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ABSTRACT

This curriculum guide, developed by an adult basic education program for disadvantaged women, provides methods for teaching mathematical concepts and operations including whole numbers, fractions, decimals, and percents. The curriculum illustrates methods and activities that bridge the gap between numerals (the symbols of mathematical relationships) and the reality upon which they are based, by providing concrete examples using manipulative materials through which the learner experiences the concepts of mathematics first hand. The methods outlined in this curriculum also address the issue of mathematics anxiety by offering learners immediate success in the subject. The 22 activities in the guide include procedural instructions and follow-up activities; 10 references are listed. (KC)

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MATH WITHOUT FEAR

A CONCRETE APPROACH TO MATHEMATICS

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Finally, I acknowledge with gratitude the many students at the Women's Program who participated in this project. We learned together in creating and refining the instructional techniques and materials described in this manual. Their enthusiasm for learning and their endless patience were and are an inspiration to me.

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INTRODUCTION

Math Without Fear: A Concrete Approach to Mathematics was developed by the Lutheran Settlement House Women's Program to create a unique and exciting curriculum guide on methods of teaching mathematical concepts and operations including whole numbers, fractions, decimals, and percents. This curriculum illustrates methods and activites which bridge the gap between numerals, which are merely symbols of mathematical relationships, and the reality upon which they are based, by providing concrete examples using manipulative materials through which the learner experiences the concepts of mathematics first hand. As learners meet success in understanding the concepts of mathematics, they are able to build on their concrete experience and transfer their knowledge and self-confidence to the learning of mathematical symbols and problem solving.

Traditional Methods vs. Experiental Learning

Traditionally, math is taught as a series of formulas to be memorized with little understanding. This method leads to the often heard student statement: "I got the right answer, but I sure don't understand how I got it." For example, the traditional method of teaching addition of fractions is through the memorization of rules for finding the lowest common denominator and then practicing the rules over and over in a rote fashion. Using this method often gives learners little real understanding.

Using the teaching techniques described in this manual, learners learn about fractions experientially before being presented with any rules, terms, or problems. For example, learners are given an empty frame and asked to fill the frame with pieces which represent halves, quarters and thirds of the frame. By placing 2 halves, 4 fourths, and 3 thirds in the same space, learners discover for themselves the real mathematical relationships behind the numeric symbols. They understand that 1/4 + 1/4 = 2/4 because they see and touch the evidence of that relationship. Once learners understand the concept of fractions, or the concept behind any mathematical operation, it is much easier and more enjoyable to learn computations.

Another problem experienced with traditional methods is that the teaching and learning of mathematics are answer-driven. The goal is finding the correct answer rather than exploring and understanding the process by which the answer is derived. Often, this approach results in teachers and learners becoming disconnected from the idea that numerals are symbols for mathematical relationships, just as words are symbols for communicating meaning. For example, the sentence "Joan jumped over the house." follows all the rules of grammar. It has a noun subject, an action verb, and a properly constructed prepositional phrase. The symbols (or words) are used correctly, but the sentence is absurd. However, for someone who has no concept of the relationships between the words "house", "jump", "over" and their respective meanings (reality) the sentence would seem correct. Similarly, mathematics is often taught without an understanding of the relationship between numeric symbols and the reality they represent. For example, learners are often asked to memorize rules for borrowing and carrying when learning addition and subtraction. Often learners have little understanding of place value. They just memorize the rule and apply it again and again. The method often leads to difficulties with actual problems, where it will be necessary to decide what operation will "work" to solve a particular word problem.

In this curriculum, learners are given concrete activities which help them explore and understand place value. One activity includes bundling and unbundling sets of toothpicks which represent different place values. Not only do learners more readily understand place value after this exercise, they find it fun and are motivated to learn more.

Another problem with traditional methods of teaching math is that those methods rely heavily on memorization. Learners are asked to memorize a formula without understanding the concept behind the formula. They may remember the formula for a short time, but, without any real comprehension, they soon forget what they have memorized. For example, learners might succeed in memorizing the times tables, but forget them when faced with additional memorizing, such as memorizing the formulas for operations with decimals or the formula for solving percent problems.

This problem has serious consequences for GED testers because the math test combines all types of mathematical computations. If learners do not comprehend the concepts, they will be much less likely to pass the test, both because their memories will be "overloaded" and because they will be less able to apply the concepts to word problems. The methods described in this manual rely on comprehension of concepts rather than on memorization and therefore enable more learners to pass the GED math test more readily.

A final and overwhelming problem with traditional methods of teaching mathematics is that those methods have already failed for many learners. It is the contention of the Women's Program that _if_ traditional methods worked, we would not teach a multitude of learners each year who believe themselves incapable of performing mathematical computations. By creating and utilizing the innovative methods described in this manual, many learners who have been unsuccessful in learning through traditional methods will experience success.

Reducing Math Anxiety

A problem often associated with mathematics instruction is the anxiety associated with the subject. Handler (90) has defined math anxiety as "induced by fear of failing when attempting to learn or to demonstrate one's learning of mathematics." Math anxiety is usually addressed through affective measures which include helping learners feel more confortable and secure in the learning situation, while the teacher continues to teach in traditional ways which may not help the student succeed.

The methods outlined in this curriculum address the issue of math anxiety by offering learners immediate success in the subject. Combined with methodology which builds on concrete experience, anxiety of failure is reduced. This approach is particularly valuable with harder-to-reach learners -- those who are afraid to try because traditional methods have failed. The questionnaire which follows (see page v) can be used to assess learners' attitudes towards mathematics at any point in mathematics instruction and/or to assess the success of the methods presented in this manual in lowering anxiety.

A final note on math anxiety: some teachers in literacy programs experience math anxiety themselves due to their own limited study of and/or experience with good teaching methodology in mathematics. These teachers should allow themselves the time to practice and enjoy the exercises included in this manual as they prepare to teach their learners. In fact, teachers should not attempt to teach from these lessons unless they have worked with the manipulatives themselves.

Critical Thinking Skills

Countless recent studies are pointing to the increasing need for critical thinking and problem solving skills for success in learning and in the workplace. The methods and activities in this manual will actively engage learners in the learning process and help them to develop their thinking and problem solving skills rather than just their memorization skills.

Further, this manual helps to solve the problem which arises when learners have simply been given information without questioning it or looking at how it relates to what they already know. "Typically, they find that if any part of a procedure is forgotten, or if a new situation has somewhat different features than their 'learned' model, success is unlikely -- further confirming their personal sense of inadequacy." (Handler, 90). When learners learn concepts experientially, they are more likely to recognize relationships and transfer their knowledge to new situations.

Building Self-Esteem

Many, many learners have achieved competence in math at the Women's Program after a lifetime of apparently failing to learn it. Many leave eager to study "higher" math at Community College. Learners who have avoided numbers for 20 years request extra math time when given a choice of subject. Most are amazed at and pleased with themselves and their accomplishments. In becoming competent in a subject they formerly believed they could not learn, they re-evaluated other self-limiting beliefs and came to expect success in other areas which had formerly appeared "too hard".

GUIDE TO USING THIS MANUAL

The activities in this manual are designed to help learners discover mathematical principles for themselves. The activities are concrete in nature; that is, learners will be solving problems with solid objects so that they can see and feel the way numbers "work". Follow-up activities, listed after each section, help learners move from this concrete level of mathematics to abstract, symbolic, paper-and-pencil problems. Many of the activities lend themselves to small group interaction.

Try it Yourself First!

Before you present an activity to a group of learners, collect the materials and try it for yourself. Try to forget that you already know how to solve a problem; try to demonstrate the answer with the objects. In other words, try to discover how the concepts "look" for yourself; this will make it fairly easy to help learners understand them.

Sequence of Instruction

Regardless of a learner's previous math accomplishment, it is important to start at the beginning. Adult learners often arrive with invisible gaps in their understanding of math. For example, a learner may be able to calculate long division without having fully grasped the concept of place value. This person's "gap" will not be apparent until s/he attempts to learn the idea of decimals, at which point the problem will be severe and not always clear to learner or teacher. It generally proves to be easier and more efficient to begin at the beginning than it is to backtrack.

Learners who have mastered elementary mathematics will, of course, move more quickly through the initial activities. Each activity contains suggestions for follow-up (practice) activities. The amount and nature of the follow-up can and should be adjusted based on the previous mathematical knowledge and learning speed of the particular learner(s).

While some concepts are based on previous ones, and therefore can only be taught after those fundamental concepts are mastered, the individual teacher has considerable flexibility in the sequencing of math instruction. Some rather unusual ideas are suggested in this manual. For instance, it is suggested that algebra be introduced early--immediately after whole number addition and subtraction. Doing so reduces learners' fears that algebra will be impossibly difficult and also relates that "higher math" to familiar, already learned concepts.

Numbers are Symbols

Numbers, like letters, are symbols. They represent some reality. For instance, the numeral "2" represents the reality of two objects. There are many symbols which will represent this reality: II, 4/2, two, and so forth. In our number system, each numeral symbolizes two things. The value of the numeral (2, 3, 4, etc.) tells us how many units are represented. At the same time, the placement of the numeral tells us the value of the unit (ones, tens, hundreds, etc.) For example, in the numeral "32", the numeral "3" describes three of a unit, in this case, tens, while the "2" describes two of a different unit, ones.

Our number system is based on the rule of tens. In fact, our number system is called the base-ten, or decimal, system. Each column has a value ten times greater than the column to its immediate right and therefore ten times smaller than the column to its immediate left. Numbers are infinite: the columns go on forever, in both directions. The rule tells us that any one column can hold no more than nine of anything. For a detailed description, see "Using Manipulatives To Teach Place Value". While the idea of an infinitely large number system is somewhat daunting, fortunately, every column works the same way as each of the others. If a learner can perform an operation (borrowing, for example) in the "ones" column, that learner should be able to do it in the "millions" column. This idea will become more clear as you do the first several activities.

Solving math problems is a series of questions and answers. Most problems can be solved in more than one way. For instance, the multiplication problem 3 times 8 can be seen as 3 sets of 8 ones (24 ones). It can also be seen as 8 + 8 + 8, or 24. Neither way is "better" than the other. One of the clearest examples of calculation as a series of questions and answers is the solving of long division problems (see "Division of Whole Numbers", page 23).

When one looks at fractions, one begins to look at numbers in relation to other numbers. A fraction is a statement about the relationship of the numerator (top number) to the denominator (bottom number). It expresses the ratio of those two numbers. It also expresses a division problem, and the answer to that problem. For example, 1/2 simultanecusly means 1 divided by 2 and provides the answer to that problem (1 divided by 2 = 1/2).

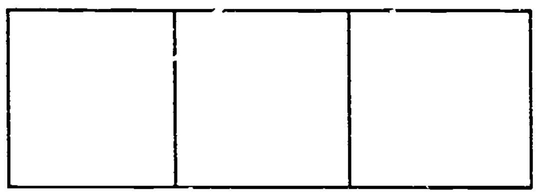
Many learners find considerable relief in being able to manipulate numbers. Many have felt as though the numbers have been manipulating them for years. With the first taste of success, many learners are able to let go of their fear that math will be too hard for them and give themselves eagerly to the effort of problem solving. Hopefully you, too, will enjoy this journey of discovery!

PLACE VALUE

Directions

For this activity, you will need the following:

Place value "mats": plain pieces of 8 1/2 by 11 inch paper, with at least three columns ruled onto it, one for each learner:



About 1000 toothpicks
About 50 rubber bands

Begin by explaining the **difference between symbols and reality**. One example to use is the symbol of the wedding band, which stands for a marriage but is not, in itself, the marriage. Another example to use is a picture of a house, which is a symbol of a house, but is not actually a house (you can't walk in the door of the picture).

The next step is to explain that **numbers are also symbols of** reality. Write the symbol "2" for the learners and explain that "this is not two". Pick up two objects and explain that "this is two", and that "2" represents the reality of the two objects. You might then show the learners that there are other symbols for the same idea (II, two, 4/2), but the reality of the number of objects does not change.

Tell the learners that our number system is based on the **rule of tens**, which means that no more than nine things can go in any one column. This is the only rule they need to remember. Assure them that this will become more clear as the activity goes on. Assure the learners that, although the activity may seem simple and childish at first, it will help greatly with higher mathematics and that it really is going to "go somewhere".

Give each learner a mat. Place a pile of toothpicks and a pile of rubber bands where each learner can reach them. Draw a mat for yourself on the board or on a piece of paper. Ask the learners to pick up one toothpick and place it in the far right column. Make a mark on your mat to represent a toothpick.

