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ABSTRACT

This document, presented in two volumes, reports on a psychology of mathematics education conference, the theme of which was "Theoretical and Conceptual Frameworks in Mathematics Education." The two volumes include 58 papers, descriptions of 4 poster and 2 video presentations, and reports of and reactions to 2 plenary sessions presented at the conference. A grade (age) level index printed at the end of volume 1 helps readers identify presentations aimed at further exploring concepts and research issues of learners of particular developmental levels. A subject index at the end of volume 1 indexes all papers by topic. (MDH)

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Proceedings of the Thirteenth Annual Meeting

North American Chapter
of the International Group
for the

Psychology of Mathematics Education

Volume 1: Plenary Lectures & Reactions,
Symposia & Papers

October 16-19, 1991
Blacksburg, Virginia, U.S.A.

SE 053 421

VIRGINIA TECH

Proceedings of the Thirteenth Annual Meeting

North American Chapter
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Editor:
Robert G. Underhill

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History and Aims of PME

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the Group and the PME-NA Chapter are:

1. To promote international contacts and the exchange of scientific information on the psychology of mathematics education.
2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians, and mathematics teachers.
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Editor's Preface

The theme for the 1991 PME-NA conference is "Theoretical and Conceptual Frameworks in Mathematics Education." We are fortunate, indeed, to have plenaries on this topic by Drs. Bauersfeld, diSessa and Eisenhart and reactions to the three plenaries by Drs. Peterson, Thompson, and Lester, respectively. The Program Committee expects that these six presentations and the Saturday panel discussion of the topic will create an atmosphere of reflection, examination and discussion on this significant issue.

Three features of these two volumes were organized in such a way as to maximize their usefulness to colleagues. First, Volume 2 was completed first so that it could be mailed one month before the conference to preregistrants. Second, a Grade (Age) Index was compiled and printed at the end of Volume 1 to help people identify presentations aimed at further exploring concepts and research issues of learners of particular developmental levels. And, third, all papers have been indexed by topic in a Subject Index at the end of Volume 1; nearly all papers were indexed twice. The Grade (Age) Levels and subject identifiers are also indicated on the upper right hand corner of the first page of each paper. Because of printing costs and the unavailability of many complete addresses, one address is given for each paper at the end of Volume 1.

I would like to extend special thanks to the members of the Program Committee, especially to my colleague Cathy Brown, for their assistance. I also am most appreciative of the efforts of the many reviewers who helped evaluate proposals in a timely manner and to Tom Hunt, Director of the Division of Curriculum & Instruction for facilitating our work through secretarial and bookkeeping support. And, finally, I'd like to express my sincere thanks to Paula Buchanan for her very able secretarial assistance this year. My job has been ever so much facilitated by her hard work, competence and good humor.

Robert G. Underhill
September 1991

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NA
Epistemology
Algebra Translation

THE COGNITIVIST CARICATURE OF MATHEMATICAL THINKING: THE CASE OF THE STUDENTS AND PROFESSORS PROBLEM

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Abstract

The Students and Professors Problem *has been researched extensively for over a decade by cognitive scientists. This paper examines some of the epistemological assumptions embedded in cognitive science methods, and presents data to suggest that these assumptions are unproductive for understanding the source of the* reversal error.

Noddings (1990) distinguishes between constructivism as a cognitive position and as a methodological perspective:

> As a cognitive position, constructivism holds that all knowledge is constructed and that the instruments of construction include cognitive structures that are either innate (Chomsky, 1968; 1971) or are themselves products of developmental construction (Piaget, 1953; 1970a; 1971a). (Noddings, 1990, p. 7)

Thus cognitive constructivism is a broadly inclusive movement admitting of divergent views as to the nature cognitive structure. Methodological constructivism is more restricted:

> As a methodological perspective in the social sciences, constructivism assumes that human beings are knowing subjects, that human behavior is mainly purposive, and that present-day human organisms have a highly developed capacity for organizing knowledge (Magoon, 1977). These assumptions suggest methods--ethnography, clinical interviews, overt thinking, and the like. (Noddings, 1990, p. 7)

The cognitive science method of fine-grained analysis of clinical interview transcripts is the principal vehicle for the now dominant constructivist psychology of mathematics (Schoenfeld, 1987). But this is the restricted *methodological* constructivism that assumes "human beings are knowing subjects, that human behavior is mainly purposive." Such assumptions help us to refabricate the psychology of mathematics as a foundationally rational domain, but they don't much help us to help children learn mathematics.

The case in point is the now famous *students and professors problem:*

> *There are six times as many students as professors at this university.* Write an equation to represent this statement using S for the number of students and P for the number of professors (Clement, 1982).

The resulting *reversal error* ($6S = P$, instead of $6P = S$) is one of the most highly investigated phenomena in the recent history of mathematics education research.

The cognitivist approach to understanding the reversal error has been to closely observe subjects, successful and unsuccessful, as they grapple with the above (or similar) problems. Transcripts of the sessions are minutely analyzed to track the problem-solving processes employed. For instance Clement (1982) has identified three kinds of solving strategies. The *word order match* strategy is to "simply assume that the order of the key words in the problem statement will map directly into the order of symbols appearing in the equation" (pp. 18-19). The second errant strategy, *static comparison*, recognizes that the conceptual content of the sentence must be accessed. But because of weak or immature notions of variable and equation, the solver is unable to encode his or her concepts in correct algebraic symbolism. Finally the successful solver uses the *operative* approach in which the conceptual content of the sentence is accessed, and the solver understands that his or her role is "*not* [to] describe the situation at hand in a literal or direct manner; ...[but to] describe...an equivalence relation that would occur if one were to perform a particular *hypothetical operation*" (p. 21).

The view of translation skill and its development that emerges from this and other studies can be summarized as follows:

1) Translation from natural language into algebraic language is an inherently semantic/conceptual rather than a syntactic task; hence it is semantic/conceptual difficulties that underlie the reversal error;
2) The syntactic translation strategies that novices apply to word order matching "in part can be described as an overextended application of the representational system of natural language to the formalisms of algebra" (Kaput, & Sims-Knight, 1983, p. 69); and
3) The standard curricular practices that support syntactic translation strategies by presenting techniques of phrase-by-phrase matching (e.g. Brown, Smith, & Dolciani, 1986) are fundamentally misconceived:

> How is it possible for students with such weaknesses to survive high school and college science courses? It appears that these students have developed special purpose translation algorithms which work for many textbook problems, but which do not involve anything that could reasonably be called a semantic understanding of algebra. Many word problems are constructed so that they can be solved through a trivial word-to-symbol matching algorithm.... While these techniques may be partly successful in many classroom situations, they are too primitive and unreliable to be trusted in any but the most routine applications. (Clement, Lochhead, & Soloway, 1980, p. 5)

There are several features of this cognitivist research program that give rise to concerns. Firstly the selection for intensive study of particular problematic translation tasks provides a means for obtaining a snapshot, as it were, of the mind in action. But the cost of this single-minded attention is that the cognitive context of such translation tasks embedded within a school curriculum of other translation tasks remains unstudied (though not unanalyzed. as the above quotation attests).

Secondly, and more alarmingly, the clinical interview is an entirely inappropriate method for investigating phenomena that may be related to syntactic processes --well-known to be unconscious and inaccessible to introspection. Think-aloud protocols only can reveal aspects of thought that are consciously accessible to the informant (Ericsson & Simon, 1984, pp. 14-15). Thus the conclusion of cognitivist research that semantic/conceptual rather than syntactic knowledge underlies successful translation is an artifact of the methods used, rather than a bona fide implication of research.

A SYNTACTIC MODEL OF ALGEBRA TRANSLATION SKILL

For the vast majority of school word problems syntactic methods of phrase-by-phrase translation can successfully be employed. In the simplest case the sentences are immediately *phrase-order-matched* (POM). For example if *J* represents John's height, and *M* represents Mary's height, *[John's height] [is equal to] [six] [times] [Mary's height]* can be translated by substituting the mathematical symbols *J*, =, 6, x, and *M* respectively for the bracketed phrases, without accessing the conceptual content of the sentence.[1]

For a second class of sentences, certain within-phrase adjustments (WPA) must be made prior to applying phrase-by-phrase substitutions. For instance the sentence *The number of diskettes is four less than the number of notebooks* must first be adjusted to *The number of diskettes is equal to the number of notebooks subtract four* before phrase-by-phrase translation can be done. We argue that such within-phrase adjustments can be accomplished without accessing the conceptual content of the whole sentence.

A third class of sentences does require whole-sentence transformation (WST) to become phrase-order-matched. The sentence *The calf weighs four times as much as the pony* has non-quantitative noun phrases (e.g., the calf); the quantitative aspect residing in the verb (weighs). In order to achieve POM form, the quantitative aspect of the verb must be parcelled out to the noun phrases, leaving the denuded "to be" verb form: *The calf's weight is four times the pony's weight.* We argue that such "massaging" of the sentence can be accomplished by syntactic means, without accessing the quantitative relationship between the variables that underlies the conceptual structure of the sentence, and, thus, that these sentences, too, can be translated by phrase order matching.

---

[1] Our position is not that translators read such sentences without understanding, but that their knowledge of the meaning of the sentence is not utilized in the translation process.

The Students and Professors problem is an exemplar of a fourth class of sentences that turn out to be not phrase order matchable (NPOM). Attempts to adjust and transform such sentences fail. For *There are six times as many* *udents as professors* attempts to make the noun phrases quantitative might result in *There are six times the number of students as the number of professors*, but then no syntactic transformations are available to relocate the verb between the noun phrases to enable phrase-by-phrase translation. It should be noted that this sentence does have a POM counterpart, *The number of students is equal to six times the number of professors*, but this form of the sentence cannot be achieved by syntactic transformations; thus reference to the conceptual structure of the sentence is needed in translation.[2]

This analysis of NPOM sentences within a broader context of algebra translation tasks leads to two possible hypotheses:

1) Either translation is inherently a conceptual task --the syntactic correlates and their pedagogical exploitation being nothing but incidental and vexatious distractions; or
2) Algebraic translation is inherently a syntactic task, and the competent translator avoids reversal errors by being sensitive to the partial products of the (syntactic) translation process, abandoning syntactic translation methods in favor of conceptual methods for NPOM sentences only.

Note that the second hypothesis is compatible with previous cognitivist observations of student protocols. That the competent translator first attempts, and then rejects, a syntactic translation strategy might well be introspectively inaccessible information. Similarly the novice translator may fail, not because of immature conceptual structures, but because of a lack of sensitivity to, or lack of security with, the partial products of the syntactic translation process, and thus miss the cue to abandon syntactic processes in favor of conceptual strategies. Following are data relevant to these hypotheses.

EXPERIMENTAL DESIGN, DATA, AND RESULTS

Our subjects in this study were 20 professors, 5 instructors, and 17 graduate students in the Mathematics Department at Louisiana State University. We reasoned that if the syntactic model presented above is correct then the amount of time required to translate sentences of the various types identified ought to vary with the amount of adjustment and transformation required. In contrast, the conceptual model of sentence translation does not predict that the different classes of sentences (controlled for word length) should require different amounts of processing time.

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[2]Actually we have discovered several types of NPOM sentences, each of which fails to be transformable for different reasons.

Following 6 warm-up sentences of the POM and WPA variety (to guard against speed up effects), a random arrangement of 2-POM, 2-WPA, 5-WST, and 5-NPOM sentences were presented individually on a computer terminal to each subject (4 "fillers" and 1 additional NPOM sentence involving an extra operation also were presented, but those data are not reported here). Preceding each sentence were definitions of the two variables to be used, and instructions to translate the upcoming sentence on the answer sheet provided as quickly and accurately as possible, and then to press the space bar. Response times to read and translate each sentence were automatically recorded by the computer.

Of the 1050 (25X42) answers given, there were a total of 61 errors, 46 on NPOM sentences, 2 on WST sentences, 13 on the filler items. Of the NPOM errors, all but two were reversal errors. Over all items, only 16 subjects scored perfectly; 26 made at least one error. Table 1 displays mean response times by problem-type for the 16 subjects with perfect scores (similar results obtain for more inclusive analyses).

Table 1

Mean Latencies (in Seconds) by Problem Type for the 16 Errorless Subjects

POM	WPA	WST	NPOM
9.2	9.4	12.2	16.8

An analysis was performed by aggregating items within problem type, and then performing a within subjects ANOVA on response latencies. Response times on the problem types differed significantly ($F(3,45) = 18.6$, $p < .0001$). Newman-Keuls post hoc analysis showed that NPOM problems took significantly longer than WST sentences ($Q(2,45) = 5.6$, $p < .01$), which took longer than WPA sentences ($Q(2,45) = 3.4$, $p < .05$) and POM sentences ($Q(3,45) = 3.7$, $p < .05$). Response times for WPA and POM sentences did not differ significantly.

CONCLUSIONS AND IMPLICATIONS

In two respects these results provide support for the syntactic theory of sentence translation. 1) The pattern of response latencies for the different sentence types matches the predictions of the syntactic theory. There would be no reason to expect such response-time differences if translation were based on conceptual analysis of sentences. 2) The high frequency of reversal errors by fully accomplished mathematicians and graduate students makes it almost inconceivable that translation is a purely conceptual task. There is nothing conceptually difficult about sentences like *There are six times as many students as professors*, even for novices (Wollman, 1983). It is much more reasonable to presume that some subtle, internal cue to access the conceptual content of the sentence has been missed.

It might be concluded that the data reported here shed new light on the translation process that can inform future research and theory. But this conclusion misses the major intention of the paper. Mathematically competent people are able to accomplish a variety of translation tasks including NPOM translations which require ultimate reference to the conceptual structure of the sentence, as well as simple POM, WPA, and WST sentences which could (in principle) be dealt with by either syntactic or conceptual means. But there is another class of complex (COMP) sentences involving multiple operations (the "filler" sentences in the instrument described above) that are easily translatable, but which may be essentially incomprehensible. For instance, sentence like *John's weight in pounds is five more than two-thirds of three more than twice the square of two less than half the cube root of Bill's weight in pounds* are easy to translate into algebraic notation, but it would probably be necessary to do the translation first, and then ponder the resulting equation, in order to be able to conceptualize the relationship between the variables. Thus adopting the position of previous reversal-error research that translation in algebra is essentially a conceptual/semantic task requires disregarding obvious and ordinary facts about translation skill. This testifies to the epistemologically-bounded nature of the theorizing that has informed reversal error research thus far, and, as argued by the first author (Kirshner, 1989a, 1989b), is part of a more general program in the psychology of mathematics to idealize algebra as a domain of rational intellection.

Pedagogically this is a matter of no small importance. Reflecting the influence of cognitivist research, the recent NCTM *Standards* calls for decreased attention to routine word problems (NCTM, 1989, p. 127). From a cognitivist perspective this serves to minimize the vexing influence of syntactic factors on the acquisition of conceptual knowledge. But despite numerous attempts to remediate the reversal error by attending to the conceptual deficits identified in cognitivist research, it has been found that "the reversal problem is a resilient one and ... students' misconceptions pertaining to equation and variable are not quickly 'taught' away" (Rosnick & Clement, 1980, p. 6).[3] If, as proposed in this paper, syntactic parsings and matchings practiced in routine word problems are the foundations of translation skill in algebra, then the disappointing results of cognitivist instruction will be replicated nationwide.

[3]An exception is the study by Clement, Lochhead, & Soloway (1980) of the effect of translating word problems in the context of writing computer programs. But they note the different functions of equal signs and variables in computer languages as compared to algebraic language (p. 11).

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COGNITIVE OBSTACLES OF DEVELOPMENTAL-LEVEL COLLEGE STUDENTS IN DRAWING DIAGRAMS

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This paper examines qualitatively some problem-solving processes used by developmental-level college students, in relation to the cognitive obstacles (diagram-drawing, algebraic, and affective obstacles) they encounter.

In this paper, we focus on "cognitive obstacles" - i.e., conceptual blocks to student understanding which can occur during problem solving, which are more than momentary difficulties.

Through carefully structured, individual clinical interviews, we have been able to describe in considerable detail the spontaneously-employed "heuristic subprocesses" used by each of 22 students, when solving an algebra word problem that may entail drawing a diagram. We also observed their responses to minimal structured suggestions, or "hints" (Bodner, 1990; Bodner and Goldin, 1990, 1991). The interviewer's script was modeled structurally on an earlier script developed for studying a different heuristic process, "think of a simpler problem" (Goldin, 1985). It consisted of six sections: (I) Introduction; (II) Understanding the terminology and concepts; (III) Presentation of the problem (*the clinician encourages free problem solving as much as possible, without interruption*):

> The length of a rectangle is one inch greater than twice its width. The perimeter is 26 inches. What are the dimensions of the rectangle?

(IV) Guided use of the heuristic process, "drawing a diagram" (*the clinician offers minimal heuristic suggestions, but only when the student cannot continue*); (V)

Presentation of diagrams (*a last resort, which was not in fact necessary for any of the students*); and (VI) Looking back. Such a structured script allows us to learn first about each student's spontaneously-employed heuristic processes; when an obstacle occurs, we provide just those minimal heuristic suggestions needed to overcome the difficulty, and learn more through observing the student's *subsequent* spontaneous behavior. The interviews were videotaped, and all 22 protocols were transcribed and analyzed. Thus we obtained for each student a detailed sequence of diagram-drawing and related competencies exhibited (a) spontaneously, or (b) only in response to suggestions, or (c) not at all.

As a result of the analysis of student protocols, we identified various kinds of cognitive obstacles, among which three were particularly prevalent:

> (1) diagram-drawing obstacles: failure to *establish or monitor an effective correspondence* between decisions motivated from a diagram (possibly an inappropriately labeled diagram), and the problem statement; this includes use of the diagram only to *organize conclusions* rather than to motivate *strategic decisions*;
>
> (2) algebraic obstacles: misconceptions about symbol meanings; misunderstandings about relationships between variables and equations; and algorithms memorized without understanding;
>
> (3) affective obstacles and unproductive belief systems: emotional interference; lack of confidence in the student's own ability resulting in ineffective executive planning and control; beliefs about "what to expect" that are unrelated to or counter to conceptual understanding.

Only a few representative examples and partial excerpts from protocols are cited here due to space limitations. For the complete transcripts of all 22 students see Bodner, 1990.

Diagram-Drawing Obstacles

The calculation of a semi-perimeter (that is, setting the sum of one length and one width equal to the numerical value of the perimeter) was the most frequently occurring student error. Nine of the 22 students committed this error at some point in their interviews. Those students who drew partially labeled diagrams may have been led to this error by their diagrams, in the absence of effective monitoring. Consider Alice's response. Initially, she had calculated a semi-perimeter and when asked to explain how she obtained her answers she replied:

> Well first you read the problem aloud and try to get an understanding of it and then if you can you try to draw a picture like to help you along. And then I drew the picture and labeled the sides. I just labeled the width 'cause it gave me the length of the other side. So if it's twice the width which is 2X and one inch greater than that is 2X plus one. Then I just worked it out because if you take the length and the width, it's like, let me do it again. . . When I looked back and noticed that I did the problem wrong, because I had only used, like, one side, one length and one side of the width, when it was actually two sides. I forgot about this side and this side. [She pointed to the unlabeled sides of the rectangle.]

Alice herself retrospectively detected her own failure to monitor the correspondence.

Millie's obstacle was that she did not spontaneously draw a diagram when solving the problem. She, too, incorrectly calculated the semi-perimeter. After she had explained her solution, the clinician asked her if she could draw a diagram for the problem. She immediately recognized her error and responded:

> I did that wrong. The perimeter would be the addition of all of these and I only added one of each. The first time I did it, umm, like I told you, I got the length equal to two times W plus one and the width is W and they give you the perimeter, which is the addition of all four sides and I only took one of the length and one of the width. But then when I, I thought I did it wrong, but then when you asked me to write the diagram I really realized what I did wrong, that I had to add two of each. If you have a diagram in front of you, it makes you, you can think a lot easier about the problem. I mean I did it without the diagram the first time and I wasn't thinking right, but as soon as I looked at the diagram and I realized, I mean it made me think right. I mean, I knew that the perimeter is the addition of all four sides but when I just went ahead and did the problem and I didn't have the diagram, I wasn't thinking, I was just doing the problem. And then when I did the diagram, I realized, you know, I just said that the perimeter was the addition of all four sides and then I didn't do it when I did the problem.

The initially unsuccessful students were eventually successful, and recognized their errors either (a) spontaneously while describing to the clinician how they had solved the problem (usually when referring to their diagrams), or (b) when prompted to draw a diagram or to label the diagram more completely. The labeled diagrams seem to have focused the students' attention on *what* they were trying to accomplish, as well as enabling them to accomplish it. Interestingly, none of the five students who labeled all four sides of their rectangles made the "semi-perimeter" error. It appears that diagram-drawing obstacles can be overcome fairly readily through suggestions and hints.

Algebraic Obstacles

One algebraic obstacle for several students occurred in interpreting the phrase "greater than" in the first sentence of the problem statement. Mary repeatedly (four times throughout the interview) wrote $L = 1 > 2W$, using the "greater than" symbol in place of the "plus" sign. Later she realized

> That is wrong. I don't know. I guess I should have a plus because if they're just saying greater and I'm just thinking greater than, less than so. . I can put it in like that.

Other algebraic obstacles involved the writing of equations. Bill attempted to write an equation for the problem by setting the sum of the four dimensions of the rectangle equal to zero. When the clinician asked him about why he wrote this he replied:

> The zero is just to keep the equation kinda' balanced. You just can't have an equal sign without anything there. . . until you put something over there.

He knew that a number of some sort was required, and zero seemed a logical "default" choice to him (instead of the value of the perimeter). Initially Mary also demonstrated considerable confusion over variable and equation usage. Her initial idea seems to have been that one shouldn't have "too many W's". Here she corrects herself:

> 26 equals two plus 4W plus 2W. See, I'll have too many. . . oh wait a minute, never mind. 26 equals two plus 6W minus two is 24 equals 6W. I have a new one now. . . . I don't know why I left it out to begin with because if I have the equation I should have used everything. To begin with I just left that whole step out completely. . . . Because, I don't know it was because I thought that there were too many, it's stupid but, too many W's. Because there was you know what I mean? There was two of them but I forgot it came back that you could add them together again because you know they could be added together.

Mandy eventually wrote a correct equation, $2(1 + 2W) + 2W = 26$; but she did not believe she could solve the problem, that is find the values of the dimensions, using her equation:

> I have no idea like how I could try to find a side like for the length or for the width. . . So, I don't understand how like to use. . . I know how to use equations but I just don't how to use an equation in this way, to find one of the four sides. . . . I'm just not, it just doesn't seem like I solved it correctly because I just took, I just guessed at how to write out the equations using what I was given. . . . Like the way I was able to get like a width out of that. . . it just doesn't seem like I solved the problem because it should have been easier than all these steps.

She had no trouble manipulating the symbols in the equation but she could not recognize the connection between the equation and what the variables represented.

A few students were able to perform the algebra correctly, but their statements evidenced algorithm memorization without full understanding of the underlying concepts. As Vicky stated:

> I took my 2X plus one and I did it times it by two, put parentheses around it 'cause it. . . I don't know. That's what you're supposed to do. . . And then I had a two left over which was with my variables and you can never have variables and constants together so I had to bring the two to the other side and since it becomes negative, I had to subtract it from 26 which gave me 24 and then I took the 6X and divided into 24 and that gave me four.

Likewise, Jane expressed herself as following rules rather than understanding what she was doing conceptually:

> I was always taught that whatever, whenever an equation says "than" whatever's after that goes first so since I know that the width is X, right? And it says twice its width, X is representing the width so I put two in front of the X which gives me. . . and I know that it's one inch greater than twice its width so I put 2X plus one for the length. . . . Then what I did I added the like terms and in front of the X is always the invisible one so that's 3X plus one equals 26. Now you gotta' get the positive one to the other side and in order to do that you have to negative one to both sides. This cancels out and it stays, 3X equals, you subtract here which gives you 25 and you divide by the three. Now your result is eight over 1/3. The reason why it's wrong is because it's a fraction.

Although the use of algebra is correct we still regard these students' "procedural" orientation as a serious cognitive obstacle.

Affective Obstacles and Unproductive Beliefs

Students' feelings about the problem, word problems, and mathematics in general, were also sometimes obstacles. Jane commented:

> I've always had problems with solving problems, these kind of things when I first encounter them. But I don't know, I kind of get like, umm. . . like I don't know I get upset 'cause I see it as a competition. Every time I try to tackle it I can't get through it. . . If I do some certain type of problem and it comes out wrong then I get discouraged and I guess that's one of my weaknesses as well.

Naomi said the problem "wasn't hard but I was aggravated because I couldn't figure it out." After solving the problem she said that she felt "relieved. . . A feeling of success, I mean it's just a little problem, but me and math don't get along so when I get a problem right then I just feel good."

Student expectations and beliefs could also become obstacles. Some of the students verbalized what sort of processes they expected to go through and the sort of answers they would find. For example, Mandy doubted her correct answers:

> I don't think it's right. . . It just doesn't seem like I solved the problem because it should have been easier than all these steps. . . from where I've had things like this in school, it just seems to me as though there's always a simple equation. Then you just plugged in numbers, but this one only gave you one number.

Many of the students expressed themselves as expecting whole number answers to the problem. As Roger explains:

> I figure this is wrong because when I adds this up it does not come evenly and because I had a mixed fraction here. . . When I seen the negative two change the negative two to 26 and I knew the eight, the 8W would go into 24 evenly, three. At that point I knew it was going to be right. This problem is going to work out right, because I figure in these kind of problems that a fraction would not help, unless, you know it's really some kind of significance within this problem.

Jane too (see earlier quote) thought a fraction had to be a wrong answer.

Our results suggest that overcoming these specific cognitive obstacles should be a more explicit goal in developing problem-solving ability in algebra.

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Adults
Evaluation
Teachers' Conceptions

EVALUATION PRACTICES OF SECONDARY MATHEMATICS TEACHERS

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A written survey was used to investigate the evaluation practices and conceptions of 201 secondary mathematics teachers (grades 7-12). About half (48%) reported using primarily tests provided by publishers to evaluate students. Over half(57%) generated computational or single step problems to test deep mathematical understanding. A follow-up survey (N=102) and interviews (N=20) confirmed that many teachers evaluation practices reflect a narrow view of what constitutes mathematical understanding.

There is no question that teachers' evaluation practices influence the nature of mathematical experiences in the classroom. Evaluation sends a powerful message to students, not only about the content that is considered important, but also about the kinds of thinking that are valued. In his review of the impact of classroom evaluation, Crooks (1988) cites the potent effect of students' expectations of what will be tested on their studying and learning:

> Examinations tell them our real aims, at least so they believe. If we stress clear understanding . . . we may completely sabotage our teaching by a final examination that asks for numbers to be put into memorized formulas. However loud our sermons, however intriguing our experiments, students will judge by that examination--and so will next year's students who hear about it. (p. 956)

The purpose of this study is to identify the evaluation practices of secondary mathematics teachers and consider those practices in light of the vision set forth in the Curriculum and Evaluation Standards (National Council of Teachers of Mathematics, 1989). Identifying teachers' conceptions of evaluation can deepen our understanding of important psychological aspects of teaching and learning by exploring what teachers think mathematical understanding consists of, what teachers believe is important for students to know, and what teachers believe students are capable of understanding. This study aims to identify these aspects as evidenced by teachers' evaluation practices and their conceptions about evaluation. Guiding the research are the following questions: (1) What are teachers' conceptions about evaluation and how are those conceptions reflected in their evaluation techniques?, (2) What is the nature of the mathematics that teachers evaluate?, and (3) Do teachers feel comfortable responding to and using open-ended evaluation items and do they appreciate the potential of such items for assessing students?

Influencing this project is a developmental theory proposed by Perry (1970) that suggests how individuals typically view their worlds. It describes a classification of individuals' beliefs about the nature and origin of knowledge and their responsibility toward those beliefs. Two of the general categories discussed by Perry include dualism and relativism. Dualism suggests that every question has an answer, that there is a solution to every problem, and that those in authority should deliver these answers and solutions. A dualist conception of mathematics is reflected in the statement, "mathematics is a fixed set of procedures to be mastered." Relativism on the other hand suggests that observed data should be interpreted in terms of validity and internal consistency, and that validity depends upon context. For example, one solution method to a mathematics problem might be preferred over another because of its computational efficiency, while the first might be easier to understand. An emphasis on mathematical processes such as problem solving, communication, and reasoning reflects a relativist orientation to mathematics and mathematics teaching.

We have little information about what teachers actually consider when evaluating students or what beliefs teachers hold about evaluation. For example, although there is a growing body of literature suggesting that many teachers and students have dualistic conceptions about mathematics and its teaching (Borasi, 1990; Brown, Cooney, & Jones, 1990), we do not know whether such a view is reflected in teachers' evaluation practices. Until such information is available, we have little basis for determining how the vision described in the Standards (NCTM, 1989) can be achieved. This study attempts to provide such information.

METHOD

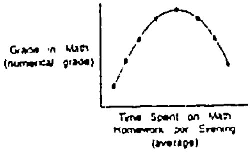
Sample and Instruments

Data about teachers' evaluation practices were obtained using two written surveys and an interview. The surveys were designed by the authors and piloted by teachers at a local high school. The first survey (Phase I) was completed by 279 mathematics teachers participating in summer (1990) inservice mathematics and mathematics education courses at colleges and universities across the state of Georgia (USA). The survey was designed primarily for secondary teachers (grades 7-12), so analysis was restricted to the 201 surveys completed by secondary mathematics teachers. The sample included 45 males (22%) and 156 females (78%). Average teaching experience was 9 years. The majority of teachers (111, 55%) taught at traditional high schools (grades 9-12 or 10-12); seventy-two (36%) taught in middle schools or junior high schools (any of grades 5-9) and 18 (9%) taught in some other type school (e.g., grades K-8 or 6-12). The survey requested teachers to respond to questions in a way that would describe their evaluation practices for their first period course (1989-1990 academic year). Table 1 summarizes the number of teachers (and percent) who reported about each of the various courses listed in the table.

Of the original 201 teachers, 102 (51%) completed a second survey (Phase II). Teachers responded to five non-traditional evaluation items (mathematics problems) that varied in open-endedness; three required some explanation or argument and two asked for the generation of a number. Figure 1 contains each of the five items. Teachers were asked to provide "ideal responses" to the items and indicate both what they thought the items tested and how likely they would be to use such items in evaluating students. A third phase of data collection (Phase III--in progress) includes interviews with 20 teachers that explore in more depth teachers' conceptions of mathematics and

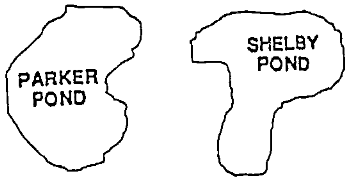
1. A researcher asked many students two questions: "What was your grade on your last math exam?" and "How many hours per night did you usually spend on math homework?" The researcher then sorted students into groups according to how much time they spent on homework. Finally, the researcher computed an average math grade for each of these groups and plotted the averages in the graph below.

   Write a plausible explanation to explain the data.

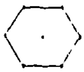
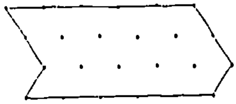


2. Theo wants to find out which pond covers the larger area, Parker Pond or Shelby Pond. He does not need to know the two areas, just which is bigger. Theo claims that all he has to do is measure the distance around each pond to find out what he wants.

   Will Theo's method work? Write a convincing argument for your answer.



3.

 IS ONE UNIT OF AREA. 

Given the unit of area shown above, what is the area of the larger figure? ________________

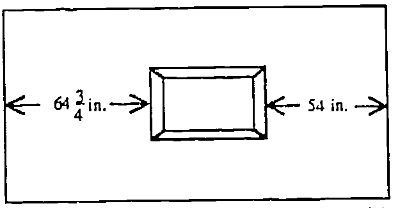
4. Gwen was given the problem 2/5 < ? < 4/7. She said that 3/6 would be between 2/5 and 4/7. The teacher asked Gwen to explain how she got her answer and why she thinks her method works. Gwen said that she chose a numerator of 3 because 2 < 3 < 4 and a denominator of 6 because 5 < 6 < 7. Gwen claimed her method always works and gave the following examples:

   i. The fraction 2/4 is between 1/3 and 3/5 because 1<2<3 and 3<4<5.

   ii. The fraction 4/9 is between 2/5 and 6/11 because 2<4<6 and 5<9<11.

   Does Gwen's method always work? Explain you reasoning.

5.



How far to the left should the picture be moved so that it is centered on the wall?

Figure 1. Items from Phase II of Evaluation Practices Survey

evaluation. One focus of the interviews is a discussion of a typical test used recently by each teacher to evaluate students.

Table 1
Courses Taught

(N=201)	MS	PA	BM	A1	G	A2+
Number of teachers(%)	41(20)	41(20)	27(13)	36(18)	26(13)	29(14)

MS=Middle School Mathematics (grades 7 and 8), PA=Prealgebra, BM=Basic/General Math (grades 9-12), A1=Algebra 1, G=Geometry, A2+=mathematics courses at the Algebra 2 level or above (e.g., Precalculus, Algebra 3, Calculus)

RESULTS

Phase I

Using data from the first survey we attempted to identify how teachers evaluated students, the sources of their evaluation instruments, and what basic conceptions of evaluation teachers seemed to have. Table 2 illustrates how the teachers' evaluation practices related to their grading procedures.

Table 2
Grading Procedures

Source	Final Exam	Unit Tests	Quizzes	Home-work	Notebooks /classwork	Class Part-icipation	Other
Percent	12.6	41.2	17.2	16.6	7.6	3.6	1.3

Almost half of the teachers (N=97, 48%) indicated that the primary source for their unit or chapter tests was textbook publishers. Further analysis led us to categorize two kinds of testers: "external" and "internal." EXTernal testers relied exclusively on external sources (e.g., publishers, local or state boards of education) for tests and final examinations; INTernal testers relied exclusively on tests created by themselves or peers. Of the 201 teachers, 39 (19%) were classified as external and 64 (32%) were classified as internal. As indicated by Table 3, the external testers were concentrated in the middle school courses (MS and PA--67%) while most internal testers taught high school courses (BM, A1, G, A2+--80%).

Table 3
External and Internal Testers

	MS	PA	BM	A1	G	A2+	Total
EXT (N=39)	31%	36%	10%	10%	5%	8%	100%
INT (N=64	8%	12%	11%	25%	19%	25%	100%

Refer to Table 1 for course abbreviation interpretation

Using the survey in Phase I we also investigated how teachers conceptions of mathematical understanding might be reflected in their evaluation practices. Specifically, we were interested in finding out the meaning ascribed by teachers to the phrase "deep and thorough understanding" of mathematics. Teachers were asked to write typical problems they believed would test (1) basic and (2) deep and thorough understanding of a chosen mathematical topic. The problems were classified according to the predicted level of student understanding necessary to complete the problem. The levels used in analysis were (1) recognition or simple computation, (2) comprehension or one step word problem, (3) application or multistep problem, and (4) non-routine or open-ended problem. Each problem was scored independently by two people, disagreements (there were only 20) were scored independently by a third person. Figure 2 illustrates some of the typical responses at each level.

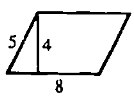
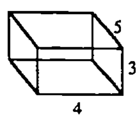
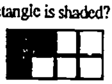
Topic	Level 1	Level 2	Level 3	Level 4
Area	Find the area of a rectangle with a width of 4 inches and a length of 2 inches.	Find the area of the parallelogram 	Find the surface area: 	Draw the floorplan of a house and determine the number of square feet in the house.
Functions	1. Is the relation {(0,5), (1,3), (0,7), (2,4), (3,9)} a function? 2. Graph $y = 2x + 3$	1. Find the equation of the line containing the points (2,3) and (-1,5). 2. Graph $y = (x-3)^2 - 2$	Find the equation of the line through (-2,3) and perpendicular to the line $2y + 5x = 5$.	Write a quadratic function f(x). Write the function that would translate f(x) vertically; horizontally; dialate f.
Fractions	1. $\frac{3}{8} + \frac{1}{4} = ?$ 2. What fraction of the rectangle is shaded? 	Mary and Joe are taking a trip of 80 miles. Mary drove 2/5 of the distance. How many miles did she drive?	Bob ate 1/4 of a peperroni pizza, 2/3 of a cheese pizza, and 1/2 of a sausage pizza. How much of a whole pizza did he eat?	Identify the activities of a typical teenager in a 24 hour period. Graphically represent the fractional parts of a day spent on these activities.

Figure 2. Typical test items generated by teachers

To measure deep and thorough understanding, more than half the teachers (57%) generated problems at either level 1 or level 2. These results were even more pronounced among the following groups of teachers: (1) teachers with less than four years experience (48/69, 70%), (2) teachers of below average students (24/33, 73%), and (3) teachers who generated problems dealing with fractions (40/46, 87%).

Phase II

Table 4 summarizes some of the interesting results from the second survey (Phase II), including teachers' performance on the problems and their indicated likeliness of using the problems to evaluate students (likleliness of use was measured using a five point Likert scale).

Table 4
Summary of Phase II Evaluation Items

(N= 101)	Item 1 (O)	Item 2 (O)	Item 3 (S)	Item 4 (O)	Item 5 (S)
Possible Scores	0-3	0-3	0-1	0-3	0-2
Average Score	2.32	2.31	.95	1.80	1.97
Avg./Maximum	.77	.77	.95	.60	.98
Likely or Very Likely to Use	54%	66%	79%	75%	85%

S=single number requested, O=open-ended response requested

Phase III

To date we have conducted 16 interviews. Detailed analysis has not been conducted on the interviews, but preliminary analysis seems to be confirming and expanding on survey results. Examples appear in the discussion section.

DISCUSSION

Mathematical Understanding

The data suggest that many teachers test a rather limited or narrow range of possible mathematical outcomes. To many teachers deeper understanding simply means solving problems involving more steps, i.e., harder computation. Although most teachers agreed that providing feedback and identifying students' misconceptions are important purposes of evaluation (86% and 88% respectively of the teachers agreed with these purposes), the problems teachers apparently use in evaluating students do not suggest that the kinds of misconceptions identified or the substance of the feedback provided are consistent with current definitions of meaningful mathematical knowledge. These results were confirmed during the interviews in which teachers consistently conveyed the notion that mathematics is a sequence of steps and implied that assessing deeper understanding means providing problems that require more steps to successfully complete. None of the 16 teachers interviewed thus far indicated that deep and thorough understanding could be assessed by asking students to exhibit reasoning abilities beyond the production of a specific answer.

Open Ended Evaluation Items

Results from the second survey further indicate that teachers prefer to use single answer problems as opposed to the open-ended ones to evaluate students (see table 4). Although teachers' perceptions about the nature of meaningful learning probably influences this result, teachers' mathematical ability may also contribute to it. Teachers were less likely to respond correctly to the open-ended items (as opposed to those requiring a single number) on the second survey. If teachers find difficulty in answering items that require the construction of counterexamples or arguments, then they will probably be less likely to engage their students in such activities. There were a variety of reasons expressed by teachers concerning why they would hesitate to use the open-ended items in testing situations, including lack of confidence in answering the questions themselves, and that the problems were too difficult or otherwise inappropriate for their students.

Conclusions

There is a national consensus that assessment should be a vehicle for curriculum reform. Although there is little doubt that tests exert a powerful influence on what teachers teach and students learn, the tests themselves are incapable of carrying the burden of reform. Unless teachers understand that new forms of evaluation reflect a better vision of what it means to know mathematics, these new forms will have little influence on curriculum change. The current study indicates that teachers will not use tasks for evaluation if (1) the tasks do not reflect their own understanding of mathematics, (2) teachers do not recognize the value of the tasks in measuring significant mathematical knowledge, and (3) teachers do not value the outcomes the tasks claim to measure. We do not know all of the reasons why the mathematics that evolves in many classrooms is dualistic in nature. It may be due in part to circumstances, i.e., student expectations, issues of fairness in grading, or the ease with which classroom activities can be managed. However, regardless of the circumstances, as long as teachers choose to communicate a dualistic view of mathematics, that is, primarily as a series of isolated steps to be applied in isolated contexts, alternative methods of evaluation will be seen as peripheral to the "real" curriculum.

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A CHALLENGE: CULTURE INCLUSIVE RESEARCH

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Currently a vast amount of mathematics education research has completely ignored the cultural diversity of subjects and failed to recognize implicit differences in treatments or contexts due to culture. Existing literature in sociology, anthropology, psychology and mathematics education is used to identify factors contributing to the problem and to provide insights for future directions. In order to develop a culture inclusive approach to mathematics education, researchers are challenged to expand their psychology-based frameworks.

Culture is an integral part of mathematics, mathematics learning, and mathematics teaching. In his presidential address to the fourteenth annual meeting of PME, Nicolas Balacheff (1991) claimed that the relevance of the psychological approach to mathematics education depends on researchers' capacities to integrate the social dimension of mathematical construction into their theoretical frameworks and research problems. We would like to add the observation that the social intercourse that occurs in the classroom cannot be separated from its cultural context. The social construction of mathematics occurs in environments that are culturally organized. Therefore, culture is an integral part of mathematics education phenomena and should be addressed in both theoretical frameworks and research design.

The Problem

Currently a vast amount of mathematics education research has completely ignored the cultural diversity of subjects and failed to recognize implicit differences in treatments or contexts due to culture. Several major factors continue to contribute to this problem:

1. Much of mathematics education research is grounded in psychology-influenced frameworks which assume that a particular cognitive model explains learning for all people. Examples can be found in research using constructivist,

information-processing, or cognitivist frameworks. Piagetian research has been replicated in a number of cultures, with the purpose of determining the deficiencies of other cultures, rather than enhancing the theory or questioning generalized application of the theory.

2. The demographic make up of the population in the United States is not reflected in the samples used by mathematics education researchers. Researchers appear to have easier access to white, middle/upper class, suburban populations. Even in urban situations, individual, homogeneous classes do not reflect the diversity that exists in the school as a whole. This situation creates a closed and dangerous cycle. Research studies based on one particular segment of the population drive the research frameworks which may be only qualified to inform research related to that particular segment of the population. Secada (1988) pointed out that research can end up legitimizing unjust social arrangements and actually indirectly causing disparity in mathematics education between cultural groups.

3. Mathematics education researchers are often not informed by related research in other fields such as social psychology, sociology, and anthropology. Equally important researchers outside mathematics education studying mathematical understanding are frequently not informed by work done in mathematics education.

4. Mathematics educators have not paid enough attention to what happens inside the mathematics classroom. Even research on learning and teaching mathematics has avoided the classroom and other social contexts. Influenced by arguments of pure research coming from psychological quarters, mathematics educators have not confronted social situations where the influence of socio-cultural factors are salient.

We challenge researchers to address these problems by adopting a cultural inclusive approach to research that is capable of capturing the wealth of information available in mathematics classrooms. A culture inclusive approach to mathematics education research should include a focus on how culture, or socio-cultural contexts, influence mathematics teaching and learning. This can be accomplished only through changes in theoretical frameworks so that research will progress beyond exclusively psychological tenets.

Insights from the Literature

A number of developmental and cognitive psychologists have recognized that culture plays a fundamental role in human cognition. This notion applies to the cognitive activity of the researcher trying to understand as well as the activity of the learner. Scholars in fields such as anthropology, linguistics, social psychology, and cognitive anthropology have also addressed the role of culture in mathematics learning. Although these studies often lack a mathematics education perspective, they do offer directions for confronting our problem. A growing number of mathematics educators have acknowledge the importance of culture in the development of mathematics, mathematics learning, and mathematics teaching. They have been mainly influenced by new developments in the philosophy and sociology of mathematics.

Developmental psychologists have been interested in cross-cultural studies as a way of testing the universality of their theories elaborated in the context of Western cultures. This approach has been challenged by other developmental psychologists who argue that such studies do not help us to understand how culture influences thought. Valsiner (1989) and Buck-Morss (1975) are among those that have criticized that approach. Valsiner, for example, proposed the creation of a cultural-inclusive developmental psychology. From his perspective, culture should be regarded as a constituting part of child development. The point here is that there are broad frameworks in developmental psychology that offer theoretical and methodological elements that can be incorporated into a framework for mathematics education in order to overcome the problems inherited from classical psychological approaches.

Researchers working from a situated cognition perspective begin with the situation in which the cognitive activity takes place, and they investigate how the situation influences individual cognition using methodologies employed in anthropological research . Through the study of the situation, the relationship of culture and cognition becomes clearer. For example, Saxe (1991) studied the situation of candy selling by children in Brazil in order to understand how the culture of candy selling organized the mathematical cognition of the sellers. Scribner (1985) was interested in how action guided the acquisition and organization of mathematical knowledge. For her, the situation and related goals regulated the action of the participant. Other researchers who have focused on the situation or context as providing information about cognition include Lave (1988), Cole and Scribner (1974), and Carraher, Carraher, & Schliemann (1985).

Both the philosophy and sociology of mathematics are crucial for the development of a culture inclusive mathematics education. Recent developments in these areas have contributed to the revival of the conceptualization of mathematics as socio-cultural phenomenon. It is important to consider how specific forms of social organization influence the construction of mathematics (Bloor, 1976; Restivo, 1983 Struik, 1942).

In mathematics education, some researchers have looked at mathematics as a socio-cultural product and at mathematics education as a social process (Bauersfeld, 1980; Bishop, 1988; Cobb,1989; Mellin-Olson, 1987; Walkerdine, 1990). In their works, these mathematics educators have called for broadening disciplinary perspectives in mathematics education in order to move beyond exclusively psychological frameworks. For Bishop mathematics is a panhuman activity. He claimed that all cultural groups have the capacity to create mathematics and, in fact, they engage in mathematical activities. There are six key "universal" activities in which mathematics is elaborated in culture: counting, locating, measuring, designing, playing, and explaining. Cobb postulated the existence of three non-intersecting domains of interpretation in the study of mathematics learning and teaching: the experiential, cognitive, and anthropological. These constructs, however, are complementary. For gaining a better understanding of mathematics teaching and learning in the classroom a coordination of these three interpretations is necessary. Mellin-Olsen argued that the different uses of mathematics in various cultures can decisively affect how members of those cultures learn school mathematics. He explained that in discussion about personal and shared knowledge, notions such as conflict and oppression are unavoidable. Therefore, he focused his work on the construction of a general theory describing the politics of mathematics education. His general theory is built on elements and assumptions borrowed from activity theory, research on language, anthropology, symbolic interactionism, communication theory, and mathematics education.

Future Directions

We are challenging all researchers to consider a culture inclusive approach to mathematics education. While we think cultural influence is important for all areas of investigation, it may be appropriate to consider different levels of involvement. At a minimal level, researchers should include descriptions of the ethnic, cultural, or social class composition of the sample even if cultural influences are not reported. At more involved levels, researchers should include culture as an independent variable in their designs, so that they can report the interactions between culture and other factors as well as the composition of the sample.

We hope that a significant number of researchers will move beyond reducing culture to an independent variable, and will address culture as an integral part of mathematics education (Valsiner, 1989; Stigler & Baranes, 1988). We encourage studies in which the primary goal is to investigate how culturally organized contexts affect the learning and teaching of mathematics.

We challenge researchers to develop an interdisciplinary, culture inclusive approach to mathematics education that borrows from current research in anthropology, philosophy, and sociology as well as psychology. Increased attention to cultural diversity will allow researchers to more accurately inform classroom practice.

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Note: An extensive, annotated bibliography related to culture and mathematics education will be distributed at the presentation.

INTERACTIONS BETWEEN COGNITION AND AFFECT IN EIGHT HIGH SCHOOL STUDENTS' INDIVIDUAL PROBLEM SOLVING

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In an exploratory study, we interviewed eight mathematically talented high school students solving a pair of related non-routine problems, and observed interactions between their cognition and their "local affect". We cite some instances of the influence of affect on executive decisions, and conjecture an important role for such affect in problem-solving success.

Recently attention has been focused on the role of affect in executive decision-making during problem solving (Goldin, 1988; McLeod and Adams, 1989).

In previous work, Goldin defines "local affect" to be the "changing states of feeling during problem solving", and treats it as an internal *system of representation* for problem solving, on a par with imagistic representation, formal notational representation, verbal representation, and a system of planning and executive control. Global affect, in contrast, refers to general feelings and attitudes, reinforced by belief structures, that solvers may bring to the problem situation but that are not so readily modified. For example, a student who is generally fearful of mathematics (global) may nevertheless, when engaged in a particular problem-solving situation, experience a variety of feelings (local) ranging from anxiety (at the outset) to surprise and satisfaction (on solving the problem insightfully). Thus, one envisions local affect as a system of *changing* emotions; some affective states include curiosity, puzzlement, bewilderment, encouragement, pleasure, elation, satisfaction, frustration, anxiety, fear and despair. Major pathways involving local affect and heuristics during problem solving may lead to positive or negative outcomes; and it is suggested that (desirable or undesirable) long-term, global affect results when such paths of local affect becoming well-established competency structures.

A similar distinction is made by McDonald (1989, p. 230), who discusses two different ways in which cognitive and emotional processes are involved in learning: "One is through the individual representation of information that is tied to emotional concerns–the emotional reactions that affect moment-to-moment conscious processing." [The other] "has to do with sociocultural influences on individuals and the way that they see themselves or the information." Mandler (1989, p. 4) suggests that a theory of emotion "should be of both general and specific interest to cognitive psychologists", while McLeod (1989, p. 246) distinguishes "beliefs" and "attitudes" (that are relatively stable and resistant to change) from "emotions" that change rapidly. This important distinction has consequences for effective mathematics teaching. The NCTM "Professional Standards for Teaching Mathematics" (National Council of Teachers of Mathematics, 1991, p. 104) discusses the need for teachers to be able to promote a mathematical disposition by facilitating students' confidence, flexibility, perseverance, curiosity, and inventiveness in doing mathematics. The assumption is that fostering these (local) feelings repeatedly in a variety of mathematical situations will foster construction of the desired (global) disposition.

