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ABSTRACT

This collection of case studies of classroom experiences in middle and secondary schools throughout the European Community describes the use of computers in mathematics education. The 16 studies are organized in four main groups: (1) experience in geometry using educational software and Logo in grades 5-10; (2) arithmetic number concepts, statistics, and probability in grades 6-7, using computers to work with numbers and simulate random events; (3) the concept of numerical function and its graphical representation in grades 10-11, including calculus, algebra, and trigonometry; and (4) new topics or new approaches to the mathematics curriculum in grades 11-12, including modeling and combining mathematics and computer science concepts. Further characteristics of the case studies include (1) the different kinds of software--games, tutorial-practice, programming languages, microworlds, spreadsheets, databases, and subject oriented flexible tools, such as MathCad; (2) the origins and nature of the teaching and learning process--group work, problem solving, exploratory activities, searching and interpreting information; and (3) patterns of classroom organization--mathematics laboratories, resource centers, and computers in the classroom. The annexes contain the names and addresses of the teachers and a list of the software packages described in the case studies. (ALF)

Comission of the European Communities

Using Computers in Mathematics Teaching

A collection of case studies

João Pedro Ponte
Fernando Nunes
Eduardo Veloso

Projecto MINERVA • Departamento de Educação
Faculdade de Ciências • Universidade de Lisboa

Using Computers in Mathematics Teaching

A collection of case studies

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Contents

Introduction

This book is intended as a collection of significant and diverse experiences concerning the use of computers in mathematics education, at middle and secondary school levels.

Computers have been regarded for the last ten years as one of the most promising factors that can influence education, and many projects both within the European Community and elsewhere have been carried out to study, their educational potential and implications. Quite sophisticated investigations have been reported in mathematics education research journals and congresses. Many proposals have been put forward in teachers' periodicals and meetings. But little is known about what is really happening in the field. What is going on in the classrooms? Are there many teachers actually using computers? What are they doing?

The experiences assembled in this collection are first of all classroom experiences. In this work, we were not interested in looking at research conducted in laboratories, usually with fairly contrived tasks and very small groups. On the contrary, we wanted to know what is happening in the actual scene of the teaching/learning process.

Some of experiences are based on the collaboration in innovative projects of teachers, teacher educators, curriculum supervisors, and university researchers. Other experiences, however, derive mainly from the initiative of the teachers themselves. In all cases we aimed at experiences in which the teacher played a clear significant role, expecting that it could encourage other teachers to try similar or different experiences of their own.

The book is thus aimed at a large public. It should be of value to all practicing teachers interested in present day educational applications of computers. Of course, it should be also of interest to inspectors, curriculum supervisors, teacher educators, preservice teachers, etc., in short, to all those who are concerned with the teaching and learning of mathematics.

Deliberately, the descriptions of the experiences avoid the use of a dense technical language and do not attempt to explain all the intricacies of the software and hardware used. It is expected, however, that the information given will be more than sufficient to allow the reader to understand what was happening in every case.

This book was also for us an opportunity to carry out an informal survey of the current situation in European countries, in which there is a variety of school systems and curricular orientations (some more sympathetic towards computers than others). We hoped that speaking with a good number of teachers and seeing classes in operation would help in identifying what are the key pros and cons of the different approaches and in devising the main lines of the most likely future developments.

The Case Studies

The case studies presented in this book come out of a selection made from a larger number of experiences. We were concerned that the selections would have significant curricular implications. Fortunately that was true in most of the case studies and the illustrative interest of the activities and the availability of detailed and precise data become the major selection criteria.

The cases were organized in four main groups, three of them related to existing curriculum areas and another to new topics likely to become important in the future. Within each group they are ordered increasingly by the grade level of the students involved.

The first group of cases deals with experiences on geometry, using both educational software and the Logo language, in a variety of ways, in grades 5 to 10. Reversing a declining trend that was apparent since the sixties, geometry is getting a renewal of attention. The graphical capabilities of present day microcomputers make them a very suitable medium of work for this subject through the creation of specific microworlds. A common feature of these case studies is the active involvement of the students in problem solving and investigational mathematical tasks.

The first case study refers to a year long experience conducted by two middle school mathematics teachers Maria José Delgado and Helena Bártolo, with their fifth grade students. With both PC compatible computers and 8 bit machines, Logo was successfully used in a quite free and unstructured way to explore the geometric concepts of the curriculum.

Case study 2 relates the work of another middle school teacher, Jean César, in France, with seventh grade students. Using Logo, he developed a microworld called *INTERDIM*, to study 3-dimensional geometry, as a natural extension to curriculum topics. The students worked in groups, constructing Logo procedures to draw and manipulate objects in space.

The next case study refers to an experience carried out in cooperation by a a university mathematics educator, João Filipe Matos, and a secondary school mathematics teacher, Isabel Amorim. Logo was used in remedial classrooms to construct specific microworlds to support students' investigations. It is particularly significant how these eighth grade students, usualy regarded as poor achievers lost for mathematics, got involved with enthusiasm and showed a positive attitude towards this kind of work.

Case study 4 describes the work carried out by a secondary school mathematics teacher, Adelaide Lister, with mixed ability ninth grade classes. In a group of four lessons the students were introduced to several educational games meant to develop the concepts of vector and translation. The scarcity of computers made it necessary to find alternative tasks for some of the students, who took their turns at the computer, an arrangement that has been seen also in other experiences. According both to the teacher and the students this was a very positive and rewarding activity.

Another case study describes the work carried out in a secondary school in the countryside of Portugal, under the leadership of Manuel Saraiva, a university teacher educator. It involved two tenth grade classes using a tool constructed in Logo for exploring and solving problems in analytic and vector geometry. The tasks were presented to the students in worksheets, discussed and worked in groups, and finally object of a written report.

Another group of case studies includes those that have to do with arithmetic number concepts, statistics, and probability. These experiences explore the possibilities of the computer to work with numbers and to simulate random events.

One case in this group describes the work led by Adrienne Ashworth, in England, on the concept of probability, using an easy to learn computer spreadsheet with sixth grade students. This experience involved all the teachers and students at this grade of the school, with the interesting feature that only a small number of computers were used in each class.

Case study 7 concerns a small but quite interesting and easy to use piece of software, Trinca-Espinhas, an educational game available in different kinds of computers. The reported experience comes from a curriculum development project, MAT_{789}, and was carried with seventh grade students, but the program is widely used with middle and secondary school mathematics classes. It deals mainly with the concepts of number, divisor, and prime. In this case the program was intended to develop a better number sense, to stimulate problem solving strategies, and to develop capacities of group work, discussion, and written communication.

The next case study refers to another experience carried out by Adrienne Ashworth and her colleagues, now with a seventh grade class. Data handling and statistical concepts were the focus of attention in this work, in which a data base was used. This activity, which was regarded as highly successful with the students, led the teachers to consider a wide range of ideas for future exploration.

The third group of case studies includes those that deal with one of the most important topics in the traditional secondary school mathematics curriculum: the concept of numerical function and its graphical representation.

Case study 9 describes the use of a tutorial program specially developed for the teaching of trigonometric functions, carried on in the Netherlands by an interdisciplinary team led by H. J. Smid with tenth grade students. This program was developed as a specific response to the difficulties felt by the teachers in teaching this topic.

The next case concerns the use of the program VU-GRAFIEK, a piece of software specially designed to study functions and graphs and promote an introduction the fundamental ideas of calculus. This program, which is rather esay to use, was developed by Piet Van Blockland, in the Netherlands and is commonly used through high school. The experience here reported refers to a tenth grade class, in which the use of the program enabled the clarification of some common misconcetions and stimulated the students to extend the exploration of some of the situations further than their teachers' expectations.

Case study 11 describes a full year long experience carried out by two Portuguese secondary school mathematics teachers, Susana Carreira and Georgina Tomé, who worked in different schools. They used a spreadsheet supported by worksheets as a basic resource to introduce through problematic situations most of the eleventh grade mathematics curriculum topics, including sequences, limits, algebraic and trigonometric functions, and derivatives.

The last case in this group refers to a series of experiences undertaken in several English schools on an initiative of a NCET project coordinated by a university mathematics educator, Kenneth Ruthven. The pocket calculators used have quite sophisticated graphical and programming capabilities, making them hard to distinguish in this respect from common microcomputers. This sort of technology, by its potentialities, price, ease of transportation, and highly personal nature, may become very significant in a near future in mathematics education.

Finally, the fourth group includes case studies that deal with new topics or new approaches to the mathematics curriculum. Two of them refer to modelling and the other report experiences combining mathematics and computer science concepts.

One of such cases describes an experience carried under the leadership of Elmar Cohors-Fresenborg, within a curriculum development project supported by the University of Osnabruck, in Germany. It aimed at the introduction of an algorithmic approach in the mathematics curriculum and used a simulation program for a Register-machine. This activity, which went along for the full academic year, constituted a sucessful introduction to new topics and ideas, such as the notion of function of several variables, appeared to promote a better understanding of usual concepts such as proportion, percentage and the ability to deal with world problems.

Another case refers to a full year long experience of integrating informatics in the mathematics curriculum done in Genoa, Italy, by a secondary school mathematics teacher, Ivana Chiarugi, with the cooperation of a university research group led by Fulvia Furinghetti. Topics usually presented in introductory computer science courses were taught to the students, as well as notions of operating systems and the use of a spreadsheet. The main idea behind this experience was to use the notion of function as a unifying concept and to introduce the study of algorithms.

Case study 15 refers to an activity using a special modelling language, VU-DYNAMO, within a government project in the Netherlands. The students, at eleventh grade, worked for about [illegible] lessons on modelling activities on the dynamics of systems of different kinds, including linear and exponential growth and decay, and discussing issues such as stable and unstable equilibrium states.

The final case study presented in this book describes an experience carried out in a group of five lessons in Belgium by a secondary school mathematics teacher, Gerda Timmermans. The twelveth grade students who were involved worked in groups of two in modelling activities concerning real world situations. They used the program MathCad, with special files specifically designed for this activity. The main idea was to use the program to do the calculations and let the students devote most of their time to reason about the situations and discuss the related mathematical concepts.

Characteristics of the Software

In these case studies we see in use a mostly uniform kind of hardware (PC compatible computers), but in contrast a large spectrum of kinds of software.

Some programs can be classified as belonging to the most traditional family of educational software, including educational games (cases 4 and 7) and tutorial-practice programs (case 9). These programs, when well designed, are not difficult to integrate into the curriculum and may serve valuable educational goals.

However, these programs are often of a limited scope. The reported experiences show how this may be overcome, devising rich and powerful educational situations. In cases 4 and 9, since the students were working in pairs, there naturally emerged a dynamic of discussion and reflection. In case 7, the fact that students were stimulated to write down and discuss strategies, created a highly participative and reflective atmosphere, much more involving than what would have been achieved with the simple running of the program. As this case shows, in using educational software, it is often of prime importance what is done before and after its actual use by the students.

In the case studies is also represented the use of programming languages, in a variety of situations. Programming activities were arried in Logo (cases 1 and 2), Pascal (case 14), and in special program simulating a Register machine (case 13). The programming language Logo was also used as a support for the elaboration of specific microworlds, namely geometrical environments (cases 2 and 3) and a geometrical tool (case 5).

Programming activities have been proposed for a long time as deserving a role in the mathematics curriculum, given the growing importance of the algorithmic aspects in mathematics. This process is just very slowly moving forward. Logo got a place in some countries in an early stage of the mathematics curriculum, but it is more valued by its geometrical features rather than by its programming capabilities.

Microworlds have also been argued for by a number of writers. The proliferation of microworlds and the possibility of designing and continuously modifying microworlds to serve the students' process of construction of knowledge seems attractive to many creative teachers. However, making microworlds certainly requires a lot of work and expertise, if they are to work properly.

General purpose programs, namely spreadsheets and data bases, were also used (cases 6, 8, 11 and 14), remarkably with quite different grade level students. These kind of programs have been extensively used in many countries, namely due to the scarcity of alternative software. As these cases show, there are places of the curriculum were they naturally fit, although requiring in some instances supplementary written support materials (and some creativity from the teacher).

Four of the experiences (cases 5, 10, 15 and 16) suggest the potential of a new generation of educational programs which is just emerging—the subject oriented flexible tools. MathCad is a program with many mathematical functions, being specially useful to perform numerical calculations. VU-GRAPHIEK is a function graph plotter with a lot of extre capabilities. LOGO.GEOMETRIA is a tool to work on classical geometry constructions and problems. VU-DYNAMO is a language designed to model real world situations. These programs combine the subject specificity with the open ended character of the general purpose software tools. Several new programs of this kind have been recently announced and have not had yet the time to make their way into the classroom. They may be expected, however, to play an important role in mathematics education in a near future.

Innovation in the Teaching and Learning Process

The experiences described in this book have different origins and nature. Some are mainly driven by research, curriculum development and teacher education projects (cases 2, 3, 5, 7, 9, 10, 12, 13, 14 and 15). Others were carried by initiative of the teachers themselves (cases 1, 4, 6, 8, 11 and 16). Many of them combine aspects of an external intervention with a high degree of initiative and responsibility from the teachers involved. However, in all the cases we see traces of a quite profitable collaboration between teachers and other partners concerned with the educational process.

Some of the experiences have a marked innovative character in the teaching methods. Others rely mostly in existing objectives and methodologies. All the experiences made strong or at least partial use of group work. These were usually small groups, with 2 to 3 elements, in which a variable degree of autonomy was assigned to the students.

In some cases, problem solving and exploratory activities were the focus of the project. In a few experiences, different groups carried different activities at the same time in the same class, what proved to be possible but required careful management from the teacher.

In some of the experiences students had to develop their skills of searching and interpreting information, both regarding the tasks in which they were involved and the actual functioning of the computers and the software. The importance of registering data and making records of the work done was also stressed in a number of cases. These activities were the starting point for lively and extended discussions, providing the basis for a deeper process of reflection on the concepts and strategies used.

One of the most critical aspects concerning the use of computers in mathematics education regards the patterns of classroom organization.

Mathematics laboratories are necessary in most cases for introducing a large number of students to the use of a single program. The free access to computers, in a informatics room or a resource centre, is indispensable if they are to be regarded a normal working tool. One computer in the classroom may be useful for demonstrating a particular idea or method or to make a quick test of a conjecture. It may be possible to combine several types of organization: (a) computer laboratories, (b) classroom, with no computers or just one or two, and (c) open spaces such as informatics resource centres. However, most of the reported cases used either a computer laboratory or a reasonable number of computers. Still very little usage was found for the classroom with one single or a very small number of computers.

In all the experiences we see the computer making a positive contribution to the teaching/learning process. The teacher acquires new roles and modifies the relationship with the students.

These, in turn, tend to become highly involved, develop new skills and attitudes, engage in cooperative work, and become more reflective.

Building up the Cases

Collecting data for this book was not an easy task. To locate teachers using computers in mathematics education in a organised way is harder than what it may seem in the first instance. In most cases, the teachers involved had some general concerns, but had not clearly specified educational objectives and learning strategies. Quite rarely there were organised accounts of students' work and a general evaluation of the experience suitable for communication to a wider audience.

An effort was made to identify relevant experiences in countries within the European Community through contacts with ministries of education, Euryclée Centres, universities and other institutions with programs on curriculum development, research, or training of mathematics teachers, teachers participating in international meetings in mathematics education and teachers that we knew as activite on the field. The cases reported were build from those who responded to our requests.

They are described having as much as possible the teacher in mind. Most attention is given to the educational objectives, the materials used and to the roles of the teachers and the students involved. Whenever they are related to research projects no detailed information is given on their objectives, instruments and specific results.

This project was carried out with the financial support of the European Community. The collection of data and writing of the case studies was carried by the project team, composed by João Pedro Ponte, Eduardo Veloso and Fernando Nunes. General advice concerning the selection of cases and the style of the reporting was provided by a project steering committee, which included Sylvan Courtois, from Belgium, Alan Greenwell, from the United Kingdom and Luis Balbuena, from Spain. Alan Greenwell was also most helpful in his editorial support.

There was a concern for including a large variety of uses, exemplifying different educational approaches, illustrating as much as possible significant roles of the computer in mathematics education. However, time acted as a limiting factor. It would have been interesting, for example, to have included case studies relating a more straightforward use of Logo, programming activities in BASIC, or adventure games. We are quite sure that in these and other areas many additional interesting experiences may also exist. For some of them we even got preliminary data, unfortunately insufficient to write an acceptable case study.

We hope, however, that the cases described in this book illustrate what is happening in the most innovative and enthusiastic schools. The use of the computer in mathematics education is not just something that is advocated from the outside. It already has a clear place in the field. And although there are many questions still to be answered, and many developments yet to come, it seems to be making a definite and worthwhile difference for the teachers and the students involved.

Case Study 1

LOGO in the Mathematics Classroom

LOGO in the Mathematics Classroom

1. The school

The activities described in this case study took place in the preparatory school (5th and 6th grades) Almirante Gago Coutinho.

The school is situated in the middle of an area of private houses, built fourty years ago in the border of the town of Lisbon. Now the town has expanded and the school and the villas are not far from the center of Lisbon. but it remains isolated from the busy streets by that wall of private villas, inhabited by old people with no children. So a large number of pupils come from distant quarters where there is an excess of students in the local school.

The teachers are also relatively old in the school, and the renewal of the body of teachers is slow.

In 1986/87 the school was selected to belong to the national project for the integration of new technologies in the Portuguese primary, middle and secondary schools (Project Minerva), in connection with the Higher School of Education of Lisbon. The preparation, execution and evaluation of the project resulted from the initiative and effort of two teachers of the school, with some support from the colleagues and also from the School of Education.

In the year 1986/87 these two teachers worked with pupils in the informal setting of the Informatics Club. They used LOGO, Word Processing and Art Studium (a graphics software). During this work, the teachers saw how active and interested pupils became when they were faced with interesting problems and challenges, and were allowed to develop some open acivities with computer and LOGO. This was a major factor in their decision to carry on the experience with LOGO in the classroom next year.

2. Aims

The main objectives of this experience were the following:

a) To create in the classroom an environment favourable to learning, where the children feel at ease to express their ideas, to acquire an independent capacity for reasoning and to build their proper knowledge.

b) To help the children to gain appropriate habitudes for problem solving, such as

- to select and organize the necessary information and data
- to clarify the objectives of the work to do or of the problem to solve
- to try different strategies and to choose the most adequate to the problem at hand
- to criticize the solutions and evaluate the outcomes.

3. Material and Resources used

3.1 Hardware

Computers used in these activities were: three Amstrad PC

1512 monochromatic and two Timex Spectrum.

3.2 Computer Softwa

IBM LOGO and Sinclair LOGO

3.3 Other materials

The activities of the pupils were based mainly in worksheets prepared by the teachers or adapted from suggestions included in a handbook written in Portuguese.

4. Overview and Curricular Context

a) This experience consisted in the using of the LOGO language in the mathematics classroom during the two and third terms of the academic year 1987/88. The first term was used to help create good habitudes of group work among the children.

b) Problem solving was also a main concern, and for the whole year 5 to 10 minutes of each lesson has been explicitly dedicated to this kind of activity.

c) Problem solving and LOGO were used sometimes to introduce and develop curriculum contents and some other times to expand and complement the curriculum.

d) Three 5th grade classes with 24/25 children and two teachers were involved in the project. One teacher had two classes and the other had one but the planning and preparation of materials were made as a team work.

5. Description of activity

5.1 Teaching and Learning Styles

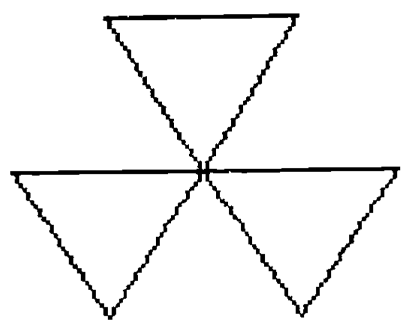
a) Children worked in groups of four or five. The groups were formed freely by the pupils but some changes have been necessary later on.
b) There was one table for each group in the middle of the room. At the back of the room, turned to the wall, there were tables for the computers.
c) In two of the three classes, the teachers were able to join two of the four weekly 50 minutes periods allowed for the discipline of mathematics For the second and third terms of the year, two lessons each week were dedicated to the learning of geometry, with the help of the computer and of the LOGO turtle geometry.
d) During these two hours, children worked in groups of two at the computer. There were not enough computers for every pupil, so the rest of the children developed parallel activities using other materials, like the geoboard.
e) In the beginning only the four main commands of turtle geometry were taught to the children. New commands were introduced naturally as soon as they were needed to solve a problem or build a procedure.

5.2 Record of activity

Follows a selection of activities proposed to the children. They correspond to the proposals usually used in LOGO, with children of this age level, after a period of open experimentation with the main commands of the language.

A.

1) Imagine a procedure that makes the following drawing

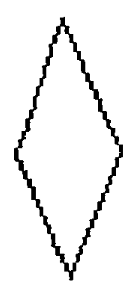


2) Try your procedure. If the drawing is not what you expected, change the procedure and try again.

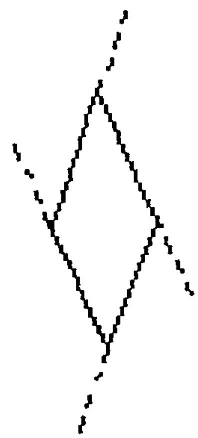
3) Using the procedure for a equilateral triangle, make a new drawing.

B.

1) Make a procedure to make a diamond



2) Record the measures of the angles you used.

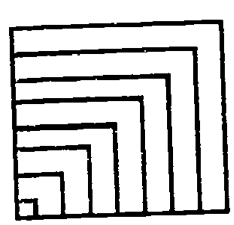


3) Try to relate the measures of the angles of the diamond.

4) Using the procedure for the diamond, make a new drawing at your own choice.

C.

1) Write procedures for making the following figures:



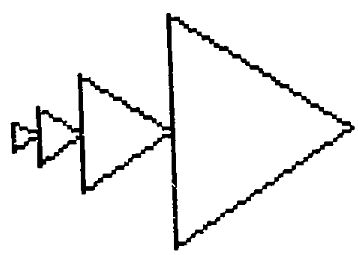
2) To make the squares, you used different procedures, each one with a different input to the command FD. Using a variable, you can make one procedure to draw all the squares. To see how this can be done, discuss it with your teacher.

3) Using the new square procedure with a variable, repeat the drawing of the squares.

4) Write a procedure with a variable to draw equilateral triangles.

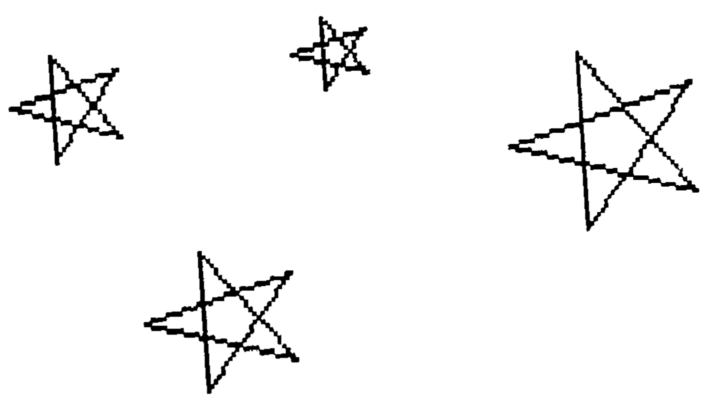
D.

1) Imagine procedures to make drawings using the same figure again and again, but with different dimensions. For instance



E.

1) Make a procedure to draw figures like the following stars



6. Conclusions

The activities with turtle geometry seem most appropriate for this level of scolarity. Children work with enthusiasm, engage often in discussions, develop the habitude of advancing arguments and do not show the passivity that characterizes other so called learning situations.

Computer activities with LOGO are well suited for group work. They are also a natural source for problem solving situations and for practice on developing and comparing strategies.

In this grade doesn't seem to exist a large incompatibility between the curriculum and this mode of using computers in the mathematics classroom. In any case, it must be said that some minor contents of the curriculum were not taught and also some computational skills were not developed to the level that is possible to attain in the normal courses.

Case Study 2

The Logo Turtle in 3D-Geometry

The Logo Turtle in 3D-Geometry

1. The School

Like all the French *Collèges* the secondary school of Jean-Jacques Rousseau at Voujeaucourt has students from the 7th grade to the 10th grade — 6 ème to 3 ème according to the French system — with age ranging from 12 to16 years old.

Voujeaucourt is a village situated in the suburbs of Montbèliard, a town famous for a strong automobile industry. The pupils' social background is very much marked by this fact although there is a mixture of professions among pupils' parents.

The school, which is 17 years old, has about 800 pupils, coming from 8 surrounding villages, and 60 teachers. It is a comprehensive school with mixed ability classes with the average of 24 pupils. The computers are no longer regarded as an innovation because they have been used by pupils since 1986. It was by that year that 6 teachers started to use computers in their classes and in different subjects such as physics, biology, mathematics, history, geography, French and English. Nowadays, the school owns 7 family computers and 9 Pc compatibles.

2. Aims

2.1. Improve students' skills in spatial geometry.

2.2. Relate different representations of the same geometrical object — plane representations, drawings in perspective and three dimensional models.

2.3. To allow pupils to foster curricular stated skills in doing geometrical constructions and drawings, and working in problem solving.

2.4. To provide opportunities for a wider range of approaches to geometry in space.

2.5. To give pupils a conceptual connection between plane and space geometry.

3. Material and Resources Used

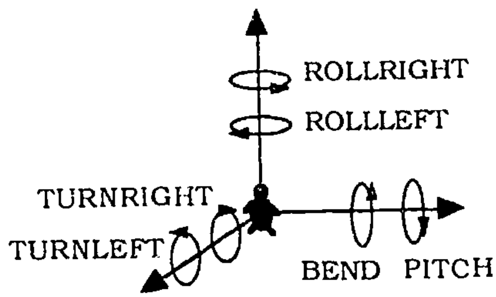
3.1. Hardware

In this experience 4 PC compatible mono-chromatic computers with 512K RAM were used.

3.2. Software

The software used was a 3D Logo turtle named INTERDIM (programmed on the Logoplus French version). This piece of software is a Logo microworld programmed by Jean César who teaches at Voujeaucourt secondary school. The turtle, besides the common Logo primitives such as FORWARD, RIGHT, etc, enables the drawing of perspective views of 3D objects. To perform this task a number of new procedures were added allowing rotations about three perpendicular axes. It is also possible to simulate the change of the position of the observer's eye and its distance from the object.

The different rotations about the three axes are indicated below, along with the corresponding English commands. The original commands are based in French.



3.3. Other Supporting Materials

Various geometric materials were used, including polyhedra models, small wooden turtles with the three axes, and worksheets. Dürer's windows, a device for perspectival drawing, can also be used.

4. Overview and Curricular Context

The work with *INTERDIM* is included in the *Logo Espace* project by the French Ministry of Education, Youth and Sports and began in 1986/87. Also included in this project are *Euclide-Espace* — a geometrical tool for space geometry — and *D3D* — a tool for technical drawing.

Besides mathematics classes, *INTERDIM* is used in other subjects such as art and design.

Regarding mathematics, this software has been used with students from 12 to 16 years old and is now used in five French secondary schools spreaded throughout the country. Periodic meetings have been held with the teachers involved in this project to exchange opinions and review progress. Because the materials are used in several grades and by a variety of teachers, there are different ways of class organization and methodologies. This description reports on the work in four 8th grade classes at the secondary school of Voujeaucourt.

In France and at this level, there are three lessons per week as far as mathematics is concerned. A fourth weekly lesson can be held if the teacher presents to the school a scheme of work that justifies this fourth lesson. It was during this lesson that the work with *INTERDIM* was carried out.

5. Description of Activity

5.1.Teaching and Learning Styles

The activities lead to the study of concepts related to polyhedra and foster pupils' spatial perception of three dimensional objects. In order to achieve this goal, a computational tool was used allowing accurate perspective representations.

Although we are focussing on the use of computers and the activities performed with them, the whole scheme of work would not be significant without other kind of activities using paper and pencil and other manipulative materials. The use of this specific software has to be viewed as another way to deepen pupils' understanding and must be linked to previous activities.

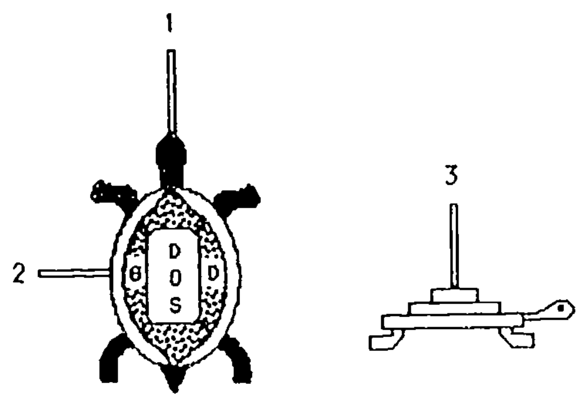
Most of the time pupils worked in groups of 2 or 3 in a classroom with 4 computers and plenty of space for drawing and manipulation. The class had 8 groups but only 4 were simultaneously in the same room. The space was necessary because of the need to relate and compare what was going outside and inside the computer. Pupils showed this need to test and validate their assumptions.

Individual work, when performing certain tasks, was also important and even homework was used.