		1

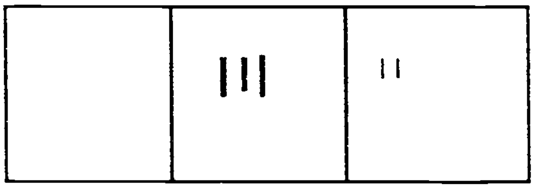
Ask the learners to pick up another toothpick and place it in the same column. Repeat the request a third time and then a fourth, each time making a mark on your mat. Ask whether anybody knows what this is called. If nobody offers "counting" prompt by saying, out loud, "five" as you place the fifth toothpick and "six" with the sixth.

Once counting has been identified, continue adding toothpicks one at a time until you reach ten. See whether the learners recognize that there are too many toothpicks in the column. If not, stop and prompt by asking whether any rules have been broken. Invite the learners to imagine how to resolve the problem. Let the learners try out various suggestions; if after a few minutes nobody has figured it out, show them how: gather ten toothpicks together, fasten them with a rubber band, and place the new unit of "ten" into the middle column. Now, instead of ten "ones", or ten single toothpicks, the learners have one bundle of ten toothpicks, or one "ten".

	1	0

On your mat, indicate the symbols for the number "10": a "1" in the tens column and a "0" in the ones column. Point out to learners that a "0" is the way an empty column is shown.

The next step is simply to get a number of toothpicks onto the mat(s). Ask the learners to continue, slowly, to do the same thing; that is, lay toothpicks into the far right column, one at a time. When the column is full, fasten ten of the toothpicks with a rubber band and put the "ten" into the middle column. Stop the learners periodically and ask "how many toothpicks are on your mat?" Then, write the answer on your mat using numerals. For instance, if the learner's mat has 32 toothpicks on it:



Your mat would say:

	3	2

Ask the learners to make the connection between the 32 toothpicks and the numeral 32, drawn in columns. Ask, "Can you see the number 32 here on my mat?"

When the learners have ten, or possibly eleven, tens in the middle column, see if they notice. If not, offer the prompt about the rule of tens. Then, see whether the learner can figure out that the solution is the same: the ten "tens" can all be connected with a rubber band and moved into the far left column. The learners will probably recognize, when asked, that there are a hundred toothpicks in this big bundle, and that they have now created yet another new unit, a bundle of one hundred, or one "hundred".

Have the learner add a few more toothpicks, so that you can demonstrate on your mat a three digit number. For example, 123:

I	II	III

1	2	3

Some learners can recognize, after this activity, what the next column after "hundreds" would be: ten bundles of 100, or 1000, in an enormous big bundle. Some may also recognize that the column before (or, to the right of) "ones" would have to be one tenth of one toothpick, or tenths.

This is the end of the basic place value activity. Leave the toothpicks bundled as they are. For the next activities, it will be helpful to have some hundreds and tens already prepared.

Most adult learners will be able to proceed directly from this point to the next activity, "Addition of Whole Numbers" (page 8). For beginning learners, the concept of place value can be reinforced by the follow-up activities listed below.

Follow-Up Activities

1. Have the learners clear off the mats and then put on the mat particular numbers of toothpicks ("Put 73 toothpicks on the mat. Now, put 106 toothpicks on the mat").

2. Other concrete activities, listed in Number Sense, Teacher's Resource Guide, pages 17-20 (Contemporary, 1990), may be useful for learners who are finding the concept particularly difficult.

3. Written place value exercises can be found in most beginning math books, including Number Sense Whole Number Addition & Subtraction,(Contemporary, 1990), Number Power 1 (Contemporary, 1988), and Mathematics Skills (Scott, Foresman, 1989).

ADDITION OF WHOLE NUMBERS

Directions

This activity uses toothpicks, rubber bands, and place value mats (see **Place Value**, page 4). Begin by having learners "count" toothpicks, (that is, lay them onto their mats, one toothpick at a time), for about five minutes. The point of this is simply to get various different numbers of toothpicks onto the mats. Alternatively, you could direct learners to place on their mats particular numbers of toothpicks. The addition activity will be most effective if a majority of learners have more than five single toothpicks in the ones column.

Direct small groups (two or three learners) to combine all the toothpicks on each of their mats onto one mat. Explain that the act of combining is the operation called addition. After this is done, check the mats and remind learners if necessary about the rule of tens. Most groups will have needed to make an additional bundle of ten when they combined the various amounts of ones. Ask the group whether anyone knows what this extra bundling is called (Answer: carrying). Demonstrate the symbols for carrying on the board:

$\begin{array}{r} 36 \\ +\underline{47} \end{array}$	Point out to learners that this problem will give them thirteen ones, so ten will have to be regrouped as a bundle of ten.
$\begin{array}{r} 1 \\ 36 \\ +\underline{47} \\ 3 \end{array}$	The new ten is then moved into the tens column. Note that we aren't carrying a "one", but "ten". This will leave three ones in the ones column. The new ten will have to be included when the tens are added, of course.

Reassure learners that it is a good idea to write the notation for carrying. Some may have been instructed not to do so and instead to remember whether or not they have carried. This has no mathematical value and allows students to become unnecessarily confused.

Next, have all the small groups combine their mats onto one mat. The point is to end up with at least 100 toothpicks. This can also be accomplished by directing learners to add some number of toothpicks that will add up to more than 100. See whether learners recognize that carrying from the tens column is handled in the same way as carrying from the ones column.

This concludes the addition activity. Many adult learners can move directly from this point to the next activity, Subtraction of Whole Numbers. Beginning learners should do several addition problems with the toothpicks and then stop at this point for further followup activity as listed below. "Addition facts" will be learned most easily and retained the longest by repeated application of them, rather than by rote memorization. That is, learners should be encouraged to count on their fingers or use toothpicks to solve problems until they no longer need to do so.

Follow-Up

1. Number Sense Teacher's Resource Guide (Contemporary, 1990) lists two other concrete activities to teach addition, on page 23, which may be helpful to learners who are struggling with the idea of addition.

2. Number Sense: Whole Number Addition & Subtraction offers semi-concrete "paper and pencil" activities on pages 10-26, especially helpful to learners for whom addition is a new skill.

3. Practice is important in mastering any new skill. Practice problems can be found in any G.E.D. or A.B.E. math book.

SUBTRACTION OF WHOLE NUMBERS

Directions

This activity uses toothpicks, rubber bands, and place mats (see Place Value, page 4). Introduce the subject of subtraction by reviewing the idea of addition (combining two amounts together); then explain that the idea of subtraction is quite different. The two (or more) numbers in an addition problem represent different amounts which can each be seen. Only the first (or top) number in a subtraction problem is this type of number. The second (or bottom) one is actually a part of the first amount. When one begins with five apples and eats (or subtracts) three of them, the three that are subtracted were part of the original five apples.

Direct learners to place 35 toothpicks (3 tens and 5 ones) on their mats. This will be the "whole" amount. Then, tell them to remove 12 toothpicks from the mat and put them back in the general pile of toothpicks. Most learners will quickly recognize that they can remove one ten and two ones from the mat without the need for any regrouping. They will, of course, be left with 23 toothpicks.

Direct the learners to remove 5 toothpicks from the 23 which are on the mat. Most learners will discover the need to "unbundle" one of their bundles of ten, and will come up with the right answer, 18. Ask whether learners know what that unbundling is called (borrowing). Finally, show the symbols on the board:

$$\begin{array}{r} 23 \\ -\underline{\ \ 5} \end{array}$$

When we try to remove five ones from the mat, we see that, in the beginning, there aren't enough ones to do that.

$$\begin{array}{r} {}^{1}\ {}^{1} \\ \cancel{2}3 \\ \underline{-\ \ 5} \end{array}$$

Having unbundled one of our tens, we now have only one of them left. Instead, we have 13 ones and can now complete the problem. Encourage learners to use these symbols in order to remember when they have borrowed. Some may have been incorrectly told not to do so at some previous point in their math career.

"Borrowing twice" can be illustrated, but should be done in small groups rather than individually because of the large number of toothpicks required. Direct learners to begin with 205 toothpicks (two bundles of 100 and 5 single toothpicks). Ask them to remove 8 toothpicks. Students will discover that they need first to regroup one hundred into 10 tens and then regroup one of the tens into ones. Do the problem on the board:

205 - 8	We know right away that we need some more ones, We look at the tens column, but it's empty.
1 1 ~~2~~05 - 8	We can use the hundreds. When we unbundle one of the hundreds, we have only one hundred left. We will have 10 tens.
1 ~~1~~9 1 ~~20~~5 - 8	Now, we can unbundle one of the 10 tens. We will have nine bundles of ten left and we now have 15 ones.

This ends the illustration of subtraction. Students should stop at this point for follow-up activities.

Follow-Up

1. Number Sense Teachers' Guide (Contemporary, 1990) provides concrete activities on pages 29-30 for learners who are struggling with subtraction.

2. Number Sense: Whole Number Addition & Subtraction offers semi-concrete "paper and pencil" activities on pages 30-49.

3. Subtraction problems can be found in virtually any G.E.D. or A.B.E. math book.

4. Students can begin to work on **word problems** at this point. One good way to work on this is to present the learners with addition and subtraction problems with the numbers eliminated. Direct the learners, working in pairs or small groups if possible, to figure out what information will be required to solve the problem and whether they will need to add or subtract. Here is an example of this:

> QUESTION: Together, what are Maria's and Debbie's coin collections worth?
> ANSWER: You need to know how much Maria's collection is worth and how much Debbie's is worth and you will add.

Another good way to help learners focus on the sense of the problem, rather than the numbers or the answer is to challenge them to write the question. Several pages of these can be found in the Number Sense series (Contemporary, 1990). For example:

> The rent is \$250/month. The utilities are \$75/month.
> ANSWER: \$325
> QUESTION: ____.

Once learners are comfortable with this type of exercise, they are ready for actual word problems involving whole number addition and subtraction, which can be found in any G.E.D. or A.B.E. math book.

INTRODUCTORY ALGEBRA

This is a good place to introduce the basic concepts of algebra. Doing this now reduces the dread of algebra that many people carry and also provides additional practice in addition and subtraction of whole numbers. There are three fundamental concepts to be taught:

1. The idea of **the unknown**. Write out several simple addition and subtraction equations with an empty box substituted for one of the terms:

$$2 + [\,] = 9 \qquad 8 - [\,] = 5 \qquad [\,] + 6 = 14$$

The learners will enjoy telling you what goes in the box. Explain that, for convenience, we don't write algebra books filled with drawings of empty boxes. Instead we use a letter, **any letter,** to stand for "some number; we don't know what it is yet". Illustrate by erasing the boxes in your equations and writing letters, instead. Try to make it clear that any letter at all is O.K.:

$$2 + d = 9 \qquad 8 - k = 5 \qquad z + 6 = 14$$

2. The idea of **balancing equations.** The most common and most easily understood illustration of this is a scale. Draw a butcher's scale and point out that, if there are five pounds of meat on each side, the scale will be even, or balanced. Add three to one side and ask learners what will need to be done to the other side to keep the scale balanced. They will recognize that three will need to be added to the other side. Now, subtract four from one side and ask what needs to be done. Students will recognize that four will have to be removed from the other side. Continue giving examples until it is clear that the scale will always be balanced, as long as the same operation is applied to both sides.

Explain that the same principle is true of an equation: the equation will stay balanced, or true, if the same operation is performed on both sides of the equal sign:

If 4 + 2 = 3 + 3 is true, then 4 +2 + **136** = 3 + 3 + **136** must also be true, even if we don't know exactly what the sum is.

3. **Isolating the unknown.** Point out that the "answer" of an algebra equation would look like "r = some number". To get to this point, it is necessary to have the unknown standing by itself on one side of the equal sign. This is the key to solving algebra equations.

