertification, and disarmament. Students can report to the class on some global issue they have researched, using mathematical concepts covered in the course.

Using these and similar mathematical problems, teachers can introduce interesting and significant information and concepts related to current issues. More important than the specific information in any problem, however, is the understanding of international issues and threats that students gain from discussing the significance of the results. Calculating the answers to the math problems raises a host of related questions: How serious is the population explosion? What are the economic, social and health costs of pollution? Are we running out of resources? What are the environmental consequences of waste in the United States and other affluent countries?

The nature of the many critical problems facing the world makes it imperative that we use new and creative ideas to help make students aware and get them involved. Relating mathematics and global issues can be one such idea that can help turn our world away from its present apparent journey toward disaster.

Resources for Further Mathematical Problems:

1. The Population Reference Bureau, Inc., (1875 Connecticut Avenue NW, Suite 520, Washington, D.C. 20009, (202) 483-1100) is an excellent source for information related to population. We have already referred to their annual World Population Data Sheet which gives a wealth of information on the world's regions and countries. They also have a "Population Handbook" which has a comprehensive summary of demographic techniques with many sample prob-lems related to the World Population data sheets. Other valuable material includes population sheets, teaching modules, and bulletins. Special data sheets and other background material with many graphs, charts, and data were produced related to the International Year of the Child, in 1979, International Women's Year, in 1980, and the 20th anniversary of the first Earth Day in 1990.

2. *Mathematics and Global Survival,* by Richard H. Schwartz (Ginn Press, 160 Gould Street, Needham Heights, MA 02194-2310; 1-800-428-GINN), 1989.

A 197-page text book with a wide variety of mathematics problems related to pollution, hunger, resource scarcity, energy, the arms race, and rapid population growth. The book includes the 1988 World Population Data Sheet of the Population Reference Bureau.

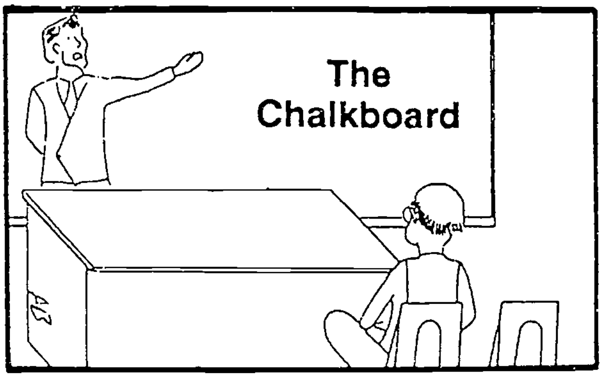
3. *Beyond The Limits,* by D. Meadows, et al, Post Mills, Vermont: Chelsea Green Publishing Co., 1992.

This book has much material with both mathematical and global applications, while providing a warning about the results of continued unrestrained growth. It uses several mathematical concepts including exponential growth, models, feedback loops, quantification of variables, and many charts and graphs.

4. *World Military and Social Expenditures*, 1991 by Ruth L. Sivard, World Priorities Publications, Box 25140, Washington, D.C. 20007.

An excellent annual publication which has much information on arms expenditures and the impact on social issues. Many charts, tables, and graphs are provided.

NOTE:
New York Times, "2 Superpowers Failing in Geography," Nov. 9, 1989, p. A20.



Edited by

Judy Cain and Joseph Browne

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This column is intended as an idea exchange. We hope to facilitate an open exchange of ideas on classroom management, teaching techniques, tips for helping students get past the usual stumbling blocks, techniques for improving student participation, etc. We know there are lots of good ideas out there, and this is your chance to share them. **Our backlog is almost exhausted!** Please send your contributions to Judy Cain.

A Class Introduction

It is generally well accepted that the first few minutes of a class should be spent informing the students of the objectives for that particular class period. Often, a few minutes should be devoted to summarizing, in general terms, what has been studied during the past few days, or weeks.