The teacher had two major roles: assign the tasks and help pupils, either when s/he was asked or when he felt it was pertinent.

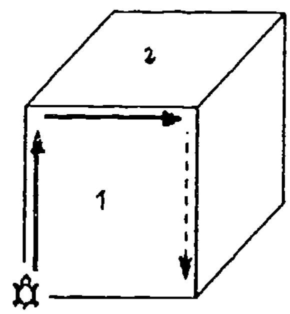
5.2. Record of Activity

After spending some lessons working with a Logo 2D turtle in order to get acquainted with the computer and the basic Logo procedures, pupils were asked to make, as homework, a turtle with three perpendicular axes and a model of a prism, using wood or other material.



Lesson 1 began with the following worksheet:

Place your turtle on your prism like it is shown below. Write down the Logo commands corresponding to the turtle *movements when it goes along the edges of face 1 followed by the edges of face 2.*

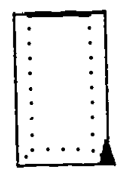


Obviously, pupils could not find the way to describe the shift from face 1 to face 2 using Logo commands, but they did it well using their own words. After the pupils were asked to draw on paper a perspective representation of the prismatic model they had chosen. The task took lesson 2 and in lesson 3 *INTERDIM* was introduced along with the basic primitives for perspective representation.

Lesson 4 was devoted for programming the chosen shape at the computer along with calculation of area and volume but firstly the pupils were asked to draw the faces assembled in order to get a plane representation of the surface. It took two lessons to end this task.

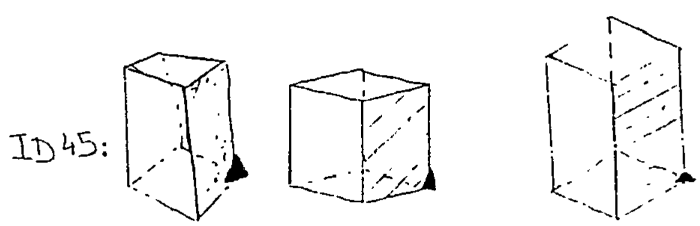
While the pupils were programming, the teacher walked around the room helping whenever necessary. Although the pupils had already the object drawn on paper, some students took time to understand the relationship between a tridimensional object and its representation on the computer screen.

In lesson 6, pupils were asked to foresee the changes that will happen on the perspective representation resulting from a rotation of the object, and to test their assumptions. For example, after the pupils have taught the turtle to draw a prism looking like the one drawn below, they were asked to preview how would it look if a roll right - *ROLLRIGHT 45* - command was given before.



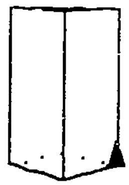
First computer drawing

With paper and pencil the pupils tried to sketch the perspectival view of the prism after this rotation of 45 degrees, marking the former visible face:



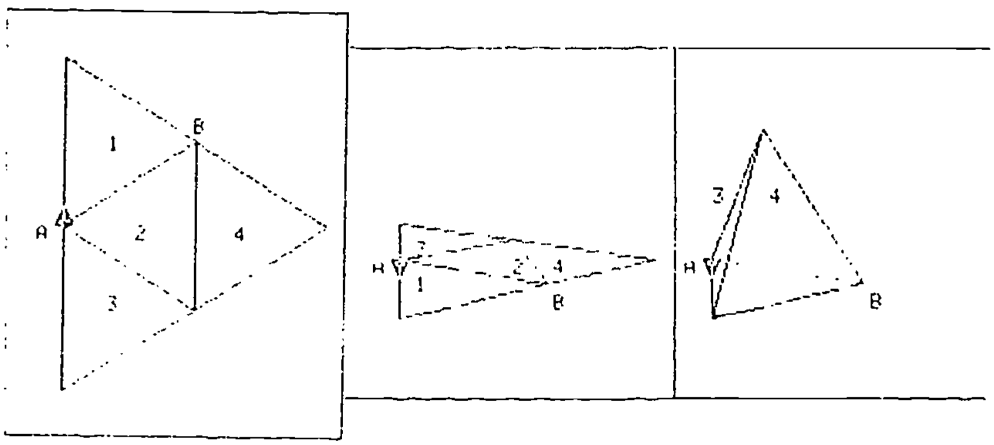
Pupils' drawings

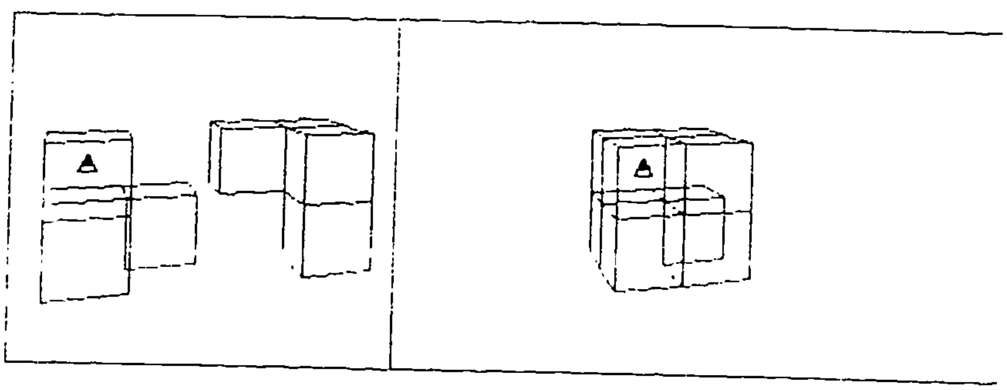
After they performed this task, the turtle was instructed to do the same drawing, allowing the pupils to test the correctness of their drawings.



Final computer drawing

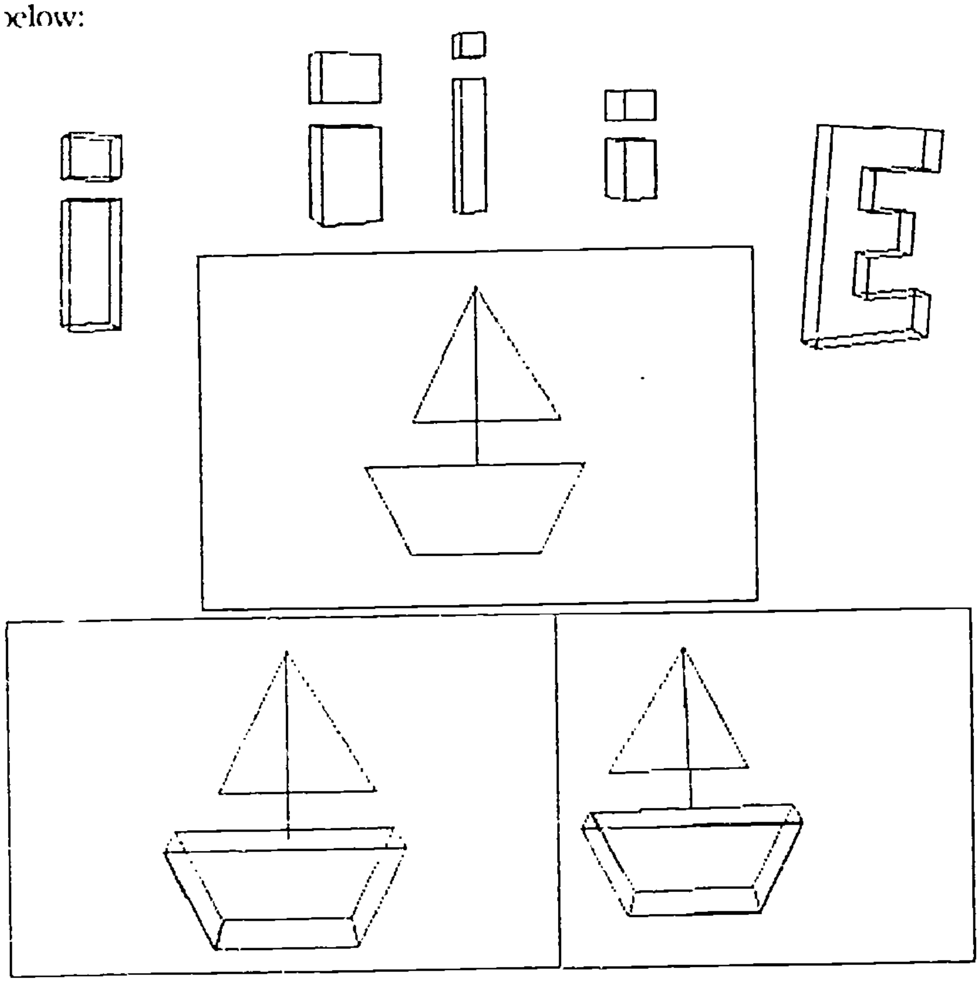
A similar study can be made with other polhyedra such as pyramids or tetracubes. For example, one group tried to assemble two tetracubes in order to get a larger cube.





40

The pupils worked also with plane objects they had programmed in Logo such as boats, houses and alphabet letters. Boats were given a third dimension in order to have perspective representations. This can be seen in the figures below:



6. Conclusions

Overall, pupils were proficient in using the software basic facilities — editing, saving and printing — and had a good response towards the proposed activities and sometimes they even proposed new developments based on the activities. In general, they did not complain when they had to make corrections or changes, which was a situation that often occurred, showing persistence while performing their work. As far as programming is concerned, some students showed difficulties in writing Logo procedures.

It seems that for this kind of experiment four is the ideal number of groups in the classroom because teachers, working within this project in other schools, noticed that group-work progressed slowly when six computers were in use. Computer malfunction was another negative aspect that teachers involved in the *LOGO ESPACE* Project in other schools reported, saying that sometimes the hardware proved to be unreliable. This was most annoying when there were six or more groups in the classroom.

Regarding the development of pupils' reasoning and spacial skills, evaluation tests were carried out with a group of project students and a comparison group in December 1986 (pre-test) and May 1987. The same tests and methodology were repeated by February 1988 (pre-test) and June 1988. From these evaluation tests some conclusions can be drawn. statistically, the project group had progressed more than the comparison group. The best results, regarding progression, were observed in the project students who had worst results in the pre-test implying that the significant results are among the pupils who showed weak logical/spacial abilities.

Case Study 3

Logo microworlds and Investigative Approaches

Logo Microworlds and Investigative Approaches

1. The School

Located in Lisbon, the secondary school D. Pedro V was founded 20 years ago and has students from the 7th grade to the 12th grade. It is a large school with more than 4000 pupils. Some of them are adults, about 1500, and follow their courses at school during the night, after their everyday work.

Regarding pupils' social background, there is a wide variety covering the whole range. All the classes are mixed ability as happens in all Portuguese schools.

Most of the 260 teachers permanent causing little change from one year to another. Both pupils and teachers live near the school.

The use of computers is widespread and began during the academic year of 1984/1985. Since then, several activities were developed using computers in various subjects such as mathematics, biology, physics and languages. Mathematics teachers played a prominent role in fostering these activities. During the last years, various curriculum development projects were carried out within mathematics classes.

2. Aims

2.1. To deepen concepts related to perimeter and area.

2.2. To increase pupils' skills in mathematical thinking and mathematical communication.

2.3. To increase students appreciation of mathematics.

2.4. To provide students opportunities to perform mathematical investigations.

2.5. To foster the capacity of students to deal with problem solving activities.

3. Material and Resources used

3.1. Hardware

Six computers IBM Pc compatible with 512K RAM and colour or mono-chromatic monitors.

3.2. Software

The Logo programming language has been often used in several educational environments. The now famous turtle geometry provides the setting for the activities developed within this case study. Indeed, turtle geometry is the most popular Logo feature. In the version used, the screen is divided in two parts: the *graphics* screen, where a small turtle moves according to the instructions which are written in the command centre, the lower part of the screen. This is illustrated on page 42.

Some Logo basic instructions are briefly described on the next page in order to help understanding the procedures used in these activities.

Box 1

fd *number*	moves the turtle **forward** a distance of *number*.
rt *degrees*	turns the turtle *degrees* to the **right.**
repeat *number list-to-run*	runs the list of instructions *list-to-run* the specified *number* of times.
if *condition list-to-run*	if *condition* is true. *list-to-run* is run, if it is false *list-to-run* is ignored; *condition* can be the result of a logical operation (or, and, not).

A Portuguese Logo version built on Logo Writer 1.0 was used to program a Logo microworld. Far from being complicated, this microworld allows the investigation about the relation between perimeter and area of rectangles. The first procedure sets a fixed value for the perimeter of the rectangles to be drawn:

to perimetro :l......................define a procedure called *perimetro* which has an input value (*l*).

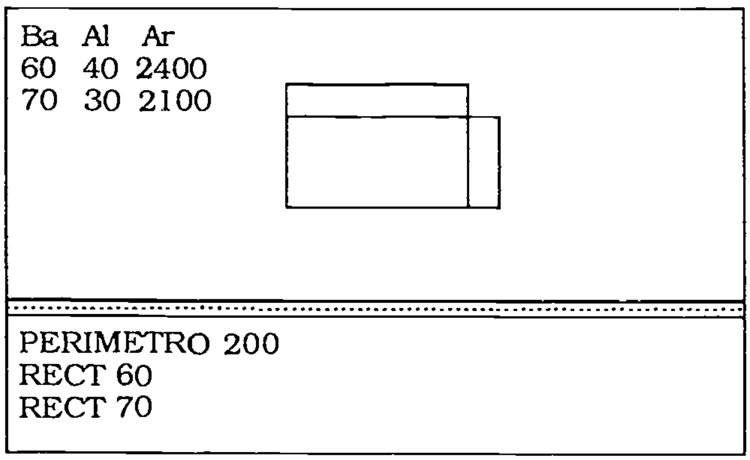
make "p :l..............................create a variable (*p*) and assign it a value (*l*).

pr (se "Ba "Al "Ar)...........print on the screen the sentence *Ba Al Ar* (stands for base, width and area).

end

The second procedure, after checking the input *b*, which has to be a positive number, draws a rectangle with the fixed perimeter and base equal to the input *b*. Note that *insert list* prints the list content without moving the cursor to the next line and *char 13* stands for a blank space.

```
to rect :b
if or :b = 0 :b < 0 [insert [I can't use this value! Choose a positive number.] insert char 13 stop]
if or :b = :p / 2 :b > :p / 2 [insert [I can't use this value! Choose a smaller number.] insert char 13 stop]
repeat 2 [fd :p / 2 - :b rt 90 fd :b rt 90]
pr (se :b :p / 2 - :b :b * (:p / 2 - :b))
end
```



3.3. Other Supporting Materials

Within this activity, pupils were given a worksheet. A full description is given below, in point **5.2.**, **Record of Activity**.

4. Overview and Curricular Context

The activities within this case study were part of the Logo and Mathematics Education Project which began in 1987/88 and is currently coordinated by João Filipe Matos of the Department of Education of the University of Lisbon. Its activities have been developed in several secondary schools in the Lisbon area. Both curricular and extra-curricular activities, in grades 7th and 8th, were carried out by several teachers within this project.

During the academic year of 1989/90 the project was extended to the Secondary School D. Pedro V. The activities described in this case study took place within an 8th grade class and were conducted by Isabel Amorim, a teacher of the school permanent staff. In this grade all classes have four fifty minutes lessons per week.

It is important to note that half of the pupils were repeating the 8th grade ; in Portugal, if the pupils do not achieve certain results it is compulsory to enrol again in the same grade, and 15 years old was the age average, too high for this grade. Mathematics played an important role in this lack of success. Most pupils do not appreciate mathematics which is viewed as a source of difficulties and a factor in future failures. An extra fifth weekly lesson was agreed by the pupils and the teacher. In Portugal it is possible to have these extra curricular lessons with pupils showing a low level of results. Usually the number of students is smaller than in regular lessons. The pupils divided in two groups, each one with 10 to 12 pupils, according to their own schedule preferences. The first lesson was carried out within this context while the second lesson took place in a curricular lesson with the whole class.

Prior to these activities, the pupils' experience in using computers was very limited. As far as mathematics is concerned, only six pupils had worked with Logo in the former academic year and the others have never used a computer within mathematics lessons.

Within this case study, the use of Logo has nothing to do with programming techniques. Pupils used the procedures to perform the activities and seldom showed the curiosity to look at the program listing which was loaded as *tools*, a facility which allows the loading of the program into the computer memory without displaying the text on the screen.

5. Description of activity

5.1.Teaching and Learning Styles

Throughout the extra curricular lesson, the students worked in groups of two or three at each computer. As stated before there were 6 computers in the computer room. The results from these activities were discussed during a regular lesson in a whole class discussion.

Students were given the tools in order to investigate a specific situation. They were free to try whatever seemed suitable to find the responses. This freedom to explore and the need to assess and discuss the consequences of decisions, is at the core of this activity.

The teacher's role was mainly to help students particularly when the difficulties were halting the students' progression through the investigation and to provide suggestions, placing new questions in order to encourage further investigative developments.

5.2. Record of activity

After spending four extra curricular lessons to get acquainted with basic logo commands and procedures, instructing the turtle to draw regular polygons, at the beginning of the lesson, pupils were given the following worksheet:

AREA AND PERIMETER

The procedure RECT allows you to draw a rectangle with given dimensions. For example, if you type PERIMETRO 200, the perimeter of the rectangles to be drawn will have the constant value of 200. If you type RECT 20, a rectangle with base 20 and perimeter 200 will be displayed on the screen as well as the values for its linear dimensions and area.

Using the procedures PERIMETRO and RECT, try the following:

PERIMETRO 200

RECT 10

RECT 20

RECT 30

Try other values and write down the results on the table:

PERÍMETRO = 200		
BASE	ALTURA	ÁREA
10		
20		
30		
40		
50		
60		
70		
80		
90		

1. What do you notice about the area? Can you explain how the area changes?
2. Which are the values for the rectangle giving the maximum area? Try with several values for the perimeter and table the results.
3. What are the connections between perimeter and area? Write down your own conclusions.

Students gave different inputs and observed the resulting rectangles, comparing the printed values variation and writing them on the table and they answered question 1. Only one group tried values not listed in the table.

By means of illustration here are some students' responses:

PERIMETRO = 200		
BASE	ALTURA	AREA
10	90	900
20	80	1600
30	70	2100
40	60	2400
50	50	2500
60	40	2400
70	30	2100
80	20	1600
90	10	900

João Gabriel

PERIMETRO = 200		
BASE	ALTURA	AREA
10	90	900
20	80	1600
30	70	2100
40	60	2400
50	50	2500
60	40	2400
70	30	2100
80	20	1600
90	10	900

Jorge - 424
Miguel - 420

1- Varia consoante a base e a altura

(It is variable according to the base and width)

51

Os valores da área são: altura x Base
A área 1º aumentou desde 900 até 2500. depois tornou a diminuir até 900 desde 2500

(Area values are: width x base. Firstly, area increases from 900 to 2500 - and then decreases - from 2500 to 900)

In order to answer question 2., pupils carried on the activities experimenting with values chosen by themselves. While some pupils tried several experiments, with different perimeter values, without a clear goal, others just made one or two trials to prove the assumption suggested in the previous question.

Dona Susana
8:43
424

Perímetro 100		
Base	Altura	Área
5	45	225
10	40	400
15	35	525
20	30	600
25	25	625
30	20	600
35	15	525
40	10	400
45	5	225

Perímetro = 200		
Base	Altura	Área
1	99	99
13	87	1131
25	75	1875
99	1	99
Perímetro 100		
39	11	429
49	1	49
45	5	225
25	25	625
5	45	225
10	40	400
15	35	525
20	30	600
25	25	625
30	20	600

... quando o rectangulo é o quadrado.

(...when the rectangle is a square)

Perimetro 100

ba	al	ar
1	49	49
15	35	525
20	30	600
25	25	625
30	20	600
49	1	49

2 Quando a base é igual à altura

(when the base equals the width)

In answering question 3 the pupils showed some difficulties. They felt confused and did not understand exactly what they were asked for. Almost every group required the teacher's help but most of them managed to wrote down their own conclusions:

A rectangle has maximum area if the base is equal to the width.

Using a fixed perimeter. the area increases or decreases according to the base changes ...area increases until it reaches the shape of a square.

Because is the same perimeter, when the base increases the area decreases and after it is the opposite.

The lesson with the whole class provided opportunities to deepen the concepts within the activities as well as a discussion about the methodology and how pupils related to it. Group work was an aspect that the students liked most. As one pupil stated:

"... to give our opinion and discuss it, seeing what is best,..., we learned to relate our ideas with others."

While performing the activities, only one group felt the need to use decimal numbers. As they claimed in the whole class discussion:

"We have tried with decimals, to see if we got a square... Because we did not know when it changes from a rectangle to a square."

The variable was viewed as discrete and, except for the group referred to above, all the pupils used integers for the input values.

6. Conclusions

Both observations made by the teacher and opinions expressed by the pupils provided information concerning these activities.

According to pupils' opinions, there were several reasons that kept them interested to the activities. The use of computers, a new technology which is widespread in the real world, and the opportunity to do "independent" work, that is, they were free to plan and experiment whatever they thought it was suitable without the usual teacher's role at the centre of the whole lesson, were evaluated as important features pervading the activities. As one student said "... *this kind of work is appealing even to students who dislike mathematics. We find it more interesting and have the will to try harder."*.

There was, as expected, much interaction between the elements of each group and even between groups and group work was another positive aspect the pupils noticed: "*to give our opinions and be able to discuss them with our partners, to see what it is better relating our views with others*".

Pupils were not familiar with group work and with investigations of this kind, so in the beginning they showed some natural difficulties. One of them had some hesitation when choosing perimeter values for subsequent trials.

The level of enthusiasm, while performing the activities, was great. This is clearly stated in the following comment: "*Although we got the wrong answer we tried again and again, because we had the curiosity of discovering...*".

It is important to state that the concepts underlying the activities - rectangle, square, perimeter, area.... - were not new to pupils. These concepts were supposed to have been studied in previous years. However, the teacher thinks that these activities fostered the pupils' understanding. For example, the first conclusion drawn by one pupil was that "*base times width is equal to the area of the rectangle*". Some pupils were puzzled when, following the instruction RECT 50 (for perimeter 200) a square was drawn by the turtle. This led to a discussion about squares and rectangles. It was agreed by the pupils that a square may be viewed as a special rectangle with four congruent sides.

Overall, it can be said that the students liked these activities and showed a positive attitude. There are clear indications proving this: the enthusiastic way in which pupils worked, either within their groups or during the whole class discussion, the views they expressed and their wish to keep on doing mathematical activities in similar contexts. This wish was fulfilled since they performed, throughout the year, more activities using an investigative approach.

Last, but not the least, it is important to underline that some pupils who were "lost" for mathematics, worked in a pleasant manner and gained an appreciation towards this subject.

Case study 4

Learning Vectors Using Computer Games

Learning Vectors Using Computer Games

1. The School

Rooks Heath High School is a mixed Comprehensive School located in South Harrow, an area in the northern suburbs of London.

The origins of the school go back to 1812, when it was first founded. In 1974 it got the present name and organization.

The school has about 800 pupils, from the whole social and ability range. There is also a mixture concerning religion. Although most of the pupils were born in England, there is a significant percentage - 40 % and rising - of students with Asian origins. The pupils are twelve to sixteen years old corresponding to 7th grade to 10th grade. The whole teaching staff has near 60 members, mostly young teachers, and the Department of Mathematics, a compulsory subject within all grades, has five teachers.

During the first two years in school, the curriculum is the same for all students. At the end of the second year in school, preparation is made for more specialised areas of study. The pupils may choose from a range of optional subjects. Most of them work towards the General Certificate for Secondary Education (GCSE) examinations which are usually taken at the end of the last year in school.

As stated in its Aims and Objectives, the school has embarked upon a reappraisal of its curriculum in response to developments in modern technology and changing patterns of employment. In this document several objectives refer to the development of pupils' skills in the handling, interpretation and analysis of information as well as to the use of available technological equipment. At the present time, the school has an IT (Information Technology) room with ten computers which can be used by every class. In addition there are six computers in the mathematics department.

2. Aims

The activities described in this case study were carried out within 9th grade classes and address mathematical concepts related to displacement vectors:

2.1. To understand displacement vectors and their representation by column vector.

2.2. To be able to identify equivalent vectors.

2.3. To understand the use of vectors to describe translations.

2.4 To provide pupils training in operations on vectors such as addition and multiplication by a scalar.

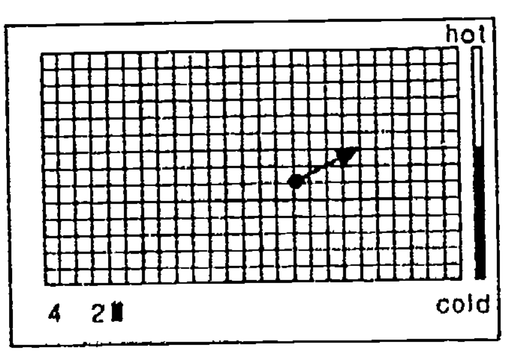
3. Material and Resources Used

3.1. Hardware

Four computers were used within the activities: three BBC and one RM Nimbus. The BBC computers are quite common in the United Kingdom. They are not IBM compatible. Nowadays, the RM Nimbus is replacing the older BBC's and although they are not quite PC compatible they are similar to them in several aspects.

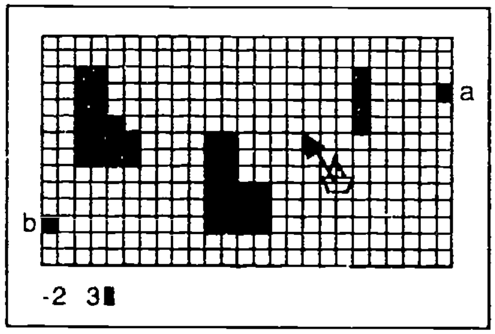
3.2. Software

Within these activities, the pupils played three educative games, Navigation, Treasure Hunt and Race, very easy to use. Essentially, the games are alike as they all aim at the same goal which is the learning of displacement vectors. In every game, the player moves a mark entering the coordinates of a vector. The mark will move according to the translation defined by this vector.

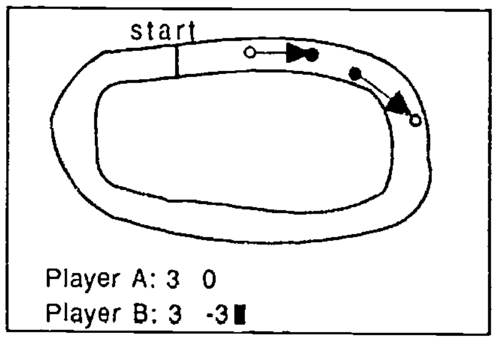


TREASURE HUNT
There is a treasure hidden on the screen. You have to find it. The distance to the treasure is displayed in a cold-hot scale.

NAVIGATION
Drive the boat from *a* to *b*. Be careful and do not hit the obstacles.



RACE
See if you can beat your colleague. Do not cross the borders or else you will stand at the same spot.



3.3. Other Supporting Materials

Worksheets, elaborated by the teacher, were used by the pupils within these activities. Besides the teacher's own ideas, these worksheets included questions drawn from a GCSE exam paper (Midland Exam Board) and from a 9th grade textbook (SMP textbook, Cambridge University Press).

4. Overview and Curricular Context

Regarding the school, the 8th grade classes are mixed ability while the 9th grade classes are from two different abilities (*yellow* or *blue*). The activities described in this case study concern the work done within 9th grade classes (the third year in the school) from both abilities and with an average of 24 pupils per class. These activities were proposed by Adelaide Lister, a teacher in Rooks Heath High School, and are currently performed in every 9th grade class.

Pupils arrive at this grade with some experience of using computers in mathematics lessons because the scheme of work, followed by every mathematics teacher, includes in the 8th grade the use of spreadsheets to investigate, for example, numerical patterns, and Logo to study the concepts of ratio and proportion. Logo is also used in the 9th grade to deepen pupils' concepts related to angles. Computers are used as well in the 10th grade. Both the IT room and the computers assigned to the teaching of mathematics have been used.

As stated before, mathematics is a compulsory subject. Grade 9 is the first year that has optional subjects, which take more than 40 % of the lessons. Vectors are a curricular item for this grade and the activities, regarding this case study, took place at the beginning of the academic year during three lessons of one hour.

5. Description of Activity

5.1. Teaching and Learning Styles

The activities lead to the study of concepts related to vectors in order to foster pupils' understanding on this matter.

Although only three lessons were dedicated to explore this subject, there was a variety of working methods. Pupils worked in groups or individually and sometimes there were explanations by the teacher to the whole class. In this case, the blackboard or a large computer screen were used. Besides the three programs described above, the software has an item in the initial menu allowing the teacher to introduce column vectors.

The class was separated into three parts, each one using the computers during one lesson on a rotating basis while the others were answering the questions on the worksheets. The pupils to work with the computers at each lesson were choosen at random. At the computers, pupils worked in groups of two, and on the worksheets the paper and pencil work was performed by the pupils individually, helping each other as several of them were seated at the same table.

Homework was also a part of the activities. At the end of the third lesson, a small worksheet was discussed to be solved at home. Later, pupils gave the solved worksheets to the teacher who, after having corrected them, returned the worksheets to the pupils.

Overall, the activities aimed at the study of vectors and underlying concepts. To convey this goal, pupils were given opportunities to work in different ways and with different materials allowing them to profit from the possible connections which can be done between the work on the computers and with paper and pencil.