To isolate the unknown (get it standing by itself on one side of the equal sign), we get rid of everything that starts out on the same side of the equal sign as the unknown. For example:

$$b + 246 = 431$$

Ask learners what we'll need to get rid of in order to show "b" standing alone. They will recognize that 246 will need to be eliminated. Ask how that could happen. If none of the learners can figure it out, tell them: we subtract 246. Then ask them how we can keep the equation true, or balanced. They should remember that the same thing needs to be done to both sides; that is, 246 should also be subtracted from 431 as well.

$$\begin{array}{l} b + 246 = 431 \\ \quad - 246 \end{array}$$

$$\begin{array}{rcl} b + 246 & = & 431 \\ \underline{-246} & & \underline{-246} \\ b & = & 185 \end{array}$$

The result of the subtraction is b + 0, or simply b, since zero added to any number equals that number, equals 185. The answer can be checked by substituting 185 for b in the original equation.

$$n - 184 = 378$$

A subtraction equation is solved using the same method, except that adding 184 is what will get rid of the "-184". One way to make this more clear is to suggest that "- anything" is that much "in the hole" or "in the ground". However far "in the hole" one is, adding that much will bring one back to even, or zero. As with the previous example, the answer can be checked by substituting the answer into the original equation and seeing if it "works": $562 - 184 = 378$

$$\begin{array}{rcl} n - 184 & = & 378 \\ \underline{+184} & & \underline{+184} \\ n & = & 562 \end{array}$$

Algebra: Jumbo Yearbook, published by ESP Publishers, Inc. is a book of black line masters which will offer learners practice in all aspects of algebra. Numbers 1 and 2 will give learners practice in writing algebraic expressions using addition and subtraction. Numbers 29 and 30 offer practice in solving simple equations with addition and subtraction.

2

2 x 1 = 2
2 x 2 = 4
2 x 3 = 6
2 x 4 = 8
2 x 5 = 10
2 x 6 = 12
2 x 7 = 14
2 x 8 = 16
2 x 9 = 18
2 x 10 = 20
2 x 11 = 22
2 x 12 = 24

3

3 x 1 = 3
3 x 2 = 6
3 x 3 = 9
3 x 4 = 12
3 x 5 = 15
3 x 6 = 18
3 x 7 = 21
3 x 8 = 24
3 x 9 = 27
3 x 10 = 30
3 x 11 = 33
3 x 12 = 36

4

4 x 1 = 4
4 x 2 = 8
4 x 3 = 12
4 x 4 = 16
4 x 5 = 20
4 x 6 = 24
4 x 7 = 28
4 x 8 = 32
4 x 9 = 36
4 x 10 = 40
4 x 11 = 44
4 x 12 = 48

5

5 x 1 = 5
5 x 2 = 10
5 x 3 = 15
5 x 4 = 20
5 x 5 = 25
5 x 6 = 30
5 x 7 = 35
5 x 8 = 40
5 x 9 = 45
5 x 10 = 50
5 x 11 = 55
5 x 12 = 60

6

6 x 1 = 6
6 x 2 = 12
6 x 3 = 18
6 x 4 = 24
6 x 5 = 30
6 x 6 = 36
6 x 7 = 42
6 x 8 = 48
6 x 9 = 54
6 x 10 = 60
6 x 11 = 66
6 x 12 = 72

7

7 x 1 = 7
7 x 2 = 14
7 x 3 = 21
7 x 4 = 28
7 x 5 = 35
7 x 6 = 42
7 x 7 = 49
7 x 8 = 56
7 x 9 = 63
7 x 10 = 70
7 x 11 = 77
7 x 12 = 84

8

8 x 1 = 8
8 x 2 = 16
8 x 3 = 24
8 x 4 = 32
8 x 5 = 40
8 x 6 = 48
8 x 7 = 56
8 x 8 = 64
8 x 9 = 72
8 x 10 = 80
8 x 11 = 88
8 x 12 = 96

9

9 x 1 = 9
9 x 2 = 18
9 x 3 = 27
9 x 4 = 36
9 x 5 = 45
9 x 6 = 54
9 x 7 = 63
9 x 8 = 72
9 x 9 = 81
9 x 10 = 90
9 x 11 = 99
9 x 12 = 108

10

10 x 1 = 10
10 x 2 = 20
10 x 3 = 30
10 x 4 = 40
10 x 5 = 50
10 x 6 = 60
10 x 7 = 70
10 x 8 = 80
10 x 9 = 90
10 x 10 = 100
10 x 11 = 110
10 x 12 = 120

11

11 x 1 = 11
11 x 2 = 22
11 x 3 = 33
11 x 4 = 44
11 x 5 = 55
11 x 6 = 66
11 x 7 = 77
11 x 8 = 88
11 x 9 = 99
11 x 10 = 110
11 x 11 = 121
11 x 12 = 132

12

12 x 1 = 12
12 x 2 = 24
12 x 3 = 36
12 x 4 = 48
12 x 5 = 60
12 x 6 = 72
12 x 7 = 84
12 x 8 = 96
12 x 9 = 108
12 x 10 = 120
12 x 11 = 132
12 x 12 = 144

MULTIPLICATION OF WHOLE NUMBERS

It is critically important to distinguish between learning how to multiply and memorizing facts about multiplication ("times tables"). All too often, these skills are confused. Sometimes the more important skill, the understanding of the nature and process of multiplication, is not even taught, in the mistaken belief that learning the times tables is the same thing.

Learning the times tables is a memorization task. It will be most easily accomplished by applying the facts, not simply by repeating the facts over and over. Virtually all learners will master these facts successfully if they are given a chart of the times tables (see previous page) and encouraged to use the chart to solve problems. In the course of this math curriculum, the times tables will be consulted hundreds of times: to solve problems of whole number multiplication and division, calculation of averages, determining area and perimeter , decimal multiplication and division, reducing and raising fractions. In short, almost every problem will require the use of them. By the time the student is ready for the G.E.D. math test, the "times tables" will have been learned, painlessly, without devoting valuable class time to the task.

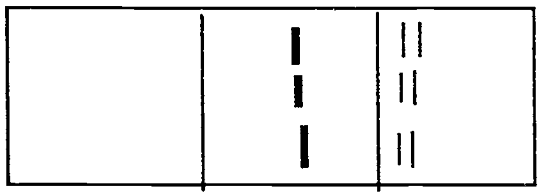
Directions

The activity which follows is designed to teach the concept of multiplication. It uses toothpicks, mats and rubber bands, as for place value (see page 4).

Begin by reviewing the operations of addition and subtraction. The important points to cover are the addition is the process of continuing two parts into a whole. Subtraction, on the other hand, begins with a whole from which a part will be removed. With addition, the parts do not have to have the same value. Introduce the idea that multiplication, today's subject, is a special kind of addition. Like addition, parts will be combined to produce a whole, and there are other similarities, as well. The difference is that, in multiplication, sets of numbers are added. That is, we will be combining groups which are the same size.

It is easiest to demonstrate the lesson if there are some previously prepared "tens". If there are none already bundled, take three minutes and ask learners to make some.

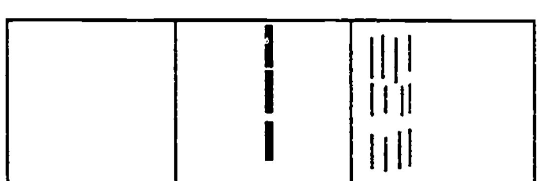
Begin by directing the learners to place three sets of twelves, or three twelves, on their mats:



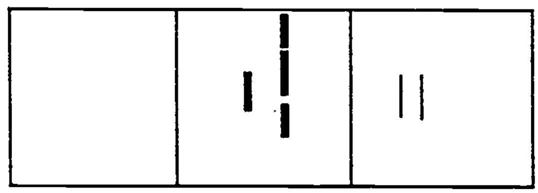
Ask learners how many toothpicks are on their mats. Then, ask them how they came up with that answer. See if you can elicit at least two different ways of counting. One is to count. One is to count the ones (6 toothpicks) and then the tens (30 toothpicks) and add them together (30 + 6 = 36). Another is to add 12 + 12 + 12 = 36. Then, show the learners the symbols for the problem they have just solved:

12 X 3 6	First, we figure out how many ones there are. The "3" stands for three sets (or groups). We ask, how many ones do we have in three sets of two ones each? We write the answer in the ones column.
12 X 3 36	Then, we figure out the tens. We ask, how many tens do we have in three sets of one each and we write the answer in the tens column.

The next problem demonstrates carrying in multiplication. Ask the learners to put three 14's on their mats. Actually, all they will have to do is add two ones to each of their 12's:



Prompt learners to check to make sure that no rules are being broken. If they don't see it independently, have them count the ones. Most will then recognize the need to combine ten of them into an additional bundle of ten and add it to the tens column:



Again, demonstrate the relationship of this reality to the number symbols which stand for it:

$$\begin{array}{r} 1 \\ 14 \\ \underline{\text{X } 3} \\ 2 \end{array}$$

Again, we count the ones first. Three sets of four ones in each set gives us 12 ones. We need to bundle up ten of them into an additional bundle of ten and put it into the tens column. That will leave us with two ones.

$$\begin{array}{r} 1 \\ 14 \\ \underline{\text{X } 3} \\ 42 \end{array}$$

When we count the tens, we need to remember that the two "1's" stand for different ideas. The original "1" stands for "one ten in each of the three sets". The new "1" stands for "one extra bundle of ten made from carrying". So, we figure "three sets of one ten in each set, plus the bundle we made by carrying", or four tens altogether.

If learners are **not** new to the idea of multiplication, skip forward to the demonstration of two digit multiplication, which begins on the next page. If learners **are** new to the idea of multiplication, it would be best to stop here and allow learners to practice what they have learned.

Follow-Up

1. The Number Sense Teachers' Guide (Contemporary, 1990) provides concrete activities on pages 38-39 for learners who are struggling with multiplication.

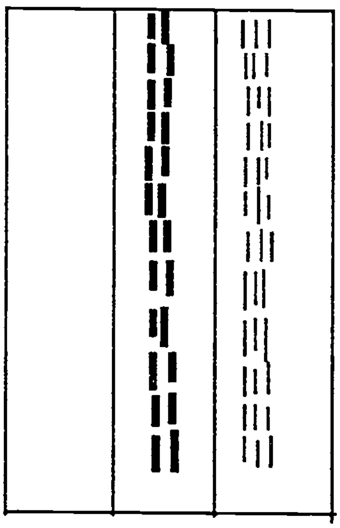
2. Number Sense: Whole Number Multiplication & Division (Contemporary, 1990) offers semi-concrete activities on this subject on pages 1-3 and 9-12.

3. Single digit multiplication problems can be found in virtually any math book.

Two Digit Multiplication

The place mats are not large enough for learners to work with ten or more sets of toothpicks. Two digit multiplication can, however, be demonstrated in much the same way as single digit multiplication. One could use toothpicks on one very long mat (perhaps a piece of newsprint) or draw a long mat on the board and indicate toothpicks with chalk lines.

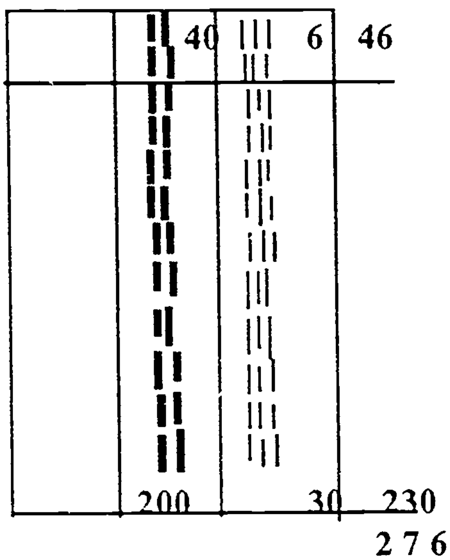
Set up the problem of 12 sets of 23 in each set:



Ask the learners to think of ways to count the number of toothpicks there are altogether (that is, solve the problem). There are several "right" answers, just as there were for single-digit multiplication.

One might combine the ones into bundles of ten, and then the tens into bundles of one hundred, and then the answer could be read right off the mat. One could add 23 + 23 + 23 ... a total of twelve times, although that would be a very time-consuming way to go about it. Finally, one could count all the ones (36 ones) and then all the tens (24 tens or 240 toothpicks) and add them together. In any case the answer will be the same, of course: 276.

In fact, when we use symbols to solve the problem, we divide the sets up, counting the number of toothpicks in one group of sets, then the other group of sets, and finally adding them all together. In this particular example, we look first at the number of toothpicks contained in the first two sets and then at the number contained in the last 10 sets.



First, we count the ones in the first two sets (there are six of them) and then the tens in the first two sets (there are 4 tens, or 40 toothpicks). We then add those results together to determine the total number of toothpicks in the first two sets (46).

Next, we add up the ones in the last 10 sets (30 ones) and then count the tens in the last 10 sets (23 tens, or 200 toothpicks). We add those results together to find the total number of toothpicks contained in the last 10 sets, which is 230.