Whereas the instructor can provide this, I have found that my students are quite capable of providing an overview. In addition, a student can occasionally suggest topics for current or future investigation. For example, in a descriptive statistics class, I might ask "Why have we been studying probability when we are in the middle of a course about statistics?" Of course, this question has probably been answered at several points in the course, but this provides an opportunity for students to reveal their impressions and review some of the fundamentals about statistics.

In a precalculus class, the first few weeks are traditionally spent reviewing selected topics such as sets and logic, the construction and field properties of the real number system, the coordinate system and others. I have asked my students at the beginning of a subsequent class the question "What have we been studying this past week?" I usually obtain a variety of results, from short answers like "proofs" (even though we had only done one or two) to short soliloquies including questions about the purpose of studying this material. This time does prove valuable in warming up the student to participate in class and provides important feedback about what the students are learning.

Alternatively, these few minutes could be spent having the student provide written responses, still providing feedback to the instructor and, most importantly, stimulating thinking.

Submitted by Ted Moore, Mohawk Valley Community College, Utica, NY 13501.

Implicit Use of the Substitution Rule

In beginning calculus the substitution rule is a powerful general rule that allows students to integrate many integrals they could not handle without it. For the special case where the only thing needed is a constant factor in the differential so that a known formula can be used, however, they can take a shortcut and use the substitution rule implicitly. Many texts never explain this, so instructors should try to wean students from making actual substitutions when only "fixing the constant" is required.

Example 1: $\int x\sqrt{1+x^2}\,dx$.

Students should be able to see that if they rewrite the problem as $\int \sqrt{1+x^2}\,(2x)\,dx$, they will have the differential they need, but will need to multiply by $\frac{1}{2}$ outside the integral to compensate for the extra factor of 2 inside. Thus, $\int x\sqrt{1+x^2}\,dx = \frac{1}{2}\int x\sqrt{1+x^2}\,(2x)\,dx$, which they can integrate without explicit substitution.

Example 2: $\int \cos^9 5x \sin 5x\,dx$.

Students should glance at the integral and arrive at the conclusion that cos $5x$ is raised to a power. The differential of cos $5x$ is $-5\sin 5x\,dx$, so everything is ready except for a constant factor of -5. Thus, they can set $\int \cos^9 5x \sin 5x\,dx$ equal to $\frac{-1}{5}\int \cos^9 5x\,(-5)\sin 5x\,dx$ and integrate without explicit substitution.

Submitted by James R. Smart, San Jose State University, San Jose CA 95192.

Factoring Trinomials

There are several methods for factoring trinomials, but the following one doesn't seem to be well known to algebra teachers and authors. Many students seem to prefer it for some of the more difficult problems.

To factor $ax^2 + bx + c$, first look for two numbers, r and s, such that $rs = ac$ and $r + s = b$ (nothing new so far). Then factor as follows:

$$ax^2 + bx + c = \frac{(ax + r)(ax + s)}{a}.$$

One or both of the quantities in the numerator will have factors which will divide out the a in the denominator, resulting in the standard factored form.

Example: Factor $12x^2 - 11x - 15$.

We have $ac = 12 \cdot -15 = -180$, so

$r = 9$ and $s = -20$. Then

$$12x^2 - 11x - 15 = \frac{(12x + 9)(12x - 20)}{12}$$

$$= \frac{3(4x + 3)4(3x - 5)}{12}$$

$$= (4x + 3)(3x - 5)$$

Submitted by Joanne V. Peeples, El Paso Community College, El Paso TX 79998.

Research in Collegiate Mathematics Education

(A new journal)

The Conference Board of the Mathematical Sciences has agreed to publish a series of annual volumes of research papers describing the state of the art in, and entitled, Research in Collegiate Mathematics Education (RCME). The volumes will be co-edited by Alan Schoenfeld, James Kaput, and Ed Dubinsky, with an editorial board that contains the Joint (AMS, MAA, AMATYC, NCTM) Committee on Research in Undergraduate Mathematics Education as a proper subset. The hope is that these volumes will serve as a showcase for the very best research in collegiate mathematics education, and as a means of advancing the field. The first volume is expected to appear in early 1994. Copies of editorial policies and submission procedures may be obtained from any of the editors.