These background considerations motivated an exploratory study to look at the interaction of cognition and local affect in a non-routine problem-solving situation. We conjectured that local affect especially influences executive decisions, and that it should be possible to observe and describe instances of such influence. We further conjectured that successful problem solvers tacitly use local affect in selecting *particular* processes, so that their affect might actually be guiding their strategies. Ultimately we are interested in the idea that metacognitive awareness of local affect can help individuals become more powerful problem solvers.

Subjects

Four high school women and four high school men from New Jersey were selected randomly from the participants in a month-long "Young Scholars" institute at Rutgers University in the summer of 1990. The students are mathematically talented, and each identified himself or herself as extremely interested in taking more science and mathematics courses at school. All had completed 11th grade, with the exception of two women who had completed 10th grade. The students returned twice to campus during the Fall 1990 semester for follow-up sessions, and it was during the second of these sessions that the interviews took place.

A "Last Day Questionnaire" distributed during the 1990 summer institute asked, "What kinds of personal traits do you think are involved in 'being good at mathematics'?". The eight subjects had responded that one needs to have "logic" or be "logical" (4), have "patience" (2), be "curious" (2), have "diligence" or be "hard-working" (2), "possess understanding" (1), "have intelligence" (1), have "ingenuity" (1), have the traits of "thinking widely", "thinking carefully", "determination", and "not giving up" (1), have "the ability to accept failure"

(1), have "a good memory" (1), "a keenness" (1), "be open-minded" (1), "be bullheaded" (1), and "be creative" (1). Five subjects thought they possessed all the qualities they themselves mentioned. Overall, subjects' "global affect" in reference to their self-perceived ability to solve mathematical problems was quite positive.

Method

In one-on-one interviews two problems were successively introduced, and the subject encouraged to "think aloud". We used non-routine problems, unfamiliar to these students, to minimize affective differences among the subjects that might be due to previous emotional experiences associated with school mathematics or with standard topic areas in mathematics. Two videocameras recorded each interview. one focusing on the subject and the other on both the subject and the clinician. All interviews were conducted by the same clinician (DeBellis).

First, Problem 1 was presented (orally); simultaneously the clinician placed two bottles of Gatorade on the table in front of the subject:

> *Problem 1.* Suppose you have two containers of liquid. Everything about the one container of liquid is the same as the other, except for color–that is, density and volume are the same. In this experiment we used Gatorade. One container held red liquid, the other container held yellow liquid. Now suppose you take one tablespoon of red liquid and drop it into the yellow liquid and mix thoroughly. Then you take a tablespoon of this new mixture and drop it back into the container that has the red liquid and mix thoroughly. The question is, which container has more contamination in it? Does the red Gatorade have more yellow Gatorade in it or does the yellow Gatorade have more red in it?

The subject was left free to solve the problem, without hints or suggestions. After a conclusion was verbalized, the clinician asked "Why?" The subject was again left free to justify his or her answer. If a subject's justification used words suggesting uncertainty or ambiguity, such as "almost", "probably", or "about", the clinician probed further, e. g.: "What do you mean by 'almost' [or 'probably' or 'about']?" After the subject verbalized a justification, the clinician asked, "Do you think your answer is correct?" The subject responded and the clinician again asked, "Why?" If the subject concluded the amounts of contamination would be equal, and justified this conclusion, the clinician asked whether that would always be the case. Finally she asked, "What happens if we don't stir the mixture? Does that change your answer?" When the subject expressed confidence or security in a solution (without affirmation from the clinician), the second problem was posed:

> *Problem 2.* Suppose you have two containers of M and M's. Each of these containers holds one hundred fifty M and M's in it. Suppose you take a

handful of red M and M's from the container and dump them into the yellow M and M's container and shake them up. Then suppose you take the same size handful of M and M's from this mixture and dump them back into the red M and M container. Which container would have more of the other colored M and M's in it? Would the red M and M's have more yellow M and M's in it or would the yellow M and M's have more red in it?

The Gatorade bottles were replaced by two containers of M and M's, each holding 150 pieces. Note the direct correspondence between the Gatorade colors and the M and M colors, and between the problem structures (with "volume measure" replaced by the discrete "number measure"). The structure of the questioning for Problem 2 paralleled that for Problem 1. The subject solved the problem freely; when a conclusion was reached, the clinician asked "Why?" When a justification was offered, the clinician asked, "Do you think your answer is correct?" The subject responded and again the clinician asked, "Why?" Some subjects spontaneously experimented with the M and M's: there was no guidance from the clinician as to how to do this, except to indicate that "handfuls" had to be the same size. Again the question, "Does stirring make a difference?" was posed. Whatever the outcome, three final questions were posed: "Have you ever seen a problem like this before?" "What did you like about this problem?" and "What did you hate about this problem?" These questions elicited some retrospective expressions of emotion.

Observations and Interpretations

Four subjects correctly concluded for Problem 1 that there would be the same amount of contamination in each container. Three of these (Subjects 2, 3, and 7) justified their answers in a valid way, while the fourth (Subject 1) responded, "my feeling just tells me." Of the other four subjects, all expressed the opinion that their answers were correct, and provided justifications. The three with correct solutions and valid justifications took far more time to achieve closure on this problem than did the others (the clock began after presentation of the problem):

Subject 1	M	2 minutes 13 seconds	
Subject 2	M	7 minutes 03 seconds	[correct solution, valid justification]
Subject 3	M	4 minutes 46 seconds	[correct solution, valid justification]
Subject 4	M	1 minutes 59 seconds	
Subject 5	F	0 minutes 57 seconds	
Subject 6	F	2 minutes 35 seconds	
Subject 7	F	6 minutes 56 seconds	[correct solution, valid justification]
Subject 8	F	2 minutes 32 seconds	

In Problem 2, Subjects 1 through 7 ultimately concluded there would be the same number of M and M's of the wrong color in each container; all but Subject 4 offered valid justifications. Subjects 2, 5, and 7 did not physically perform any experiments with the M and M's; Subjects 3 and 6 performed one experiment before reaching this conclusion; while Subjects 1 and 4 performed two experiments. Unfortunately the videotape recording Subject 8 ran out; she had performed two experiments to that point indicating equality, but never articulated this as a firm conclusion.

The following excerpts illustrate some representative instances of local affect interacting with executive decision-making. We let "..." denote a pause by the subject, and " ***** " an omitted portion of the transcript. In the first problem, Subject 2 has concluded the yellow bottle will have more red in it.

[Clinician:] Why? [Subject:] Because you take a teaspoon of this (points to red Gatorade) and put it in there (points to yellow Gatorade) then when you ... diffuses, then you take the amount back up, there will probably be a couple of reds still in there ... so when you put it back in there (pointing to the red container) you're only adding a certain amount of yellow and there will be couple of reds still in there so there won't be quite as much yellow ... oh, I know what you're saying ... you're saying it's equal ... (pause) [C:] Why do you think I'm saying anything? [S:] (ignoring the question, sits back in chair and smiles) Yeah ... it's equal ... I understand what you're saying ... [C:] (gestures, shaking head) I'm not saying anything. [S:] Ummm ... (points to the red container) see it's hard to think out loud ... [C:] Yeah, I understand that ... but that's okay ... [S:] Ugggh ... if you put some of this (points to red Gatorade) in there (points to yellow Gatorade), it's gonna diffuse and you take some of it back in there (points to yellow Gatorade then to red Gatorade) ... no, all of it's going to go ... (pause) let's say this was ten and that was ten (subject points to red, then yellow bottle), let's say hundred ... ***** ... so it would be equal. [C:] Do you think your answer is correct? [S:] Yeah. [C:] Why? [S:] By my example of ... if this was a hundred and that was a hundred ... (pauses) oh ... oh ... it's not equal. Okay ... ***** ... Take back one eleventh of it (points to the yellow), put it in here (points to the red) ... this would have ... [S looks up at C and smiles], this is confusing ... I like it ... a hundred (points to the red), ten in there (points to the yellow), so you would be leaving just over nine ... take back ... ***** ... so it would still be the same ... I think ... see, you know, I don't wanna look like a fool.

Noteworthy is the way this subject twice ignores the clinician while *attributing to her a point of view*; this construct seems to help him express his first insight, at which he smiles. His pleasurable affect then appears to cause him to reflect on the insight; his expression "ugggh" suggests a letting go of anxiety, and marks his strategic decision to try a special case, assuming particular amounts of red and yellow liquid. His concern about "looking like a fool" also motivates him to retain some tentativeness in his conclusions, and to monitor further their validity.

Subject 7 has responded to Problem 1 by saying that the yellow bottle has more red in it; the clinician has asked "Why?" and "What are you thinking?"

[S:] Let's see ... the yellow would have more contamination unless the amount of red that you took out was a half of tablespoon ... yeah ... 'cause if you took out a tablespoon, the only way that they could be equal contamination would be ... if you took out a half a tablespoon of red and half a tablespoon of yellow and put it in there, and each would have one half tablespoon of contamination. But since you shake it up, you can't ... ***** ... for some reason, it doesn't sit right, though. (S stops, puts her hand to her mouth, sits back on her chair, speaks very softly) A little bit of red that got out of there (points to yellow container) and put it back in there (points to red container, mumbles under her breath) ... they're equal. (smiles, looks at C) they're, yeah, no (squints her face and covers her mouth again) ... if you take into consideration the amount ... I guess it doesn't matter (pauses, looks at C) ... now I'm thinking they're equal, because ... ***** ... I want to say they are equal. Is that right? (C shrugs, S appears frustrated, exclaiming and sitting back in her seat) 'Cause I can't explain it! ... the amount of red that you take out in the tablespoon ... part of it ... most of it ... is yellow. Okay, most of what you take out is yellow ... and the yellow that you take out equals the red that's remaining in there (referring to the contamination in the yellow container).

We observe the subject's puzzlement, soft speech, and smile as she reorients away from her initial commitment that only a transfer of exactly half a tablespoon of red liquid could achieve equality. Note also how this subject's frustration at her difficulty in explaining her conclusion appears to have served her well, motivating her to articulate an explanation.

Subject 5 had reached an erroneous conclusion on the first problem. When Problem 2 was presented, she initially responded that that the container of yellow M and M's had more red in it.

[C:] Why? [S:] This one (points to yellow container) if you stir, if you stir this up perfectly, you would take back some of the red M and M's. (stops) Oh! Okay! (excitement) All of a sudden I ... (stops to think) [C:] What just happened there? [S:] (S ignores C). If you take out a given amount ... a certain percentage of M and M's that you added so ... um ... okay! if you take the handful out (gestures as if taking a handful out of the reds) and put it in here (mimics dumping it into the yellow container), redefines problem) and shake it up (gestures as if to shake) a certain percentage of the stuff you gave here ... ***** ... so it would be the same. And the same goes for that! (S points to Gatorade bottles, displaying confidence).

The display of excitement accompanies the subject's "aha!" experience. Her positive affect, happy but not quite elated, appears to increase her determination to regroup, to reorganize the problem, and to see her reasoning through to its conclusion–nothing is going to stop her until she finally has it. And when she does, she confidently transfers her analysis back to Problem 1.

But not all the affect we observed had consequences that were positive (from a mathematical point of view). Subject 6, in solving the first problem, concluded that there will be more red in the yellow Gatorade.

[C:] Why? [S:] (giggles) Because since you're taking the red first, and you're putting in the yellow and mix it up, you have red and yellow mixed up. When you take another tablespoon so then you're putting red back into the red, so it's not really a full tablespoon of yellow. [C:] Do you think your answer is correct? [S:] Yes. [C:] Why? [S:] Because I'm confident.

This subject's feeling of confidence *substitutes* for an analysis, rather than encouraging her to investigate further. Her executive decision, inspired by her affect, is to stop considering the situation as a problem, and to cease to engage.

Conclusion

We have seen examples in which affect appears to guide problem-solving choices, and where powerful problem solvers use it effectively. However, affect can also have negative consequences, even in strong students. The goal of achieving effective use of local affect for mathematical problem solving needs considerably more research attention.

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NA
Teacher Development
Teacher Beliefs

CHANGING INSTRUCTIONAL PRACTICE: A CONCEPTUAL FRAMEWORK FOR CAPTURING THE DETAILS[1]

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This paper discusses the conceptual framework and methodology of a longitudinal investigation of teacher change within the context of an instructional reform project. It describes the aspects of the instructional environment and teacher knowledge and beliefs that are being monitored as well as the various data sources and perspectives from which information is being gathered.

Over the past several years, recommendations for the reform of mathematics instruction have been remarkably consistent (National Council of Teachers of Mathematics, 1989; 1991; National Research Council, 1989; Silver, Kilpatrick, & Schlesinger, 1990). Reformers agree that mathematics classrooms should be places where meaning making is paramount, where students take an active role in constructing their own knowledge, and where mathematical communication is as important as obtaining the correct answer. This vision is very different, however, from the way in which most classrooms currently operate. A number of studies have reported that mathematics lessons typically follow a predictable sequence of activities, most of which emphasize rules, procedures, memorization, and right answers (e.g., Stodolsky, 1988). Moreover, the vision represents a radical departure from the manner in which most practicing teachers learned mathematics and learned to teach mathematics (Ball, 1988). Clearly, both teachers and their classrooms will need to undergo some fairly profound transformations if they are to create new instructional practices that answer the reformers' calls. The purpose of this paper is to present the conceptual framework and methodology of a longitudinal investigation of teacher change within the context of an instructional reform project.

The present research is part of QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning), a large, multi-year project that shares the above vision of how mathematics classrooms should be transformed. The goal of QUASAR is to foster and study the development and implementation of enriched mathematics instructional programs for students attending middle schools in economically disadvantaged communities. Toward that end, a set of six geographically, ethnically, and intellectually diverse sites began developing and implementing

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unique approaches to teaching high-level mathematical thinking, reasoning, and problem solving in the Fall of 1990. The programs at each site are school-based and involve a partnership between the school faculty and leadership and one or more resource partners (typically faculty from a nearby university). Implementation of these programs will continue and expand over the next several years.

The QUASAR Documentation Effort

Instructional change is influenced by variables operating at a variety of levels within the schooling environment (McLaughlin, 1990). As such, QUASAR's documentation strategy systematically examines three interwoven components: the social and organizational context for instructional change (e.g., the school climate, the collaboration between teachers and resource partners), the development and implementation of the mathematics programs, and self documentation produced by site-based participants (see Stein, 1990). This paper focuses on the second component, the documentation of the classroom implementation of the mathematics programs.

Classroom documentation serves a variety of purposes within the QUASAR project. First, observations and descriptions of mathematics lessons provide specific instructional instantiations of the broad principles on which QUASAR is based. Although the project has provided a broadly stroked picture of the kinds of instructional activities and conditions that should exist at project sites, the development of specific instructional programs has been left to the individual sites. Second, classroom documentation data complements other project data. For example, students in QUASAR classrooms are periodically assessed with respect to their understanding of a variety of middle school topics and their performance on problem solving tasks. Descriptions of classroom instruction contribute information on the nature of the mathematical tasks and instruction activities to which students have been exposed. Possible relationships between changes in student understandings and instructional activities can then be explored. Similarly, the project is systematically collecting data on staff development activities at each site. Hence, possible relationships between staff development experiences and teachers' instructional practices can also be examined.

Finally, QUASAR classroom documentation expects to contribute to the extant knowledge base on teacher change. Although teachers are the chief mediators of most school improvement efforts, history suggests that helping teachers to alter their practice is not easy (Cuban, 1990). Moreover, recent studies are beginning to document the difficult nature of teaching in a manner compatible with the spirit of the mathematics reform movement (e.g., Grover, Gill, & Kaduce, 1991). All teachers, of course, must struggle to overcome tendencies formed by the way they were taught mathematics (Ball, 1988). *New* teachers can learn to experience mathematics and themselves as learners of mathematics in new and exciting ways; they fare less well, however, when they attempt to create an appropriate instructional role for themselves, including how to deal with the often unpredictable contributions of students (Schram, Wilcox, Lappan, & Lanier, 1989). *Experienced* teachers carry the baggage of established practice and prior experience with reform efforts, many of which have been diametrically opposed to the goals of the new reform (e.g., direct instruction and the back-to-basics movement) (Cohen & Ball, 1990). QUASAR documentation research can contribute to a small but growing body of work that examines the processes by which experienced teachers' practice changes as they are confronted with new ideas about mathematics, about how students learn mathematics, and about better ways to teach mathematics.

Conceptual Framework

The conceptual framework guiding the classroom documentation work is informed by recent recommendations for the reform of mathematics education (i.e., NCTM, 1989; 1991) and research in a number of areas including the cognitive aspects of teaching (e.g., Clark & Peterson, 1986; Leinhardt & Greeno, 1986), teacher knowledge and beliefs (e.g., Brophy, in press; Shulman, 1986), and research on mathematics teaching and learning (e.g., Cobb, Wood, & Yackel, in press; Fennema, Carpenter, & Lamon, 1988). The above recommendations and research suggest that it is important to systematically monitor specific aspects of the *instructional environment* and of *teacher thinking* that are expected to change as mathematics programs are implemented and progress is made toward facilitating high-level thinking and reasoning.

Four main variables form the nucleus of our framework for describing changes in the *instructional environment*: mathematical tasks, classroom discourse, intellectual environment, and the nature of instructional formats. **Mathematical tasks** are a central feature because it is through engagement with such tasks that students are provided with opportunities to think about concepts and procedures, connections among mathematical ideas, and applications to other domains and real world contexts. Mathematical tasks also implicitly carry messages about what is worthwhile mathematical activity. Consequently, we attend to several features of the tasks that occur in QUASAR classrooms including goals (implicit or explicit) for students' learning or understanding, the degree to which the tasks focus students' attention on doing mathematics as opposed to following preestablished procedures, and the kinds of communication that the tasks foster. Recent research emphasizes the role of classroom **discourse**--the way that mathematical ideas are exchanged-- in how students develop and refine their knowledge (e.g., Lampert, 1988). Our framework includes attention to various aspects of classroom discourse including the extent to which students are encouraged to explain and justify their thinking rather than simply supply the "right" answer, the representations and technological tools that teachers and students select or invent, and the extent to which students are encouraged to initiate problems and to question the teacher and one another. The framework's attention to **intellectual environment** examines hidden classroom norms that may influence students' ideas about mathematics and themselves as learners of mathematics. It is based on the epistemological consideration of who possesses intellectual authority in the classroom: Does task presentation suggest an outside author of knowledge (e.g., teacher, text) or does it encourage students to view themselves as constructors of knowledge? Finally, the framework incorporates descriptions of the **instructional formats** (paired learning, small group work, whole-class discussion) used in QUASAR classrooms, including attention to the assignment of roles to group members, teacher monitoring of group work, and peer interactions.

Since the teacher is central to decisions made about the instructional environment, *teacher thinking* constitutes another broad area for the systematic study of change. Our framework for documenting teacher thinking includes the following variables: Teacher knowledge and beliefs about mathematics as a discipline, beliefs about instructional practice, and beliefs about how students learn mathematics. A host of findings suggest that teachers' own understandings of a subject matter influences their instructional approach, impacting both what they teach and how they teach it (Brophy, in press; Stein, Baxter, & Leinhardt, 1990). Teachers' beliefs about mathematics and about how students learn mathematics are similarly influential (e.g., Thompson, 1984). Our focus on teacher knowledge and beliefs explores both how they change during the course of the project and how they act as a filter through which teachers interpret project goals and activities.

Methodology

A methodologically eclectic approach to classroom documentation is being employed. Interviews, observations, paper and pencil instruments, and classroom artifacts form the data base. In addition, we employ the qualitative research approach of triangulation which calls for gathering information on a specific phenomenon from a variety of sources.

Instructional environment. The mathematics classrooms are being documented from a number of perspectives. The most visible and labor-intensive consists of three 3-day observation sessions occurring in the fall, winter, and spring of each school year. The purpose of each of these sessions is to gain a detailed understanding of mathematics instruction in a particular teacher's classroom at a particular point in time. We are also collecting data to gain insight into instruction over the course of the year. These data include teacher self-reports (paper & pencil) on their instructional objectives, pedagogical techniques, and content coverage; and teacher-provided classroom artifacts (e.g., teacher-made tests, student work).

The classroom observations include both an analytic examination of the mathematical content and pedagogy of the lessons and an ethnographic-style investigation of what it is like to be a student in the classroom. Two observers take detailed field notes, one focusing on the overall mathematics instruction, the other on two pre-selected target students. Both observers' fieldnotes are guided by pre-designed observation guides. The mathematics observation guide consists of qualitative questions grouped by the main themes: tasks, discourse, environment, and formats. The target student observation guide consists of questions about the students' behaviors and their level of engagement during the various phases of the lesson. The aim is to chronicle the development of the lesson through the eyes of the student, thus "personalizing" the observations and providing detailed information regarding how students are responding to the lessons. After the observation, the observers write narrative summaries (of the lesson and the target students respectively) and, using videotape and their fieldnotes as data, answer the questions on the observation guides. In addition to these qualitative accounts, the observers complete a quantitative evaluation of the lesson on a series of anchored rating scales.

The observers were selected on the basis of a set of qualifications that included a strong background in mathematics education, psychology, or a related field, a demonstrated competence in their ability to analyze instructional events from both pedagogical and mathematical content perspectives, prior experience observing classrooms and conducting interviews, and their understanding of the ethnic or multicultural nature of the community at the site (many of the observers are residents of those communities). In some instances, Spanish-English bilingual skills were also required because the population included a high percentage of students whose native language is Spanish.

The observation reports are complemented by interview data from a variety of project participants. The mathematics observer conducts a pre- and post-observation interview with the teacher, asking questions about the teacher's objectives for the 3-day sequence and his/her evaluation of the lessons. The target student observer conducts a post-observation interview with 6 students from each observed class. The interview is focused on the students' perceptions of their mathematics class in general (e.g., students brainstorm about "what it takes to get a good grade in Mr./Mrs. ___'s math class") and of the 3-day observational period in particular (e.g., students respond to the question, "What do you think was the main thing that you were supposed to learn during these past 3 days?"). In addition, the target student observer conducts semi-structured interviews with the principal, the resource partner(s), and the site facilitator. The interviews focus on these individuals' perceptions of mathematics instruction in the observed classrooms. All of the above data is organized to provide information on the four main variables outlined in the conceptual framework.

Teacher thinking. Two inventories elicit information about the teachers' thinking. One inventory, consisting of 30 Likert-style statements, focuses on teachers' beliefs about mathematics and how it is best taught and learned. The second inventory consists of 10 problem situations and focuses on the teachers' knowledge of mathematics and pedagogical skills in dealing with student responses to problem situations. Teachers are asked to provide examples of full- and partial-credit student responses to five of the problems and to give a rationale for their assignment of points. On the remaining five problems, teachers comment on how they would respond to students who have answered the problems in a particular manner (usually exhibiting some misunderstanding). These inventories are administered once per year during the course of the project. Additional information about teachers' knowledge and beliefs is gathered from a variety of informal sources (e.g., teacher journals, teacher-made or teacher-selected tests, lesson plans).

Expected Contributions

Given the relatively weak instructional specification of the reform's vision to date (Cohen & Ball, 1990), grounded examples from QUASAR classrooms should be useful to the field of mathematics education as it seeks to specify promising practices. Even with more detailed portraits of exemplary insruction, however, the reform's recommendations will not be implemented unless teachers undertake the complex, long, and often difficult process of creating a meaningful instructional practice. The present research, combined with other longitudinal work on how novice teachers learn to teach mathematics (e.g., Schram, Wilcox, Lappan, & Lanier, 1989; Jones, Brown, Underhill, Agard, Borko, & Eisenhardt, 1989), should provide insight into the process of becoming a skilled, knowledgable, and thoughtful teacher.

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Impact on Students

TOWARDS A CONSTRUCTIVIST PERSPECTIVE: THE IMPACT OF A MATHEMATICS TEACHER INSERVICE PROGRAM ON STUDENTS[1]

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A constructivist-oriented inservice program provided teachers of mathematics (K-12) with intensive two-week summer institutes and weekly classroom follow-up. Pre- and post-program data on student outcomes indicate that, along with transformations in the nature and quality of mathematics activity in the classroom, students' beliefs about learning mathematics changed and elementary students' attitudes toward mathematics improved. Although instruction focused more on conceptual understanding and less on computational skill, standardized test scores assessing routine knowledge did not drop.

The Educational Leaders in Mathematics (ELM) Project was an inservice program that provided teachers of mathematics (K-12) with intensive two-week summer institutes and weekly classroom follow-up during the succeeding academic year. While the project predated the NCTM Professional Standards for Teaching Mathematics (1991), its goals--to stimulate and support teachers' development of instructional practices informed by a constructivist view of mathematics learning--were consistent with the vision the Standards proposed. An instrument designed by ELM staff to assess participants' classroom practice after one year's involvement in the program (Schifter and Simon, 1991) determined that 99% of them implemented new instructional strategies and approximately half developed a practice informed by a constructivist epistemology (Simon and Schifter, in press). In general, students' rote learning of facts and practice of routine algorithms was deemphasized; instead students were encouraged to generate their own ideas and communicate them to one another.

This paper discusses the impact of ELM on the students of these teachers. We were interested in the effect of the program on: 1. students' attitudes toward mathematics, 2. students' beliefs about mathematics learning, 3. students' performance on standardized tests, and 4. the nature and quality of the mathematical activity in the classroom.

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Impact on Students

Many teachers engaged in innovative inservice programs such as ELM feel what they perceive to be contradictory pressures. On the one hand, they are aware that traditional instructional approaches do not promote the levels of understanding and interest among their students that an alternative practice could inspire. On the other hand, they feel that they are held accountable for students' scores on standardized tests of computational ability and they must prepare their students for those tests. Yet this study and others (Cobb et al, 1991; Carpenter, et al 1988; Heid, 1988) are beginning to show that as teachers change their focus to student construction of mathematical concepts--emphasizing problem solving, communication, and reasoning--not only do assessments of attitudes, beliefs, and conceptual understanding indicate positive change, but standardized test scores do not drop. These results ought to allow more teachers, with support from their school districts, to become involved in inservice efforts directed toward implementation of the NCTM Professional Standards for Teaching Mathematics without fear of diminished computational skills and lowered test scores.

Methodology

In examining the program's impact on students, we employed qualitative and quantitative methods that included both formal and informal approaches: data was collected through surveys, standardized tests, and teachers' reports of student change

For the three cycles of instruction (1985-1988), surveys and standardized mathematics tests were given to parallel classes (grades four and above for the surveys) of participating teachers at the end of the academic year prior to entering the program and again at the end of the following academic year. The students surveyed were thus not the same individuals from one year to the next, but they were taking the same course with the same teacher. As a consequence, surveys and tests were included only for classes of teachers who taught the same course (e.g. third grade heterogeneous, sixth grade remedial, honors precalculus, etc.) two years in a row. Between pre-test and post-test, teachers participated in a two-week summer institute and then received weekly follow-up visits (September to May) from ELM staff.

In April 1988, ELM teachers who had entered the program between 1985 and 1987 were requested to respond in writing to the following question:

> What changes have you observed in your students as a result of your involvement in the ELM Project? (Include all types of changes: positive, negative, and neutral.)

Response items were consolidated and categorized.

Results

Survey items about feelings toward mathematics and the importance of mathematics were combined to calculate a general attitude score. Two-tailed t-tests were run to compare pre- and post-program survey responses.

Attitude scores for elementary students (grades four through six) calculated from 171 pre-program surveys and 179 post-program surveys showed a highly significant increase ($p. < .001$). Looking at specific items that comprised the general score, the following items changed at a level of $p < .005$:

> It is fun to work math problems. I'd rather do math than any other kind of homework. Math is one of my favorite classes in school. It is interesting to do story problems. Math helps me learn to think better. I like to explain how I solved a problem.

For secondary students responding to the questionnaire, there were 295 pre-program surveys and 303 post-program surveys. The composite general attitude scores indicated no significant change from one year to the next.

Beliefs about learning mathematics were assessed from survey items for which students responded to the following question: To do well in mathematics, how important are these? For elementary students, the following items increased in importance at a level of $p < .05$:

> Checking your own answers; being able to explain what you did; drawing diagrams; luck; being creative; trying new things to see how they work; seeing connections between things you've learned; trying different ways to solve problems even if you're not sure how to solve them; opinions.

The following items decreased in importance at a level of $p < .05$:

> Working problems quickly; reading the textbook; writing down what the teacher says in class.

Survey scores for the following items indicated no difference between pre- and post-program surveys:

> Neatness; asking questions in class; memorizing; thinking logically.

For secondary students, the following items increased at the level of $p < .05$:

> Being creative; trying new things to see how they work;

and the following items decreased at the level of $p < .05$:

> Reading the textbook; writing down what the teacher says; thinking logically.

Teachers of all grades administered standardized tests which evaluated routine and computational knowledge of mathematics. Like most of the standardized tests available, they could not, in our view, adequately measure conceptual understanding and problem solving abilities. Three hundred eighty pre- and 388 post-program elementary students and 290 pre- and 303 post-program secondary students took the tests. Two-tailed t-tests were used to compare pre- and post-program scores. No significant differences were found for the total group, or for elementary and secondary students analyzed separately.

To consider the nature and quality of mathematical activity in the classroom, we solicited observations of changes in student behavior from sixty-one ELM teachers. The thirty-five responses included reports of both positive and negative effects, but the former were overwhelmingly in the majority. Following is a list of the effects which were reported by at least five teachers. The number of teachers reporting the observation is noted in parentheses.

Students:

show greater ability to express mathematical ideas and to defend their point of view (16); express more interest and/or enjoyment in mathematics (13); listen to and respect others' ideas (9); show greater cooperation among themselves (9); willingly use concrete manipulatives to solve problems (8); take risks/share their strategies with the class (8); understand that there is more than one way to solve most problems (8); depend more on each other and less on the teacher (8); participate more in class (8); probe for understanding (6); are more confident, competent problem solvers (6); understand more (6); are more confident in math (5); and experience more frustration (5).

Discussion

Although teachers' observations of their students need independent corroboration, when taken together with the survey data some tentative conclusions may be drawn. We can categorize student change into three broad areas: cognitive, affective, and social.

Cognitive change described by teachers involved greater facility with mathematical ideas, greater ability to communicate about mathematics, and deeper understanding of mathematical concepts. They reported that students were becoming more competent problem solvers who understood that there is more than one way to solve most problems.

These reported changes are consistent with survey responses concerning beliefs about mathematics learning. Both elementary and secondary students' scores increased for items such as, "It is important to be creative," and "It is important to try new things to see how they work." Rote behaviors such as "writing down what the teacher says" became correspondingly less important.

Two results concerning student beliefs seem particularly puzzling: first, although "luck" continued to be considered relatively unimportant (the mean remained low), it increased in its perceived importance for elementary students. This may be attributable to the change in the nature of mathematical activity in the classroom. If pre-ELM assignments were largely computational exercises, then "luck" would have played little or no role; success was dependent on careful repetition of a known algorithm. But teachers participating in ELM gave non-routine problems where trying out different strategies was appropriate, and some students might have identified hitting on a successful strategy as a matter of luck. And second, while elementary students' response to the item "It is important to think logically" did not change, secondary students' response to this item decreased (although the mean still remained high). Perhaps this was due to the fact that it is generally held that mathematics helps to develop and requires logical thinking. If, prior to their involvement in ELM, teachers tended to emphasize this, students might have come to identify "logic" with mechanical or routine solutions and it would be expected that the pre-program measure for this item would be as high or higher than the post-program measure.

Affective change. Teachers reported that their students now expressed more interest in and enjoyment of mathematics, and that they demonstrated more confidence in solving problems and in doing mathematics generally.

The attitude survey scores for elementary students supported the teachers' observations. After their teachers had participated in ELM, elementary students more frequently reported that it was fun to work mathematics problems, that they liked to explain how to solve problems, and that mathematics helped them to think better.

Among secondary students, responses to the attitude survey did not change. A possible explanation is that older students' attitudes toward mathematics were more firmly set as a result of more schooling. Informal discussion among elementary and secondary teachers indicates that school structure also effects the potential for change. Elementary teachers, who have the same students for the entire school day, report that after attending the summer institute, instruction changed in many of their subjects. Thus, they were able throughout the day to communicate beliefs about learning and to convey expectations of student behavior consistent with their goals for their mathematics classes. To secondary students, mathematics classes taught by ELM participants tended to be the odd experience.

Impact on Students

Social change. Among teacher-reported changes, it is interesting to note how many of their observations concerned changes in social behavior. Teachers wrote that students showed greater cooperation among themselves, listened to and respected one another's ideas, and depended more on one another and less on them Students were more willing to take risks and to share their ideas and strategies with their peers, and in general more willingly participated in classroom activities.

These developments reflect changes in the social organization of the classroom: students often worked in pairs or small groups and were responsible for their own and each other's understanding. By listening to and valuing students' mathematical ideas, teachers worked to shift the locus of authority from the all-knowing instructor (or textbook) to students' reasoning processes.

Conclusions

Teachers participating in ELM tended to increase their attention to problem solving and conceptual development, deemphasizing computation and memorization. As a result, student beliefs about mathematics learning came to include an appreciation for the values of creativity and experimentation. And elementary students developed more positive attitudes toward mathematics.

Yet standardized test scores did not change. This result should help allay concerns that greater attention to understanding and problem solving, particularly considering the additional time allotted to conceptual exploration, will lead to a decline in computational skill. The related concern that instructional changes of this magnitude will result in lower test scores for the first year or two, as teachers learn the ropes, has also been expressed. However, these test results indicate that even during the initial change process, computational skill is not necessarily sacrificed. For teachers and school administrators who wish to engage in teacher development efforts along the lines of the NCTM Standards, this should come as encouraging news.

Aside from those shifts in attitudes and beliefs described above, the results of our standardized tests could not tell us whether students were constructing stronger conceptual understandings. That the future of educational reform is tied to the development of ways of measuring such complex processes is increasingly widely recognized.

In addition, many--perhaps crucial--questions arose for us which can only be addressed through longitudinal studies. Does the teacher's, and her students', enthusiasm wear off as more time passes? What happens to students who have project teachers several years in a row? Do secondary students' attitudes begin to change after two or three years of constructivist-oriented mathematics instruction? What are the differences between elementary and secondary schools that are reflected in different responses to the project?

Finally, future research must more closely examii e change in teachers' conceptions of mathematics, and of learning and teaching, and relate such change to cognitive and sociological studies of students and teachers in classrooms.

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PROSPECTIVE SECONDARY TEACHERS' SUBJECT-MATTER KNOWLEDGE ABOUT FUNCTIONS AND GRAPHS

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This study deals with prospective secondary teachers' subject-matter knowledge about functions and graphing. A set of tasks was designed to assess the elementary knowledge that prospective secondary teachers have about functional relationships represented by verbal descriptions and by graphs. The primary objective of this study was to document evidence of the relationships between elementary knowledge about functional relationships and constructing graphs that represent these relationships. The findings suggest that the pre-service teachers' knowledge of functions and graphing was incomplete and particularly fragile with respect to certain classes of functions.

CONCEPTUAL FRAMEWORK

The study of teacher subject-matter knowledge in the context of functions and graphing is an important topic within the conceptual frameworks dealing with research on teaching (Brophy, in press; Shulman, 1986) and research on functions and graphing (Leinhardt, Zaslavsky, & Stein, 1990). Teacher subject-matter knowledge has received a great deal of attention of late and is an important component of the conceptual framework dealing with research on teachers and teaching. Shulman (1986), who identified teacher subject matter knowledge as the "missing paradigm" in research on teaching, has inspired much of this work. In order to describe the relationships between teacher subject-matter knowledge and instructional practices, it is necessary to examine the elementary knowledge that teachers have about a particular mathematical topic. The mathematical topic of functions, graphs, and graphing has also received considerable attention of late (Leinhardt et al., 1990). A significant amount of research concerned with students' understanding of functions and graphs has been completed and further studies dealing with instructional aspects have been suggested (Leinhardt et al., 1990). This study may be located within the intersection of these two conceptual frameworks - in the subset of teaching that deals with subject-matter knowledge and in the subset of functions and graphing that deals with verbal descriptions and graphs of functions.

The current reform movement in mathematics education suggests that teacher subject-matter knowledge is an important component of the new view of mathematical competence. The emphasis is being placed on examining various representations of a concept and developing connections between those representations. Solving multi-step problems and utilizing appropriate representations in the solution process replaces the memorization of isolated facts and displays of algorithmic dexterity. Classroom teachers are encouraged to convey to their students the processes in which mathematics is discovered and communicated.

It has been suggested (Ball, 1988) that this view of what it means to know and do mathematics is very different from the mathematics instruction of both current and prospective teachers. Stein, Baxter, & Leinhardt (1990) suggest that " the subject-matter knowledge necessary to support the instruction that will foster this new view of mathematical competence remains underspecified" (Stein, Baxter, & Leinhardt, 1990, p.641). They argue (Stein et al., 1990) that the realization of this new view of mathematical competence will not take place without systematic attention to subject-matter knowledge and " how both current and desired levels of teacher knowledge impact instructional practice" (Stein et al., 1990, p.641).

The significance of a study on functions and graphs can best be described in terms of the view that "graphing can be seen as one of the critical moments in early mathematics" (Leinhardt et al., 1990, p.2). They describe these "critical moments" as sites within a discipline when the opportunity for powerful learning that is different from other learning episodes takes place. Two key features of the "critical moments" are that they are usually unmarked in the curriculum and that they are fundamental to the development of more sophisticated mathematical knowledge. The study of teacher subject-matter knowledge about functions and graphs provides the opportunity to examine this critical site of learning in the context of knowledge of the content and organization of the topic.

Research Question

What relationships can be documented between elementary knowledge about functional relationships and constructing graphs that represent these relationships?

METHOD

Subjects

The data source for this study consists of six secondary math-education majors. All of the prospective teachers were enrolled in mathematics methods classes at a major state university located in the south-eastern United States. Three of the students were scheduled to do their student-teaching during the upcoming quarter and three of the students had recently completed ten weeks of student-teaching at area high schools. The six students were randomly selected from the class. There were 4 female students and 2 male students who participated in the study.

Tasks

A set of five tasks was designed to assess elementary knowledge about functional relationships and graphs. The tasks focused on documenting evidence of subject-matter knowledge that prospective teachers have about functional relationships represented by verbal descriptions and by graphs. The teachers were asked to do the following:

1. Match a graph to a situation presented verbally;
2. a) Construct a graph from a situation and
   b) Construct a situation from a graph;
3. a) Answer a series of questions dealing with a specific situation and culminating in the construction of a graph;
   b) Do the same as (3a) but begin with the graph;
4. Choose a particular representation (equation, table,

graph) and use that representation to solve a problem presented in the verbal description;

5. Task 5 is the same format as task 4 with the exception that the set of problems were weighted toward a particular representation in task 5 and neutrally weighted in task 4.

Each of the first four tasks consisted of linear, quadratic, and exponential functions. In task 5, the questions were constructed such that a particular representation was salient and the class of function (linear) remained constant. In addition, each of the teachers participated in a card-sort task based on those described in the literature (Chi, Feltovich, & Glaser, 1981; Silver, 1979; Stein, Baxter, & Leinhardt, 1990). The set of tasks was administered individually by the researcher. For each of the tasks, the teachers were asked to "describe the relationship between the quantities in your own words" prior to choosing or constructing a graph or situation. They recorded these descriptions on their paper. After completing the problems in each task, the teachers described their strategies to the researcher. These were audiotaped and later transcribed.

Data Analysis

The set of five tasks provide a variety of data sources to assess the prospective teachers' subject-matter knowledge. The first two tasks may be characterized from the literature on functions and graphing as either interpretation or construction tasks. Tasks 4 and 5 provide the opportunity for the teachers to use a variety of representations to solve problems about situations. Of interest is the consistency of their descriptions "in their own words" with the choice or construction of a graph; the direction (situation-to-graph or graph-to-situation) that provides evidence of greater understanding; the choice and variety of representations used to solve problems; and the breadth and depth of knowledge revealed by the series of questions in task 3.

The card-sort task provides the opportunity for the teachers to categorize the cards based on a variety of criteria. There were several dimensions in which students could sort the cards:

representational format (tables, graphs, equations, ordered-pairs, arrow diagrams); the mathematical relationship depicted by several representations (all representations of $y=x$); and whether or not the mathematical relationships were functions. This task provides a backdrop in which information from the other tasks may be interpreted. The use of a variety of data sources is necessary to triangulate subject-matter knowledge about functions and graphs.

SOME RESULTS

For the card-sort task, three of the prospective teachers used the function vs. non-function distinction as an initial categorization. Within these two classes they grouped all of the cards that represented a particular mathematical relationship. These were the teachers who had recently completed their student-teaching in the area high schools. The other three teachers also grouped together the cards that represented a particular mathematical relationship. All of the teachers had a great deal of difficulty deciding what to do with the arrow diagram that represented a one-to-many situation. Everyone matched the one-to-many diagram with the graph of $y = x^2$.

Task 1 (interpretation) proved to be much more difficult than task 2 (construction). All of the teachers were able to correctly describe the situation in their own words. However, three of the teachers basically made all of the wrong choices for the graphs. They chose a cubic graph for a linear situation and a linear graph for an exponential situation. All of the teachers indicated a degree of uncertainty between the two graphs that depicted position vs. time and velocity vs. time. The axes were not labeled and the graph of velocity vs. time was requested in the situation. In the construction task (2), teachers were much better at constructing the graph from the situation than visa versa. Although they could describe the mathematical relationship depicted by the graph, in most cases their situations were not very clear. While most of the teachers performed well on the tasks dealing with linear functions, almost everyone confused exponential and quadratic functions. In this regard they were fairly consistent across the first three tasks.

Task 3 was designed as a series of questions about the variables described in the situations or depicted by the graphs. The teachers had the opportunity to " discover" the mathematical relationship by answering the questions and noting the relationship between the previous answer and the subsequent question. While they were able to describe the situation in their own words, they continued to rely on surface characteristics to answer the questions and depict the graph. No one described the relationship between the variables as " ________ is a function of ________". Instead they focused on whether or not the dependent variable increased or decreased as the independent variable increased (my words - not theirs). There was also no indication that they made the connections between specific features of the situations and the key points on the graph. In all of the situations, the initial value of the dependent variable (# of minks, area of a pizza, and # of lisrelites) was not identified as the y-intercept on the graph of the function.

Tasks 4 and 5 were designed to provide the students with a choice of representations for solving the problems. In these tasks the students were very successful at solving the problems. Of the three representations presented iconically to the students, the tabular representation was chosen least often. The graphical representation was used appropriately as a tool for problem solving (maximum number in a quadratic situation, point of intersection, and interval during which one quantity is greater than another quantity). The equations or algebraic representation was also used appropriately to determine specific values of the dependent and independent variables.

CONCLUSION

The primary objective of this exploratory study was to collect descriptive data about prospective teacher's subject-matter knowledge as it relates to the topic of functions and graphs. The results of this study support the need for further research. In particular, future studies should examine the relationship between the subject-matter knowledge of prospective secondary mathematics teachers and their instructional practices during student-teaching and in their own classrooms. While it is impossible to predict exactly how these prospective teachers will present functions and graphing in their own classrooms, it is unlikely that their presentations will reflect all of the conceptual connections and powerful representations that characterize rich well-organized subject-matter knowledge.

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Secondary
Functions
Representations

A FRAMEWORK FOR FUNCTIONS: PROTOTYPES, MULTIPLE REPRESENTATIONS, AND TRANSFORMATIONS

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Abstract. This paper summarizes several years of research culminating in an approach to teaching functions which emphasizes: 1) families of prototypic functions with associated actions in human activity; 2) coordinating the use of multiple representations in representing and acting on functions; and 3) transforming functions with an emphasis on the consistency in the actions of transformations across prototypes and representations. An important part of this pedagogical approach is the use of a multi-representational software tool.

Introduction

The concept of function has been identified as central to the secondary mathematics curriculum in several curricular reform documents including the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). This paper reports on an approach to the study of functions that has been successfully used in a number of classrooms (Rizzuti, 1991; Vedelsby and Confrey, in progress; Smith, 1991) and which combines several key features: 1) families of functions, called prototypes, are described through characteristic actions and operations related to particular human contexts and activities; 2) functions are represented through multiple forms including tables, graphs, algebraic expressions and calculator procedures; and 3) students learn to fit these prototypes to particular data through stretching, translating and reflecting. In this paper, each of these key features is discussed with brief examples. The discussion of multiple representations and transformations is presented using Function Probe© (Confrey, 1989; Confrey and Smith, 1988).

Functions Defined

Historically, two traditions in the development of functions were witnessed. Functions were viewed as: 1) the covariation between quantities. As one quantity changes in a predictable or recognizable pattern, the other also changes, typically in a differing pattern. Thus, if one can describe how x_1 changes to x_2 and how y_1 changes to y_2 then one has described a functional relationship between x and y; 2) a correspondence between values of two quantities. If one can describe how to find y (or f(x)) given a particular value for x, then one has described a functional relationship. Due to the heavy emphasis on algebraic expressions and manipulation in the secondary mathematics curriculum, the correspondence approach dominates current presentations of functions. However, we see both approaches as invaluable to the process of learning functions and seek to develop a more balanced approach to the function concept in our curriculum .

Typical definitions of function describe a relationship between two quantities, one identified as the domain and one as the range, such that each member of the domain is associated with exactly one member of the range. We will accept such a definition in this paper. However, we believe that the rejection one-to-many correspondences as functions is relatively arbitrary and is curricularly overemphasized due to the tendency to select easily measured standardized tests items. In our teaching, we spend relatively little time on the distinction between relations and functions. Our primary goal is to have students recognize and develop flexible ways to portray and act on a variety of relationships between varying quantities. With this emphasis on operational and relational concepts, we believe, for example, that it is more important to understand the reasons behind the tendency of students to resist intuitively certain kinds of "monster" (Lakatos, 1976) functions (discrete sets of points, constant functions, multiple rule functions etc. (Vinner, 1983)) than simply to label this resistance as evidence of misconceptions. The constant function, f(x)=c is a particularly interesting example of a "monster" function, for it defies students' intuitive sense that quantities should be covarying. Such examples should be incorporated gradually as they become useful in modelling applications or as the need arises to describe commonalties between well-behaved functions (the ones they want to accept) and "monster" functions that distinguish them from other kinds of relationships.

We place our work within a constructivist tradition, seeking to map and follow the construction of students' ideas rather than imposing a more singular approach. This openness of constructivism is often interpreted to mean that no curricular design can be offered. We take issue with such an assumption. A well-designed curriculum will invite students to explore a variety of approaches to functions, develop and expand their concepts in ways compatible with these experiences, and encourage them to construct connections between their own experiences and the common usages of these concepts by their larger community. However, we believe such a curriculum must be based on an understanding of student methods. Although much of what we describe in this paper and many of the design features of Function Probe result from examining student methods, due to space limitations few detailed descriptions of this work will be given. Descriptions of and references to the supporting work are provided in the final section of this paper.

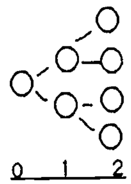
Prototypes

We introduce students to a variety of families of functions, called prototypic functions, each having a range of identifiable operational characteristics. These families include: linear (including absolute value, step functions), inverse, quadratic, high degree polynomials, exponential, and trigonometric. Algebraically, these are: $f(x)= x$, $f(x) = 1/x$, $f(x)= x^2$, $f(x)= x^n$, $f(x) = a^x$, etc.

To connect a prototypic function to characteristic operations and actions, we use contextual problems designed to help student create and identify appropriate actions. For example, one way to introduce the exponential function is to use the context of a cell splitting, building the relationship between a constant splitting action and exponential growth. Alternatively, the idea of change through the identification of a constant ratio can be investigated using a bouncing ball. Students are asked to predict the height of a ball on the nth bounce when dropped from a given initial height. In classroom situations, we often give them tennis balls and let them work in groups to create their experimental data. Compound interest provides another approach. Students typically understand that the amount of growth is a constant proportion of the amount present (i.e. the interest rate). They must deal with the issue of how this understanding of the magnitude of growth can be transformed into a way to predict the total amount present after a given number of years. A fourth approach builds on the idea of growth through similarity. A particular example we use is a nautilus shell, where succeeding chambers are similar in shape and grow on each other.

Once a student can coordinate these varying types of experience and their associated actions with the more generalized concept of an initial amount followed by repeated multiplicative growth, s/he possesses the fundamental attributes of the prototype of exponential functions. We have found that the careful exploration of several contextual problems which encourage the development of the types of action appropriate to exponential functions gives students the power to recognize situations in which these functions are appropriate. For example, students must learn to sense that the act of splitting cells can be identified with the operation of repeated multiplication per iteration, so that there must be an initial amount, a constant multiplier per iteration and a way to keep track of the number of iterations (in this case, the exponent). These kinds of experience will lead to a need, on the students part, for negative and fractional exponents, making their introduction both meaningful and necessary (Confrey, 1991). Note that this treatment of the exponential also parallels one possible treatment of the linear function prototype, that of an initial value and an action of repeated addition (or subtraction) using a constant value giving $y= b+ x\ (m)$ where b is the initial amount, m is the amount which is added per iteration and x is the number of iterations.

The Use of Multiple Representations



Becoming familiar with a functional prototype, such as the exponential, requires one to develop generalized procedures that allow one to "recognize"[1] its appearance in diverse representations, to operate with these different representations and to coordinate and contrast the actions across the representations. Thus in the cell splitting example, we would encourage students to create a picture such as the one shown on the right as a legitimate functional representation for exponentials. Many less standard representations have also been used.

In our teaching, we work extensively with four conventional representational forms: tables, graphs, algebraic expressions and calculator keystrokes using the multi-representational software tool Function Probe. Each representation yields its own insights into functions such that no one can be subordinated to another. This is in contrast to typical secondary mathematics texts and most current software programs which rely almost entirely on algebraic expression, subordinating other representations to either secondary forms or merely displays of data.