The final step is to add the 46 toothpicks from the first two sets to the 230 toothpicks from the last ten sets to find the total number (276 -- the same as before, of course).

Leaving the mat as it is, with the numbers still written on it, demonstrate the same problem, showing the number symbols:

$$\begin{array}{r} 23 \\ \underline{\times\ 12} \\ 6 \end{array}$$

The first step is to figure out how many ones there are in the first two sets? Three sets, each with two ones, will contain altogether six ones. Write the "6" here and point out the "6" on the toothpick display.

$$\begin{array}{r} 23 \\ \underline{\times\ 12} \\ 46 \end{array}$$

The second step is to figure how many tens there are in the first two sets?" There are four tens, 40 toothpicks. Altogether there are 46 toothpicks in the first two sets of 23. Write the "4" here and point out to learners the "40" and the "46" on the toothpick display.

$$\begin{array}{r} 23 \\ \times\ 12 \\ \hline 46 \\ 30 \end{array}$$

The third step is to figure out how many ones are in the last ten sets. Ten sets with three ones in each would be 30 toothpicks. Write the "30" here and point out the "30" on the toothpick display.

$$\begin{array}{r} 23 \\ \times\ 12 \\ \hline 46 \\ \underline{230} \end{array}$$

The fourth step is to figure out how many tens there are in the last ten sets. Ten sets, each having two tens in it, would make 20 tens, or 200 toothpicks. Altogether there are 230 toothpicks in the last ten sets. Write this in here and point out the 230 on the display.

$$\begin{array}{r} 23 \\ \times\ 12 \\ \hline 46 \\ 230 \\ \hline 276 \end{array}$$

Once the number of toothpicks in each group of sets is counted, all we have to do is add the two amounts (which are called partial products) to find out the total number, or total product. The answer, as we have already figured out, is 276.

Generally speaking, the zero in "230" is understood, but not written. Erasing the zero in "230" will make the problem look more familiar to you and the learners who have been exposed to multiplication previously. This exercise will help to explain, however, why the partial products seem to "move" to the left during two and three digit multiplication.

This concludes the multiplication activity. Like any other new math skill, learners will need practice to become comfortable with multiplication.

Follow-Up

1. <u>Number Sense: Whole Number Multiplication & Division</u> (Contemporary, 1990) offers semi-abstract activities on pages 13 - 18.

2. Drill can be found in virtually any math book.

3. Working on word problems involving multiplication will further reinforce the concept of multiplication and offer practice with word problems at the same time.

4. The calculation of the area and perimeter of rectangles and triangles requires only the skills of whole number addition and multiplication. These subjects are generally covered as part of geometry and, as such, can be found in any G.E.D. math book. Introducing these topics at this point offers two advantages over waiting until the traditional time for presenting them. One is that learners will receive useful practice in the whole number operations addition and subtraction, before going forward to a new skill, division. The other is that learning these topics at this point reduces the dread of geometry that many learners have.

The Cambridge G.E.D. Program presents area and perimeter at this point of instruction, and is a very useful resource. This topic can be found in the High School Equivalency Examination (Cambridge, 1987) on pages 447 - 449.

DIVISION OF WHOLE NUMBERS

The Basic Concept of Division

Directions

While it would be possible to use the place mats to demonstrate division, it is easier simply to use the toothpicks and rubber bands (see **Place Value**, page 4).

Begin by explaining the idea of dividing. One familiar example is dealing out a deck of cards. If one deals out four hands, one has divided the 52 cards four ways. This is the idea of 52 divided by 4. Another familiar example is dividing food (cookies, for example) among children. If one has ten cookies and two children, each child will get five. This is the same as ten divided by two.

Direct the learners to begin with a "pile" of 24 toothpicks (two bundles of ten and four single toothpicks). Explain that this is the whole, or total, that is the beginning point of a division problem. Ask the learners whether they can separate these 24 toothpicks into 2 even piles of toothpicks. They can, of course, putting 12 into each pile. In fact, they will not have to unbundle the tens to do so; they will simply put one ten and two ones in each pile. Show the learners the symbols for the problem they have just solved:

$$2\overline{)24}^{\;1}$$

The first question is, are there enough tens to go around at least once (or, are there two or more tens?) Yes, in fact there is one ten in each pile.

$$2\overline{)24}^{\;12}$$

Next, we ask, are there enough ones to go around at least once? Yes, in fact, there will be two in each pile.

The next example illustrates the idea of a remainder. Tell learners to begin with 34 (three bundles of ten and four single toothpicks). Direct them to divide this whole into three even piles. Learners will quickly discover that each pile will have one ten and one single, but that there is one toothpick left over.

Ask learners if they know what that means and what could be done. Many learners will recognize the idea of a remainder. Some will volunteer that the left over toothpick could be split into three equal parts, with a third going into each pile. Tell them that there are two correct ways to indicate the situation:

$$\begin{array}{r} 11\ \ r1 \\ 3\overline{)34\quad} \end{array} \qquad \text{OR} \qquad \begin{array}{r} 11\ 1/3 \\ 3\overline{)34\quad} \end{array}$$

The next example illustrates the need to unbundle, and will help learners to understand where they should write the answer to such a problem. Tell learners to begin with 24 toothpicks (two bundles of ten and four single toothpicks). Ask them to divide these toothpicks into three equal groups. Most learners will recognize the need to remove the rubber bands, and use the two tens as 20 ones.

$$\begin{array}{r} 0\ \ \\ 3\overline{)24} \end{array}$$

This time, when we ask are there enough tens to go around at least once, the answer is "no". There will be no tens in our answer. We need to unbundle them. Ask learners how many to unbundle. The answer is "all of them", unlike subtraction, where we unbundled only one.

$$\begin{array}{r} 08 \\ 3\overline{)24} \end{array}$$

Once we unbundle those two tens, we have 24 ones, twenty that result from the unbundling and the four thatwe started with. These, of course, can be split into three - equal piles of eight each.

The basic concepts of division have now been shown. Learners previously familiar with division can move on to The Process of Long Division, page 25. Learners for whom division is a new concept may benefit from further work with the basic concepts:

1. The toothpicks can be used to solve other, different problems.

2. There are four other concrete ways to illustrate the concept of division described in the Number Sense Teacher's Guide (Contemporary, 1990), pages 46 and 47.

3. Number Sense: Whole Number Multiplication & Division (Contemporary, 1990) offers semi-concrete "paper and pencil" activities on pages 21-23 and pages 26-27.

The Process of Long Division

To illustrate the question and answer process of long division, write a problem on the board or on paper, and ask the learners to imagine the "whole" as an extremely large pile of toothpicks, which are to be split up, or divided, into five piles. Describe the large pile: we have six bundles of 10,000 toothpicks, no bundles of 1,000, two bundles of 100 each, four bundles of ten and five single toothpicks.

$$5\overline{)60245}$$

$$\begin{array}{l} \;\;\;1 \\ 5\overline{)60245} \end{array}$$

First, we look at the bundles of 10,000. Are there enough to go around at least once? Yes, in fact, just once. Before we move on, we need to know how many of them we'll need to unbundle and use as bundles of 1,000.

$$\begin{array}{l} \;\;\;1 \\ 5\overline{)60245} \\ \;\;\;5 \end{array}$$

We figure this out by first calculating how many of bundles of 10,000 we have used. We ask, how many would one in each of five piles use up? Five sets of one each is the same as five times one, or five.

$$\begin{array}{l} \;\;\;1 \\ 5\overline{)60245} \\ \;\;\underline{-5} \\ \;\;\;1 \end{array}$$

Knowing that we started with six, and have used five, we subtract the number already used (five) from the original number, and discover that we have one bundle left. All we can do is unbundle it.

$$\begin{array}{r} 1 \\ 5\overline{)\,60245} \\ -5 \\ \hline 10 \end{array}$$

We now have ten bundles of 1,000. The next step is to add to that any bundles of 1,000 that were part of the original whole. In this particular example, there weren't any to begin with, so we have only the ten that came from the unbundling.

$$\begin{array}{r} 12 \\ 5\overline{)\,60245} \\ -5 \\ \hline 10 \end{array}$$

We discover that we can put two bundles of 1000 in each pile.

$$\begin{array}{r} 12 \\ 5\overline{)\,60245} \\ -5 \\ \hline 10 \\ -10 \\ \hline 0 \end{array}$$

To find out whether we have any bundles of 1000 left, we first figure out how many we used. We used two bundles in each of the five piles, or 2 X 5. In other words, we used ten bundles of 1000. When we subtract them from the original 10 bundles, we find that we used them all.

$$\begin{array}{r} 120 \\ 5\overline{)\,60245} \\ -5 \\ \hline 10 \\ -10 \\ \hline 02 \end{array}$$

We then consider the "hundreds". We started with two of them and there aren't any to be added to that. Two is not enough to put even one of them in each pile, so we will not have any bundles of 100 in our answer piles. Point out to learners that the zero in the answer gives important information: the fact that there are no "hundreds" in the answer.

$$\begin{array}{r} 1204 \\ 5\overline{)\,60245} \\ -5 \\ \hline 10 \\ -10 \\ \hline 024 \\ -\quad 20 \end{array}$$

When we unbundle the two hundreds, we will have twenty tens. When we add in the four tens we started with, we have 24 tens. They can be divided among our five piles, four in each pile. Four of them in each of five piles equals 20, which is the number we used up.

$$
\begin{array}{r}
12049 \\
5\overline{)60245} \\
\underline{-5} \\
10 \\
\underline{-10} \\
024 \\
\underline{-\ 20} \\
45 \\
\underline{-\ 45} \\
0
\end{array}
$$

When we find the difference, by subtracting, between the 24 tens that we started and the 20 tens that we have distributed among the five piles, we find that we have four tens left, which must be unbundled. We will then have 40 ones. We will add in the five ones that we started with, making 45 single toothpicks in all. These can be distributed among the five piles. Nine will go into each pile, and that will use up all 45 toothpicks. We now see that when our "whole" is split five ways, there will be 12,049 toothpicks in each pile.

Learners who have previously been exposed to division can usually proceed from this example to performing almost any division problem. They may need some help with estimating (see below) in order to perform two and three digit division. Learners who are learning division for the first time should stop here for some practice in single-digit division:

1. Number Sense: Whole Number Multiplication & Subtraction (Contemporary, 1990) offers semi-concrete "paper and pencil" activities on pages 28-33.

2. Problems for drill can be found in virtually any math book.

Two (or More) Digit Division

The process of dividing a whole by a number greater than 9 is the same process as illustrated above. However, in order to begin the process, learners will need to use the skills of rounding off and estimating. This is covered in a number of math books. GED Math (Contemporary, 1989) offers a fairly thorough explanation on pages 28-30. Some enrichment activities are offered in the Number Sense Teacher's Guide (Contemporary, 1990).

Once learners are comfortable with rounding off and estimating, explain that this will help them estimate answers to division problems. An example is given below:

$$37\overline{)294}$$

Learners should understand at this point that 37 cannot be divided into 2, or into 29, and so we must estimate how many times 37 can be divided into 294. Using the skill of rounding off, we ask how many times 40 can be divided into 294. This will be approximately the same as the number of times 4 can be divided into 29, or 7 times.

$$\begin{array}{r} 7 \\ 37\overline{)294} \\ 259 \\ \hline 35 \end{array}$$

We will check whether 7 is the exact answer in the process of multiplying 37 by 7. Since 259 is **smaller** than 294, we know that seven is not too large a number. If it were too large, we would try one number smaller (in this case, "6"). Because 35 is **smaller** than 37 (in other words, the number of toothpicks left over cannot go around again), we know that 7 is not too small a number. If the remainder were larger than the divisor, we would try one number larger (in this case, "8"). In other words, 7 R 35 is the answer.

Follow-Up

1. Problems for drill can be found in any A.B.E. or G.E.D. math book.

2. This is a good place to re-introduce word problems (see page 12 for suggestions about word problems with addition and subtraction.) Use the same approach for multiplication and division problems. The Number Sense series (Contemporary, 1990) has several pages where learners are challenged to write the question, which helps them to understand questions that other people have written. Word problems using whole numbers can be found in all GED and ABE math books.

In addition to the above, learners will have many opportunities to practice division, along with the other three operations, by studying the various topics presented in the next chapter.