Ed Dubinsky, Purdue University, 1395 Mathematical Science Building, W. Lafayette IN 47907-1395.

James Kaput, Department of Mathematics, University of Massachusetts, North Dartmouth MA 02747.

Alan Schoenfeld, School of Education, University of California, Berkeley CA 94720

Software Reviews

Edited by Shao Mah

Title: MathWriter - The Scientific Word Processor for the Macintosh (V2 0)
Authors: J. Robert Cooke, E. Ted Sobel
Distributor: Brooks/Cole Publishing Company
Pacific Grove, California
Price: Professional Version - $398.00 US
Education Version - $99.95 US
System Requirements: Professional Version 68020 - 68030 based Macintosh with 2MB of RAM and a hard disk
Education Version: 1MB of RAM (using Finder rather than Multifinder)

MathWriter is a scientific word processing software package that comes with three 3.5 inch disks and a user's guide. It was designed to "make the writing of mathematics-laden scientific and technical manuals less daunting" and it certainly does just that. MathWriter not only has improved and increased the number of features of traditional word processors, but it makes the inclusion of graphics and mathematical expressions (in what-you-see-is-what-you-get format) a much less encumbered operation. Extensive menus allow the beginning user to access its features, many of which can also be gained by keyboard commands if so desired.

Several of the nontechnical word processing features of MathWriter are noteworthy. Input aids allow creation of reusable default or stationery tem-plate files, and also use of floating windows, an auto save to protect files, and an on-line help menu which is easy to understand and remarkably comprehensive. Document review is possible through a revision tracking feature and will automatically display typeface changes of additions and deletions. When revising text, the *Replace All* command cannot be undone and this may be viewed by some users as a criticism of the program. However, it is simple to repeat the process and replace what was changed with the original. Another feature is the hidden windows that provide a place for notes but do not disrupt the page layout. It is possible to incorporate minidocuments called sidebars within the text and the main document will automatically flow around these sidebars. Any graphics which are included can easily be resized or cropped and may have the text flowing through if so desired. There is automatic numbering of the lines of text (screen and print) and also of figures, equations, tables, footnotes, etc. with cross-references and automatic updates when editing.

The most exceptional improvement of MathWriter over my previous word processor is the inclusion of technical features, most notably the full integration of mathematical expressions into text. Tables and mathematical expressions are automatically formatted and can be edited within the text material with ease. Templates, palatte tools, and fonts can be displayed in floating windows so as to allow fluent incorporation of mathematical symbols. Another feature that may be used to increase the speed of technical writing is a floating window of frequently used user-defined expressions. Access may be gained to these by clicking on the expression or by typing a pre-designed simple abbreviation. A thesaurus for synonyms is included in the program as well as a supplementary word list of math, science and engineering terms in the spell-checker.

The only difficulty I have had to date was in implementing the automatic update of numbered equations when editing within the guided tour. This however has not been a problem when used in the technical writing I have done. It should also be noted that I used the professional version of the program on my Macintosh SE even though this was not recommended. It was suggested that the response would be sluggish, but MathWriter is still a vast improvement over what I used previously.

Given MathWriter's easy-to-use and polished program and it's virtually error free, comprehensive and easy-to-read guide, I would recommend it for anyone producing scientifically intensive material. Be sure to check out this scientific word processor before purchasing any other word processing program!