1 We place "recognize" in quotations here to emphasize that functions are not "in" a representation, rather that we construct ways by which we associate various representations with those actions we associate with a prototypic function. However, once we have constructed those connections, we will "see" the function in the representation.

Tables, for example, are perhaps one of the most under-utilized resources for the exploration and creation of functions, particularly in the rich environment they provide to construct and explore covariation. If a student can fill two columns in a table, creating an arithmetic sequence in one column and a geometric sequence in another and then describe the relationship as the change of x_1 to x_2 being constant addition while the corresponding change of y_1 to y_2 being constant multiplication, s/he is demonstrating a significant grasp of exponential functions. We have also found the table window in Function Probe to be particularly appropriate for drawing attention to such issues as rates of change, the rates of accumulation, the need for interpolation and extrapolation, (See Nemirovsky's work (1991) on how these ideas can create a bridge to elementary calculus concepts.), and the maintenance of functional relations (by linking columns) during sorting and editing without the necessity of specifying a formal algebraic relation.

Emphasizing the independence of the various representations has allowed us to reconceptualize how actions across multiple representations can be coordinated. For example, whereas most graphical software is algebra-driven -- changes in graphs can only be made by changing the algebraic parameters -- Function Probe allows one to transform a graph directly through mouse actions. This direct graphical action allows two-way communication between representations -- one can change the algebra and observe the change in the graph or one can change the graph and see the change in the algebra. This flexibility tends to minimize some of the perceptual ambiguities reported in Goldenberg, Harvey, Lewis, Umiker, West, and Zodhiates (1988). For example, the graph for the equation $y= 3x-6$ [$y= 3(x-2)$] can be formed from the prototype, $y = x$, by: 1) stretching the graph until its slope reaches 3 then translating it down (vertically) until its y-intercept is at -6; 2) translating it down until its y-intercept is at -6, then stretching it horizontally until its slope is 3; 3) translating it horizontally until its x-intercept is at 2, then stretching it vertically until its slope is 3; or 4) translating it down until its y-intercept is at -2, then stretching it vertically until its slope is 3. Predicting algebraic outcomes from graph actions, and graph outcomes from algebraic actions becomes a significant and multi-directional undertaking. Using such transformations with point-sets, in combination with the table, can contribute dramatically to students' insights in this area and provides an example of how interactions among three of the representations can be woven together.

A final example of using multiple representations comes from teaching inverses. Most students leave secondary courses knowing that one can get an inverse of a graph by reflecting around the line y=x and an inverse of an equation by reversing the x and y and solving for y. We have found that the calculator on Function Probe, which provides a keystroke record and allows one to build buttons, can provide a unique contribution to this understanding. If a student builds a function as a set of keystrokes, to create an inverse is simply to undo that set of keystrokes. For example, if a student has evaluated the function $y= 7(3^x)+ 9$ using 2 as the value for x, s/he would likely have entered: 2 $\boxed{a^x}$ 3 = * 7 + 9 = (getting 72 as the answer). Generalizing this to a button yields, j1: ● $\boxed{a^x}$ 3 = * 7 + 9 =. The inverse can be seen as undoing these actions: 72 − 9 = ÷ 7 $\boxed{\log_a x}$ 3 = (getting 2 back as an answer). Making this into an inverse button produces: j1 inverse: ● − 9 = ÷ 7 $\boxed{\log_a x}$ 3 =. Two results are significant: 1) creating an inverse, becomes identified with undoing a procedure; and 2) the log function in any base becomes seen as a way to undo an exponential with the same base, i.e. the notational inverse of a^x. The strength in this example is amplified when students work such a problem using multiple representations, creating from their outcomes a convergent and secure understanding of inverse functions.

Transformations

The development of multi-representational approaches to functions through contextual problems can appear to make the study of each prototype overly independent. Functional transformations are an important means of uniting these approaches. Algebraically, the transformations we use can be coded as: $y = A\,f(Bx + C) + D$; that is, as a linear transformation on the variable x and followed by a linear transformation on f(x). Students learn that although all of the prototypes behave quite similarly under these transformations, the uniformity of that behavior is not necessarily obvious. This becomes a major issue to be explored in the course.

Transformations are initially introduced through vertical stretches and translations of the identity function $y=x$, creating the class of linear functions. Because of the equivalence of vertical and horizontal translations on lines, we use the absolute value function to introduce the distinction between a vertical and horizontal translation, for example, whereas the graph of $y = (x-2)+6$ is identical to $y=x+4$, it is not the case that the graph of $y = |x-2| + 6$ is the same as either $y = |x+4|$ or $y=|x|+4$. Although the distinction between a horizontal and vertical stretch can be seen in the absolute value function, it becomes more apparent in the step function $y=[x]$, particularly when introduced in relation to an appropriate context. We have used a parking garage fee structure as an example, showing that horizontal stretches (which affect unit time intervals), are clearly distinct from vertical stretches (which affect costs per time unit). This context allows one to explore separately the effects of each parameter in $y = A\,f(Bx + C) + D$.

Two approaches to transformations are used, each with its own strengths and weaknesses. One, called "function building," starts with $y=x$ and builds the final function, step-by-step. The initial series of transformations creates the linear function: $y = Bx + C$. The action of the appropriate prototype is then applied to this function, creating $y = f(Bx+C)$. This is more straight-forward for some prototypes than others. For example, taking the absolute value of $Bx + C$ is simply a reflection of the portion of the graph below the x-axis about the x axis. Squaring a linear function involves a similar process. The effect of applying a trigonometric function to a given line is considerably less obvious. For all functions, however, the final transformations, stretching $f(Bx+C)$ by A and translating $Af(Bx+C)$ by D, occur in the vertical direction. Thus it is essential for students to build an understanding of how the x-intercept of the line $y = Bx+C$ is transformed under each prototype.

A second approach starts with $A(f(Bx + C))+ D$, emphasizing the importance of seeing C/B as the horizontal translation, B as the horizontal shrink, A as the vertical stretch, and D as the vertical translation. These individual transformations must be carried out in order. This approach is typically emphasized for trigonometric applications where each of these factors has a differing effect: phase shift (C/B); periodicity (B); amplitude (A); and (adjusted) initial position (D). If relied upon too early, this approach may encourage students to memorize the ordered transformations before building an understanding of their distinctions, stating, for example, that the horizontal translation is C rather than C/B or failing to build the distinctions between when they can and cannot combine vertical and horizontal stretch. It has the advantage that all prototypes can be viewed in terms of their movements on the plane under to these four transformations. A powerful outcome occurs when students learn to coordinate both approaches, developing a sensitivity to the distinctions in the formula while being able to revert to function building from a line when in doubt or when using a new prototype. The use of transformations has been repeatedly identified by students as a major strength of the courses.

A final example illustrates that transformations need not depend on algebraic manipulation: A student is given a problem where annual tuition at a university was, in some previous year, $12,000 and has been increasing at a rate of 8% per year. She initially creates the graph of the function, $f_1(t) = 12{,}000 * 1.08^t$. She is then asked to find two forms of graph actions that will transform the graph of a new function, $f_2(t) = 6{,}000 * 1.08^t$, into her previous function. She can do this through either a vertical stretch by 2 [algebraically, $f_1(t) = 2 *(6{,}000*1.08^t)$] or a horizontal translation by 9 [algebraically, $f_1(t) = 6{,}000 * 1.08^{t+9} = 6000 * 1.08^t * 1.08^9 = 6000 * 1.08^9 * 1.08^t = 6000 * 2 * 1.08^t$]. The equivalence of a horizontal translation and a vertical stretch provides an important insight into the structure of exponential and logarithmic functions. A sign that the student has understood the contextual implications of such transformations would be recognizing the equivalence of seeing: 1) $f_2(t)$ as representing a halving of the initial tuition of $f_1(t)$; or 2) $f_2(t)$ as representing an equation for the initial year being nine years earlier than the initial year for $f_1(t)$.

Conclusions

In this paper, we have described an effective framework for teaching functions. Although the paper does not report on the specific research findings which led to the development of the methods or to the design of the software, references are provided below. Research results, however, are seldom sufficient to lead to the development of a complete curriculum but do provide the conditions under which such development might be undertaken. The principles which underlie this work include the need to: 1) allow students to develop concepts of function which support both covariation and correspondence approaches and create the possibility of focusing on rate of change and accumulation per unit time; 2) develop families of functions built around prototypes which are tied to human actions and operations; 3) encourage the use and exploration of multiple representations in both traditional and non-traditional forms; and 4) value the integration of these diverse families of functions through transformations. We do not wish to imply that with such a framework, our investigations of functions are complete. We are currently engaged in an extensive pursuit of the schemes which underlie each of these families of functions. In the case of exponential functions, for example, we seek to understand its roots in a form of multiplication which is not repeated addition (Confrey, 1991, 1990). For quadratics, we are finding schemes about the ideas of symmetry, dimensionality and rates of change based in arithmetic progressions (Afamasaga-Fuata'i, 1991). This work, combined with research on how teachers develop insight into these ideas (Piliero, in progress; Vedelsby and Confrey, in progress), and how groups of students interact around these ideas (Smith, 1991) leave ample room for further investigations.

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High School
Algebra
Variables

UNKNOWN AND VARIABLE IN ANALYTICAL METHODS FOR SOLVING WORD ARITHMETIC/ALGEBRAIC PROBLEMS

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The tensions that the concepts of unknown and variable provoke in the students are the main focus of attention in this experimental study. The use of algebraic symbols is interpreted as an answer to the need for representing and operating on the more and more complex unknowns that (theoretically) the solution (through logical analysis) of arithmetical word problems presupposes.

I. INTRODUCTION

This study forms part of a large project on the solution of word problems in algebra. Relevant previous studies have been presented in P.M.E.N.A. (see Filloy, 1987; Rubio, 1990a and also Filloy-Trujillo Roshlander, 1987 and the now classical work of Krutetskii, 1976, on general skills). In this study the difficulties that appear in the teaching of strategies of logical analysis for solving certain types of arithmetic-algebraic problems are contrasted with Krutetskii's statement of the need to have certain general mathematical skills.

II. THE STUDY

<u>The study is composed of the following stages:</u>

1.- The classification of the problems according to their difficulty of logical analysis.
2.- The design of the teaching sequence on the basis of 1) the tensions that the concepts of unknown and variable provoke in the subjects. Natural tendencies are used in considering

unknowns as variables, producing what in other studies we have called "polisemia" of the x (the unknown in a linear equation -see Filloy PME- XIV-1990). The teaching strategy, then, proposes varying the data in order to obtain a multitude of problems that are equivalent from the logical point of view, and also, 2) to propose solutions that will vary during a trial and error test (what we call arithmetical exploratory analysis).

3.- Development of the strategy of teaching in a class with 16 year old pupils.

4.- Observation of this teaching strategy in the classroom.

5.- On the basis of the previous stage, all the teaching procedures are refined in order to go on, firstly, to using Spread-Sheets for carrying out the exploratory arithmetic part and secondly, using clinical interviews to relate the students difficulties with the difficulties of logical analysis of word problems carried out theoretically in the first point.

6.- Description and interpretation of the difficulties arising from the logical analysis that either appeared in class or in the clinical interviews.

7.- Description of the cognitive processes involved in the exploratory, analytical and problem resolution phases.

<u>Theoretical framework</u>

The theoretical framework of this study is based on the concept of <u>local theoretical models</u> (Filloy, 1989, 1990) in which the object of study, in this case, the solution of word problems in algebra, is focused on through three interrelated components: 1) <u>models of the teaching of algebra</u>, 2) <u>models of cognitive processes</u>, 3) <u>models of formal competence</u>.

In this study on the solution of word problems in algebra, the <u>local theoretical model</u> takes as the basic idea, the logical analysis of problems. In turn, this analysis is taken as a general model of reasoning for seeking the solution of arithmetic-algebraic problems. The logical analysis shows the way to the solution presupposing the answer (the magnitudes or quantities) is given, and deriving logical consequences from the "gives" relationships between the "given" magnitudes or quantities (the data of the problem), (see J. Klein, 1968 for his description of the analysis). The logical analysis of the problem shows the interrelationships implied by the relations between the data and the unknowns.

In its teaching component, the *local theoretical model* takes into account the analysis of problems through explorations of an arithmetic type, be they 1) using a trial and error procedure, Rubio (1990a, 1990b) (whose historical antecedents can be found in the false position methods which use, in the case of certain problems, are used spontaneously by some pupils), or else 2) starting with a logical analysis that is directed straight at the solution of the problem, focusing its interpretation and representation in a basically arithmetic way; this logical-arithmetic analysis is what allows the discovery of the relations of the problem that lead to the solution.

In its cognitive component, the *local theoretical model* links the representation of symbols with the processing of information and, thus, with memory. On the processing of classroom observation, it appears that the information processed in the memory (short and long term) when a logical-arithmetic analysis is carried out (in many types of algebraic word problems) requires adequate recovery, based on the installation in the memory of the numerical facts produced by the exploratory arithmetic analysis. In this way the path from STM to LTM, when the problem to be solved is processed, tends to become less saturated (Rubio, 1990c, Filloy, 1987) than when an attempt is made to represent the problem algebraically from the outset. The latter requires higher order processing of the information, as more information is concentrated in less symbols and their relationships, making comprehension of the secondary unknowns difficult, among other things. This can be seen in the studies carried out on algebraic errors which show difficulties in semantic interpretation, for example, Booth (1984), Trujillo

(1987), Rojano (1985), Rubio (1990). The first results of the study show promising directions for further research on teaching algebraic problems with verbal sentences constructing and/or recovering algebraic concepts using arithmetic or logical methods of exploratory analysis. On this basis, a model of formal competence of an ideal user of the conceptual apparatus of elementary algebra is generated, when the latter is employed to solve the usual arithmetic-algebraic problems in high school (Junior and Senior).

SOME RESULTS

1.- As a consequence of a teaching strategy that requires the pupils to vary the solutions to the problems, 1) we observed a tendency to thinking that any datum is possible. When asked for the invention of various problems similar to the one that has been solved, 2) the pupils tended to center on taking the data to calculate solutions instead of proposing solutions to calculate the data.

2.- Another tendency observed in the pupils when they were carrying out the logical analysis of the problem, was that they could not accept the operativity of the unknowns; that is, when they attempted the analysis, they tended to employ or to give values to the unknowns and could not manipulate them as such. Even in problems with concrete objects, when they attempted the analysis, the pupils could not follow through the corresponding process of thought (although they could accept them separately without problem). This is due to the fact that they cannot follow the logical implications that derive from thinking about something unknown, such as a number of children (syntactically this would be linked to the incapacity to accept the uninterrupted operations that arise when certain types of semantic errors are committed (Booth, 1984; Trujillo, 1987; Rubio, 1990).

3.- A natural tendency to handle a single unknown in problems that can imply handling two or more unknowns was found, in contrast with the classical teaching strategies that tend to use two unknowns in the solution of such problems (this was also reported in Filloy, 1987).

4.- Great difficulty in being able to represent one unknown in terms of another was observed, even in subjects who had overcome the difficulty expressed in 2).

5.- The possibility of carrying out an exploratory, arithmetic analysis makes it easier to show, and to make explicit if desired, the relationships between the unknowns and the data, making the logical analysis of the problem possible.

6.- We noted that arithmetic analysis led to easier access to the algebraic interpretation of the problems and to giving it meaning.

7.- By making the natural tension between the notion of unknown and that of variable explicit, through teaching (in the clinical interviews as well), various states of development were observed, which depended more on the degree of progress in the possibility of carrying out increasingly complex logical analysis and not so much on progress in the utilization of the symbolic elements of algebra and in the exploratory arithmetic analysis.

FINAL DISCUSSION

The results of the exploratory phase of this study seem to imply that the earlier concepts of algebra simplify the logical analysis and make it possible for the learner to solve word problems that he or she wouldn't be able to solve just with a logical analysis based on the use of the Arithmetical Signs System. But, it also shows that the learner needs to master late stages of the arithmetic-logical analysis to be able to carry on with latter phases of the mentioned algebrization programme. Furthermore, contrasting with its logical counterpart, (that generates unsurpassable obstructions), arithmetical exploratory analysis of word problems appears as a bridge from arithmetical to algebraic competences.

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Primary
Representation
Small Group

AN ANALYSIS OF STEPHANIE'S JUSTIFICATIONS OF PROBLEM REPRESENTATIONS IN SMALL GROUP INTERACTIONS

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This report describes one component of a three-year longitudinal study involving Stephanie, a third grader, who has been observed doing mathematics in small-group settings since grade one. Data in the form of videotape transcripts of small-group problem-solving explorations, indicate how Stephanie represents certain mathematical ideas in social situations, and how her ideas, methods, and attitude change over time.

Studies over the last twenty years comparing male and female achievement in mathematics have revealed that females do not participate in advanced mathematics to the same extent as males and that achievement in mathematics is higher, on average, for males than for females (Fennema, 1990). Currently, researchers are acknowledging that before educators can know more about mathematics and gender, they need first to know more about the characteristics of learners who do or do not succeed in mathematics, as well as their schools and classrooms (Leder & Fennema, 1990).

The predominant mode of analysis for much of the research on mathematics and gender has been to focus upon the behaviors of teachers as they interact with children. Koehler (1990) reports two studies that involve small groups and gender differences (Webb, 1984; Webb & Kenderski, 1985). In these studies, male out-performance of females was explained by three factors with relationship to student-student and teacher-student interactions in the classroom: (1) males received more explanations than females; (2) females did not receive answers for their requests for help; and (3) both males and females asked for help most frequently from males.

The results of these small group studies suggest that the analyses focused on the responses of teachers to student questions and students to each other's questions. The setting seemed to be "authority centered", that is, the justification for correctness of solutions came from some authority within the classroom rather than the logic of the situation. What is unclear from these reports is the extent to which small group organization was usual classroom practice. Cobb, Wood and & Yackel (1990) argue that in order for mathematical communication to be shared, the classroom norms must be "taken-to-be-shared" by all members of the community.

Clearly, there is a need for research to be conducted in classrooms in which mathematical conversations and open sharing of problem-solving strategies are regularly a part of the child's mathematical environment. In particular, some questions emerge that have potential implications for gender studies conducted in learning environments which encourage students sharing, asserting and defending their mathematical ideas. Before addressing the gender issue directly, research needs to take place which focuses on the construction of mathematical knowledge for all individuals in learning environments which value student discussion and sharing of solutions.

This report focuses on the study of a female student, Stephanie, who has been observed over a three year period dealing with a variety of mathematical situations in small-group settings. We encountered Stephanie for the first time in March 1989 as a first grader. She and three other children were working together on a mathematics problem. Careful viewing of the videotape of this classroom episode revealed that Stephanie exhibited considerable *assertiveness* (persistence in stating and defending her position) by insisting that her solution be considered and calling for *justification* (the provision of evidence to support one's position) for the solution presented by another group member. What was particularly noteworthy about this episode was that Stephanie was the only girl in this four member group. These preliminary observations of Stephanie raised a number of interesting questions.

Guiding Questions

1. What criteria did Stephanie and the other children use to validate or to reject their own ideas and the ideas of others?
2. What role did *assertiveness*, the ability to state and defend one's own position, play in Stephanie's communication of her ideas?
3. Were there patterns of assertiveness and/or validation for children engaged in these small-group explorations that emerged over time?

Theoretical Framework

The study is based on the view that children, given the opportunity to be challenged by problem situations, cycle through various steps as they build representations of those situations (Davis, 1984; Davis & Maher, 1990). Careful analyses of videotape transcripts of children doing mathematics enables a detailed study of how children deal with mathematical ideas that arise from the problem situation. By following these episodes over time, researchers can learn much about how children's thinking has developed.

Design

The setting in which our observations take place is the Harding School, a K-8 district which has participated in a mathematics teacher development project since 1984 (Maher, 1988; Maher & Alston, 1990). This research was conducted in classrooms where children have worked in small groups on problem-solving explorations since their entry to grade one. In this environment, children are encouraged to relate abstract or symbolic ways of working with more concrete representations, using pictures or actual physical models.

This paper will present a fine-grained analysis of three videotape transcripts of classroom small-group problem-solving activities, with the accompanying student written work, spanning grades 1-3. Specifically, this analysis will focus upon Stephanie's building of representations of problem solutions as well as her ability to validate and reject her own and the ideas of others. It will also explore Stephanie's patterns of assertiveness and how these served to facilitate or impede her own problem-solving.

Results

The three small-group episodes will be presented in chronological order.

Grade One: Four children, Stephanie (St), Gerardo (G), Aaron (A) and Sean (S), were working in a small group to solve the following problem:

> The kangaroo jumped six times. If the rabbit jumps four more times, he will have jumped as many times as the kangaroo. How many times has the rabbit already jumped?.

St: (to *Gerardo*) Listen! These two boys are going to have to figure this one because you and me already figured this one out...Ok? So I'm going to read you the question and you're the two who are going to figure it out. Aaron...you didn't answer a question and Sean you didn't...

Notice Stephanie's insistence upon fairness within the group. She asserts that she and Gerardo have solved a problem, so the next problem should be solved by Aaron and Sean. She is orchestrating a division of labor.

G: So it would be six...because the kangaroo jumped six times!

Gerardo provides an answer with an explanation that is insufficient for Stephanie who now demands justification for the answer of six.

St: Wait a second buddy...you can't just say six!

G: It's gotta be six.

Gerardo reasserts his answer with no justification for his choice. Therefore, rather than pursue Gerardo's assertion, Stephanie decides to validate this problem with her own concrete model.

St: Wait...I think we should read this word (*the word is already*) Wait for us buddy (*to Gerardo*) remember, you can't just jump to conclusions like "I know this"... wait...let's just try these five...no six...jumped six (*Stephanie groups six unifix cubes and takes away four of them*) six and four...two! Put two over here...1,2. We did it. It's two! Do you want to go over the problems and figure out if they're right?

In this episode, Stephanie asserted and justified her own solution with a physical model. She was clearly searching for justification, but did not actively pursue the reasoning of others. This and other observations made in grade one formed the basis for further study of Stephanie's monitoring of group problem-solving and her need to justify all paths of solution. Stephanie consistently demonstrated confidence in her mathematical thinking and refused to accept solutions at "face value".

Grade Two: Four children, Stephanie (St), Dana (D), Michael (M) and Sean (S), worked together to determine how many single units would be needed to construct a base 4 cube (4 x 4 x 4 dimensions). The children were given a set of base 4 blocks which included units, longs of four units, flats of sixteen units and a cube of sixty four units. This excerpt begins with Stephanie monitoring the selection of a solution strategy. Note the progressive sophistication of her methods for choosing the group's solution strategy, and her need to justify her method for the following problem:

How many small blocks do we need to make the big block?

St: Let's decide on a way we all want to do it.

D: Would you like to do it by tracing the sides?

Dana planned to trace all six faces of the cube to find the number of units.

St: (*joining*) Or would you rather do it like this...

M: Give me that! (*referring to his cube*)

St: Wait a second. Trace the box and then you'd be able to fill the box and figure out the picture.

Stephanie wanted to trace one face of the cube and rebuild the cube inside her traced outline.

S: Yeah! Wouldn't that be easier?

St: Let's vote, who wants to do it my way? *(Sean raised his hand)*.

St: Who wants to do it Dana's way? *(Mike and Dana raised their hands)*.

D: It's two against two.

St: Two against two...so we'll have to do odds and evens. Evens is mine, odds is hers. Ready Dana...one,two...I won! We're doing it our way.

As they began to use Stephanie's strategy, she provided a more legitimate means of justification for her method of solution.

St: You know Dana why you can't do it that way..because you have to work around in the middle... in order to get in the middle...if you were going to do it your way what would happen is you would be doing the outside and you wouldn't be doing the inside. What I'm saying is you have to get all the blocks. *(Dana nods)* But what you were doing was this...you were going around it *(referring to the surface of the cube)*. If you do that there is no way you can fit the blocks inside *(referring to unseen blocks at the center of the cube)*.

Stephanie used an explanation which compared surface area to volume, and pointed out that Dana's method would not account for the blocks which were not visible to the eye. Thus, she had replaced voting strategies and the "odds and evens" method for selecting the best strategy with a well conceived explanation of her method of solution. This transcript indicates that Stephanie was simultaneously monitoring her own construction of a solution and those of her classmates, then judging which method was best.

Grade Three: Stephanie continued to refine her explanations and group monitoring techniques. She (St) worked with Dana (D) on the following problem:

> Stephen has a blue shirt, white shirt and yellow shirt. He also has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

Note that Stephanie subtly monitored her classmate's work as well as her own by drawing a simple diagram to keep track of her outfit combinations.

D: ...how many different outfits can he make?

St: We.. why don't we draw a picture?

As Stephanie and Dana drew their pictures (See Figures 1 and 2), we see them focusing on the pieces of data that deal with numbers of shirts and pants and their colors. In so doing, they searched for a way to map their knowledge representation into the data representation of the problem.

D: Ok...he had a white shirt *(The girls drew pictures of shirts)*.

St: So I'll make a white shirt..*(Notice that Stephanie was checking her representation as she drew her picture to match the problem data).*

D: A blue shirt...

St: I think I'll have to use the big marker for this one...you know color it in blue *(decided to color the shirt blue to match the problem data).*

D: And a yellow shirt *(The girls drew another shirt).*

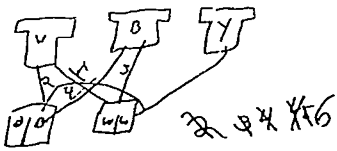

Figure 1 - Stephanie's work

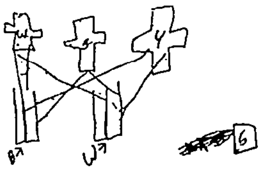

Figure 2 - Dana's work

At this point, Stephanie suggested that the data be coded by assigning the first letter of the color rather than coloring the piece of clothing.

St: Why don't we just draw a Y, a B and a Y (sic) instead of coloring it in? *(Began to use a one letter code to represent garment color).*

D: That's what I'm doing...

St: W, B, Y *(she put a letter in each shirt to denote color).* Ok, he has...

Stephanie indicated that she had begun to construct a representation of the relevant knowledge and proceeded to draw two pairs of jeans, blue and white.

St: Alright let's find out how many different outfits you can make. Well, you can make white and white so that would be one...I'm just going to draw a line...*(connected each shirt/pant combination with a line and attached a number label to each of these lines).*

Later Stephanie and Dana were questioned by the instructor about their purpose in using connecting lines. Stephanie replied, "So we could know if we already matched that (any shirt and pants combination). So we don't get more that we were supposed to." For Stephanie, the connecting line strategy was a way to check one's work by avoiding the repetition of combinations as well as a method for obtaining those combinations (Figure 1).

Concluding Remarks

Over the course of several years, classrooms such as those at the Harding School, which embody sharing ideas and providing explanations to justify those ideas, are an important asset to children's construction of mathematical knowledge. In grades two and three, Stephanie continued to build models to justify her solutions. She demonstrated competence in assisting other group members as they pursued and refined their own strategies. One pattern that manifested itself in grade two and was reinforced in grade three was greater attention to multiple justification of arguments provided by others. She listened to the ideas of others, and rejected or integrated them into her representation of the problem solution.

In this longitudinal case study, Stephanie's criteria for the validation or rejection of ideas seems to have been "sense-making". Some ideas she was able to accept because they mapped into her representation of the problem situation, others did not and were rejected. This learning environment of open sharing provided Stephanie with the opportunity to further develop her own mathematical knowledge by allowing her to share her ideas with others. In this forum, she demonstrated her ability to assert her mathematical beliefs and justify them in a variety of ways (physical models, comparing strategies, drawing diagrams). A modification and refinement of her approach was that in pursuing her own problem-solving strategies, she simultaneously referred to and monitored the strategies of other group members.

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Elementary through College
Theoretical/Conceptual/Frameworks
Multiplicative Structures

A THEORETICAL-CONCEPTUAL ANALYSIS OF U.S. AND SOVIET STUDENTS' UNDERSTANDING OF MULTIPLICATION

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This paper reports on an assessment of conceptual structure in two groups (from the U.S. and Soviet Union) whose formal mathematics instruction differed significantly. The Soviet group experienced three years of a curriculuum which was explicitly designed to develop conceptual structure using a Vygotskian psychological approach. Differences reflecting that approach were found in the psychological structure of multiplication, the concept investigated.

The Soviet research was conducted during the fall of 1990 in cooperation with the Academy of Pedagogical Sciences of the USSR. Results were compared with results obtained earlier in similar investigations conducted in the United States (Schmittau, 1988, 1989, in press). Data from written instruments and clinical interviews were obtained from 40 American and 24 Soviet subjects.

The Soviet subjects were drawn from the fourth, fifth, and upper secondary forms of neighborhood public schools in both urban and village settings. All had learned mathematics during their first three years of schooling using experimental materials developed by V.V. Davydov (1975), which were designed to promote conceptual integration and the development of real number concepts in measurement contexts.

The U.S. subjects were drawn from a highly selective university and from middle class public secondary schools in the eastern United States. Seventy-five percent of the U.S. secondary students were identified as high achievers in mathematics, fifteen percent as average, and ten percent as low achievers. Soviet lower form subjects were evenly divided between average and high achieving, and all upper form secondary students were rated as average. Despite the age and achievement rating disadvantages of the Soviet subjects vis-a-vis their American counterparts, no disadvantages with respect to conceptual structure on the part of these subjects were in evidence. In fact, Soviet fourth and fifth-form school children often gave evidence of powerful relational understandings not found in U.S. secondary and university students.

The investigations were designed to assess the psychological structure of multiplication for real number instances. Psychological structure is described by Ausubel as the "cumulative residue of what is learned, retained, and forgotten" (Ausubel, Novak, & Hanesian, 1978, p.129). The psychological structure of a concept which has been meaningfully learned typically reflects a progressive restructuring of knowledge under more general and inclusive higher order concepts.

Hence, the investigations sought to determine whether knowledge structures were hierarchically integrated or fragmented (indicating meaningful or rote learning, respectively), and whether connections were established along formal, conceptual, or prototypic lines. This latter consideration reflects the current level of our understanding of category structure, which was modified by Rosch's (1973) challenge to the classical view of organization according to genus and differentia. Numerous studies have corroborated her findings of prototype effects in a wide range of perceptual and semantic categories (Lakoff, 1987).

To assess for prototype effects, a ratings instrument based on the work of Rosch and following the design of Armstrong, Gleitman, and Gleitman (1983) was administered. Using a scale of 1 to 7, subjects were asked to rate instances of multiplication for their degree of membership in the category, with a "1" indicating a high degree or "best exemplar" and a "7" a low degree or "poor exemplar" membership status. The ratings data suggest that for U.S. subjects the instance "4 x 3" functioned as an exemplar for the category. Organization around a cardinal number prototype was confirmed by triangulation across two additional data sources: a) subjects' explanations of the meaning of multiplication and b) clinical interviews on the manner in which the various instances were understood as multiplication.

While all American subjects rated the cardinal number instance "1", no such clear prototype emerged from the Soviet ratings.

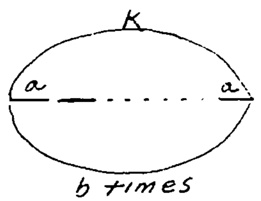
RATINGS TASK -- Mean Ratings

	U.S. Studies		Soviet Study	
Instance	University	Secondary	4th Form	5th Form
4 x 3	1	1	1.8	1.9
2/3 x 4/5	2.3	2.3	-	2.9
ab	1.9	1.7	2.3	2.5
(2x+y)(x+3y)	2.7	2.8	2.7	3.4
(-5) x 2	2.2	1.9	-	2.8
(-3)(-2)	2.6	2.2	-	3.1
$\sqrt{2} \cdot \pi$	3.4	2.8	-	-
A = bh	2.3	2.3	1.7	1.8

Indeed, for the lower form students especially, the instance of rectangular area was rated as more representive of multiplication than the instance "4 x 3". In addition, every American subject defined multiplication at the operational level as the repeated addition of positive integers, while the Soviet students described it as an action which they flexibly applied across numerical and algebraic domains, extended to lengths of line segments, and expressed with rows of squares whose repetition generated rectangles. It seems probable that these differences reflect, first of all, the influence of the theory of activity in Soviet psychology; and second, cardinality emphases in elementary education in the United States as opposed to the Soviet emphasis on measurement contexts in the development of the real numbers (Davydov, 1975; Minskaya, 1975).

Clinical interviewing confirmed that for the American students the conceptual structure of multiplication for real number instances was not only organized around a prototype, but that this prototype functioned as a rudimentary concept to which other instances of the category were linked with difficulty or not at all. For the Soviet subjects, however, the conceptual structure of multiplication appeared to be highly integrated--lacking the prototype effects found in the U.S. studies--and organized around a conceptual base of greater generality. The expanded conceptual base promoted relational learning and allowed for the subsumption of other instances of the category and their meaningful integration into cognitive structure. At the same time, the understanding of multiplication as an action connected with the children's action schemas and allowed their understanding to develop to a high level of abstraction and generalization--to proceed, in fact to algebraization.

These differences were obvious, even in the case of those instances, such as "ab", which appeared to have meaning for both the U.S. and Soviet subjects. Typically, the American students substituted small positive integers for "a" and "b", thereby effecting a reduction to the prototype. The Soviet children were more likely to represent "ab" with a schema which constituted a representational embodiment of the action of repetition understood at an abstract level. The following schema, in which $a \cdot b = k$ is illustrated, was provided by Soviet children who were beginning their fourth year of schooling.



Other differences between the two sets of subjects were observed which related both to autonomy and to the use and extension of the knowledge base. While American students preferred easier tasks and small whole numbers, the Soviet children dismissed these as uninteresting, choosing large numbers and decimals and preferring tasks such as multiplying fractions or binomials, which they had not yet encountered in the classroom. Even the youngest students were successful with such tasks, consistently demonstrating their ability to extend their knowledge in the zone of proximal development identified by Vygotsky.

Space considerations preclude a description of the many differences which surfaced between the Soviet and American students' mathematical understandings. However, the following incident illustrates the relational nature of the Soviet children's knowledge. A fifth-form girl, having successfully accomplished all tasks connected with the preceeding instances, stated that "(-5) x 2" was "the classic example" of a times b" and that one "could put any variant in as a and b." She announced that she was changing to "1" her rating for "ab". She changed all of her other ratings as well, and proceeded to diagram her understanding of multiplication, explaining how it was ordered under the general schema for "ab". The U.S. subjects, by way of contrast, subordinated their understanding of "ab" to that of the multiplication of two cardinal numbers.

Although space considerations limit not only the reporting of results, but the full development of the lines of inquiry of the study as well, it is important to point out that the results do not merely present yet another chronicle of American students' misconceptions (vis-a-vis the more adequate conceptualizations of their Soviet counterparts). Rather they reflect <u>the psychological organization of the category itself</u>. The works of both Vygotsky (1962) and Ausubel (1978) allude to the important consequences of these organizational differences.

The first consequence concerns the difference between the psychological and the logical structure of the concept under investigation. We did not find knowledge connected at the formal level for any of the subjects, whether Soviet or American. The second concerns the fact that for U.S. subjects the psychological structure was not only prototypic but <u>organized around the most conceptually restrictive instance in the category</u>. Given the predominant tendency toward assimilation into the existing cognitive structure (Ausubel, 1978), it was not advantageous for the American subjects, that after 8 to 14 years of schooling multiplication should remain for them at the most rudimentary level of understanding, organized around a conceptual base of insufficient generality to support the subsumption of other instances of the category. As one tenth-form Russian student observed, multiplication as repeated addition of positive integers "does not apply to [other instances such as] irrational numbers; for irrational numbers you need to use area, and for fractions it is easier to use area as well." In fact, the Soviet students' relational understanding of both area and irrational products was far superior to that of their U.S. counterparts, for whom irrational numbers themselves often had no meaning.

Finally, given the stability of the conceptual framework, once it is established, to function as a subsumer for new knowledge (Ausubel, et al. 1978), the extent to which the establishment of an inadequate conceptual framework limits meaningful learning becomes obvious. The difficulty of integratively reconciling later multiplicative understandings based upon area considerations is reminiscent of similar difficulties engendered by the Greeks' isolation of number from magnitude, which extended historically throughout most of the succeeding two millenia.

By structuring the elementary mathematics curriculum around measurement rather than cardinality, Davydov has developed <u>from the very first years of schooling</u> a conceptual base of sufficient generality to subsume other instances of the category. Rather than being faced (as are U.S. children) with the necessity of a total reorganization of psychological structure around newly introduced concepts of greater generality (a Herculean cognitive task), the Soviet children's pedagogical experiences work <u>with</u> rather than against the prevailing cognitive tendency toward assimilation.

As the Soviet results indicate, multiplication for real number instances is not a category subject to an invariance of natural development, but is instead modifiable through the application of principles derived from the psychology of learning. Davydov's curriculum materials reflect not only Vygotsky's (1962) emphasis on the importance of the conceptual framework, but his cognitive developmental theory of the internalization of action as cognition.

In addition to developing and refining Vygotsky's ideas over six decades of research, Soviet psychologists have also proved their applicability to the learning of mathematics. In so doing, they have demonstrated a) that mathematics education can profitably tap its foundations in order to inform its practice, and b) that there may be, after all, nothing quite so practical as a good theory.

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BUILDING A REPRESENTATION OF MULTIPLICATION: A DESCRIPTION AND ANALYSIS OF ANGEL'S RECONSTRUCTION[1]

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This paper describes and analyzes the mathematical activity of a fourth grade boy, Angel, as he develops and explains pictorial constructions related to multiplication. It focuses on his creative reconstruction of a problem situation in which he builds and maps his numerical to his pictorial representation, developing a complex, multi-level problem situation involving multiplication.

Davis (1984) describes thinking about a mathematical situation as involving a series of components that include constructing a representation of the input data, building a representation of relevant knowledge that may be used to solve, or attempt to solve, the problem, and mapping between the representations of data and knowledge. According to Davis, checks are made along the way as the learner attempts to develop this mapping and other knowledge may be entered, causing certain representations to be modified or rejected. When the learner is satisfied with what has been constructed, various strategies and procedures associated with the particular knowledge representation may be applied to carry out the solution to the problem at hand. Examples of such constructions can be found in the problem solving activity of Brian, as he is engaged in problem solving activities that encourage the building of representations (See Davis & Maher, 1990 and Maher, Davis & Alston, 1991). Studying in detail children's problem solving in environments that provide the stimulus and tools for constructions, suggests that children can develop meaningful understanding of mathematical ideas from an early age (See, also, Maher & Martino, 1991; Maher & Alston, 1989; and Landis & Maher, 1989). In the domain of multiplicative structures, Vergnaud (1983) suggests that children from an early age develop understanding through engaging in meaningful problem activities. His position is that children's problem solving should be carefully analyzed in order to facilitate this development.

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The purpose of this paper is to describe and analyze, in detail, the mathematical activity of a fourth grade boy, Angel, as he develops and explains particular constructions. In particular, the focus is on the process by which he reconstructs a picture of three fish tanks, each containing a different number of sea creatures, to build a representation of a multiplicative relationship.

Methods and Procedures

Angel's instruction in mathematics in grades 1 through 3, as a member of an urban, New Jersey classroom, had depended almost entirely on memorizing number facts and procedures. Lessons, for the most part, consisted of the children working individually to complete sets of practice exercises. Two task based interviews were administered to Angel within a one-week period of time in November. A variety of materials were available during the tasks including Unifix cubes, base ten blocks, and paper and pencil. The interviewer was guided by prepared protocols to explore whether the student could accomplish the following: (1) communicate procedural knowledge of multiplication; (2) construct and explain ideas about multiplication using concrete materials and/or pictures; and (3) select from a group of pictures those that seemed appropriate to use in learning about multiplication and to explain these choices to the interviewer. As a part of each protocol, the interviewer asked Angel how he would explain what the various problems might mean to a child who had not yet been introduced to multiplication in school.

Results

Angel's Thinking about Learning Multiplication. In each interview, when Angel was asked about how a young child learned about multiplication, he immediately referred to procedures and learning facts. In the first interview, Angel was asked how he would explain what 4 times 7 means to a 2nd grade girl, Nina.

Int: She's probably heard some of the big kids talking about times.... and she wonders.... 4 times 7...What does that mean?

Angel: I'd get my "trapper keeper" (matrix chart of multiplication facts) and it's got all that stuff that will teach you. Four (pointing down an imaginary vertical side of a chart) and then go seven (over) and then come to (28 in the matrix cell)....

Angel was asked again at the beginning of the second interview how he might teach multiplication.

Int: What would you do to help children first start thinking about multiplication?
Angel: Write them some easy problems, with the ones or with the zeroes and let them add it.
Int: And let them add it? Give me an example.
Angel: Like nine ... or one times one equals one ... Three times zero is always zero. I'd do it step by step like that.

Episodes from Interview One. Throughout the interview, Angel defended his solutions by explanations based on addition and/or counting, often constructing and reconstructing the numerical representation of the solution.

Example One: Angel's response to the first question, *What is 4 times 7?*, was to immediately write out 4 x 7 = 28, apparently as a memorized fact. When asked to defend this, he spontaneously used the base-10 blocks, counting out four groups of seven small cubes and then counting the total. He then regrouped the cubes and traded the two sets of 10 small cubes for two "longs", indicating that this also represented 28. The interviewer questioned Angel about how he might explain 4 times 7 to Nina, the 2nd grade child.

Int: Is there any other way you could show her? What if she didn't know how to read that chart (the "trapper keeper" matrix referred to earlier)?
Angel: Count on your fingers. Can I write on here?
Int: Sure (Helping Angel reach a blank sheet of paper).
Angel: 14, two times (writing two 14's, one above the other), 8 (writing 8 below the two 4's). Two (writing 2 below the two 1's).
Int: Now what did the 14's stand for?
Angel: The 7's. Four 7's and that's this (pointing to one 14) two times. Make it like 7, 7, 7, 7 (writing the sevens vertically with a plus sign before the fourth 7)...14... 14...28.

As the interview continued, Angel was asked to solve other problems, including 12 times 4 and 4 times 12. For each problem, Angel first computed the answer numerically by adding.

Example Two: For 12 times 4, after adding 12 four times for a sum of 48, Angel was asked how he would explain the problem to Nina. He counted out 40 Unifix cubes and 8 small base-10 wooden cubes, grouping them as 4 stacks of 10 Unifix cubes and one group of 8 wooden cubes.

Angel: I'd take all these apart (holding up a stack of 10 cubes) and then I'd count them: 1,.....(pointing to each individual cube in the four stacks of 10), 40... and then I'd say that these do the same thing (placing the 8 wooden cubes on top of one of the stacks of 10)....41, 42, 43, 44, 45, 46, 47, 48. She'd (Nina) probably count on her fingers.

When the interviewer questioned Angel as to what these 48 cubes had to do with 12 times 4, he removed the eight wooden cubes and added 2 more Unifix cubes to each of the four stacks of 10.

Angel: Twelve. Four 12's...and those are the 8 that I had...2, 4, 6, 8 (pointing to the wooden cubes and then touching the 2 cubes that he had attached to each of the 4 stacks of 10).

Angel said immediately that 4 times 12 was "the same thing" as 12 times 4. When asked if there was any difference between the two, he first said 12 groups of 4, but then rewrote the addition as follows, whispering to himself and counting on his fingers as he wrote vertically:

$$\begin{array}{r} 16 \\ +\ \underline{16} \\ 32 \\ +\ \underline{12} \\ 44 \\ +\ \underline{\ \ 4} \\ 48 \end{array}$$

Int: Well, that says 12 times 4, so why 16?
Angel: I just made it different, but I come up to the same answer like the four of these (pointing to the four 12's) together be 48 and I get two 16's out of that.
Int: OK, but how did you get the two 16's? That's what I don't understand.
Angel: Off the 48. You make the same answer. [Perhaps indicating successive subtractions of two groups of 16 from 48].

Angel seems here to be inventing a new notation to represent actions of decomposition and partitioning of products from partial sums. When questioned about how Nina would understand 4 times 12 from this, Angel wrote twelve 4's on the paper and began to count.

Angel: 8 (placing fingers on two 4's)...16 (pointing to two more 4's and then holding up four fingers) 17, 18, 19, 20.

He continued with this process, using his fingers to indicate each 4 in turn, until he finally announced: "48... 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 (pointing to each 4). There's 12." In this instance he seems to be representing skip counting by fours.

Episode from Interview Two. In the second interview, Angel was asked to select pictures from a set that he could use to help young children to learn about multiplication. Among his choices was a picture of three fish tanks, each containing a different number of sea creatures, which had been intended to serve as a distractor. Angel's reconstruction of the picture along with his numerical representations give us further insight into his thinking about multiplication (See Figure 1).

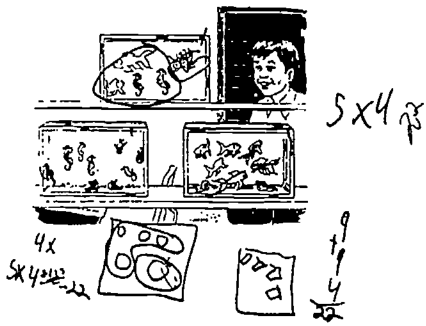

Figure 1. Angel's reconstruction of sea creatures with his numerical representations

Int: How would you use this one for multiplication?
Angel: Have to count - 1, 2, 3, 4, 5. (Uses his pen to point to each of the fish in the upper tank and then looks at the next tank.) - 1, 2, 3, 4 - (pauses for a moment) - 5, 6, 7, 8, 9. (points to the fish in the lower left tank.) ... Can I draw another box?
Int: You can do anything you want to.

Angel drew a box at the bottom of the page, crossed out the five snails in the bottom left tank of the picture, and drew five circles in his box. He drew a second box. Using his pen, he pointed to the creatures in the bottom right tank, appearing to count them. He marked three fish and one snail in the tank and then drew four squares in the second box.

Int: (Pointing to the second box) OK - Now tell me what this stands for?
Angel: The fishes.
Int: Oh, so you took those fishes and put them down here?

Angel: Yes - (pointing to the upper tank and counting its contents to himself) - five - (pointing to unmarked creatures in the original lower left tank) - Nine - (moving to the tank on the right and pointing to each unmarked item) - 10, 11, 12, 13 - (then pointing to each circle in the bottom left box) - 14, 15, 16, 17, 18 - (finally pointing to each square in the bottom right box) - 19, 20, 21, 22.

Int: (Pointing to the right tank and the box beneath it) So you put these fish in this box? - And these ones? (Pointing to the bottom left box) Where did these come from?

Angel pointed to the left tank, indicating that the circles represented the creatures crossed out in the tank above. He then began to point to the different tanks and to count the creatures. He first wrote the vertical addition of 9, 9, and 4 shown in Figure 1. After completing the addition, Angel paused and then wrote 5 X at the bottom left side of the page. He raised his eyebrows, paused again, moving his mouth as well as his head, eyes, and the fingers of his left hand. After several seconds, glancing at his addition problem, he finished writing the symbolic statement as indicated in Figure 1.

Int: OK. Show me what you did. 5 times 4 (pointing to what he wrote)
Angel: And 2 plus that - a half of - a half of a 4.
Int: So 5 times 4 was what?
Angel: 20.
Int: And then you added?
Angel: 2....and you got 22.

When questioned by the interviewer, Angel circled the single fish in the top tank and the single circle in the left bottom box, explaining that these represented the two that he had added.

Conclusions

In the first interview, Angel produced symbolic mathematical statements that were remotely related to multiplicative relationships. His responses were based on numerical representations of the solution. This is not surprising; his experiences with mathematics in school were symbol driven. During the second interview, in constructing his pictorial representation, Angel modified a picture which included three fish tanks, each with a different number of sea creatures. His new construction contained two more tanks into which he "transferred" some of the creatures in order to develop groups of four. He then constructed a numerical representation to accompany his explanation of his production (five times four plus one half of four.) Angel's reconstruction of the pictorial representation of the fish tank problem was then mapped into his representation of numbers of sea creatures. As he built his "reconstructed" tanks, he modified his numerical representation to produce a more complex multi-level problem. What is particularly interesting in Angel's problem solving, is the process of his reconstruction. The symbolic representation of numbers of fish and numbers of tank had meaning for him. Angel, monitoring and building the representation, checked the input data and how it mapped between the two representations of the problem situation.

The study suggests the power of a child's thinking when given an opportunity to build task conditions in an open-ended problem situation. It provides insight into the powerful reasoning of a child engaged in a task that provides opportunity to create and connect different modes of representation.

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Teachers
Concepts
Integration

UNDERSTANDING THE INTEGRATION CONCEPT BY THE TEACHERS OF ENGINEERING SCHOOLS.

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This document discusses the result of a research done with teachers of engineering schools, by studying their ways of conceiving the integration concept when faced with specific situations of continuous variation, capturing their explicative models tied to their notions, intuitions and practice that emerge in the continuous variation.

INTRODUCTION

The problems in the transmission of knowledge in Calculus is related to the explicative model chosen in the Scholastic Mathematical Discourse (SMD) [4]. This model respond more to the demands of the Calculus formal system then the requirements of a significations system, based on the notions, intuitions, and experiences of the teachers and students faced with specific situations of continuous variations.

This last system is practically absent in the SMD, aspect that could damage the efficiency in the transmission of knowledge in Calculus. This research is aimed at exploring the ways of thinking the ideas in Calculus under the scheme of a signification system, to analize the explicative models that may derive from them, to draw didactic situations that will allow, given the case, an efficient transmission of knowledge of Calculus.