APPLICATIONS OF WHOLE NUMBER OPERATIONS

There are several reasons to teach students some or all of the following math lessons before going forward to "part-numbers" (decimals and fractions). In the learning of them, students will have many opportunities to practice the whole number operations they have just learned. Practicing in a variety of applications is more interesting than pages of drill. Applying concepts helps students understand why something like multiplication may be useful in their lives. Finally, some of these lessons ("powers", for instance) are fairly simple ideas, yet ideas that have previously confused and frightened students. Mastering them gives students increased self-confidence at an early stage.

The Contemporary GED Math book (Contemporary, 1989) and the Cambridge Comprehensive Program for the High School Equivalency Examination (Cambridge, 1987) are both organized this way. Contemporary presents calculation of mean and median and predicting the next number in a series immediately following the chapter on whole numbers. Calculations and word problems are provided.

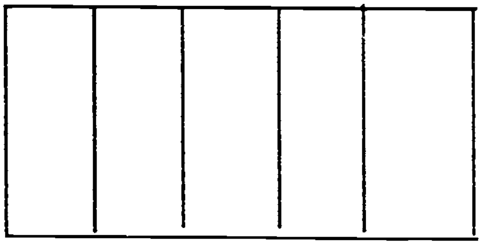
Cambridge's High School Equivalency Examination (Cambridge, 1987) also presents calculating mean and median at this point. In addition, Cambridge presents using the distance and cost formulas, calculating powers and square roots, area and perimeter problems, and calculating the volume of a rectangular solid. Cambridge's accompanying Exercise Book for the Mathematics Test (Cambridge, 1987) is organized the same way and provides four additional pages (pages 3-6) on these whole number applications.

In addition to these specific whole number applications, both Contemporary and Cambridge present specific problem-solving techniques and introduce students to types of problems found on the GED test at this point. Both books demonstrate "set-up" problems and multiple step problems at this point. Cambridge also introduces item sets at this point (Contemporary offers this after the following chapter, on decimals). Both in the texts and the accompanying exercise books, many word problems utilizing only whole number operations can be found.

DECIMALS: THE BASIC CONCEPT

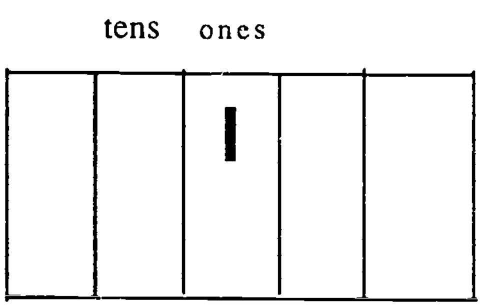
Students who thoroughly understood place value (see page 4) will have little trouble understanding decimals. Decimals operate, and look, more like whole numbers than do fractions, and are easier to learn than fractions, initially.

Demonstrate the basic idea of decimals with one bundle of 100 toothpicks (that is, ten bundles of ten each) and either a mat or a set of columns drawn on the board. For this activity, it would be better to have five columns available:

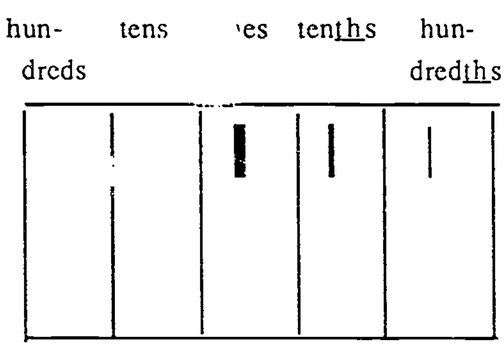


Begin by reviewing the lesson on place value. Remind learners that the columns go on in both directions forever and that every column "works" alike. Review with them the basic operation: the rule of "tens", the fact that as we move from column to column toward the left, each column is worth ten times **more** than the one before it and that, conversely, as we move from column to column toward the right, each column is worth ten times less than the one before it. In other words, when we make a new, larger, unit or bundle, we place it to the left and when we "unbundle" into smaller units, we place them to the right.

Now, show learners the bundle of 100 and explain that, for the purpose of this lesson, this (the bundle of 100) is not worth 100, but will be defined as "one". Label a column in the middle "ones", and also label the one on its left "tens" Help the learners agree that the big bundle belongs in "ones" and, if you had ten of them all fastened together, that unit would belong in the tens column.



Take off the outermost rubber band, leaving ten bundles of ten toothpicks each and ask learners which column these bundles should go into. They should be able to see that the smaller bundles would go into the column to the right of "ones". The next step is to determine what value those bundles have. Try to prompt learners to discover it by asking them how many of the bundles were in the original "one". When they recognize that there were ten, most learners will respond correctly when you reply something like, "then one of them is equal to one ..?" When they have named "tenth", label the column. Point out the very important "th" in the word "tenth" and point out the difference between the size of a ten (the size of ten of the original bundles) and a ten<u>th</u> (the original bundle **divided by** ten).



Now, take one of those smaller bundles and remove its rubber band. Again, ask learners where units of this size (one toothpick) should go. The answer, of course, is in the column just to the right of the "tenths" column. Again, prompt learners to discover the name of the column by asking how many of this size unit were in the **original one.** Label the column and, again, point out the "th" in "hundred<u>th</u>s".

Ask learners to imagine one toothpick broken into ten parts (or break one into ten parts). Ask where those little bits should go (in the next column to the right, of course). Again, prompt them for the column's name (thousandths). Most learners get the idea at this point.

Looking at the complete diagram, remind learners that units get larger moving to the left and smaller to the right. Point out the apparent contradiction that the names of the columns **seem** to be getting larger as they go from tenths to hundredths to thousandths. Help them understand that the sizes are actually getting smaller. One way to do it is to refer once again to the toothpicks. Try to help learners see that the more ways something is divided, the smaller each part is. One concrete example of this is that if a pizza is divided among five people, each gets a smaller piece than if the same pizza is divided between two people.

The last basic concept relating to operations with decimals is that of the decimal point. Once again, remind learners that the number system goes on in both directions forever. The decimal point is our way to know where we are in that system. It sits between the ones and the tenths (put one on your diagram). It **always** comes between ones and tenths. Even when it appears to be moving, it isn't really: it stays in its place. It could be called a place-marker, in fact.

Students come to the subject of decimals with many misconceptions which must be unlearned. Tell them, just in case, that the decimal point does not mean **anything** except that the ones column is to its immediate left and the tenths is to its immediate right. It does **not** mean that numbers to one side act differently from numbers to the other.
It does **not** mean "stop" (that is, borrowing and carrying go right across it).
It does **not** mean that some special math should be performed.

Follow-Up

1. One enjoyable and concrete way to reinforce the meaning of decimals is to have the class play Decimal Blackjack, a game distributed by Education Plus (1975), listed in the References.

2. Number Sense: Decimal Addition and Subtraction (Contemporary, 1990) offers "paper and pencil" work that is semi-concrete in nature (pages1-18).

3. Most G.E.D. and A.B.E. math books offer some work on the concept of decimals.

DECIMAL ADDITION AND SUBTRACTION

Learners who have firmly grasped the concept of decimals and the operations of addition and subtraction will probably be able to perform decimal addition and subtraction with very little additional demonstration or explanation. Only one new concept needs to be learned: how to use the decimal point as a place marker.

Review the importance of lining up whole number addition and subtraction problems in such a way that ones are added to (or subtracted from) ones, tens to tens, and so forth. Since the decimal point is our place marker, it is the decimal point that is the key to this lining up. That is, if the decimal points of all the numbers are lined up with each other, all the columns will be correctly in line. For example, given the problem 2.5 + 12.67 + .429, the first step is to line up the decimal points:

.	If learners are at all uncertain, have them write the decimal points first, in a line.
.	
+ .	

2.5	Once the decimal points are lined up, fill in the numbers.
12.67	It may help learners to write in zero's into the empty
+ .429	hundredths and thousandths places.

1	
2.500	Once this lining up has been completed, proceed to add
12.670	or subtract in just the same way as for whole numbers.
+ .429	Reassure learners that borrowing and carrying work the
15.599	same way as for whole numbers; the decimal point is only a place marker, and all the columns work the same, no matter what side of the decimal point they fall on.

One aspect of using decimals that occasionally confuses learners is where the decimal point is placed in a whole number. For example, how would one do the problem 34 - 7.6?

Drawing the mat with its columns may help to illustrate this:

tens	ones	tenths
3	4	

Ask learners to identify the digits in the number "34" (three tens and four ones) and write them in the correct columns.

3	4	. 0

Ask how many tenths are in the number (zero), and write that in. Finally, add the decimal point between the ones column and the tenths column. Our number now reads "34.0". In other words, the decimal point must come after the last whole number. If learners are unconvinced that "34" and "34.0" are the same, ask whether "$5" and "$5.00" are not two different ways of writing the same thing. The problem stated above (34 - 7.6) would therefore look like this:

$$\begin{array}{r} {\scriptstyle 2\ 13\ 1} \\ \cancel{3}\cancel{4}.0 \\ \underline{-\ \ 7.6} \\ 26.4 \end{array}$$

Again, borrowing is done in exactly the same way, no matter which column is being worked in.

Follow-Up

1. Learners who are struggling with decimal addition and subtraction will benefit from solving additional problems with manipulatives. The toothpicks will work for this, but be careful to define which size bundle will equal "one". The game pieces from <u>Decimal Blackjack</u> will also work.

2. The decimal system most familiar to adult learners is money. Most adult learners know with certainty how much change to expect when they pay for a $3.75 purchase with a $5.00 bill. Money is therefore an excellent way to demonstrate addition and subtraction with decimals. It is necessary to clarify first that, in our money system, "one" is a **dollar**, not a **penny**. A dime is a "tenth" (of a dollar), a penny is a "hundredth" (of a dollar), and a ten is ten (dollars).

3. Any G.E.D. math book will offer pages of drill in decimal addition and subtraction, and will present word problems that call for the application of these skills as well.

DECIMAL MULTIPLICATION

Multiplication and division with part-numbers (decimals and fractions) may be the exceptions to the general rule that all math concepts should be demonstrated in a concrete way. While it is possible to do so, the demonstration can be more confusing to learners than it is helpful. These operations are fairly simple to perform and learners can generally master them without much difficulty if they thoroughly understand the operations of multiplication and division.

Explaining decimal multiplication requires only showing some examples.

12.25 X .5	The first step is to line up the problem. Explain to learners that this is very different from addition and subtraction.
12.25 X .5	Instead of lining up the decimal points as we did for addition and subtraction, we ignore them in lining up a multiplication problem. We write them in the problem, because we will use them later.
12.25 X .5 6125	After lining up the problem, we perform the multiplication just as though the decimal points were not there. At this point we need to think about the fact that the answer to this problem (12 times one-half, actually) could not be in the range of 6000. Our answer will be much smaller. We need to put a decimal point in the answer.
12.25 (2) X .5 (1) 6.125 (3)	Placing the decimal point in the answer follows one fairly rule: the total number of decimal places (columns to the decimal point which have numerals in them) in the answer must equal the total number of decimal places in the problem. In this example, there are two decimal places in 12.25 and one decimal place in .5, so we must have a total of three in the answer. Simply placing the decimal point in the answer, three columns from the right, completes our calculation.

Follow-Up

1. The Number Sense Teacher's Guide (Contemporary, 1990) offers three concrete ways to demonstrate decimal multiplication, which may be helpful to some learners.

2. Number Sense: Decimal Multiplication & Division (Contemporary, 1990) offers semi-concrete, simplified paper and pencil work on pages 1-17, followed by word problems containing decimals on pages 18-20.

3. All G.E.D. and A.B.E. math books contain some drill work on decimal multiplication.

DECIMAL DIVISION

As with multiplication, division by part numbers (decimals and fractions) is usually easier for learners to do than it is to demonstrate on a concrete level. There are two new ideas which learners will need to understand, after which most will be able to perform calculations involving decimal division.

The first, and traditionally most confusing idea, is what is usually presented as "moving the decimal point". Actually the decimal point does not "move"; the decimal point is a place marker which always appears between the ones column and the tenths column. The symbols which look like "moving the decimal point" actually represent the movement of numbers between columns, which is a quick way to multiply and divide by tens. To demonstrate, draw a place mat.

Use a number such as .5 as an example. Write it on the mat.

tens	ones .	tenths
		5

Next, show learners what happens when the "5" is moved one column to the left. The number now is 5.0, ten times larger than the original .5. Remind learners of the "rule of tens" (see Place Value, page 4). The columns increase by ten as numbers move to the left.

tens	ones .	tenths
	5	0

If learners are not certain of this idea, move the "5" one more column to the left, where it will represent "50". Moving it back to the right will, of course, divide the number by ten. For instance, moving the "5" from tens to ones changes the value from 50 to 5.