Reviewed by Marilyn MacDonald, Red Deer College, Red Deer, Alberta, Canada

Send Reviews to: Shao Mah, Editor, Software Reviews
The AMATYC Review, Red Deer College, Red Deer, AB, Canada T4N 5H5

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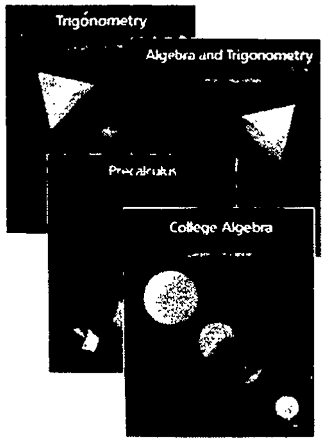

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The Problem Section

Edited by Michael W. Ecker, Ph.D.
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Solutions by Robert E. Stong, Ph.D.
Mathematics Department
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Greetings, and welcome to still another Problem Section!

This particular column you are reading is the first completely under new editorship in the person of Prof. Joseph Browne. I've worked with fine editors—Etta Mae Whitton, Jay Huber, Don Cohen—during the period since I founded this department 11 years ago. In particular, I believe Joe is the first *AMATYC Review* editor to have been a long-time problem-solver and problem-editor himself. Welcome to Prof. Browne, and congratulations with thanks to retiring editor Prof. Don Cohen for his contributions the years of his stewardship.

I don't know how many serious problem-solvers we have, but you should be told again of Dr. Stanley Rabinowitz's new compilation, *Index to Math Problems 1980-1984.* It contains over 5,000 problems from the period 1980-1984 in two dozen journals, including this one. It is indexed more ways than one has a right to hope for, including by journal, by authors (proposers and solvers), by problem type, and so on.

I know Stan from our days on the committee on American Mathematics Competitions. I am always impressed with the energy and dedication he brings to his efforts in problem-solving. So, I knew his *Index to Math Problems 1980-1984* would be authoritative even before I received it in mid-1992. Last price I have listed is about $50 to individuals. Contact Stan at MathPro Press/ P.O. Box 713/Westford, MA 01886.

Now, on to our regular business by reminding ourselves that the *AMATYC Review* Problem Section still seeks not mundane exercises but lively and interesting problems and their solutions from all areas of mathematics. Particularly favored are teasers and challenges of an elementary or intermediate level that have some applicability to the lives of two-year college math faculty and their students. We welcome computer-related submissions, but bear in mind that programs should supplement, not supplant, the mathematics.

To submit material for this department, send your new problem proposals only, preferably typed or printed neatly with separate items on separate pages, to the Problem Editor at the home address shown below. If you have a solution to your proposal, please include it along with any relevant comments, history, generalizations, special cases, observations, and/or improvements. Include a mailing label or self-addressed envelope if you'd like to be assured a reply.

All solutions to others' proposals *except Quickies* should be sent directly to Dr. Stong at his home address. Addresses are:

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If you see your own problem printed later, please send Prof. Stong a copy of your solution, regardless of whether you sent me one originally with your proposal or not. This is not mandatory as policy, and you will automatically be listed among solvers (unless problem is starred, indicating no solution submitted with proposal). However, this is the safest way to assure that your approach will be considered for publication as a featured solution. *Thanks!*

Quickies

Quickies are math teasers that typically take just a few minutes to at most a half-hour to an hour. Solutions follow the next issue. All correspondence to this department should go to the Problem Editor, not the Solutions Editor.

Quickie #4 Revisited, Solution by Steve Plett: What is the smallest possible value for the sum of a positive real number and its reciprocal?

Express the number as $1+x$, with $x>-1$, so the sum is $(1+x)+1/(1+x)$. This can be rewritten as $(2+2x+x^2)/(1+x)$ or $2+x^2/(1+x)$. The smallest value of the latter is 2, the minimum being achieved for $x=0$ – that is, when the positive real number used is 1.

Bob Stong noted an even simpler solution: The result flows from the equivalence of $x+1/x \geq 2$ and $(x-1)^2 \geq 0$ for $x>0$. (Multiply the first by x).

Quickie #5. It is obvious that $a+1/a = b+1/b$ if and only if $a=b$ or $a=1/b$. Prove it – without calculus.