5.2. Record of Activity

Lesson 1 began with a short introduction to displacement vectors on the blackboard with an example very much like the question on the next page to appear later on a worksheet.

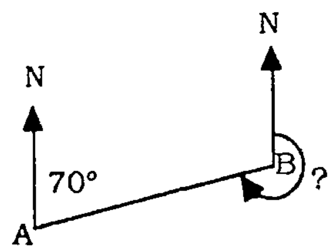
1. Mark a point P. An aircraft travels from P along the vector $\overrightarrow{PQ}$ (50 km, bearing 75°). Draw the vector $\overrightarrow{PQ}$ to scale, using 1 cm to stand for 10 km.

Another short introduction, this time to column vectors, followed using a large screen and the computer program. After the first four groups, of two pupils each, began the work on the computers while the others started to work on the worksheets. The following questions are part of the worksheet which has the first question written above

2. From Q the aircraft travels along $\overrightarrow{QR}$ (40 Km, bearing 20°). Draw a line at Q pointing North, measure the angle of 20° and draw $\overrightarrow{QR}$ to scale.

3. Measure the length and bearing of the vector $\overrightarrow{PR}$.

4. If the bearing of $\overrightarrow{AB}$ is 70° calculate the bearing of $\overrightarrow{BA}$.

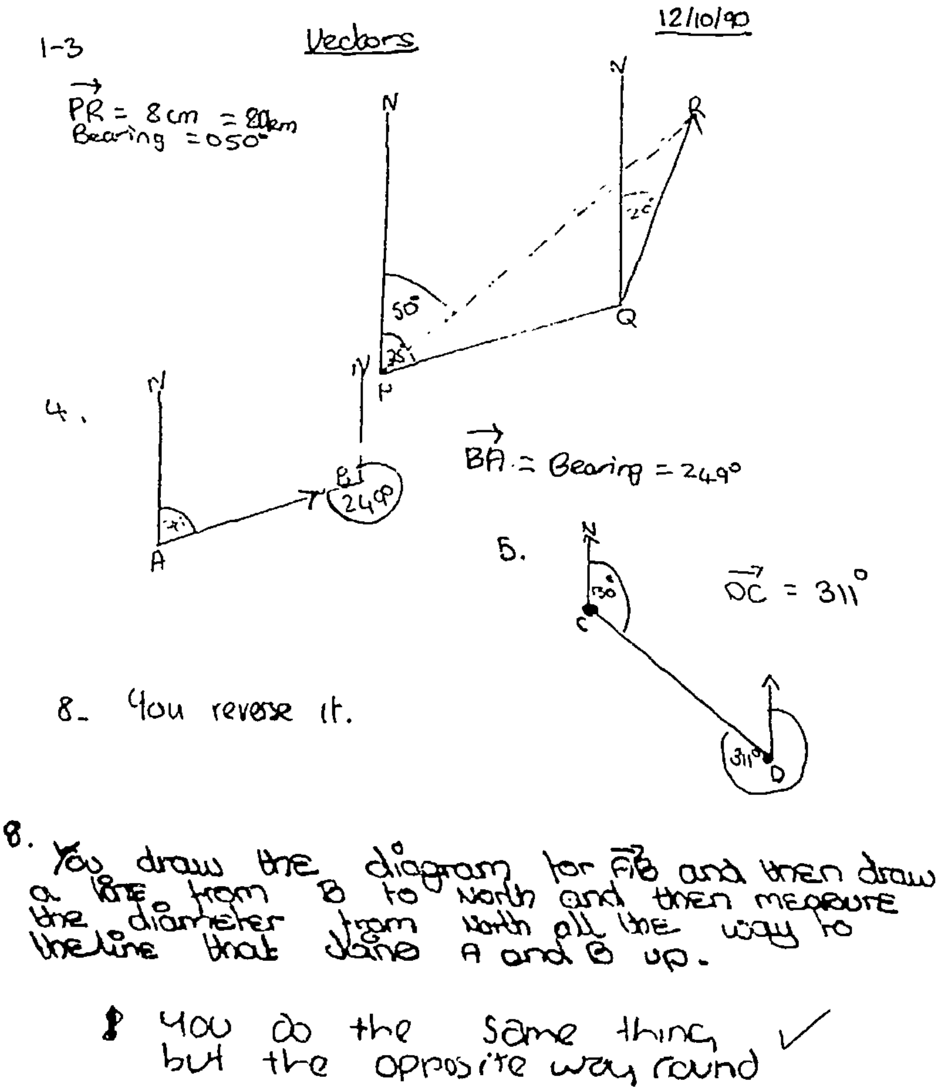


5. Draw a sketch of a vector $\overrightarrow{CD}$ whose bearing is 130° and calculate the bearing of $\overrightarrow{DC}$

...

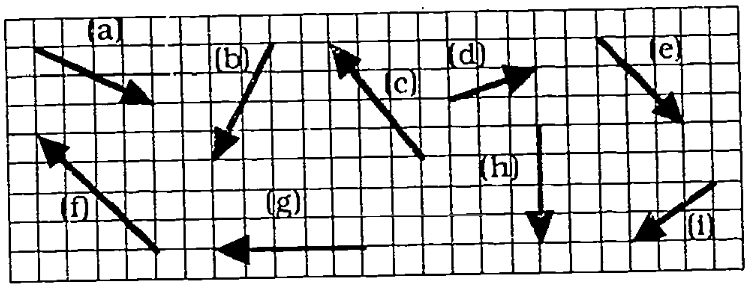
8. Can you find and explain the rules for working out the bearing of $\overrightarrow{BA}$ when you were told $\overrightarrow{AB}$?

Some pupils' work on these questions are illustrated below:



At the beginning of lesson 2, pupils rotated. The pupils who have worked with the computers in the first lesson, started the work with the worksheets and another eight pupils moved to the computers while the last third did go on the work with the worksheets.

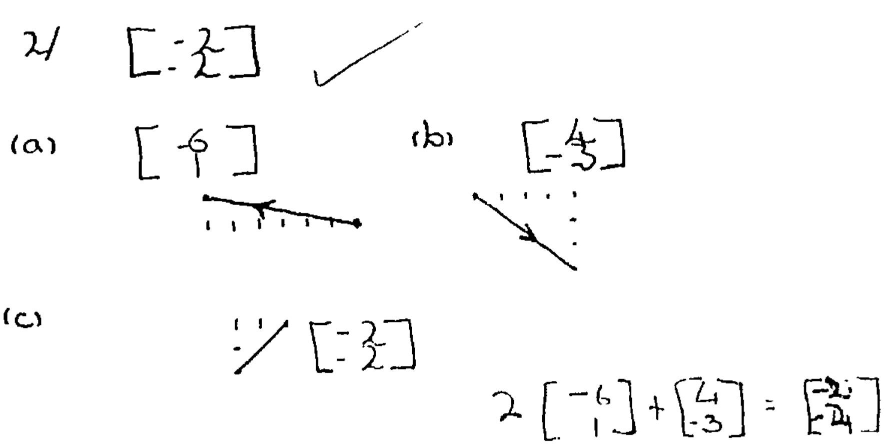
1. Write down the column vector of each of these...



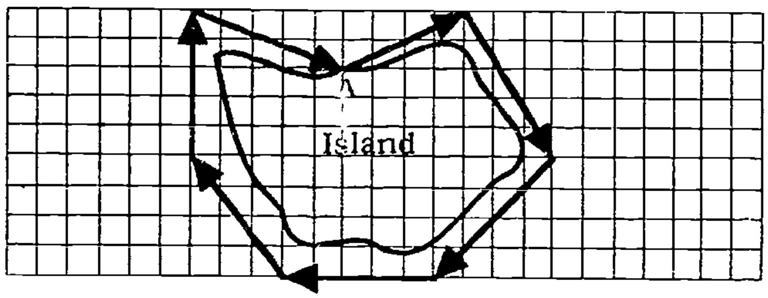
2. *Work out* $\begin{bmatrix}-6\\1\end{bmatrix}+\begin{bmatrix}4\\-3\end{bmatrix}=\begin{bmatrix}\ \\ \ \end{bmatrix}$

Now draw these vectors and show that your calculated resultant vector is correct.

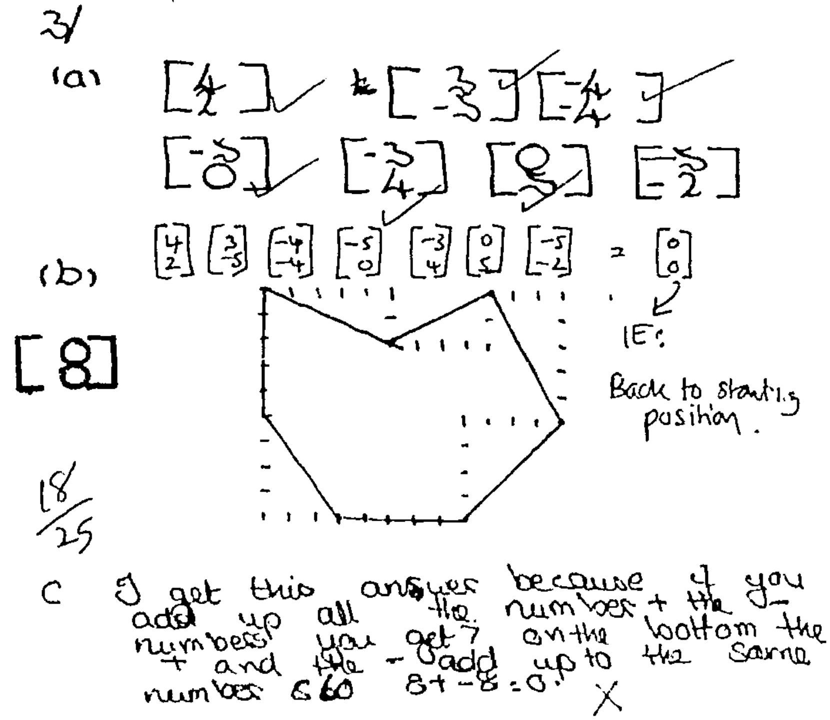
Although pupils did not explicate their reasoning in question 2.,they managed well to answer correctly these questions as it is shown in the sample below:



3. This map shows a boat trip round an island. The journey starts and finishes at A.
(a) Write down each column vector in the journey.
(b) Add all the column vectors together.
(c) Can you explain why you get your answer?

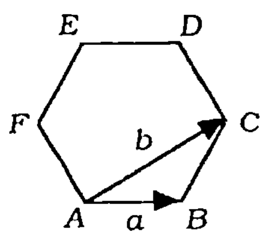


Like in the previous questions, the only difficulty pupils felt was answering *(c)*. This can be viewed on pupils' work and teachers' remarks:



6,

At the end of lesson 3, preparation for homework was done on the blackboard during a whole class discussion. The pupils copied into their books the following worksheet (the last part of the worksheet was drawn from a GCSE exam paper of 1989):



regular hexagon

$\vec{AB} = \underset{\sim}{a}$

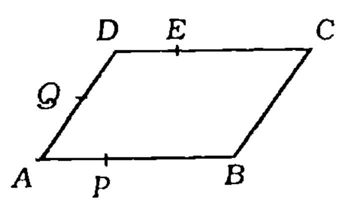
$\vec{AC} = \underset{\sim}{b}$

$\vec{BC} = b - a$

$\vec{AD} = 2\,(b - a)$

$\vec{AE} = 2b - 3a$

$\vec{AF} = b - 2a$



ABCD is a parallelogram.

$AP = \frac{1}{3}AB,\ AQ = \frac{1}{2}AD$

$DE = AP$

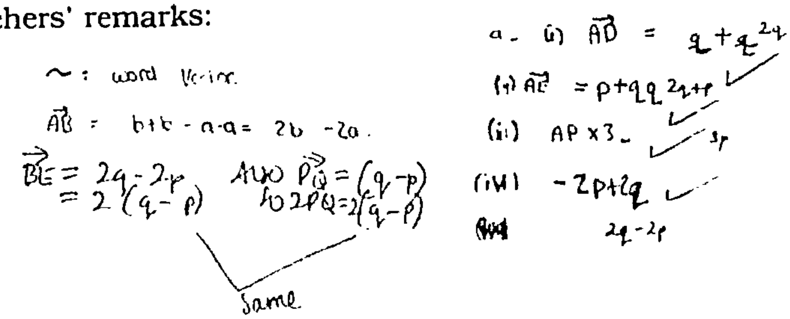
$\vec{AP} = \underset{\sim}{p},\ \vec{AQ} = \underset{\sim}{q}$

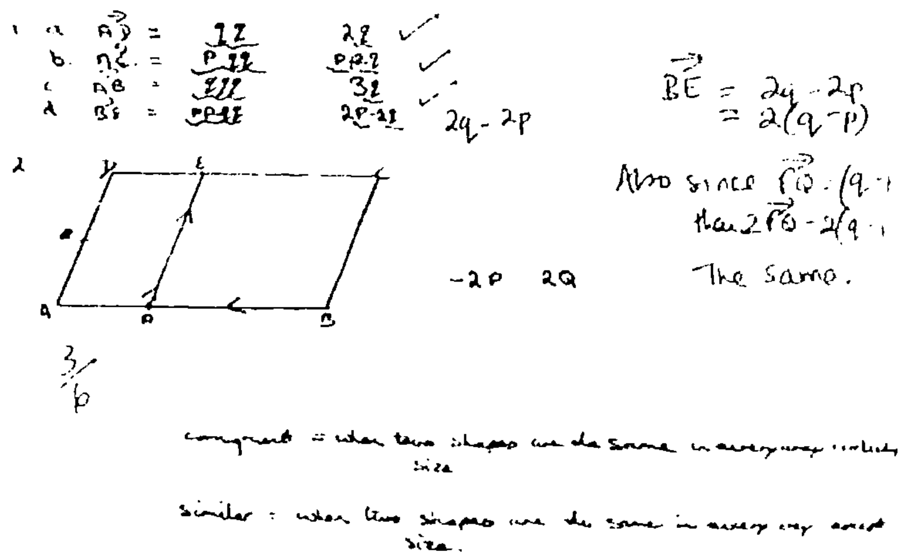
(a) Write these vectors in terms of $\underset{\sim}{p}$ *and/or* $\underset{\sim}{q}$*:*

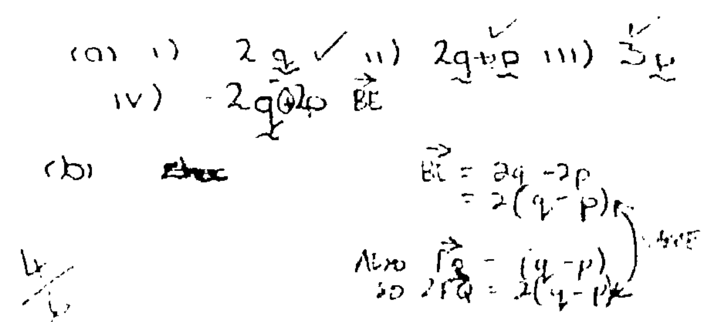
(i) $\vec{AD}$, (ii) $\vec{AE}$, (iii) $\vec{AB}$, (iv) $\vec{BE}$.

(b) Show that $\vec{BE} = 2\vec{PQ}$

As it is written before, pupils returned the solved worksheets to the teacher, and later were given them, with the teachers' remarks:







6. Conclusions

Overall, Adelaide Lister is very positive towards the outcomes of the activities. The pupils showed interest while performing them.

Regarding the use of computer games, and quoting the techers' opinion, th made the work more fun and interesting, more lively and worthwhile for the students. The pupils who began the work with the computers in the first lesson showed better results solving the worksheets than the others. They were more confident, asked less questions and got more right answers at the first time. Often, the pupils who started the activities answering the worksheets showed the need to review the answers after they had been using the computer games.

The single Adelaide Lister's criticism is related to these conclusions:

"This would be more effective if we had more computers. I would prefer to use the computer games with the pupils first and only then go onto the worksheets after"

Case Study 5

Learning Cartesian and Vector Geometry with a Computer Tool

Learning Cartesian and Vector Geometry with a Computer Tool

1. The school

This work was carried out at school Amato Lusitano, a secondary school with about two thousand students in Castelo Branco, a major town in a rural area well in the countryside of Portugal. This state suported school, is one of three existing in this town, and covers 7th to 12th grade students, that is, those that normally would be 12 to 18 years old. However, since students may be required sometimes to repeat an year, it is not uncommon that they are 19, 20, 21, and still attend secondary school.

The school offers several courses oriented towards university, attended by the majority of the students, such as electronics, informatics, art & design, business and sport, and also more technical oriented courses, with lesser attendence, in fields such as mechanics, business and agriculture.

The school is known by its quite young and innovative teacher body. It has 142 teachers, most of them on the permanent staff.

The students involved in the experience were in two 10th grade classes (aged 16 to 19). One class, with 27 students, was in the university track, taking a specialization in informatics. Another, with 16 students, was in the technical-oriented branch, with concentration in agriculture. In both classes, roughly, two thirds of the students lived in Castelo Branco and one third in the neighbouring villages. Only two students, both on the informatics class, had parents with higher education degrees.

2. Aims

The major objectives of this activity were to promote in the students, in their study of Cartesian and Vector Geometry:

a) the construction of the relevant concepts on this topic,

b) the ability of formulating and dealing with problem solving situations,

c) an understanding for the need and utility of proofs,

d) new attitudes and concepts about mathematics and about their role as students, and

d) the ability to work in groups, at their own pace.

3. Material and Resources Used

3.1. Hardware

This activity used 8 PC type computers of different brands (Amstrad, Philips, Unisys) and 2 printers. Six of the computers had a hard disk. All this equipment was available in a single computer room.

3.2. Software

LOGO.GEOMETRIA is an open computer tool designed to deal with classical Euclidian Geometry constructions. It is written in IBM LOGO, and is composed of several modules, each of which appropriate for a particular kind of problem. It is possible to construct points, lines and other two dimentional figures, and to perform operations such as 'calculate the length of a line segment' or to 'measure an angle' and to 'indicate if a point belongs to a line' (a full description is given in Veloso, 1989).

The program was complemented with new procedures to cope with the content covered in this activity.

When the program is loaded, it apears a cooordinate system, with axes x and y and origin O. At the botton a blanking cursor and 8 empty lines wait for the introduction of commands (see Box 1 on the next page).

3.3. Other Supporting Materials

The work of the students at the computer was based on worksheets. Typically, they had some structured questions leading to problems. Three examples of worksheet questions:

(a) Construct a line r with general equation

$$2x - 5y + 200 = 0$$

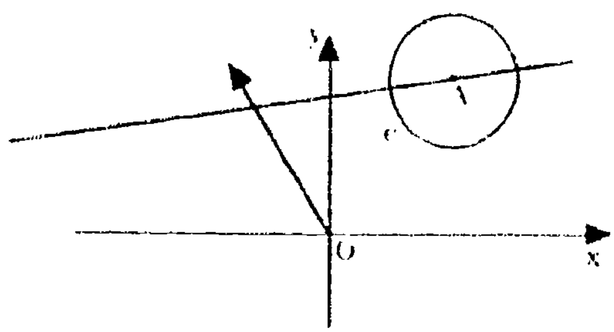
Find a relationship of the line slope with the coordinates of its versor. Consider other cases of your choice. Test your conjecture.

(b) Construct the circle passing through points A(100,40) and B(50,20), and whose centre is on the line with equation y = x. Draw and explain all you have done.

Box 1

As the student constructs points, lines, circles or vectors, those are kept in memory and drawn on the coordinate system. For example:

FAZ.PONTO "A [40 50]	draws a point at (40,50)
FAZ.RECTA "r [[-100 30] A]	draws a line passing through (-100,30) and A
FAZ.V "vect [120 60]	draws a vector making an 120 º angle with the X axis, with module 60
FAZ.C "c [A 20]	draws a circle with centre A and radius 20



Other commands include random constructors, operations and geometrical constructions. For example:

TRIACASO	draws a random triangle
DISTANCIA [A B]	gives distance between points A and B
BISSECTRIZ "r [p q]	draws the angle bissector of the smaller angle between lines p and q

(e) Construct a circle passing through point A(10,20) and B(20,10), tangent to the x-axis. Are there any other circles satisfying the requirements of the problem? Which ones? Draw and explain what you have done

4. Overview and Curricular Context

Cartesian Geometry is an important topic of the mathematics curriculum in most countries. In Portugal, basic vector concepts and Cartesian representations are introduced at grades 7 to 9, associated with the study of the simplest functions. Later on, in grade 10, lines and other geometric figures are studied using a vector approach which includes the notions of base and scalar product. Conic sections are studied in grade 12.

There were 2 introductory worksheets. They were designed to introduce the program LOGO.GEOMETRIA and dealt with problem solving situations involving concepts already studied in former grades, such as similar figures The remaining worksheets concerned new material, directing students' explorations and activities. Worksheets 1 to 4 dealt with operation with vectors, bases, scalar product and related problem solving situations. Worksheets 5 to 8 dealt with the line and the circle and related problems. In many of the problems, the students were asked to write down explanations for the methods they used.

This project was conducted as an experiment on the possibilities of a piece of software, LOGO.GEOMETRIA as an environment to support students' learning of this topic and to develop new attitudes and working habits in mathematics.

It was carried in collaboration by one university mathematics educator, Manuel Saraiva, and two secondary school mathematics teachers, Arminda Lourenço and Carlos Salvado. The university researcher provided the original proposal for the project and conceived most of the activities

These were afterwards discussed with the teachers and modified according to the indications got from the development of the experience.

5. Description of the Activity

5.1. Teaching and Learning Styles

In this activity the students, using a geometry computer tool, worked from activities and situations proposed given in worksheets, formulated conjectures and reflected on their strategies.

Every week, students had a two-hour class in the computer room. They had also one-hour classes in their regular room (2 for the agriculture and 3 for the informatics students). In the regular classes the teacher explored the work done with the computer, doing synthesis, formalizing, and proposing practise exercises. Sometimes new concepts were presented, to be further explored with the computer.

The informatics students, who were experienced in using the computer, were introduced to the program LOGO.GEOMETRIA in 2 classes. The agriculture students, who had little previous contact with the computer needed 3 classes. The instructions for the use of the program were introduced according to needs of each group, and the students were given a reference card with the main commands.

From this point on, the students did work based on the 8 remaining worksheets containing situations to explore and problems to solve. They were required to write what they had done, why they done it, and what were the difficulties they had met.

The students worked in groups of 2, 3 or 4 on the computer. Usually, one of them was at the keyboard and another was taking notes to fill in the report later. These roles rotated frequently, although some students tryed to avoid, if possible, the task of report writing.

All the classroom activities, with and without computers, were conducted by the regular teacher, although now and then there were some interventions of the researcher. A few times the teacher interrupted the activities to stress particular points from which discussion developed.

In the beginning of the computer classes the teacher distributed the software, working disks, and the worksheets. The loading of the program was at first made by the teacher but, in time, the students took over that task quite confidently.

The pedagogical setting for this work was markedly innovative. It was explicitly intended that students, through the resolution of problems would try different ways to solve them, using the graphical capabilities of LOGO.GEOMETRIA. The practical nature of the work was expected to develop in students a feeling for the real content of concepts, so often just memorized.

The classroom organization and the requirement of writing explanations for the methods used, provided the students with many opportunities to discuss with colleagues and reflect on their strategies. The justification of reasoning was meant as a beginning of local conceptual organization.

5.2. Record of the Activity

The two student groups were quite different, not just in computer experience, but also in mathematics achievement. The agriculture students, were much weaker. In the beginning, for example, some of them had problems with some basic geometrical concepts .

It is not surprising that while the informatics students worked on all the 8 worksheets, at a rate of about one per class, the agriculture students went much slower, and only compled the first four. However, in the facility in handling the computer, these students progressed very quickly, and in the end of the experience they were as handy as the informatics students.

In the situation (a), indicated above, some of the groups were satisfied with the discovery that the slope is given by the quotient of the y- by the x-coordinate of the directing vector (note that FAZ.R.EQG and ESC.COORD.V are comands of the program) :

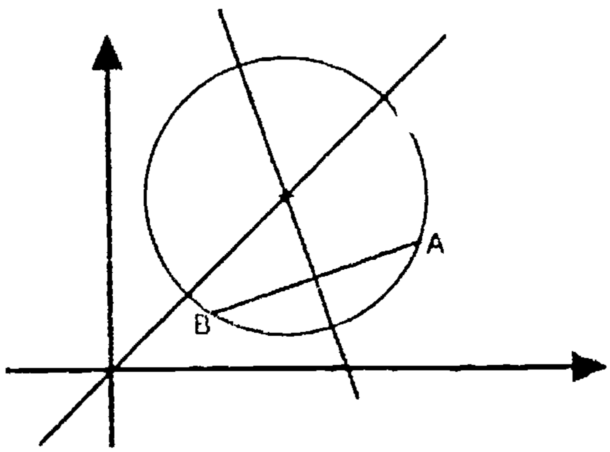
> "...We constructed line r, as required, from the general equation using FAZ.R.EQG "r [2 -5 200]. Then we constructed its directing vector and asked its coordinates through ESC.COORD.V. We asked the slope of the line, which was 0.4 (a number equal to 40/100, the coordinates of the directing vector with y/x, that is, the y-coordinate divided by the x-coordinate.)"

Others, tried with other lines and established this relationship with basis in more cases. Some, used the random lines constructor to verify it:

> "To make sure that this was true for any line we

constructed a random line and went in the same way to relate the slope with the coordinates of the directing vector, and found the same thing."

One of the groups that answered problem (b) presented the following sketch and report:



"We draw the given points, transformed the reduced equation in the general equation, and draw the line. Then, we draw a perpendicular line, b, to the given line, passing through B. We got the perpendicular bisector of the line segment [AB], which we intersected with the given line. From the intersection point we found the distance to A and B, thus getting the radius. And then we draw the required circle and verified that the given points were on it."

Only three groups presented their reports for situation (c). One of them got the correct solution but the explanation is quite confusing, failing to explain how they found it. Another group gave a clear explanation for a wrong solution. They did not verify it. The third group presented the circle of radius 10 with centre at the intersection of the perpendicular bisector to [AB] with the line parallel to the y-axis passing through A, which satisfies the problem.

From the work of this group, evolved an interesting whole class discussion. The students realised that the conditions of the problem represented a simple and very particular case. They formulated the more general case, and concluded that there was no obvious geometric methods to solve it. However, an alternative analytical strategy was sketched.

6. Conclusions

The students were quite pleased with this experience, although some of them, who were used to work with computers, complained about the low resolution of the monitors and the slowness of the program.
The following excerpt illustrates one student's view about the work done:

> "LOGO.GEOMETRIA is not just 2 hours in which we learn to use the computer in a different way... it is the most positive experience in all these years that I have been in school. Why? The computer should be associated to all the disciplines, but since it begun with mathematics why not to continue, as we learn practising. LOGO.GEOMETRIA ... is not just new mathematical notions but a new notion of mathematics... mathematics is not seen as something with just one way of going, because in LOGO.GEOMETRIA we have several ways to solve the problems that were given to us. What pleases me is to attempt to get the solution through the easiest way, although

sometimes I fail it..."

The students become much more confident in their work. Even in the regular classroom (without computers) they began making more questions than usual. They become stronger in defending their own ideas, their own methods.

The students got a better idea of the role of error in the mathematical activity, of the need to make several attempts and reflect upon them — a very different notion of mathematics indeed.

The availability of the program LOGO.GEOMETRIA did not reduce the tendency of some of the students to use analytical approaches, instead of the geometric ones: they start making the computations to find the solution and then verify geometrically the result.

It was interesting to observe that the students do not just like easy things. They also enjoyed working in hard problems that make them think, and commented positively about that.

The facility with which weak students such those of the agriculture class were able to learn how to deal with an open program as LOGO.GEOMETRIA gives a strong indication of the power of this kind of programs for a wide range of educational situations. It also indicates that the apparent trouble that students sometimes appear to have with computers must be attributed mainly to lack of a proper introduction and access to this medium.

There were not major difficulties in conducting this experience. Both teachers did not have a strong background of dealing with computers, but this was naturally overcome by the presence in each computer session of the researcher and the other teacher. Their interventions, however, were minimal, and mostly in response to students' requests.

It is often assumed that with students of this level, especially if they have plans to go to university, it is very difficult to create an innovative environment, based in experimentation and practical work. This experience showed students getting quite involved and interested, and learning a good deal.

Case Study 6

Experimenting with Probability

Experimenting with Probability

1. The School

Hodgson High School is a co-educational, mixed ability school for pupils aged 11 to 16 years old corresponding to grades 6th to 10th.

It is situated in Poulton-le-Fylde, Lancashire, on the North-West part of England, a semi-rural area where the main activities are related to industry, namely tourism, and agriculture. The school serves the small towns of Thornton Cleveleys and Poulton-le-Fylde as well as several surrounding villages.

The school has 850 pupils in classes with about 30 pupils each, and 50 teachers. The majority of the pupils, after five years, leave school to follow courses at colleges and many of them go on to Universities or Polytechnics. In 1989, the pupils in year 11 achieved results above national averages in the General Certificate of Secondary Education (GCSE) examinations.

Parents, mainly from middle class origin, show a great deal of interest and involvement in school activities.