The second concept to be mastered is the idea of proportionality. This will be important several more times in this curriculum. The idea is this: if two numbers are related by division, such as a division problem (or a ratio or a fraction), that relationship is held constant if both numbers are either multiplied or divided by the **same** number. An example is shown below:

$4\overline{)8}$ The answer to this division problem is, of course, 2. If **both** of the numbers are multiplied by 3, we have the problem $12\overline{)24}$ and the answer is still 2. If we divide both of those numbers by 4, we have the problem $3\overline{)6}$ and the answer is still 2.

You can give any number of similar examples until learners are convinced of the truth of this fact: that as long as both parts of a division problem are multiplied or divided by the **same number**, the answer (or, value) is unchanged. You might recognize this as a different way to express the reducing of fractions.

Once these two concepts are mastered, students are ready to divide by decimals. It helps to distinguish between problems where decimals appear only in the dividend (inside number) and those where decimal places appear in the divisor (outside number).

$4\overline{)32.4}$ Solving the first type requires no skill. One of the basic rules of the number system applies here: **all columns work alike.**

$\begin{array}{r} 8.1 \\ 4\overline{)32.4} \end{array}$ The division is carried out just as though the decimal point were not present, although the place marker is kept in the problem and in the answer. This kind of problem can be demonstrated with toothpicks, if you wish: given a whole consisting of three tens, two singles, and four tenths, separate them into four equal piles.

The slightly more difficult type of problem to solve is the one where the divisor (outside number) contains one or more decimal places. This type of problem can be thought of a giving us a whole which is to be divided up into less than one pile. It doesn't make sense.

.5) 52.35

In this particular example, 52 toothpicks and part of another are to be divided into half a pile. As it stands, it doesn't make sense.

.5) 52.35

The first step is to multiply the outside number by ten until it is a whole number. The easy way to show this is to use the idea that shifting numbers to the left is the same as multiplying them by ten. We show this symbolically by "moving the decimal point". In this particular example, we will have a whole number, 5, after "moving the decimal point" once.

.5) 52.3 5

Proportionality tells us that in order to keep the answer, or relationship, the same, we now need to multiply the inside number by the **same amount**, in this case, ten. This establishes where the decimal point belongs in the answer. At this point, all we have to to is divide.

Having mastered division by decimals allows learners to solve problems which previously could only be written as having remainders. By adding zero's, division can be continued until it does come out even, or until a repeating pattern emerges. This is a good place to explain the idea of "rounding off", in order that the answer can be expressed as accurately as is needed.

```
   .333
3 ) 1.000
     9
     10
      9
      1
```

In this example, the pattern becomes clear fairly quickly. The rounding rule is that, if the final digit is five or greater than five, round up. If less than five, as in this example, round down. The answer to this example, then, could be expressed as .33 or .3 or .333, depending on the degree of accuracy needed in the problem.

```
   .66
3 ) 2.000
    18
     20
     18
```

In this example, the last digit is five or greater, so we will "round up". The answer could be expressed as .67 or .667 or .6667, depending on the problem.

Follow-Up

1. Number Sense : Decimal Multiplication & Division (Contemporary, 1990) presents the process of decimal division in a slow, simplified fashion on pages 21-37. The "paper and pencil" activities include some which are semi-concrete.

2. All G.E.D. math books offer some drill on decimal division, including word problems requiring the application of the skill.

3. Before moving on to fractions, learners should spend some time reviewing all operations with decimals. The G.E.D. math books also have a section of mixed practice. Some books present fractions before decimals (Cambridge, for instance) and some present decimals first (Contemporary, for instance), so one must take care that problems assigned for the purpose of decimal practice do not also contain problems with fractions. The G.E.D. Mathematics Exercise Book (Contemporary, 1988) offers problems with decimals, with fractions, on pages 18-23, including 13 problems which are in the format of the G.E.D. math test.

4. Having learned operations with decimals, learners now have the skills necessary to solve additional geometry problems (see page 22). Using 3.14 to represent π , learners can solve problems of circumference and area of circles, and also volume of cylinders. Some examples can be found in The Cambridge ...High School Equivalency Examination (Cambridge, 1987) on pages 498-499.

5. Learners now have the skills to create and solve word problems involving money -- in dollars and cents. Several "real-life" approaches now become possible.

One good way to do this, in a class setting, is to use a menu (see page 43 for one example). Divide the class into small groups of two or three learners each. Give each group one copy of the menu. Challenge the groups to create one or two word problems, using the information on the menu, for the other groups to solve.

The only rule for this exercise is that the group which creates a problem must also be able to solve it. This will eliminate the possibility of a group creating a problem for which information is insufficient to solve it. When the groups have finished working on their own problems, have the groups work to solve each others' problems. Alternatively, the solutions to the problems could be a homework assignment.

Other sources of information besides menus could be used. Any printed material with prices on it will work: shopping circulars, catalogues, a parts list. You might challenge the learners to find their own examples of written materials with decimals on them. This would have the additional advantage of heightening the learners' awareness of the presence of numbers and math problems in their everyday lives.

Pizza

Select from White or
Traditional Styles

Personal	$3.60
Each Additional Topping	$.60
Small	$4.80
Each Additional Topping	$.90
Large	$6.50
Each Additional Topping	$1.25

TOPPINGS...

Pepperoni • Onions
Fresh Mushrooms • Sausage
Tomatoes • Black Olives
Anchovies • Broccoli
Artichoke Hearts
Spinach • Four Cheeses
Peppers (red & green)
Eggplant

Fresh Pasta

Served with House Salad (or)
Soup and Italian Bread.

Marinara	$5.50
Bolognese (meat sauce)	$6.25
Aglio & Olio (oil & fresh garlic)	$6.00
Gnocchi	$5.75
Pasta Primavera	$6.25
Linguini with Clams	$5.50
Tortellini in Cream Sauce	$6.50
Penne Arrabiata	$6.25

(Meatballs or Sausage with Any Above $1.25)

Char-Broiled Specials

Hamburger	$3.00
Cheeseburger	$3.50
Grilled Chicken Breast:	
Plain .. $3.85 Barbecued	$3.95
Marinara and Mozzarella	$4.25
Chiamicello with our Special Marinade	$4.25
Grilled Eggplant with Tomatoes and Mozzarella	$3.45

Beverages

Pepsi • Mountain Dew • Diet Pepsi
Slice • Root Beer • Iced Tea

Small ...... $.75 Large	$.95
Extra Large (take-out)	$1.35
Pitcher	$2.50
Coffee (regular & decaffeinated)	$.65
Tea	$.65
Juices & Sparkling Water	$1.00

THE MEANING OF FRACTIONS

Teaching Approach

The activities illustrating fractions are based on a manipulative called Fraction Circles (or Fraction Squares--the two can be used interchangeably) which is manufactured and distributed by the Ideal School Supply Company. Worksheets are included which will guide learners in the discovery of each concept involved in working with fractions. A worksheet could be used by an individual; however, these activities are ideal for small group work. Giving each small (three to five) group of learners one copy of the worksheet and one set of Fraction Circles to share will counteract the tendency of learners to work individually.

Proper Fractions as Part of a Whole

The first activity illustrates the meaning of fractions, seen as parts of one "whole". Give each small group a box of the Fraction Circles and a copy of the worksheet entitled "Proper Fractions as Parts of a Whole" (see page 45). Give them time to guide themselves through the activity. Encourage "faster" learners to take care not to "lose" other members of their group by explaining that the assignment is to be sure that every member of the group understands what is going on. Encourage them to move through the assignment slowly, paying careful attention to "how it looks".

After learners are finished, review the activity to be sure all the critical ideas were discovered. These include the fact that fractions, unlike whole numbers and decimals, are symbolized by two numbers. Each of the two numbers stands for something different and it is the relationship between the two numbers that defines the fraction.

With some prompting, learners should be able to tell you that the bottom number (the technical term, denominator, is not especially important at this point and is frightening to many learners) stands for the number of pieces the whole is divided into. It therefore represents the size of the piece.

PROPER FRACTIONS AS PARTS OF A WHOLE

NOTE: ANSWERS MAY BE EXPRESSED IN NUMBERS **OR** WORDS. THERE MAY BE MORE THAN ONE RIGHT ANSWER FOR SOME ITEMS.

1. Find the one piece that completely fills the frame. This is "one" or "the whole".

2. Find the two pieces that, together, fill the frame completely.
   What is the name of one of these pieces?

3. Find the three pieces that, together, fill the frame completely.
   What is the name of one of these pieces?
   What could you call two of these pieces?

4. Find the four pieces that, together, fill the frame completely.
   What is the name of one of these pieces?
   What could you call two of these pieces?
   What could you call three of these pieces?
   What could you call four of these pieces?

5. Find the six pieces that, together, fill the frame completely.
   What is the name of one of these pieces?
   What could you call three of these pieces?
   What could you call five of these pieces?

6. Find the eight pieces that, together, fill the frame completely.
   What is the name of one of these pieces?
   What could you call three of these pieces?
   What could you call six of these pieces?

This is a good place to explore the denominator's effect on the value of the fraction. Ask learners, "What happens when the bottom number gets larger? Does the fraction get larger or smaller?" Many will guess "larger". Demonstrate that the fraction gets smaller. Consider the difference between 1/2 and 1/4. Only the bottom number has changed (gotten larger). Encourage learners to look at the one-half next to the one-fourth. They will readily see that 1/2 is larger. One way to explain the "sense" of this is to remind them of the following: four people will each get a smaller piece of pizza than if that same pizza is divided between two people.

Next, explore the symbolism of the top number (numerator). Most learners, when prompted, will be able to tell you that the top number stands for the **number** of pieces of a given size. Again, explore the effect of changing the size of the top number. Consider, for example, the difference between 1/3 and 2/3. It will be clear to most learners that, if the size of the piece (bottom number) does not change, then more of those pieces is a larger amount and fewer of them is a smaller amount.

If learners appear to have a firm grasp of these concepts, you might choose to proceed directly to the next activity, "Fractions as Part of a Group" before doing the follow-up work listed below. If not, stop now for reinforcement and practice.

Follow-Up

1. Several concrete methods of illustrating proper fractions as part of a whole, using a variety of manipulative materials, are described in <u>Number Sense Teacher's Resource Guide</u> (Contemporary, 1990) on pages 76 - 78.

2. <u>Number Sense: The Meaning of Fractions</u> (Contemporary, 1990) has several pages of semi-concrete work about proper fractions as part of a whole on pages 1 - 4.

Fractions as Part of a Group

This next activity explores another way to look at fractions: as part of a group. It also introduces the idea of improper fractions, mixed numbers, and the calculations needed to convert one to the other. Because more different ideas, and more difficult ideas, are being explored, learners may need help to complete the worksheet.

This is the one fraction activity that uses toothpicks rather than the Fraction Circles. Give each working group one copy of the worksheet, "Fractions as Part of a Group" (see page 48) and fifteen toothpicks. Give them time to complete the activity, but stay close by in case they need some help.

The first idea that learners will encounter is that of fractions as part of a group. Some will stumble on the first challenge, dividing eight toothpicks into two groups. Forgetting what they know about the meaning of a denominator, they may identify one group of four as 1/4 rather than 1/2. Generally, once over this hump, they will be able to proceed through the next two steps, which explore fourths and eighths. This last step, identifying one toothpick as 1/8, is crucial to the rest of the activity.

The next idea is that of improper fractions, a rather poor term, since they are no more or less "proper" than any other form of fraction. Many learners will stumble here (step 4 on the worksheet), when asked to name 12 toothpicks, where eight is still the "whole". If help is needed here, ask learners "what is the value of one toothpick, where eight toothpicks equal the whole?" Ask for the value of two toothpicks (2/8), then three (3/8), then four (4/8), until they understand that 12 toothpicks must be 12/8, or 12 of the eighths.

The second part of step 4 asks for "another name" for the twelve toothpicks. This is **not** a request for reducing, which will be covered in the next activity, although 6/4 (or 3/2) is not a "wrong" answer. It is an effort to have learners discover that eight of the toothpicks could be grouped together as "1", leaving 4/8, and that one could call the 12 toothpicks 1 4/8.