Solutions by Bella Wiener, University of Texas-Pan American, Edinburg, TX, and Ken Boback, Penn State U., Wilkes-Barre Campus, Lehman, PA.

Multiply by ab to obtain $a^2b+b=ab^2+a$; transpose and factor by grouping to get $(a-b)(ab-1)=0$.

Bob Stong noted that $x+1/x=b+1/b$ is a quadratic in x. Hence, there cannot be more solutions than the $x=b$ and $x=1/b$ ones obtained by inspection.

Quickie #7. For k and n whole numbers, demonstrate that $(kn)!/(n!^k)$ is integral.

This is just a multinomial coefficient counting the number of ways of choosing, from kn items, n items to be of one color, a second n items to be of a second color, and so on, up to a k-th set of n items being of a k-th color.

Quickie #6. Prove that there exist irrational numbers a and b such that a^b is a rational number.

Consider $\sqrt{2}^{\sqrt{2}}$ and $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$. If the first number is rational, we are done. If not, the second number, being equal to 2, gives the desired result.

For more on this, see the new Quickie #9 to follow below.

More on Quickie #6: I got a great letter from Prof. Ken Sydel of Skyline College regarding Teaser (Quickie) #6. Ken pointed out that the usual existence proof omits telling whether $\sqrt{2}^{\sqrt{2}}$ or $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is the rational number. The question is answered by the Gelfond-Schneider Theorem; see Niven's *Irrational Numbers*, p.134. The answer is that the second is the rational one since the first expression is transcendental. Ken suggested another oldie as a companion follow-up problem:

Quickie #9: Proposed by Ken Seydel. Prove that there exist non-real complex numbers a and b such that a^b is real.

New Problems

Set X Problems are due for ordinary consideration October 1, 1993. Our Solutions Editor requests that you please not wait until the last minute.

Of course, regardless of deadline, no problem is ever closed permanently, and new insights to old problems – even Quickies – are always welcome.

An asterisk on a problem indicates that the proposer did not supply a solution with the proposal.

Problem X-1*. Proposed by Leonard M. Wapner, El Camino Community College, Torrance, CA, who writes: "My colleagues and I have puzzled over this problem for some time now and we would like to share it with others."

Does the sequence $\sin^1 1, \sin^2 2, \ldots, \sin^n n, \ldots$ converge?

Problem X-2. Proposed by the Problem Editor, Penn State U., Lehman, PA.

Consider polynomials of the form $p(x) = ax^3 + bx^2 + cx + d$, where a,b,c,d are arbitrary real constants. In terms of these constants, characterize all such polynomials p that are invertible as functions.

Problem Editor's Comment: I've proposed for the *College Mathematics Journal* a potentially more involved but similar problem for odd fifth-degree polynomials.

Problem X-3. Proposed by Jim Africh, College of DuPage, Glen Ellyn, IL.

If $a_1, a_2, \ldots, a_n$ are n real numbers that sum to 1, show that the sum of their squares is at least $1/n$.

Problem X-4. Proposed by J. Sriskandarajah, University of Wisconsin, Richland Center, WI, and modified slightly by the Problem Editor.

Find the sum of the finite Arithmetico-Geometric Series

$$ag + (a+d)gr + (a+2d)gr^2 + \ldots + (a + (n-1)d)\, g\, r^{n-1}.$$

For which values of r is the corresponding infinite series convergent? For such r, derive the sum of the infinite series. Hence or otherwise evaluate

$-4-2 + 3-5/2 + 7/4-9/8 + 11/16 - \ldots$ (arithmetico-geometric series).

Problem X-5. Proposed by the Solution Editor, University of Virginia.

Find the factorizations of the polynomial $x^4 - 6x^2 + 1$ into irreducible factors in the ring $\mathbb{Z}\left[2\sqrt{2}\right][x]$. Hint: There are three such factorizations.

Problem X-6. Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA.