Computers are used within several subjects. Within mathematics, there are 4 computers assigned to the department and their use is an integral part of the curriculum. It seems that, as far as mathematics is concerned, there is a lot of work going on. The head of the mathematics department made a study of the software available to relate suitable programs to particular mathematical topics. The use of computers in mathematics lessons is current practice.

2. Aims

Deepen pupils' understanding of the use of computers by relating it to probapility and data analysis. The method of approach will be:

2.1. Collection of data.

2.2. Revision of tally and bar charts, frequency tables and diagrams.

2.3. Make and test predictions.

2.4. Understand likely, unlikely, certain, impossible, fair, unfair, evens.

2.5. Probability of 0, 1, 1/2.

2.6. Listing possible outcomes

2.7. Simple probability.

2.8. Use of spreadsheet.

3. Material and Resources Used

3.1. Hardware

This experience used 3 RM Nimbus computers in a normal classroom. Although not totally, these computers are similar to the PC compatibles.

3.2. Software

The spreadsheet GRASSHOPPER was used in the lessons described in this case study.

There are a variety of spreadsheets commercially available. These are fairly similar except for graphic facilities. Spreadsheets have been designed to handle large sets of numbers that need to be organized and with which calculations can be done. **Every spreadsheet is a matrix of cells, each cell** named after the corresponding column and line. In these cells data -- a text, a number or a formula -- can be entered. Using the formula facility, the content of a cell can be automaticly recalculated when the contents of other cells are changed.

GRASSHOPPER is an easy to use spreadsheet that provides a variety of computing and graphing facilities, including the drawing of frequency graphs, which proved to be important in this case.

3.3. Other Supporting Materials

Different materials were used in the classroom experiences, including an assortment of dice, coloured cubes and packs of cards.

4. Overview and Curricular Context

All the aims to be attained are within the English mathematics curriculum for the 7th grade.

The scheme of work was suggested by Adrienne Ashworth, the head of the mathematics department, and all the mathematics teachers agreed to run the activities in their own classes. Being a mixed ability school students from all abilities were involved. This means that 180 pupils performed the activities - 2 parallel groups of 90 pupils, each one with a high, a low, and a middle ability class.

A school *Inset* day (teacher training) was devoted to this item for all mathematics teachers.

It was suggested 5 double lessons would be needed to cover this topic.

The plan that was followed is explained below, although some aspects of detail and time needed varied from class to class.

5. Description of Activity

5.1. Teaching and Learning Styles

The whole scheme aims to study the concepts related to probability. The collection of data from practical experiments, its organization and representation using computers, were the guidelines for the activities. This allowed more time for discussion and to draw conclusions.

The toss of a die or some other suitable activity using spinners or counters provided the source of data which was collected and represented by the pupils, in charts and tables, as well as making and testing predictions, and listing possible outcomes.

When performing the experiments and entering data into the computers, pupils were organized in groups and the teacher walked around the classroom helping where necessary. A brief explanation of the workings of the spreadsheet was given to the whole class. The first group then entered their data under strict supervision. They in turn supervised the next group and so on.

The teacher role was also to introduce questions for further analysis and make suggestions about the data organization. In a second phase, and after the computer provided the summed data for whole class, the class performed as a group discussing the outcomes. Once again the teacher introduced questions for whole class or group discussion acting as a coordinator.

Pupils wrote down their results individually, drawing graphs where necessary, as well as using the computer for further analysis.

5.2. Record of Activity

In lesson 1 some topics were reviewed. Students discussed and recalled basic probability concepts.

In lessons 2 and 3 students worked in groups of three. There were three experiments running simultaneously on thirteen or fourteen tables: tossing a die (or two dice) and recording the score, selecting one coloured cube from a bag and replacing it (the bag contained 10 pink, 4 blue and 2 yellow congruent cubes), cutting a pack of cards recording whether a picture or numbered card. Pupils were not told what was inside the bags containing the coloured cubes. The order of colours listed in their charts was often different from the order listed on the spreadsheet. This occasionally led to data being ascribed to the wrong column on the spreadsheet. If this had remained undetected the class totals may have been in error. Not all the experiments took the same length of time. This fact required carefully attention in order to harmonize the development of the experiments.

The toss of a biased die is another possible suggestion for further data collection. These dice would had a coin stuck on one of the faces, from the inside.

An introduction was made, explaining the experiments and reminding the pupils how they should write up their work. Some questions relating to what they thought would happen were posed encouraging students to have an investigative approach to the subject.

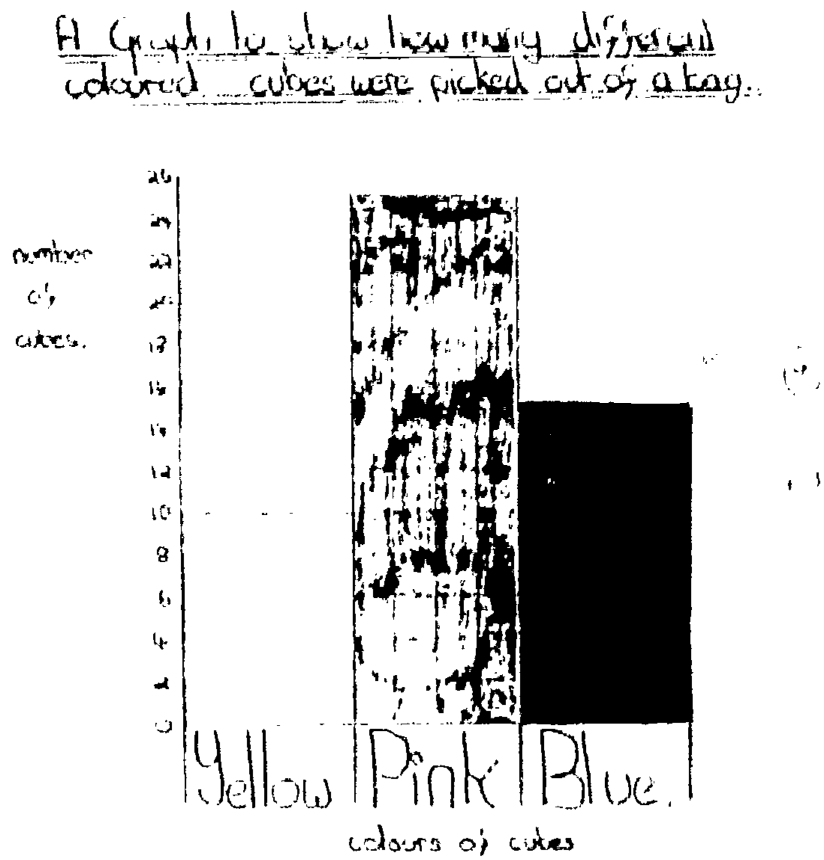
Each group worked through these three experiments in a cycle. Each experiment was performed fifty times by each group and the results were recorded in the students' exercise books.

The following two figures show two types of charts made by Rochelle Poulter for the recording of the experiments of her group

Rochelle Poulter

A tally chart to show how many different coloured cubes were picked out of a bag.

Colour	Tally	Frequency.
yellow	卌 卌	10
pink	卌 卌 卌 卌 卌	25
blue	卌 卌 卌	15

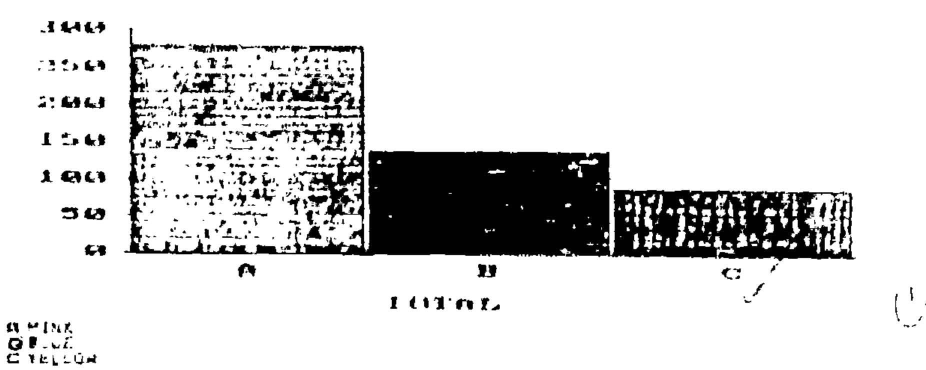


Later they keyed in their results at the computer assigned to that particular experiment, returned to their places before going onto a second and subsequently third experiment. Consequently, all pupils had done the three experiments and the spreadsheets contained the number of groups times fifty experimental results. Meanwhile, the teacher walked around the classroom helping where necessary.

There were three computers available one for each experiment. Teachers had set up spreadsheets in order to ease students' work avoiding extra computer manipulation. This detail can be significant in classes where students do not have experience of working with computers.

By the end of lesson 3, spreadsheets including bar charts were saved and printed in order to provide all students with whole class data.

:	A	B	C	D	E
00:	NAME	PINK	BLUE	YELLOW	TOTAL
01:	AH DR VS	27	14	9	50
02:	TH VP HE	34	10	6	50
03:	HH AB SC	31	12	7	50
04	KT JH DJ	21	16	13	50
05:	KT AN EW	29	12	9	50
06:	EW SE AS	29	11	10	50
07:	HN HN AZ	32	13	5	50
08:	NS ES RF	25	15	10	50
09:	SS CH JJ	23	17	10	50
10:	[illegible] SL CS	27	16	7	50
11:	TOTAL	278	136	86	500



In lesson 4 each pupil was asked to write up one experiment, but ensuring that within the group all 3 experiments were recorded. This was a time for group discussion where pupils helped one another. The following suggestions were offered:

Title
Aim/Introduction
Frequency table of own group's data
Bar chart of own group's data
Computer print out (spreadsheets and bar charts)

Students were then asked to compare, comment on and explain the differences between their group's data and the whole class data.

To help with conclusions, some possible questions were provided:

Dice
1. What is the smallest score? Why?
2. What is the largest score? Why?
3. Wich score is most likely to occur? Why?

...

Cubes
1. What were the colours?
2. Which colour was selected the most?
3. Which colour was selected the least?
4. What can we say concerning the number and the colours of cubes inside the bag?

...

Cards
1. Which of the two choices occurred the most?
2. Is there any relation between the ratio of picture cards:total cards in the pack and that ratio for the data collected?

An example of a written report by one of the pupils follows:

16/3/90 Selectacube. Rochelle Poulter.

We were given a bag with some different colourd cubes in. We did not know how many or what colours the cubes were in the bag. We pulled cubes out of the bag 50 times without looking and recorded it on a tally chart. We then put our information on-to a spreadsheet in the computer. The computer totaled the scores up to check that we had added it up right. (2)

Conclusions

Looking at the graph there could have been more pinks than blues and more blues than yellows. I can not be certain of it until I have had a look at the other classes results. (1)

Final Conclusion.

After looking at the classes results, I am certain that there were more pinks than blues and more blues than yellows. (1)

Selectacube

1. The colours were pink, blue and yellow.
2. The colour pink came up the most.
3. The colour yellow came up the least. (1)
4. I think that there were about 6 pinks, 4 blues and 2 yellows because it fits my conclusion. ✓ (1)

In lesson 5, the concepts underlying all the activities were discussed in order to increase the students' formal understanding. Vocabulary such as *likely, unlikely, certain, impossible, fair, unfair, evens,* used naturally in the context of previous lessons was formalized and this was extended to simple probability. For example, probability of 0, 1, 1/2, P(a 2 when a die is thrown) = 1/6...

6. Conclusions

Teachers think that pupils have learnt a great deal, showing enthusiasm towards this activity.

This experiment, firstly carried out by Adrienne Ashworth in a 7th grade class, was sufficiently successful for Hodgson High School to have a version of it as part of their first year (grade 6th) syllabus.

One teacher said, referring to the inset day course on this activity, that the only trouble was it did not go on long enough.

Teachers indicated that the pupils from all abilities had been enthusiastic and had clearly enjoyed the work.

In an article (*Spreadsheets and the National Curriculum*) published in the number 16 issue of Lancashire Maths Newsletter, Tony Mozley, an advisory teacher who participated in the experiment, stated that one conclusion to be drawn was:

> " ...that one teacher with one computer could use spreadsheets in this way, perhaps as a part of group work where just one group was doing this work and others were doing something completely different. It also showed that spreadsheets could be used at this relatively early stage without having to teach the ins and outs, that is, the computing side of spreadsheets".

Case Study 7

Developing Number Sense with a Computer Game

Developing Number Sense with a computer Game

1. The School

This experience was carried out at school D. Pedro V, a secondary school in a very populated area of Lisbon. One of the largest schools in the country, attended by approximately 4000 pupils, from the 7th to the 12th grade, this school operates in three daily shifts. From a social perspective, and due to the various types of neighbourhoods that surround the school, pupils are from very diversified strata of society.

The school has about 300 teachers and has been traditionally one center for field activities of preservice teachers and also for inservice training. It was one of the first schools to be included in the Project MINERVA, the Portuguese national project for the use of new information technologies in basic and secondary education.

2. Aims

The main objectives of this experience were the following:

a) To develop a better number sense in the pupils, through the use of a computer game as a complementary approach to problem solving activities concerning natural numbers — divisors, primes, and so on.

b) To allow pupils to try and compare different strategies, then to choose the most appropriate.

c) To develop the capacities of observation, of criticizing the results of different trials, of learning from the errors.

d) To develop the capacities of discussion in small groups, of reasoning, of advancing arguments and of criticizing the arguments of other students.

e) To develop the capacity of taking notes, of making written reports.

3. Material and Resources Used

3.1. Hardware

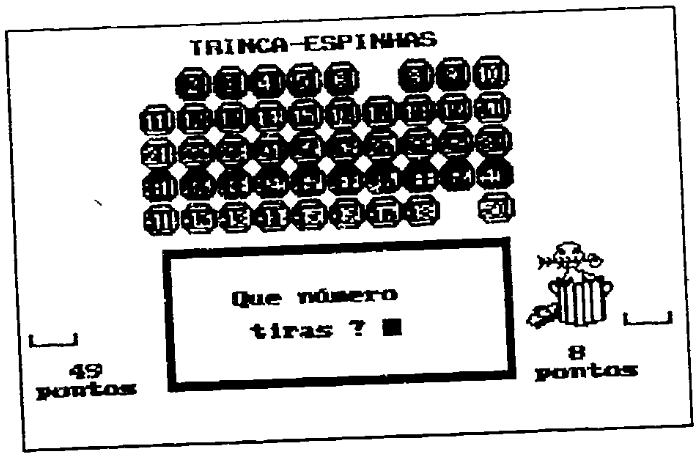
One classroom with 5 PC compatible computers (Amstrad 640K RAM with colour monitor).

3.2. Software

"TRINCA-ESPINHAS" is a computer game widely used in the Portuguese middle and secondary schools. The game is played by the student or a group of students against the computer, represented by a popular figure called Trinca-Espinhas. The rules of the game are very easy:

- one number less than 50, say 20, is chosen by the player, and the list of numbers from 1 to 20 is shown in the screen;
- the player will be selecting and pulling out from the list, one at a time, numbers (that have at least one divisor in the list); after each one, TRINCA-ESPINHAS will take for himself the divisors

of the selected number then add these together as his score;



The game had just began. The student picked 49, and Trinca-Espinhas picked the divisors 7 and 1, making the total of 8 points.

• the game ends when the player has no more numbers (with divisors) to chose; TRINCA-ESPINHAS will then pick the remaining numbers;
• to find the winner, the player's mark is the sum of the selected numbers and TRINCA-ESPINHAS gets the sum of the divisors picked by him.

3.3. Other supporting materials

Pupils were asked to make a written report, after playing the game, and one piece of paper with guidelines for the report was given to them.

4. Overview and Curricular Context

a) In this case study, the activities with the program were integrated in a long-term project — MAT_{789} — of curricular development for the grades 7, 8 and 9. In the first year, 1988/89, the project included two 7th grade classes of the school D. Pedro V and in 1989/90 two 8th grade classes and two new 7th grade classes, one in the School of Amadora and the other again in the school D. Pedro V.

b) The experience described in this case is the use of TRINCA-ESPINHAS in the two 7th grade classes (about 25 pupils in each class) of D. Pedro V, but in the other school the experience went on in the same way. The game was used in the beginning of the school year, after spending some time in problem solving activities with natural numbers (natural numbers, divisibility and primes are curricular contents of the 6th grade, in the Portuguese preparatory school).

5. Description of the Activity

5.1. Teaching and Learning Styles

a) Children worked in groups of 2 or 3 at each computer. The class was divided into halves, due to the number of computers available. One half played the game on the computer and the other half was occupied in other activities. Work with the program in the classroom occupied only one two hour period, and each group used the computer only for one hour.

b) TRINCA-ESPINHAS has a first page of instructions and is an easy game to play (but not to win); so no previous explanations were needed and children begin immediately to play.

c) Pupils were asked to take notes during the game: record of winners and losers, marks, best strategies, and so on. At the end, a written report to be presented one week later was requested as home work and some written guidelines were given to help make the report.

d) Children were allowed to play the game outside the classroom, in the computer club of the school.

e) In the next period after the classroom activity, children were asked to describe the best strategies to win the game.

f) Reports were commented on and some pupils were asked to make an improved version.

g) Later, in an evaluation written test, one of the questions referred to the game, asking the best strategy to win in a given case — 20 numbers.

5.2. Record of the Activity

a) Some extracts of written reports from the pupils:

> " In the first game we played with 20 numbers, in the second with 25 and in the fourth with 30, because with 20 we thought it was easier; after that we raise the numbers, because we won the first game."
>
> ...
>
> [in what concerns strategy] "we played first with the maximum prime number, and after we took the numbers with less divisors, and like this until the end." (João)

> " In the first try we played with 16 numbers because we want to begin with few numbers to understand the game, and we won, 87 marks to us and 49 to Trinca-Espinhas. In the second try, we played with 20 numbers and we won again 107 against 103. In the third try we played with

30 numbers and by lack of attention we lose, 186 for us and 279 to the Trinca-Espinhas."

...

[in what concerns strategy] "we have chosen the greatest number with less divisors, so the Trinca-Espinhas didn't get many divisors and didn't win the game" (Ana Teresa)

b) Views expressed by the students on the best strategies:

" we take a large number with few divisors" (David)
" for instance, when we play with 40 numbers, after taking the 37 we could take a prime number times 2" (Vasco)
"that is not good, 2 is a precious number" (Flávio)
(*there follows a discussion about the question of 2 being a precious number. one says that 2 is precious for 10, another says that for 10 you have the 5, but 5 could go before,...)*
"2 is precious for every even number; I will take the large prime squared, so for the play with 40 numbers I will take the 25" (Isabel)

c) Teacher's own views:

" The game is very interesting; some 8th grade pupils go on playing the game one year after they played for the first time"
" The discovery of strategies is a difficult task for the children and they need to get some help"
" Two or three children of my class discovered by themselves some good strategy to begin the play but some others never won the game"
" It is essential that they can play the game outside the classroom"

6. Conclusions

It seems that this type of game is very helpful as an important complement to the usual problem solving activities on natural numbers. The way pupils speak and reason naturally about different properties of numbers — divisibility, prime numbers,... — when they are playing the game is an indication that they are developing an improved number sense.

Playing this game involves the pupils in constant intellectual activity, trying their best to discover a good strategy to win and reflecting on the consequences of each move. As they play in small groups, and the game is not easy to win, they are naturally learning how to cooperate in order to find the best strategy.

Asking for written reports was a distinctive feature of the use of TRINCA-ESPINHAS in this experience, and revealed itself as a good way to develop the ability to communicate ideas and to use the language of mathematics

This is the kind of activity with computers that has many positive aspects:

- children like it very much,
- the teacher doesn't need to spent much time in the planning of the activity,
- the game is easy to play and every children will understand the instructions immediately, no further explanation is necessary,
- it's not easy to win the game, so it's a good challenge to the children, and this is one of the reasons they like it,
- it deals with a interesting subject of the curriculum, and serves very well one of the objectives: developing the sense of number,
- doesn't take much time in the classroom, and the students can go on by themselves outside the classroom.

Case Study 8

Databases and Numerical Relationships

Databases and Numerical Relationships

1. The school

Hodgson High School is a co-educational, mixed ability school for pupils aged 11 to 16 years old corresponding to grades 6th to 10th.

It is situated in Poulton-le-Fylde, Lancashire, in the North-West part of England, a semi-rural area where the main activities are related to industry, namely tourism, and agriculture. The school serves the small towns of Thornton Cleveleys and Poulton-le-Fylde as well as several surrounding villages.

The school has 850 pupils in classes with about 30 pupils each, and 50 teachers. The majority of the pupils, after five years, leave school to follow courses at colleges and many of them go on to Universities or Polytechnics. In 1989, the pupils in grade 10 achieved results above national averages in the General Certificate of Secondary Education (GCSE) examinations.

Parents, mainly from middle class origin, show a great deal of interest and involvement in school activities.

Computers are used within several subjects. Within mathematics, there are 4 computers assigned to the department and their use is an integral part of the curriculum. It seems that, as far as mathematics is concerned, there is a lot of work going on. The head of the mathematics department made a study of the software available to relate suitable programs to particular mathematical topics. The use of computers in mathematics lessons is current practice.

2. Aims

The aims of this case study concern the handling of data and using it as a basis for reasoning, discussion or calculation.

2.1. To collect data, hence construct a database.

2.2. To interrogate a database.

2.3. To use a graph facility (scattergraph) to discover connections between sets of data.

2.4. To use calculators and spreadsheets in order to calculate numerical relationships.

2.5. To foster pupils' skills in using decimal numbers and approximate values.

3. Material and Resources used

3.1. Hardware

The computer used within the activities described in this case study is a RM Nimbus, a computer that is not quite PC compatible but is similar to it in several aspects. All the activities were developed with three computers in the classroom.

3.2. Software

A database program called GRASS was the software used by the pupils.

Database programs are designed to help the collection and management of information or data.

A database is an organized collection of related information. You probably deal with databases every day, for instance your address book and phone book. Each of these databases organizes data for easy storage and retrieval, or access. The specific aspect of computer databases is the electronic way in which the data is recorded and accessed.

There is a variety of database programs commercially available. GRASS was designed to be used in educational environments and it is easy to use by pupils of the earlier secondary school grades. It provides important facilities concerning these activities such as graphical features. From the data entered in the database, scattergraphs can be drawn.

3.3. Other Supporting Materials

Worksheets developed by the teacher, measurement tapes, and calculators were some of the materials designed for pupils to perform the activities described in this case study.

4. Overview and Curricular Context

These activities were performed by pupils of a low ability class during 1989/1990. This class was a 7th grade class (year 2 of the English secondary school). In the first term of

1990/1991, other classes performed the same task. This work was set up by Adrienne Ashworth, the head of the mathematics department.

During six lessons, the pupils collected, represented and analyzed information in order to discover relationships between sets of data drawn from measurements on their bodies. While performing the activities, they had to deal with curricular topics such as decimal numbers, approximate values, graphical representations, and the use of calculators and databases. It should be pointed out that the use of databases to enter and retrieve data is explicitly mentioned, for this grade, in the National Curriculum in England. To create a database is also mentioned as appropriate to the Attainment Targets of the National Curriculum.

Although the activities here described concern only the work done in six lessons, they can be easily extended. Indeed there is a wide range of possible relations to be investigated and a tool designed for numerical manipulation would also be useful. Spreadsheets fit in this kind of work. Data would be transfered from the database to the spreadsheet and pupils would use facilities of the spreadsheet (graphing, formula, copy, average,...) in order to do more exact and deeper investigation.

5. Description of Activity

5.1. Teaching and Learning Styles

During these six lessons, the class was organized in different ways, according to what appeared to be suitable at a particular time, and the pupils worked both individually and in groups of three or four. Sometimes the discussion involved the whole class.

The teacher's role was to introduce the questions, provide help when necessary, and encourage pupils to have an investigative approach to the proposed questions or to the problems that came up while developing the work. This proved to be important because, within these activities, a number of questions, both related to procedural difficulties and to reasoning issues, arose in the course of the activities.

5.2. Record of Activity

In lesson 1 the topic covered was scattergraphs. Pupils learned what they are and how they can be drawn.

Lesson 2 began with a question: the students were asked by the teacher their opinion about a possible relationship between one person's height and the circumference of the head. More precisely, the teacher introduced the following equation:

height = 2.9 x circumference of head

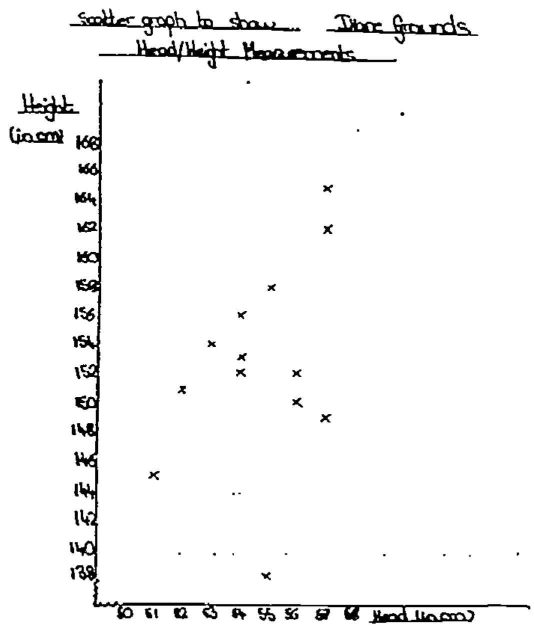
Pupils reacted saying it was nonsense. But was it actually wrong? How can one be sure? All the students measured each other and built scattergraphs of height/head and, much to their surprise, found out that the points could roughly be seen to lie on a straight line. That impression was increased when the same graph, with a different arrangement of the axes, was viewed in the computer.

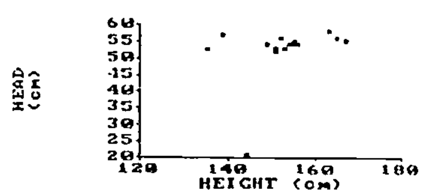
By means of illustration, a table and a graph drawn by a pupil and the corresponding computer graph are shown on the next page.

Year 5 Set 3
Data - Head/Height

Diane Grounds

Name	Head (in cm)	Height in cm
Matthew	51	145
Richard	57	149
Jonathan	54	156
James	57	162
Kelly	62	151
Diane	55	158
Nosarene	54	152
Anthony	64	156
Ian	54	154
Caroline	55	138
Kerry	56	150
John	53	164
Peter	56	154
Christopher	54	153
Chris	57	165
Duncan	57	135





After becoming aware, in lesson 2, of the relationship between head and height, lesson 3 was devoted to begin an investigation about further possible connections. Firstly, pupils proposed the measurement of span, waist, hipground (from the hip to the ground), wrist, plane (finger to finger outstretched), foot, cubit (elbow to fingers extremity), ar ' hand (width across hand - insertion of thumb to fingers extremity). Keeping in mind the construction of a database, details were discussed. Regarding this matter, it was agreed that the fields would include those named above plus name, sex, height and head.

In lesson 4, pupils set up their own groups. All measured each other and entered the data in the database. There were some difficulties concerning numerical approximations. Sometimes, pupils did not agree when measuring. It was a good opportunity to talk about approximate values and how to handle them.

Occasionally, pupils measured from the wrong end of the measurement tape causing errors. In these cases the mistakes were usually detected by the other pupils in the group.

Before entering the data into the computer, everyone made a table with their own data:

Diane Grounds

Data sheet

Fields	Data
Name	Diane
Sex	Female
Height	167cm
Head	55cm
wrist	15cm
Foot	24cm
Span	20 m
Hand	8cm
Cubit	43cm
Plane	152cm
Waist	73cm
Hipground	106cm

The pupils keyed their data into the computer. At this stage, some of them found difficulties when the fields in the database and in their tables were in a different order.

:	A	B	C	D	E	F	G	H	I	J	K
00:	NAME	HEIGHT	HEAD	WRIST	FOOT	SPAN	HAND	CUBIT	FLARE	WAIST	HIPGROUND
01:	DUNCAN	139	57	15	23	17	14	37	141	65	85
02:	PETER	155	55	10	24	20	8	43	156	68	90
03:	CHRIS	165	56	16	25	19	15	44	163	82	91
04:	JOHN	155	54	14	20	17	8	43	158	69	92
05:	MATTHEW	144	21	13	15	17	7	36	147	57	84
06:	KERRY	152	56	15	24	20	9	40	132	72	83
07:	NASAFENE	[illegible]	53	15	23	18	11	40	149	66	100
08:	KELLY	[illegible]	52	14	23	16	12	42	154	63	96
09:	CAROLINE	155	53	11	21	17	8	39	137	52	36
10:	DIANE	167	55	16	24	20	8	42	152	73	106
11:	JONATHAN	149	54	15	24	19	9	43	150	54	89
12:	RICHARD	151	52	16	22	20	9	40	156	62	83
13:	IAN	154	54	15	26	22	9	42	144	42	93
14:	JAMES	163			26	15	11	42	160	93	100
15:	ANTHONY	151	-		24	20	9	35	138	71	78

Lesson 5 was devoted to interrogation of the database. Several questions were posed by the teacher or by the pupils. They were given the following worksheet:

Database Worksheet Name: Duncan Gould

1 Name the tallest person in the class. Diane

2 What is that persons height? 167cm

3 Name the tallest boy in the class. Chris

4 What is his height? 165cm

5 Name the tallest girl in the class. Diane

6 What is her height? 167cm

7 What is the average (mean) height of the pupils in the class? 153cm

8 What is the average (mean) height of the boys in the class? 153cm

9 What is the average (mean) height of the girls in the class? 152cm

10 What is the median height of the pupils in the class? 153cm

11 What is the median height of the boys in the class? 154

12 What is the median height of the girls in the class? ~~154~~ 152

13 Name all the pupils the class whose height is less than 158cm. Christopher, John, Jonathan, Nasafene, Kerry, Anthony, Richard, Kelly, Matthew, Duncan, Caroline

14 Does the tallest person in the class have the biggest handspan? No

15 Does the tallest person in the class have the biggest feet? no

Make up five more questions like the last two and find the answers.