FRACTIONS AS PART OF A GROUP

1. Count out eight toothpicks. This is the "whole".
   Divide the toothpicks into two groups.
   How many toothpicks are in each group?
   What could you call one group?

2. Divide the whole (eight toothpicks) into four equal groups.
   How many toothpicks are in each group?
   What could you call one group?
   What could you call two groups?
   What could you call three groups?
   What could you call four groups?

3. Divide the whole into eight equal groups.
   What could you call (or, what is the value of) one toothpick?
   What could you call four toothpicks?

4. Add four toothpicks to the original eight. Eight is still the "whole".
   If eight is "one", what could you call twelve toothpicks?
   Can you think of another name?
   HINT: REMEMBER WHAT THE VALUE OF ONE TOOTHPICK IS.

5. Add two more toothpicks. Eight is still the whole.
   What different names could you use for 14 toothpicks?

6. Add one more toothpick; you should now have 15 toothpicks.
   WE'RE GOING TO CHANGE THE DEFINITION OF A WHOLE!
   Five toothpicks are now "one whole".
   How many "ones" do you have?

7. Remove three toothpicks.
   Where five toothpicks equal "one", what could you call 12 toothpicks?
   What else could you call it?
   HINT: CAN YOU FIGURE OUT THE VALUE OF ONE TOOTHPICK?

8. Remove four toothpicks.
   What would you call eight toothpicks where five equals "one whole"?

If learners are not clear about the relationship between these two expressions, move the toothpicks back and forth several times, from 12 single toothpicks to the grouping together of one whole (1 4/8). Given this understanding, learners should be able to proceed through the activity with little or no assistance.

Once learners have completed the activity, demonstrate the calculation involved in converting a mixed number to an improper fraction and vice versa, explaining the connection between the number symbols and the toothpicks. An example is shown below:

12/5 = ?	This is one of the problems that was illustrated in the above activity. Learners will probably remember that the first step was to determine how many "whole" groups
12/5 = 2 and ?	of five could be formed. The question, "how many groups of five are there in twelve" is another way to state 12 divided by 5. The answer, of course, is two groups. The
12/5 = 2 2/5	remainder, 2, represents the two toothpicks that were "left" after gathering the two wholes together. Each of them had the value of 1/5, together this remainder has the value of 2/5.
2 2/5 = ?	To write a mixed number in the form of an improper fraction, the question is "how many fifths are in 2 2/5"
2 2/5 = 10/5 + ?	The first step is to discover how many fifths are contained in the two wholes. Each whole contains five of the fifths, by definition, and we have two of them. Two sets of five is another way of saying two times five. By multiplying the denominator by the number of wholes,
2 2/5 = 10/5 + 2/5	we find there are 10/5 in two. Finally, we add in the 2/5 and discover that altogether there are 12 of the fifths represented in 2 2/5.

If learners seem to be having a difficult time with this, do several examples on the board or on paper, while they work the same problem with toothpicks.

Follow-Up

1. Number Sense: The Meaning of Fractions (Contemporary, 1990) offers semi-concrete "paper and pencil" work on fractions as part of a set, improper fractions, mixed numbers and comparing fractions on pages 5 - 17.

2. Most G.E.D. and pre-G.E.D. math books offer some written work on the meaning of fractions.

EQUIVALENT FRACTIONS

Learners typically bring a great deal of fear and shame to this subject, and it needs therefore to be approached with care. Learners need to be reassured that reducing fractions to lowest terms is **not** a moral imperative. In fact, reducing fractions is not at all useful in performing calculations or solving problems, but is simply a means of expressing the **answer** in a more convenient (but mathematically insignificant) way. It is important to remember that the G.E.D. test is a multiple choice test; therefore, while it is critical that learners understand the concept of equivalent fractions and come to recognize common equivalent fractions, reducing per se is not at all important.

This activity, like the first fraction activity, uses the <u>Fraction Circles</u>. Begin by giving each working group (or individual) a box of <u>Fraction Circles</u> and a copy of the worksheet "Raising/Reducing Fractions". Give the groups time to work through the activity.

Following the activity, see whether any of the learners have discovered the calculations for raising or reducing a fraction: multiplying or dividing by the **same** number. If they haven't, review the idea of proportionality that was introduced in the chapter on "Decimal Division". A fraction is a division problem: 1/2 means one (one whole) divided by two. It therefore follows the same rule as any division problem. One may multiply or divide both parts of the problem by the **same number** without changing the relationship between the numbers. Here is an example:

The first thing we do when asked to reduce 4/8 is to look for one number which can be divided **evenly** into both parts of the fraction.

$$\frac{4}{8} =$$

One might use the number four. 4 divided by 4 is equal to one. Eight divided by four is equal to two.

$$\frac{\cancel{4} \div 4}{\cancel{8} \div 4} = \frac{1}{2}$$

RAISING/REDUCING FRACTIONS

1. Before you begin these exercises, identify the different pieces:
   Each red piece is _______ Each green piece is _______
   Each blue piece is _______ Each purple piece is _______
   Each yellow piece is _______

2. Place a one-half piece on the table.
   How many quarters cover one-half entirely?__________
   1/2 = ?/4 ________
   How many sixths cover one-half entirely? ___________
   1/2 = ?/6 ___________
   How many eighths cover one-half entirely? ________
   1/2 = ?/8 __________

NOTE: WHEN YOU USE MORE PIECES (OR, HIGHER NUMBERS) TO EXPRESS THE SAME FRACTION, IT IS CALLED "RAISING" THE FRACTION TO HIGHER TERMS. THIS IS BECAUSE THE **NUMBERS** ARE HIGHER, ALTHOUGH THE **VALUE** HAS NOT CHANGED.

3. Arrange six eighths in the frame.
   Can you cover it with fourths? _______ How many?________
   6/8 = ?/4 __________

4. Arrange four sixths in the frame. Can you cover it exactly with another size piece? ___________ 4/6 = __________

NOTE: EACH TIME YOU USE FEWER PIECES TO EXPRESS THE SAME FRACTION, (OR, LOWER NUMBERS), IT IS CALLED "REDUCING" THE FRACTION TO LOWER TERMS. THIS IS BECAUSE THE **NUMBERS** ARE LOWER, ALTHOUGH THE **VALUE** HAS NOT CHANGED. WHEN YOU HAVE USED THE FEWEST POSSIBLE PIECES TO COVER AN AREA, YOU HAVE "REDUCED" THE FRACTION TO "LOWEST TERMS".

5. Use the pieces to **reduce** the following fractions:
   4/8 = 3/6 = 2/8 =

6. Use the pieces to **raise** the following fractions:
   1/3 = 2/4 = 1/2 =

One might also use the number two. Dividing each part of the fraction 4/8 by two would give the result 2/4. This could then be reduced by two a second time, giving the result 1/2. The only difference is that, in the second case, two calculations have to be done rather than one. It is reassuring to learners that it is equally valid to divide by a small number (2 or 3) several times as to try to figure out the largest possible divisor.

Raising a fraction is similar, and will become necessary when adding fractions with unlike denominators. An example of raising fractions is shown on page 57.

Follow-Up

1. The <u>Number Sense Teacher's Guide</u> (Contemporary, 1990) offers additional concrete demonstrations of equivalent fractions on pages 79 - 80, which may be helpful for learners who are having a difficult time with the concept.

2. <u>Number Sense: The Meaning of Fractions</u> (Contemporary, 1990) offers semi-concrete work on equivalent fractions on pages 18 - 24, which may be helpful for slower learners.

ADDING AND SUBTRACTING LIKE FRACTIONS

This activity explores adding and subtracting fractions and mixed numbers with like denominators. It includes the concept of borrowing from a whole number in the subtraction of mixed numbers.

Again, begin by giving each working group a box of Fraction Circles and the worksheet titled "Adding and Subtracting Fractions -- Part One"(see page 55). Give them time to work through the activity. Your help may be needed as learners come to the first problem requiring borrowing from a whole number (1 - 1/4, at the end of the first row of problems). The way to demonstrate this with the Fraction Circles is to begin with the whole, then cover it with four of the fourths. Once learners see that 4/4 has the same value as 1, and can therefore be substituted for it, they will probably not have difficulty solving the problem. All they will need to do is remove one of the four pieces.

When the activity is finished, check to see that learners have discovered why numerators should be added and subtracted, but the denominators remain unchanged. The answer will vary, but should include the fact that the denominator stands for the size of the piece, and the size doesn't change if one adds more pieces the same size or removes pieces of the same size.

Learners will need an explanation of the symbols for "borrowing" from a mixed number to solve a problem like 1 - 1/4. For an example, let's consider the problem:

4 1/4 - 1 3/4	Learners will quickly recognize that they will need more fourths to solve this problem. Most will understand that the place to get them is from the "4".

4 1/4 = 3 + 4/4 + 1/4 - 1 3/4 = 1 3/4	Explain that you will use only one of the wholes. See if learners can tell you that one whole will be equal to 4/4 and that after you cut up one of the wholes into fourths, there

will only be three of them left. That 4 1/4 is now written as 3 wholes plus four fourths (obtained from the whole that was cut up) plus the one fourth we began with.

ADDING AND SUBTRACTING FRACTIONS
PART ONE

1. Before you begin these exercises, identify the different pieces:

   Each red piece is __________ Each green piece is _______

   Each blue piece is __________ Each purple piece is _______

   Each yellow piece is ________

2. Use the fraction pieces to explore different ways to express the following:

   2/6 = _________ 2/4 = ____ = ______= ________ 2/8 = _______

3. Place one third in the frame. Then add another third. How much do you have? ____________________

4. Place one sixth in the frame. Then, add two more sixths.

   1/6 + 2/6 = _________ Can you call this by another name? _____

5. Place six eighths in the frame. Then, remove three of them.

   6/8 - 3/8 = ________

6. Use the fraction pieces to solve the following problems:

1/4	4/6	5/8	2/3	1 1/3	1/2	1/2	1
+ 2/4	- 1/6	+1/8	-1/3	+ 1/3	- 1/2	+ 1/2	- 1/4

3/4	1 3/4	2/6	7/8	1
- 1/4	- 1/4	+ 2/6	- 3/8	- 2/3

$4\ 1/4 = 3\ 5/4$	When we combine all the fourths together, we see that we have five of them. Now we can subtract.
$\underline{-\ 1\ 3/4} = \underline{1\ 3/4}$	

Follow-Up

1. The Number Sense Teacher's Guide (Contemporary, 1990) describes one conceptual demonstration of adding and subtracting like fractions on page 88, which may be useful for any learners who did not achieve a complete understanding of this from the activity above.

2. Number Sense: Fraction Addition & Subtraction (Contemporary, 1990) offers simple, semi-concrete work on addition of like fractions on pages 1 - 5 and on subtraction of like fractions on pages 27 - 30.

3. All G.E.D. and pre-G.E.D. math books offer some pages of drill on these skills.

ADDING AND SUBTRACTING UNLIKE FRACTIONS

This activity illustrates finding a common denominator. It will be more effective and less threatening if the words "common denominator" are not used prior to the activity, as this is another lesson to which many adult learners bring a history of perceived failure. Simply distribute one box of <u>Fraction Circles</u> and one copy of the worksheet titled "Adding and Subtracting Fractions -- Part Two" (see page 58).

Give learners plenty of time to experiment with this one. Many will falter, initially, in their efforts to solve the first problem with unlike denominators (1/2 + 1/3). The apparently unsuccessful solutions they try will still provide them valuable practice in exploring what will "work" (and what won't). Many learners will notice that, if they cover the 1/2 piece with three of the 1/6 pieces, and cover the 1/3 piece with two of the 1/6 pieces, the problem is quite easy to solve. If they can't figure that out and seem to be getting frustrated, suggest, "Is there any one size piece that will cover each of these pieces exactly?"

Once all have completed the activity, help them connect the solving of the problems with the number symbols:

$$\frac{1}{2} = \frac{\quad}{?}$$

$$+\frac{1}{3} = \frac{\quad}{?}$$

The first step, with the <u>Fraction Circles</u>, and with number symbols, is to discover what size piece, or what denominator will "work" for both fractions. There is nothing special, mathematically, about the lowest one, except that calculations are a bit simpler. A denominator that will work, like 12, but which is not the lowest one, will still yield a "right" answer, but it may need to be reduced.

$$\frac{1}{2} = \frac{?}{6}$$

$$+\frac{1}{3} = \frac{?}{6}$$

Learners are frequently at a loss as to where to start. The simplest way to find a denominator that will always work is to multiply the two denominators together. In this example, the answer, 6, also happens to be the lowest common denominator.