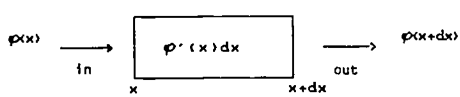
We present here, the study of cases with mathematics teachers in engineering schools under a controlled teaching experience focused on the concepts of integration. The study consists in an epistemological analysis of integration concept associated to the genesis of the concept, comprehended in two stages: the old Calculus texts and the treatises on the movement of fluids. Recognizing a construction pattern of the integral tied to signification of the accumulation notion allows to view Integral Calculus (IC) in two directions: based on a movement system, where the most important aspect lies in the *acumulation notion and the accumulated value* reconstructing elements of the quantities that vary continuously, and, the other, based on the theoretical system of IC, where the most important concept of the general integration theory is the primitive function [2]. Both visions of IC are analyzed in the expiicative models of the teachers and students placed in continuous variation surroundings.

SOME ASPECTS OF THE INTEGRATION PATTERN AND ITS SIGNIFICANCE.

The basic idea of the pattern consists in observing or rather recognize the difference between an invariant state and its adjacent states in the movement system of a particle or a continuous fluid

$$\varphi(X+dX)-\varphi(X)$$

where $\varphi(X)$ represents the invariant state and $\varphi(X+dX)$ its adjacent states. The difference expresses the last position of the particle or the local acumulation of the fluid



$\varphi(x+dx)-\varphi(x)=\varphi'(x)dx$
accumulation

The last position and the local variations are obtained by the variation of in the whole system's process, which is recognized by the difference $\varphi(X+dX)-\varphi(X)$; this difference represents, on one hand, the displacement portion needed to reach its last position and, on the other, the accumulated portion of the fluid. If we join or integrate these portions we will find the final position of the paricle and analogously the total accumulation of the fluid at the end of the process. When recognizing

$\varphi(x+dx)=\varphi(x)+\varphi'(x)dx$, we know $\varphi(x)=\varphi(a)+\int_a^x \varphi'(x)dx$ and by recognizing $\varphi(x+dx)-\varphi(x)=\varphi'(x)dx$, we know $\varphi(b)-\varphi(a)=\int_a^b \varphi'(x)dx$.

Within the mathematical analysis requirements,the model can be reformulated for functions

$F:\mathbb{R}^k \longrightarrow \mathbb{R}$ and functions of $F:\mathbb{C} \longrightarrow \mathbb{C}$, in the following

manner:

''...if the function $F(x)$ complies with $F(x+dx)=F(x)+f(x)dx+o(dx)$, so that $f(x)=F'(x)$, in the $[a,b]$ interval and $o(dx)$, represent an infinitesimal order bigger than dx, which is equivalent, in the limit context, to $O(dx)/dx \longrightarrow 0$ when $dx \longrightarrow 0$. Hence the integral can be defined as

$$\int_b^a f(x)dx=F(b)-F(a) \text{ [2].}$$

The ideal conditions of F(x) for the model to take place when faced with different physical and geometrical situations are that F(x) be of the type C^{∞} or be analitical [2].

An aspect to mention on the way of thinking about the integration by the professors and the impact of this pattern on the student activities for the comprehension of the integral in regard to continuous variation situations, is that the discussions on integration starts precisely with the "unknown" quantity (primitive function) that has to be found, requiring this to recognize its vatriation (derivative function) in the situation's context to finally know its integral (know quantity). This symptom can be distinctive from the traditional scholastic discourse on integral Calculus, in which, it is normally asked about the integral of an arbitrary functions starting from a definition of the integral. This matter leads necessarily to a different discourse, that we will describe.

PLAN OF THE STUDY

We worked with 13 professors of engineering schools, in 4 sessions of 10 hours every 15 days, within the activities of the Mathematical Teacher's Formation Program [4], in a controlled teaching atmosphere with a clinical approximation of stidies of cases.

The discussion on Calculus was based on the *continuous variation idea*, comprehending a mathematical content recognized in the *study of fluids movement*. Under this view, the outlining of Calculus is defined as follows:

"*...we try to express an explicit form able to interrelate all the variables involved in a movement system to study a fluid of a specific nature, generally expressed by $F(x,y,z,...)=U$, but by starting to recognize its local variations, that depend on the specific situation of the system, expressed by $F'(F,dF,d^2F,...)=0$ ...*" [4].

In that ense the important thing was to study the respective variations $F(X+H)-F(X)=DF$+something, and from the DF variation and from the integration or accumulation of the variation $\int DF$, to establish the nature of the $F(X)=U$ quantity.

The elements to explore in this surroundings, in the ways of thinking of the teachers,were the regularities of the concepts in regard to different situations, the common construction patterns and using analogy to recognize a situation to know a new one, all these with physical, geometrical, and analitical attributes.

RESULTS OF THE STUDY.

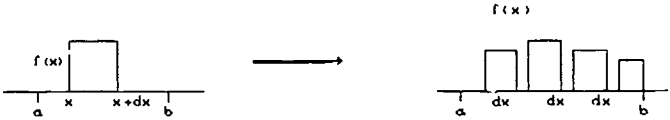
The analysis was performed by interviewing the teachers, based on reading their explicative models of integration, considering their mathematical discourse generated by fluid, surface and volume situation. It is important to mention that the need to study integration in the ancient didactics, in the text of Calculus, and in the original treatises on fluid movement, was due to the fact that our investigation wanted to have a reference to understand the ways of thinking of the teachers in regard to integration when faced with the different situations we have mentioned.

We will describe the results by means of two paradigmatic packets, which present the following situations: the necessary elements that have to be incorporated to the specific situation to recognize an instrument of continuous integration in the discrete of the continuous and from there the invariant of that what changes. This packet relates to the second, the one that describes the notion and the necessary intuitions to understand the theorems of divergence and rotational, which consist of thinking in a "principle" of continuity and conservation to express the accumulation of the fluid and the work of its displacement, all these through a notion of accumulation, configurated in "Taking the differential element" [1 y 2].

The previous appreciations point out that the way of thinking the integral are more related to the specific situation that to the integration concept; either to the definition associated to the Riemanniano apparatus [3], or to the definition of the primitive function, as it was seen in a parallel phenomenon between a teacher (of the 13) and a student of a course in Calculus in several variables - both persons do not know each other- but they both established the same explicative model on the Fundamental Theorem of Calculus. The explanation is as follows: ''... *when considering an unknown quantity that varies* *in respect to a parameter F(x)=U, we need to recognized its variation, which is possible with the lineal terms of the Taylor Series, that is*

$$F(x+dx)=F(x)+f(x)dx, \text{ where } f(x)=F'(x)$$

then f(x)dx is the variation of F(x), which can have a geometrical interpretation as follows:



Each rectangle has dx as a base and F(x) as height, but as the variation of F(x) is continuous, there are no space between the rectangles, i.e., a region forms that is covered by the rectangles f(x)dx, that is, $F(b)-F(a)=\sum f(x)dx$, but this sum is continuous, that is it joins in continuous form the f(x)dx quantities, from this region we can observe that height varies in the f(x) form, if we choose one point on each height we can draw a curve will be y=f(x)...''

Both recognized the known expression

$$\int_a^b f(x)dx=F(b)-F(a), \text{ where } f(x)=F'(x) \text{ in } [a,b]$$

its "construction is inverse" from how it is traditionally explained.

CONCLUSIONS

Our research point out that holding on to these significances could help to create a mathematical discourse that could make the transmission of the knowledge of Calculus easier, a transmission based on system of significations, captured from the intuitions, notions, and experience from both the teachers and students, in specific situations of continuous variations by studying their ways of thinking and that in order to do this study it was necessary to create a background of thinking through the genesis of the concept, its didactic in the ancient texts and its significance in the treatises on fluids movement. The investigation itself suggests didactics situations that focus more an the specific situation of continuous variation than on the concept, as it is in the accumulation notion and not in the derivative function and/or "Riemann sum".

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MATHEMATICAL PERFORMANCE OF NON-MATH MAJORS AT THE COLLEGE LEVEL

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In 1989, more than 1.5 million students at colleges and universities in the United States were enrolled in remedial mathematics courses. In 1990, the number of students enrolled in such courses had increased. The majority of these remedial courses review arithmetic computation and algebraic manipulations, content with little applicability to other academic areas, and content with which students have limited difficulty. Data from studies conducted at Boston University and Michigan State University indicate that remedial courses are not targeting students' learning difficulties, nor preparing them for further study of mathematics and/or mathematics-related subjects. What is needed is a means of identifying specific difficulties college students have with mathematics, and providing appropriate instruction that targets those difficulties.

The problem of inadequate preparation for college mathematics is pervasive. From 1975 to 1980, enrollment in remedial mathematics courses at colleges and universities in the United States climbed 72%, while the total student population increased by only 7% (Chang, 1983; Coleman & Shelby, 1982). A report of a survey of 500 institutions of higher education, conducted by the National Center for Education Statistics, stated that enrollment in remedial courses increased in 1983-84 at 67% of the colleges that offered such courses. Of the freshmen in public institutions of higher learning, 27% were enrolled in remedial mathematics courses; at private colleges and institutions, 15% were enrolled in remedial mathematics courses; and at colleges with open admission policies, 30% of the freshmen were taking remedial mathematics courses (Evangelauf, 1985). In 1984-85, 86% of all U.S. colleges and universities offered courses in remedial mathematics and 35% of all college freshmen were enrolled in such courses (Akst & Ryzewic, 1985). In 1985-86, there were more than 800,000 students in U.S. colleges and universities enrolled in remedial mathematics courses. In 1986-87, the number increased to more than a million. The 1989 report of the National Research Council states that "each term nearly three million students enroll in post secondary mathematics courses. About 60 percent study elementary mathematics and statistics below the calculus level" (p. 51).

At the same time that there are growing numbers of students in remedial mathematics courses, there are increasing demands on students to take more mathematics and/or mathematics-related courses. These demands stem in part from the expanding number of career options in technology fields that require additional study of mathematics, and the fact that at comprehensive universities, virtually all programs now require some university-level mathematics (National Research Council, 1989; Akst & Ryzewic, 1985; Leitzel, 1983).

While more mathematics is being called for at the college level, there is limited help for students who have learning difficulties in mathematics. Traditionally, remedial or basic mathematics courses at colleges and universities are one or two semesters in duration, and focus on the reteaching of arithmetic and algebraic computational skills (National Research Council, 1989; Akst & Ryzewic, 1985; Chang, 1983). Underlying this content focus is the assumption that all students with low achievement in mathematics require review of the same skills, primarily arithmetic computations and algebraic manipulations, to the same degree of sophistication, and at the same instructional pace. For many students, much of the content is review of skills in which they are already proficient (Greenes, 1987b). For other students, the remediation is a duplication of high school instruction (Steen, 1986); instruction with which they were previously unsuccessful (Hirsch & Goodman, 1986; Kelly, Balomenos & Anderson, 1986).

As a consequence of the inappropriateness of the content, remedial and basic courses have tended to be terminal mathematics experiences which, rather than restimulating and preparing students for possibly pursuing mathematics-related careers, have effectively closed off this option. What is needed is a better understanding of the difficulties college students experience with mathematics in order to provide appropriate remedial instruction.

From September, 1985 to September, 1987, the Mathematics Education Department and the Center for Assessment and Design of Learning of the School of Education at Boston University, with support from the U.S. Department of Education's Fund for the Improvement of Post-Secondary Education (FIPSE #G 00854104), developed the Probe Assessment of Mathematical Abilities (PAMA). The mathematical concepts and skills that PAMA identifies are those same skills that academicians in the natural and physical sciences, and the social sciences, have identified as requisite to the successful study of mathematics-related subjects.

More than 2,200 students at Boston University, the University of New Hampshire, and Pine Manor College (Chestnut Hill, Massachusetts), participated in the design of PAMA. During the development phase, project staff gained greater insight into those mathematical skills related to the topics of arithmetic, algebra, graphs, and applications (problem solving) with which college students have little or no difficulty (scores of 80% or better on test items), moderate difficulty (scores between 60% and 80%), and great difficulty (scores of 60% or less). These topics are identifed below.

Undergraduate's Math Performance

<u>Little or No Difficulty</u>

Arithmetic
- Compute with whole numbers.
- Multiply and divide integers.
- Add and subtract decimals.
- Find the percent of a number.

Algebra
- Substitute values for variables.
- Solve equations with one variable.

Graphs
- Identify the coordinates of a point that lies on the intersection of grid lines.
- Interpret bar graphs.

Applications
- Read, understand, and obtain data from prose.
- Solve word problems with action sequences.

<u>Moderate Difficulty</u>

Arithmetic
- Multiply decimals.
- Add and subtract integers.
- Compute with fractions.
- Convert whole number percents greater than 100 or less than 10 to decimals.
- Compute with measurement units.

Algebra
- Use a variable to express a direct relation.

Graphs
- Distinguish between two lines given a prose description of the relation.
- Recognize the function of the scales and title of a graph.

Applications
- Solve special case word problems taught algorithmically (e.g., mixture, distance-rate-time)

<u>Great Difficulty</u>

Arithmetic
- Divide decimals.
- Convert percents with fractions to decimals.
- Find the number when the percent and percentage are known; find the percent one number is of another.
- Compute with percents.

Algebra
- Use a variable to express an inverse relation.
- Write equivalent equations.
- Solve proportions.
- Write equations to express relations.
- Solve equations with two variables.

Graphs
- Interpolate.
- Extrapolate.

Applications
- Solve word problems for which data must be obtained from a graph by interpolation or extrapolation.
- Solve variations of special case word problems taught algorithmically (e.g., break-even analysis).
- Solve word problems in unfamiliar settings.

Based on the PAMA study, most college remedial mathematics courses do indeed reteach concepts/skills with which students have least amount of difficulty, and give little or no attention to moderate or most difficult topics. PAMA was designed to assess student performance on topics of moderate and great difficulty.

PAMA (Greenes, 1987a), a computer-presented assessment, consists of two parts. Part A contains Sections I-V: Compute with Integers, Fractions and Decimals; Compute with Percents; Solve Equations; Interpret Graphs; and Translate Words to Symbols. Part B contains Sections VI and VII: Solve Word Problems and Solve Word Problems with Graphs. The problems in Sections I-V are independent of one another. The problems in Sections VI and VII are grouped by applications setting. Each setting has a target problem and a set of related probe questions. Probe questions explore understanding of the solution process and are only presented when the target problem is not solved correctly. All items are multiple choice with five-choice response formats. At the conclusion of either Part A or Part B, a Student's Report is presented immediately. Students leave the assessment with knowledge of their mathematics strengths and weaknesses.

Despite the fact that college students study algebra in high school, they appear to have difficulty with equation solving, graph interpretation, and words to symbols translation, concepts and processes basic to an understanding of algebra and presented early in the traditional Algebra I course. The performance of undergraduate students at Boston University and at Michigan State University on PAMA sections III, IV, and V, provide additional evidence of this continuing difficulty.

Section III: Solve Equations is a 15-item section that assesses various equation solving skills. Students must recognize equivalent equations, solve equations with one variable, and solve systems of two equations with two variables. Section IV: Interpret Graphs is an 8-item section with two graphs. One graph is a linear function that does not contain the origin; the other graph is a pair of linear functions in which the lines do not contain the origin and do not have the same slope. Students must read and interpret the graphs, interpolate, and extrapolate. Section V: Translate Words to Symbols is a 16-item section that assesses students' abilities to identify symbolic representations (expressions) of mathematical relations presented in prose. The relations assessed in this section are: 1) maximum, minimum, at most, at least; 2) less than, more than; 3) n times as many as; and 4) n times with more than, less than, or n times. Both direct and inverse relations are presented.

PAMA is administered to all undergraduate students enrolled in the Introduction to Education course in Boston University's School of Education each academic year at the beginning of the Fall semester. The majority of students in the course are freshmen. In

1988-89, 1989-90 and 1990-91, all students enrolled in the course had completed Algebra I or its equivalent (e.g., an accelerated, advanced algebra that combines Algebra I and II), one-third of the students had completed Algebra II, and two-thirds of the students had completed Geometry in high school

TABLE 1 - BOSTON UNIVERSITY - PAMA DISTRIBUTION OF SCORES

	1988-89 (N=46)			1989-90 (N=79)			1990-91 (N=92)		
	Males	Females	Total	Males	Females	Total	Males	Females	Total
Section III. Solve Equations (15)									
Scores 0-5	0	1	1	2	2	4	0	4	4
6-10	0	16	16	2	21	23	5	26	31
11-15	7	22	29	12	40	53	12	45	57
Section IV. Interpret Graphs (8)									
Scores 0-4	1	11	12	1	21	22	1	18	19
5-8	6	28	34	15	42	57	16	57	73
Section V. Translate Words to Symbols (16)									
Scores 0-5	0	2	2	1	2	3	1	5	6
6-10	1	9	10	0	13	13	5	14	19
11-16	6	28	34	15	48	63	11	56	67
Mean			29.76			30.1			29.23
Standard Deviation			6.64			5.21			6.63

Table 1 shows the distribution of scores on Sections III, IV and V by gender for the three years, 1988-89, 1989-90, and 1990-91. The total number of items on the three sections is 39. The means and standard deviations are given for the totals for each of the years.

Of note is the consistency of performance from year to year, and between males and females. What is perplexing is the item difficulty. Graph interpretation items were the most difficult; translation items, least. Item difficulties for the line graph items that required interpolation and extrapolation were in the range 0.21 - 0.69, with the most difficult item requiring both interpolation and extrapolation. While graph interpretation was expected to be difficult, because of little or no instruction on this topic at the high school level, little difficulty with translation was unexpected. Algebra programs do not provide much instruction in words to symbols translation. Yet, the item difficulties in Section V, Translate Words to Symbols, were in the range 0.69 - 0.91.

Table II shows results of administration of PAMA Sections III, IV, and V to 16 undergraduate students enrolled in College Algebra (an Algebra II course that uses the Casio FX 7000g graphing calculator) at Michigan State University in the winter of 1991. All 16 students had completed a remedial mathematics course focusing on Algebra I content at the

University prior to enrolling in College Algebra. Fifteen of the 16 students had completed Algebra I, 12 had completed Algebra II and 12 had completed Geometry in high school.

TABLE 2. MICHIGAN STATE UNIVERSITY - PAMA DISTRIBUTION OF SCORES

	1990-91 (N=16)		
	Males	Females	Total
Section III. Solve Equations (15)			
Scores 0-5	0	0	0
6-10	1	3	4
11-15	7	5	12
Section IV. Interpret Graphs (8)			
Scores 0-4	1	3	4
5-8	7	5	12
Section V. Translate Words to Symbols (16)			
Scores 0-5	0	1	1
6-10	0	1	1
11-16	8	6	14
Mean			30.44
Standard Deviation			4.51

The mean for the Michigan State group is similar to the means of the three Boston University groups, despite the fact that the Michigan State group had the remedial course in mathematics. The smaller standard deviation for the Michigan State group may be attributable to the small sample size and/or to the fact that the Michigan State group is more homogeneous in mathematics experience (i.e., all students took the same remedial course and were enrolled in the same college algebra course).

Mathematics education faculty at Boston University and Michigan State University are investigating further college students' understanding of mathematical relations presented in graphical form (lines and curves), how this understanding develops, and why college students who are able to recognize expressions of mathematical relations presented in prose, have great difficulty recognizing the same mathematical relations in equations.

Concurrently, attention will be paid to the degree of continuing need for students to perform the essentially algorithmic tasks in Section III, Solving Equations, as we watch symbol manipulating utilities becoming increasingly available. Also, one might expect the level of students' performance in interpreting graphical information to increase as their experience with graphing utilities becomes more commonplace.

As demonstrated by the Michigan State group and others, students frequently exit remedial courses with no greater skill in mathematics than they had prior to the course, and no confidence in their abilities to reason mathematically (National Research Council, 1989). What is needed is an instructional program that provides in-depth development of only those concepts and skills with which students demonstrate difficulty.

In 1989, the PAMA development group was funded by FIPSE for three years to develop mathematics instructional materials for college students in remedial courses. Seven modules are being written, each using an "active" fill-in format, and focusing on a specific mathematical concept and related skills as identified in PAMA. The modules are being designed to engage students' interest, to encourage them to read analytically for relevant information, and to enhance their understanding of the mathematical ideas by demonstrating application of the mathematics to the solution of problems. Applications have been selected from the physical and natural sciences, the social sciences, and the arts. Each module is designed to be used independently of the others, and may serve as the content focus of a short-term mini-course, or to support existing remediation programs. Students will be assigned to modules that target their specific deficits. An Instructor's Guide will summarize the content of each module, identify common student misconceptions, suggest implementation techniques, and provide additional information related to the applications. The modules are: 1) Graphs and Their Interpretation, 2) Decimals and Decimal Computation, 3) Integers and Integer Computation, 4) Rational Numbers, 5) Proportionality, 6) Variables and Equations, and 7) Problem Solving.

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Age Level: Adult
ID #1: Naturalistic Inquiry
ID #2: Data Analysis

NATURALISTIC INQUIRY OF EXISTING VIDEOTAPED/TRANSCRIBED DATA SETS: A PROCEDURE FOR ANALYSIS

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The secondary analysis of a naturalistic inquiry will be examined. The study, a secondary analysis of videotapes and transcriptions, examined teacher attributions of success and failure and other beliefs exhibited while solving mathematical problems. A detailed procedure developed for constant comparative analysis of videotaped/transcribed data will be described and its effectiveness in generating emergent themes will be discussed.

Introduction

Often data collected for a particular study are not looked at as a source for examining additional research questions. However, when research is designed using existing data beneficial outcomes can occur. Among them new questions can be posed with the old data or original questions can be re-examined using alternative methods of analysis (Glass, 1976).

This paper will describe a naturalistic inquiry employing secondary analysis. A brief description of the purpose and design of the "parent" study and a rationale for using secondary analysis in a subsequent inquiry follow.

Data Source and Rationale

The Problem Solving and Thinking Project (PSTP; Schultz, 1991), the primary research for the study being examined, was designed as a naturalistic inquiry of middle school mathematics teachers for the relationship between their metacognitive activity and knowledge and their problem-solving ability. PSTP thus adhered to the basic assumptions of the naturalistic paradigm which in turn established the paradigm for any inquiries conceived from this work. The decision to conduct a secondary analysis on PSTP data grew out of an interest in teacher beliefs and the prospect of examining them emerging out of teachers' actions and comments rather than professed beliefs solicited in response to self-report instruments. Teacher beliefs were viewed as central to understanding teacher instructional behaviors and were believed to be the motivating forces for them. Attributions of success and failure were considered to be critical in influencing teachers' disposition toward mathematics and were believed to be related to other beliefs regarding their personal perceptions of themselves as problem solvers. The research design was compatible with the author's perspective and the data offered opportunities to explore research interests. This research direction would also enhance the primary study by elaborating on the aspect of metacognitive knowledge through its focus on a specific set of beliefs.

The resulting study, a naturalistic inquiry, involved the secondary analysis of data generated by PSTP, in the form of videotapes and verbatim transcriptions of two teachers engaged in individual and small group problem-solving protocols. The investigation examined a specific set of beliefs as one aspect of metacognitive knowledge, including attributions of success and failure and related beliefs about value of task, persistence, goal expectancy, and competence (Najee-ullah, Hart, & Schultz, 1989).

Naturalistic Inquiry and Secondary Analysis

The advantages and limitations of secondary analysis have been discussed at length by many (Burstein, 1977; Boruch & Reis, 1980; Miller, 1982). Its virtues and limitations have however, been examined primarily for the quantitative analysis of massive data sets such as national samples. Many of the virtues and limitations can be applied to all research regardless of design or paradigm. However adherence to certain characteristics fundamental to the naturalistic paradigm becomes difficult if not impossible for a number of reasons.

Naturalistic inquiry is characterized by a number of inherent features (Lincoln & Guba, 1985) which, due to the use of secondary analysis, may be addressed in a limited manner or may be impossible to address at all. This was the case with the secondary analysis of PSTP data.

Using the PSTP data meant that data selection was limited to what was collected by the primary study. Within those limitations however, data selection was purposive. Respondents were selected who seemed to represent the best source of information related to the set of beliefs being examined. There was less latitude in deciding the most useful and relevant form of the data. That too was determined by the primary investigation which included videotape and verbatim transcription data. The most significant infraction of naturalistic inquiry requirements was that data collection and data analysis could not be simultaneous. Instead, these activities were separated by time and most importantly the inability of the researcher to interact with the respondent; therefore findings are devoid of concurrence with respondents. Despite these constraints, the overwhelming advantage was that the researcher was able to devote essentially all energy and resources to the transformation and analysis of the data. A detailed procedure developed to process videotaped/transcribed data using the constant comparative method will be described below.

Procedure for Analysis

The study used the constant comparative method of analyzing data (Lincoln & Guba). This method involves the examination of data for categories of emerging patterns and themes. Categories are further divided into groups. As these thematic incidents emerge during analysis, they are coded. The code defines the incidents and identifies its group membership. The constant comparative method requires that the coded incidents be compared to incidents within the same and different groups within the same category. It is this process that begins to generate theoretical properties of that category. The constant comparison process motivates the thinking leading to describing and explaining categories (Lincoln & Guba), categories that the investigator has constructed and those that have emerged as categories used by the subjects.

In the study being examined the category of "beliefs" was identified. This category was further divided into three groups of "task", "strategy", and "self". Any belief emerging during analysis was coded and grouped, not merely the set of beliefs of interest.

The procedure for analysis included ten phases: videotape selection, viewing sequence, initial coding, characterization, summarization, transcript correction, intermediate coding, classification, translation, and synthesis. The essence of each phase will be extracted and described, referring to the study to provide context and clarification. The procedure is characterized by recursion and repetition to identify incidents and themes and to clarify their significance. Within each phase, analysis activities are repeated and should be exhaustive, clarifying previous impressions, revealing relevant information, and continuing until no additional information is revealed.

Videotape Selection

Videotapes were defined in categories according to their purpose. It was then determined which videotapes would provide the information most relevant to the focus of the inquiry. Thus, videotape selection served as a process which refined the focus of the inquiry. The selection of tapes was dependent on whether the focus of the inquiry would be the relationship between teacher beliefs/problem-solving ability and performance or an investigation of the relationship between teacher beliefs/problem-solving ability and mathematics instruction.

Viewing Sequence

Those factors which influence the order of viewing tapes should be considered and established. A chronological order by respondent was established for reviewing videotapes having the same definition (i.e. individual problem-solving protocols; first pre then post for respondent 1) to reinforce changes occurring over time. Respondent order was considered arbitrary, yet once established was maintained throughout the analysis.

Initial Coding

All incidents observed in the tapes which appear to relate in any way to the broad focus must be identified. These will be refined and perhaps discarded over the course of analysis yet it is necessary to include them at this stage. Tapes should be viewed several times to obtain a sense and an atmosphere of what has transpired. Viewing while following the transcripts will then help to begin to clarify and define critical incidents. Notes can be jotted in the transcript margins. This viewing is necessary to begin to sort through the data for relevant information. Reading the transcripts without the distraction of the tapes may reveal relevant incidents that simultaneous viewing may miss. Previous impressions are then checked by viewing tapes and transcripts simultaneously.

Characterization

By this point in the process, certain types of incidents will be found to recur. Therefore there is a need to develop a method for recording similar incidents. A chart of cells with headings broadly defining the incidents can be constructed. The cells would include a list of line numbers and a tape code to keep track of where within a transcript and for which tape it occurred. For instance, a cell headed "persistence" including lines 219-239A meant that a belief related to persistence occurred within lines 219 to 239 of the transcript for the pre interview problem-solving protocol. Such a chart was created for each respondent.

Summarization

A summary of the general activity relating to the focus should be prepared for each tape. The summarization phase is necessary to provide a cohesive picture of the incidents as they occur within the tapes, for up to this point the emphasis has been on incidents in isolation. Summarizing can also help to make similarity among incidents apparent. It was at this stage of the analysis that the properties of the broad belief groups of "task", "strategy", and "self" emerged and belief definitions began to replace the more intuitive judgements of previous phases.

Transcript Correction

Transcript errors may be found that are critical. Entire sections may not have been transcribed or misinterpretations may have occurred which alter the tone or intent of certain statements. Many of these corrections may have been "penciled in" during previous viewings but need to be included. These may change line numbering. If original un-numbered transcripts were typed using a standard word processing software the Ethnograph (Seidel, 1987), a data management software program, can be used to convert, number, and print corrected transcripts.

Intermediate Coding

This phase is similar to the initial coding phase in its steps, yet the observed incidents are more refined, more specifically than broadly defined. Concurrent viewing of the tape and reading of the transcript is performed with notes being jotted in margins. Notes define specific incidents and can be written to identify the group they fall within. Again, reading transcripts alone may reveal additional incidents otherwise obscured by the rapid dialogue of the videotape. The final set of transcript codes should now be reviewed with the tape for confirmation.

Classification

Now that incidents have been defined a method for recording them must be developed. Incidents can be recorded in a manner similar to those in the characterization phase except the charts will contain more specific information. For instance, each page of cells identified a group (task, strategy, or self) and each cell was headed to define the beliefs that were listed within. Thus a page headed "strategy" may include a cell defined "usefulness" and include 613-639D meaning the belief about the usefulness of a strategy occurred within lines 613 to 639 of the post small group protocol. A separate set of charts would be created for each respondent.

Translation

At this point, coded transcripts must be synthesized to generate more specific patterns and themes. This process began by translating the coded incidents to the Ethnograph software (Seidel). Codes from each transcript and the classification charts were used to transfer this information. Blocks of transcript text are marked by the program using beginning and ending line numbers headed by abbreviated codewords of up to 10 letters or less that include a group and incident definition designation. Blocks of texts which define different incidents may appear nested within another block.

Synthesis

Coded transcript segments similarly defined may now be used to generate patterns and themes. The Ethnograph (Seidel) facilitates the search for codes. Specific transcript files are selected along with the codewords. The program will then print all segments corresponding to the codewords and these segments can be further examined for patterns and themes. The constant comparative method of analysis requires that the coded segments be compared to segments within the same and different groups of segments for the same category. In this study, beliefs was the only category and the incidents falling under the groups of task, strategy, and self were the specific set of beliefs being investigated.

Summary

Despite the limitations expressed regarding a naturalistic inquiry using secondary analysis, it clearly has its advantages. Such analysis allows for new questions to be asked

Age Level: NA
Identifier # 1: Complex Systems
Identifier # 2: Research Methodology

THE STUDY OF COMPLEX SYSTEMSAPPLIED TO MATHEMATICS EDUCATION

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A complex system as an organized total can be studied by means of a research methodology with an epistemological background. The starting point is a leading question, then a scale of phenomena and a time frame is defined. Data, observables and facts are clearly distinguished. A system as such has limits, elements and structures and its own dynamics which are studied at different levels of analysis according to different levels of processes. Given the leading question: "What are the major trends in mathematics education in 1991?" a first analysis of mathematics education as a complex system has been attempted.

PART I: THE STUDY OF COMPLEX SYSTEMS -A RESEARCH METHODOLOGY

1. Introduction

García (1986) proposes a basic research methodology for the study of complex systems. If we suppose that mathematics education and related areas have characteristics of a complex systems, this methodology could be relevant for a better understanding of our academic endeavours.

In the first part of this paper, a brief description of Garcia's methodology is presented. In the second part, it is applied to explore mathematics education in the light of the leading question: "What are the major trends in mathematics education in 1991?".

2. Epistemological Background

According to García (1986), a global or complex system is a set of elements together with constitutive factors, interrelations and interactions with other systems in what he consideres a first aproximation of a definition. The study of such a system is interdisciplinary work, done in a conceptual framework with epistemological foundations. The term "system" here is not used in the same way as in engineering, econometrics or computer science and is not related to system analysis as commonly used in these disciplines. The models developed in those contexts are not applicable here, because a complex system is much more than a mere set of related elements.

A complex system is not given or defined a priori, but it can be defined in the context of a leading question. The proposed methodology is antiempiricits, but not antiempirical, because observations are interpreted and observables are not considered the basis of all knowledge, accessable by pure, neutral perception. Hanson (1965) affirms that all experience is charged with theory and Piaget showed in the light of his genetic psychology from an epistemological viewpoint that there are no "pure observables" -there is always a previous construction of relations in the observer and "observables" are forms of organized data. Knowing means establishing relations between data in a social, intersubjective environment.

From this antiempiricist viewpoint we have to distinguish data, observables and facts, whereby observables are interpreted data, facts are relations between observables. The interpretation and organization of observables and facts requires previous conceptual schemes or theories. The term "theory" is used in a very broad sense as a set of affirmations and assumptions in which a researcher establishes hypotheses and makes or refutes inferences. This way the researcher sets up an empirical field with an epistemic frame and an empiric domain (Piaget, García, 1982) in which he selects data and interpretes them to establish observables and facts. For example, an educational researcher which studies learning will select or interprete data according a learning theory.

3. Components of a complex system

A complex system is a piece of reality which can be investigated in its different aspects. As a point of departure, a leading question is asked. Then the limits, elements and structures of the system which will be studied and eventually make up the components of the system, are selected.

3a. Limits

Real complex systems often lack limits -they have to be drawn more or less arbitrarily. If what is left out influences the "inside" of the system, we speak of surrounding or contour conditions.

3b. Elements

Elements of systems are generally themselves complex subsystems which interact. They are selected according to a spatial scale of phenomena which determines the location and extension of events which coexist or interact-and a time frame as a temporal scale for the study.

3c. Structures and processes

The structures of a system are given by the relations between its elements as an organized total which is kept in a state of stationary fluctuation by means of dynamic regulation processes. The main objective of the proposed method of analysis of complex systems is primarily the study of these processes, not the states of a system in any given moment. This emphasis in processes is sometimes refered to as "genetic structuralism".

The processes describe changes in the system and occurr at different levels which again require different levels of analysis.

Three levels of processes can be differentiated:

Level one processes are observed and measured on a local, regional basis by means of polls, interviews, explorations in a merely descriptive way. All observations are made within a conceptual frame. At the second level, there are metaprocesses which explain level one processes and third level processes are of a more predictive nature and determine the processes at the lower levels. Associated to levels of processes are the corresponding levels of analysis with local, national or international dimensions.

In the study of the dynamics or evolution of a global, complex open system without clearly defined limits and affected by surrounding conditions, the imbrication or overlapping of structures is a basic approach. For example, the learning of mathematics can be studied at an individual, classroom, local regional, national or international level. Each structure at a given level becomes part of a subsystem at superior levels.

PART II: MATHEMATICS EDUCATION AS A COMPLEX SYSTEM, A FIRST ANALYSIS

As a leading question we chose: "¿What are the major trends in mathematics education in 1991?". This will enable us to apply the methodology of studying complex systems to a "metaresearch" problem about the nature of mathematics education. To begin our research we have to restrict ourselves to a limited portion of reality in order to be able to establish a system with its elements, internal relations and contour conditions. We limit ourselves to consider mathematics education as dealing with teaching and learning mathematics.

This definition of limits of the system, requires a selection of a scale of phenomena we will study. This scale could be as follows:

Teaching Learning	
One-to - one basis (individual learner) Group - one teacher (-classroom situation)	— 1st. Level
Local school system Regional education system	— 2nd. Level
National education system International comparison	— 3rd. Level

The temporal scale is determined by the question itself, the present time, even if some of the subsystems could have a different time scale, specially if we use the analysis of developments over a period of time to explain the "state of the art".

Elements of the complex system may be special problem areas considered subsystems like the individual students, the teachers, the mathematics curriculum, the researcher in machematics education, all of them in different, but related domains.

Our scale of phenomena is applicable to all subsystems since we can study the role of an individual student in a teaching-learning situation on a one-to one basis, in a group situation and so on. We can also study on the way the mathematics curriculum is affected by each phenomenon of the scale or how each phenomenon affects the mathematical contents. Structural relationships go very often both ways.

It is now possible for example, to make observations, within the subsystem "teachers", of a level 1 phenomenon, -a classroom situation. In order to convert the collected data into "observables", our theoretical predisposition will affect the interpretation of the data. We would make observations about teaching techniques, contents, student-teacher interaction, selfperception of teachers and so on.

If we adapt a constructivist point of view, the teacher is no longer considered as the main actor in the classroom, he is a guide or monitor of the construction of a student's mathematical schemes based on previous knowledge. Teaching is not the transmission of knowledge, but a guidance in the recons--truction of concepts by the learner, taking into account the epistemological obstacles which have to be overcome (Herscovics, Bergeron, 1989).

If we make an analysis from the standpoint of activity theory, observations would focus on the role teachers play in the human activity within the social group "classroom". The zone of proximal development (Vygotsky, 1978) can be interpreted as a location in the interaction between teachers and students in which new understanding can arise. Teachers and learners work together on problems which students alone could not solve. Sometimes the term construction zone is also used in this context as a mediator between the thoughts of teachers and students or a shared activity in which interpsychological processes occur (Newman et. al. 1989).

We can also adapt an information-processing paradigma (Nason, Cooper, 1988). The teacher takes the role of a disseminator and facilitator of learning, as well as a diagnostician and an introspective professional who analyses and evaluates his behavior continuously, but most of all he would conceived as a scientist who understands how each student processes the information being taught.

If we believe in the metaphorical nature of thinking (Wenzelburger, 1991), we put special attention to the teacher's discourse in the classroom and use of metaphors to initiate thought processes and to communicate knowledge. We have to take also into account the basic underlying metaphors teachers use to describe the teaching-learning of mathematics -"education is like growing plants" or "the mind is a muscle", the "conduit" metaphor or the "teacher as a builder of knowledge" metaphor.

Each theoretical frame will provide us with an apparently different collection of observables and facts to identify structures within the subsystem and the global complex system "Mathematics education". A complimentarity principle (Pathee, 1982) may be applied in order to avoid false dichotomies (Hilton, 1977). Such a principle requires simultaneous use of descriptive modes that are formally incompatible-contradictions are accepted as an irreducible aspect of reality.

The leading question makes possible a thorough analysis of each suggested subsystem according to the scale of phenomena and time frame. This requires the efforts of a research team with an interdisciplinary approach. The research methodology we discuss here puts special emphasis on interactions of phenomena from different domains. It is not intended to discover "given" facts and list data produced by isolated groups of specialists -rather a systemic view is adopted, more appropriate to complex phenomena from an interdisciplinary stand point. Researchers in mathematics education would work together with teachers, administrators, psychologists, historians, mathematicians, in order to complete the picture of the major trends in mathematics education at the present time.

The systems approach discussed here, applied to a meta research question in mathematics education, is in accordance with tendencies in modern science to search for relations, interactions and structures in order to move away from "dissecting" phenomena into isolated parts. A general awareness that the "whole is larger than the sum of its parts" is a consequence of the general theory of systems (Bertalanffy, 1972) on which Garcia's work is based.

To think of mathematics education as a complex system of interrelated elements with limits and structures is potentially useful to reach a better understanding of our discipline.

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EWG/avo.

THE CONCEPTUAL-REPRESENTATIONAL ANALYSIS OF CHILDREN'S EARLY ARITHMETIC

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CANADA

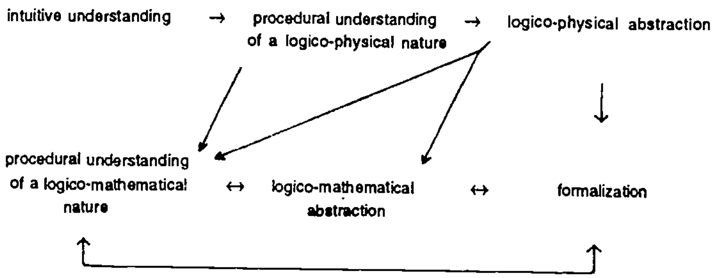
We explore the interface between two models: the two-tiered extended model of understanding, developed by Herscovics and Bergeron; and the unified model of problem-solving competence based on cognitive representational systems, proposed by Goldin. The context for the exploration is children's early arithmetic.

Research on the construction of conceptual knowledge, and research on mathematical problem solving, have both advanced significantly in recent years; in part, through the development of more sophisticated theoretical models. In the study of conceptual development, several *models of understanding* have been proposed, based in large part on observation of children's early arithmetic through numerous structured, individual interviews (Herscovics and Bergeron, 1983, 1984, 1988). In the study of problem solving a unified *model of competence* was proposed (Goldin, 1983, 1987, 1988), based mainly on observations made in more advanced mathematical domains, with the goal of providing a framework for detailed descriptions of mathematical problem-solving processes. Here we explore the interface between the two-tiered extended model of understanding of Herscovics and Bergeron (1988), and the unified model of problem-solving competence proposed by Goldin. The context for our exploration is children's early arithmetic. We are interested in whether ideas drawn from problem-solving research can help characterize conceptual development at this early stage.

Two Cognitive Models

When children construct basic mathematical concepts such as "cardinal number", "ordinal number", "addition", etc., the complexity of the different aspects of understanding they achieve is difficult or impossible to describe using classical concept-formation theory based on exemplars and non-exemplars. The idea of a *conceptual scheme* proves more useful. This is defined (Bergeron and Herscovics, 1990) as a network of related knowledge, together with the problem situations in which the knowledge can be used. The extended model of understanding of Herscovics and Bergeron is designed to identify systematically the components involved in the construction of conceptual schemes for early arithmetical concepts. It involves two tiers; one referring to the preliminary physical concept, the second to the emerging mathematical concept. The model may be viewed schematically as follows:

Understanding of the physical pre-concept



The emerging mathematical concept

This framework, while not claimed to describe the development of understanding of all mathematical concepts, proved adequate for the analysis of many aspects of children's early arithmetic. For example in discussing the concept of number, on the first tier "intuitive understanding" would include qualitative visual approximation/estimation (by the child) of whether one set of objects contained "more", "less", or "the same" as another set; "logico-physical procedures" would include activity such as placing physical objects from two sets into one-to-one correspondence; and logico-physical abstraction would include the mental operations leading to conservation of quantity (or more precisely in this context, of plurality). Symbolic representation comes into play in the second tier, with the advent of "logico-mathematical procedures". These involve steps such as counting, taken with number-words or symbols rather than (or at the same time as) physical objects. On this tier, the results of abstraction include conservation of *quotity* (which refers to number-words or symbols) as opposed to the first-tier conservation of *plurality* (which refers to physical amounts). On this level we also have the ultimate formalization of mathematical ideas involving "number", making use of symbol-systems for numeration (and for arithmetical operations); ideas for which prior procedural understanding has developed to some extent. Formalization also includes axiomatic properties of "number", and concepts involving mathematical justification (or at a more advanced level, mathematical proof).

The model for competence in mathematical problem solving of Goldin is based on the idea of cognitive representational systems internal to problem solvers, as distinct from (external) task variables and task structure (Goldin and McClintock, 1979). Such a cognitive representational system is comprised of a (not necessarily well-defined) class of signs or characters, together with weys of combining these into configurations, and higher- level structures which can manipulate and transformation configurations. Five kinds of internal cognitive representational systems are proposed: (a) a verbal/syntectic system, involving words, grammar, and syntax: (b) imagistic systems, including internal visual/spatial, auditory/ rhythmic, tactile/kinesthetic, and other non-verbal representation of objects, attributes, and relations; (c) formel notational systems, involving mathematical symbols and rules for manipulating them; (d) a system of heuristic planning and executive control, which encompasses strategic competencies as well as capabilities that are often termed "metacognitive"; and (e) an affective system, making possible the changing states of feeling that occur during problem solving that can influence decision-making. An important feature of the model is that representations of any one kind can stand for or *symbolize* those of any other--for example, words can symbolize visualized objects, or mathematical notetion can stand for kinesthetically-encoded sequences of physical procedures. Systems of these five types are seen as developing over time through three stages of construction: (1) inventive-semiotic, in which characters in a new system are first given meaning in relation to previously-constructed representations; (2) structural developmental, where the new system is "driven" in its development by a previously-existing system, which functions as a kind of "template" for growth of the new system; and (3) autonomous, where the new system of representation can function independently of its precursor.

The model was motivated by the desire to describe complex mathematical problem-solving processes; but it would appear to contain features that are helpful in describing the development of conceptual understanding. If we take children's early number concepts as an example, we can regard qualitative visual approximation/estimation by the child (e.g., of whether one set of objects contained "more", "less", or "the same" as another set) as a complex, *problem-solving* process involving (a) verbal representation (making use of the terms "more", "less", etc. and symbolizations for these terms previously constructed); (b) considerable internal visual/spatial processing; (c) at least some elements of executive control and decision-making; and (d) an affective component reflecting the child's pleasure (or lack thereof) in the task, the child's interaction with the investigator, etc. These processes can then be analyzed into subprocesses, in a manner analogous to that which is possible in studying problem-solving strategies: we can discuss the (visual) seperation between the two sets of objects, the mental construct of a "measure" of their size (e.g., in terms of a portion of the visual field), and the act of (visual) comparison of such "measures". Likewise, the ability to carry out procedures such as placing physical objects in one-to-one correspondence, involves the construction of complex, internal kinesthetic configurations, which eventually enable abstraction to take place.

We see that the two models, though developed for different purposes, are capable of addressing some of the same phenomena. Our long-range goal in beginning the present investigation is to achieve a full synthesis between the models for understanding and the model for problem-solving competence. This would enable us to describe the learning of more advanced mathematical concepts, as well as to understand the constructive learning process in greater detail; it would assist us in understanding why some problem solving *results in the construction of important new knowledge*, while other problem solving (though perhaps equally successful in reaching the problem goal) does not.

In our examination of what is known about children's early arithmetic, we have identified some key points of contact between the two models.

1 *The relationship between the physical pre-concept tier, imagistic systems of cognitive representation, and developmental sequences:*

First we distinguish carefully between *external* representation (a structured environment with which the child is interacting that may include, for example, actual physical objects to manipulate), and *internal* imagistic representation (a theoretical construct to describe the child's inner cognitive processing). This is reminiscent of Piaget's term "interiorization", which refers to the child's ability to *re*-enact mentally a sequence of actions or operations. Next we ask, why is the physical tier a *pre*-concept? The answer is that in order for the child even to *ask the question* that leads to a meaningful construction (i.e., for the situation to be a problem), it *needs* to have constructed certain internal, imagistic representations. Consider for example, in the case of addition, the question "How many do we have all together?" In order that "How many" be meaningful (i.e., for the words to *represent* something), there must be initially imagistic configurations for sets of discrete objects, and for counting operations. For "all together" to be meaningful, the needed representations may include physical partition (the two separate sets under discussion), physical transformation (the act of moving the objects), disjoint union (the joining of the two sets into one set), and so on. The construction of such imagistic representations *requires* interaction with external physical objects. Thus the physical tier serves as a *pre*-concept because, for concepts of early mathematics, conceptual understanding *necessarily* involves imagistic configurations as precursors, which in turn require external, physical configurations for their construction.

Furthermore, the young child does not yet have an elaborate formal notational system of cognitive representation. Thus during the semiotic and structural-developmental stages of representational development, it is necessary to build on imagistic configurations if anything at all is to be built on; formal configurations cannot substitute for the imagistic.

For more advanced mathematical concepts, imagistic representation *can* be, but is not *necessarily*, a precursor to formal representation. This is a crucial difference between early mathematics and later development. For example, multiplication can be meaningfully introduced as repeated addition (logico-mathematical procedure), using the formal notational system of cognitive representation (for addition) as a precursor to construct new kinds of formal configurations (the notation and accompanying procedures for multiplication). Physical models (such as rectangular arrays) can follow later. We emphasize that *we are not saying this is the best way* to introduce multiplication; only that it is possible from a cognitive point of view. Indeed, we would argue that an important goal of mathematics education should be the development of powerful imagistic systems of representation; never- theless, the use of formal representation as the precursor to further formal representation in mathematics inevitably becomes more frequent as the mathematics becomes more advanced.

2 The relationship between the emerging mathematical concept, formal systems of cognitive representation, and heuristics:

We noted above it is possible to engage in problem solving *with or without* constructing significant new mathematical knowledge. Thus we would like to characterize *when* it is that problem solving results in such construction. Even in children's early arithmetic, we believe it is possible to identify the emergence of complex *heuristic strategies* such as trial and error, subgoal decomposition, etc. The "counting on" strategy for addition, when meaningfully constructed, involves subgoal decomposition. In certain *didactic situations*, as discussed by Brousseau (1981), such strategies are invented or assimilated and brought to bear when the problem goal itself provides a *reason* for the construction of new knowledge. Formalization (formal representation) occurs meaningfully when it assists in achieving such a problem goal.

However, it is also possible for children to use what seem to be heuristic strategies non-meaningfully. In work on addition (Bergeron and Herscovics, 1990), children were shown a cardboard strip on which were glued 11 chips in a row. In front of their eyes, 6 chips were covered at one end. While these were hidden, the children were told, "We are hiding 6 chips. Can you continue counting from here on?" All of the children counted "7, 8, 9, 10, 11." When they finished counting, they were asked, "How many chips are glued on this cardboard?" Most said they did not know. In response to the question, "Why don't you know? We just finished counting," the answer was forthcoming, "We didn't count those [the hidden ones]." In this context, "counting on" was something the children *could do*, but only as a meaningless procedure [more precisely, as a *verbal* procedure rather than a *heuristic* procedure to solve a problem]. The starting point of counting had not been established as the cardinality of the initial hidden subset: the children thus learned "reciting on", but did not associate what they were doing with cardinality. The task was not a didactic situation, in the sense of providing a learning outcome.

3 *The role of affect:*

In general, children *like* to play with the physical objects that serve to assist in the construction of imagistic representations. Using the term "affect" in its broadest sense, we conjecture that affect serves an important role in both tiers of the extended model of understanding, facilitating and guiding the construction of cognitive representations. To provide just one example (Herscovics and Bergeron, 1986), Montréal kindergarten children were observed counting sets of objects "visually" (with the eyes, or nodding with the head), without physically partitioning the objects or touch-counting. When asked why they did not use these other methods, some of them answered, "C'est trop bébé!" (It's too babyish.) Having the choice of several counting procedures, they purposely chose the more difficult one--even though they made more mistakes with visual counting. They selected procedures they felt were more sophisticated, making the problem more challenging, as a way to enhance their self-image. This illustrates how affect can actually be a determining factor with respect to the "heuristic planning/executive control" system of cognitive representation (i.e., with respect to what are commonly called "metacognitive" processes).