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As stated before, a number of questions were proposed by the pupils. Here is a sample of the questions which were answered with the help provided by the database facilities:

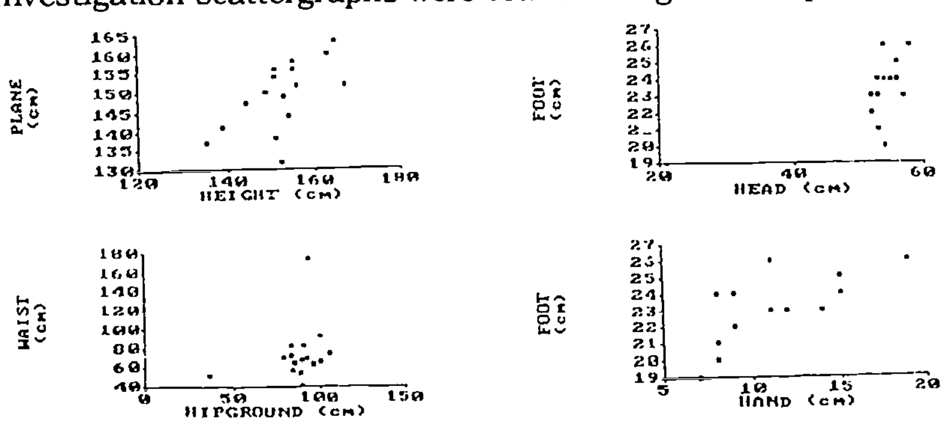
How many pupils have:
— Cubit greater than 40 cm?
— Wrist less than 12 cm?
— Hand equal to 8 cm?

How many girls have feet greater than 25 cm?
How many boys have span greater than 25 cm?

How many pupils are
— taller than 165 cm and have feet bigger than 27 cm?
— taller than 165 cm or have feet bigger than 27 cm?

Does the tallest pupil still look the tallest when seated?

The investigation about numerical relationships within the sets of data went on to the next lesson. In order to help the investigation scattergraphs were drawn using the computer.



The total number of hypothetical relationships is large and those which were found in each class represent only a small number of them. Moreover, they differed from class to class. By means of illustration, below are some of the equations that pupils wrote:

height = 3.8 x cubit
span ÷ hand = 2.24
height ÷ plane = 1.02
plane ÷ cubit = 3.9

When doing the activity. the pupils used calculators to find the relationships. It was also a opportunity to handle and discuss decimal numbers and decimal places.

6. Conclusions

Observing pupils' achievements, and noticing that the work was firstly carried out with a low ability class, one can say the pupils have profited from this activity. They worked with important mathematical ideas and topics having the opportunities to discuss them in a concrete situation. From the teacher's point of view, the pupils liked to work in this way and she noticed enthusiasm when they were doing the tasks assigned.

The wide range of possible developments enabled by this particular activity is also a very positive aspect. As several classes have carried out this work, a conclusion that can be drawn concerns the variety of relationships the pupils can find, from class to class and even from group to group within the same class.

Although the activities have stopped in this stage it would be worthwhile to go further on next year in order to extend pupils' knowledge. This is Adrienne Ashworth's opinion. If the same class will perform the activities the investigation can be deepened and there are new approaches which can be developed such as the study of the rates of change. It is planned to use spreadsheets with such purpose, using their scattergraph, formula, copy and average facilities. The spreadsheets can be used even at earlier stages of these activities, as long as they have available facilities similar to those used in the database.

In-service teacher training is already in progress at Hodgson High School in order to implement these activities.

Case Study 9

Computer Assisted Instruction in Trigonometric Functions

Computer Assisted Instruction in Trigonometric Functions

1. The School

These lessons in trigonometry with computers using the program "Goniometrics" were conducted in the Christelijk Lyceum in Delft, the Netherlands. This is a urban catholic school for pupils from the 7th to the 12th grade (13 to 18 years old) that accept pupils for the upper and middle streams of the three that follow the primary school.

From nine years ago the Christelijk Lyceum has been equipped with computers (Apple II). They were used mainly in programming courses and for the teaching of text processing. With the support of a Dutch project related to the introduction of new technologies in the schools — the NIVO project — a network of 15 PC compatible computers was installed in 1989. These computers are mainly used in courses on the MS-DOS system, text processing, spreadsheets and data bases, for all pupils from grade 9 on.

2. Aims

The main objectives sought with the use of this software in the teaching of trigonometry were:

a) to help pupils to become more active in mathematics lessons,

b) to enhance the understanding of some parts of the subject of trigonometric functions.

The trigonometric functions were selected as a subject for this experience because it is was viewed as distinctively a difficult one.

3. Material and Resources Used

3.1. Hardware

One classroom with 15 PC compatible (Phillips) computers was used for instruction in this school.

3.2. Software

The software used in the lessons of trigonometry was a program called "Goniometrics", a set of interactive exercises on trigonometric functions, equations, inequalities and graphs. The software can be used to cover 4 to 6 lessons.

3.3. Other Supporting Materials

The software is self contained, and only a single sheet with notations was given to the pupils. No worksheets were used, only the general textbook is used as a resource.

4. Overview and Curricular Context

The program used in these lessons has been developed in the Technical University of Delft since the end of 1984, and is the result of a project involving three groups of people - two experts in mathematics education, two researchers in the psychology of learning and two experts in the technical aspects of Computer Assisted Instruction. Resources only allowed work with one school. It was importante to the project leaders that:

- it should be an average school, with a number of computers large enough to allow one class to work at the same time with them;
- the mathematics teachers and the school should be interested in participate in the project.

The mathematics teachers were consulted about the topic to be selected as a subject matter. Trigonometry was chosen. This subject is taught in the tenth grade (16 years old pupils) in the stream preparing for the University. It is seen as a difficult subject by the teachers. The software was developed from 1985 to 1987, subject to some conditions:

- the work with the computer would be integrated in the normal curriculum, without any change in the usual order of the topics; some lessons would be replaced by lessons with the computer;
- the most difficult questions and topics in the subject matter would be assigned to the lessons with the computer.

5. Description of the Activity

5.1. Teaching and Learning Styles

a) The four to six lessons with the computer were part of a unit of 14 lessons on the subject matter of trigonometry. Each lesson had a normal duration of 50 minutes. About 25 pupils worked in groups of two in each computer.

b) Questions and exercises were presented on the screen and in the keyboard the pupil typed short answers, mostly numerical. Comments or further explanations, and sometimes an extra-screen with some information and other examples were presented on the screen as a feedback. The last lesson of the program is more open, requesting from the pupil a higher level of decisions. The consequences of an answer is made visible in the graph or in a formula, and the pupil must decide if he/she is moving in the right directions or if something must be corrected immediately.

c) The pupils were free to choose who they work with and a good learning environment and working atmosphere existed in the classroom, with the pupils discussing with each other the answers to be given and the reasons for this or that feedback from the computers.

d) The role of the teacher was significantly changed in the computer lessons, as compared to the one in the lessons without computers in the same subject. The teacher was now a supervisor, and was much more able to grasp the pupils' difficulties, to help or encourage pupils and to ask questions in order to further enlighten the problems in discussion.

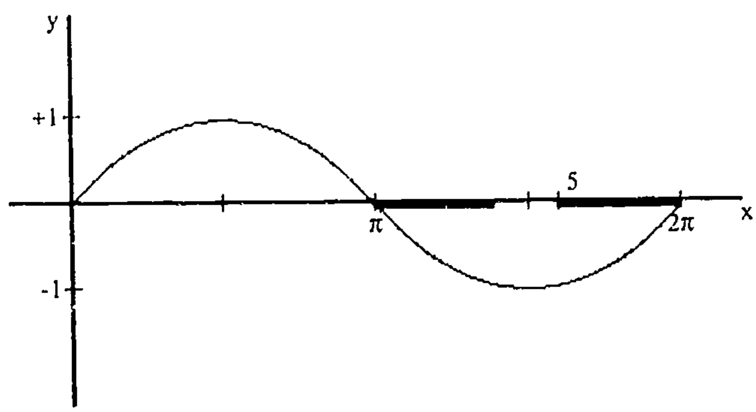
e) In the last two lessons with the computer some students were much slower than others in solving the questions posed by the computer, and it was necessary to give the faster learners other interesting and more complex questions and problems to be worked without computer.

5.2. Record of Activity

In the first two lessons with computers pupils are requested to do a number of exercises in order to sharpen their knowledge of the graphs of the functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ on the interval $[0,2\pi]$. The properties of symmetry of these graphs are pointed up through convenient exercises, like for instance the following one:

- Given the value 5 of x, for which value(s) on [0, 2π] the function f(x) = sin (x) assumes the same or opposite value as in x=5?

One graph is presented to the pupils on the screen (the figure is not a reproduction of the screen)



and the children are taugh' ' use the symmetry in the graph, indicated by the line segments of equal length (in bold in the graph). In this way, pupils can easily find that the solution for the same value is $\pi + (2\pi - 5) = 3\pi - 5$. In a similar way, they can use graph symmetry to find the x-value where the function has an opposite value.

In the first exercises the whole figure is shown to the pupils, but after more is left to the pupil. At the end, exercises without the graph are proposed to the students, and only in case of wrong answers the graphs are shown.

In the third lesson exercises to solve simple linear equations and inequalities are proposed to the students. The fourth lesson deals with transformation of functions and graphs, namely through the study of functions of the type $f(x) = d + c \sin(bx + a)$, with $b>0$, and the equivalent functions for the cosin. In this way, the transformations studied are: horizontal translation, magnifications with respect to the x-axis and to the y-axis, and vertical translation. Another type of problems proposed could be expressed by the phrase "looking for the formula", because the pupils are requested to discover the function corresponding to a given graph.

6. Conclusions

Pupils had a moderately positive response to the computer lessons. Most of them did not consider these lessons annoying, and they declared that the work was not harder than before. But, as told by the team of researchers, that was the case. Pupils felt they have actually learned with the computer lessons. These are some quotations from the pupils interviews:

> " It was more varying, and when he (the computer) said it was wrong, you could do it again by yourself."
> " It is nicer to work together with other pupils, you learn more from it, you can explain things to each other."
> " I found things rather slow, if you have a good answer it was explained to you again why your answer was good. I found that boring."
> " We did not have to ask the teacher often for something, only if you used the wrong notation, but most of the time, no."
> " Well, on the whole I did like it, it went rather fast sometimes, but I did like it, it was after all something different than the usual explaining in the class, yes, I did like it."

It seems that there is some improvement in the achievement of the pupils in this subject matter when taught in an interactive way with computers. But when the subject is essentially more difficult, as in the case of transformation of trigonometric functions and graphs, more time must be spent in the topic, even if computers are used.

Teachers were very happy with the computer lessons. They recognized that these lessons were much more easygoing and causing less tension than the others, and that the pupils worked much more than before. In their view, 20 to 30% of the lessons could be organized this way, if there was hardware and software available. Also, teachers considered that the computer lessons combined well with the lessons without computer and that the subject matter was well dealt with.

Case Study 10

Graphic Calculus

Graphic Calculus

1. The school

This work was carried out in the Het Waterlant-College, a catholic school situated in Amsterdam. This school has 1200 pupils and the social level of the pupils is low or medium.

Three levels of the secondary education in the Netherlands — Gymnasium, Havo and Mavo — are present in this school. The pupils in the class adressed in this study belong to the first one and will normally undertake university studies.

2. Aims

The main objectives adressed with the activities described in this case are:

1) To help students develop aptitudes for establish connections between the analytical description of functions and their graphic representations.

2) To encourage students to do investigations in mathematics, for instance looking for specific relations within a family of functions or analysing the effect of a parameter of changing a parameter within a function.

3) To facilitate the exploration and demonstration of the funda-

mental ideas of calculus.
4) To provide students and teachers with a greater insight into the mathematical processes in elementary calculus courses.

3. Material and resources used

3.1 Hardware

MS-DOS IBM compatible computers with at least 512 K memory and colour monitors.

3.2 Software

VU- GRAFIEK is a MS-DOS program developed by Piet Van Blokland after the original design and implementation for the BBC computer by David Tall.

VU-GRAFIEK is a powerful graphics program with extensive possibilities, including looking for the formula, drawing graphs, area computing, equation solving (several methods), Taylor polynomials, curves in space, parametric representation of curves, first and second order differential equations, functions of two variables and complex functions. In addition it is very user friendly and pupils begin to use it fully almost from the first contact.

3.3 Other supporting materials

There is a handbook for the program VU-Grafiek and the students receive worksheets with the proposals of the activities.

4. Overview and curricular context

This kind of program can be used in many different situations, grades and subject matters. Namely, this software has been used by several teachers in different schools. It has been used extensively in the experiments of the new mathematics programs in The Netherlands, mainly in the upper secondary school. There are also experiences with children 14 and 15 years old, and in this case the program can be a tool to be used at the very beginning of the study of functions. Later on it can be used for the study of trigonometrical functions and also for the study of surfaces in the 3-D space and their defining functions. Some parts of it are useful at the university.

5. Description of activity

5.1. Teaching and learning styles

a) VU-Grafiek has been used in many different situations. One situation was one computer for the whole class, with the teacher discussing with the pupils. Another common situation is one computer for every two students. There is a report of a teacher using VU-Grafiek in an oral examination.

b) With this program, with appropriate support materials, students can take initiatives and explore unexpected situations outside the normal curriculum limits.

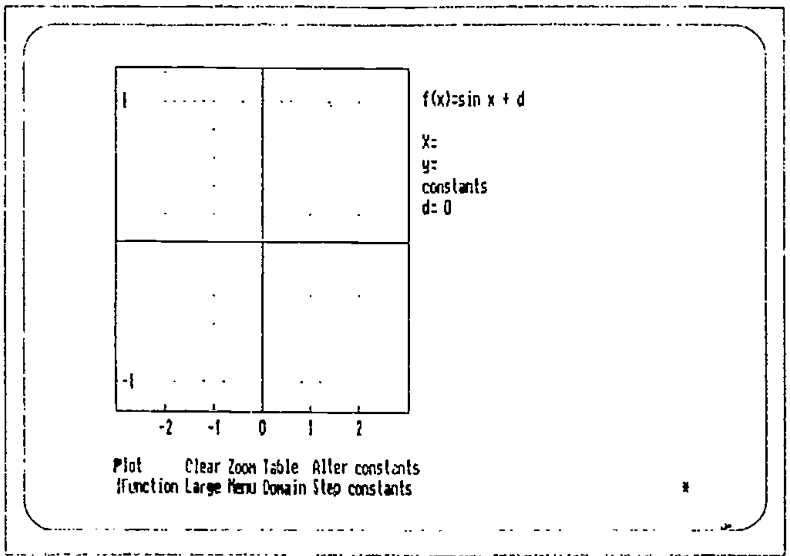
5.2. Record of activity

There follows some extracts from an actual worksheet that was used in the school for a lesson with 10th grade pupils. Pupils worked in groups of two in each computer and had some previous lessons in trigonometry. In this activity they studied the role of the various parameters in the family of functions $f(x) = a \sin(bx+d)$.

For information we insert some examples of the screens obtained from the computer.

f(x) = sin x + d

- erase the screen
- insert function f(x) = sin x + d
- d = ? ; program is asking for a value for parameter d

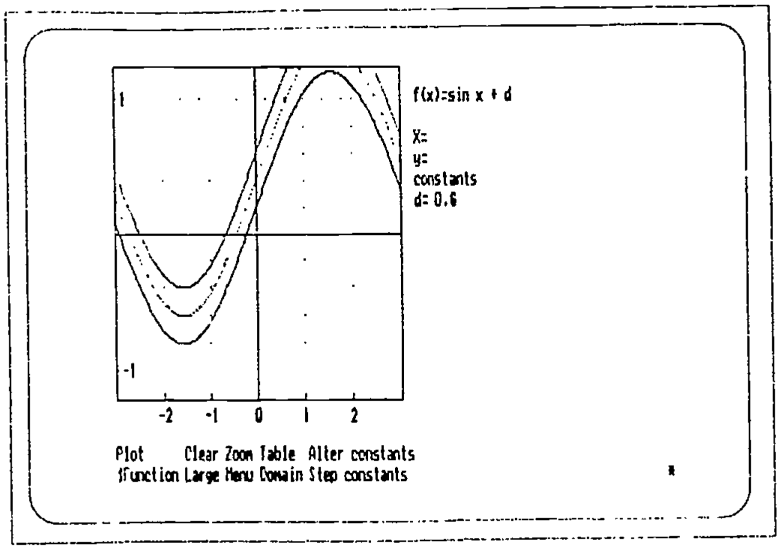


- put d=0 and press <return>

You have two options for working with a parameter. With the option **A** (Alter constants), you can draw a new graphic for each case (value of d). With the option **S** (Step constants), you can ask for a set of graphs one after another.

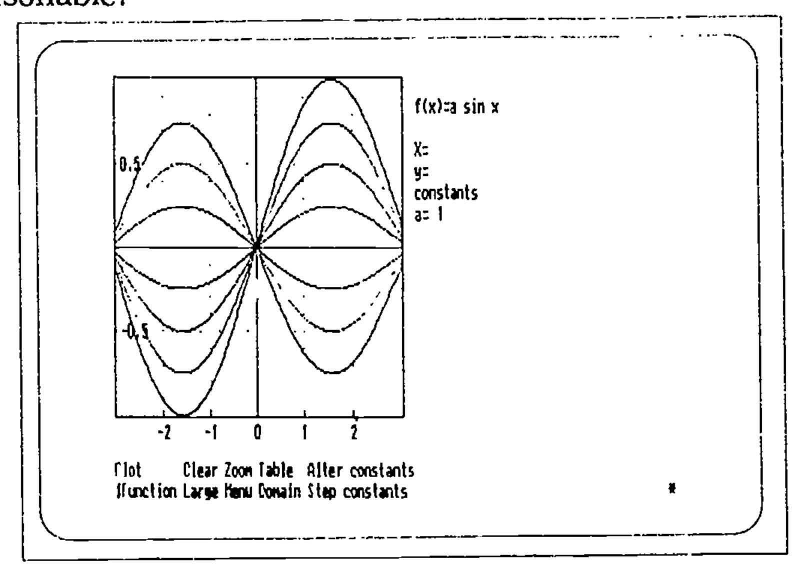
- choose option **S**, put d=0.6 and ask for 4 graphs,
- observe that in the upper right corner appears the parameter value corresponding to the curve the program is drawing,
- repeat the same operations with other values and investigate the different cases.

What is the role of parameter d?



f(x) = a sin x

- insert f(x) = a sin x
- put a=0. Do you think the result is unexpected or reasonable?



• ask for a sequence of graphs and adapt the axes if necessary

• investigate other cases (for instance, with negative values for a)

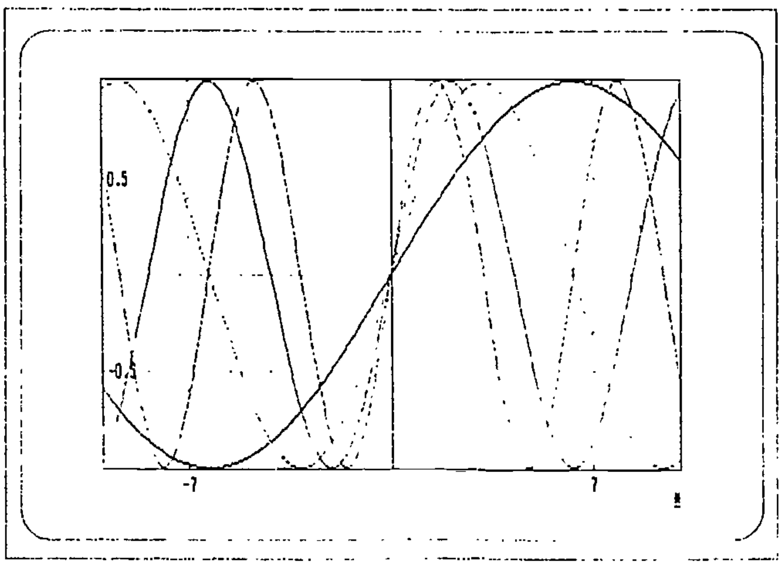
What is the role of parameter a?

---

f(x) = sin bx

• do the same kind of investigation as for the other families

• use option **L** (Large) to observe the intersections with the axes

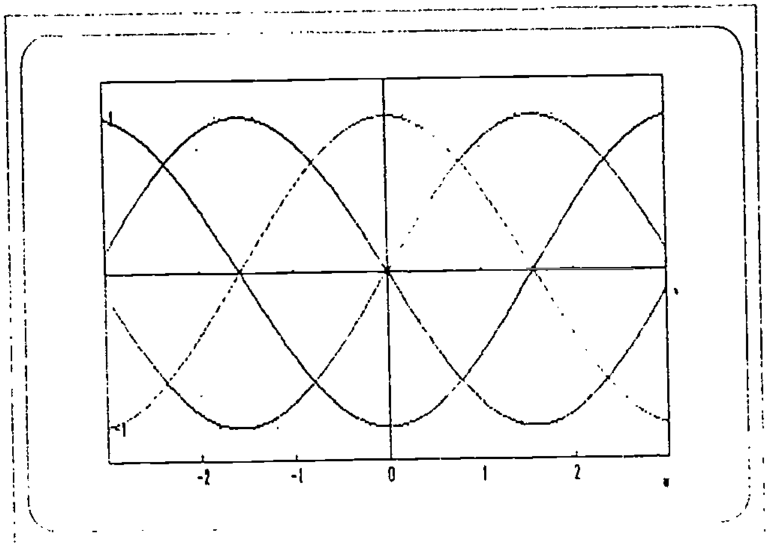


What is the role of parameter b?

---

f(x) = sin (x + c)

• same investigation as before; use stepvalues of $\pi/2$



What is the role of parameter c?

f(x) = a sin bx; f(x) = a sin (x + c); f(x) = a sin (bx + c)

- Study the functions with 2 or more parameters at the same time; for a good point of departure give value 0 to the parameters.

6. Conclusions.

The program is very easy to use and many times the pupils extend the exploration of one situation further than it is expected by the teacher. Some common misinterpretations are clarified by the use of the program. Teacher Heleen Verhage observed a class where the pupils studied the function **f(x)= ax + b**, trying to understand the role of the parameters **a** and **b**. She made the following comments:

"Annemarie and Karin worked earnestly and with pleasure. For this type of work, half an hour is enough, many things happen in a short period, the work is not a placid one as when drawing a graph with pencil and paper.
The class was a success, in the sense that the pupils worked hard and in a meaningful situation. For the pupils of MAVO-4 [15 years old], we would think that this is an easy subject, but many misinterpretations appear that are only clarified with the use of a graphic program. Perhaps we could avoid these mistakes if we had began this kind of work sooner."

Case Study 11

Problem Solving and Spreadsheets

Problem Solving and Spreadsheets

1. The Schools

In Portugal, secondary schools have pupils from 12 to 18 years old corresponding to grades 7th to 12th. Pupils with different abilities are together in the same classes. This case study describes an experience undertaken with 11th grade pupils.

Mem Martins is a small town in the suburbs of Lisbon mainly inhabited by people working both in the suburbs and in the Lisbon area. The secondary school of Mem Martins is 7 years old, has 2000 pupils, mostly from working and middle class origin, and about 200 teachers. Few teachers live close to the school and there are quite a number of staff changes at the end of each year. This interferes with development work within the curriculum.

The secondary school Veiga Beirão is situated in downtown Lisbon, surrounded by commercial and business quarters of the city. Both pupils and teachers do not live near the school. It is an old school — about 40 years old— with 600 pupils in day courses and the same number in night courses. Unlike the school of Mem Martins, a reasonable number of the 120 teachers are members of the permanent staff.

Regarding computers and the way in which they are used, there are sharp differences between the two schools. Five years ago (1985/1986) the first computers arrived to the school Veiga Beirão. Only the mathematics teachers showed interest and developed some activities although there was no real continuity. In the school of Mem Martins the computers were introduced during the year 1987/1988. Since then several teachers, namely history, mathematics, languages and health teachers, used them in extra-curricular activities as well as in normal classroom lessons.

2. Aims

2.1. To develop pupils' skills in problem solving and oral communication within mathematical subjects.

2.2. To increase pupils' abilities in group work and in other ways of work organization allowing pupil activity to be the centre of classroom work.

2.3. To allow the pupils to become more independant in developing solutions to problems.

2.4. To deepen pupils' understanding about mathematical concepts related to sequences and functions.

3. Material and Resources Used

3.1. Hardware

In these experiences were used PC compatible (Amstrad 1512) computers, 6 in Mem Martins and 4 in Veiga Beirão, with mono and colour monitors; there was one Epson LX-800 printer in each school.

3.2. Software

The spreadsheet SuperCalc 4 was used in both schools.

There is a variety of spreadsheets commercially available. Most spreadsheets are quite similar except for graph-drawing facilities; these are often very important in education. They have been designed to handle large sets of numbers that need to be organized and from which computation can be done. Every spreadsheet is a matrix of cells, each one named after the corresponding column and line. In these cells data — a text, a number or a formula — can be entered. Using the formula facility, the content of a cell can be related to the other cells in order to calculate the required value. Supercalc 4 is a powerful spread-sheet available throughout Europe with several graphic facilities namely the drawing of line, bar and X-Y graphs.

3.3. Other Supporting Materials

Worksheets were developed and included in a book written in Portuguese by the teachers who conducted this activity (*Quod Novis*, Susana Carreira and Georgina Tomé ed. Associação de Professores de Matemática/Minerva 1989).

4. Overview and Curricular Context

The choice of the software — a spreadsheet — was justified by two major reasons: it is a powerful tool with facilities that can easily be used for the learning of mathematics, and in particular fits very well in the 11th year Portuguese curriculum which is mainly the study of functions ~d sequences.

Two classes, one in each school, have been involved in the activity. Pupils' age ranged from 16 to 18 years old. The two teachers set up the scheme of work and developed all the materials (worksheets, evaluation tests and curricular organization) together. The work was supported by the Portuguese national MINERVA Project.

The year began with the general concepts regarding functions — classification, monotony-order-, etc — without the use of computers.

The computer is particularly suitable for the study of sequences, this was the subject that was chosen to begin the computer work, reversing the usual curricular order in which functions appear before sequences. The study of functions was done as an extension of the previous study of the sequences. The concept of the limit of a sequence helped students to understand the concept of the limit of a function at a point. In this kind of approach, the software proved to be most suitable providing opportunities to draw conclusions concerning extensions and restrictions of a function.

The computers were used by several classes in the schools, so they were not always available. It was at Mem Martins where this fact was most annoying.

5. Description of Activity

Problem solving was present throughout all lessons and a similar approach for each curricular topic was made. A problem solving situation was introduced and a period of exploration and discussion followed in order to draw conclusions.

As the whole curriculum was studied based on the spreadsheet, seventeen problems were used to introduce the mathematical topics.

5.1. Teaching and Learning Styles

The use of problem solving as a methodology and the spreadsheet as a tool provided opportunities of discussion within mathematical subjects. There was also a new organization in the classroom, shifting the focus from the teacher to the students' activity. The teacher should act as a coordinator.

At Mem Martins there were 6 computers available. The class was divided in 6 groups of 3 or 4 pupils. Each group had one computer to do its own work allowing all groups to use them at the same time. In Veiga Beirão there was a change within the first weeks. Since there were only 4 computers available, groups of 5 pupils showed difficulties to find a suitable organization. It was then decided to change the number of pupils in each group in order to make them smaller. While 4 groups were working with computers the others were doing activities within the same subject and related to the activities they had done or would do later using the computer.

During the period of discussion and exploration for each problemam solving situation, the teachers acted as coordinators encouraging the students to go into deeper analysis, proposing questions and challenges, and helping them in order to allow the discussion to go on until a mathematical model has been found. This search was followed by worksheets with questions related to the concepts implied in the model. Paper and pencil work had an important role in the activities carried out. The discoveries made were then formalized and linked to the curricular topics.

After students have tried to solve the problem, the different strategies developed were compared in order to promote the exchange of opinions and the improvement of final results. The exchange occurred spontaneously without the teachers' interference.

Outside the classroom, the time was mainly spent conceiving and building teaching materials and correcting every worksheet done in the classroom.

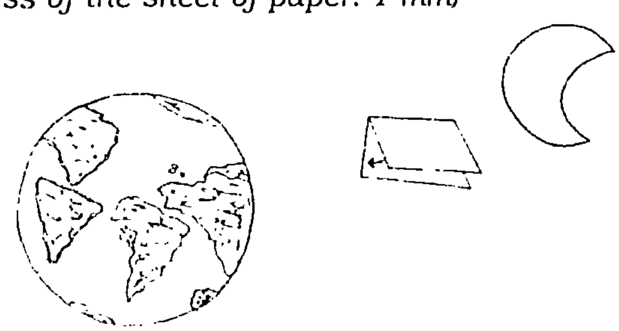
In every evaluation test, a problem similar to the those presented in the computer lessons was included. It was the part that pupils found easier and liked most. Although planned, a evaluation test using computers never took place.