Let n and k be fixed integers with $0 \leq k \leq n$ and let $a_i = C(n+i,k)$. (This is a binomial coefficient or number of ways of choosing k things from $n+i$.)

Find a formula for a_4 in terms of a_1, a_2, and a_3.

Set V Solutions

Do Not Pass GO

Problem V-1. Proposed by Travis Thompson, Harding University, Searcy, AR, and edited by the Problem Editor.

Suppose we have a linear array of properties (as in the Parker Brother's game of Monopoly), said array of length N + 1, with property #0 being GO and property #N being JAIL. (This part is not as in Monopoly.) Suppose your token starts at property #1. You toss a fair coin. If heads, you advance your token one to the right; if tails, one to the left.

What is the probability of reaching GO before reaching JAIL? Generalize your answer so the token starts at property #k ($0 \le k \le N$).

Solved by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Commuity College, Utica, NY; Mike Dellens, Austin Community College, Austin, TX; Mark de Saint-Rat, Miami University, Hamilton, OH; Stephen Plett, Fullerton College, Fullerton, CA; and Nelson Thomas, Texas City, TX.

Comment: Michael Andreoli, Miami Dade Community College North, Miami, FL and Frank Soler, De Anza College, Cupertino, CA both noted that this is known as "The Gamler's Ruin Problem" and that solutions may be found in many probability books, including William Feller: *An Introduction to Probability Theory and Its Applications,* volume 1, pp. 342-345.

Let $p(k)$ denote the probability of reaching GO before reaching JAIL if you start at position #k. Then $p(k)$ clearly satisfies the properties: a) $p(0) = 1$; b) $p(N) = 0$; and c) $p(k) = (1/2)p(k-1) + (1/2)p(k+1)$. One may readily verify that $p(k) = (N-k)/N$ satisfies all three. As an alternative, let $p(N-1) =$ and inductively show that $p(N-k) = kx$ using rule c), this being known for $k = 0$ and $k = 1$. Then $1 = p(0) = Nx$ gives $x = 1/N$ and so $p(k) = (N-k)/N$.

Collection Agency

Problem V-2. Proposed by the Problem Editor.

You are attempting to complete a collection that has n distinct objects. There are infinitely many copies of each of the n items, and the probability of getting any one of the n types is always $1/n$, regardless of previous draws. On the average, how many items must you draw to complete your collection, which is to say, get at least one of each type? Can you give a nice asymptotic closed-formula expression for this? Generalize to collecting at least any m of the n types ($m \le n$).

Solved by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Frank Soler, De Anza College, Cupertino, CA; and the proposer.

Comment: Michael Andreoli, Miami Dade Community College North, Miami, FL notes that this is known as "The Collector's Problem" and that a solution appears in Feller's book on pages 224-225.

For Bernoulli trials with probability p of success, the expected number of trials to get a success is $1/p$. (For a proof of that, take $k = 1$ in problem V-3.) Having collected k of the n objects, we find the probability of drawing an uncollected object is $(n-k)/n$ and the expected number of draws to get the new one is $n/(n-k)$. The expected number of draws to get m different objects is

$$n\,[1/n + 1/(n-1) + \dots + 1/(n+1-m)].$$

The expected number to get at least one of each of all n objects is

$$n[1/n + 1/(n-1) + \dots + 1/2 + 1/1\,] \sim n[\ln(n) + 1/(2n) + \gamma]$$

where γ is Euler's constant. Note: The unusual term $1/(2n)$ in the asymptotic formula is suggested by Charles Ashbacher, who notes that it gives significantly improved approximations.

Batter's Streak

Problem V-3. Proposed by Mike Dellens, Austin Community College, Austin, TX.

Player X has a .300 lifetime batting average. On the average, how many times theoretically must he come to bat to get one hit? ...two hits in a row? ...n hits in a row? (Assume the independence of each at-bat and ignore walks.)

Solved by Frank Soler, De Anza College, Cupertino, CA and the proposer.

Comment: Michael Andreoli, Miami Dade Community College North, Miami, FL notes that this is discussed on pages 322-324 of Feller's book.