Conclusion

This initial exploration of two theoretical models has found several important points of contact in the domain of children's early arithmetic. The theory based on cognitive representational systems proves helpful in lending precision to, and elaborating on, learning processes described by the components of the extended model of understanding.

The authors have also begun a theoretical discussion addressing more advanced mathematical concepts, in the context of exponentiation and the exponential function (Goldin and Herscovics, 1991). The goal of achieving a synthesis between models of understanding and of problem solving appears to be deserving of further effort.

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High School
Communication
Function

CONTEXT AND STUDENT TALK ABOUT FUNCTIONS IN A HIGH SCHOOL ADVANCED MATHEMATICS CLASS

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This paper reports part of an ethnographic study in an high school precalculus class where students used materials designed to foster communication. It reports the nature of student talk during instruction on mathematical functions and how talk differed in each of two distinct instructional contexts.

Introduction

The NCTM Curriculum and Evaluation Standards (1989) lists learning to communicate mathematically as one of five primary goals for all students. Spoken language, an essential part of communication, is used for representing mathematical ideas (Janvier, 1987; Lesh, Post, & Behr, 1987), and as a vehicle for instruction. To date, very few studies address the role of talk in mathematics classrooms. This paper, which is part of a larger ethnographic study, is about the nature of talk during instruction of mathematical functions and how talk is influenced by two distinct instructional contexts.

The study was conducted in a suburban high school pre-calculus class of 24 students in which the teacher integrated materials from The Language of Functions and Graphs (Swan, 1987) into the standard, textbook driven (Dolciani et al, 1980) curriculum. In the functions and graphs activities (LFG) students created and interpreted solutions to problems about functional relationships that were represented by verbal descriptions of situations, Cartesian graphs, and tables. Students' work in small groups was followed by full-class discussion. In 'book math' the teacher, Mr. Dennis (not his real name), presented material through posing problems on a topic, by questioning, and by eliciting appropriate answers to guide students to solutions. Mr. Dennis used the same style to go over student-selected problems from the homework.

Data and Analysis

Data were fieldnotes and audio-recording of 10 weeks of classroom observations, written work of students, audio-recordings of teacher and student interviews, written pre and post tests, and an anonymous questionnaire.

Three general categories were used as a framework to analyze talk in this classroom:

1. the focus of talk during discussion of problems

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2. the extent to which talk was oriented to a group purpose, and
3. aspects of students' knowledge that were exemplified.

Triangulation of multiple sources of data were used to support or disconfirm assertions that developed during data analysis.

Results

I will use two instances of full-class talk, Picking Strawberries, an activity from LFG, and a problem to identify the domain and range of a function defined by its graph (See Fig. 1) from the standard curriculum, to illustrate how talk differed between these two contexts. Picking Strawberries shows a man in a field. A balloon says, "The more people we get to help, the sooner we'll finish picking these strawberries." Students were to sketch a graph of the time it takes to pick the patch as a function of number of people.

5.

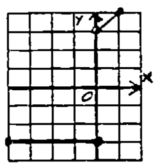

Figure 1.

<u>Focus of Talk</u>

Problems played a central role in talk in both contexts as almost all discussion of mathematics took place in the framework of working problems. Yet the focus of talk about problems differed between contexts. In LFG, talk was <u>on</u> the problem, while in book math, even though talk was about problems and their solution, it focused on correctness of definitions, learning a specific method of working problems, and a specific way of thinking about the topics.

In LFG students usually spent considerable time defining the problem situation. In Picking Strawberries, students talked mainly about how to define the picking situation and implicitly about the match between the problem and the graph. Initially they offered five potential graphs and explained their reasoning. As the discussion continued, they refined their definitions of the problem situation. They discussed whether the number of people should be a whole number, whether the number of people or the time to pick could ever be zero, whether they must consider people to pick at a constant rate or whether their rates could be averaged. They discussed whether the graph should touch each axis, whether it should be curved or straight, whether it should consist of dots, and to a lesser extent, the validity of notation used to represent their ideas on the graph.

In book math students talked about problems, but talk focused on correctness and 'getting it'. When students called out answers to questions posed by the teacher while he presented new material or went over homework, they appeared to be 'filling in the blanks'. At other times, they tested the correctness of their understanding by asking the teacher specific questions, such as how to write a specific set description or whether to use "and" or "or".

Sometimes students posed highly specific variations to problems, which when answered, could provide insight into whatever they were puzzling about. For example, after it had been established this graph represented a function and the domain and range had been determined, Alex checked his understanding of open circles.

Alex:	Uh, the open circle's mean that it's (pause). If those were closed circles right there and you do the vertical test, do you get a function? (both circles are open)
Tchr:	No.
Alex:	It wouldn't be a function?
Tchr:	No.
Alex:	It's a function 'cause they're open?
Tchr:	Yes.

Alex seemed to be checking that he heard correctly and whether he had 'got it' in dealing with open circles.

Sometimes students asked what might be conceptual questions, but seemed satisfied with short responses from the teacher. John asked what seemed to be a question about multiple functions defined on the same domain. He asked, " If you have that domain and range, can't you end up with a different graph somehow?" When Mr. Dennis suggested John was simply uncomfortable because he didn't know the specific rule for the function, John said, "Okay," and didn't pursue it further. When students chose homework problems to be discussed in class, it is likely they were checking the answer or the solution process. Infrequently they asked about concepts, and very rarely students posed problems that were an extension of that under discussion.

Orientation to a Group Purpose

Most full-class discussions centered around working problems, but students had a different .entation to the group in each context. In LFG talk seemed oriented toward a group outcome of resolving the problem. They focused on investigating and discussing their ideas about a common problem. At one point Doug, prompted by Ned's comment that one was the smallest number of people possible, said you couldn't have .2 people either. Several students agreed. Later in the discussion, Doug returned to the issue and said:

> If we are just considering 1.2 and all that, it could be that if you just had 0, 1, 2, 3, 4, 5 people, and the time. It could be just be the dots, It would go on and off either in a curve or in a straight line down. It would just be dots, the number of people.

When Mr. Dennis asked him why he said curve or a straight line, and Doug said he believed it would go down in a straight line. John said he had something to add and that it should be curved because if you had one person and it took ten hours, then two people would take five hours. A lot of students began to talk at once; with at least some dissenting. John continued with a new argument, comparing the differences in the relative increases in total time when adding an additional person to 1 picker and to 100 pickers. The students' talk was often directed to other students, and they seemed to be engaged in a joint effort of solving the problem at hand.

In book math, talk was directed from individuals toward the teacher, and less frequently toward each other than in LFG. Students asked for specific homework problems to be worked, sought answers to highly specific questions on how to do a process or on the correctness of an answer. They posed alternative problems to test their understanding, and the queried the teacher about concepts they did not understand. Students' talk was oriented to meeting individual needs. Students seemed to have in common the purpose of 'getting it', but not to be engaged in a common group purpose.

Aspects of Students' Knowledge Exemplified

In general, talk in these two contexts was oriented to exemplify different aspects of students' knowledge. In LFG talk was focused on what students knew, whereas in book math, it was oriented to display what they did not know.

In Picking Strawberries, students made conjectures about the solution, explained their reasoning, and defended or debated the merits of solutions or suggestions made about the problem under discussion. For example, five students gave their graphs for the strawberry problem; each was different in some aspect. All were asked to explain some part of their graphs. Giving their views on the problem at hand was a common feature in each activity from LFG. When students defended their positions, as John did in the example above, or debated the merits of a position, such as the role of average rates, their differences were viewed more as disagreements about how to proceed since talk seemed focused on the solving the problem rather than on whether students understood a process.

Tentative solutions, even when incorrect, contributed to the resolution of the problem. Three of the first graphs offered for Picking Strawberries were linear. Yet talk about these three graphs and their three different ways of handling end behavior of the function resulted in a rich discussion of many points, such as John's justification of curvature of the graph. Ned's description of how he left a gap at zero because the smallest number of people was one led Doug to raise the issue of a discrete domain. Because ill-formed conjectures contributed to solving the problems, they were view positively as constituting knowledge rather than illustrating the lack of it.

In book math, however, the situation differed. Filling in the blanks, of course, displayed what the students knew, and many students participated, but often several would call out answers and a chorus of the correct answers would gradually strengthen while other responses dropped out. It seemed that some students listened for the trend before participating. Since the talk focused on correctness or a single way of doing things, it oriented students to focus on what they did <u>not</u> know, so that participating incurred a risk of exposing their failure. To ask for a homework problem to be worked was to announce it was not fully understood. Questions checking their solutions or finding out how to fill in the gaps in their understanding focused on what students did not know. While it could be argued that students sometimes asked questions to show off, it appeared that only one student engaged to any extent in this behavior.

Accounting for the Difference

I account for the differences in student talk between the two instructional contexts by the nature of the curriculum, the source of the tasks, and to a lesser extent, the teacher as a novice in using the materials of the <u>Language of Functions and Graphs</u>.

<u>Nature of Curriculum</u>

The materials in LFG contributed to the difference in students' talk. They were specifically designed to be used in a collaborative way. Students worked in small groups and were asked to come up with a common solution. The materials included suggestions, which Mr. Dennis followed, about how to run full-class discussions to facilitate sharing without devaluing any student's ideas. Students were asked to express their ideas and to justify their reasoning. The tasks were inherently open-ended.

In book math, the activities were heavily dependent on the text. The tasks set forth by the text were highly compartmentalized. Problems had single correct answers. Connections between sections were implicit. The tasks students were given were individual. Each student turned in their own homework, and all work on tests and quizzes was carried out individually.

The social organization inherent in the tasks as defined in these two contexts was important since students considered working together important. On an anonymous questionnaire given at the end of my time there, students were asked "How important or unimportant was working in a group? Why and in what way?" Nineteen of the 24 students answered "very important" or "important" to the question, and 11 cited the value of collaboration as a reason.

Source of Activities

Differences in student talk between contexts might be explained by the source of the activities. I had provided the LFG materials to Mr. Dennis, and while he was careful to speak to students of their value and role in class, both he and the students considered them mine. Several students indicated on the questionnaire that the activities were 'add-ons' to the class. Since all grading depended on book math, surely students were more likely to focus on correctness and to engage in getting "answers" that might prove useful on quizzes and tests than if they considered them as recreation.

Teacher as a Novice

Finally, the teacher was using LFG for the first time, but had taught book math from this text for several years. As in any new situation, neither of us knew what to expect from the students in LFG. Sometimes Mr. Dennis (and I) was very surprised by the richness of students' responses. On such occasions, Mr. Dennis responded by simply repeating each phrase the student had just said, a style of replying that very effectively prompted students for more of their ideas. Since this kind of responding occurred more frequently when he seemed puzzled or surprised by students' responses, his use of it may be a result of his inexperience with these kinds of materials.

In book math, on the other hand, Mr. Dennis was an experienced teacher who had used this text for several years and knew what he wanted for the students. Thus, Mr. Dennis gave students more cues and much more direction.

Discussion

Students' talk in this class raises important issues related to learning mathematics: what it contributes to teaching, what is their understanding of what it means to study mathematics, and how they feel about doing mathematics. Students' talk is important as a diagnostic tool for the teacher. Talk that focuses on what they do not know can provide a teacher with useful information about how confused students are, but when students explain their reasoning, as in LFG, the teacher can learn both the nature of their difficulties and the richness of their thinking.

Focusing on tentative solutions and wrestling with partially formed— possibly incorrect— ideas is closer to what those engaged in mathematical problem solving do and might provide students with a more accurate picture of what it means to do mathematics.

If classes discuss 'conjectures' rather than 'answers', even students traditionally perceived as poor in math can participate without risk to their self-esteem since there is less risk in exposing their thinking. If students' responses, even though not completely correct, are viewed as valuable, and if other students provide support through joint efforts at final resolutions of the problems, all students might feel that they are succeeding. More students might participate in mathematics.

Finally, working in groups or orienting the class toward a group outcome might help students succeed at math. On the questionnaire, several students gave the need for support as one reason why they felt working in groups was important. One student added, "and the 'smart' kids were not always right," a powerful argument for how orienting the class toward groups contributes to students' self-esteem and possible success in mathematics classes.

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Middle School Teachers
Questioning
Inquiry

OPEN TO QUESTION: AN EXAMINATION OF TEACHER QUESTIONING

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This study examines the questioning practices of three mathematics teachers attempting to adopt an inquiry approach to mathematics instruction. Analyses of classroom transcripts and teacher journals suggest that although teachers may ask many questions during instruction, their practice remains traditional in many ways.

Ushering in a new paradigm is never an easy task (Kuhn, 1963, Confrey, 1988) Although reform efforts in mathematics education abound (NCTM, 1989; NCTM, 1991; NRC, 1989), the transition from the traditional classroom which presumes a transmission view of knowledge to a classroom where students construct knowledge from genuine mathematical inquiry and discourse is exceedingly problematic.

The constructivist view of mathematics learning (von Glasersfeld, 1983) asserts that discourse is a universal and critical feature of concept development in mathematics. For discourse to occur, there must first develop a "consensual domain" (Maturana, 1978) whereby discussants implicitly acknowledge shared assumptions.

Richards (in press) describes communities in which qualitatively different mathematical discourse occurs. This discourse includes **research math**, or the spoken mathematics of professional mathematicians and scientists; **inquiry math**, or the mathematics of "mathematically literate adults"; **journal math**, or the language of mathematical publications; and **school math**, or discourse consisting mostly of "initiation-reply-evaluation" sequences and "number talk". The distinction between inquiry math and school math is fundamental in the appraisal of the success of present reforms in mathematics education.

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Research Framework

Because the inquiry approach presumes an emphasis on questioning as the impetus to dialogue, we have chosen teacher questioning as a focus for the current study. Our framework for describing types of questions originates with Brousseau's (1981) description of educational social situations and their corresponding cognitive functionings. These situations include **action**, **formulation**, **validation**, and **institutionalization** (as described in Balacheff, 1990; Laborde, 1989, and Cobb et al., in press). We see Brousseau's situations initiated in questions of the type which promote the milieu.

In the table below we identify each of these question types, describe the type, and offer sample questions. As we found many different types of formulation questions, we differentiate among them using sub-types: formulation/what, formulation/how, formulation/different, and formulation/thinking.

Question Type	Description	Example
Action	Poses problems for students to solve.	How long would his property be? Could you use your base 10 pieces to show ...?
Formulation	Asks that students make their interpretations and conceptualizations explicit	(a)**What**: What can you tell me about ...?; What does parallel mean? (b)**How**: How did you do that? (c)**Differently**: Did anyone see it differently? (d)**Thinking**: How did you decide...?; How did you know...?; What were you thinking when...?

Open to Question

Validation	Asks students for justifications for their solutions.	Can you go up to the overhead and prove that it's a hexagon?
Institu-tionaliza-tion	Asks students to recognize or confirm an official truth.	Did you notice that the 2nd train was the same as a hexagon?; Can we use another word to describe the area around (a rectangle), how about distance?

In addition to the above categories of questions, we identified **factual recall** questions which ask students if they remember information discussed earlier, **repeat** questions in which the last teacher-spoken question is repeated, **repeat student response** questions in which the teacher repeats the last spoken student statement with inflection; **general assessment** questions which ask how well students are understanding generally, and **management** questions which pertain to classroom management and organization.

Research Methodology

The subjects in this study are three middle school mathematics teachers who are part of an on-going project involving an effort to implement many of the recommendations contained in recent documents such as the NCTM Standards (1989). They were videotaped for three consecutive days in October of 1990, January of 1991, and May of 1991 as they taught a 6th-grade lesson. They also kept journals throughout the year.

Two sources of data are examined in this study: (1) transcripts of the first two lessons videotaped in the Fall and Winter, and (2) teacher journals. Transcripts were analyzed in committee by two researchers and two graduate students as to question type, with independent judgments for selected segments compared until close consensus was reached

Frequencies were summed over the two days in each cycle for each teacher. Journals were examined for teacher reflections about their questioning and pertinent passages were identified.

Results and Discussion

Transcript analysis shows that all teachers asked many **formulation/what** questions and **institutionalization** questions and few **formulation/how** questions, **formulation/thinking** questions, or **validation** questions. All teachers also repeated questions often, repeated students responses with inflection often, and asked relatively few **action** questions. There are comparatively fewer factual recall questions and general assessment questions. The number of management questions varies among teachers and reflect differences in the degree to which classroom discipline was a problem. Most apparent is the observation that the teachers asked many questions. Total number of questions asked by each teacher over the two lessons for Fall and Winter ranged from 254 to 109 questions.

School Mathematics

With regard to **formulation/what** questions, an average of 39% for all observations were questions of this sort. These questions tended to be quite leading and typically required one-word responses. They seemed to be used as a vehicle for calling student attention to what the teachers saw as relevant information, such as in "What would the 100th train look like?" **Repeat** questions tended to be **formulation/what** questions also. **Institutionalization** questions comprised 13% of all questions asked, tended to be rhetorical, and generally served as a technique for teachers to transmit information in a question form.

It is interesting to note patterns in the sequence of questions in the discourse. The sequence often began with an **action** question followed by an extensive series of **formulation/what** questions and then brought to closure with an **institutionalization** question. This sequence is reminiscent of the traditional discourse in which teachers initiate, students respond, and then teachers evaluate and summarize for closure.

Inquiry Mathematics

Teachers asked very few questions that attempted to probe student thinking **Formulation/how** questions account for only 2% of the total questions, **formulation/differently** questions are 4.5% of the total questions, and **formulation/thinking** questions are 0.8% of the total questions **Validation** questions in which students are asked to justify their solutions using either formal or informal proof consisted only of 0.8% of the questions asked. There were few changes in the numbers of these questions asked from Fall to Winter Overall, the questions which one would associate with genuine mathematical discourse are conspicuously absent.

Probing questions posed many challenges for teachers. In their journals, teachers discuss a number of issues which make asking such questions problematic Teachers felt that students were not well-prepared for open-ended questions which probed their thinking:

> It makes it tough when you move to a setting that allows for a more open-ended approach. . . I think I am discouraged from asking these kinds of questions from the poor quality of response I get on them. . . Once the kids have success, they will try harder and it won't have to be structured the same way.

Teachers used student lack of preparation and classroom management issues to justify a need for greater structure in the activities and explorations. Structure was often interpreted to mean the use of questions which were "set up" for students success. One teacher, towards the middle of the year, expresses a concern about such structure:

> I need to make sure I'm not structuring too much It is easy to be too leading and feel ok about it because the kids seem happy.. (I see) how hard it is to ask questions and wait in silence and how easy it is to fill the silence with direct instruction.

Other teachers, over the year, show increasing awareness of the limitations of their questioning:

> I was asking lots of questions. But as I wrote down the questions it seemed that almost none of them were probing student thinking. Rather, on many of them I had a specific answer in mind.

Analysis of the most recent cycle of classroom observations will indicate the extent to which these teachers' insights are associated with changes in classroom practice.

Conclusion

It is our observation that inquiry-based curriculum and teacher questioning do not necessarily result in inquiry math discourse. In spite of the efforts of curriculum developers and teacher educators to encourage teachers to foster such discourse, instruction still bears many of the characteristics of school math. Although the teachers in our study religiously eschew the didactic approach to instruction in favor of teacher questioning and student problem-solving, an analysis of the frequency and types of questions asked indicate that the ensuing discourse is "school math". We maintain that unless teachers change their fundamental epistemologies, they will continue to negotiate classroom norms in which the teacher is the director and the students passive players in a theatre where the pose is the problem.

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Elementary
Spatial Visualization
Video Games

SPATIAL LEARNING IN ONE VIDEO GAME

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Mathematics can be learned in out-of-class activities. Video games are examples of highly stimulating environments that might be exploited for mathematics instruction if we knew what mathematics is used, and how that mathematics is used, by game players. This study is a beginning investigation of that question.

Children of all ages choose to play video games of many types. Do video games merely provide recreation or do some offer enjoyable training that supports mathematics learning? This study of one child as he played TETRIS was aimed at conceptualizing important research questions.

Theoretical framework. The evolution of spatial learning proceeds at two different levels: perceptual and conceptual. Perception refers to a situation in which the senses gather static information from the environment and transmit that information to the brain, analogous to a camera taking a picture. However, perception is not simply transmission of a copy of an object (e.g., Del Grande, 1987). Instead, perceptions of static space are constructed. Thus, development of perception seems to require the organization and coordination not only of the activity involved in gathering that information but also of coded and stored sensory information from prior experiences.

Conceptions involve mental operations which consist of transforming what is observed (Montangero & Smock, 1976). Representations of transformations are possible only when conceptual development interacts with the perceptual image; clear progress in representing transformations can be found around 7 years of age. Generally speaking, learning spatial concepts seems strongly related to attempts at representing spatial transformations.

Spatial ability is a cognitive skill which involves the ability both to perceive spatial relationships and to manipulate visual material mentally. McGee (1979) identified two distinct factors of spatial abilities: orientation and visualization. Spatial orientation tasks rotate or translate an entire object. These activities require a person to see that the pattern arrangement of a structure is maintained even though the direction or angle of inspection has been changed. A visualization task requires an understanding of how the parts of a structure can change position in relation to each other and yet not violate the way the pattern connects. A classic example is the visualized paper folding task in which a person must anticipate what a pattern will look like when it is folded.

Game description. TETRIS is a puzzle video game in which different geometrically shaped game blocks fall down, one after the other into a 10 x 20 unit game field. The shape of the block that falls is randomly selected by the computer. Each block is formed from four small squares (i.e., tetrominoes), analogous to the well known pentominoes. For our purposes, the shapes will be called by the following names: 4-bar, 4-square, L, reverse L, T, Z, reverse Z.

The object of the game is to keep the blocks from piling up to the top of the game field. To do this, one can (a) translate a playing block left or right and (b) rotate it as it falls. As horizontal lines are filled, those lines are erased from the playing field and points are awarded. A bonus is given for completing four rows (the maximum possible number) simultaneously. Play continues until the blocks pile up to the top of the game field.

At all times during the game two playing blocks are visible, the one that is currently in play and the one that will appear next at the top of the playing field. To become expert, one must visualize the placement of the current playing block in order to plan for the placement of the next block. As players plan the placement of both the current piece moving down on the screen and the piece which will appear next at the top of the screen, changes in the board must be mentally constructed for various placements of the pieces (i.e., visualization). Since there is not time to generate physically all possible transformations on a piece falling down the screen, players need to generate at least some of those transformations mentally (i.e., orientation) in order to use their time efficiently. Mental imaging of the placement of the current playing block in the playing field is necessary in order to "plan ahead" for the positioning of the next block (i.e., spatial visualization). Thus, TETRIS requires the development of a metacognitive skill, "planning ahead," as well as both spatial orientation and spatial visualization skills. Because TETRIS has a built in time factor, players are rewarded for their ability to plan ahead in the placement of pieces. This "looking ahead" strategy can be considered analogous to the "looking back" strategy frequently mentioned in discussions of problem solving.

Procedures

Subject. Carl, a seven-year-old Caucasian male, was interviewed and videotaped twice for approximately two hours each time. Sessions were held in August and January. Carl had been playing TETRIS for about six months prior to the first session. During the observations, he received no training on either the game or transformational geometry terminology or concepts.

Method. At the beginning of each session, Carl was asked a variety of questions concerning his understanding of the game rules and the seven game blocks. During the first session, Carl first played the game four times by himself. Then he "played" one game by telling one of the researchers where to place the pieces. This change in Carl's role was selected to determine if his strategies changed when he was relieved of the burden of the physical manipulation of the control device and when the time factor was not as critical.

At the beginning of the second session, the Figure Rotations Test from NLSMA was administered. This test was chosen because it matched the orientation aspects of the game. Carl then played two games by himself, with one of the researchers watching and asking additional probing questions. He began a third game by telling one of the researchers where to place the pieces. After placing 54 pieces, his frustration level at the speed of play was so great that he was allowed to complete this game (191 additional pieces) by himself.

First Session

Pre-game questioning. Carl drew five of the seven playing pieces correctly. He described the Z and reverse Z blocks, but he was unsuccessful at drawing them. While drawing, he asked if he should "draw the blocks that could be changed around." Additional questioning revealed that Carl seemed to view the same block oriented in two different ways as two separate figures. He was aware that the shapes had been turned, but once they were turned, he no longer recognized them as the same shape. Thus, he was unwilling to use a common descriptor for a block in different orientations.

Observations. Carl always placed the first block against the left wall and then positioned the next 2 or 3 blocks from left to right. During play, there were cases when Carl appeared to mentally select a position for the playing block, rotate and translate the piece so that it would fit into that position, and then rotate the piece again through a complete 360° turn. As he rotated the piece, he would observe other openings in the lower portion of the playing field and occasionally reevaluate his original decision and move the piece to a new position. Carl regularly performed this ritual with the L, reverse L, and T blocks.

Carl used only the B button on the control device (for counter-clockwise rotations) during the first two and one-half games. Then, for no apparent reason, he switched in the middle of the third game to the A button (for clockwise rotations) and continued to use that button through the end of the fourth game. During his explanations of the rules Carl had said the "button A moves the block to the right and button B moves the blocks to the left." Though he did not use the more conventional terms, rotate or turn, Carl did make the appropriate clockwise and counter-clockwise turning motions with his hands to illustrate these concepts.

When Carl assumed the role of direction giver for the placement of pieces, he gave directions in terms of what a figure would look like once it was turned. For example, Carl's explanation on how to orient the L block was to leave it as be (no turn), put it in the L position (90° counter-clockwise turn), put it in the hangman position (90° clockwise turn), or put it in the body or bed position (180° turn). He described turns of the T block with similar everyday terms, but he realized that the Z and reverse Z blocks had only two possible positions and did not generate iconic descriptions for their placement.

Second Session

Testing. Carl's responses on the figure rotations test were very good. On each of the 14 items, there are 8 figures given; each figure must be classified as a rotation or non-rotation of the item stem. Of the 112 responses, Carl correctly classified 107.

Pre-game questioning. Carl easily drew the seven blocks, though he continued to use different descriptors for each position of a block. His explanations of the directions for the game were clear and complete.

Observations. Carl used only the A button on the control device throughout the second session. He explained that "I only need one button."

During the game in which Carl told the researcher where to position pieces, his explanations were clear, to the point that the game did not need to be paused to ask for clarification on where a particular piece should be placed. When Carl was probed about why he placed a piece in a particular position, he sometimes said that it was because "the next piece goes here." During similar questioning in the first session he never mentioned accommodation of the next piece.

Relative frequencies of pieces that Carl rotated through 360° were similar in the two sessions. In session one, 70% of these pieces were L, 10% were T, and

20% were Z; in session two, 63% were L, 13% were T, and 25% were Z.

Conclusions

Carl's insistence on creating iconic descriptors for different orientations of most of the blocks has potential implications for instruction. In particular, if Carl's behavior is indicative of that of other children, then it is possible that young children who are just beginning to develop operational understanding have not built connections that will allow them to view, for example, a rectangle as still a rectangle once the figure is rotated. Dual-coding theory, which proposes that visual representations may be generated from verbal cues as well as visual cues corresponding to objects or events, suggests that the use of verbal connectors in conjunction with visual connectors might assist a child in developing stronger images of object transformations.

Carl's explanation that he positioned one piece in preparation for the next piece seems to support the notion that Carl had begun to plan ahead. Although this strategy does not seem to be developed well in school mathematics instruction, it may be one that mathematicians utilize regularly (a strategy that may allow them to become expert in the field). The instantaneous feedback that is provided in electronic games such as TETRIS provides a dynamic learning environment for the practice of such a strategy.

Carl's tendency to rotate a figure through a complete 360° turn, even after he had apparently decided where to place the block suggests that he was utilizing both perceptual and conceptual reasoning while playing the game. This might be important if other children also demonstrate similar reasoning.

Although the patterns observed in Carl's play are idiosyncratic to one child's organization of schemata, the importance is that organizational patterns appear to have been formed. For example, more frequent rotation of the L block suggests that this piece was more difficult for him to visualize. It is our opinion that continued study of additional subjects might reveal groups of organizational patterns that might have significant pedagogical implications.

Hypotheses raised. A variety of questions are suggested from these data. Do most children begin by placing the first block against a wall? Do other children initially position pieces across the bottom of the playing area? Will players choose to use both rotation buttons? Are there groups of children who prefer one type of rotation over the other? If so, what characterizes these groups? Will older or more experienced players interchange the use of the A and B buttons more frequently? Does the asymmetrical shape of the L and reverse L blocks cause equal placement difficulty for players who are developmentally more mature? Do many children use real-world objects to describe the rotation of the blocks? If so, does this have implications for geometry instruction? Will the repetitive visual exposure to rotation and translation of blocks in this game provide a sufficiently rich setting that will enhance the acquisition of spatial concepts? Does TETRIS help students learn to plan ahead? If so, is there payoff for performance in content such as solving equations, performing geometric constructions, or creating and organizing the steps of a mathematical proof?

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Post Secondary
Parameterization
Geometry

ON THE UNDERSTANDING OF VARIATION
A TEACHING EXPERIENCE

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This paper deals with a research that pretends to explore the strategies that favor the understanding of the representation of parametric curves in the plane. We report the results of a three years long teaching experience with college students where we explore and explain the difficulties and strategies that students have when faced with problems that involve parameterization.

§ 1. ABOUT THE RESEARCH PROBLEM

Students at college level have difficulties with understanding and graphing curves, specially when they are given in a parametric representation. Because of the importance of this material for the understanding of other mathematical concepts as area, curve length and the solution of differential equations and their use in application problems, we tried in the present study to find out why they have such a difficulty.

These difficulties can arise from situations related to different kinds of representation in several contexts, with problems in the transfer of information from one kind of representation to another or with the understanding of the concepts of variable and variation.

In dealing with parametric representation, students are usually faced with three different situations: They can have the parametric equations of a curve and be asked to graph the curve, or they can have two different curves for each dependent variable and be asked for the curve's graph or they face a verbally stated problem that can be solved using parametric equations. We are interested in finding out the strategies that students use in all the situations and in differentiating between the strategies that are independent of the context in which they face the problem and those closely related to the context.

§ 2. ABOUT THE TEACHING EXPERIENCE

We worked during three semesters with Mathematics students who are taking a mandatory course in Analytic Geometry. We chose this population because they have already taken a pre-calculus course where emphasis is given to the handling and graphing of one variable real valued functions, so they know how to graph.

We designed a small questionnaire to try to find out what they thought when faced with parametric equations and what they did with problems that require parameterization for their solution. We studied the answers to this questionnaire and then interviewed the students about their difficulties.

We then gave them two lessons on parametric curves, particularly in how to graph them and on some methods to deal with problems. After the lessons we gave them another questionnaire where we found that most of them still had problems. We gave them four more lessons emphasizing the qualitative reasoning associated with the construction of the graph of the curve and some methods to deal with problems. After these lessons a new questionnaire was given to them and we found that most of them were able to graph the curve but still were not very successful in dealing with the problems.

After a year, because we wanted to analyze what they had assimilated from the teaching experience, we chose a male student and a female student who had been successful in solving the problems after the teaching sessions, and by means of a new questionnaire and a clinical interview, we analyzed the strategies they used to solve the problems and we tried to isolate the different episodes in their reasoning.

First we found that when faced with parametric representations of curves, students can graph each of the dependent variable with respect to the independent variable, but cannot find the graph of the interrelated dependent variables, unless they are able to eliminate the parameter, and they do not perceive the interrelation of the dependent variables. They think of them as two separate equations not dealing with the same problem and they do not understand why they have to put the information in a single graph. We also found that when faced with problems that require parameterization they cannot break them in components, even the simple ones; they always try to find a relationship between the two variables involved in the problem. Even if you tell them to separate the problem, they are not able to find a third variable on which the other two depend.

In the first questionnaire we asked the students some general questions about what they think when they find the word parameter and how can they explain in words what a line is and particular questions dealing with the graph of a curve when the parametric equations are given in a problem.

Some typical answers were,

"A parameter is:

a) a way to measure something",

b) something that relates one thing with another",

c) a constant that can take any value", or

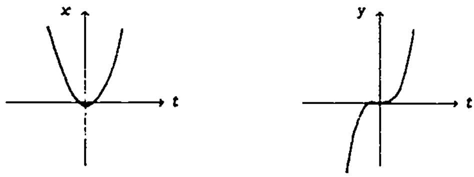
d) something that you can change and as you change it you find different points on a line".

"A line is:

a) '*algo que va derechito derechito*', (it is something that goes straight).

b) Something that connects two points

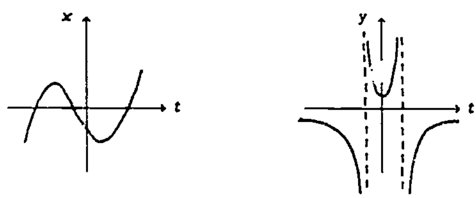
When they have the parametric equations, all of them eliminated the parameter and graphed the curve. When they could eliminate the parameter they didn't graph the curve. When they have two curves, most of them tried to find out and explicit relationship in algebraic terms, and then they eliminated the p-rameter. For example when given



They write $x = t^2$, $y = t^3$ then $t = \pm\sqrt{x}$, so $y = \pm x^{3/2}$ and graph the curve.

In a first discussion session about their answers we confirmed the difficulties already mentioned.

In the second questionnaire we tried to force them to think about the problem in an essentially geometrical context, so we asked them to graph a curve from two arbitrary graphs for which no algebraic expression could be found. For example, we gave them



and found that most of them read some points from the graph, and made a table showing some important points:

t	x	y

and they showed them in the x-y diagram. But if the curve was not easy, as in the example given, they didn't know how to join those points.

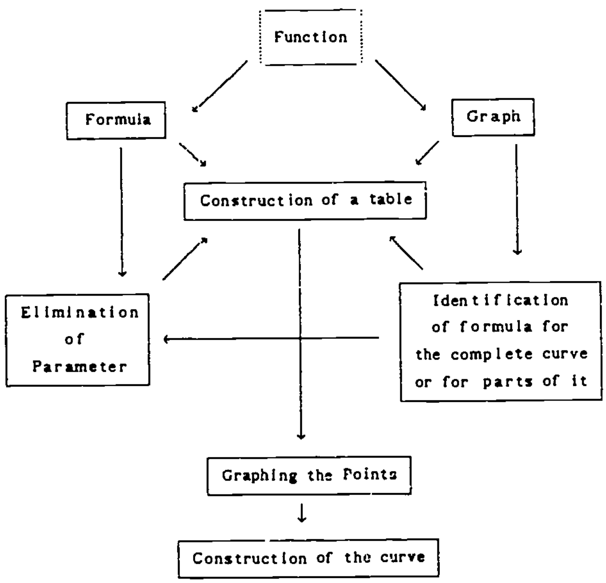
In the last questionnaire we wanted to find out if the qualitative techniques had been learned. However, we recognized that although some of them could graph the curve, they only applied the techniques without any understanding of the strategies.

Once we recognized these problems we designed a research protocol to isolate the main strategies.

§ 3. ABOUT THE STRATEGIES AND THE LEARNING EPISODES

Since our objective was to analyze the representation strategies, in the recent interviews we searched for elements to explain why the students cannot build graphs of parametric curves.

We found a sequential order in their strategies shown in the following scheme.



As we see there are two kinds of general strategies: one is of algebraic nature, the other is of geometrical nature, but both of them are based on a numerical and quasi numerical approach.

We observed that students feel more confident if they can find a relationship between the variables. For example one of the students thought that it was always possible to eliminate the parameter from the equations, and that given a difficult curve one can always break it in parts so one can find piece wise relationships and eliminate the parameter from each of them. The other student could not tell for sure if a point was part of the x-y graph if he didn't have a formula for it.

The qualitative or geometric strategy was not present until we talked about movement of a point. So it seems that the numerical strategy is independent of the context, the algebraic depends on the context but works in two different ways: as a resource for the numerical strategy and as a tool that is self sufficient to solve a problem, and the geometric is not spontaneous and is closely related to the idea of movement.

The interrelation of the dependent variables was not evident for them until we made explicit reference to the idea of movement, and even then, one of the students couldn't see it. It may be that the understanding of parametric curves is made easier when it is closely associated with the concept of movement.

After an analysis of their answers we think that although they can deal with one variable problems, they do it mechanically. They do not have a clear concept of variable, and this difficulty is made more evident when several variables are involved. We are trying now to explore the student's response when dealing with these kind of problems when they are directly related with concrete physical problems and to relate the results with their understanding of the concept of variation within a computer environment.

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Teachers
Life Histories
Vygotsky

A VYGOTSKIAN FRAMEWORK FOR EXAMINING MATHEMATICAL ATTITUDES AND THE NCTM *STANDARDS* THROUGH LIFE HISTORIES

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Life histories are used to explore attitudes toward mathematics and the NCTM *Standards* (1989). This methodology has been a valuable tool in exploring attitude development. The conceptual framework for this research is grounded in Vygotsky's (1930-4/1978) social interactionist theories and mathematical attitude research. A model of mathematical attitude development is presented and discussed.

In this PME paper I will demonstrate how life histories can be used to study mathematical attitudes and how these relate to the NCTM *Standards* (1989). Vygotsky's (1930-4/1978) social interactionist theories and mathematical attitude research, especially Fennema (1989) and Reyes (1984), provide the conceptual framework for this work. This framework and the analysis of mathematical life histories guided the creation of my model of mathematical attitude development (see Figure 1). Directly or tangentially, the framework and the analysis support the idea that students' attitudes develop when they interact with other persons and their environment. This constructivist (actively creating knowledge) view also appears to be part of the framework supporting the *Standards* (1989).

Vygotskian conceptual framework. Vygotsky's emphasis on how culture influences learning, provides a broad conceptual framework that can take us beyond a strictly cognitive focus and challenge us to examine the learning and teaching of mathematics in the context of cognitive, affective and social dimensions. His Zone of Proximal Development (ZPD), the region between a person's current and potential achievement, is very helpful in gleaning relevant information from life histories.

Attitude. I see attitude as a "way of thinking, feeling and behaving." This broad multi-dimensional definition goes beyond most definitions and suggests affect is only one part of attitude. When attitude is viewed simply as liking or disliking, the cognitive and behavioral components of attitude are frequently overlooked. The formation of an attitude is a complex process involving the interaction among many factors such as family, socialization, schooling experiences, and relationships with mentors (see Taylor, 1988 & 1990 for further discussion).

Life histories and the *Standards*. My work researching the mathematical life histories of twelve outstanding teachers, I believe illuminates the essence of the *Standards*, as well as the context in which teachers apply the *Standards*. I found many of the goals and specific areas for increased and decreased attention recommended in the *Standards* to be complementary with events in the participants' lives and with Vygotsky's theories.

For example, Curtis told of a negative fourth grade math experience that affected his mathematical development, especially his attitude. He remembered "multiplying two six digit numbers... and I never could get all the rows straight! It was just terrible. I got bored with it, and

While learning multiplication was an experience which was fun and exciting for Karen, Bill, and Tom, it was not for Curtis. It concerned Curtis that in his elementary mathematics classes the product or answer was overly stressed, rather than understanding the process. This concern is also discussed in the *Standards*; a process-orientation is strongly emphasized.

Like Curtis, Joe's earliest schooling memory with mathematics was not a pleasant one. "I remember being embarrassed in the third grade, b ng at the chalk board, trying to do one of the very simple addition carrying problems, and I just, for some reason, could not do it. It was not a very good experience, [but] a humiliating one. I'll always remember that, always! That may have set the whole tone for the way I viewed math. I don't know. That really sticks out in my mind." Joe also remembers his elementary mathematics as "a lot of memory work, a lot of computation on paper, and not a whole lot of application, I'm afraid." He further feels the mathematics that was stressed involved "the mechanics of doing things, as contrasted with understanding why you are doing things." When Joe became an elementary teacher, his negative elementary experiences motivated him to provide his students with useful math experiences with concrete objects. He stressed understanding the processes involved and the usefulness of mathematics.

Bill's elementary arithmetic experience also affected his development as an educator. "I still have a real clear image of something I use today in my own teaching. The image is looking at a page of exercises in a text book, this could be in a fourth or fifth grade book, seeing on the page at the top a whole lot of arithmetic problems that are written out explicitly, add two numbers, multiply two numbers. Then at the bottom lower quarter of the page are the story problems: suddenly you don't see numbers, but you see words... I know my own feeling at that time was that the problems at the top were the easy ones; you were just asked to do some manipulation on a couple of numbers. The hard ones were at th bottom; you had to read the words, formulate the problem mathematically and then solve the problem. Those were the story problems."

Students are often able to do rote lower level tasks as was emphasized in Bill's elementary class, yet when they are challenged to apply their mathematical knowledge to solve problems they often have difficulty. In Bill's words, "I know that in my own classes today it is the very same way. People are good at working math problems when they are just stated mathematically, but it's the problems with the words in them that the people just shuddered about. I realize this was the outlook I had back then. I see it in my own students today and I try to dispel it. I give them a lot of word problems and I tell them this is what math is all about. It's not multiplying two or three digit numbers together, but it is taking the problem in the real world that's given to you in English, visualizing that problem, what is being asked, casting it in a mathematical form, and then solving it. That is a completely different process! Some people can multiply four digit numbers in their head, but when it comes to translating story problems they may be useless. Conversely, some people are good at the modeling aspect but are very slow at doing calculations. That's my own attitude about math. It is not there to just simply do manipulations on numbers and symbols; it is you never get the right answer if the rows are not straight!" Curtis felt that he and many of his "classmates got turned off to mathematics at that point." Yet, he remembered that before he was "taught" multiplication in school it intriqued him. "I actually figured out what multiplication was on my own when I was in third grade... I thought that was neat! I used to play marbles and I remember putting them in rows of four and I figured it out by looking at the rows." The *Standards* (1989) would not support assigning a tedious six digit multiplication problem. In fact, they suggest teachers place decreased attention on "complex pencil-and-paper computations" and isolated treatment of such computations. The purpose of computation is to solve meaningful problems. Therefore, we are challenged to reduce the computational emphasis so often used and focus more on "the thoughtful use of operations and number relationships" (p. 47).

Developing an understanding of the underlying concepts of multiplication is important. Curtis's marble story illustrates his conceptual understanding of multiplication. The *Standards* (1989) also emphasize the importance of linking concepts to the paper-and-pencil procedures. This was not "taught" in Curtis' class.

While the *Standards* challenge us to demphasize drill, it is important to keep this recommendation in perspective. Some students are particularly fond of drill activities and even find them "exciting" and "meaningful." For example the earliest mathematical memories for three of the mathematicians in my study involved basic arithmetic drills Karen's memory was a pleasant one. "I remember standing up and having to say 9 x 1 = 9, 9 x 2 = 18, etc.... I just thought it was so much fun doing that. I never thought it was boring. I never thought it was dull, even though I know that a lot of kids [now and then] think it is boring. I just thought it was fun! Exercising the memory."

Bill remembered learning the multiplication tables in fourth grade. "I can clearly remember learning the multiplication tables. There was a big bulletin board in the corner and down one side were all the students names. and across the top were the multiplication tables from 1 to 12. As soon as students passed a test in the multiplication table they put an X up on the board. I can still see that... It was something everybody had to do and some people finished sooner than others. I can't really picture any tasks we had to do. I guess it was an exciting challenge to be working on the fours table and see way down at the end the table of 12s, and to realize that there was a sort of unknown territory out there was kinda exciting to me. It was that kind of thing that motivated me, seemed to be a good incentive to keep going and get to the end of the tables as quickly as I could."

Tom also felt that it was fun doing basic arithmetic. His earliest mathematical memory was during the addition, subtraction, multiplication, and division "era", but it took place at home working with his dad and a slide rule. He enjoyed doing basic mathematics with his dad and a slide rule. "I could usually come up with the right answer most of the time. Maybe that is why it was fun." One wonders if the basic facts would have been so much fun for Karen, Bill, and Tom if they weren't so successful.

not valuable unless you are taking real problems given to you in English, perhaps by a non-mathematician, and using math to solve it." Today, Bill is an applied mathematician. It seems clear his interest in applying mathematics developed more strongly over time. Possibly his early experience during his elementary school days triggered this development. Bill's emphasis on applications and problem solving is advocated in the *Standards*.

Priscilla's response, when asked "what would you say is your earliest memory regarding math?", involved a detailed description of an influential teacher and some of the activities in her class. "That's an interesting question. Let me think. I don't think I have any real memories about it until I was in junior high, seventh grade. I really don't have a lot of memories of anything until junior high.... When I was in seventh grade I had a wonderful math teacher. His name was Mr. Sweat. We played after school and at lunch time. We would sit around with him and do things like discover unique patterns with numbers, like the nines, and tricks for getting the multiplication done faster or division faster, and all those really fun things. Those were puzzle solving, but it was still eminently obvious how it [the experience] could be used in real life and how it described real things." Mr. Sweat appeared to be a teacher who was "ahead of his time" and who was teaching a curriculum advocated in the *Standards*.

Seeing the usefulness of mathematics was very important not only to Priscilla and Bill, but also to the other ten participants. In fact the four social scientists in the study all elected to not pursue math when they did not perceive it was useful. Fennema (1981) and the *Standards* (1989) have documented the importance of the perceived usefulness of mathematics. Some students stop taking mathematics when they do not perceive it as useful to them (Sells, 1979; Fennema, 1981). In the *Standards* usefulness is exemplified by mathematical connections and applications.

Concern over the way "school mathematics" is, and has been, taught is not unique. Tauskky-Todd (1980) enjoyed studying and using mathematics on her own, yet was not very interested in the "school math" she was studying as a secondary student after World War I. In her words "The work at school was really not that difficult if one applied oneself to it, but it was so uninteresting that you could not wish to apply yourself. I felt there was another mathematics" (p. 313). This other mathematics was the one that she was pursuing on her own and with her father at his vinegar plant; it had meaning and relevancy for Olga Tauskky-Todd, and it was connected to her life.

Nancy, a mathematician in my study, also found the math she studied at home to be especially interesting. Her father was a "jack-of-all trades, like many laborers. His primary job was sheet metal. He was always laying plans on the metal, and I was always tagging along with my Dad and he would fold the metal up and come up with these nice boxes like an air conditioner box. I'd see it laid out, then I'd try to visualize what it would look like when he finished with it. He was always planning this out on paper and I always thought that it was neat! So, I would duplicate that behavior with cardboard and stuff like that. Being that we were not from a well-to-do family we had to invent our games.... If we wanted to play anything we would have to build our own objects." It is clear that Nancy's interest in spatial visualization and geometry developed at home as a young child helping her father.

It is not uncommon for children to develop an early interest in an area through their active involvement with a parent or relative. For Nancy, Tom, and Olga Tauskky-Todd, interactions with their fathers facilitated their mathematical interests. John-Steiner (1985) and Vygotsky (1930-34/1978) have discussed the importance that a relationship with a significant adult can play in the development of a person's interests. These adults may or may not be family members. For Priscilla, her teacher facilitated her mathematical interest. Each participant in the study had significant mentors.

Einstein's early interests were encouraged by his family and a friend of the family. His Uncle Jake introduced him to mathematics and his mother introduced him to music and literature. Max Talmey, a poor Jewish medical student who came to dinner at the Einstein home in Southern Germany when Albert was twelve, brought with him a number of books on science which he showed to Albert. "And more significantly Max followed up Uncle Jake's teaching of algebra with a book on geometry. With Talmey's assistance Albert worked through Speiker's Plane Geometry and later went on to teach himself the elements of calculus" (Schwartz, 1979, p. 30). The interaction with older people fostered Einstein's early interest in mathematics and science.

Not all children develop an interest in an area through their active involvement with a person. Some children, such as Curtis with his marbles, may have an experience that facilitates their mathematical interest.

Model Presentation. My working model of attitude development emerged from the life history research and is, I believe, very much in line with the five essential goals of the *Standards*. These goals assume that students' should not only be able to solve problems and reason mathematically, but also to become confident and value mathematics, communicate it effectively, make connections and become aware of how mathematics has impacted their lives. I believe these goals illuminate the importance of and the connections among thoughts, feelings and behaviors (the components of attitude). Therefore, the goals of the *Standards* suggest we be concerned with attitudes.

It appears that attitude change, specifically mathematical attitude change, is often a function of significant social interactions. Below is a model that provides the underlying conceptual framework that illustrates the attitude development process. Elements of this framework include: attitude which is viewed as a complex construct including thinking, feeling, and behaving; and the ZPD which is defined as "the distance between the actual development level as determined through independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1930-4/1978, p. 86).

The double arrows in the model are needed to show the complex interactions. In accordance with Vygotsky's emphasis on the importance of the environment, particularly the culture and other persons within the environment are depicted as shaded areas surrounding and affecting attitudes. Thus, a person's attitude is affected by his/her environment. This includes experiences within the environment as well as the culture of the environment itself. Other persons are also a part of the environment.

This model emphasizes the larger cultural context within which an individual's development occurs, first on a social level, between people, then on an individual level as internalization occurs. The arrow through the ZPD depicts the meta-awareness an individual develops when s/he bridges his/her ZPD. Meta-awareness involves reflecting on one's thoughts, feelings, and behaviors. Arrows are included from meta-awareness back down to attitude to represent the continual interactions people experience. Therefore, an individual can repeatedly bridge his/her ZPD to a meta-awareness state and then have an attitude that is further developed. For an individual such as Curtis, Nancy, and the others discussed, this means that their attitudes toward mathematics, including their feelings, thoughts, and behaviors have changed.

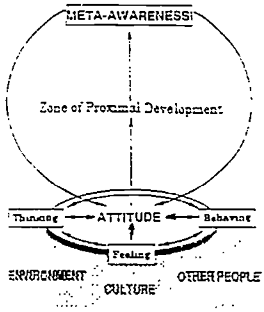

Figure 1 - A model of mathematical attitude development depicting the components of attitude, bridging the ZPD to meta-awareness and factors influencing this process.