5.2. Record of Activity

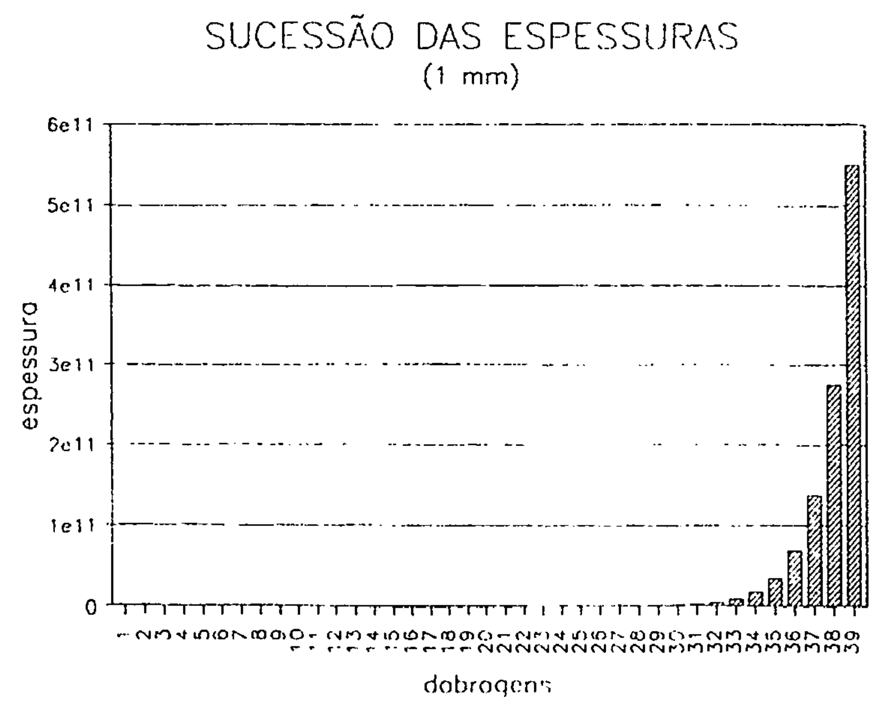
One example: **From the Earth to the Moon**

Imagine that you can fold a sheet of paper as many times as you want. Knowing that the distance from the Earth to the Moon is 380.000 km. how many times do you have to fold the sheet of paper in order to reach the Moon?

(Thickness of the sheet of paper: 1 mm)



When this problem was introduced several pupils tried to guess the correct answer. Fifty was the smallest number mentioned during this first discussion. After a while, and when the pupils' opinion seemed to be established, the need to check the truth arose and the spreadsheet was set up by the pupils with two columns: the number of folds and the corresponding thickness. The initial surprise was overcame by the rapid sequence increase, clearly evident in the computer (*espessura* and *dobragens* are thickness and folds in Portuguese):



The following worksheet was given to the pupils in order to deepen mathematical concepts like:

- sequence
- subsequence

- order
- upper and lower bound of a sequence
- limit of a sequence
- geometric progression and its common ratio

1. Define a sequence e_n whose terms represent the successive thickness of the sheet of paper, after each fold:
1.1. by a recursion formula
1.2. by a formula that describes the nth term of the sequence.

2. Compute e_1, e_2, e_3, e_{38} and e_{39}.

3. If the initial thickness of the sheet of paper was 0.1 mm, what will be the first term of the sequence? In this case, how many times would it be necessary to fold the sheet in order to reach the Moon?

4. Is there any term equal to 512? Or equal to 80? Why?

5. Draw, using a spreadsheet, a graph with the first terms of the sequence.
5.1. What can you say about the sequence after you have examined the graph?
5.2. Prove that the sequence is monotonic and indicate the kind of monotony.

6.1. Compute, using the spreadsheet, the ratio and the difference between consecutive terms (e_{n+1}/e_n and $e_{n+1} - e_n$).
6.2. Compute, without the spreadsheet, the same ratio.

7. Let E be the set of the terms of the sequence.
7.1. Is it possible to find a lower bound of E?
7.2. And a upper bound?
7.3. Is it possible to find a range bounded by two horizontal lines containing all points of the graph?
7.4. Is the sequence bounded? Why?

8. Is $v_n = 4^n$ *a subsequence of* e_n*? Why?*

9. What is the limit of e_n*?*

Items 1. and 2. were answered during the introductory lesson and it took two more lessons to work through the whole worksheet. The recursion formula was quickly found as there is a straight connection with the task that the pupils performed at the computer.

The computers were available in case the pupils should like to use them. For example, they enlarged the spreadsheet with two columns where the ratio and the difference between consecutive terms were calculated. As stated before, every question was discussed both with the pupils of their own groups and with members of other groups.

6. Conclusions

Based on direct observation of students' work, worksheets and evaluation tests, the teachers have drawn conclusions concerning the activities.

After overcoming initial apathy, enthusiasm and the will to go on remained constant all year long. The teachers think both the methodology and the use of computers were responsible for this students' behaviour. Even in more formal lessons the students had an active role communicating their own views, certainties and doubts.

In the first term, the progression throughout the curriculum was below the average for the 11th grade classes. The students took a certain amount of time to get acquainted with the software. Moreover, the methodology, based on group work and problem solving, was a novelty to them as far as mathematics is concerned. After the first term a inversion took place and there was a rapid progression, faster than the teachers expected. One hour extra (6 per week instead the usual 5) helped to overcome this difficulty and in the end all the curricular topics were covered.

A similar change happened in the students' autonomy. In the first lessons, the pupils showed great need for the teachers' support, as soon as a doubt arisen the teachers' help was required. After this period they developed ways to deal with the difficulties, managing to work through the whole problem requiring little or no help from the teacher. Obviously, it was during the first phase that teachers had additional work while students were trying to adapt themselves to this methodology.

There was a significant increase in the students' ability to make clear their reasoning, both orally and in written form.

One of the teachers involved in the activity had another 11th grade class where computers were not used and thus the problems using computers were not presented. Although the curriculum organization was the same, the outcomes in this class were below that obtained by the class in which the computers were used.

The methodology generated opportunities to discuss connections within mathematical subjects, links between them, and relations of mathematics with the real world. In fact, the students were surprised to find that mathematics has applications in physical phenomena, for example. As mentioned above, the 11th grade curriculum was fully studied and besides the students had opportunities to face subjects outside curriculum boundaries that arisen in discussions and they found interesting. Students said it was a very rewarding approach and an enjoyable experience.

Case Study 12

Graphic Calculators in Mathematics Teaching

Graphic Calculators in Mathematics Teaching

1. The schools

The activities described in this case study took place in several schools, from a set of twenty four belonging to six areas: Bury, East Sussex, Essex, Leicestershire, Northamptonshire and Suffolk, corresponding to 6 teaching groups. Within this set we can find a diversity concerning pupils' social background and age, schools environment and dimensions. In order to ilustrate, two schools are described below.

Newmarket Upper School is located in the Suffolk town of Newmarket and provides secondary education to pupils aged between 13 and 18 years old. They come both from the town and from the surroundings which is mainly a rural area. The town is known as a centre for breeding and training racehorses. Along with this there are light industrial and agricultural enterprises. The school is a comprehensive school and has 660 students covering the whole social and ability range.

Huxlow School is a comprehensive school in Northamptonshire and it serves the small towns of Finedon and Irthlingborough and the surrounding villages. Light industry and agriculture are the main activities within the school area. The school has 620 pupils from 11 to 18 years old, and a wide range concerning social background and ability is represented.

In both schools the Head of the Mathematics Department is involved in the use of graphic calculators. The students involved in this activities were aged between 16 and 18 from A-level courses in Mathematics. An A-level course is a two-year option course in Mathematics, for students in the upper secondary stream. The number of one hour lessons per week is around 4 and the topics covered are, among others, algebra, coordinate geometry, trigonometry, functions, calculus and statistics.

In Newmarket Upper School, 15 students in each year-group took A-level courses in Mathematics and in Huxlow School this course is taught in a consortium with two neighbouring schools to 40 students in each year-group.

2. Aims

Although for each subject specific goals are to be attained, there are general aims to consider regarding the use of graphic calculators:

a) To foster pupils' skills in mathematically thinking and reasoning.

b) To allow students to use spontaneously computing facilities, making it part of the normal mathematical activity.

c) To allow a variety of approaches for the same problem (numeric, algebric, graphic).

d) To emphasise the exploration of mathematical relationships rather than present symbolic manipulation as an end in itself.

e) To encourage students to evaluate their own ideas and conclusions.

3. Material and resources used

The models of calculators used were the CASIO fx-7000 (5 teaching groups) and the Hewlett-Packard 28 C (1 teaching group). Both these models have graphing and programming facilities.

Each student was given a graphic calculator on permanent loan or they were freely available in every mathematics lesson. In this case, and to be permanently available, pupils could obtain them at the school library or at a resource centre.

Box 1

The graphic calculators have also programming facilities which can be used by the students. The following program, to appear later on in this case study, calculates, and displays on the screen, the sucessive terms of a series which is a sequence of the sum of the terms of an arithmetic progression with first term *A* and common difference *D*.

```
"A"? —> A ............................. ask for number A, put in memory A
"D"? —> D
A —> U ................................ store A in U
U —> S
Lbl 1 ................................. mark the top of a loop
" SUM= " .............................. display on the screen: SUM=
S◢ .................................... display the value stored in S
U+D —> U .............................. add U to D and store the result in U
U+S —> S
Goto 1 ................................ jump to the top of the loop
```

4. Overview and Curricular Context

The activities described in this case study have been developed within a NCET - MESU (National Council for Educational Technology - Microelectronics Education Support Unit) project coordinated by Kenneth Ruthven from the Department of Education of the University of Cambridge.

The Graphic Calculators in Mathematics project ran from July 1988 to July 1990 involving over 30 teachers and 1000 students in 24 schools. Its main goal was to develop, trial and evaluate teaching approaches wich exploit the potential of the graphic calculator and produce a professional development pack for dissemination to a wider audience.

During the two years, a newsletter - *Graphvine* - and several supplements with specific and general topics were published. These supplements were elaborated by working groups of project teachers and included proposals of activities to be worked in the classroom, for example, suggestions of worksheets. Some issues were dedicated to provide an alternative to the official manuals of the graphic calculator models explaining and exploring basic features. The project teachers met periodically to exchange ideas and review progress.

The activities within the project took place in upper secondary mathematics courses, mostly A-level. A variety of mathematical topics and subjects were studied using calculators: numeric patterns, sequences, series, algebra, statistics, graphic representations, including polar and parametric graphs, graphic patterns, locus, polynomials, functions and trigonometry including identities.

5. Description of activity

5.1.Teaching and Learning Styles

Project teachers were free to plan the work of their classes, having occured both common and different approaches. For example, programming was not given the same stress by all teachers. In any case, common facets can be identified. Symbolic (or algebric) manipulation, available only in one model, was little used and learning took place more privately and informally. This is not to imply that group work, peer discussion and interacting do not occur. During lessons students were free to explain their own reasonings thus helping each other.

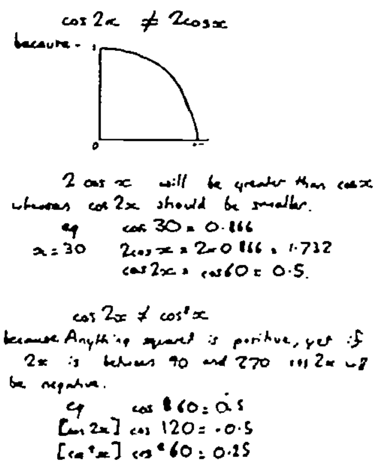
5.2. Record of activities

In a 11th grade class, students are beginning to develop the trigonometric identities for double- and compound-angles such as cos 2a and cos (a+b). They had already made use of calculators exploring transformations of graphs and the periodic properties and symmetries of simple sine and cosine functions, and are aware of simple interrelationships, for example $sin^2+cos^2=1$.

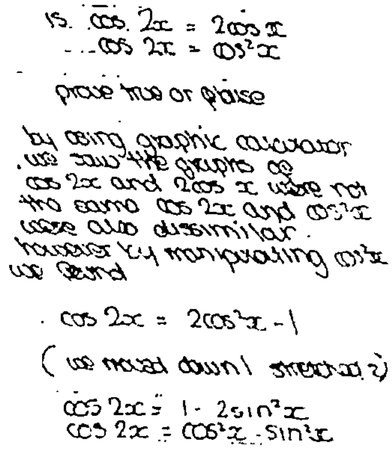
The teacher asked the whole class if there are suggestions for equivalent forms of *cos 2a*. The most popular suggestions are $2\cos a$ and $\cos^2 a$. Then, the students were asked to evaluate these suggestions, in groups of three or four, and write down their reasoning. In case both the suggestions proved to be false, they should search for a correct one.

Different strategies were developed by the students. In the first phase, while some of them selected a particular value for a and calculated the cosine values for each case, others draw the graphs for each function. The calculators were used to find the cosine values and to draw the graphs of the functions. In the second phase, almost every groups used the graphic calculators.

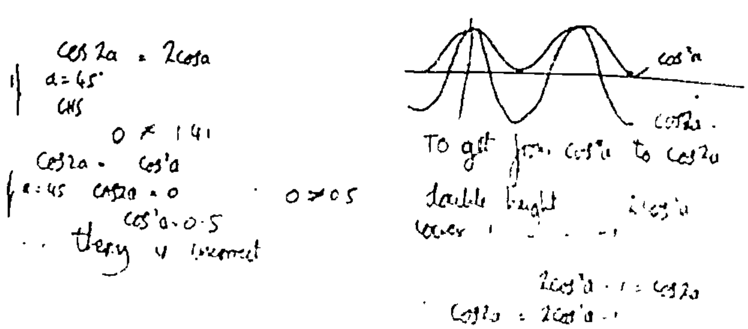
The students' work is illustrated below:



student 1



student 2



student 3

The following activities took place at the Bury Metropolitan College in a 11th grade class and were designed by Barbara Richmond and Peter Normington. These activities aimed the study of arithmetic and geometric progressions as well as the corresponding series.

On the course of developing the activities supported by worksheets, the students worked in group and discussed their results which were summarised and documented in a follow-up discussion.

After they have generated arithmetic progressions, using calculators, the students were asked to try the following in their calculators:

-2 —> A EXE

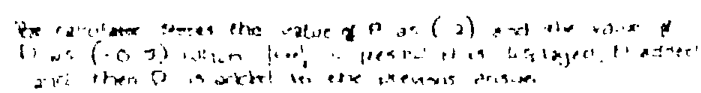
-0.5 —> D EXE

A EXE

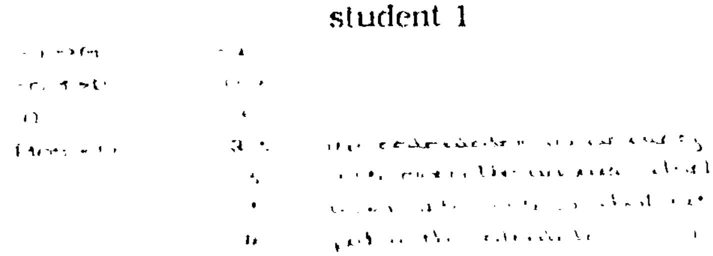
Ans + D EXE

Keep pressing EXE. Explain in your own words what the calculator is doing.

Here it is a sample of students' work regarding this particular worksheet item:



student 1



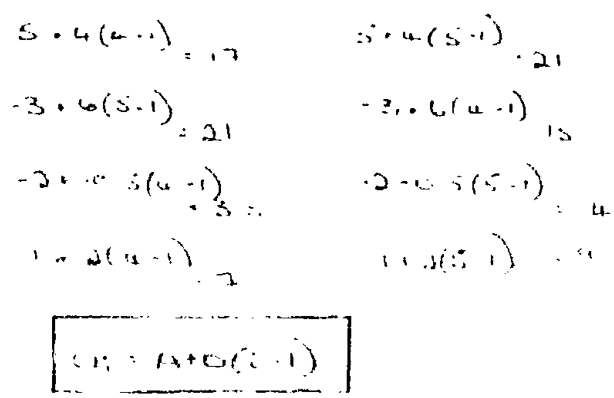
student 2

```
"A"?—>A
"D"?—>D
A◢
Lbl 1: Ans+D◢
Goto 1
```

This program, as well as the way to store and run it, was listed in the next worksheet item. The students tried it using several values for *A* and *D*. Information concerning notation and terminology of arithmetic functions were written on the worksheet. For each sequence, a formula for u_i in terms of *i* had to be found and later generalised:

For the sequence with first term a, and with adding-on number d (which we call the common-difference), can you find a formula for the ith term?

Spotting a pattern or devising a formula for each sequence and then generalising were two strategies that students employed:



student 1

$U_4 = 7$ $U_5 = 9$ $U_{10} = 19$ Formula $2i - 1$

$U_4 = 17$ $U_5 = 21$ $U_{10} = 37$ Formula $4i + 1$

$U_4 = 15$ $U_5 = 21$ $U_{10} = 51$ Formula $6i - 9$

$U_4 = -3.5$ $U_5 = -4$ $U_{10} = -6.5$ Formula $-0.5i - 1.5$

general formula $\frac{Di - D + A}{A + D(i - 1)}$: $A + D(i - 1)$

student 2

After this study about arithmetic progressions, the students were introduced to the concept of series.

a) *A series is the sum of a sequence. Take the arithemetic progression with 1 as the first term and 2 as the common difference, and write down the values of u_1, u_1+u_2, $u_1+u_2+u_3$, etc. Make a table of values as follows:*

n	u_n	*sum up to and including* u_n	(u_1+u_n)	$n(u_1+u_n)$
1	1	1	2	2
2	3	4	4	8
3	5	9	6	18
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

b) *Repeat a) for the sequences you worked with. Do you notice anything?*

c) *Can you write down a formula for the sum of the first n terms (wich we denote S_n) of these sequences?*

d) *Can you write this formula in terms of the first term* a *and the common difference* d?

$$S_n \times 2 = n(d_1 + a - d + u_n)$$
$$S_n \times 2 = n(u_1 + u_n)$$
$$S_n \times 2 = n(a + u_n)$$
$$S_n \times 2 = n(a + dn + a - d)$$
$$S_n = \frac{n(a + dn + a - d)}{2}$$

$$S_n = \frac{n(2a + dn - d)}{2}$$
$$S_n = \frac{n(2a + d(n-1))}{2}$$

student 1

$$S_n = \frac{n(U_1 + U_n)}{2}$$
$$S_n = \frac{n(U_1 + A + d(i-1))}{2} = \frac{n(U_1 + (a + d(i-1))}{2}$$
$$= \frac{n(U_1 + (a + n - 1)d)}{2}$$
$$= \frac{n(a + (a + (n-1)d)}{2}$$
$$= \frac{n}{2}(2a + (n-1)d)$$

student 2

As illustrated above, the students coped well with this activity, although a few had to be led towards comparing columns three and five when doing the investigation in questions a) and b). A follow-up discussion of the proof of the formula for the sum of arithmetic progressions brought out the reason for considering (u_1+u_n) in the investigation.

By this time the arithmetic progression was formalized and later on the concept of geometric progression was introduced using a program, similar to the program which was used for the arithmetic progressions. A formula for the ith term was found.

A	1	2	3
R	2	3	4
	1, 2, 4, 8, 16	2, 6, 18, 54	3, 12, 48, 192

A = Starting value
R = Common Multiple

The starting value is A and this is continually multiplied by R.

$u_i = AR^{i-1}$

At this point, some students tried to find a formula for the sum of the geometric progression and applied the same approach they had used concerning the arithmetic progression. It proved to be difficult!

Based on the concepts studied, the students began to work within another worksheet focusing the ideas of convergence and the limit of the sum of a series.

At the beginning of the worksheet, the students were asked to enter into their calculators the program (see box 1 for details):

```
"A"?—>A
"D"?—>D
A—>U
U—>S
Lbl 1:"    SUM= ": S◢
U+D—>U
U+S—>S
Goto 1
```

$$u_i=a+(i-1)d$$

$$S_n=\sum_{i=1}^{n} u_i$$

After they had calculated the sum for several number of terms within various arithmetic progressions, the program was modified in order to find the sum of terms of geometric progressions. Again, the students had the opportunity to try the program with several progressions and some questions were proposed in order to investigate:

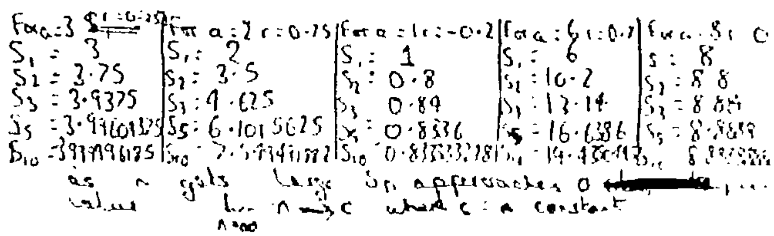
Do you notice anything about S_n as n gets larger?
What meaning can you give to $\lim_{n \to \infty} S_n$?

Which is the important number which governs the behaviour you notice in the sum sequence? Is it a or r?
What is the condition for $\lim_{n \to \infty} S_n$ *to exist?*

Students found their way through the investigation, although there were differencies between them. Some showed more facility than others. For example, it was necessary to encourage some students to calculate the sums for more than 10 terms in order to investigate the limit and be sure of convergence.

This approach allowed students' assumptions providing opportunities to test them. One student came up with the condition a (first term) $< r$ (common ratio) for the convergence of a geometric progression, and was encouraged to test this out.

Below and on the next page are printed some students' ideas of convergence to a limit and comments on the condition for convergence.



N°'s gets closer & closer together until they can't go any further and it stops at a certain no.

The N° has a limit by which it cannot continue past probably due to the infinite n° of digits it can take in decimal

As n gets larger S_n approaches a number but never reaches it.

A = 1   R = 2
1 , 3 , 7 , 15 , 31 , 63 , 127 , 255 , 511 , 1023
2047 , 4095 , 8191 , 16383 , 32767 , 65535

A = 2   R = -1.5
2 , -1 , 3.5 , -3.25 , 6.875 , -8.3125 , 14.46875
-19.703125 , 31.5546875 , -45.3

A = 3   R = 1
3 , 6 , 9 , 12 , 15 , 18 , 21 , 24 , 27 , 30

A = 0.5 , R = 1.1
0.5 , 1.05 , 1.655 , 2.3205 , 3.05255 , 3.856 , 4.74

A = 2   R = 0.9
2 , 3.8 , 5.42 , 6.878 , 8.1902 , 9.37 , 10.4 , 11.4
12.25

R Governs the behaviour A is fixed

a) A = 1 , R = 2 get larger & larger until calc could not take any more

[illegible]

R is important number   [illegible] → series converge
r = 1 → alternate [illegible]
r > 1 → no limit
r [illegible] → no limit

Overall, the students enjoyed working through these activities and the introduction to programming promoted some students' involvement in using the calculator.

6. Conclusions

Several conclusions regarding the use of graphic calculators, the way students relate to them, can be drawn from the work developed.

The initial period of becoming familiar with graphic calculators can be frustating at times but, as they were always available, students usually managed to overcome it. At first, students experienced difficulties in breaking habits established on traditional calculators. For example, the necessary adaptation to the reverse Polish notation used on the Hewlett-Packard was not easy for most of them. There was a considerable mismatch between the informal concepts employed by the students and the formal language of the calculator in working with symbolic manipulation.

Although a small number of students were reluctant to use any calculator, traditional or advanced, because they felt they were *losing control* of the mathematics, by the end of the first term nearly all the students were making confident and spontaneous use of the calculating and graphing facilities and by the end of the first year two thirds of the pupils were making spontaneous use of the programming facility.

Students who had previously used traditional calculators continued to use them for calculating, for example, the manipulation of fractions, even if they had became proficient in using graphing facilities on graphic calculators. This happened often in two project classes where the teacher showed reservations about the use of graphic calculators.

Regarding teachers' opinions, they were keen about the use of calc·ʼ·ıtors, showing enthusiasm and encouraging the use of graphic calculators in their schools and local areas.

Comparing the work in a class where every student has one calculator, and another one with a single computer in the classroom, the teachers noticed that students used graphing and programming facilities in the first setting in a wider manner than in the second environment. Even in classes where access to calculators was more limited they also think that access to graphic calculators is more supportive of small group exploratory approaches.

By the end of the first year students were tested on describing a given graph in symbolic terms. The results in the project group were significantly higher than that of the comparison group, formed by students without access to graphic calculators or computing graphing. This test underlined other significant fact: females outperformed males in the project group while the inverse occured in the comparison group.

Proving the succes that calculators had among students, they started to use them in other lessons and for private study and several pupils purchased their own graphic calculators.

In the Autumn 88 issue of *Graphvine*, it is stated that:

> "*Sketching a graph, for example, or finding numerical solutions to an equation, are important component routines within the process of problem solving. Using the calculators to do this quickly and reliably frees students to focus on the strategic and tactical elements of a problem-solving task. Equally, the graphic calculator can be used as a*

learning aid. permitting the adoption of graphically-mediated appraches to new mathematical concepts and strategies. Many students found that such approaches made mathematical ideas more accessible in early stages of development, and complemented the traditional symbolically-mediated approaches at later stages."

Case Study 13

Integration of Algorithmic Thinking into the Mathematics Classroom

Integration of Algorithmic Thinking into the Mathematics Curriculum

1. The school

This work was carried out at the Gymnasium in Bad-Iburg, a town of 10.000 inhabitants 20 km south of Osnabrück in north-western Germany. This gymnasium is a state school, the only one in this town. It covers seventh to thirteen grade students, that is, those that normally would be 12 to 19 years old. It has 650 students.

The german "Gymnasium" is the type of secondary school which is orientated towards university. In a rural area like Bad Iburg, about 25% of the students in one grade are going to a gymnasium. The decision to do so is made by their parents at the end of grade 6th (in a kind of primary school for all children) consulted by the teachers.

The Gymnasium Bad Iburg has 61 teachers on the permanent staff.

From 1987 to 1990 in each year, two of four parallel classes (of about 25 students) were involved in the experiment. Since the beginning of the school year 1990 all four parallel classes in grade 7 are thought to be following this new approach.

2. Aims

The main objectives of the activities described in this case study have been

a) To use the integration of algorithmic thinking in the mathematics curriculum to help students develop a sound understanding of the concept of function (of several variables) and related algebra concepts.

b) To give pupils, through the use of appropriate materials, a perception of computer operation and of the mathematical outlook of the automation of procedures.

3. Material and resources used

3.1 Hardware

10 MS-DOS compatible computers were used.

3.2 Software

The software used was a simulation program for a *Register-machine* (RM). Before using the computer, in the first part of the school year, pupils work with concrete or imagined RM's (see below, § **4.**).

3.3 Other supporting materials

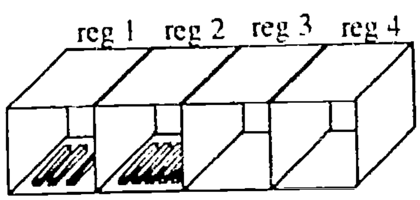
A textbook for pupils is available (an English edition is in preparation). There is also a detailed teachers' handbook. In the classroom, pupils receive worksheets. The instruction material "Dynamische Labyrinthe" is used to represent an algorithm as a kind of working flow chart.

4. Overview and curricular context

This experience is included in a curriculum project developed in the Osnabrück University during the past three years (1987-90), that will last until the end of July 1993 and is financed by the Ministry of Education of the Niedersachsen State (in the Federal Republic of Germany). One main objective of the project is to build and experiment learning situations leading to the integration of algorithmic and axiomatic thinking into the mathematics curriculum for grade 7 and 8 and the construction of a new curriculum and textbooks for the students.

In the beginning of the school year, 7th grade pupils are introduced to the subject through the use of some models of register-machines.

A concrete and unsophisticated model of a *register-machine* is a box with several compartments, called registers.



One of the very first activities that could be proposed to the pupils is the following:

Children are asked to imagine a robot that is only able to execute two elementary operations: i) to take one stick from one register and ii) to put one stick in one register. They are asked to put 3 sticks in register one and 5 sticks in register two and to give instrutions to the robot in order that, after some operations, the number of sticks in the register one is to be equal to the number of sticks that were in both registers before. This is just the beginning of a long process where the children, after playing with this RM as a first step, will go on building adding - machines with counters and other blocks and inve ing small computer programs (based on the referred elementary operations) for solving problems. It is important, for the didactical approach, that the students learn to represent an algorithm in different forms of (external) representations. It has been shown by research on the cognitive structures of algorithmical thinking that the pupils have individual differences in the structure of the mental model which is induced by the offered (external) representations of an algorithm.

By the middle of the school year, in a typical situation in the classroom, pupils will solve a given problem in two different but mutually illuminating ways: they define a function, normally with several variables, to mathematize the concrete situation expressed by the problem, and they build one computer program as a simulation of the same situation.