Consider Bernoulli trials with probability p of success and let E be the expected number of trials to obtain k consecutive successes. Then E satisfies the equation

$$E = p^k k + \sum_{j=0}^{k-1} p^j (1-p)(1+j+E)$$

where the sum considers strings starting with j successes ($j<k$) followed by a failure. One may readily solve for E to obtain

$$E = 1/p + 1/p^2 + ... + 1/p^k = (1-p^k)/[p^k(1-p)].$$

Cubism

Problem V-4. Proposed by Stephen Plett, Fullerton College, Fullerton, CA.

Define the cubeness of a rectangular solid (or rectangular parallelepiped) with edges a, b, c ($a \le b \le c$) by $Q = a/c$. Call a rectangular solid Pythagorean if its space diagonal and edges a, b, c are all positive integers. Prove that there exist Pythagorean rectangular solids with cubeness arbitrarily close to 1. Are there such solids with cubeness arbitrarily close to 0?

Solved by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Mark de Saint-Rat, Miami University, Hamilton, OH; and the proposer. Partial solution by Robert Bernstein, Mohawk Valley Community College, Utica, NY.

The box with edges $a=1$, $b=2m$, and $c=2m^2$ has diagonal $2m^2+1$ and cubeness $1/(2m^2)$ which tends to zero as m increases. Now consider a box with edges $a=x-2$, $b=x+1$, $c=x+1$ and with diagonal y. Then $3x^2 + 6 = y^2$. The theory of the Pell equation says this has an infinite number of solutions. The smallest is given by $(x,y) = (1,3)$, and if (x,y) is a solution, so is $(2x+y, 3x+2y)$. The cubeness of these boxes is $(x-2)/(x+1)$ and tends to 1 as x increases.

Impossible Compression Ratio

Problem V-5. Proposed by Frank Flanigan, San Jose State University, San Jose, CA.

Consider monic real polynomials P(x) with all zeros real as mappings $z \rightarrow P(z)$ of the complex z-plane. Which of these mappings compress the y-axis toward the origin in the sense that $|P(i)| < 1$? (Here, $i^2 = -1$ as usual.)

Solved by Mark de Saint-Rat, Miami University, Hamilton, OH; Stephen Plett, Fullerton College, Fullerton, CA; and the proposer.

If $P(z) = \prod_{k=1}^{n} (z - r_k)$ has real zeros r_k, then

$$|P(i)| = \prod_{k=1}^{n} \sqrt{1 + r_k^2} \geq 1 .$$

Thus, no such polynomial can contract the y-axis.

Comments on Earlier Problems

Nora S. Thornber, Raritan Valley Community College, Somerville, NJ notes that the condition $u_{xx}u_{yy} - (u_{xy})^2 = 0$ in problem U-5 is the defining relation for a developable surface $z = u(x,y)$. These surfaces have been studied extensively.

For problem S-2, Robert Bernstein, Mohawk Valley Community College, Utica, NY observes that it is easy to see that the number of ways to get a sum less than $n + 1$ on k rolls of an n-sided die is the binomial coefficient $C(n,k)$. The sequence of k partial sums is just a choice of k different numbers from 1 to n listed in increasing order.

Book Reviews

Edited by John J. Edgell, Jr.

THE WAITE GROUP'S NEW C PRIMER PLUS, Mitchell Waite and Stephen Prata, SAMS, a division of Prentice Hall Computer Publishing, Carmel, Indiana, 1990, 654 pages, (softbound), $29.95, ISBN 0-672-22687-1.

The grade assigned to this book is extremely dependent on the desired use. If you are a novice programmer and wish to use it as a how-to book, then it is as good as can be found. However, if your goal is to find a textbook for an introductory course in C programming, or you are an experienced programmer wishing to learn C, then it is probably not strong enough.