Educational Implications and Conclusion. Vygotsky has been called a "genius" who lived ahead of his time. A Vygotskian perspective presents an integrated theoretical framework which looks at the whole rather than dwelling on the parts. In this age of fragmentation and specialization it is important to keep the complex picture in mind. This view is one which is complementary to the *Standards* emphasis on the importance of mathematical connections, usefulness, communication, reasoning, problem solving, and attitudes.

In Frye's (1989) words implementing the *Standards* implies the use of: "Words like explore, communicate, construct, use, and represent, stress the involvement of students on the active "doing" of mathematics. Words like collaborate, question, express, value, share, and enjoy, bring a new flavor to the work of the students. Words like reflect, appreciate, connect, apply, and extend, build a new attitu ward mathematics and its uses" (p. 59).

Further, a Vygotskian perspective supports the importance of alternative teaching strategies such as using cooperative groups, providing opportunities for significant peer interactions, and posing problems beyond students' understanding. This maximizes learning and facilitates students bridging their zones (ZPDs). Vygotsky believed that "creative imagination grows out of the play of young children" (Williams, p. 117). This perspective also stresses the importance of play and our cultural environment in students' development. It is the interactions among one's thoughts, feelings, and behaviors (attitude) emersed in culture that is significant.

The stories, model and thoughts presented in this paper offer ways of extending Vygotskian thought to mathematics education and the affective domain. Mathematical life histories can be used as a tool to explore mathematical attitudes and the *Standards*, as well as in alternative classroom assessment.

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Running Head
Mathematics Teacher Researcher

Age level NA
Identifies #1: Teachers Psychology of Learning

Identifies #2: Math Teacher Researcher

THE MATH TEACHER AS RESEARCHER: A CHANGE IN PERSPECTIVE WHICH IDENTIFIES THE MATH TEACHER'S PSYCHOLOGY OF LEARNING AS A NEGLECTED AREA FOR MATHEMATICS LEARNING

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Abstract

A case is presented for the teacher-researcher. As researcher, the teacher is made aware of his/her psychology of learning. The narrative is the method used to promote an understanding of the teacher's psychology of learning. Two narratives, one by a writing teacher, and one by math teacher, are examined by the technique of phemonological exegesis. The findings indicate that their psychology of learning, the exploration of error, influences the course content and the students learning of of that content.

That university researchers should collaborate with elementary and secondary school teachers has been suggested by Noddings (1988) and others. More recently, the National Council of Teachers of English has awarded its 1990 David Russell Award for distinguished research in the teaching of English to Nancy Atwell, a former 8th grade teacher, for her book <u>In the Middle: Writing, Reading, and Learning with Adolescents</u>. In her acceptance speech, Ms. Atwell described her award as representing an acknowledgement that "observations and reflections of classroom teachers count as research" (Rothman, 1990). Implicit here is the notion that teachers can and should be the generators of classroom research.

Indeed, with the increased interest in teacher professionalism, a growing number of teachers are undertaking critical analyses of classroom practices. In many cases, according to <u>Education Week</u> reporter Rothman (1990), these teachers are also writing up and reporting their findings. These efforts, teachers say, have given them a deeper understanding of how students learn and how teachers can contribute to learning. Led by such interest, a number of organizations such as the A.F.T. and the N.C.T.E. have created grant programs to sponsor teacher research. As Rothman notes, research should be seen as part of the teaching <u>act</u>.

The issue of teachers' perspectives toward knowledge, their view of themselves, and their students as knowers has recently been acknowledged as important to pedagogical thinking (Lyons, 1990). The method employed for understanding the teachers' perspective is the narrative: narratives produced by teachers promote an understanding of the psychology of learning that may be missed in more analytical forms of research. Indeed, as Tappan and Brown (1989) note, the authorship of narratives (stories) provides a new vision of the relationship between developmental psychology and education.

Individuals give meaning to their experiences by representing them in narrative form. In fact, narratives play a role in helping us to understand human actions, both the actions of oneself and the actions of others. That is, learning how you work helps you to see how others work. It is in the awareness of one's own biases toward knowledge, learning, and education revealed in a narrative that objectivity can be found and understanding communicated. In this way, teachers may be restored to the role of "reflective practitioners" (Rothman, 1990). When they are reflective practioners, part of the "content" must be the teachers' psychology of learning, not an aspect usually recognized by traditional non-teaching researchers.

This presentation will take a developmental approach. First we will look at the work of teacher-researcher Mike Rose, whose narrative Lives on the Boundary (1990) provides a model of a teacher's psychology of learning. The next step will be to look at the ongoing narrative of a Piagetian researcher (supported by federal and university funds), who is also a mathematics teacher, and more recently, a mathematics teacher-researcher.

The method for pursuing such a developmental approach is phenomenological exegesis, that is, a line by line reading of a text that encourages "asking the right question." The key idea behind this approach is two-fold: first, the setting forth of details allows the voice of the text or narrative to come through (which does not happen so clearly with the act of abstraction); second, the process of working through the text by accurately describing it allows for an understanding of text (and not explanation, which is often the end result of analytical criticism). Therefore, first the readers let the text speak; then the readers dialogue with the text so that understanding may emerge.

To illustrate, we have selected passages from three chapters of Mike Rose's Lives on the Boundary. In the first "text", Rose is himself a student in an urban, economically disadvantaged school; in the second "text", he is a new teacher in a non-traditional setting (instructing Vietnam War veterans); in the third, he is a more mature teacher working in a university remediation program.

An exegesis of the first "text" (see Narrative 1) reveals Rose in the "role of mediocre student, the survival mechanism he has developed in an academic setting which seems to be alien and perhaps hostile territory. In this foreign terrain he understands that he does not know how to do some things and does not know how to do some thing "the right way" (error free). He believes he is responsible for these academic deficiencies; he is the author of his own "faulty and inadequate ways." As a result, Rose's defense is to construct his own ignorance by the self-concocted "magic" of using only half of his mind's capabilities. He sabotages himself to conform to the expectations made of him: he lives down to those expectations and in that way renders himself "average." Therefore, at this stage of his development, Rose's psychology of learning is that of limitation as survival, not growth. Certainly that kind of thinking is a non-national, induced magic, as all living organisms generally survive by growth.

In the second "text", Rose as a new teacher has now become part of an academic authoritarian system which decrees what constitutes acceptable levels of learning. In this system, errors constitute a rational scientific explanation of students' deficient performances. However, a nagging sense tells Rose that mere mechanics that are either "right" or "wrong" cannot truly indicate a received education. The standard "proofs" of educational acceptance - being free from error -may not be true indicators and may in fact restrict students' true abilities. Thus, Rose's psychology of learning has started to shift. He still feels the burden is on him, as he did as a student, but now he is in the position of both imposing and lifting the burden. Indeed, he would blame himself if his notion of learning remained that of making students "free of error," for his students would be doomed to failure. It is here that he must try to restore or reclaim expectations more appropriate to a true learning environment.

In the final "text", as Rose works with his adult remedial students, his perceptions about the notion of mistakes shift even further. Rather than regard them in a pejorative way as indicators of deficiency, he sees mistakes as forms of communication which render his students' worlds and expectations. This allows a dialogue of understanding between teacher and students to develop: it isn't just the teacher telling them what they need to know to be acceptable but the students educating the teacher through their texts about the stories of their lives. As Rose suggests, mistakes are a map through the landscape of their lives; like life itself, mistakes are just part of an ongoing process. If so, students can begin to feel like they have some control over their learning. In this way, errors provide understanding for both students and teachers. With these boundaries between teacher and student lessened, their worlds may intersect so that a shared academic community may emerge. Consequently, the next stage in Rose's psychology of learning is based upon collaboration: he listens, helps, and facilitates as students try to achieve modest but desired goals.

A developmental sense of the psychology of learning also can be gleaned by tracing Mermelstein's narrative (see Narrative 2). Like Rose, he has constructed a psychology of learning from his classroom experiences. Also, as with Rose, it is ongoing.

In the first "text," (see Narrative 2, paragraph 1) Mermelstein while teaching students mathematics and employing the lecture method, at the same time makes use of Piaget's clinical method of questioning on conservation of quantity tasks. On the one hand he tries to explain mathematical content to the students and on the other hand he attempts to understand the students' quantitative thinking. Mermelstein's psychology of learning vacillates between the reduction of error illustrated by his explanations of mathematical content and his exploration of error with the clinical method of questioning on Piagetian quantitative tasks.

In the second "text" (paragraph 2), Mermelstein demonstrates the role of trial and error in his conservation research to resolve problems in the conservation tasks. Thus, his psychology of learning now represents trial and error as a method for understanding.

In Mermelstein's third "text" (paragraph 3), it is the presence of non-aggressive humor in the mathematics classroom which encourages trial and error learning. At this stage of Mermelstein's thinking, the importance of humor to relax students is explicit while the importance of errors or mistakes as vehicles for understanding is yet to be made explicit.

In the fourth "text" (paragraph 4), after a period of reflection, Mermelstein integrates non-aggressive humor, mistakes, and a sense of community. Humor takes the worry out of being mistaken and communicates a sense of caring, thereby creating a feeling of community. These reflections have been shared not only with students but with other colleagues at professional meetings. Mermelstein has progressed from where he used trial and error learning in a mathematics classroom only with Piagetian conservation experiments to trial and error learning in his own conservation research, and to finally focusing on trial and error (mistakes) in a mathematics classroom while using this classroom as source for research problems. Clearly, Mermelstein's psychology of learning views the making of mistakes as central for learning of mathematics.

In the final "text" (paragraph 5), Mermelstein generates a research problem in which his college students are collaborators. These college students, in an attempt to understand the source of their math anxiety as well as the anxiety of elementary school children, tutor 5th grade students in an neighboring elementary school, thereby enlarging the scope of the caring community.

In summary, Mermelstein's psychology of learning provides a sense of community in which teacher and student have "listened" to each other and learned each other's point of view. Not only are one's own mistakes accepted and examined in this context but others' as well. In this way an understanding of mathematical ideas may be provided.

When a teacher's psychology of learning stresses the students' "understanding" or discovery, it emphasizes exploration of error or mistakes. On the other hand when a teacher's psychology of learning stresses explanation, it seeks to minimize students' error and thereby precludes understanding. Yet understanding provides the global context in which an explanation makes sense.

Therefore, what a mathematics teacher-researcher uncovers is that there is an interaction between his psychology of learning and the mathematical context, that is, mathematics content is not independent of the observer presenting that material. Teachers can discover this important perception only by observing and recording their own development of a psychology of learning. As we encourage students to be the foremost interpreters of their own experience, so should we allow teachers, in the role of researchers, to be amongst the foremost voices in the complex arena of educational psychology.

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"I JUST WANNA BE AVERAGE"

And the teachers would have needed some inventiveness, for none of us was groomed for the classroom. It wasn't just that I didn't know things-didn't know how to simplify algebraic fractions, couldn't identify different kinds of clauses, bungled Spanish translations-but that I had developed various faulty and inadequate ways of doing algebra and making sense of Spanish. Worse yet, the years of defensive tuning out in elementary school had given me a way to escape quickly while seeming at least half alert. During my time in Voc. Ed., I developed further into a mediocre student and a somnambulant problem solver, and that affected the subjects I did have the wherewithal to handle: I detested Shakespeare; I got bored with history. My attention flitted here and there. I fooled around in class and read my books indifferently-the intellectual equivalent of playing with your food. I did what I had to do to get by and I did it with half a mind.

The tragedy is that you have to twist the knife in your own gray matter to make this defense work. You'll have to shut down, have to reject intellectual stimuli or diffuse them with sarcasm, have to cultivate stupidity, have to convert boredom from a malady into a way of confronting the world. Keep your vocabulary simple, act stoned when you're not or act more stoned than you are, flaunt ignorance, materialize your dreams. It is a powerful and effective defense-it neutralizes the insult and the frustration of being a vocational kid and, when perfected, it drives teachers up the wall, a delightful secondary effect. But like all strong magic, it exacts a price.

RECLAIMING THE CLASSROOM

It would not be until later in my career that I could methodically challenge these assumptions; at this early stage in my development as a writing teacher I had to rely more on the feel of things. It just didn't make sense that not knowing the delicacies of usage or misplacing commas or blundering pronouns and verb forms or composing a twisted sentence indicated arrest at some cognitive-linguistic stage of development, a stage that had to be traversed before you could engage in critical reading and writing. Such thinking smacked of the reductionism I had seen while studying psychology at UCLA. Besides, I had never gotten some of this stuff straight, and I turned out okay. It seemed that, if anything, concentrating on the particulars of language-schoolbook grammar, mechanics, usage-would tremendously restrict the scope of what language use was all about. Such approaches would rob writing of its joy, and would, to boot, drag that veterans back through their dismal history of red-penciled failure. Furthermore, we would be aiming low, would be scaling down our expectations-as so many remedial programs do-training to do the minimum, the minimum here being a simple workbook sentence free of error. The men had bigger dreams, and I wanted to tap them.

CROSSING BOUNDARIES

Slowly something has been shifting in my perception: the errors-the weird commas and missing letters, the fragments and irregular punctuation-they are ceasing to be slips of the hand and brain. They are becoming part of the stories themselves. They are the only fitting way, it seems, to render dislocation-shacks and field labor and children lost to the inner city-to talk about parents you long for, jobs you can't pin down. Poverty has generated its own damaged script, scars manifest in the spelling of a word.

This is the prose of America's underclass. The writers are those who get lost in our schools, who could not escape neighborhoods that narrowed their possibilities, who could not enter the job market in any ascendent way. They are locked into unskilled and semiskilled jobs, live in places that threaten their children, suffer from disorders and handicaps they don't have the money to treat. Some have been unemployed for a long time. But for all that, they remain hopeful, have somehow held onto a deep faith in education. They have come back to school. Ruby, the woman who wrote the passage that opens this section, walks unsteadily to the teacher's desk-the arthritis in her hip goes unchecked-with a paper in her hand. She looks over her shoulder to her friend, Alice: "I ain't givin' up the ship this time," she says and winks, "though, Lord, I might drown with it." the class laughs. They understand.

It is a very iffy thing, this schooling. But the participants put a lot of stock in it. They believe school will help them, and they are very specific about what they want" a high school equivalency, or the ability to earn seven dollars an hour. One wants to move from being a nurse's aide to a licensed vocational nurse, another needs to read and write and compute adequately enough to be self-employed as a car painter and body man. They remind you of how fundamentally important it is-not just to your pocket but to your soul as well-to earn a decent wage, to have a steady job, to be just a little bit in control of your economic life. The goals are specific, modest, but they mean a tremendous amount for the assurance they give to these people that they are still somebody, that they can exercise control.

NARRATIVE 2

After conducting Piagetian research on quantitative thinking in children and teaching at the college level, I returned to teaching mathematics to elementary and junior high school children in two inner New York City schools. In addition to the traditional mathematics, many of the children were provided with Piagetian conservation experiments: conservation of quantity, length, and number. My purpose here was to provide them with an understanding of the foundations for mathematics reasoning. The clinical method of questioning used for the Piagetian conservation experiments provided an opportunity for a dialogue between students and myself.

After two years of teaching (1973-75), a government grant for continuing conservation research led me to further explorations regarding conservation of liquid quantity and liquid volume. It was while examining the children's responses to these experiments that questions arose regarding the appropriateness of the existing Piagetian conservation of quantity experiments. My research at this time was punctuated by trial and error experimentation. I would try one approach, discuss it with a colleague discard it, and then try another approach. As a result of such trials and errors slowly an understanding of the conservation problem emerged. What was implicit for me was that mistakes were necessary ingredients for understanding.

After four years of research (1975 - 1979), I returned to the mathematics classroom in a private school (grades 6-12). Many of these students had learning difficulties in mathematics or "hated" math. I noticed that humor seemed to relax them and free them to concentrate. Nonsense humor seemed to relieve their tension and make it easier for them to learn. During my time at the school I sensed the students' reluctance to put their work on the black board. They needed to be correct or right. Unless there were guarantees of correctness they refused to show their work, only their answer.

In 1983, after four years at the private school, I taught students who also had difficulties in mathematics at the College of Aeronautics. At the college of Aeronautics I had more time to reflect on my activities in the classroom. The writing about humor in the classroom forced me to articulate a relationship between humor, the making of mistakes, and learning. The conflict generated by society's need to curb mistakes and the individual's need to make mistakes I defined as math anxiety. Further, because mistakes are "I" openers and because humor takes the worry out of being mistaken, mistakes are "all right." This liberating notion promotes a caring relationship among students themselves and between teachers, thus enhancing learning.

Most recently this attempt to relieve math anxiety in college students as well as in elementary school students has resulted in a 5th grade class from a neighboring school "to be tutored" by the college students in my math class. The eagerness with which both groups interacted with each other holds forth considerable promise.

Age level: Undergraduate
Identifier #1: Randomness
Identifier #2: Statistical Thinking

NOVICES VIEWS ON RANDOMNESS

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Novices and experts rated 18 phenomena as random or non-random and gave justifications for their decisions. Experts rated more of the situations as random than novices. Roughly 90% of the novice justifications were based on reasoning via a) equal likelihood, b) possibility, c) uncertainty, and d) causality.

Much of the prior research on randomness has focused on people's ability to generate and identify strings of random characters (Falk, 1981; Wagenaar, 1972). The major finding has been that people hold non-normative expectations about the production of random strings. For example, a random sequence of heads and tails typically contains longer runs than people expect would occur by chance. These studies have recently been criticized on a variety of accounts, including the argument that since a random sequence cannot be rigorously defined, it makes little sense to speak of people's inability to generate one (Ayton, Hunt, & Wright, 1989).

"Randomness," in fact, comprises a family of concepts. In this study we explore in particular the use of the word as it is used in the phrases, "random phenomenon," "randomizing device," and "random sample." In this sense, randomness is a collection of abstract models which can be applied to various situations. Sometimes we identify these models closely with some physical system, like a coin toss, or blind drawings from an urn filled with balls. In actuality, such physical systems are imperfect instantiations of some "ideal" random-generating system that is only realized in the abstract. Thus, we don't talk about flipping a coin, but flipping a "fair" coin.

Randomness, as an application of an ideal model to some phenomenon, is best thought of as an orientation we take toward, rather than as a quality that belongs to, the phenomenon. This meaning is inherent in the notion of a model. When we apply a model to some situation, we do not regard the model as isomorphic to the target situation as a whole, but only to certain aspects of the situation. This view of randomness explains why most experts are not bothered by the idea of "psuedo-random" numbers. These numbers are produced in perfectly determined ways, yet remain unpredictable to those who do not know the seed and multiplier used to produce a particular sequence. Such a system is not random, except in regards to the orientation adopted by the observer.

While admitting that the notion of randomness is ambiguous and complex, we maintain that variants of the concept are nevertheless at the heart of probabilistic and statistical thinking, and that people's beliefs about randomness must be figured into attempts to teach these topics (Falk, 1991; Falk & Konold, in press; Pollatsek & Konold, 1991). In this article we present preliminary results of an exploratory study of people's subjective criteria of randomness. We asked both novices and experts to categorize a variety of situations as either random or not random, and to give rationales for categorizing each situation. Our primary objective was to identify, in the justifications of the novices, defining features of random and non-random situations.

Some potentially critical features of randomness for the novice have been suggested by Nisbett, Krantz, Jepson, and Kunda (1983), who found that subjects are more likely to employ statistical reasoning to an event when it a) involves a repeatable process with a finite set of symmetric outcomes (e.g., rolling a die), b) consists of outcomes that are produced via a mechanism that is associated with chance (e.g., blindly drawing from a set of well-mixed objects), and c) has been identified within the culture as largely unpredictable and capricious (e.g., the weather).

Method

Twenty subjects (twelve women and eight men) were recruited from undergraduate psychology courses at the University of Massachusetts. Subjects were given 18 cards on each of which was written a brief description of some situation (see Table 1) and were asked to sort the cards, one at a time, into "random" and "non-random" piles. After placing a card in a pile, they were asked to give a brief justification for their categorization. The sessions were videotaped. The same sorting task was given to five experts, four of whom teach graduate-level statistics in psychology departments; the other is a statistician.

Results and Discussion

Randomness Judgments

A basic question is whether salient features of the items were predictive of subject categorizations. Table 1 shows the percentage of novices and experts that categorized each item as random. Experts categorized more items as random than the novices (62% compared to 53%). The largest differences occurred with situations involving real-world phenomena. For example, 80% of the experts judged Item 12, which involved occurrences of earthquakes, as random compared to 20% of the novices.

Item	Group	
	Novice	Expert
1. Whether or not a planted seed germinates.	35	40
2. The number showing up on a die that has already been rolled but that you can't see.	95	80
3. The number of tomatoes you get in your serving of tossed salad at a restaurant.	35	40
4. The winner(s) of next week's megabucks state lottery.	95	100
5. Selecting one of a variety of available flavors of ice cream given that the stranger in the line in front of you is doing the selecting.	80	40
6. Selecting one of a variety of available flavors of ice cream given that you are doing the selecting.	5	0
7. The number of heads that occur in 100 tosses of a fair coin.	85	100
8. Dividing a group of players into two basketball teams such that one team is not obviously better than the other.	0	20
9. The next gear a car with 5 speeds is shifted into given that it is currently in 4th gear.	20	50
10. Whether or not it rained in Amherst on April 3, 1936.	45	50
11. Whether it will rain tomorrow in Amherst.	35	60
12. Whether a large magnitude earthquake occurs in Boston before one occurs in Los Angeles.	20	80
13. Picking a white marble from a box that contains 10 black and 10 white marbles.	100	100
14. Picking a white marble from a box that contains 10 black and 20 white marbles.	70	100
15. Saying the first thing that comes to your mind.	30	40
16. Whether or not you get the flu in the next month.	40	80
17. Whether or not you get exposed to the flu in the next month.	65	40
18. The outcome of the fifth flip of a fair coin that has landed with heads up on the previous four flips.	100	100

Table 1. Percentage of novices and experts who rated each item as random.

The items can be grouped into "real" (Items 1,3,5,6,8-12,15-17), and "stochastic" situations. The stochastic items correspond roughly to those that Nisbett et al. (1983) would rate high on their three features as summarized above. Table 2 shows the mean percentage of items of each type that were rated as random, along with the standard deviations over subjects. As can be seen in Table 2, a higher percentage of stochastic items than real items were classified as random by both experts and novices. This is not surprising given that many of the real items were chosen because they seemed characteristically non-random.

	Group			
Item type	Novice		Expert	
	Mean%	SD	Mean%	SD
Real	37.3	22.4	43.3	33.0
Stochastic	90.8	14.8	93.3	9.1
Symmetric	97.5	11.2	95.0	11.2
Non symmetric	77.5	34.3	90.0	22.4

Table 2. Mean percentage of random ratings by experts and novices as a function of item type.

The stochastic items were further broken down into those with symmetric outcomes (2,4,13,18) and non-symmetric outcomes (7,14). This feature seemed to make little difference in the categorizations of the experts. However, the novices were more likely to rate a stochastic situation as random when its outcomes were symmetric (97.5%) than when they were non-symmetric (77.5%). This finding is born out in the analysis of subjects' justifications.

Analysis of Justifications

Subject justifications were transcribed from the videotapes, and various response categories were developed to capture basic rationales that were used repeatedly by novices. Table 3 shows the number (and percentage) of justifications of the various types for both the novices and experts. Below we describe these categories and provide examples from the transcripts.

Justification	Group	
	Novice	Expert
Equally-likely	64 (17.2)	3 (3.3)
Possibility	63 (16.9)	1 (1.1)
Uncertainty	82 (22.0)	25 (27.2)
Causality	128 (34.4)	20 (21.7)
Model	11 (2.9)	17 (18.5)
Other	24 (6.5)	26 (28.3)

Table 3. Number (and percentage) of various types of novice and expert justifications.

Equally likely. According to the "equally-likely" justification, a phenomenon is random only when each of its outcomes have the same probability. This reasoning, which mirrors early historical development (Zabell, 1988), is exemplified by responses of Subject 9 on Items 13 and 14. Brief item descriptors appear in parentheses.

13. (10/10) "Random. You have an equal chance of getting white or black."
14. (10/20) "Not random. You have a greater chance that you'll pick white."

This reasoning, used rarely by the experts, was used by novices to justify 17% of their categorizations, and was not limited to stochastic items. For example, Subject 6 categorized Item 1 (Seed) as random because, "Each seed has an equal chance of growing or not growing." Subject 13 categorized Item 9 (Gear) as not random because: "Usually you are going to go to a 5th or a 3rd. First and second don't have the same chance."

Multiple possibilities. According to the justification of "multiple possibilities," a phenomenon is random when there is more than one possible outcome and is not random when there is only one possible outcome. In justifying a "random" categorization, subjects typically noted that any of the multiple outcomes were possible. Responses by Subject 6 are shown below as examples.

9. (Gear) "Not random. Has no choice — it has to go into 5th gear."
11. (Rain tom.) "Random. It may or it may not."

Justifications based on possibility were rare in the case of the experts (only one instance). This reasoning, as well as the equally-likely rationale, may be related to an informal interpretation of probability that has been described as the "outcome approach" in prior research by Konold (1989a; 1989b).

Uncertainty. According to the "uncertainty" justification, a phenomenon is random when there is no prior knowledge about the outcome, and thus no ability to predict. When prediction is possible, the phenomenon is non-random. This justification, exempliefied below by responses of Subject 20, was used in 22% of the novice and 27.2% of the expert categorizations.

10. (Rain '36) "Non random, because there is a way to predict the weather."
18. (5th flip) "Random. There is just no way to determine what is going to happen."

Causality. According to this justification, situations are random when no causal factors can be identified, and thus there is no potential to control the result. If causal factors are present, and/or control is possible, the situation is considered non random. For the novices, this was the most commonly-used justification (34.4%), and was also used frequently by the experts (21.7%). The examples below are statements made by Subject 18:

1. (Seed) "Not random, because it depends on soil and all kinds of other things."
7. (# Heads) "Random, because I have no control over what the coin is going to do."

The four categories of justification described above were developed on the basis of analyses of the novice justifications, and for this reason account for a higher total percentage of the novice than the expert justifications (90.6% vs 53.3%). Based on a separate analysis of the expert justification, we added a fifth rationale, as described below.

Model. By this reasoning, the randomness of a situation is established by comparing it to some standard model of randomness. In the case of Expert 3, situations were frequently compared to a "box model."

4. (Lottery) "Random. It is determined by a random device, or a pretty good approximation of one."
5. (Stranger ice) "Non random. He does it by some kind of rule, unknown to you, but you don't have any serious box model."

As might be expected, the experts used this rationale more frequently than the novices (18.5% compared to 2.9%). However, even with the addition of this response category, roughly 28% of the expert justifications did not fit into any of the five categories. Several of the experts expressed their dissatisfaction with having to categorize items as either random or not random. They tended to view randomness as an entity that can be present in degrees, rather than as a categorical attribute, and described several of the situations as consisting of both random and non-random components.

Indeed, an important idea in statistics is the notion that scores or measures can be decomposed into two sources of variation: systematic (explained), and random (unexplained). One of our objectives in future analyses of these data is to identify aspects of novice thinking that present barriers to the development of this "component" view of phenomenon. Subject reliance on "possibility" and "equal-likelihood" are two possible barriers that we are currently exploring. Seeing randomness in terms of possibility might lead students to overgeneralize the concept, viewing any situation as random as long as there is more than one possible outcome. On the other hand, reliance on equal-likelihood restricts the notion of randomness. Introducing students to a wider range of probabilistic situations, including ones in which outcomes are not equally likely, is an approach we are currently testing which may help students develop probabilistic intuitions that can be successfully transferred to statistical thinking.

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K-12
Black & Math
Math Failure

TOWARD A FRAMEWORK FOR ANALYZING THE UNDERACHIEVEMENT OF AFRICAN AMERICAN STUDENTS IN MATHEMATICS

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Disparate results of research on the issue of Blacks and mathematics are synthesized. A comprehensive framework for analyzing the underachievement of African American students in mathematics is presented.

Few, if any, would deny that current mathematics education programs do not work for the African American (black) students. One consequence of this failure is the long-standing and continuing under participation, underachievement, and underrepresentation of Blacks in the mathematical sciences (Anick et al., 1981; Matthews, 1984). Many attempts have been made to explain why black students have not had much success with mathematics, but these efforts often suffer from the use of false assumptions, faulty logic, or the lack of a comprehensive framework for examining the issue. The purpose of this paper is to propose a comprehensive framework analyzing the performance of African American students in mathematics.

Research And Speculation Concerning Blacks and Mathematics:

One "explanation" for the relatively poor performance of Blacks in mathematics stems from an old opinion, still widely held, that Blacks are an inferior race, with low intellect; scarcely capable of abstract reasoning or learning. This belief is further reinforced by data from tests of "intelligence" such as IQ tests. Because Blacks usually score lower than Whites on these tests, some researchers conclude that Blacks are less intelligent than Whites, and that the lower scores for Blacks must be due to inferior genes (Jensen, 1969). Set in this belief, some teachers and school officials just attribute the difficulties that black students encounter in any academic task or subject, like mathematics, to low intellectual endowment, genetic handicaps (Jensen, 1969), or innate learning difficulties (Coleman et al., 1966).

A major weakness of this explanation is that it ignores or belittles significant environmental and school-related factors which affect learning and intellectual performance. Scarr and Weinberg, (1976) demonstrate the paramount importance of environmental factors in the development and performance of African American children. Flynn's (1987) analysis of the data compiled from a major study of IQ, also reveals the influence of "potent", "unknown environmental factors" on IQ test scores. According to Flynn, "the hypothesis that best fits the results is that IQ tests do not measure intelligence but rather a correlate with a weak link to intelligence" (p. 171)

Thus, granted that IQ may indeed influence mathematical performance, to understand why many black students do poorly in this subject, one would have to look far beyond speculations based on race.

Apart from race, other factors have been blamed for the underachievement of Blacks in mathematics. Matthews (1984) referred to "three clusters of variables" pertaining to parents, students and schools, that are believed to influence black participation and achievement in mathematics, Factors within the parent cluster include parents' levels of education, attitudes towards mathematics, beliefs about their children's ability in mathematics (McBay, 1990), child-rearing practices (Bell, 1975), socioeconomic status (Bond, 1981), home language, and culture (Orr, 1987). It is still unclear how much each of the factors contributes to underachievement because there are minority groups who face similar barriers, whose children nonetheless do very well in school (Ogbu, 1990).

Within the student cluster, the major factors are attitudes towards mathematics, self-concept with respect to mathematics, and perception of the usefulness of mathematics. There is evidence of positive correlation between self esteem and mathematical achievement (Reyes, 1984). Self-esteem is also a strong predictor of whether a black student will take advances math courses in high school (Griffin, 1990). Some studies show that, for the past twenty years or so, black adolescents have been registering "moderate to high levels of self esteem" (Graham,1988). Yet, neither achievement nor even course taking patterns in mathematics match the levels of self esteem found among African American students! More research is needed to determine whether the observed disparity between performance and the level of self esteem is a sign of "self-delusion", or, perhaps, an index of unrealized potential in mathematics.

In the third cluster of factors are the mathematics curriculum, teacher attitudes toward black students, teacher expectations of black students, and classroom processes. Each of these is very important because each by itself can significantly influence learning outcome. Take teachers' expectations for instance. It is well known that many teachers have low expectations of black students.

(Oakes, 1985). Because of this, black students are often tracked into non academic classes, where they are taught less material, at a slower pace, in ways that are not conducive to the development of the intellect (Braddock II and McPartland, 1990).

Certain classroom processes can also limit the attainment of black students in mathematics. Some studies suggest that black students, like their white counterparts, start school with the cognitive skills they need to succeed in mathematics (Ginsburg and Russell, 1981: Entwisle and Alexander, 1988). By the end of the first grade, however, black and Hispanic students are already falling behind white and Asian students. Although the reasons for this phenomenon are not yet known, findings from the study by Entwisle and Alexander strongly indicate that teachers' judgement of Blacks' "personal maturity" and "conduct" may be critical factors in the students' mathematical performance.

There have been some notable attempts to integrate the disparate findings concerning the mathematical performance of Blacks. In two related papers, Reyes and Stanic (1985, 1988) present a model to explain differences in mathematics achievement based on the race, sex, and the socioeconomic status (SES) of students. They attribute the differences to (1) school factors - teacher attitudes, mathematical curricula, and classroom processes (2) student attitudes and achievement-related behaviors and (3) societal influences that send different messages about the aptitudes and expected levels of achievement for students based on race, sex and SES

Clark (1988) points out how important student behavior and attitude are to school success. Citing the results of his research on home and community influences on school achievement, Clark asserts that a disadvantaged student will succeed to the extent that he or she spends about 35 hours a week engaged in "constructive learning activity."

Cummins (1986), critically explores why minority students fail in school, and why various attempts made in the United States to reverse the trend have been unsuccessful. He concludes that minority students fail because they are disables by school/minority student and school/minority community relations that are exclusionary rather than collaborative, a transmission-oriented pedagogy that confines students to a passive role, and assessment processes that do not serve the interests of minorities

Gentile and Monaco (1988) use a psychological construct - "learned helplessness" - to shed some light on the nature of mathematical underachievement. Learned helplessness sometimes develops in people who have been exposed to "uncontrollable failure experiences". In their study (Monaco and Gentile, 1987), the two scholars show how frequent exposure to uncontrollable failure in mathematics creates conditions that tend to produce more failures.

Powell (1990) also proposes a model, based on learned helplessness theory, to explain the low achievement of African Americans in mathematics and science. Many young African Americans, she asserts, learn early in life that they just don't do well in mathematics and science. As a result, many blame themselves when they fail in math, and attribute their lack of success to low intelligence, thereby paving the way for learned helplessness syndrome.

From cross-cultural research comes the assertion that even though IQ, SES, language and culture may influence school achievement, none of these factors can explain the poor performance of African American students in mathematics (Ogbu, 1978, 1989, 1990). Ogbu states that Blacks, like involuntary, maltreated minorities in other societies, develop "ambivalent or oppositional social identity vis-a-vis the social identity" of the dominant group. This can make adjustment to the school culture and success rather difficult for black students.
All these factors are taken into account in the framework presented below.

A Framework for Analyzing Black Underachievement in Mathematics

The structural elements of the framework are:

The Society at Large (SL)	The African American Community (AC)
The School System (SS)	The African American Student (AS)

The relationships among these elements are represented by the multidimensional variables - A, B, C, X, Y, Z, as follows:

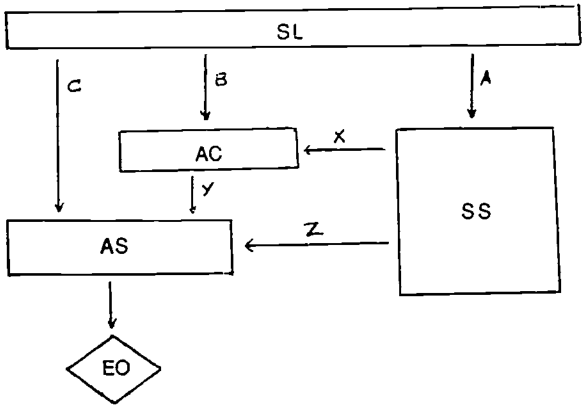
A between SL and SS	X between AC and SS
B between SL and AC	Y between AC and AS
C between SL and AS	Z between AS and SS

Each of these relationships may be strong or weak, and may have a positive or negative effect: one may strengthen or weaken another. When the cumulative effect of these relations is positive, it is enabling to the African American student and, as a consequence, produces good educational outcome in mathematics (EO). Otherwise the student is disabled, resulting in poor educational performance.

Variable A includes (1) societal "theories" about the cognitive capability, "educability" and "inferiority" of black students; and (2) differential school funding based upon race and/or SES.

B includes: (1) racism in the social, economic and political arenas; (2) societal view of Blacks in world history, and especially African Americans in U.S. history; (3) societal view of black culture, and (4) status of Blacks.

Figure 1
A Framework for Analyzing the Underachievement of African American Students in Mathematics



C includes: (1) media images and messages concerning African Americans in general, and black males in particular; (2) racism
X includes: (1) school/black community relations
Y includes: (1) "entry behavior"; (2) "effective entry characteristics" (Bloom, 1971, p 14) of black students.
Z includes: (1) curriculum (2) quality of instruction; (3) teacher expectations; (4) teacher/black student relations; and (5) opportunity to learn.

Discussion

Variables A,B, and C exert a powerful influence on black students' learning and achievement even though Blacks have very little control over them. They pose, for the black student, a formidable barrier to learning and educational attainment by attacking his self concept and confidence. Thus any attempt to enhance black achievement in mathematics has to find a way to neutralize the effects of these variables.

Within this framework, Cummins' empowerment model involves only the variables A, B, X and Z. Variables A, C, and Z incorporate the model developed by Stanic and Reyes, while C and Z address Powell's "learned helplessness".

The model also suggests possible paths to successful mathematics education of African American students. Cummins (1986) implies that black students will succeed in academic work if X and Z are positive. This implication is supported by the work of Hilliard (1990), who describes instances of successful learning that occurs when individual teachers ignore the negative messages of A, B, and C, change the way they relate to black students and communities, and institute good curriculum and instruction. In fact , when school effects (Z) are strongly positive, black students succeed in spite of the negative effects of all other variables. Ogbu's research, cited earlier, also shows that black students can achieve academic success, if they , like many immigrant minorities, rise above the negative effects of A,B, and C, and develop behaviors and characteristics, (+Y), that facilitate adjustment to school culture and learning.

It is clear from the foregoing discussion that it is erroneous, or at least far too premature, to ascribe the current underachievement of black students in mathematics to anything but the intolerably difficult circumstances under which they have to learn. Blacks are, perhaps, the only people whose cognitive abilities are routinely questioned and ridiculed even in the popular press. The assaults on their self esteem are relentless; so much so that even representatives of foreign governments get in on the act, and denigrate black people with impunity, without a formal protest! It is remarkable that African Americans achieve as much as they do under these hellish circumstances. Perhaps, this is why Anderson (1990) confidently asserts that "if minority (black) students were encouraged to attain scholarship and achievement in mathematics as widely as they are encouraged to attain stellar achievement in sports, their performance in mathematics would shock this country" (p. 265)

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STIMULATING ACTION RESEARCH ON TEACHING MATHEMATICS THROUGH THE USE OF EXPLICIT FRAMEWORKS: TEN YEARS OF OPEN UNIVERSITY EXPERIENCE

Presenters: David Pimm and John Mason

Since 1982, the Centre for Mathematics education has supported the professional development of teachers of mathematics at all age levels from Kindergarten to Tertiary through the provision of undergraduate course materials, videotapes of classrooms and mathematical and didactic packages designed to stimulate teachers' thinking and awareness.

Producing materials for study at a distance, or for use by other tutors, is relatively easy. What distinguishes our materials is the underlying approach: the provision of initial frameworks to stimulate teachers to become more aware of opportunities in their classrooms for altering their behaviour.

The format will be a participatory-workshop with time for discussion, and will include as much of the following as time permits. Reference will be made to ways in which frameworks have been used by teachers taking our courses in order to research their own practice, and to develop their teaching.

THE USE OF FRAMEWORKS

Language frameworks focus on the particular affinity of language with mathematics in both expressing thoughts, meanings and images in order to communicate with others, and in the linguistic nature of much mathematics in creating the reality of which the language then speaks.

A framework such as *See, Say, Record* focuses on an implied temporal ordering of images and perceptions, spoken utterances, and written records, and carries with it messages about relative priority and sequence in mathematics teaching. Conversely, focus on the nature of language patterns illustrates the importance of gaining access to the mathematics register as a critical component of learning mathematics, with teachers attending to the particular discourse patterns that pupils have to acquire in order to participate in particular mathematical areas.

This dual, but shifting, emphasis on both symbol and referent (related to the persepctives labelled as 'metonymic' and 'metaphoric' in the language of

Roman Jacobson) marks an important interplay for a teacher in working in any of the main mathematical areas of number, algebra or geometry.

Mathematical Frameworks

Specialising, Generalising and Conjecturing are fundamental processes in mathematical thinking, which have been isolated and promoted by many authors. They are introduced via short mathematical questions, and related to past experience, particularly of getting stuck, in order to constitute a framework which activates the corresponding activity for teachers and pupils.

Mental imagery acts as a mediator between written/spoken mathematics and mathematical ideas. It is illustrated through mathematical activity, and invoked in our methodology for personal action research into teaching.

Psychological Frameworks

The psychological divisions of psyche into cognitive, affective and enactive dimensions reflects two thousand years of informal and formal psychological research. We have found it useful to recast these for use in a fresh framework for use by teachers to study their teaching, to provide a structure for preparing oneself to teach a mathematical topic. Attention is drawn:

> to the language patterns which pupils will be expected to have integrated into their thinking, and connections to language with which they are already familiar, and to techniques which they need to master (and ones which they tend to construct for themselves), complete with inner incantations which drive those techniques;
>
> to the original questions which people wanted to answer and which gave rise to the topic as we now know it, and to a variety of contexts in which that topic appears;
>
> to images and fuzzy 'senses of' one would like pupils to associate with the topic, as well as confusions and obstacles which pupils encounter.

Through mathematical activity, the use of the distinction between *giving an account of* something and *accounting for it* will be demonstrated and reference made to applications both in mathematics itself and to the conduct of personal action research into teaching.

ANALYZING AND DESCRIBING STUDENTS' THINKING IN GEOMETRY: CONTINUITY IN THE VAN-HIELE LEVELS

Michael Shaughnessy and William Burger--Oregon State University
Angel Gutierrez and Adela Jaime--Universidad de Valencia
David Fuys--Brooklyn College

This Symposium is dedicated to our colleague Bill Burger, researcher, mentor, and friend. We all miss him very much.

While the results of first efforts in van Hiele research generally confirm the validity of the model for describing students reasoning processes in geometry, several unanswered questions have emerged. Is there a way to describe a students' progress through the van Hiele levels as a continuum, so that the model accounts for students who are acquiring more than one level at a given point in their geometric development? Can some combination of clinical and traditional methodologies be used to devise a reliable, yet flexible and valid, test for measuring students' van Hiele levels? This symposium will be a research-workshop on some new approaches to assessing van Hiele levels. Participants will actually become co-researchers with the presenters, investigating these two questions during the symposium.

Overview

The van Hiele model has provided a framework for investigating children's and adolescent's thinking in geomtry (the levels), and also has suggested a pedagogical model for teaching geometric concepts (the phases). Within the past decade, research based on the model indicates that the description of thought processes in geometry is a fertile area for the interaction of psychologists and mathematics educators alike (Usiskin, 1982; Mayberry 1983; Shaughnessy & Burger 1985; Senk 1985, 1989; Burger & Shaughnessy 1986; Crowley 1987, 1990; Fuys et. al. 1988; Wilson, 1990; Gutierrez et. al., in press). Thus, the van Hiele model provides a particularly useful framework to investigate the crossroads between theory and practice in teaching and learning geometry. There is concurrent interest among both teachers and researchers on the potential usefulness of the model for providing both diagnostic information about students' thinking in geometry, and also prescriptive information about how to redesign the geometry curriculum to facilitate students' geometric development.

The first early work on researching the van Hiele levels focused on attempting to identify the existence of these reasoning levels in students, to validate the model, to describe level indicators of reasoning, and to use the five pedagogical phases in teaching experiments to help move students through the levels of reasoning. In all of this work, tasks were developed to allow students to reason in a geometric environments. Some of these tasks were purely paper and pencil tasks, some were interview tasks of a more open ended nature. The results of these first efforts generally confirmed the validity of the model for describing students reasoning processes in geometry (Usiskin, 1982; Burger & Shaughnessy, 1986; Fuys et. al. 1988). However, several unanswered questions emerged from this first series of research efforts. Among them are two that we wish to address in this symposium.

First, the van Hiele levels do not appear to be entirely discrete. Several of the researchers mentioned above found that students often flip-flopped between levels from one task to another, or even within the same task. Also, many students seem to have a "preferred level of reasoning" on certain tasks. That is, they may prefer to respond in an analytical way when they are perfectly capable of verifying some argument by deduction (second Level preferred over higher levels), or they may respond purely visually when they could just as well have talked about properties of shapes or relationships among those properties had they been required to do so (first level preferred over higher levels). Thus, the process of determining a students' van Hiele level is much more complicated than just assigning a single level on a few tasks. There are also task variables and content knowledge variables, so that students who reason at a level on one task do not necessarily exhibit that same level of reasoning on a subsequent task. This raises the question: Is there a way to describe a students' progress through the van Hiele levels as a continuum, so that the model accounts for students who are acquiring more than one level at a given point in their geometric development? This view presupposes that their are passages between the levels, and that students can reason partially at one level, and partially at another.

A second question that has been researched more recently is the problem of devising a suitable test to assess van Hiele levels, the development of valid, reliable tasks. Both pencil and paper tests and clinical interview tasks have demonstrated certain strengths and weaknesses in van Hiele research. The former may sacrifice detail and/or reliability for convenience and speed. The latter while lending itself well to detailed probing can prove cumbersome and time consuming to administer to large numbers of students. Perhaps some combination of both methodologies is needed to devise a reliable, yet flexible and valid, test for the van Hiele levels.

Van Hiele Continuity

In this symposium we will focus on describing the continuous development of students' passage through the van Hiele levels. Gutierrez and Jaime (in press) have developed a method of analyzing students' written responses to geometric tasks that includes a first attempt to quantify the passage between levels. After an introductory phase which recaps the attempts of several projects (both in the USA and in Spain) to identify students' van Hiele levels, the participants in this symposium will be put to work in a research-workshop. Participants will be given the opportunity to analyze and discuss students' responses on tasks, both pencil and paper tasks and audio taped tasks, while learning about and using the analysis approach of Gutierrez and Jaime.

After group discussion, the presenters will share their own interpretations of the task results with the participants. The symposium will conclude with suggestions for merging the best parts of both methodolocial approaches—interview and paper and pencil—for researching students' continuous passage through the levels. This approach has recently proved quite valuable in obtaining a more accurate picture of a students' van Hiele levels.

Conduct of the Symposium

The symposium will evolve in three phases.

Phase 1.

Mike Shaughnessy and Bill Burger

Shaughnessy and Burger have planned the introduction and overview for the symposium. Prototype instances where students are between van Hiele levels on certain tasks will be presented. Sample student responses to particular tasks will be used to introduce the dilemma one faces when a student appears to be between levels. Responses to tasks presented in taped interviews (developed in the USA) and pencil and paper tasks (developed in Spain) will be considered. The interview tasks are similar to the pencil and paper tasks, but not all are identical. Some general comparisons of the two different methodologies for determining van Hiele levels will be mentioned. (Time: about 30 minutes)

Van Hiele Continuity

Angel Gutierrez and Adela Jaime

Gutierrez and Jaime will provide a brief background on their research, and desribe their scheme of "levels and types" for quantifying the passage of students between van Hiele levels. In this scheme, a students' response to a particular task is assigned both a van Hiele level and a "type" of answer. The types reflect both the strength and clarity of the predominant van Hiele level on that task, and also the mathematical completeness and accuracy of the response. The types are quantified to indicate how complete a student's aquisition of a level is. This scheme will also makes allowances for task variables and the potential range of thinking levels that may be used to answer a particular question. Each task can be pre assigned a potential "range of levels" of response. Thus, a question could be answered at, say, van Hiele levels 2,3, or 4 (using 1-5 numbering), and if a student answers it at level 3, a complete aquisition of level 2 is assumed, a partial aquistion of level 3 (depending on the "type" of answer) is assumed, and no aquisition of level 4 is inferred on that task. Using the "types"it becomes possible to quantify a students responses and to talk about a student's "degree of aquisition" of each of the four (1-4) van Hiele levels.

Gutierrez and Jaime will give specific examples of students' responses that they have coded by levels and types, and the corresponding degree of acquisition of the levels, in order to provide the necessary information for the second phase of the symposium in which the participants themselves will "do" some van Hiele research using this scheme. (Time: about 30 minutes)

Phase 2.

Participants will be given taped responses of a student's work on a geometric task(s). The participants will be asked to evaluate the "type and level" of the student on each task. The participants will work in pairs on this activity, first noting their own responses, and then interacting with their partner.
(Time: about 30 minutes)

In the second part of phase 2, the participants will share the results of their pair-wise analysis with the large group. The symposium organizers will also share their own analyses of the same task(s). (Time: about 30 minutes)

Phase 3

Van Hiele Continuity

David Fuys

David Fuys will play the role of reactor. His remarks will be partly devoted to methodological considerations, with special attention to a "marriage" of certain aspects of the interview and paper & pencil methodologies. When post hoc structured interviews are administered to students after they have answered paper and pencil van Hiele tasks, the in depth probing allowed in the interview format may help to clarify a students' true acquisition of the van Hiele levels. Fuys will discuss examples of students where this was indeed the case. He will also reflect on the process of attempting to quantify the passage between van Hiele levels, and in particular, the process in which the participants have engaged during phase 2.
(Time: about 30 minutes)

The final part of the symposium will be devoted to open discussion about the process of researching van Hiele levels, focusing on the method of quantifying a student's passage between and through the continuum of levels. (Time: abut 30 minutes)

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Grade: N/A
ID #1: Teacher Education

School Division Expectations: How are they communicated? What kind of responses do they elicit?

Bob Underhill, Virginia Tech, Organizer and Presenter
Pat Agard, Virginia Tech, Presenter
Kari Cox Beaty, Virginia Tech, Presenter
Doug Jones, University of Kentucky, Presenter
Hilda Borko, University of Colorado, Moderator

Untangling the complexities of learning to teach necessitates a deeper understanding of communications networks among central administrators, principals and teachers. In this symposium, we will examine the styles of two central administrators and follow the chair of reactions and impressions of subordinates as goals are translated and implemented at subsequent levels in two elementary schools.