ADDING AND SUBTRACTING FRACTIONS
PART TWO

1. Before you begin these exercises, identify the different pieces:
   Each red piece is ________ Each green piece is __________
   Each blue piece is ________ Each purple piece is __________
   Each yellow piece is ___________

2. Use the fraction pieces to explore different ways to express the following:
   4/8 = _____ = _____ = ______ 4/6 = ______ 1/4 = ______

3. Use the fraction pieces to solve the following:

3/8	5/6	1 3/4	1 1/3	1/2	5/8	1
+ 2/8	- 2/6	- 2/4	+ 1/3	+ 1/2	- 5/8	- 2/3

4. Use the fraction pieces to solve the following:

1/2	1/2	3/4	1/2	2/3	1 1/3	1 1/2	1 3/4
+ 1/3	+ 1/4	- 1/8	- 1/4	- 1/2	- 1/6	- 1/8	+ 1/8

$$\frac{1}{2}\frac{(\times 3)}{(\times 3)} = \frac{3}{6}$$

$$\frac{1}{3}\frac{(\times 2)}{(\times 2)} = \frac{2}{6}$$

$$+ \quad \frac{5}{6}$$

The next step, determining the numerators, uses the previously learned skill of raising fractions. Ask learners, "What did we multiply the 2 by to get a bottom number of 6?" When they respond, "3", ask, "If we multiply the bottom number by 3, what do we need to do to the top number to keep the value of it the same?" Learners should recognize that the answer is that one must do the **same** thing to the top and bottom number, in this case, multiply by 3. Repeat the prompting, if necessary, for 1/3.

Most learners will be able to repeat this process for other problems, although, like all new skills, they will need practice before they can do so with ease.

Follow-Up

1. Number Sense: Fraction Addition and Subtraction (Contemporary, 1990) offers simple, step-by-step exercises in adding and subtracting with unlike denominators on pages 20 - 24 and pages 31 - 33. Subtraction of fractions requiring borrowing is covered on pages 34 - 44.

2. Again, drill can be found in any G.E.D. or pre-G.E.D. math book.

3. Before proceeding to multiplication and division of fractions, it will probably be worthwhile to have learners practice their skills in adding and subtracting fractions by applying them to word problems. These can be found in all A.B.E. and G.E.D. Math books.

MULTIPLICATION WITH FRACTIONS

As was true for multiplying and dividing with decimals, multiplying and dividing with fractions are much simpler to do than to understand. In particular, multiplication is simplicity itself: all one does is multiply the numerators together, then multiply the denominators together.

$$\frac{3}{5} \times \frac{3}{7} = \frac{9}{35}$$

3/5 X 3/7 is equal to 9/35. Multiplying the two numerators together (3 X 3) gives us the numerator 9. Multiplying the two denominators together (5 X 7) gives us the denominator, 35.

Cancelling is yet another application of proportionality and actually consists of reducing, in advance. Again, one can divide a top and a bottom number by the same number. Here, learners discover that any one top and any one bottom number may be divided by the same number to make the multiplication easier. Cancelling is optional, of course; one would still get the "right" answer if it were not done.

$$\frac{2}{3} \times \frac{5}{8} =$$

Looking at the problem, we may notice that 2 and 8 can both be divided by 2.

$$\frac{\overset{1}{\cancel{2}}}{3} \times \frac{5}{\underset{4}{\cancel{8}}} = \frac{5}{12}$$

After we do that, the multiplication is a bit easier. If we hadn't cancelled, we would have gotten 10/24 for the answer, which can be reduced to 5/12.

The only other work which may be required is to write any whole or mixed numbers to their fractional forms. Learners have previously learned to convert mixed numbers to improper fractions, but may need review of the skill. A whole number is written as that number divided by (or "over") one: 12 would be written as 12/1.

One thing about fraction multiplication which can be confusing for learners is one of language in word problems. That is, that a fraction **of** some amount represents a multiplication problem. Remind learners that "5 **of** some number" represents multiplication and so does "3/4 **of** some number".

Follow-Up

1. Two concrete models of multiplication of fractions are described in the <u>Number Sense Teacher's Guide</u> (Contemporary, 1990) and may be useful for learner's who are curious or confused.

2. Semi-concrete and simple step-by-step practice is offered in <u>Number Sense: Fraction Multiplication</u>, which may be useful for slow math learners.

DIVISION WITH FRACTIONS

While this operations is very difficult to demonstrate, and has traditionally confused learners, it need not do so if learners thoroughly understand fractions and the operations of multiplication and division. It is a good idea to begin with a review of two already learned ideas: first, that multiplication and division are opposites of each other and, second, that in a problem written in the form "24 ÷4", the first number (in this case, 24) represents the whole which must be split up and the second number (in this case, 4) represents the "action", the dividing.

When one inverts the "action number" (in the above example, 4 would become 1/4) one is then performing the opposite action. In other words, 24 **X** 1/4 yields the same result as 24 divided by 4. This is the idea behind dividing by a fraction: once one inverts the second number, one can multiply and have the answer. The same rules apply to dividing with fractions as apply to multiplication: whole and mixed numbers must be written in fractional form and cancelling is acceptable **after** the divisor has been inverted. An example is shown below:

3 2/3 ÷2/3	The first step is to write the mixed number, 3 2/3, as a mixed number, 17/3.
$\frac{17}{3} \times \frac{3}{2}$	After that, we invert the action number, 2/3 and change the operation to multiplication.
$\frac{17}{\cancel{3}_{1}} \times \frac{\cancel{3}^{1}}{2} = \frac{17}{2}$	We can cancel, if we like, in order to avoid having to multiply by 17. We divide each of the threes in problem by three and the solution becomes clear. The answer is 17/2, or 8 1/2.

Once this has been explained, along with an example or two, most learners are able to proceed on their own. Like all new skills (and especially ones like this where several steps are required), practice is necessary.

Follow-Up

1. The Number Sense Teacher's Guide (Contemporary, 1990) does give two pictorial models of fraction division on page 100, which may be useful for students who are particularly curious or particularly confused.

2. Number Sense: Fraction Multiplication & Division presents semi-concrete and very simple step-by-step work, which will be useful for slow math learners.

3. G.E.D. and A.B.E. math books will contain problems involving division with fractions, 'ncluding word problems.

4. Before proceeding to the next topic, it would be a good idea to review all operations with fractions (and possibly operations with decimals, as well). Mixed word problems with fractions, with decimals, and with both can be found in most math books. Additional problems, including word problems in the format of the G.E.D. test can be found in the G.E.D. Mathematics Exercise Book (Contemporary, 1988), on pages 24 - 29, and The Exercise Book for the Mathematics Test (Cambridge, 1987), on pages 11 - 18.

6. Cookbooks are excellent sources of "real-life" fraction problems. Most ingredients are measured in fractions of cups and fractions of teaspoons or tablespoons. Making more than one "batch" of something would require multiplication of these fractions and dividing a recipe would require division. Learners could be divided into groups and given a cookbook or xeroxed page(s) from one. Each group could then be challenged to create one or more word problems for the other group(s) to solve. The only rules are that the problem must contain at least one fraction and that the group who makes up the problem(s) must be able to solve it (them).

ALGEBRA, SECOND ACTIVITY

This is a good place to continue with algebra. Learners now have the skills necessary to solve problems involving multiplication and division. Algebraic notation of multiplication and division is slightly different from what learners are familiar with. Learners will need to be informed that the "times symbol" (X) is not used. Instead, multiplication is indicated by the appearance of a number right next to an unknown or right next to some amount in parentheses: 3b or 3(15). The "sense" of this is that 3 fifteens is the same thing as 3 times fifteen. Division is indicated as a fraction: t/4 means "some number divided by four".

Review the introductory algebra lesson on page 13, or do that lesson now if it was not done previously. Remind learners that multiplication and division are opposites of each other. Just as subtraction was used to solve an addition problem, so division is used to solve a multiplication problem. Just as addition is used to solve a subtraction problem, so multiplication is used to solve a division problem. Examples of each are shown below. Remind learners of the principle of balancing equations: whatever is done on one side of the equal sign, must be done on the other side as well.

$3n = 27$	Help learners understand the meaning of the equation: three times some number equals 27. Most learners will see that n must equal nine, without solving the problem in a formal way.
$\frac{3n}{3} = 27$	Ask how they could come up with nine and many will figure out that nine is 27 divided by 3. Remind them about subtraction being used to solve an addition problem, and confirm that division is used to solve this multiplication problem. The reason it works is because 3/3 equals one, so the expression 3n/3 becomes 1n, and one times any number equals that number.
$\frac{3n}{3} = \frac{27}{3}$	In order to keep the equation balanced, or true, we must also divide 27 by 3. Having performed the division on both sides, we have our answer: n = 9.

$\frac{g}{8} = 4$	A division problem is similar. Again, begin by making sure that learners understand the meaning of the equation: some number, divided by 8, equals four. Again, many learners will know that the answer is 32 without any formal solving, and some may be able to figure out that this is achieved by multipliying four by eight.
$\frac{8(g)}{8} = 4$	Confirm that this is the case, and again show the "sense" of it: 8/8 equals one, so once again we have one times g or simply g.
$\frac{8(g)}{8} = 8(4)$	As always, what is done to one side of the equation must be done to the other, so we are left with our answer: $g = 32$.

Follow-Up

Algebra: Jumbo Yearbook (ESP Publishers, Inc.) offers black line masters in all aspects of algebra. Numbers 3 and 4 will give learners practice in writing algebraic expressions using multiplication and division. Numbers 31 and 32 offer practice in solving simple equations with multiplication and division. G.E.D. Math books also offer this kind of equation.

PROBABILITY, RATIO, AND PROPORTION

Probability, ratio and proportion problems are merely additional forms of operations with fractions. Learners who have worked through the previous activities on fractions should not have difficulty with these problems. They require no special conceptual presentation: all the concepts involved have been presented already. Learners should be able to follow the explanations and work on this material in any of the G.E.D. books. If a learner is struggling, the Number Sense: Ratio and Proportion (Contemporary, 1990) book, with its simpler, slower presentation may be helpful.

While there are very few problems on the G.E.D. that specifically require these skills, there are two compelling reasons to teach them at this point. One is that they serve as an excellent application and reinforcement of the recently learned fraction skills. The second is that proportions offer an excellent method of problem-solving for a large variety of problems, including some "algebra" and "geometry" problems. The use of proportions is an especially useful way to solve percent problems; therefore, it is very helpful if learners understand proportions before moving forward into percents.

PERCENTS

The Meaning of Percents

Learners who thoroughly understand place value and decimals will not struggle very much with percents. Pictorial explanations, such as are provided in Number Sense: The Meaning of Percent are very helpful. These relate percents to previously learned ideas, such as ratios, decimals, and fractions. It is worth spending a fair amount of time working through the meaning of percent. Having done so, performing problems with them will not be difficult. It is crucial that some of this work focus on defining the aspects of which part of a problem is the "whole", which the "part", and which the "percent".

Percent Problems

One of the most conceptually valid ways to solve any percent problem is to use a proportion to do so. The concept is this: that, by definition, a part of some whole is equal to the percent/100. Or, in mathematical terms:

$$\frac{\text{part}}{\text{whole}} = \frac{\%}{100}$$

In other words, the definition of percent is, in itself, a statement of proportion. Learners who can readily define the part, the whole, and the percent, and who have mastered the solution of proportion, should be able to solve virtually any percent problem without further explanation. This is the method presented in G.E.D. Mathematics (Contemporary, 1989).

For students who really cannot master this method, it may be useful to try the "percent circle". This is something of a gimmic: an aid to memory rather than a conceptually based approach. It is thoroughly covered in Number Sense: Percent Applications (Contemporary, 1990).

"HIGHER MATH"

Learners who have obtained a thorough understanding of the material presented thus far will score well on the G.E.D. test. Although there are a few problems on the test which apparently require more advanced skills, many can be solved with proportions. Without higher skills, no learner will get a perfect score on the G.E.D. test; however, a perfect score is rarely the learner's goal.

On the other hand, many learners will have begun to enjoy the study of mathematics by now and will want to study further. They should, at this point, have the tools to do so. Using any G.E.D. math book, they will almost be able to teach themselves, provided a teacher or tutor is available to answer questions. They have a solid foundation for this study, having learned how whole and part numbers work and what algebraic notation means.

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