Many of the example programs tend to be "cutesy", with few longer than a half-page in length. New material is introduced at a very slow pace. There are exercises at the end of each chapter that allow for coding of the new material, but again the level of difficulty is low. There are very few that could be considered sufficiently challenging to be assigned as programming assignments in a college level course. All of which are excellent for the beginner with little or no programming experience, as this places the readability at a very high level.

This book is an excellent fit for a particular niche, but somewhat of a square peg in a round hole if used in any other situation.

Reviewed by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA 52402.

SYMBOLIC COMPUTATION IN UNDERGRADUATE MATHEMATICS EDUCATION, edited by Zaven A. Karian, Mathematical Association of America, 1992, 181 pages, (softbound), ISBN 0-88385-082-6.

The noise that teachers of mathematics are hearing in the background is the inexorable march of symbolic computation into the classroom. With the increase in capability and drop in price, it has reached the point where the consensus is that some mathematics programs are now arguably irresponsible to their students if they do not at least introduce symbolic computation.

This collection of papers points out many of the peaks and pitfalls one faces when attempting to introduce the students to abstract computation on a computer. Although there is a section dealing with reviews of the current material, it appears last and most of the problems presented are independent of the hardware and software being used. A few of the papers also discuss the integration of the graphing calculator. All of the articles are well written and easy to follow.

The wide range of difficulties discovered points out how complex the psychology of human-computer interactions really is. Sometimes it appears as rich as human-human interactions. The triad of teacher-computer-student is fast becoming the new "eternal triangle."

If you have any plans to integrate symbolic computing into your program, read and study this book first. You and your students will thank you for it.

Reviewed by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA 52402.

BEGINNING ALGEBRA, (5th ed.), by Alfonse Golbran, PWS - Kent Publishing Co., Boston, MA, 1991, xiii + 523 pages. ISBN 0-534-92443-3.

Audience

Golbran intends *BEGINNING ALGEBRA, 5th ed.*, to be used as an "introduction to the fundamentals of algebra for students with little or no background in the subject." The text is well written for that purpose. Golbran's text should be considered for use in the following courses: Advanced Eighth Grade Algebra, Mainstream High School Algebra I, Introduction to College Algebra. (Also, see **(POSITIVE) REMARKS** below.)

Format

BEGINNING ALGEBRA, the *5th ed.* is formatted a little differently from other texts in a positive way. The material is divided not only into chapters divided into sections, but some sections are divided into subsections, each followed by many problems dealing specifically with the topic of the subsection. (Perhaps most texts do have this same division of material, but most texts don't supply problems relevant to a subsection *immediately* following the subsection.).

Also, *BEGINNING ALGEBRA, 5th ed.*, provides sufficient and working definitions, plenty of topic relevant examples (graded in difficulty), and a wealth of problems (graded in difficulty) following each chapter, section, and subsection.

(Positive) Remarks

- Golbran's examples and problems deal predominantly with the algebraic operations that are essential to mastering the topics being discussed.
- The examples and problems are indicative of the algebraic operations necessary to solve further problems in algebra.
- No cloudiness is involved in the examples or problems that may mislead or confuse students about the objectives of a section's topic.
- *BEGINNING ALGEBRA, 5th ed.*, provides an abundance of clean concise, "topic-closed" problems. (For this reason, the text would serve well as a refresher text for High School Algebra II or College Algebra students.)
- The book would be an asset to any learning lab.

Reviewed by Jefferson A. Humphries, Southwest Texas State University, San Marcos, TX 78666.

Send Reviews to: Dr. John J. Edgell, Jr., Editor, Book Reviews, *The AMATYC Review*, Mathematics Dept., Southwest Texas State University, San Marcos, TX 78666.

Answer to **The Dead Mathematicians Society Crossword Puzzle**, p. 51

E	I	N	S	T	E	I	N			A	R	C	H	I	M	E	D	E	S		
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We should be able to talk about our careers with our friends, but the trouble with insisting on discussing mathematics is that it diminishes the supply of friends.

Peter Hilton

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