As social institutions, schools create and sustain patterns of professional culture and social organization. Leaders at each level attempt to influence behavior at lower levels and persons on lower levels interpret those influences through their unique personal and professional filters. Individuals will behave in ways which reflect their commitments to shared goals by seeking to implement the letter or the spirit or both. They may also do what is minimally required so as to look okay (Lacy, 1977; Desforges & Cockburn, 1987). Eisenhart, Behm, & Romagnano (1991) have explored some of these issues within two frameworks for training professional teachers, and Goodman (1985) and Britzman (1986) have explored some of these issues in the process of becoming teachers.

In this symposium, several members of the NSF Learning-to-Teach Mathematics research team will examine two central-level administrative styles and examine how the perceptions and actions of administrators influence ultimate classroom behaviors and perceptions of teachers in two elementary schools. The following format will be used:

10 minutes - Doug Jones - Overview of NSF project/broad context
20 minutes - Bob Underhill - Division level context
10 minutes - Pat Agard - Elementary School Context 1

10 minutes - Kari Beaty - Elementary School Context 2
10 minutes - Hilda Borko - Comments
Remainder - Audience Discussion

Project Context - Doug Jones

The Learning to Teach Mathematics project (NSF: MDR 8653476) was designed to investigate the experiences of a small group of teachers as they were learning how to teach. The researchers studied four beginning middle school mathematics teachers for two years, their senior year in a K-8 certification program (Year 1) and their first year as full-time teachers (Year 2). During Year 1, each teacher had four 7-week long field placements, 3 of which were in the middle grades; all placements were in the same school system. In an effort to understand the teachers' orientations and possible influences on their development as middle school mathematics teachers, a wide range of data concerning background, university experiences, and classroom experiences were gathered during both years of the study (see Jones et al., 1989 and Borko et al., 1990 for details concerning data collection and analyses). This symposium focuses on data collected during Year 1 and examines possible influences on beginning teachers of administrative styles and the ways in which goals are expressed and implemented at the central administrative level, the building level, and the classroom level. Interviews concerning the sociocultural climate of the schools and social organization of mathematics teaching that were held with the beginning teachers, their university supervisors, their cooperating teachers, mathematics department chairs, building administrators, and central administrators were supplemented with artifacts from the teacher education program, the schools, and the school system.

School Division Context - Bob Underhill

In the elementary schools, there were two line-and-staff tracks as follows:

Mathematics	*Non-Mathematics*
Assoc. Supt.	Dir. of Personnel and Staff Dev.
Math Supervisor	Dir. of Elem. Admin.

Principal Principal
Teachers Teachers

The leadership style of the Associate Superintendent could best be described as "persuade, coax and support." And the leadership style of the Director of Personnel and Staff Development could best be described through an "accountability" model.

The Associate Superintendent had a mathematics background. He really wanted to improve mathematics teaching and learning. He worked closely with the mathematics supervisor to provide considerable in-service and resources; their shared goal was to motivate teachers to change.

The Director of Personnel and Staff Development believed that the way to get change was to mandate it. If you want a particular approach to teaching (a variation of the effective teaching model), train the teachers and require evidence of its use in administrative evaluations. The follow-through was provided through the office of the Director of Elementary Administration.

School No. 1 - Pat Agard

The principal viewed himself as an instructional leader. He sought to understand division-level administrative and teacher perspectives and to provide resources and support for implementation. He valued the use of manipulatives to teach mathematics, so he purchased many manipulatives and encouraged teachers to requisition and use them. He accepted the effective-teaching mandate, believed in the value of the model, and included evaluation of its components in his teacher evaluations.

The classroom teacher at the 6th grade level used virtually no manipulatives. She believed the effective teaching model to have considerable merit and used it in her daily instruction.

School No. 2 - Kari Beaty

The principal viewed himself primarily as a manager. He thought manipulatives were mainly for use with primary grade children, so the message from "downtown" was lost for the intermediate grades. On the other hand, he was well aware of the effective-teaching mandate and carried through with its required use in his administrative teaching evaluations.

The classroom teacher used virtually no manipulatives. He valued the effective teaching model and used it in most of his lessons.

Reaction - Hilda Borko

The trends within the school division on the selected set of issues presented will be summarized and attention will be drawn to the probable impact of leadership styles, how expectations are communicated, and how the perceptions of principals and teachers effect implementation at the school and classroom levels. A discussion will ensue concerning the power and importance of these issues in teacher education research and, as time permits, in teacher education program design.

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THE NATURE AND PURPOSE OF RESEARCH IN MATHEMATICS EDUCATION: IDEAS PROMPTED BY EISENHART'S PLENARY ADDRESS

Frank K. Lester, Jr.[1]
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It is an honor for me to have been afforded the opportunity to react to Professor Eisenhart's ideas concerning the value of conceptual frameworks for educational research. She has established herself as a leading advocate for the research tradition of ethnography and its application to education in general and, in particular, to mathematics education (Eisenhart, 1988). Indeed, in her earlier writings as well as in this paper, she has added some much needed clarity to the ongoing discussion of the underlying assumptions, goals, and methods of ethnographic research. But in my opinion her contribution extends far beyond this. She is (implicitly at least) forcing us as mathematics educators to come to grips with two fundamental questions: *What should mathematics education research be about?* and *How should we go about the business of doing research in mathematics education?* As the incoming editor of the Journal for Research in Mathematics Education I will be faced with questions such as these when: choosing reviewers, assessing their comments, responding to authors and, ultimately, deciding to accept or reject manuscripts. Thus, it may not be surprising to learn that these two questions were uppermost in my mind when I began to think about the sort of reaction paper I would prepare.

Knowing that I hold her views in high regard, it should come as no surprise that this paper does not offer a counterpoint to the positions and arguments she puts forward. Rather, I intend to do two things: (1) discuss several issues raised by Eisenhart about which I agree almost completely, and (2) pose two questions generated by her ideas.

POINTS OF AGREEMENT

Among the several issues Eisenhart discusses, four are central to my interest in the nature and purpose of mathematics education research. These issues relate to: (1) the nature of frameworks for research, (2) the constraining nature of theoretical frameworks, (3) the nature of conceptual frameworks, and (4) the importance of interdisciplinary research. Each issue is discussed in turn in the following paragraphs.

The Basic Nature of Frameworks

Eisenhart insists that some kind of framework (i.e., "skeletal structure") is essential to the research process. I agree wholeheartedly and wish to suggest that the extremely slow pace at

[1] I am indebted to my colleagues, Peter Kloosterman and Diana Lambdin Kroll, for their helpful comments on a draft of this paper.

which we mathematics educators have been able to move our field forward may be due in large part to the lack of clearly described frameworks, conceptual or otherwise, in much of our research. During my nineteen years of university teaching I have had the pleasure of working with many very good doctoral students. It is no exaggeration to say that as they begin their doctoral dissertations almost all of them have very little, if any, understanding of what it means to have a conceptual or theoretical framework for their research. Of course they are not at fault for this condition since they are rarely taught anything about frameworks in their classes and seminars, and they only very infrequently see evidence of explicit frameworks in the published research papers they are asked to read. I attribute this unfortunate state of affairs in large part to the fact that there is no well-defined research tradition within mathematics education to guide them in conceptualizing their studies. Further, I suggest that over the years the best doctoral research has been conducted at those universities in which the mathematics education programs have been willing to ground their research in traditions that have been clearly established in other disciplines. (By and large, it is at these same institutions that the best faculty research is done.) I will add a bit more about the importance of research traditions in the discussion of the next point.

The Constraining Nature of Theoretical Frameworks

In her argument against the appropriateness and usefulness of theoretical frameworks, Eisenhart points out that such frameworks often are "used by academics to set a standard for scholarly discourse that is not functional outside the academic discipline" (p. 6). I agree with her to some extent, but the issue at hand may really be a matter of research tradition, not one of the appropriateness or usefulness of theoretical frameworks. For quite a long time (at least since the days of Thorndike), mathematics educators have looked to the research traditions established in experimental psychology (and more recently to its offspring, cognitive psychology) for guidance in determining what the important research questions are and how they should be studied. In my view, the frameworks used by psychologists have often not been functional for studying questions of fundamental interest to mathematics educators (cf. Kilpatrick, 1985). But this is not a shortcoming of frameworks! Rather, it is a problem of perspective. As I have noted elsewhere, "a researcher who has taught mathematics and studied it seriously will necessarily have a different perspective about the nature of mathematics . . . than someone who has neither taught nor studied mathematics in any depth. It is natural that non-mathematicians would introduce views about the nature of mathematics that are quite different from those held by mathematicians or mathematics teachers" (Lester, 1988, p. 116). Thus, when a theoretical framework becomes non-functional, the problem may actually stem from the researcher having adopted a research tradition that has a very different way of looking at problems related to mathematics learning and teaching than is customary.

That mathematics educators borrow frameworks from elsewhere is both natural and inevitable because mathematics education itself is not a discipline. Rather it is a field of inquiry that borrows freely from well established disciplines such as history, philosophy, psychology, anthropology, and sociology, among others. We mathematics educators, then, must take care to give ample attention to the perspectives and assumptions underlying a particular discipline before we decide to use it to investigate questions of interest to us.

Justification vs. Explanation

Eisenhart describes a conceptual framework as "a skeletal structure of justification, rather than a skeletal structure of explanation" (p. 10). Furthermore, it "is an argument including different points of view and culminating in a series of reasons for adopting some points . . . and not others" (p. 10). I think this distinction can be an extremely useful one for mathematics educators inasmuch as it suggests that justification should (for now at least) be of paramount importance to educational research. Heretofore this has not been the case. It may be the case that researchers in our field have been too concerned with coming up with good "explanations" and not concerned enough with justifying why they are doing what they are doing. In my experience reviewing manuscripts for publication and advising doctoral students about their dissertations, I have consistently found that the most glaring weaknesses in the research are often lack of attention to clarifying and justifying why a particular question is proposed to be studied in a particular way and why certain factors (e.g., concepts, behaviors, attitudes, societal forces) are more important than others. Eisenhart's discussion of the nature of conceptual frameworks and the advantage of them over theoretical or practical frameworks (see pages 10-14) is quite lucid and almost compelling (my reservations are raised in the last section of this paper).

But there is more to what Eisenhart is suggesting than simply recommending that researchers devote more attention to providing good arguments to support their research studies. In fact, she is arguing that the very purpose of our research efforts needs to be reconsidered. I have a bit more to say about this later in this paper.

The Importance of Interdisciplinary Research

In her plenary paper and elsewhere (Eisenhart, 1988), Eisenhart argues convincingly for collaborative, interdisciplinary research efforts in mathematics education. If her recommendation is taken seriously, it could have far-reaching implications for doctoral programs in mathematics education. As I see it, since mathematics education has borrowed, borrows now, and will continue to borrow liberally from several disciplines, it seems essential that the training graduate students receive must include direct and substantial attention to the research traditions of several disciplines (anthropology, psychology, and sociology are the most prominent examples, but history and philosophy wouild also need to be considered). But, it is unreasonable to expect graduate research programs in mathematics education to provide adequate preparation in conducting research based on so many different traditions. Consider the case of ethnographic research, which has arisen largely from anthropology. I become very worried when one of my students announces to me that her dissertation will be an ethnographic study. It is worry enough that I know so li e about doing this sort of research, but this is not the source of my concern. Rather, it stems from the likelihood that the student will have had at most one seminar related to conducting ethnographic research. Is she an ethnographer and can she be expected to do a truly first-rate ethnography? I think the situation is very much the same as calling someone a carpenter based simply on the person having read about what carpenters do, having an interest in carpentry, and (possibly) having hammered a few nails into some boards. "True" carpenters are trained in the traditions of carpentry by working for long periods of time (usually years) with other carpenters; that is, by serving as apprentice carpenters. Our doctoral students rarely serve any kind of real research apprenticeships and so they have no opportunity to develop a sense of any legitimate research tradition, much less multiple traditions. In his thoughtful discussion of the nature of ethnographic research in education, Wolcott notes, it is "useful to distinguish between anthropologically informed researchers who do ethnography and educational researchers who frequently draw upon ethnographic approaches in doing descriptive studies" (Wolcott, 1988, p. 202). The former types of individuals would expect to be interested in a broad cultural context, an expectation arising from having been trained in a research tradition that too often is alien to the latter types[2].

Another concern is that even being reasonably well-versed in the techniques of a research tradition does not make an individual an ethnographer, a historian, a philosopher, or a specialist in whatever discipline is being drawn upon. Much more is involved. For example, familiarity with the special language that often is associated with a tradition and awareness of the underlying assumptions and purposes of research within the tradition help define what it means to do research based on that tradition (cf. Eisenhart, 1988; Wolcott, 1988). Consequently, unless the researcher has developed a good sense for these kinds of things, there is the danger that the research will not be particularly well-informed. In a review of Alan Bishop's recent book, Mathematical Enculturation: A Cultural Perspective on Mathematics Education (Bishop, 1988), Jeanne Connors points out that problems often arise from uninformed interdisciplinary dialogue. In particular, she notes that when researchers in one field borrow ideas from another, the results are often unsuccessful. She suggests that the lack of success often stems from the fact that:

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[2] A similar situation also develops when mathematics educators attempt to use research methods borrowed from disciplines such as history, philosophy, sociology, etc.

A researcher in one field is not always aware of the issues surrounding, or the current status of, a particular paradigm in another. Every discipline is dynamic. A theoretical framework is posed, examined by scholars, elaborated upon, and then may be discarded in favor of newer ideas. Unless the "borrower" is aware of this disciplinary debate, the result can be the application of an outmoded idea to a new field, where it may very well be accepted, and perpetuated, by naive readers (Connors, 1990, p. 462).

Connors goes on to suggest that Bishop used a largely discredited anthropological theory to inform his analysis (viz., Leslie White's science of culture) and that "anthropology has moved away from the 'easy' answers of the first half of the century and is beginning to realize that the 'real' world is messy, complex, and impossible to model as simplistically as White had hoped" (p. 462). Does this mean that Bishop's conclusions are wrong or misguided? Perhaps, perhaps not. The point is that when researchers borrow theories from another discipline, they should be aware of the status of those theories within that discipline. Thus, my enthusiasm for interdisciplinary inquiry is tempered by my concern that, however well-intentioned, the inquiry may be naive and ultimately fruitless.

I suspect that Eisenhart would not disagree with my concerns about interdisciplinary research. In fact, her remarks in her paper make it clear that she is calling for <u>collaborative</u> interdisciplinary research of the sort that apparently is taking place in the "Learning to Teach Mathematics" project. This collaboration involves a team of researchers, each with her or his own special expertise, working together to try to better understand the changes that take place in the process of moving from being a prospective mathematics teacher to being a certified mathematics teacher.

SOME QUESTIONS

In the preceding section I have discussed several areas about which I am in basic agreement with Eisenhart. It should be clear that instead of attacking her ideas and positions, for the most part I have simply elaborated upon them. In this section I raise two questions that seem central to the theme of her paper.

<u>What Role Should Theory Play in Research?</u>[3]

At the beginning of this paper I suggested that Eisenhart's ideas should cause us to think seriously about what it means to be a mathematics educator and to engage in research in mathematics education. Central to this deliberation is the concern about the role of theory in research. I am a bit worried that some readers of Eisenhart's paper will interpret her remarks against the use of theoretical frameworks as meaning that it is acceptable, perhaps even a good

[3] I am grateful to my colleague at Indiana University, Thomas Schwandt, for sharing with me his ideas about the role of theory in educational research. These ideas served as the basis for this section of the paper.

thing, for research to be atheoretical. Consequently, I think it is reasonable to make a few comments about the role theory should play in research in mathematics education. For a start, I don't think Eisenhart is suggesting that good research can (or should) be completely atheoretical[4]. Instead, she is arguing against the sort of rigid, blind adherence to a theory that characterizes much theory-based research. In an indirect way she is also arguing for having theory play a different role in educational research than it has played historically. Martyn Hammersley, an ethnographer, insists that it is the duty of sociologists "to attempt the production of well-established theory" (Hammersley, 1990, p. 109). Furthermore, he argues that this "gives us the best hope of producing effective explanations for social phenomena and thereby a sound basis for policy" (Hammersley, 1990, p. 108). Thomas Schwandt, a philosopher of education, argues that Hammersley is suggesting that theory development "is the raison d'etre for the practice of social inquiry" and "to talk of theory is not simply to talk of some feature of scientific investigations, but to talk of a pervasive and dominant intellectual orientation to social . . . inquiry" (Schwandt, in preparation). Moreover, this view has been the dominant position among educational researchers for some time. Thus, to question, as Eisenhart does, the importance of theory development in mathematics education is tantamount to questioning the very purpose of research in the field.

The debate about the role of theory should be a lively and interesting one as the community of researchers interested in issues and problems related to mathematics education begins to think seriously about the nature of research in the field. It is clear that some notions will be discarded in favor of others— this is after all one way that progress is made. But, let us hope that when the debate is settled we are not left with the feeling that the baby has been thrown out with the bath water.

Do Eisenhart's Notions about Frameworks Apply to Traditions Other than Anthropology?

In an essay about the relations between the history and philosophy of science, Thomas Kuhn writes:

> The final product of most historical research is a narrative, a story, about particulars of the past. In part it is a description of what occurred . . . . Its success, however, depends not only on accuarcy but also on structure. . . . In a sense to which I shall later return, history is an explanatory enterprise; yet its explanatory functions are achieved with almost no recourse to explicit generalizations. . . . The philosopher, on the other hand, aims principally at explicit generalizations and at those with universal scope. He is no teller of stories, true or false. His goal is to discover and state what is true at all times and places

[4] Garrison (1988) provides an interesting and somewhat compelling argument that it is impossible for scientific research to be atheoretical. A similar, if not the same, argument might be made for educational research.

rather than to impart understanding of what occurred at a particular time and place (Kuhn, 1977, p. 5).

Later in the same essay, Kuhn compares and contrasts the processes involved in writing research articles in physics, history, and philosophy. He states that "I have myself, at various times, written articles in physics, in history, and in something resembling philosophy. In all three cases the process of writing proves disagreeable, but the experience is not in other respects the same" (p. 8).

A part of Kuhn's message is that a particular discipline can be distinguished from others in some fundamental ways. It seems to me that disciplines differ with regard to:

- The nature of the questions asked within a discipline.
- The manner in which questions are formulated.
- The way the content of the disciplines is defined.
- The principles of discovery and verification (justification) allowed for creating new "knowledge" within a discipline.

With this in mind I begin to wonder about the applicability of Eisenhart's ideas to various research traditions. For example, it seems quite possible that as appropriate as Eisenhart's ideas may be for research conducted in the tradition of anthropology, they may not apply to some other traditions. In order to stimulate discussion about this question, I will end this reaction paper by identifying four broad types of research questions for mathematics education.

Four Types of Research Questions

When we consider seeking guidance from experts, like Margaret Eisenhart, in other disciplines, in order to pursue answers to the important questions in our field, it may be useful to think of four types of questions: *What was? What is? What would happen if?* and *What should be?*

Type I: What was? Questions of this type might be addressed by means of historical inquiry, a research tradition that has been all but ignored in mathematics education. Illustrative of the sorts of questions that might be addressed are *What forces led to the creation of the NCTM Standards documents?* and *What was the place of mathematics in the development of the public school system in the United States?* Individuals interested in historical inquiry will find Kaestle's discussion and the readings that accompany it quite useful (Kaestle, 1988).

Type II: What is? There are at least two sub-questions associated with this question: What is going on in _____? and What is the state of affairs with respect to _____? A number of research methods developed within several different traditions seem appropriate for investigating type II questions; notably, ethnographies from anthropology, case studies from psychology (among other traditions), and surveys from sociology. But Type II questions should not be limited to these traditions alone. Philosophy, for example, might be drawn upon as well. Recent research by Cobb, Wood, and Yackel (1991) is a case in point. They used analogies from the philosophy and sociology of science to help them understand students' motivations, emotions, and beliefs as they develop in the classroom.

Type III: What would happen if? Quasi-experimental research methods developed in experimental psychology (as well as in other "behavioral and social sciences") have been used for quite a long time to study questions of this type. Oftentimes questions of this type arise from efforts to identify different (e.g., more effective, more efficient) ways to offer various aspects of instruction (e.g., What would happen if students worked together in small groups?). In particular, questions involving the standard comparison of treatments are of this type (e.g., Which is better: treatment A or treatment B?). It is clear that quasi-experimental methods are not the only ones suitable for addressing questions of this type; ethnographic techniques, which are referred to throughout Eisenhart's paper, as well as other methods can also be useful.

Type IV: What should be? Philosophical methods can be of great value for questions in which an attempt is to be made to make a case for a particular position. For questions of this sort, arguments from analogy and the method of examples and contrasts, two fundamental tools of the philosopher, would be invaluable (Scriven, 1988). For example, the philosopher's tools could be used to argue for or against the statement, "Problem solving should be the focus of school mathematics."

Mathematics education, then, is a field of inquiry concerned with a very wide variety of types of questions, and to a certain extent these types determine the nature of the research that can be conducted. Mathematics education researchers should not be expected to become experts in the use of all, or even many, of the daunting array of research methods available. However, it is vital that we: (1) recognize that our field needs to draw upon many research traditions; (2) acknowledge that the most effective research programs are likely to be those characterized by applications of "disciplined eclectic" (Shulman, 1988, p. 16); and (3) actively seek to collaborate with researchers who have been trained in traditions different from our own.

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CONCEPTUAL FRAMEWORKS FOR RESEARCH CIRCA 1991: IDEAS FROM A CULTURAL ANTHROPOLOGIST; IMPLICATIONS FOR MATHEMATICS EDUCATION RESEARCHERS

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Some of you are probably wondering: What is a conceptual framework and why all the fuss about whether you have one for your research project? Is it simply politically correct to have a conceptual framework or is there more to it? Perhaps some interpretivists out there are wondering whether concerns about conceptual frameworks aren't just another means for positivists to reassert their way of doing things in educational research. And you closet positivists are secretly hoping it's so.

I am a qualitative researcher, an anthropologist of education, an ethnographer, and someone firmly committed to the value of explicit conceptual frameworks for educational research. In this paper, I hope to give you some idea why I feel this way and what difference conceptual frameworks, particularly those informed by some recent work in cultural anthropology, might make in mathematics education research.

What is a Framework?

According to my dictionary, a "framework" is defined as a "skeletal structure designed to support or enclose something" (The Random House Dictionary of the English Language, 1979). As used metaphorically by researchers to "support or enclose" their investigations, frameworks come in various shapes and sizes; may fit loosely or tightly; are sometimes made explicit, sometimes not. In this paper, I will compare three kinds of frameworks--theoretical, practical, and conceptual. I use the comparison to suggest the special potential of conceptual frameworks. In the second part of the paper, I will argue for the importance of including particular elements in conceptual frameworks for current research in cultural anthropology. The elements I focus on are derived from a set of issues--which I refer to as the "structure/agency problem"--that is at the center of current debates affecting all the social sciences and philosophy. In the third and final section of the paper, I will suggest how these elements might also be valuable for conceptual frameworks in mathematics education research.

A Note on the Research Process

Before beginning the main part of my discussion of frameworks, I need to say a few things about how I conceive of the research process, so you can understand how I think frameworks fit into it. As I see the research process, it has three primary governing conceptual steps--by which I mean key steps that demarcate the study and require considerable mental planning.[1] First, a researcher must decide what is to be explained by the study (establish the research problem). In mathematics education, the range of research problems in need of explanation is broad: Why do girls eschew mathematics in greater numbers than boys? Why do U.S. students score lower than those from Japan or Hong Kong on international comparisons of mathematics test scores? What is the best way for students to learn and appreciate mathematics? What kinds of instructional changes can be stimulated and supported with policy initiatives, what kinds with site-based (locally-specific) initiatives? It's unlikely that a researcher would attempt to provide explanations for all these research problems in one study; instead, he or she selects one problem to concentrate on.

Deciding on the research problem does not automatically determine the perspective, or angle, from which the investigation will proceed. Each problem listed above could be investigated from numerous perspectives. For example, the researcher might choose a discipline-based perspective, e.g., one from psychology, sociology, or anthropology; a practice-oriented perspective, e.g., a formative or summative evaluation; a philosophical perspective, e.g., a positivist, interpretivist, or critical epistemology; or a pedagogical perspective, e.g., a constructivist or foundationalist approach. In the second conceptual step of the research process, the researcher must decide what perspective to use. At this point, an explicit framework becomes important: It is the (metaphorical) structure that defines the perspective taken and thereby guides the data collection for the study. The framework is composed of ideas or "concepts," i.e., abstractions, such as self-esteem, interactive thinking, culture, social organization, or pedagogy. These abstractions and their assumed interrelationships stand for the relevant features of a phenomenon, as defined by the perspective. In selecting a perspective/framework, the researcher is deciding upon the abstractions and relationships that will be used 'to enclose or support' the study and, in

[1] I have identified the steps separately and in a chronological order for the sake of clearly describing them, not because they do or must proceed in exactly this way.

turn, the data that will be collected.

To sum up the second step by way of example, a researcher might select a perspective from psychology that focuses on self-esteem as a framework for studying the research problem: Why do girls eschew mathematic· in greater numbers than boys? For the same research problem, another researcher might select a framework from sociology that focuses on peer group socialization. In making the first selection, the researcher has decided to rely upon the abstraction, self-esteem, and to collect data about self-esteem and its differential impact on boys' and girls' attitudes and achievement. By choosing peer group socialization instead of self-esteem, the second researcher has decided to focus on such things as differential peer group norms for boys and girls and their influence on attitudes and achievement (cf. Shulman, 1988). In broad strokes, this is how frameworks "work" in the research process.

The third conceptual step in the research process begins when data analysis begins. At this point, the researcher must decide how to reduce the empirical data collected into meaningful categories, how relationships among categories of findings will be specified, and what form the explanation for the empirical data will take. Depending on the epistemological perspective chosen in step two (e.g., positivist or interpretivist), the originally specified research framework may or may not continue to serve as a guide for data analysis and explanation, but some framework--some coherent way of thinking about how to organize and interpret the data--must.

Recent critics of research practice have argued that an adequate explanation for empirical results must convincingly show that the data occur as they do <u>because of</u> the processes described by the explanation, and not accidently or coincidently (Liston, 1988). To meet this requirement, the researcher cannot simply describe or identify data in terms of a framework, nor unquestioningly accept a predetermined framework, as either would be to assume, rather than to demonstrate, that an explanation derived from the framework is adequate.

In brief then, I consider some kind of framework basic to both the second and third conceptual steps of the research process. With this background about the research process, I'd like to turn to the three kinds of frameworks: theoretical, practical, and conceptual. What are they and how are they used?

Kinds of Frameworks

Theoretical Frameworks

A theoretical framework is a structure that guides research by relying on a formal theory; that is, the framework is constructed by using an established, coherent explanation of certain phenomena and relationships, e.g., Piaget's theory of conservation, Vygotsky's theory of socio-historical constructivism, or Newell and Simon's theory of human problem-solving. In the second step of the research process (described earlier), the research problem would be rephrased in terms of the formal theory selected for use. Then research hypotheses or questions would be derived from the research problem qua theory, relevant data would be collected, and the findings used to support, extend, or revise the theory. In selecting a theory as the basis for a research framework, the researcher is deciding to follow the programmatic research agenda outlined by advocates of the theory. That is, she or he is choosing to conform to the accepted conventions of argumentation and experimentation associated with the theory. This choice has the advantage of facilitating communication, encouraging systematic research programs, and demonstrating progress among like-minded scholars working on similar or related research problems. Researchers testing the applicability of Piaget's theory of conservation in different settings and with different people, for example, work together with a shared set of terms, concepts, expected relationships, and accepted procedures for testing and extending the theory.

However, there are some disadvantages associated with the programmatic use of theoretical frameworks. Howard Becker, a fieldwork sociologist and ethnographer, has recently summarized the value of relying on theory and one of its drawbacks--that important information may be omitted or ignored when researchers rely too much on formal theory to guide their work:

> Whenever scientists agree on what the questions are, what a reasonable answer to them would look like, and what ways of getting such answers are acceptable--then you have a period of scientific advance...[at] the price, Kuhn is careful to point out, of leaving out most of what needs to be included in order to give an adequate picture of whatever we are studying, at the price of leaving a great deal that might properly be subjected to investigation, that in fact desperately needs investigation, uninspected and untested. (1991, p.3)

Dan Liston (1988, p. 324), a sympathetic critic of radical theories of schooling and a teacher educator, has argued (following Crews, 1986) that scholars who use Marxist theories of schooling (e.g., Bowles and Gintis, Apple, Carnoy and Levin), tend to address and explain research problems by theoretical decree, rather than with solid evidence to support their claims. John Van Maanen, another fieldwork sociologist and ethnographer, has lodged the objection that data collected under the auspices of theoretical frameworks have to "travel," by which he means that (unfortunately, from his point-of-view) data must be stripped of their context and local meaning in order to serve a theory.

> Events must be specified, simplified, patterned, and to a large degree stripped of their context if they are to travel well and serve as fodder for formal theory. Such is true for all description, of course, but theory itself can be a formidable taskmaker. (1988, p. 131)

Another difficulty with the use of theoretical frameworks is the tendency for them to be used by academics to set a standard for scholarly discourse that is not functional outside the academic discipline. Conclusions produced by the logic of theoretical discourse about educational practice, for example, are often neither practical nor helpful in day-to-day practice. House (1991) makes the following pertinent observations about the relationship between the concerns of academic disciplines and those of practitioners.

> A discipline is composed of groups and subgroups of scholars linked together through common communication--journals, meetings, associations, informal contacts, e-mail....At the center...are the leading authorities of the disciplines, the Cronbachs and Campbells, if you will. Those at the center are the gatekeepers who influence the others. The discipline changes as people in the field argue and debate new ideas....All [theories] in the field are subject to change over a period of time, subject to the critique of the group, so there is no certain foundation of knowledge, just the continual debate, dialogue, and argument, the disciplinary [theoretical] discourse.
>
> So we end up with disciplines in which there is theory which is often irrelevant to the experience of practitioners. Some of this theory is...necessary for [academic] legitimation. [But, if] one waits until all the debates are over to do the work, then one will wait forever. (pp. 3-5)

Practical Frameworks

It is just this kind of irrelevance for practitioners and practical matters that has led some researchers, like educational evaluator Michael Scriven, to object to theoretical (disciplinary) research as the model for educational research and to suggest practical frameworks as an alternative. Scriven's low regard for the value of theoretically-driven social science research to educational practitioners is clear:

> ...practical problems are defined by reference to several parameters concerning which the basic scientist gathers no data and rarely has any competence. These include the not-entirely-independent parameters of cost, ethicality, political feasibility, the set of practicable alternatives, system liability, and overall practical significance. (1986, p. 54)

Scriven's alternative is what he calls a "practical research approach" that would focus research efforts on "problems that really pay off for practitioners," and relegate "the search for...theoretical understanding...to a secondary position by comparison with the search for improvements" (p. 57). He further compares theoretical and practical research as follows.

> Let us consider...the difference between the ivory-tower research approach to [a] particular problem and the practical research approach. The problem...is...how to improve the teaching of handicapped children....I have frequently posed this problem to groups of educational researchers....In all cases, the results are about the same. What one must do, they suggest, is find out--from the literature or by developing a theory--which variables control the outcomes in question and then modify those variables. I ask: Is there any way to find that out besides the ways that researchers have been trying for decades? Well, basically, No, they say; except to do it better: the literature search, the design, the run, the data crunch. But there is a much better way, and the fact they do not think of it immediately shows how far we have come from commonsense. You must begin by identifying a number of practitioners who are outstandingly successful at the task in question; you must then use all the tricks in the book to identify the distinctive features of their approach...; you then teach new or unsuccessful practitioners to use the winning ways and retest until you get an exportable formula. (pp. 58-59)

A practical framework, then, guides research by using "what works" in the experience or exercise of doing something by those directly involved in it, e.g., in the case of educational research: by using "what works" in teaching, administering, trying to change schools, being the helpful parent of a school-aged child, as a "kernel" idea or action that, if extended to other teachers, etc., could help to alleviate some educational problem. The study is structured to determine key features of the practice, and whether, or in what circumstances, a practice (behavior, technique, strategy, way of thinking, style of teaching, etc.) works as expected or envisioned. This kind of framework is not informed by formal theory but by the accumulated practical knowledge (ideas) of practitioners and administrators, the findings of previous research, and often the viewpoints of politicians or public opinion. Research hypotheses or questions are derived from this knowledge base, and research results are used to support, extend, or revise the practice. In selecting practice as the basis for a research framework, the researcher is deciding to follow conventional wisdom as understood by people who are stakeholders in the practice.

Although this approach has at least one obvious advantage over a theoretical framework--the problems and the discourse are those of people directly involved, it shares some of the same drawbacks. Like the work based on a theoretical framework, the existing knowledge base--in Scriven's example, the accumulated wisdom of practitioners and interested lay persons--will constrain the topics of study, the data collected, and often the conclusions drawn. Again, there is the danger that conclusions will describe the data in terms of preexisting practitioner knowledge rather than provide convincing evidence that a particular teaching practice is best, all else considered. Further, results obtained from research based on practical frameworks are expected to "travel," as Scriven indicated. This is another dangerous situation. In the absence of theory, there is no systematic way to think about how well, or under what conditions, the results might or might not travel; there is also no readily available discourse to explain why the practice works or why anyone else should adopt it. Proponents would be in the position of imposing a practice on the (slim) grounds that it worked somewhere else.

Another more serious and perhaps more subtle difficulty with practice-driven research is one shared with research guided by a theoretical framework of extreme interpretivism.[2] Like extreme interpretivism, practice-driven research depends on the insiders' perspective--in Scriven's example, the insiders' perspective would be constituted by the views of various stakeholders in educational practice. Whereas insiders know the behaviors and ideas that have meaning to people like themselves who regularly participate in the practice, they are unlikely to recognize the patterns of group life of which their

[2] "Interpretivism" is used here, following Eisenhart, 1988, Howe & Eisenhart, 1990, and Howe, in press, to refer to the epistemological position that privileges the "insider's perspective" on the meanings and implications of social events and arrangements.

actions are a part (Eisenhart & Borko, 1991, p. 147). Insiders rarely consider the structural features and causes of social practices or the norms which they unwittingly internalize and use in communication and action (Howe, in press, following Fay, 1975). These features, causes, and norms are part of the taken-for-granted backdrop of insiders' lives. Because insiders take these constraints for granted, practical frameworks--built up as they are from insiders' perspectives--tend to ignore macrolevel constraints on what and how insiders act and how they make sense of their situation. I will return to this point when I take up current issues in cultural anthropology.

Conceptual Frameworks

A conceptual framework is a skeletal structure of justification, rather than a skeletal structure of explanation based on formal logic (i.e., formal theory) or accumulated experience (i.e., practitioner knowledge). A conceptual framework is an argument including different points of view and culminating in a series of reasons for adopting some points--i.e., some ideas or concepts--and not others. The adopted ideas or concepts then serve as guides: to collecting data in a particular study, and/or to ways in which the data from a particular study will be analyzed and explained.

Crucially, a conceptual framework is an argument that the concepts chosen for investigation or interpretation, and any anticipated relationships among them, will be appropriate and useful, given the research problem under investigation. Like theoretical frameworks, conceptual frameworks are based on previous research and literature, but conceptual frameworks are built from an array of current and possibly far-ranging sources. The framework may be based on different theories and various aspects of practitioner knowledge, depending on exactly what the researcher thinks (and can argue) will be relevant to and important to address about a research problem, at a given point in time and given the state-of-the-art regarding the research problem. For example, researchers developing a conceptual framework might build an argument for assessing the power of several different theories or explanations for an important research problem, such as why U.S. minority students are, as a group, less successful in school mathematics than their mainstream counterparts. In this case,[3] a list of currently relevant theoretical propositions

[3] My ideas here are adapted from Denzin who calls this approach "theoretical triangulation" (1978, pp. 297-301, following Westie, 1957).

and practitioner explanations would be compiled; their strengths, weaknesses, and appropriateness described and assessed; and an argument built for making some subset the focus of empirical investigation. Then, data would be collected to determine which propositions could be supported by empirical evidence. Finally, a record would be made of those propositions that passed and failed the empirical tests, and a theoretical system reformulated based on all the findings of the empirical tests.

The arguments of a conceptual framework also must be timely; that is, they should reflect the current state-of-affairs regarding a research problem. For this reason, conceptual frameworks may have short shelf-lives; they may be revised or replaced as data or new ideas emerge.

To illustrate the preceding points: In the NSF-sponsored study, "Learning to Teach Mathematics," that I am conducting of novice mathematics teachers with Hilda Borko, Cathy Brown, Bob Underhill, Doug Jones, and Pat Agard, we have developed a series of conceptual frameworks that draw on specific ideas from cognitive psychology, mathematics education, and educational anthropology (see especially Brown, et al., in press). To build our first framework, each of us consulted the literature in our respective fields (Borko-psychology; Brown, Underhill, Jones, and Agard-mathematics education; Eisenhart-anthropology) and wrote position papers on the concepts or ideas we considered most relevant to the research problem (which is: What kinds of changes occur as mathematics education students become mathematics teachers and what or who influences the changes?). As a group we read each others' papers and debated the merits of each idea for our study. We discarded some ideas (we couldn't study everything) and, for those retained, tried to organize them in a coherent way. The resulting framework guided the data collection during Year 1 of our two-year project (see Brown, et al., in press, and Eisenhart & Borko, 1991, for more information about the content of our framework). At the end of Year 1, we reconsidered the framework, revised and refined it in light of the data we had collected and new ideas that were emerging in our respective fields. The (temporarily) "chosen" ideas were then categorized into the six boxes represented in Figure 1 and, in their present incarnations, are serving as guides for the data analysis in which we are presently engaged.

Conceptual frameworks then, like the one represented in Figure 1, intentionally are not constructed of steel girders made of theoretical propositions or practical experiences; instead they are like scaffoldings of wooden plank that take the form of arguments about what is relevant to study and why--in our case, about novice mathematics teachers--at a particular point in time.[4] As changes occur in the state-of-knowledge, the patterns of available empirical evidence, and the needs with regard to a research problem, used conceptual frameworks will be taken down and reassembled.

Relative to theoretical or practical frameworks, conceptual frameworks facilitate more comprehensive ways of investigating a research problem. By coordinating concepts from anthropology and psychology in the conceptual framework for our Learning to Teach Mathematics project, for example, we were able to investigate a broader range of potential influences on novice teachers than would have been possible using a theoretical framework from either discipline alone (for more information on our collaboration, see Eisenhart & Borko, 1991).

Similarly, and unlike either theoretical or practical frameworks, conceptual frameworks routinely accommodate both outsiders' and insiders' perspectives. Because conceptual frameworks (merely) outline the kinds of things that are of interest to study from various sources, the argued-for concepts and their interrelationships--regardless of their source--must ultimately be defined and demonstrated in context in order to have any validity. Users of conceptual frameworks, then, must adopt what Norman Denzin (1978), another fieldwork sociologist, refers to as a "sensitizing approach":

> If I choose a sensitizing approach to measuring intelligence [for example], I will leave it nonoperationalized until I enter the field and learn the processes representing it and the specific meanings attached to it by the persons observed. It might be found, for example, that in some settings intelligence is measured not by scores on a test but rather by knowledge and skills pertaining to important processes in the group under analysis. Among marijuana users intelligence might well be represented by an ability to conceal the effects of the drug in the presence of nonusers. (p. 16)

This sensitizing feature of conceptual frameworks encourages the researcher to tack between the concepts advanced or assumed and the meanings given or enacted in context.

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[4] It is also important to note here that our analysis strategy depends on some additional decisions not reflected in Figure 1. For example, we decided to focus on "critical incidents" as a means to identify the sources of influence on the novice teachers. We also decided to focus on "case profiles" as a means to identify changes in the novice teachers over time. These methodological decisions and the way they have been integrated with the substantive elements of our conceptual framework are described in Borko, et al., in press, and Jones, et al., in preparation.

In this way, outsiders' and insiders' presuppositions, as well as their respective interpretations, have a place in the research project.

The inclusive and sensitizing features of conceptual frameworks also make it less likely that researchers who rely on them (compared to those who rely on theoretical or practical frameworks) will draw unwarranted conclusions or offer unsupported explanations for their empirical results. Westie summarized the advantages of these features as follows,

> [Use of a conceptual framework] minimizes the likelihood that the investigator will present to himself [sic] and the world a prematurely coherent set of propositions in which contradictory propositions, however plausible, are ignored. (1957, p. 154, quoted in Denzin, 1978, p. 300)

In other words, it minimizes the likelihood that empirical evidence will be explained by decree, convention, or accident. In sum, then, I find conceptual frameworks better suited than theoretical or practical frameworks for research in applied areas such as education, at least at this point in time. Because of the various perspectives and disciplines that can be brought to bear on educational issues and the seriousness of educational problems, research frameworks that outline and enable comprehensive, inclusive, sensitive, appropriate, useful, and timely approaches to the problems of the day would appear to be especially valuable. In the next section, I turn to one potentially useful conceptual framework that is currently being constructed for research in cultural anthropology.

The Structure/Agency "Problem" as a Basis for One Conceptual Framework in Cultural Anthropology

Epistemology

At this point in historical time and space, many social scientists, including cultural anthropologists, are grappling with what is sometimes referred to as the "structure/agency problem," where "structure" is defined as constraining or enabling macrolevel forces--outside individuals but affecting what they do--and "agency" as (microlevel) individual intentions. The structure/agency problem derives from the insights of postpositivist and postinterpretivist epistemological debates. The root of the current debate is the definition of human nature and is described in broad strokes by Howe (in press) as follows:

> ...a theory of human nature specifies the kinds of beings that a theory of social scientific explanation has for its subject matter. Positivism, with its "spectator view" of knowledge [e.g., humans as molecules in motion] and

Humean conception of causation [where causes and effects have no conceptual connection], encourages a view of humans as passive and determined by exogenous causes; interpretivism, with its constructivist [self-created] view of knowledge and intentionalist conception of causation [where human intentions can "cause" things], encourages a view of human as active and self-creating....The correct view, or so I shall argue, acknowledges elements of truth in both of these views but rejects each as one-sided.

Intuitively, human beings are neither wholly passive and determined nor wholly active and self-creating. Instead, they exhibit these two characteristics in varying degrees...."[H]uman nature" is partially determined by how humans see themselves and partially determined by things of which they are unaware and over which they have no control. Accordingly, insofar as interpretivism remains trapped within the first perspective and positivism, within the second, neither view can give an adequate account of human nature. (p. 10)

Later in the same article,

> [A new "compatibilist"] conception of human nature...concedes to the natural science model mechanistic (e.g., structural-functionalist) accounts of human behavior, preserving a place for the self-determined, "active" side of human nature. On the other hand, it concedes to interpretivism intentionalist accounts of human behavior, preserving a place for the self-determined, "active" side of human nature...[and] insofar as human behavior is an admixture of active and passive ingredients, a conception of [social science] explanation should capture both. (p. 12)

Following Howe then, an adequate social science explanation should (now) aim to account for both structural forces (positivism's 'exogenous causes') and human agency (interpretivism's 'self-created constructions'). To develop such an explanation with empirical evidence--as required in (empirical) research, frameworks for research must accommodate, and guide investigations and interpretations of, both structure and agency.

Cultural Anthropology

In cultural anthropology, the structure/agency problem can be phrased as: How is it possible to represent the embedding of richly described local cultural worlds (including individuals' intentions and a third concept, "culture," the anthropologists' favorite) in larger impersonal systems of political economy (Marcus & Fischer, 1986, p. 77)? In summarizing recent trends in anthropologists' experimentation with ethnography, Marcus and Fischer explain the "problem" further:

> ...one trend of experimentation is responding to the imputed superficiality or inadequacy of existing means to represent the authentic differences of other cultural subjects; the other is responding to the charge that interpretive anthropology, concerned primarily with cultural subjectivity [insiders' perspectives], achieves its effects by ignoring or finessing in predictable ways issues of power, economics, and historic context. While sophisticated in representing meaning and symbol systems, interpretive approaches can only remain relevant...if they come to terms with the penetrations of large-scale political and economic systems that have affected, and even shaped, the cultures of ethnographic subjects almost anywhere in the world. (p. 44).

Later, Marcus and Fischer suggest why the task is difficult:

> This would not be such a problematic task if the local cultural unit was portrayed, as it usually has been in ethnography, as an isolate with outside forces of market and state impinging upon it. What makes representation challenging and a focus of experimentation is the perception that the "outside forces" in fact are an integral part of the construction and constitution of the "inside," the cultural unit itself, and must be so registered....(p. 77)

In other words, processes of communication and meaning are thought to be constitutive of structures of political and economic interests and these interests, in turn, both enable and constrain individual intentional processes.

At the present time, debates among anthropologists about these issues are self-consciously taking place in the absence of grand theories. As in other social sciences, literary criticism, architecture, and even the natural sciences to some extent:

> [the] authority of "grand theory" styles seems suspended for the moment in favor of a close consideration of such issues as contextuality, the meaning of social life to those who enact it, and the explanation of exceptions and indeterminants rather than regularities in phenomena observed....(Marcus & Fischer, 1986, p. 8)

The need for conceptual frameworks that can more adequately address "structure," "agency," and "culture," and guide research including these elements, in cultural anthropology is exemplified in the limitations of many familiar works, including Shirley Brice Heath's educational ethnography, Ways with words (1983).[5] In the book, Heath tells a story of literacy teaching and learning in three Southern (U.S.) communities. Her study was framed by an implicit theory of cultural difference; that is, consistent with cultural difference theory, she expected that reasons for children's differential performance in school could be found in differences in the ways of life and speaking (the cultures) they learned at home. By revealing the many ways in which the cultures of the three communities were different and how cultural elements learned at home matched (or did not) those expected at school, she intended: a) to explain the sources of children's early school success and difficulty: and b) to help teachers find appropriate ways to bridge the home-school gaps she found. Heath achieved both her goals, but in the book's Epilogue she acknowledged that the understandings and changes she helped produce were not sustained by the teachers for very long. She noted that the focus of school district policy changed, apparently eliminating the opportunities and rewards for teachers that had enabled their involvement in the kind of work she (and they) had begun and believed in. By ending the book with this discussion, Heath seemed to recognize some role for structure in the explanation of her findings, but it was a role that the theory of cultural difference had not prepared her for and could not account for. Heath's theoretical framework also was not able to account for individuals who did not fit the school performance profile predicted by her cultural difference analysis, nor for subgroups within each community that might have constructed an oppositional culture or resisted the dominant position within the group. Finally, she used "culture" to mean "tradition," as if "culture" had no dynamic or emergent characteristics. In Ways with words, individuals were always following their community's traditions (culture), as if tradition fully determined their intentions and actions. Use of this kind of cultural difference or cultural determinist theory is very common in educational anthropology and is analogous to the economic determinism of "structuralist correspondence" theories such as Bowles and Gintis' (1976), in which individuals are always following the dictates of their class position (see also Foley, 1991). (In work inspired by psychology of course, there is a corresponding tendency to focus on the processes of individuals as deterministic.)

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[5] I find Heath's book very powerful and wish to acknowledge its considerable contribution to educational research. I use it here for illustrative purposes because so many people in education are familiar with it.

The special contribution of Paul Willis' ethnography of schooling, Learning to labor (1977), for educational anthropology was his conceptual framework in which "structure," "culture," and "individuals" were conceived of and investigated as separate though interrelated phenomena. For Willis, "structures" are relatively fixed, enduring, and broadly constraining features of a society--features such as class stratification or patriarchy; "individuals" are viewed as semi-autonomous from structure (i.e., capable of considering or reflecting upon structures) and thus potentially able to choose (actively produce) their own "cultural" response to structures, where "culture" is conceived of as a medium in which individuals act and interpret the world as given and, simultaneously, as the medium through which structure passes in and out of individual lives (see also Foley, 1990; Holland & Eisenhart, 1990).

The relevant implications of this work, for my purposes in this paper, are that an adequate conceptual framework for research in cultural anthropology, including educational anthropology, must now include: 1) some conception of the structures that exist and have existed over time, recognizing that they act both to constrain and to enable the actors situated within their influence (these structures might include class and racial stratification, patriarchy, and the social organization of academic disciplines, e.g., mathematics); 2) some conception of the cultures that serve as mediums within which individuals and subgroups respond to the structures surrounding them (these cultures might include class culture, peer group culture, the culture of teaching, the culture of being a student, or the culture of school subjects or (specifically) mathematics; and 3) some conception of the meanings and actions of individuals (including individual "voices," individual intentions, and subjectivities).

Implications for Mathematics Education Research

Conceptual frameworks that direct attention to structures, cultures, and agency in this way have some important implications (I think) for research in mathematics education. For example, the activities and discourse through which children (and teachers, parents, etc.) construct their understandings of mathematics would have to be viewed and investigated as deeply embedded with historical and social contradictions and inequalities. If structures, e.g., class stratification, patriarchy, or academic disciplines, are conceived of as enduring constraints on and resources for the activities and discourse of individuals, then it is not adequate tc study classroom teaching and learning in isolation or without reference to these structures. Although individual actions will be much more fluid and variable than the surrounding structures, they will always be affected by them. I think we would also be asking our research programs to help us to understand how teachers and students rely, often inadvertently, on these structures in teaching and learning mathematics. In addition, we might ask: How are teachers and students (as groups or individuals) responding to these structures? What cultures of mathematics or of schooling are their mediums for interpreting the mathematical school work they are doing? To what extent are novice mathematics teachers learning that judgments of their competence as teachers depend on acquiring the characteristics of existing (conservative) teacher culture (cf. White, 1989)? And, to what extent are students learning that assessments of their school competence depend on acquiring the characteristics of existing (conservative) student culture?