This curriculum project has been accompanied by a research on the cognitive structures of algorithmical thinking, conducted by Dr. Inge Schwank, from the Forschungsinstitut für Mathematikdidaktik, in Osnabrück.

The project started in August 1987 with two 7th grade classes in the same school and has been expanded every year. In 1989-1990, twelve 7th grade classes in five schools were involved and in 1990/91 not only those classes are followed in the 8th grade as twelve new 7th grade classes have began the experience

In the 7th grade, the mathematics curriculum in this State includes statistics and probability (2 weeks), geometry (4 weeks) and algebra (32 weeks) In the classes that are following the proposed curriculum of this project, the whole subject of algebra and statistics is replaced by the new algorithmic and functional approach.

The project is carried on at the University of Osnabrück and headed by prof. Cohors-Fresenborg.

5. Description of activity

5.1. Teaching and learning styles

a) The classroom in the Gymnasium Bad Iburg is a pleasant and well illuminated room with several rows of tables and a large space in the back. In this space, in one row and against the wall, ten computers are placed in such a way that two pupils can work in each of them.

b) In a typical session, the teacher — in this case Dr. Christa Kaune — will give the pupils one worksheet with some problems to solve. Pupils will work mainly individually, at their tables, but sometimes will discuss their findings with the colleagues that are seated nearby. They are very interested in their work and eager to find a solution — for example, a function of several variables modelling a given situation.

c) After some time, the teacher will ask if someone has found a solution for the first problem, and some pupils answer "yes" and go to the blackboard and write their functions, while their colleagues go on trying to find their own solutions. The teacher asks the pupils to criticize the solutions written on the blackboard, and each one is scrutinized, in a disciplined but lively general discussion, with good contributions from many pupils.

d) Finally, several different solutions are accepted and one is chosen for computer work. In the next step — to build a computer program representing an algorithm to compute the value of the function — pupils will work in the computers in groups of two and will take that solution as the chosen model to be represented by the computer program.

c) If the worksheet was not completely worked out in the classroom lesson, the pupils will finish it as homework, and their findings will be discussed in the next lesson.

5.2. Record of activities

1) The following is an actual worksheet proposed to 7th grade pupils in the Gymnasium Bad Iburg:

Worksheet

To be used in a school party we have to buy bottles of coca-cola, mineral water and orange juice in a drink market. In this shop the prices of beverages are the following

1 box of coca-cola (12 bottles)	0.85 DM per bottle
1 box of mineral water (12 bottles)	0.45 DM per bottle
1 box of orange juice (12 bottles)	1.15 DM per bottle

For each box a deposit of 6.60 DM will be requested.

a) Write the expression for a function that computes the expenses of the listed beverages. You must explain the meaning for each variable included.

b) After looking at the needs of previous parties, the school party committee decided to buy 252 bottles of beverages including: 7 boxes of mineral water, 8 boxes of coca-cola and the remaining bottles of orange juice.

Write one expression to compute the charge for all beverages to be paid to the shop when the bottles are purchased.

c) Write a RM program that allows the shop to compute the charge for the beverages. You dont need to consider any other type of beverage. Identify the input and the output in your program.
Using the program, compute the cost of the beverages for the school party.

d) The party commission counts on 480 guests. If the admission price is fixed at 2.50 DM per person, are the income from admissions enough to compensate for the party expenses, if we take in account the cost of the beverages and a charge of 456 DM to pay for the staff (waiters, waitresses etc)?

e) The party committee decided to adjust the admission price in order to balance the party expenses, but without any profit. Using the previous data, compute a new admission price, but with an increase over the previous price not larger than what is needed.

2) There follows a problem from a written test, and the answer from pupil Fabian:

Question 4

In a written test, pupils were requested to write a program for the following problem: $(x_1, x_2, x_3) \longrightarrow (2(x_2 + x_3), 2x_2 + 2x_3)$.
Here are two selected answers.

Martin: $(_1S_1)(_2S_2A_1)(_3S\ A_2)(_4S_4A_1A\ A_2A_3)$

Elisabeth: $(_1S_1)(_2S_2A\ A\ A_1)(_3S\ A_2A\ A)(_4S_4A\ A_3)(_5S\ A)$

1 4

a) Verify the correctness of the answers.

b) What is the fastest program?

note: the answers of Martin and Elisabeth are written in the code of the register-machine language; if we have for instance two counters, 1 and 2, and if we count bakwards one by one in counter 2, we write S_2 (S means subtracting); if after this we count forward, one by one, in counter 1, we write A_1 (A means adding); so, for one loop, we will write (S_2A_1); but, to indicate that we made as many loops as necessary until register 2 is zero, we will write $(_2S_2A_1)$

Fabian answer

a)

a) Martin $(X_1, X_2, X_3) \rightarrow (0, X_2, X_1) \rightarrow (0, 0, X_1, X_2)$
$\rightarrow (0, 0, 0, X_2 + X_1) \rightarrow 2(X_1 + X_2)$ [illegible]
Martins Lösung ist richtig, außerdem hat
er das Distributivgesetz angewendet ✓

Martin's solution is correct and moreover he test the law of distribution.

Elisabeth $(X_1, X_2, X_3) \rightarrow (0, X_2, X_3) \rightarrow (0, 0, X_1, X_2, X_3)$
$(0, 0, 0, X_2 + X_1, X_2 + X_3) \rightarrow$
$(X_1 + X_2, 0, 0, 0, 2X_2 + 2X_3) \rightarrow$
$(X_1 + X_2) 2X_2 + 2X_3$ ✓

Auch Elisabeths Lösung ist richtig, sie
hat exakt die vorgegebene Lösung heraus-
bekommen. Und auch sie hat das Distributivgesetz
nicht angewendet ✓

Elisabeth solution is correct, because she obtained the initial result, but she did not tested the law of distribution.

b)

b) M $RS_M(X_1, X_2, X_3) = X_1 + 2X_2 + 2X_3 + 5(X_2 + X_3)$ ✓

$= X_1 + 9 \cdot (X_2 + X_3)$ Rechenweg!

E $RS_E(X_1, X_2, X_3) = X_1 + 4 \cdot X_2 + 4X_3 + 3(X_2 + X_3) + 4(X_2 + X_3)$

$X_1 + [illegible] (X_2 + X_3)$ ✓

Martins Rechenweg ist schneller er hat durch Anwendung des Distributivgesetzes den kürzeren Weg gefunden. Dieses habe ich durch ausrechnen der RS nachgewiesen. Elisabeths Rechenweg dauert etwas länger. Sie hat sich [illegible] an die vorgegebene Lösung gehalten und so damit ihr Rechenweg länger. ✓

Martin's computations are faster because he used the law of distribution and found the shortest way. I was able to check this computing the RS.

Elisabeth computations are slower because she did not restrain herself to the initial results.

note: RS means *computing step counter function*, a function used to compute the speed of a program [illegible] ... It is the complexity of the algorithm).

3) Another question from a written test and some answers for item 4):

"1) In the program ($_1$ S_1 A_2 A_2 A_2 one ")" is missing. To get a the program sintax correct there is more than one way to place the ")". Indicate these ways.

2) What functions are computed in each program?

3) For each program, write the analitical expression of the corresponding function.

4) Compare the bugged program with the phrase "Er bat mich zu grüßen." (He asked to salute). Explain why in school mathematics it makes sense to work with grammar as we have done."

Katharina: "The phrase is ambiguous. The bracket can be put in two places. Also in mathematics there exist programs that can be interpreted in different ways. In order to understand each other [...] we need grammar. We are able to justify easily why something is wrong, because it doesn't follow this or that rule. [...] Thinking of this phrase as an example, we can say that it is ambiguous because one comma is missing [...]. What I said could also be applied to programs."

Ines: "We must realize how important is punctuation. For instance, if we forgot to place one parenthesis our program becomes ambiguous. RM machine doesn't know how to compute and gives an error signal. Another possibility is that the machine computes the expression inputed by us but not in the way that we intend to compute it."

6. Conclusions

In the classroom, pupils enjoy the work they are doing and show a great confidence when they deal with some concepts considered difficult for pupils of this grade.

The following extracts shows a teacher's viewpoint on the experience:

"Pupils know important concepts in the domain of computer programming. They know how to use, in the elementary environment of RM language, interaction and tool programs. They dominate debugging, optimization of progams (to write functions to measure the complexity of algorithms), modular programming, notions of sintax and semantics, relations between natural language and programming language.

[...]

The concept of function, in my opinion, is well consolidated. Pupils are able to write or manipulate functions of several variables, without any dificulty. Most important is that they do this without any fear. For me, this aspect is a new discovery in the teaching of mathematics.

[...]

Classroom ambience may be described as animated but serious. Many pupils are deeply involved in the lesson and their interest is explicitely shown. Some pupils are always making questions, aiming a better understanding of the subject of their study, and evaluating their learning achievements.

[...]

In the computer room, in my opinion, noise is sometimes too great [...] I am sure that the noise produced by the majority of the pupils [...] is the result of the enthusiasm put in challenges raised by their activities."

Pupils really understand the concept of functions (of several variables). In what concerns algebra, the experience has shown that pupils are able to use symbolization. As they learn these curricular matters through the use of computers and their programming, which includes the syntax and semantic of programming languages and the complexity of algorithms, this additional items of the proposed curriculum do not cause problems in what concerns the fullfilling of the "normal" curriculum.

"Pupils are much more successful in the normal fields (like proportions, percentage, word problems)" as compared with control classes, according to the leader of the project.

Case Study 14

Integrating Informatics in the Mathematics Curricula

Integrating Informatics in the Mathematics Curricula

1. The School

This experience was carried out at the scientific Liceo G. D. Cassini, a post-compulsory education secondary school, in which students range 14 to 19 years old. It awards a diploma of *maturitá scientifica*, preparing for university studies. This school is situated in the centre of Genoa, having about 1100 pupils and 90 teachers.

In the first year of this experience (1989/90) participated 82 students from 3 classes. They had ages from 14 to 16 years old and were attending the first year of secondary education. The experience continued in the following year, with the same classes (now in the second year), considering new topics, concepts, and procedures. And new first year classes begun to be involved as well.

There were no special criteria in the selection of classes to participate in the experience, which can be regarded as representative of liceo population. The students come from an heterogeneous socio-economical environment, most of them belonging to middle class families.

The two first years of Italian scientific *liceos* (forming the *biennio*) provide a completion and a deepening of the subjects of the previous compulsory school level and provide the introduction of Latin as a school subject.

As in most Italian *liceos*, mathematics is regarded as an important subject. In the first year, students have 5 hours of mathematics classes and in the second year they have 4 (but 5 in the experimental classes involved in this experience).

2. Aims

Computer science ideas were used to foster mathematical objectives. In a first phase, from September to December, the students learned how to work in the computer environment. It was sought that they would draw out on knowledge acquired at previous school levels to:

a) make conscious use of numerical calculations;

b) develop abilities such as singling out and building relations and correspondences, generalizing simple situations, reasoning on abstract concepts, and using a rigorous language;

c) use written materials such as textbooks and manuals.

In a second phase, from January until the end of the school year, it was sought that students would:

d) recognize the variables of a problem and describe its solution through an algorithm;

e) use the computer as a "non intelligent" instrument and show flexibility in the use of programming languages.

3. Material and Resources Used

3.1. Hardware

In this experience were used 10 IBM PC XT computers, printers, overhead projector and data show.

3.2. Software

The software used was the spreadsheet Lotus 1 2 3 and the programming language Turbo Pascal.

Lotus 1 2 3 is one of the most widely used commercial general purpose programs. As in all spreadsheets, it enables to handle numerical data, generated by algebraic or transcendental functions and relate data generated by different expressions. It also permits to draw the respective graphical representations.

Turbo Pascal is a quite popular version of this programming language enabling the construction of all sorts of mathematical and non mathematical algorithms. It is widely used in introductory informatics courses at secondary and university level.

3.3. Supporting Materials

For the computer science topics the students used photocopied notes prepared by the teacher, including a mini-glossary and worksheets. Also used was a MS-DOS pocket manual. For the mathematical topics they used the regular textbook (Batelli & Moretti, Matematica sperimentale, Le Monnier).

4. Overview and Curricular Context

This experience was conducted by Ivana Chiarugi, a secondary school mathematics teacher, as part of the activity of the GREMG (Gruppo Ricerca Educazione Matematica Genova, with Prof. Fulvia Furinghetti as director), which is carrying out the project "Integration of Computer Science and Mathematics".

In Italy, the Ministry of Education is developing a new curriculum for the first two years of secondary school which contemplates the introduction of elements of computer science within the mathematics program. Since 1985, a widespread program on teacher training in computer science has been undertaken. Now, teachers are encouraged to design and realize experimental courses, officially supported.

In this experience, the mathematics contents included set theory and logic, binary relations, and tabulation of functions. The computer science contents involved notions about the MS-DOS operating system, use of the spreadsheet Lotus 1 2 3, algorithms and programs, Pascal language, and notions of sequence, decision, cycle.

5. Description of the activity

5.1. Teaching and learning styles

Students worked in groups in the computer laboratory (in one hour sessions). They worked as a whole group in the classroom and, in the afternoon, worked individually at home.

In the initial phase of work, the individual evaluation was not particularly stressed, and was mainly carried orally. Later on, and having in mind the final evaluation, items concerning computer science topics were included in the written tests.

In learning the MS-DOS basic commands, the material was first delivered to the classroom during the regular classes. Afterwards, in the computer lab, the students were given a worksheet specifying several tasks, such as format a diskette and print a directory file.

This activity was organized as an experience of "survival in the management of computer resources". It was intended that the students should get used in decoding the information contained in written instructions. They had to read the task, look up in a mini-glossary the definitions of the new words, and check out in the manual the commands necessary to execute it.

Lotus 1 2 3 was used for 1 hour a week for collective work, plus the time required for individual revision. For example, in one activity the students were given the sequence which the first term is 1/a and the kth term is obtained subtracting 1 from the product of the (k-1)th term with a+1. Then, they were asked to construct with paper and pencil the four first terms and describe the behaviour of this sequence. Afterwards, they were asked to use the spreadsheet to organize a table to compute the first 40 terms of the sequence for a=2,3,4,5,6,7,8,9, study the behaviour of the sequences and try to give an explanation for the anomalies eventually found.

The aim of the activity with Lotus was to provide an easy way of introducing the computer, in a non sophisticated environment, before introducing the elements of Pascal. The mastery of Lotus was not a primary objective but mostly a mean of introducing informatics.

In the second phase of the experience the subject of functions was studied in the language of geometry (properties invariant under given transformations), in the language of algebra (representation of various situations and symbolic manipulation), and in the language of informatics (including algorithms and its representation in flow charts — in Italy also called Nassi-Schneidermann charts — and in a programming language).

The problem then become to express a transformation process through a language with certain characteristics just using the concept of sequence. This involved a careful study of the variables of the problem. Then, the problem was translated into a Turbo Pascal program. The analysis of algorithms was afterwards enlarged, including the concepts of decision and cycle, and the respective translations.

In this second phase, the work was not so much guided step by step as in the first one. It was deemed advisable that the students should be encouraged to develop the ability to organize their study and to assemble the materials they have been provided with.

Both regular mathematics classes and laboratory classes had as objective to foster mathematical abilities and knowledge. A significant example of this integration was the stress on the informal approach to real numbers, the problem of their representation and of approximation. The laboratory activity stressed the need of facing these topics and thus added motivation to learning. Analogously, the more sophisticated concepts of structure and closure of operations are approached (informally) in the informatics environment.

The most relevant change in the regular class pattern was in the discipline. Students become more noisy and, at times, insubordinate. They like to discuss among then, even when it is necessary to be quiet and follow the lesson of the teacher.

5.2. Record of the activity

In the work with Lotus 1 2 3, the first task given intended to allow an overview of the possibilities of the program. The second was conceived to present the use of graphics and complete the picture regarding the notions in the students' manual. Tasks three and four required the search for more complex formulas for generalizing, promoting besides a complete autonomy in using the computer. Task 5, indicated in the previous section, was regarded as conclusive.

The computer was called upon by the students merely to perform repetitive tasks. And the teacher used it to stress the difference between label and value and the difference between absolute and relative copy. Emphasis was set on the following observations: (a) fractions are approximated by a decimal number, (b) approximations of different numbers can produce the same result, and (c) 'very large' and 'very small' numbers (in absolute value) are not accepted by the computer which goes in overflow and underflow. It was observed that the computer is only able to represent numbers expressed as fractions whose denominator is a power of 2. Otherwise it will approximate and cause problems of propagation of errors.

It is felt that new and old ideas are learned quite in the same way. Nevertheless the motivation offered by the computer makes a difference in some topics.

Students did not encounter particular difficulties. The major problem is that they are not able to carry out an individual reflection after the lesson if they do not have a computer at home.

6. Conclusions

This project endeavoured to assemble ideas coming from books, given at teacher training courses, and arising from the teacher's experience and reflection, in order to set up a workable curriculum. Particular attention was given to choice of the notions and concepts to teach and the selection of exercises (in a gradual level of difficulty).

The main objectives of this experience were prevalently mathematical: to deepen the understanding of mathematical concepts and ideas, emphasizing an algorithmic viewpoint. There is a change in the traditional focus from 'existential' to 'computing' and problem solving mathematics.

The students considered the work interesting but difficult. In particular, the second part, specially the concept of cycle, needs to be developed again in the second year.

According to the teachers, the experience showed that, as in regular mathematics classes, the students can do autonomous work in routine exercises but, except the few gifted students, they often need the intervention of the teacher to be able to cope with more difficult problems.

The teachers felt that there was the need of a good cooperation of the colleagues who will be in charge of these students in subsequent years. If it is not taken into account that they have followed a different curriculum in these two years, they will be seriously handicapped. They also felt that the central authority of education should make compulsory innovative curricula contemplating computers. In their view, the experience can be regarded as successful because:

—it answers the need of updating mathematics teaching on the light of new technology.

—It fosters a general reflection and re-styling of mathematics teaching;

—It adds motivation to mathematics teaching;

—It allows to try new approaches to difficult concepts such as real numbers, approximation, closure and structure;

—It leads to consider the heuristic methods and problem solving activities usually not much considered in Italian mathematics classes.

Case Study 15

Modeling in the Teaching of Matematics

Modeling in the Teaching of Mathematics

1. The school

The modeling classes referred in this study were conducted in the Hermann Wesselink College a protestant school situated in Amstelveen, a wealthy suburb of Amsterdam. The social level of the pupils is medium to high. The scholl has 1100 pupils.

Four levels of secondary education are present in this school - Gymnasium (direct access to the University), VWO, HAVO (higher general school ending in the 12th grade) and MAVO (lower general school ending in the 9th grade). The pupils in the class addressed in this study belong to the VWO and will normally undertake university studies and are at the age of 17.

2. Aims

The activities described in this case study have the main objective of helping students:

1) to develop the capacities of mathematizing concrete situations, through the building and testing of mathematical models;

2) to gain experience in the work with various kind of models, appropriate for the quantitative analysis of phenomena.

3. Material and resources used

3.1 Hardware

One classroom with 15 MS-DOS compatible computers with colour monitors were used.

3.2 Software

Dynamo is a language developed for big computer systems in the Massachussets Institute of Technology by Forrestor, in the end of the fifties. Later a version for PC was developed, called Micro-dynamo. The program VU-Dynamo is based on this language and was designed by Piet Van Blokland at the Free University of Amsterdam with the aim of being well suited to use in education. VU-Dynamo allows the user to make mathematical models to study systems of various kinds. Pupils can work with pre existing models, modifying the equations, constants or the initial data, or can build models from scratch. The output of the program are graphs and tables describing the behaviour of the model.

3.3 Other supporting materials

Students use a handbook including various kinds of models, with proposals and suggestions for activities with each model

4. Overview and curricular context

This activity is part of a government project called PRINT. In this project several experiments are being conducted in order to evaluate the possibility of some modifications and the suitability of new themes in the curriculum. For mathematics there are two projects. One is statistics and the other is System Dynamics. This means a method of simulation for the study of the structure and behaviour of systems involving feedback loops. This is a new trend in the teaching of mathematics in secondary education that is now being studied in several countries.

This Dutch experiment — with the aim of verifying if the package VU-Dynamo is well suited to the teaching of that subject-matter in 11th grade classes — started in 1988/89 with two 11th grade classes. This year (1989/90) two new classes entered the experiment, with six teachers now involved in teaching System Dynamics using VU-Dynamo.

The experiment has been conducted in the stream A (technical) of the Dutch secondary school system; next year it will be expanded to the stream B (humanities).

5. Description of activity

5.1. Teaching and learning styles

a) The subject is taught for approximately 20 lessons. In the first 10 lessons, students work on the first models of the handbook, developing the activities proposed. These consist chiefly of changing the values of parameters and making other small modifications. Teachers follow the work, make new proposals, and conduct class discussions where appropriate. Pupils spent most of their time on the computer, working in groups of two, experimenting, giving and changing values and asking for graphs and tables.

Through the experiments with different models, students begin to understand that several models have the same structure.

to distinguish between stable and unstable equilibrium and to recognize the difference between linear and exponential growth or decay.

In the second series of 10 lessons a more complex model is built and studied and the teacher gives a more detailed explanation of how a complex kind of model is built. In these lessons students begin to be more confident and take more initiatives on the modification of the models included in the handbook.

b) The computer and VU-Dynamo software are used as tools, and the programming aspect is not stressed at all. The emphasis is put on the process of modelling, in its several aspects and phases.

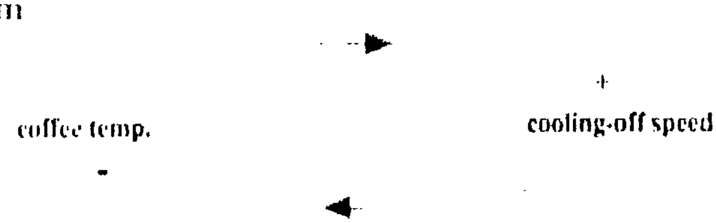
5.2. Record of activity

a) The cooling-off of a cup of coffee

This is one of the first models investigated by the students. The concrete situation to be studied is the cooling-off of a cup of coffee. It is natural to suppose that the decay of the temperature of the coffee depends on:

- the temperature of the coffee
- the temperature of the room
- the shape and other characteristics of the cup

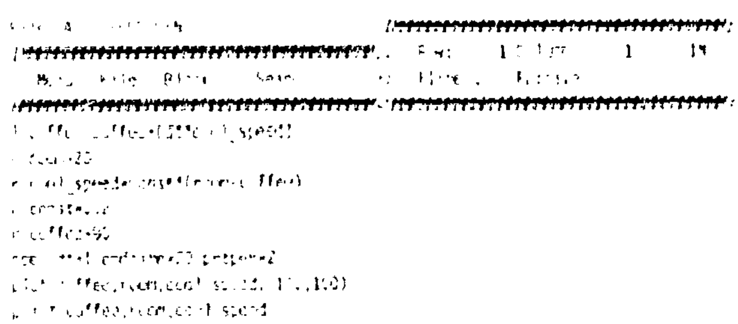
The model built in the VU-Dynamo program follows Newton's law, that says that the cooling-off speed is proportional to the difference between the coffee temperature and the room temperature. So we can see that the higher the temperature of the coffee the faster is the cooling-speed, but at the same time, the faster the cooling-speed the lower the temperature of the coffee. One or more of these feedback-loops are always present in the models studied in the handbook. This feedback-loop can be represented by the diagram



The equations describing the cooling-off process are

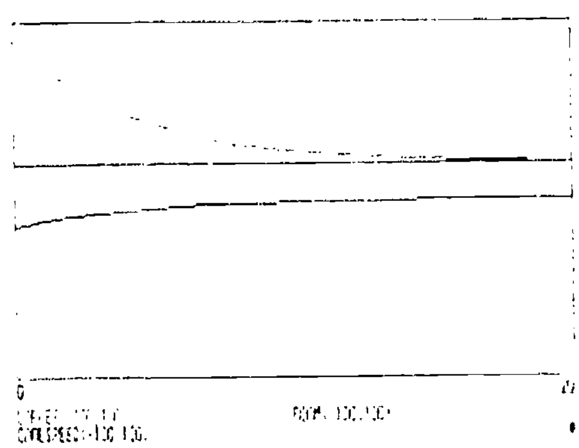
coffee = coffee - dt * cool_speed
cool_ speed = constant * (coffee - room)

In the screen, the program to simulate this process appears as follows:



Notes.

- We see that in this example the initial (line with letter **n**) coffee temperature is set at 90 degrees, the room temperature is fixed at 20 degrees and the constant of the cooling-off speed is fixed at 0.2 (lines beginning with letter **c**).
- The lines beginning by **l** are reserved for the expressions giving the level of one variable as a function of time. In this case, coffee temperature as a function of time.
- The line beginning by **r** denotes the expression defining the rate of change
- The lines beginning by the words SPEC, PRINT and PLOT mean the data the student is asking the program to compute and show. In this case, student defines the step dt as 1 second, and the upper limit of the time as 20 seconds. He is asking for the printing of the coffee temperature and of the cool_speed every two seconds (prtper=2) and also for a plot of the coffee and room temperature and of the cool-speed in the range (-100,100).



After a <return> by the user, the requested table is displayed:

TIME	COFFEE	ROOM	COOL_SPE
0.0000	90.000	20.000	-14.000
2.0000	64.600	20.000	-8.960
4.0000	49.672	20.000	-5.734
6.0000	38.3[illegible]	20.000	-3.670
8.0000	31.744	20.000	-2.349
10.0000	27.516	20.000	-1.503
12.0000	24.810	20.000	0.962
14.0000	23.0[illegible]	20.000	0.616
16.0000	21.9[illegible]	20.000	-0.394
18.0000	21.261	20.000	0.252
[illegible]	20.8[illegible]	20.000	-0.101

In this example, students can change the initial temperature of the coffee, the temperature of the room and also the value of the constant and investigate how they influence the process. Sure, they can also modify the equations, if they want to try a different model of the cooling-off process.

b) **An ecological model**

Towards the end of the twenty lessons, a more complex model is investigated by the students. In the Kaibab plateau, the development of a deer population, threatened by mountain lions, depends on the food available on the plateau. Students study how we can progress from a simple model to a very complex one with several feedback-loops, and learn the process of designing such models step by step. In this case, the steps are five:

1) Only the deer population is considered.

2) We add a constant number of lions and they live by hunting deer.
3) The number of mountain lions is variable — they are born and die — and the government creates a premium on killing lions.
4) We add the food to the model.
5) The amount of food is variable, as a result of extreme grazing. So the killing of the lions, increasing "too much" the deer population, resulted in extreme grazing, less food and a decrease of the number of deer in the Kaibab plateau.

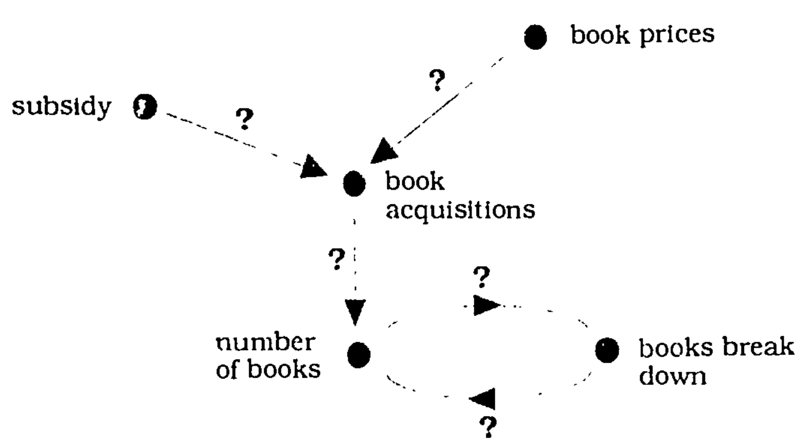
5.3 A written test

After 20 lessons, students did the following written test:

The library

The number of books owned by a library is not always the same. Each year there are books damaged or not returned. Other books are taken out because they are replaced by new ones in better conditions. The library receives also a subsidy for the acquisition of new books.

For this situation we can design the following flowchart:



As you see in the diagram, the number of books is a function of two variables, the book acquisition and books break down. The first one is a function of the subsidy and of the book prices. Question marks can represent a + or a -.

•**1** Make a copy of the flowchart, replacing the question marks by the correct signs.

One library has 5000 books at 1st January 1990. Experience has shown that each year, for several reasons, 20 % of the books are removed completely from the library. City Hall gives a subsidy of 15000 Dutch gulden each year. The average price of a book is 30 gulden.

•**2** Make a forecast of the evolution of the number of books . Justify your prediction.

•**3** Copy the following grid and fill the blank spaces

year	books	out	in
0	5000		
1			
2			
3			
4			

When we are dealing with a dynamic model, as in the case of this set of books, VU-Dynamo can be a good support.

•**4** Write a dynamo-model called BIEB1 to simulate the situation of this library. Make a forecast of the results with a graph and a table. Let the model compute for 20 years. Run the model and revise your answers to question 3. Save the model with the name BIEB1NN, where NN represent the two first letters of your last name.Print the progam in paper, write your name and give it to the teacher.

•**5** Make another choice for the initial number of books. for instance 10000 and 15000. Study the development of the books. Write your comments.

•**6** The number of books seems to stabilize around 2500, for any initial number of books. Why 2500?

•**7** With a new management, books taken out of the library decreases to 15%. What is the new level of stabilization? Explain your answer.

•**8** Use now the new model and revise your answer to question 7.

Every year the prices of the books raise 5%, and the subsidy increases 500 gulden.
To understand what are the consequences to the number of books we need to make some changes in the program

•**9** Change the initial model BIEB1 in order that it includes the book price rises and the growth of the subsidy of 500 gulden per year.

•**10** Find, for the modified model, the new results for the number of books of the library. Write your conclusions.
Save the model under the name of BIEB2NN. Print the model in paper, write your name and return it.

The City Hall decides to give at the 1st January 2000, an extra subsidy for the number of books in the library on 2010, January 1 to be the same as on 1990, January 1.

• **11** Use the model BIEB2 to find the value of the subsidy.

6. Conclusions

The students are motivated doing modeling with VU-Dynamo. They ask for more time to have the possibility of working more deeply with the program.

Pupils are eager to work with this new subject matter that is to some extent difficult. They feel this type of work as a challenge, and this is also true for the teachers, because for them this subject is also new.

The use of the computer with this program allows the study of concrete problems not analytically solvable, because the computer makes numerical approximations. Another advantage is that the results are shown not only in tables but also as graphs.

One interesting feature of this computer use is the changing of perspective, from the solving of equations — in this case differential equations — to the set up of the equations needed to investigate a given problem situation. This is the case when students build and modify models of the situation with VU-Dynamo.

Case Study 16

Linear and Exponential Growth with a Modeling Tool

Linear and Exponential Growth with a Modeling Tool

1. The School

The *Sancta Maria Instituut* is a comprehensive secondary school located in the Flemish part of Belgium, at Deuene in the suburbs cf Antwerp. It is an ancient school founded 160 years ago. Nowadays, the school has five hundred students, ranging from the 7th to the 12th grade, in classes of an average number of 20 pupils. The grade range corresponds to the Belgian secondary level and pupils are aged from 12 to 18 years old. The students' social background is mixed but mainly the parents are from the middle class and several work in commercial businesses. The pupils intend to follow their studies at the University.

Regarding the 80 teachers currently teaching in the school not all of them have full time occupation within this school and several teach part time in other schools as well.

The *Sancta Maria Instituut* is a free school, one of the 75% of the Flemish secondary schools which are not attached to the state or to local authorities. It is run by Catholic Education, a branch within the free schools. The school is for girls only; this is quite common within the Catholic Education schools.

The school management is free to plan the purchase of the equipment and to draw up policy regarding educational issues.

Five years ago, the first computers were used in school activities and presently there are three computer rooms equipped with either 10 or 16 computers, and rooms with one or two computers assigned to specific subjects.

The computers are currently used in informatics. mathematics and sciences classes. Three mathematics and three sciences teachers use computers within their lessons on a regular basis.

2. Aims

Particular and more general curricular goals were drawn by the teacher for these activities.

a) To foster the understanding of concepts related to functions such as domain, limit of a function, asymptotes and derivative.

b) To deepen pupils' understanding on monotony, namely linear and exponential growth.

c) To be aware that linear and exponential functions can model and approximate real world phenomena.

d) To enlarge the understanding of linear regression.

3. Material and Resources used

3.1. Hardware

During the performing of the activities 10 PC compatible computers with 640K RAM and mono-chromatic monitors were used.

3.2. Software

MathCad is a program that features several facilities. It can be easily used for modeling. It deals, for example, with variables, formulas, text and plots, showing some similarities to a spreadsheet.

To illustrate the MathCad facilities, an example is described. This example is the file that students worked with (*GROEI1*), when they first studied linear and exponential functions.

The following problem was written at the beginning of the file:

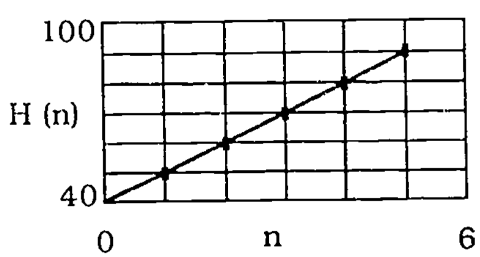
"Each of two brothers purchased a pony, on the same day. One month later they compared the weight of the two animals. Hans' pony increased 10 kg whereas Wim's increased 25%. Two months after the date of purchase they again compared the weights and they were equal. Han's pony was 10 kg more than in the previous month and Wim's pony showed an increase of 25 %.

Search for the weight two months after the ponies were bought."

n := 0 .. 5

H (n) := 40 + n.10

n = H (n) =



G := [ ]

< Fill in >

< Press F9 and check your result >

The variable *n* is defined in order to assume the integer values from 0 to 5. Below is a plot area where a line graph is drawn. *G* is defined as a matrix, to be completed by the pupils with the various weight values. The message on the lower corner on the right instructs Mathcad to perform the calculations, printing tables with the values of *n* and *H(n)*.

3.3. Other Supporting Materials

The students were given several worksheets reproducing the MathCad files screens, where the problems were written along with incomplete tables and graphs.

Measurement tapes were also used.

4. Overview and Curricular Context

Gerda Timmermans was the teacher who designed and carried out the activities, during the academic year of 1988/1989, with several 12th grade classes, usually with 20 students aged 17 years. Nowadays, she is working at the institution in charge of the introduction of informatics into schools, all levels but universitary, belonging to Catholic Education, the Pedagogical Steering Group for Informatics in Catholic Education.

The people working within this institution, responsible for grades 10 to 12, did a survey trying to identify the mathematical subjects and software more suitable for the integration of informatics in the curriculum of mathematics. The development of materials and the implementation of teacher training courses followed the survey. The activities described in this case study resulted from this work.

In Belgium the number of weekly mathematics lessons is variable, from two to eight, according to the university courses that pupils intend to follow after leaving secondary school. Some classes spent nine lessons of 50 minutes to perform the activities using this software, covering topics such as statistics, whereas others only took three lessons. This case study focuses on the first five lessons, corresponding to the aims stated in part **2**.

All the mathematical themes studied in these activities are related to linear and exponential functions which are curricular subjects within the Belgium curriculum.

Pupils were introduced to several problems, some of them based in real world situations, and each problem had a corresponding file, already set up by the teacher, which modeled the problem. The students keyed into the computers the suitable data for later calculation.

Pupils' prior experience in using computers included the introduction to the *PASCAL* programming language, during the informatics classes in the 11th grade. Besides this very few students had prior experience in using computers at the school or even at home.

5. Description of activity

5.1.Teaching and Learning Styles

All the activities were performed during regular lessons, in a computer room or in a normal classroom. When using the computers, the students were in groups of two and each group had one computer available. This was the most common environment. The need for discussing results, that the various groups obtained, led to whole class debates.

Although all the students were working within the same subject, the problems they were trying to solve could be different. Later, they exchanged the problems in order to work out solutions for every proposed problem.

The time spent by the students at the computers doing routine tasks was reduced to a minimum. They only had to key in numerical data, use the arrow keys to move the cursor and the F9 key to instruct the program to do the calculations. The remaining time was dedicated to reason about the problems, discussing several issues.

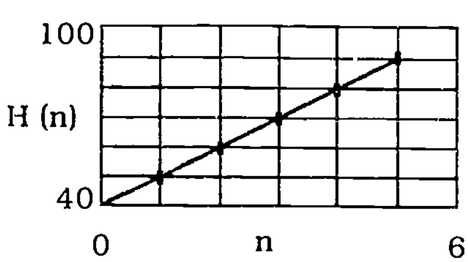
Helping the pupils whenever necessary and fostering students' discussions and mathematical communication were the main teacher's roles during the lessons.

5.2. Record of activity

In the first lesson, pupils were given the problem about the weight of the two brothers' ponies (see 3.2.). After filling matrix *G*, and pressing the function key F9, the screen displayed:

n := 0 .. 5

H (n) := 40 + n.10



$$G := \begin{bmatrix} 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \end{bmatrix}$$

< Fill in >

n =	H (n) =
0	40
1	50
2	60
3	70
4	80
5	90

< Press F9 and check your result >

After some comments on the linear increase, the second part of the problem was approached in a similar way. Below the statement of the conditions, students were asked to calculate the values for monthly weight, to key them into the computer and verify their assumptions. The concept of rate of change was underlined and the students were asked to verify their values for every month. This kind of approach provided the opportunity to discuss the similarities and differences between linear and exponential growth, as well as the analogies with arithmetic and geometric progressions.

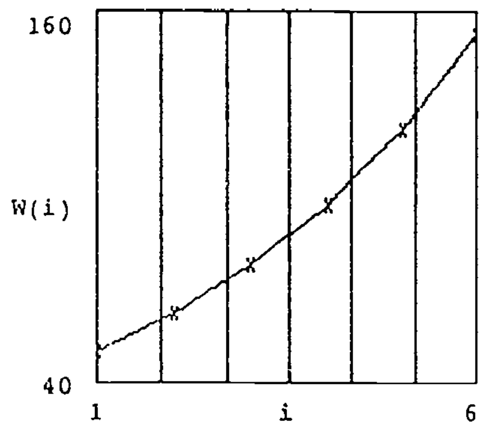
> Wim's pony has a different kind of increase. Lets study it.
> The initial weight was 40 kg. A month later was   kg.
> If we divide the weight at the end of the first month by the initial value, we obtain the RATE OF CHANGE corresponding to the first month.
>
> $a := 40$ $a =$
> $b := a + a \cdot 0.25$ $b =$
> $c := b + b \cdot 0.25$ $M := \begin{bmatrix} \\ \\ \\ \end{bmatrix}$ $c =$
> $d := c + c \cdot 0.25$ $d =$
> $e := d + d \cdot 0.25$ $e =$
> $f := e + e \cdot 0.25$ $f =$
>
> < Fill in >   < press F9 >
>
> The rate of change corresponding to the second month is:
>
> $F2 = \frac{c}{b}$ $F2 =$
>
> ...

The mathematical model was then defined as a discrete function with a given domain.

g := 1.25

i := 1 ..6

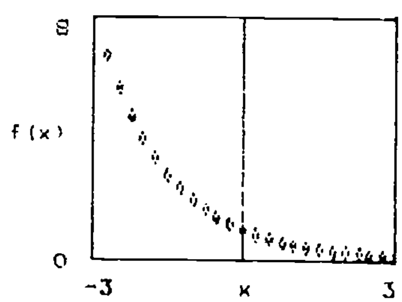
W (i) := a · gi

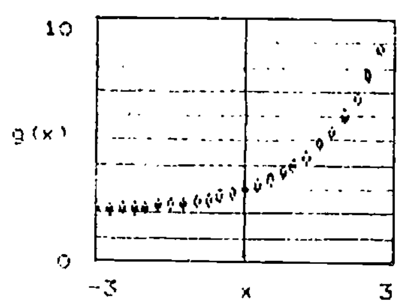


Later, students gave several values to the variables involved - initial value, time step, weight increase (linear) and rate of change (exponential) - and plotted the graphs. They were asked to change the context to population growth.

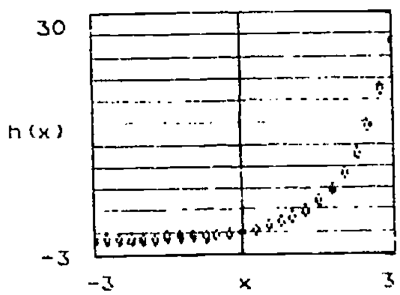
A more formal approach to exponential functions followed focusing on asymptotic behaviour. Regarding the following graphs, pupils were asked to write down the equation for each asymptote.

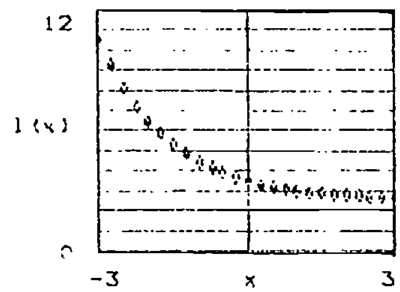
x := -3, -2.75, ..3 $\qquad f(x) := \left[\frac{1}{2}\right]^x \qquad g(x) := 2^x + 2$





$$h(x) := 3^{x} - 1 \qquad l(x) := \left[\frac{1}{2}\right]^{x} + \left[\frac{5}{2}\right]$$





In the next lessons, students were given several problems based in real world situations which can be modeled by exponential functions. They spent two lessons solving these problems and a third lesson to discuss the issues which arose in the previous lessons. Each problem had a file for students use, while solving them. Pupils choose the order for solving the problems, thus not all solved the same problem at the same time and sometimes groups helped each other in order to clarify colleagues' views.

Here is a list of the proposed problems along with specific questions and comments on possible further developments:

1 - Cellular Division

Every cell in our body , and there are millions, has its origin in a single cell. This increase is possible through cellular division. One cell generates two cells with identical DNA structure. Find out the function that models this phenomenon, how it increases and how many cells are there after ten divisions.

Instead of having one cell at the beginning, assume they are

32 and compare the two situations.

This problem caused some discussion, especially in the last part. At first, the students were surprised about the big difference in the number of generated cells between the two settings, for equal time.

2 - Population Increase

By the year 1650, the world population was approximately 500 millions and was increasing 0.3% each year.

Is the growth exponential? What is the value of the rate of change?

In 1970 , the population of the world was 3,600 millions. what was the rate of change then?

Back to 1650, how many years did the population take to double?

When did the population reach 3,000 million? Since this year, how many years did the population take to double?

The Earth has, approximately, 4 thousand million ha of cultivable land. According to experts' opinions, each human being needs 0.25 ha to provide the necessary food. In 1965 there were 3 thousand million people and the yearly increase was 1.8%. When do we reach the maximum population?

Pupils were given graphs with the Carr-Saunders/Wilcox and the United Nations estimations, since 1650, as well as the U N previews, till the year 2,000, and the real population increasing, from 1800 to 1970. They were able to relate theoretical approaches with the reality.

Half-Life of a radioactive substance

The archeologists can tell the age of a fossil skeleton from the number of radioactive atomic nuclei.

$h(i) := N_0 \cdot 2^{-i}$ $\quad\quad$ $i := t_i / p$

N_0 — *number of radioactive nuclei at the initial instant.*

t_i — *amount of time since the initial instant.*

$h(i)$ — *number of radioactive nuclei at time t.*

p — *half-life of the radioactive substance.*

1. For a Radium isotope p = 22 years.
If there are now 1,000 radioactive nuclei, how many would it be 2, 44 and 150 years later?

2. Within a substance, there is a radioactive element with p = 1590 years. From the analisys of the substance, we know that N_0 = 118. How old is the substance if N (i) is 59? And if N (i) is 113?

3. Plutonium-239 has a half-life of 24,400 years. Plutonium is the most important substance for nuclear power stations and atomic bombs. How much time does it take for 1 tonne of Pu-238 to be reduced to 62.5 kg?

The next example was about electrical condensers and the discharge time. As with the other problems there is also an exponential law, in this case for the variation of intensity along the time. Within the files, pupils were provided with the formula expressing the law and the meaning of each variable:

$$I(t) := I_0 \cdot 2^{\frac{-t}{T}}$$

I_0 — *initial intensity.*

T — *time required for intensity decreasing to half.*

t — *time (independent variable)*

As the electrical charge, Q, is $I \times t$, it was possible to approach the problem from this view. Pupils were asked to calculate the total condenser charge by computing the areas of rectangles defined in the function graph. To implement this approach, students were given another file. A new function, I_i, was defined in order to compute approximate values for the electrical charge. This function was defined from a partition of the interval [0, 70]:

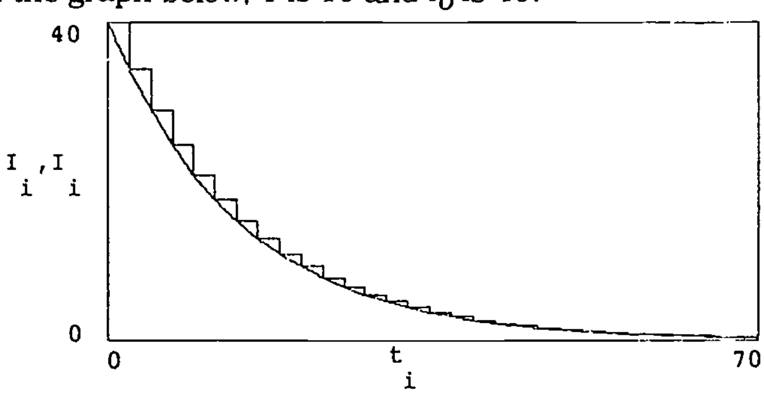
$i := 0 .. n$ (number of intervals) $n \equiv 30$

$dt := \frac{70}{n}$ (time step)

$t_i := i \cdot \frac{70}{n}$ (lower end point of each interval)

$I_i := I_0 \cdot 2^{\frac{-t_i}{T}}$ (intensity at each lower end point)

In the graph below, T is 10 and I_0 is 40:

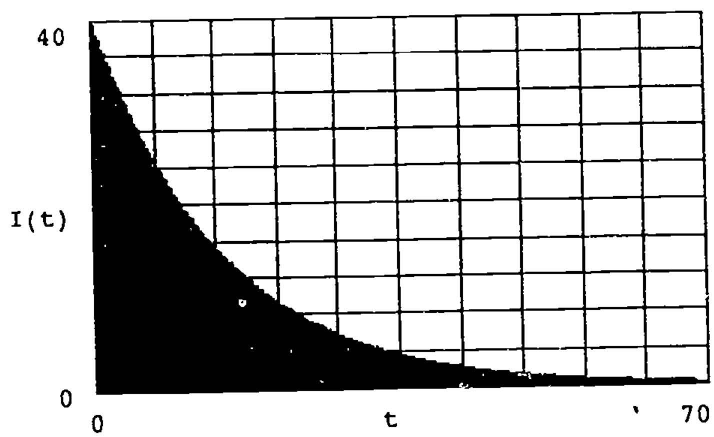


As the software includes facilities such as the defining of series and integrals, it was possible to calculate Q as a series:

$Q := \sum_i I_i \cdot dt$ $Q = 620.848$

The value of Q decreases as n tends to infinity. This fact was observed and discussed by the pupils. They agreed that the limit was a definite integral. In order to validate the assumption, Q was defined as an integral and $I(t)$ plotted.

$t := 0, 0.2 .. 70 \qquad Q := \int_0^{70} I(t)\,dt \qquad Q = 572.57$



The last problem was an opportunity for students to deepen concepts related to linear regression while they were trying to find a mathematical model for another real world example.

Besides a MathCad file (*GROEI8*), students were given, within a worksheet, print outs of the file and a copy of twelve photographs of a starfish. These photos were taken at twelve different moments of its development. In order to fill in both in the file and on the worksheet, students measured, in each photo, the length of an arm from the centre. The length, expressed in milimetres, was multiplied by two because the first values are very small.

There follows, on the next page, the first part of the file, followed by a student graph and the computer graph.

starfish	date	length
1	25 Jun	10
2	28 Jun	18
3	2 Jul	36
4	10 Jul	60
5	13 Jul	72
6	17 Jul	96
7	22 Jul	180
8	28 Jul	240
9	1 Aug	320
10	17 Aug	520
11	10 Sept	660
12	28 Sept	1080

$$N := \begin{bmatrix} 10 \\ 18 \\ 36 \\ 60 \\ 72 \\ 96 \\ 180 \\ 240 \\ 320 \\ 520 \\ 660 \\ 1080 \end{bmatrix}$$

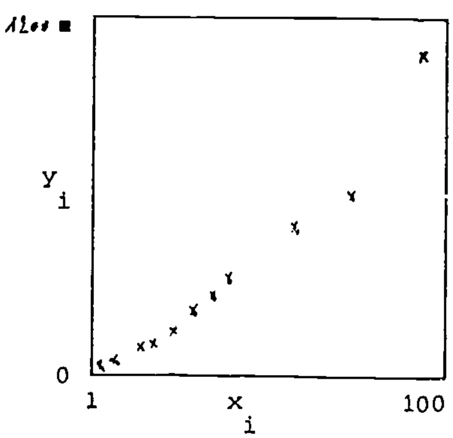
$i := 1 ..12 \quad j := 1 ..2$

$$M := \begin{bmatrix} 10 & 0 \\ 18 & 3 \\ 36 & 7 \\ 60 & 15 \\ 72 & 18 \\ 96 & 22 \\ 180 & 27 \\ 240 & 33 \\ 320 & 37 \\ 520 & 54 \\ 660 & 78 \\ 1080 & 96 \end{bmatrix}$$

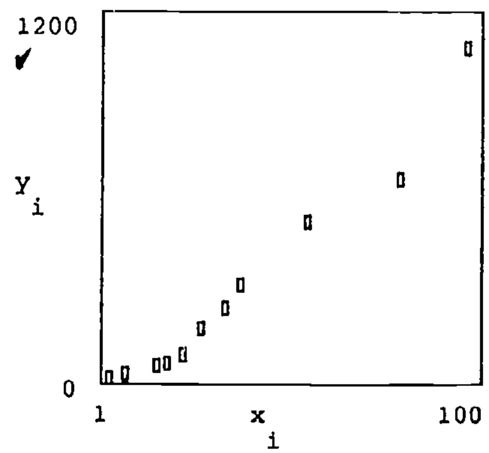
origin ≡ 1

$x := M^{<2>}$ Number of days after the first measurement

$y := M^{<1>}$ Length



student' s graph



computer graph

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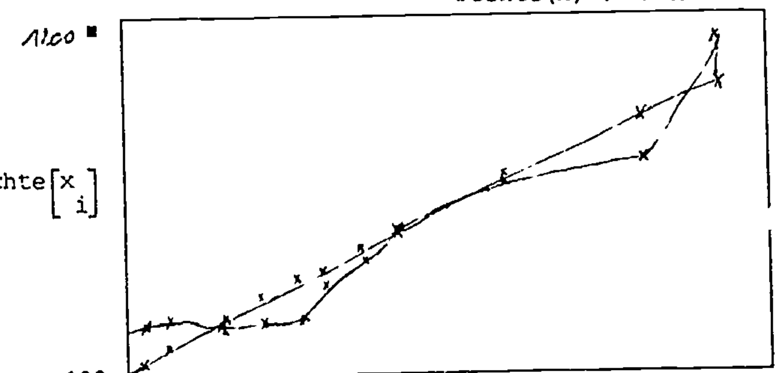
The next step was to find out the line that best fits the distribution. Using the built-in facilities for linear regression, *slope* (x_i, y_i) - slope of regression line for data vectors x_i and y_i - and *intercept* (x_i, y_i), it was easy to find the equation of the line. To illustrate this activity, a copy of the worksheet follows.

LINEAIRE REGRESSIE

m := slope(x,y)

b := intercept(x,y)

rechte(x) := m·x + b



verschil(x,y) := (rechte(x) - y)

x_i =	y_i =	rechte[x_i] =	verschil[x_i, y_i] =
0	10	-74,809	-84.809
10	~~11,569~~	-41,581	-60.581
3	18	0,346	-35,609
7	36	86,334	26.334
15	60	118,562	46,562
18	72	161.533	65,533
	96	215,248	35,248
22	180	279,705	39,705
27	240	328,676	8,676
33	320	505,305	-14,695
37	520	763,133	103,133
54	660	956,504	-123,496
78	1080		
96			

m = 10,743 b = -74,809

De regressielijn heeft als vergelijking f(x)= 10,743 x - 74,809

Concerning linear regression, one last question was proposed:

Within a set of ten students, two written tests in mathematics and physics gave the following results:

mathematics	*7*	*5*	*6*	*3*	*8*	*10*	*4*	*6*	*9*	*4*
physics	*7*	*4*	*7*	*5*	*6*	*9*	*5*	*5*	*8*	*4*

Write down the equation of the regression line.

What is the most likely result to be obtained in physics by a pupil who scored a 6 in mathematics?

Again, this activity was performed with little or no difficulty as it is similar to the previous task.

Students carried out these activities over three lessons. As stated before, not all of them were performing the same task. A fifth lesson was devoted to summarize the results. It was an opportunity to discuss and clarify the several issues concerning the problems. Every pupil was free to communicate her views to the whole class, results achieved and doubts in order to have some feedback from the colleagues and teacher.

6. Conclusions

Students had had some unsatisfactory work with computers in the previous year. Consequently, when computers were suggested as a tool for learning mathematical concepts, a lack of enthusiasm was noticed by the teacher. As the activities went on, the pupils' first opinion towards computers began to change. They were pleasantly surprised and although pupils felt that the methodology was not easier, they found it appealing and worthwhile. The whole atmosphere during the lessons was pervaded by students' interplay and active work. In the last lesson, the discussion focused on many mathematical themes as well as their connections with real world phenomena. Students were able to discuss these issues in a deeper way than usual, maybe because the computer was used as a tool, allowing the students to devote more time to do other activities rather than the monotonous computing exercises.

This is confirmed by Gerda Timmermans. After the first lessons, pupils' speed of progression throughout the activities increased along with pupils' facility in handling the means to solve the problems. She also thinks that pupils had a better understanding about the concept' related to exponential functions, resulting from the methodology used during these lessons. Overall she thinks that the time spent doing the files had a very rewarding outcome.

Besides Gerda Timmermans, no teacher tried these activities but several showed interest in using a similar kind of approach and are currently working towards this goal.

Annexes

1. Names and adresses of the teachers involved

Case study 1
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1500 LISBOA
Portugal

Case study 2
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Case study 3
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1600 Lisboa
Portugal

Case study 4
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Middlesex HA2 9AG
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Case study 5
Manuel Saraiva
Universidade da Beira Interior
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6200 COVILHÃ
Portugal

Case studies 6 and 8
Adrienne Ashworth
Hodgson High School
Poulton-le-Fylde
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United Kingdom

Case study 7
Leonor Cunha Leal
ESE de Setúbal
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Case study 9
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University of Technology
Faculty of Mathematics
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2628 BL Delft
The Netherlands

Case studies 10 and 15

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Geerdinkhof 561
1 [illegible]03 RK Amsterdam
The Netherlands

Case study 11

Susana Carreira
Departamento de Educação
Faculdade de Ciências da Universidade de lisboa
Av. 24 de Julho, 134, 4º
1300 LISBOA
Portugal

Case study 12

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United Kingdom

Case study 13

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Fachbereich Mathematik/Informatik
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Genova
16132 Genova
Italia

Case study 16
Gerda Timmermans
NVKSO
Guimardstraat 1
1040 Brussels
Belgium

2. List of software publishers

Case study 1

Sinclair Logo and LogoWriter are registered marks of Logo Computer Systems Inc.

Case study 2

INTERDIM is published by:
IREM de Besançon
Faculté des Sciences et des Techniques
Route de la Gay, La Bouloie
25030 BESANÇON CEDEX
France

Case study 3

Logo Writer is a registered mark of Logo Computer Systems Inc.

Case study 4

Published by Cambridge Micro Software
Netherall Software, Maths Topics 1, BBC Version

Case study 5

Logo.Geometria is published by:
Gabinete de Estudos e Planeamento
(a/c Dra. Maria de Lurdes Fragateiro)
Av. Miguel Bombarda, 20, 4°
1093 LISBOA CODEX
Portugal

Case studies 6 and 8

The Grass database and the Grasshopper spreadsheet (machine versions BBC, Master and RMNimbus) are available from:
Newman College, Birmingham, United Kingdom

Case study 7

Trinca-Espinhas is a public domain software available from:
Pólo do Projecto Minerva do DEFCUL
Av. 24 de Julho, 134 , 4º
1300 Lisboa
Portugal

Case study 9

Program designed and published by:
H. J. Smid
University of Technology
Faculty of Mathematics
Julianalaan 132
2628 BL Delft
The Netherlands

Case studies 10 and 15

Graphic Calculus (Englisn, Spanish, French and Dutch versions) and VU-Dynamo (English, german and Dutch versions) are published by Walters-Noordhoff, Groningen and available from:
VU Soft
Geerdinkhof 561
1103 RK Amsterdam
The Netherlands

Case study 11

Supercalc IV was developed by the Micro products division of Computer Associates International and published by:

Computer Associates
2195 Fortune Drive
San José CA 95131
USA

Case study 12

A professional development pack will be published during 1991:

Ruthven, Kenneth (editor), Graphic calculators in Advanced Mathematics, National Council for Educational Technology, Coventry, United Kingdom

Case study 13

Registermachine is available from:

E. Cohors-Fresenborg
Universität Osnabrück
Fachbereich Mathematik/Informatik
45 Osnabrück, Postfach 4469
Germany

Case study14

Lotus 1 2 3 is a registered mark of Lotus Development Corporation.

Turbo Pascal is a trademark of:

Borland International Inc.
4585 Scotts Valley Drive
Scotts Valley, CA 95066
USA

Case study 16

Math Cad is a registered mark from:

MathSoft Inc.

1, Kendall Square

Cambridge, MA 02139

USA

Using Computers in Mathematics Education

A collection of case studies

Computers have been regarded for the last ten years as one of the most promising factors that can influence education, and many projects both within the European Community and elsewhere have been carried out to study, in one form or another, their educational potential and implications. Quite sophisticated investigations have been reported in mathematics education research journals and congresses. Many proposals have been put forward in teachers' periodicals and meetings. But little is known about what is really happening in the field. What is going on in the classrooms? Are there many teachers actually using computers? What are they doing?

This book is intended as a collection of significant and diverse classroom experiences concerning the use of computers in mathematics education, at middle and secondary school levels.