Related questions might include: To what extent are "active" (enthusiastic, conscientious, well-intentioned) mathematics teachers merely "making do" (merely tinkering, or doing what Hatton, 1989, has recently referred to as bricolage, following Levi-Strauss) with what is available within a limited and fixed structure of schooling and curriculum (see also Kutz, 1990)? To what extent are mathematics teacher educators doing the same thing within the context of their university or college work (cf. Eisenhart, Behm, & Romagnano, 1991)? To what extent do these conservative learnings (constructions), along with enduring structural features of schools, constitute teachers' and students' "resistance" to (decisions to refuse to act in accord with) innovations such as those proposed by the NCTM standards? Is there any potential or some "language of possibility" (Giroux & McClaren, 1986) in these constructions that would enable the desired changes?

It would also be important to discover why individual students are doing the particular work they are doing in school, e.g., do they have worthy motives in doing it? Do teachers have worthy motives in guiding it? Who is advantaged or disadvantaged in the process?

I believe these questions are very important ones for mathematics educators to answer. I also believe that answers to these questions can be obtained, at least in part, by using ideas from cultural anthropology to build conceptual frameworks to guide the work that mathematics education researchers do.

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If We Want to Get Ahead, We Should Get Some Theories*

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*I am "borrowing" the essence of the title from the justifiably well-known paper by Karmiloff-Smith and Inhelder (1974/75).

Abstract

In education, or in the learning sciences generally, theory is in a poor state. We have not reached deep theoretical understanding of knowledge or of the learning process, and it is important that we recognize this. Even more importantly, our community does not seem particularly intent or armed to change the situation. This paper is aimed at raising the issue of intent, arguing for new dedication toward theory. It is also aimed at a modest contribution to our toolkit for a more theoretically attentive practice of education research.

Introduction

I view the educational research community as demonstrating only minor concern for theory and its development. That should not be so. Minimally, I hope with this paper to spur discussion of the issue; at best, I hope to participate in building a consensus about the importance of theoretical thinking to our goals, and about what kind of theoretical thinking makes most sense.

My approach will be personal and more than usually assertional for two reasons. First, I hope to raise issues provocatively and relatively sharply. Second, there are deep and complex epistemological issues here that I simply cannot enter into in any great detail. I recognize I will mostly be staking ground rather than uncovering, explicating or settling the issues involved.

Theory has a somewhat deservedly bad reputation in educational circles. The relation of theory to practice is problematic. Many times the best practitioners don't have any explicit theory at all. Alternatively, it may not be at all clear that the theory they espouse "does the work" in their good practice, as opposed to their practical expertise. Others with the same theory may not be nearly as good at teaching. Some of the best, or at least, *best known* theories, such as Piagetian stages, have often seemed to put a straightjacket on instruction rather than offering many productive suggestions. To practitioners, and all too often for researchers as well, "in theory" is more a lazy lament that some expectation has gone awry rather than an appeal to some felt-to-be necessary and well-elaborated set of ideas.

Along the same lines, theoretically inclined researchers seem often to ignore the most obvious common sense. They do "silly things," if they *do* anything at all, and discover those things don't work. Or they do clever things and hide their cleverness behind theoretical claims that just do not seem refined or appropriate enough to catch their own cleverness.

I want to claim that whatever might ail both theory itself and its relation to practice is not incorrigible. For many enduring reasons, theoretical development is a principal hope for the future. An uncertain relationship between theory and practice should be viewed as an indicator of too little and insufficiently sharp theoretical thinking rather than an indicator that theory is not useful.[1]

I advocate cultivating community skills and predilections for theory. In this I am certainly not alone, although I feel I am in the minority.

I begin assuming that there is face value in having good theory, and assess the current situation in that light. Then I examine in more detail the standards by which my judgments are made. At that point, I will briefly return to buttress the assumption that theory is valuable and not just an

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1. There was no sharp boundary between Aristotle's ethics and his physics. After Newton sharply formulated his physics, it is clear to us that it helps specifically with designing effective and efficient automobiles, but it should not be expected to decide whether it is our right to pollute.

annoyance. Finally, I turn to how we might pursue being more theoretical. These last suggestions are particularly important as they help define the practice of being theoretical as I see it, and also provide some doable steps short of directly inventing deep and excellent theories.

The State of the Art

Baldly, I think the state of the art with respect to theory is, indeed, quite poor. There are two sides of this. First, there is no general agreement at the level of theories of learning or instruction. There just aren't any strong, broadly respectable and workable theories around. Tom Romberg commented on one of the most thoroughly researched areas, children's arithmetic, in a collected volume that represented the state of the art in 1982:

> This copious literature has lacked an implicit body of intertwined theoretical and methodological beliefs that permit selection, evaluation, and criticism. (p. 1)

His hopes that the situation was imminently to change, on the "route to normal science," have not been realized. As evidence, I note that several of the contributors to that volume have moved strongly away from their orientation at that time, and the rest have not converged into anything like the common frame Romberg hoped would emerge. In areas closer to my own, like "misconceptions" and conceptual change in science, I am willing to be even more aggressive in asserting the theoretical backdrop is fragmented, diverse, and, if for no other reason than that, unsatisfactory.

I strongly believe that there were theoretically interesting threads in 1982, as there are now.[2] Several of the participants in the volume noted above had and have what I judge to be insightful theoretical frames. Case and Steffe, et al., have, in their particular areas and in their own ways, done Piaget one better. Vergnaud's theoretical work on conceptual fields and "theorems in action" is related to some of my own thinking, and appeals to me. The computationally-oriented VanLehn and Greeno (Greeno, vintage 1982!) bridge to another powerful community of theoretical thinkers who deserve attention and respect.

Yet, the list is awkwardly long if it is to represent strong and broad theoretical lines. The list is also labelled mostly by individuals who, for the most part, are the only ones pursuing their theoretical lines. There is *enormous* diversity of styles and aesthetics evident, even if I limit myself to what is represented in that one volume. All these facts show severe limitations in what the research community can claim about its theoretical state.

Rather than theories, there are broad communities with similar and, arguably, strong meta-theoretical commitments. Certainly there is an unmistakable family resemblance among "Pittsburgh school" computationalists, although you must chose among ACT*, SOAR, etc. Closer to home, many call themselves constructivists these days. However, constructivism is not a well-developed theory, or even a class of theories. It lacks specificity, to take one obvious and important measure. It never really comes down to saying, as far as I can tell, exactly what and when people will learn. That is why Case, Steffe, von Glasersfeld, myself and others who are, in some ways, dyed-in-the-wool constructivists all pursue different theoretical lines.

Social constructivists, who are increasingly visible in the cognitively oriented education community, or those who advocate a situated view of cognition, also share meta-theoretical commitments. Yet there is precious little that even claims to be a compactly articulated theory,

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2. Looking at the contributions, it's striking how little, in some sense, the situation has changed in 9 years.

as opposed to an elaborated point of view, and I am skeptical about the well-formedness and clarity of these views.

So, we have precious little in the way of "hard core" theory. I am not demeaning pre-theoretical or "mere" meta-theoretical points of view. As a matter of fact, I expect that theories can only emerge as elaborations of these points of view, so we *need* to cultivate them as a means to better theories. But they are not the theories we need.

There is no shame in the fact that we do not have broad and deep theories. I believe theory development about learning and instruction is among the deepest and most difficult topics of contemporary investigation. That anyone has only paltry theories to offer is disappointing, but not surprising.

The second feature of the contemporary landscape of theory development is less cosmic than the inherent difficulty of understanding knowledge and its development, and our current "pre-Galilean" state with respect to this. That feature is, therefore, perhaps more something about which we can and should immediately do something. The general level of theoretical awareness and concern in education and learning-oriented communities is quite impoverished. In the extreme, investigators don't know or care that they have no systematic framework to guide their work, let alone a theory. They feel the most schematic principle deserves the name "theory."

I have been particularly struck with both the lack of theory and the lack of concern and critical judgment with respect to theory in the context of reviewing papers for journals. The influence of experimental psychology is strong. Experimental methods are well-developed, and there are good criteria for having adequately carried out an experiment. Reviewers are attentive to the aptness of particular statistical tests and general experimental design principles. Even most standard paper organizational formats derive from what is needed to present an experiment coherently. Or course, this is not troublesome except in contrast to the way theoretical ideas are handled. Ad hoc criteria abound, if any are applied at all. As I suggested, I think quite incoherent or simply unclear points of view are proposed as theories. Almost anything may get past reviewers theoretically, while experiments are thoroughly vetted for cultivated community practices and standards. Experimentally, confounds in experiment 1 are acknowledged and inevitably lead to a revised control in experiment 2. Theoretically, I long for the day that we similarly acknowledge familiar gaps in our positions and invoke standard repair strategies for future work.

I can cite a couple of other points at which the lack of concern for theory is vexing to me. I find it amazing that graduate school requirements are filled with "methodology" courses, while I've not yet heard of one that focused on the development of theory. That indicates a feeling that theory is either too easy to deserve attention, or else it is hopeless, at best an art that only the tiniest fraction of researchers will develop.

I also find that the way literature is cited betrays a deeply empiricist and a-theoretical bent. Articles are cited as "X showed that Y," where Y is some easily statable fact. My own reading of these articles is almost always full of nuance. They might have suggested terms for analysis and interpretations of data, but it is hardly ever compellingly clear that their terms of analysis are optimally appropriate, or that very different interpretations might not be as apt. Almost all the work in providing other interpretations and, more important, pursuing the meanings of terms, their integrity and general utility is left to the theoretically reflective reader. Similarly, much research provides phenomena without explanations. Experts do this; novices do that. Any theoretically inclined reader wants to know *why*?

In a nutshell, not many people care much about theories. Standards of practice are sorely lacking.

How Do We Know Theoretical Work When We See It?

Given the diversity of standards of theory, I feel obligated to elaborate mine. All my educational training was in physics, which may be the best developed empirical science in terms of theories. There is danger in saying any social science should be in any respect like any physical science, but standards do not arise by fiat.

I take three things from my experiences with physics. Each of these provides a "place to look" and a "judgment to make" with respect to the state of theory in an empirical science. The first has to do with the "texture" of theories, their scope and structure as complex systems of knowledge. The second concerns how the quality of theory may be judged by the quality of data that is acquired in its service. The third concerns some signs that indicate genuine theoretical progress over common sense.

Theories are richly interconnected collections of ideas and are substantial precisely because of their unusual integration. I learned from physics how much it takes to create an adequate theoretical frame. This is not done in a day of thinking or in a flash of insight. It is not explained in a paragraph or two. When scientists seem to have flashes and create revolutions, usually it is easy to see how much his/her own work and that of the community has gone into preparing for the "flash." It is trivial, I think, to understand how even Einstein's stunning "de nova" creations were tied in many and deep ways to cumulative work. And filling out the system or cleaning up the foundations has typically taken at least decades, if not generations.

Fundamental physical theories are as rich and compelling (to those who hold them) as world views. They are intricately connected to a stunning degree. There are many ways to present them, yet there is such a solidity in their interconnected nature that, among adherants, some experiments at least have entirely unambiguous interpretation and cleanly prescribed results.[3] Every Newtonian knows the outcome of billiard ball collisions.

That kind of clarity sometimes allows decisive experiments within the general theoretical frame.[4] Consider that so many scientists can agree that a little quiver of a meter reading can mean a theory of stellar evolution has been substantially confirmed. Here, I'm thinking of the detection of less than a score of neutrinos which has recently contributed vital substantiation to hypotheses related to stellar evolution and super novas. That "little quiver" (metaphorically) represents the detection of a neutrino, a massless particle that travels at the speed of light and can easily penetrate the earth. The quiver rests on a strong fulcrum consisting of a stunningly reliable understanding of the contexts of quivering, a transparent understanding of so many interconnected, invisible but theoretically sensible ideas (like neutrinos), and a web of thousands of experiments in which basic facts of quantum mechanics, relativity and particle physics have not given us enough pause for concern that one would ordinarily think the experiments were even about those fundamental theories. The fulcrum is so strong that it can be leveraged to confirm a theory about stars, where we have never been. How remarkable!

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3. I am not talking about paradigms being overthrown (or confirmed) by critical experiments. Instead, I am more referring to experiments whose outcome are so obvious that no practitioner would bother performing them except to illustrate a fundamental point to a student. It would be extremely unlikely that a competing theoretician would bother trying to upset a theory on these core grounds.

4. Again, these are decisive *within* the paradigm.

I will be critical of learning theories until they have some similar integrity. As a consequence, for a long time to come we will be able, if we chose to, to critique the adequacy of given and proposed theories. We should chose to do so as a means of advancing our understanding.

A practical implication of this position is that it should be natural and acceptable, if not expected, that those advancing theories should spend as much time explaining the limits of their ideas as expounding them. Much more than where the theory is empirically weak (e.g., what experiment should be done next), this means exploring where it is conceptually weak, where it is unsharp, hard to articulate, in danger of incoherence, and so on. Only if we lower our standards substantially do these critical pursuits not seem worthwhile. Only if we pretend we are much farther along than we are can it be seen as a sign of weakness to discuss these issues with respect to our theoretical proposals.

There is no data without theory. As much as science involves experiment, it is not a purely inductive enterprise. This is so obviously true in contemporary physics that it hardly bears remarking on. If one didn't have a very well-developed notion of what those invisible neutrinos were all about, the "data" of meter twitching I remarked on above would not be data at all. The whole rationale for the experiment and set of observations would not exist, nor would the fabric of reasoning that makes the observations informative. Nobody would have been looking for the quiver, and it would have been incomprehensible if they had accidentally seen it.

There are two things that tend to undermine the influence of the above observation. First, scientific formulations in physics *look like* empirical generalities that one could stumble on by doing a lot of measurements and finding a pattern in the results. One just has to measure a bunch of forces, masses and accelerations and find out that, reliably, $F = ma$. Or you make a bunch of resistors and "discover" Ohm's Law. Why can't we find the laws of learning by correlating parameters? I have only space for a "one-liner": It made no sense and would have been impossible to measure forces or mass before at least some features of the theoretical framework of which they were part existed. Measuring X requires a lot of commitments about the nature of X, the very first, but highly non-trivial part of which, is to believe X exists.[5]

The power of intuitive or commonsense knowledge also undermines the appreciation of how important and necessary theoretical frames are in the production of data. That is, common sense, or some slightly refined species, can substitute for a theoretical frame so easily that we just don't notice it. Every one of us is full of intuitions about the mind and learning. Some of these are cultivated by the language we inherit -- "concepts," "beliefs," even "to know" and "knowledge" -- that have adequate purchase on the world to justify their everyday use. Some roots of these frameworks are probably more private, extrapolations of our own experiences in thinking and learning, or extrapolations of what we observe in others. We can, in these intuitive frames, "observe things" and draw fairly adequate conclusions under some circumstances. For example, we are not outstripping the power of common sense when we say with conviction, "He doesn't *know* I went out with his girl friend."

It is common to say any observation implies a theory. Observations certainly imply a framework of ideas, but not at all a deep theory by the standards implied above. (Hence a-theoretical empiricism does not mean without a framework, but without an adequate scientific one.) The problem is that intuitive frames are not powerful enough to constitute sufficient theories of the mind in general and of learning in particular. We should draw them out when we rely on them, and critique and refine them to produce more scientifically adequate frameworks.

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5. See diSessa, 1991-b, for an articulation of what might be involved in thinking to measure a quantity and carrying that process out.

Since theory, in some respects and on some occasions, defines data, we can sometimes judge the quality of theory by the quality of its data. I provide a brief and clearly elliptical example where I judge the problem with learning data is in the theory on which the data depends. In this case, the problem seems to me to be both the clarity and integrity of the ideas themselves, and also hidden intuitive presumptions that, when brought to light, seem dubious.

Some "theories" of learning provide that learning occurs when the learner is disequilibrated by new ideas or observations that compete, in some sense, with old ones. I think the commonsense roots of such ideas are evident. Everyone knows the feeling of being presented with "destabilizing" information that doesn't jibe with our current take on the world. We all, also, sometimes follow that feeling with a consideration of the circumstances of our knowing what we think we know, and we sometimes "resolve" the difficulty by realigning our existing "beliefs." Some likely inadequacies of this kind of theory (as sketchily as I've presented it) are not hard to find. First, it is drawn from a particular class of experiences where we have reflective access to our epistemic state: We are aware something is wrong. I take it as the right minimal assumption that this awareness is only possible in certain circumstances where our meta-awareness of knowing processes is above a certain threshold. Second, we must also consider the generality of the processes by which we "decide" to reorganize our beliefs, and the means by which "we" carry out that reorganization. Indeed, the sense of self that is indisputable in commonsense thinking about thinking is hardly something we can, to be theoretically self-conscious, take for granted. Sometimes we can act as an agent on our thoughts in a semi-reflective way. Sometimes, I am quite sure, we cannot. More technically, we could ask what exactly constitutes the state of disequilibrium. If we deprive ourselves of the common sense that says "I've had that feeling!" how do we describe in any generic terms what constitutes that feeling, especially in such a way as to apply to every event of learning? I could also enter into discussion of the empirical limitations of such theory. To put it crudely, there are such a host of details about learning that depend on the specifics of the knowledge to be learned and the individual as he/she comes to the learning context, that it seems unlikely that disequilibration can possibly account for them. If disequilibration uniformly exists, I believe there must be hundreds of different kinds of it. At least, this is a thing to be seriously worried about.

Respectable theory, when we get it, cleanly transcends common sense. My last point of extrapolation from physics to our expectations for theory in education really follows from discussion of the above two points. Unless we can unambiguously point to how we have transcended -- in generality, precision, clarity, and justifiability -- the intuitive sense of mechanism we all build in daily life observing and thinking about psychological matters, we just won't have adequately prepared theoretical ground. I'll pick one focus for this exposition, but I think the point is much broader. Commonsense vocabulary just won't do the job of providing the technical terms of a theory of learning. When we stop with "beliefs," "knowledge," "concepts," and so on, even with a few phrases of elaboration, we are on extremely shaky ground.

To put an edge on this, physics theorizing has always involved ontological innovation. The "force" in Newton's theory is a new entity that simply does not exist in common sense. Even mass took on a much refined interpretation to make sense in that theory. More evidently, quantum wave functions did not exist before quantum mechanics. My presumption is that we will not have adequate theoretical purchase on learning until concepts, facts, beliefs, skills, and all the rest of our common sense about knowledge and learning become reinterpreted within a fabric of more precise and less intuitively loaded terms. Please, do not mistake: I'm not appealing for obscure language, or for proliferation of new words. I'm appealing for the clarity that can come with ontological innovation.

Defending Against "Social Science Is Different"

I have three defenses against the claim that the above is simply an unwarranted extrapolation from physical to social sciences, which I can only briefly pursue. First, I believe all of those foci are epistemological, not just saying "cognition should be like physics." That is, they can be given motivation independent of their appearance in physics. I don't think, for example, that the theory dependence of data is at all unique to physics. I do believe that transcending commonsense frameworks is an important task to pursue, and a reasonable measure of success for any empirical science.

Second, let me demonstrate the care involved in selecting these points to extrapolate by listing characteristics I do not extrapolate.

1. *Mathematics.* I deliberately did not pick mathematization as a core characteristic to extrapolate. In the first instance, I believe explanation is a higher priority goal than mathematization. As well, I don't believe the mathematics of mind descriptions will be very much like the mathematics of physics; I expect it will be more like the formalisms of computation. This is, of course, a long story of its own, but it at least means simpleminded expectations about the form of knowledge and learning theories are to be guarded against.

2. *Sense of mechanism.* I don't believe the basic sense of what terms and forms are explanatory can be imported from physics. In particular, I don't expect that reductionist accounts, for example, a purely "brain science" approach to mind, will prove successful. The distinction between correlation and explanation is fundamental to any science, and deciding which is which is not a matter to prejudge on the basis of other sciences. My advocacy of theory in this paper is precisely to say we must do this for ourselves.

3. *Methods.* Every science needs its own methods adapted to its own theories and to the observational circumstances available to it. We can't blindly appropriate empirical techniques that work for sciences that have much more theoretically sound, or simply different, ontologies. In contrast to physics, I believe "empathetic techniques" that use (carefully and with many qualifications) our ability to sense our own thinking, and react instinctively to aspects of others' may be quite helpful. We don't have recourse to this in most areas of physics (though we do, in some degree, in our kinesthetic senses for the case of Newtonian mechanics).

Third, I explicitly recognize the many arguments against expecting theories in social sciences to be at all like those of physical sciences: "Social sciences are too complex and contingent to admit of theories of the sort we find in physical sciences." Or, "Social sciences are and must be fundamentally interpretive, not predictive." Without pretending to argue the points, I note that I simply have not found the arguments compelling for reasons like the following:

1. Such claims are too often simply assertional, without providing a theoretical basis for the meaning of the "fundamentally differentiating attribute," or how it opposes its supposed antithesis in the physical sciences.

2. Even if the distinctions turn out to be well-founded, one has the obligation to explain why they bear on the possibility of good theories. I don't see why the observer's being like the observed means that there can be no clean conceptualization of the observed.

3. Claims of intrinsic difference between social and physical sciences often are drawn from caricatures of physical science, far from what I experienced as a physicist. My experience of physics was of highly integrated explanatory systems that involved important ontological innovation. It was not of "narrow and mechanized prediction."

Similarly, to think that physical systems are easy to observe simply does not jibe with the fact that the appropriate thing to "observe" may be a wave function! There was plenty of argument and "interpretation" around in the early stages of any of the foundational physical theories.

Physical theory deals with systems of 10^{23} particles and chaotic systems that are, in some ways, strictly unpredictable. How, exactly, is the complexity of human systems fundamentally different so they are intractable by theory that resembles, only in some basic epistemological senses, physical theory?

4. Many of these claims seem to be simple restatements of the fact that we don't have good theories, drawing the conclusion, somehow, that we *can't* have such theories. "History shows that learning theory has had a poor track record in its application, in education." Of course it does. It also shows this has been true of every field of inquiry before it developed deep scientific foundations.

I've explained and, to some extent, justified my standards and judgment that we don't have excellent theories yet, but that they might be achievable. It is possible to think we are so far from *that* kind of theory that applying such standards to educational or psychological theory is ludicrous. I think, in contrast, that we may develop a tremendously helpful set of at least interim, if not absolute, standards and heuristic moves to advance our understanding out of the realization that we are not done yet. Realism is almost always the best policy. Although it is exciting to believe we're on the edge of really major breakthroughs, if we have not made them already, it is probably more important to have a cultivated sense of how far we have actually gone, and how far and in what directions we need to move. I prefer to avoid accepting "wimpy" epistemological standards that claim social sciences just won't ever and shouldn't strive to meet at least some strong standards in some respect like those physics has achieved.

As I have indicated how difficult I believe it is to achieve deep theoretical understanding, I am quite sure we will never achieve it if we don't set our minds to it. This is a kind of Pascal's wager I'm prone to accept: Unless there are compelling reasons to abandon searching for deep understanding that is in some ways like what we have in physics, we ought to pursue it.

Do We Really Need Theory?

I've treated, however briefly, claims that we *can't* reach the kind of theory in social sciences that has been achieved in physical sciences. In this section, I consider what we get from theory to bolster our resolve that it will be worthwhile before getting on with the program. Much that can be said about this will sound familiar and commonsensical. Yet I believe it bears reviewing in view of the apparent undervaluing of theory in the educational community. Of the many things that could be said, I'll select only a few.

The Scientific Power Principle.

Theoretical scientific understanding reliably yields capabilities that far surpass what we can attain by experience or intuitively-based empirical methods. Physics (lasers, nuclear energy), biology (recombinant DNA techniques), medicine (controlling viral and bacterial infections), technology (materials engineering, semiconductors and computer technology), and so on, all repeatedly show that theoretical advance is the linchpin in spurring practical competence. Even when a great deal of experiment and much engineering must be done, theoretical advance defines the parameters of experimenting (e.g., the terms of materials science), and establishes entire engineering domains (e.g., modern electronics emerged out of the basic quantum and materials principles that suggested the transistor could work).[6]

It is true that many aspects of our lives are entirely adequately handled by experiential or "purely empirical" approaches. You don't need Euclid's Axioms or General Relativity to navigate your house. Reading *Consumers Reports* and finding there a statistically reliable correlation between the measured reliability of a car and its brand is probably all you need to figure out which car to buy to have the best chance at getting a durable product.

Sometimes things are not so easy. Generating adequate power for our planet is not so easy. Building machines that fly is not so easy. I strongly believe designing for human competence, ranging in my immediate concerns from designing instruction to designing information machines for comprehensibility and effective use, is not so easy. I don't think it even needs argument that getting the most from our intelligence is a worthwhile pursuit. There is plenty of value, hence motivation for spending the time and effort to understand learning well.

"Because It's There"

One needn't be so practical about pursuing deep understanding. I believe our field is dealing with almost timeless questions. Physics approaches questions like: What are space, time and matter, and what accounts for their structure? Does the universe have an end; how could it? How did this all start? In the same way, I believe we all deep down want to know things like: How do we know? What are the limits of human knowledge? Why are people different from other animals; what does it mean to be intelligent, and are there fundamentally different types of intelligence? Such questions deserve deep answers. These are grand enough pursuits to make me very happy when I feel I've taken a small step. Realizing the scope of one's goals give meaning to the enterprise beyond the limits of present understanding.

Cumulativity in Science and Overcoming Barriers.

I have suggested already that theory is important to the infrastructure of science independent of implications for practice. "There is no data without theory." I suggested that developing standards and being critical of our explicit or implicit theoretical commitments is a prime method of improving our scientific understanding. I wish to point to two general and important infrastructural issues here.

The first is cumulativity. I hear echoes of Allen Newell's (1973) "You can't play 20 questions with nature and win."[7] His sentiments strongly parallel mine. One can't simply collect ad hoc hypotheses about what might influence what, and it is boringly non-cumulative to identify one after another little experimentally valid "phenomenon." Science needs a broader woof and warp. It needs breadth in order to supply focus. One simply must take stabs at overarching views so that the pieces fit into a larger context -- or don't, in which case we need another theoretical stab.

My reference to neutrino detection above can make another point. The "strong fulcrum of well-elaborated theory" I described in that story can disconfirm as well as confirm. For example, scientists might measure a tiny shift in the orientation of an orbit to (possibly) disconfirm Einstein's theory of relativity. It *has* to be that way, if Einstein is right, no ifs or buts. In a sea of "phenomena," of correlations without rigid underlying causal mechanisms, of heuristic but

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6. diSessa (1991-a) describes some details of how the engineering context of learning theories might relate to the theories themselves.

7. Or see the first chapter of Newell, 1990.

commonsense ideas about knowledge and learning, no such disconfirmations are possible. There are always exceptions and extenuating factors. We don't know when exactly our hypotheses must apply, nor exactly what they predict. To take a case I introduced above, I believe that current disequilibration theories of learning are not disconfirmable. (Perhaps they are tautological, which is not the worst status possible.) Until we know exactly what disequilibration is, what processes generate it, and what processes are available to "select" a new view, and "change beliefs," we will always be able to fiddle with our characterization of a learning event to make it look like disequilibration.

Problems with a Theoretical Approach

I hear a couple of "Well, OK, but..." reactions to my line of argument to which I would like to respond. The first is the feeling that only special individuals, the Einsteins, Newtons, maybe the Piagets and Skinners, and so on, create theories. I am comfortable that grand moves might always be associated with individuals. Still, a field is not all grand moves. As I suggested, I believe almost every paper I have reviewed for journals could have been improve and clarified -- putting its results and non-results in clearer relief -- by some hard thinking about its hidden or missing theoretical commitments. I think small steps at clarity, generality, even to better fix the present state of the art, can accumulate. This may be more plausible to those who habitually see theory as always coming in identifiable, "world shattering" chunks after I make some suggestions (in the section on Some Almost-Practical Steps) about small things we can do on the way to more adequately addressing the theoretical side of the requirements of science in our community. Even if we accept the grand move hypothesis about theory, our community has a much better chance of cultivating or attracting individuals who can make those moves if we are more theoretically aware and intent. Perhaps we would be better at noticing and judging important theoretical moves in the making.

I anticipate one other reaction. It is easy to imagine that if theory-building becomes a more popular sport, journals will be filled with incomprehensible jargon and unsubstantiated speculation that now tends to characterize "theoretical" work. But I'm advocating "better" as much as "more." Future theorizing should be constrained by significant advances in a critical sense, which would prune away idle speculations. Indeed, as I suggested, the first signs of a more theoretical orientation will much more likely be self and other criticism and recognition of limits rather than just more theory.

Cultivating a Theoretical Turn of Mind: Some Almost-Practical Steps

The premise of this section is that the pursuit of theory is an excellent thing to do short of producing encompassing and revolutionary theories, as usually catch our attention. I've collected a short, ad hoc list of steps we can take toward becoming better theoretical thinkers. Many of these reflect things I've said above.

These heuristics for the development of theory are actually a fairly critical part of this essay. First, this is really the place I begin to define what I mean by theoretical thinking, short of standards for "having arrived." I hope it is evident that I have a broad interpretation of theoretical thinking, and I would argue that is appropriate. Second, if appeals to be better-oriented theoretically are to have any effect, they had better have particular, doable moves associated with them. I hope to get from this section reaction from colleagues on what they think constitutes theoretical work, and whether it is important and doable (or done!).

Some of these suggestions, especially the later ones, specifically single out students. I don't mean to imply that those suggestions are only for students, or that students shouldn't expect to get anything from the other suggestions. I do mean to emphasize the importance of students' training in changing a field, and also to point out some steps toward theoretical thinking that I think are either particularly easy or particularly important.

Almost every proposition we car formulate these days is as false as it is true. Try to understand why and when they are both true and false. This is a heuristic I've cultivated myself in reviewing journal submissions. It helps us discover the hidden contextual dependencies of our ideas, hence helps to define their real generality. It combats "confirmation bias." In addition, it asks us to be more explicit about what we mean so that one can make sure we have explained what our terms mean, rather than relying on inarticulate instincts that apply ideas only where we know already they work. The heuristic can be also used to be clear on the contexts in which our ideas have their intuitive roots. Armed with that, we can understand both a bit more about why and when our claims might be valid and adequately specified.

Is learning always best done in groups? Almost certainly not. Is cognitive apprenticeship the right method to learn any material? Can't be. Are novices always concrete and experts always abstract? Not a chance.[8] For all the social roots of individual cognition, I am confident there are also individual roots of social cognition.

If you can't decide, take a line and push it until it breaks. I frequently tire of papers that list all the possibilities of how the world might be configured to explain a phenomenon. Sometimes, anyway, we should be able to make good guesses that cut away broad ranges of possibilities and hence have important consequences. These are guesses that are worth pursuing in an extended way, in contrast to meandering among the many possibilities. For example, in my work with intuitive physics, I have quite deliberately made the decision to assume that such knowledge comes in identifiable bits, "atoms of cognition" if you like. I am quite aware I have precious little evidence to establish that fact, but I expect only to know whether or not, and in what way, it is true *if* I develop an elaborate theoretical scheme that defines precisely what "knowledge in pieces" means, and can draw extensive implications.

A complementary heuristic is to understand when you have made such a commitment, as opposed to believing every aspect of your thinking is justified by the weight of evidence. Many of our working assumptions are simply not justified in this way. It's worth our taking cognizance of that fact.

Arrange your work to be thematic, cumulative. I don't think it happens without effort that each of us (and, perhaps, communities of researchers as well) plots a coherent line. I think it is particularly easy to have an empirical program that does a little of this, a little of that, and moves on. Experimental methods seem much more transportable than theory. Yet, if we are to develop theory, we shall have to work coherently at it.

I see too much opportunism in the way research topics are approached. Mental models, "misconceptions," or collaboration become "hot topics," and many jump in. But they are also as likely to leave in a year or so as to make a deep mark. Of course, we must all decide when a line

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8. I've applied this heuristic systematically in thinking about differences we instinctively apply to naive versus expert knowledge. This has become articulated criticism of some of the "expert/novice" literature. See, for example, sections on "concrete and abstract" and on "generality and specificity" in Smith, diSessa & Roschelle (in preparation).

is progressing and where the new opportunities lie. But we should also select our foci carefully enough that we believe an extended effort will be rewarded.[9]

Question ontologies; refine categories. I've transplanted my suspicion that deep theoretical advances are always accompanied by seeing the world in new and different terms into this heuristic. What are "concepts" or "entrenched (or any other kind of) beliefs"? What is "metacognition," "a community practice," "an educational activity"? Questioning the analytic and empirical meaning and adequacy of these categories expresses skepticism about the precision of nearly commonsense ideas that substitute in much current work for what should be technical terms in well-developed theory. Questioning meanings also expresses a feeling that a pursuit of what we instinctively mean by these words can be clarifying. Of course, this could become an armchair game. The enterprise works best in the context of empirical study that tests the work more refined terms might do for us.

I find myself questioning my instinctive categorization of instances all the time. It would be a worthwhile enterprise to catalog strategies for making these tests. Such questioning episodes turn frequently into pursuing clearer meanings for terms -- operationalizing them or framing them better in order to afford both easier classification of instances and also clearer import of classifications that have been made.

Make the most of "what we know for sure." Physics has a few things that it knows for sure. Symmetry considerations are among them. As well, it knows that all physical interactions must be local in space and time. Although things "we know for sure" may seem general and bland, in the hands of the best physicists they have proved amazingly powerful and particular. They seem, especially in combination, nearly to "deduce" particular physical laws.[10] Surely we must have, or should be looking to find, similar principles in education or learning psychology. What are they? I'll leave this heuristic open as a good litmus test concerning how we think our field is or will ultimately be organized. It might be that most readers will simply not know what I am talking about. Or, alternatively, they have their list, or believe there can be no such list.

Let us think what appropriate empirical work, data collection and analysis, might be like to serve theory building. I am convinced that our arsenal of empirical methods are skewed tremendously toward confirming or disconfirming hypotheses that are assumed to be well-formulated rather than toward building an adequate basis for making hypotheses, or testing the well-formedness of our ideas in contrast to testing their truth or falsity. I believe empirical work can play a vital role in developing theory, but this role and methods that fit it are undervalued and underdeveloped. I would love to see a good course and text developed around empirically grounded theory development.[11]

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9. Early in my professional formation, I was influenced by Howard Gruber's concept of a "network of enterprise" (Gruber, 1981) to describe how creative individuals manage to pursue a sufficiently diverse yet cumulative, and mutually reinforcing set of lines of inquiry. I sat down and designed my near-future network. I believe, in retrospect, that was an important step for me.

10. Feynman (1965) wrote a beautiful little book on this. I have also been tremendously impressed by the work of scientists like E.P. Wigner, and Einstein in this regard.

11. Perhaps I am defensive, but I believe some of my empirical work has been misunderstood as not-so-good theory confirmation, when I view it as more-than-usually-conscientious data sensitivity for the purpose of theory motivation, specification and development.

Cultivate a sense that explanation is the name of the game. When people begin to play the game of science, their first glimmers of understanding it are that science is about finding the way things are. Science finds "that" X or Y. More deeply, I take it that science is explaining "why and how" things can be the way they are. Of course, there should be a few "thats" in science, that F equals ma, for example. But these "thats" must have thousands of "and therefores" following them. In general, observations must be carefully placed in an explanatory web.

I think versions of this primitive "that" orientation are insidious and long-lived. As I mentioned, many too many papers talk about the existence of a phenomenon without pursuing underlying mechanism. In education, prescription substitutes to an amazing degree for adequate understanding of underlying mechanisms. To parody, "We know that to teach well, one should *do* X." I find this in some degree even in some of the best work in the field, or at least in the field's (if not the investigator's) take on the work. Reciprocal teaching inappropriately becomes a principle rather than a technique.

Taking instructional prescription as mechanism is essentially a category error. Instruction is an area of complex design. I don't expect deep principles of learning will often if ever show themselves on the surface of an effective design. Of course, this fact makes our job harder -- we must both understand the principles behind instructional interventions, and we must understand the contexts of application of those principles well enough to know that the principles are truly involved and do the central work we might claim for them.

I've been struck by a characteristic of most of the most creative and deep thinkers (of course, in my judgment) I have known. They are constantly on the alert for interesting phenomena, where, perhaps, a fundamental piece of the world breaks through its mundane presentation, or, as interesting and likely, where we find a deep intuition confounded. They take the time to look again, recreate, modify, and make a proposal for both an explanation and for why the phenomenon is puzzling in the first place.

In some respects, this behavior seems unprofessional. It is amateurish because these individuals frequently have no specialized interest or knowledge about the phenomenon at issue, why bottled water fizzes in a particular way, or how geological formations of a particular sort might have come into existence. But I have come to feel that these entertaining little escapades are both telling and important. They tell us that being alert to the odd moments when nature reveals herself to us is a high priority enterprise. It is an enterprise of observing, reflecting and explaining, which some people cultivate or do naturally. These people have likely acquired some generally useful skills with respect to this enterprise, and probably find it both entertaining and profitable to exercise even away from their domain expertise.

I find the instruction in cognitive science and education unusually devoid of such spontaneous pursuits. Too often students are expected only to be "library indices" to sanctioned data, knowing the results of the field, thinking to observe and comment on only things others have declared comprehensible or empirically tractable. Students don't think much about their own experiences in learning, or what they make of others', except as filtered by the sanctioned state of the art. Though the focus of this little, perhaps dubious, indicator of a more general theoretical orientation may be misplaced, I find similar indicators again and again in deep thinkers. These are almost never reflected in our training.

Create Mini-Theories. There is a slightly more professional version of the activities described above. That is to formulate little mini-theories about important issues in the field, and use them to accumulate and refine ideas about what must or might be true. The criteria for these mini-theories are not ad hoc. First, they ought to be about important things, so the time spent on them is worth the effort specifically concerning conclusions (as opposed to the process orientation, above). It also helps a lot if they are counter-intuitive, to test the strength of our "knee-jerk" dispositions that arise from implicit theoretical orientations. Frequently, mini-theories occur to me in the process of thinking, "That seems strange, but there's something appealing about it, and it might explain some very puzzling phenomena."

I find these are the kind of things from which programs and theoretical ideas grow. For example, my own "theory" of intuitive physics arose from two at the time counterintuitive (to me) mini-theories. One was that cognition is radically unsystematic. As I put it to myself, every idea is a different form. The second was to presume that we could identify a large set of what appeared to me at the time to be a few cute little intuitions you could trick people into displaying, and that, in fact, causality was constituted of a whole body of such entities, rather than being localized in general principles of cause. The latter seemed particularly counter-intuitive at the time, but I could not see how to dismiss it out of hand. And since causality had proved so elusive, maybe people were looking in the wrong place. These mini-theories developed into a fairly elaborate theoretical and empirical program, of which they are still good motivators or hooks to explain the gist of the program (diSessa, in press-a).

A recent mini-theory of mine is that the robustness of scientific "misconceptions," which is touted in the literature about them, is mostly constructed in encounters that are intended to expose and overcome them. This contrasts with presumptions that misconceptions are inherently stable, and hence *must* be attacked. Instead, people may only formulate positions when asked to. But once asked, they can build rather resilient ideas out of what might otherwise be fleeting impressions. We may then be doing exactly the wrong thing in "attacking" misconceptions. I wouldn't pretend to defend this statement scientifically at this point, but it will orient some of my thinking, and I believe it *might* turn into a collection of defensible claims. One of the properties of this mini-theory is that it challenges some of my own presumptions, as well as those I feel others have inappropriately taken up in their work. So now the game is: What could this mean? Could we demonstrate that it is definitively false, thus simply drop it?

Formulating and pursuing mini-theories strikes me as not only a reasonable practice for professionals, but, with guidance, a good and tractable finger-exercise for students.

Redescribe, redescribe, redescribe. Students particularly suffer from the feeling that the world presents itself directly to them, that intuitive characterizations define exactly the circumstances in which we can use those terms and descriptors. This is profoundly false. Our future colleagues need to understand this and need to play a better game of formulating and judging descriptions as soon as possible. I am especially fond of redescribing educational practices that students find instinctively repellent in terms that they use to describe good practices. We propagate attitudes rather than clear conceptions about instruction by only using words that sound laudable (or the reverse) to describe particular practices. Of course, redescription is not only to get students to rethink judgments and their bases, but to articulate and refine the meanings of the terms that seem clear and apt, but may not be either well-defined nor apt.

Cultivate a sense for the "big issues" in the field. I've underlined how difficult yet central I believe theoretical considerations are, and how important it is to generate a coherent program to make advances. Students especially need to know where the field is, how to measure the latest fads, and how, in general, to calibrate progress they or others might make. It is often "schoolish" and vapid to announce what a field is about. The first chapter of textbooks that explain "what physics is," or psychology, are usually crushingly boring and uninformative. Yet the responsibility of keeping track of our advances on a large scale is critical, and we should not shirk it.

Identify, practice (and give students opportunities to practice) basic theoretical moves. The subproblem here is a particularly interesting one. What *are* basic theoretical moves? This is the parent problem of several of the above suggestions. Identifying basic theoretical moves not only defines the practice of being theoretical, but it also explains in a more explicit way what is or should be meant by theoretical work and what are central as opposed to peripheral parts of it.

For example, the heuristic "redescribe" tells us that the terms in which we describe the world are as important an object of study as finding the "right" propositions using the terms we already have. Heuristic strategies of evolving more precise and powerful descriptions are thus a central set of moves in making theoretical advances, of which "do it" (redescribe) is the crudest.

A basic move I find myself rehearsing explicitly and self-consciously for students embarked on their theses might be called the "characterize, systematize, re-examine loop." Typically, one immerses oneself in data, using whatever initial predilections for analytic frameworks one has at one's disposal. Usually one comes out having found a number of critical phenomena -- happenings that can be somewhat effectively characterized in available terms and seem also to be critical in one way or another. Then, one takes the terms of description, categories, and implied or conjectured relationships among them and tries to complete and systematize the story. What could a generic characterization of such knowledge be? Why might this relation hold? Is it an example of a more general relation, or what co-requisite (but undescribed) circumstances might make the relation more comprehensible and "necessary"? With a more articulated, complete and more evidently causal story to tell, we need to return to the data. Can we see the differentiation of contexts implied? Is there, in fact, only one critical feature, or is the phenomenology of our data much more diverse than we presumed? Do the new categories developed in the second phase help make better sense of the data?

The second phase is one students especially need coaxing to do. It's not an obviously workable tactic in an empirically dominated world view. It seems rather rationalist -- how can we find ambiguity in terms, extend items to "a full list," and so on, without looking at the data? Yet, this is where theory originates or is iteratively improved. We not only can, but we must be analytic and systematic in reordering existing perceptions and observations, in sharpening the meanings of categories that define how we see things, in completing fragmentary patterns, which gives us new eyes to check the data.

Summary

Theory is a tough goal to maintain in the face of the state of the art in learning and instructionally oriented investigation. It would be easier if we could just "bail out" and think we were more like "literary critics" of practice, or artisans fabricating all-the-time better, but unprincipled artifacts. I think we should face up to the fact that it is very likely we could, if we chose to, be a science in the making, however limited our present powers. If we do not critique our work by high standards, then we will certainly delay obtaining the kind of power deep scientific understanding might bear.

I have tried to advance an image of theory building that is incremental and heuristic as much as it is a set of simple, hard standards by which we will know when we are done. In fact, I've really avoided the "standards" view for the most part, except to give a sense for why I judge we are not far along on the path to excellent theory. The heuristic view of theory building is especially important given that no one can say with much certainty how much future learning theories will look like the excellent theories we know in other domains. It is also simply more important to know how to move things forward than it is to know when you are done. So, theory-building can be hard-nosed in its goals, but at the same time generous and truly exploratory in its active parts.

As a community, I am arguing we should exercise more effort in and attention to theoretical matters. We should cultivate a critical capacity to understand modest advances at the same time we recognize the many types of limitations of existing theories. I think we should share and systematize methods to improve our frameworks. Most especially, I urge we scrutinize, articulate and refine the theoretical moves we've all intuitively developed and found powerful. We should do this for the benefit of our students, for our colleagues, and, especially, for ourselves.

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I owe a debt to my mentors for teaching me what I have thought valuable enough to try to convey here. I wish to thank Steve Adams and Jack Smith for extensive and thoughtful commentary on a draft that influenced the form and content of this paper. Ehud Baron similarly stimulated my thinking about the topic with his reactions.

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Appendix: A Theoretical Orientation

I deliberately avoided discussing particular theories or theoretical orientations, for the most part, in the body of this essay. This was to avoid contentious detailed issues that could easily obscure the main points. But, theory-building is not a "meta" exercise. Scientists must take on or develop an orientation toward theory, and find the classes of theorizing they believe appropriate to the subject matter they are investigating. I am advocating that we articulate and advocate particular lines. I wish to do a little of that here.

Every researcher develops a particular "sense of mechanism" about what the basic principles operating in a domain are like. I believe this is a precious personal and community resource that guides observation and generalization, but it needs explicit consideration. If you think theories will look like prescriptions, that's what you will develop. If you think "thick descriptions" are explanatory, you won't develop other kinds of explanations. If you believe that a particular social relationship can define learning, or that no description of knowledge "in the head" is relevant to learning, you won't pay attention to the structure of content domains.

My instincts are that we must develop mathematical-computational theories of mind and learning. I am drawn to current attempts to do this on several accounts. First, there are at least languages of analytic precision in play. This also builds in some strong mechanisms for testing the ambiguity or sufficiency of the ideas involved, and for surpassing reliance on intuitively attractive, but "magical" ideas about the way things may work that common sense provides us in abundance. There is plenty to criticize about most present computationally formulated theories, but I don't see the sense in denying the ways in which they are attractive.

On the other hand, I don't yet insistently couch my own ideas in these terms. This is a judgment that we haven't got the mathematical-computational foundations quite right yet. Most directly, the best developed theories in this area (and they are better formed by many standards than "theories" belonging to many other traditions) just don't, in my judgment, reach the issues or touch the empirical phenomena I am most interested in pursuing, mainly those dealing with conceptual change and long-term conceptual and intellectual development.

The crux of this lack of contact, I believe, is that current theories just do not get to the heart and power of knowledge. More specifically, I believe there is a tremendous diversity to the kinds of knowledge and systems of knowledge that one can find. Essentially all computational theories are much too "flat" and uniform, to my taste, suggesting much more uniformity than I believe exists. I believe I perceive many different subsystems of human knowledge that have very different properties, which properties I don't know how to describe in the terms of these theories. (Or better, I don't see how the precision of the theories improves the apparently looser descriptions I make outside of them.)

This leads directly to a general program for studying thinking and knowing. It is roughly at the level of knowledge itself, though one needs to have at least a minimal sense of computational mechanism in order to see how pieces of knowledge relate to one another, and how the system functions dynamically. The basic plan is, roughly, to develop a sense for the grain size of knowledge elements and of their rough individual properties, but then the real business is to describe the system properties of these elements. How "densely" are the elements interrelated? Are they tightly interconnected and used almost always in contexts of the same other elements, such as elements of a skill that are activated only in patterned sequences of that skill deployment? Or are they very loosely interconnected and fluid in their composition in particular thought contexts? Can we describe the functions of the particular system at issue and how they join with other systems to perform more complex functions? Are there mechanisms that produce levels of systematicity other than those that have to do with performance? For example, do some core set of ideas in some sense derive the rest, though derivation is not the usual mode of operation of the system?

I've developed two exemplars of knowledge system analysis. The first is my analysis of intuitive ideas in physics.[12] Roughly, my claims are the following. Intuitions about causal mechanism reside in a large system of fairly simple elements that are only loosely connected. The function of the system is to provide judgments of how adequate a description explains why one should expect a particular thing to happen. The elements are configurations of circumstances that "just happen" and need no further explanation. Trying to figure out how a physical system works or what will happen is trying to find an optimal description of the situation in terms of these causally primitive elements, and one that best matches the conditions under which each of the elements is understood to apply.

This knowledge system does "judgment." It does not solve problems per se, or even specify very much about how an individual improves his current best decomposition of a problem situation into causal primitives. As for levels of systematicity, the system is mostly ad hoc, consisting of individual abstractions that are particular to some class of situations and just don't apply to others. Typically, only a few primitives apply to a problem situation, and connections of the elements are also mostly ad hoc, determined by the situation instead of general patterns of use of multiple elements.

On the other hand, there are some higher level systematicities that are useful to know. There are a few families of primitives that share a "base vocabulary" of descriptive terms. In some cases, a family of causal primitives share a central common abstraction, for example, one abstracted from agentive interaction: a "willful" (in some sense) agent, a patient, and a legitimized, but always directed "influence type." Pushes and pulls are canonical examples. Some of these families are important in identifying problems in learning, such as the need to undermine an entire class of primitives and support a new class.

This knowledge system analysis has educational implications. The principal one is that conceptual change is a system issue. It is hopeless to believe you have found the core of intuitive "misconceptions" and can argue the core away for students, leaving the conceptual field free for new conceptions. Instead, the whole problem must be conceived as an elaborate reorganization (not replacement). One must attend to system issues in learning, not just "one-at-a-time concept learning." In addition, knowing the existing intuitive primitives constitutes knowing the basic resources that must be reorganized, and establishes particular targets of difficulty, but also opportunities to build on some particularly apt corners of the naive system. "Engineering" is an appropriate metaphor for instructional design, since the richness, generativity and diversity of the naive system means there will likely be many opportunities and possibilities, no one "right way to construct the new system."

The knowledge system of causal judgments I have described is really a system of problematic descriptions. They are problematic because they prescribe the "deep causal structure" of a situation, which may frequently not be immediately evident. On the other hand, people also have "strong and reliable" descriptive capabilities, for example, in the area of spatial organization, and possibly dynamic spatial configurations. This is a different kind of system that may be the intuitive base of more mathematical ideas rather than physical ones. It is one I intend to study in future work.[13]

The second area in which I have developed a knowledge system analysis concerns understanding complex computational artifacts -- programming languages. In this context I claim to have

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12. A quick sketch of these ideas is available in diSessa, (1983) or diSessa (1988). A thorough treatment will appear in diSessa (in press-a).

13. See diSessa (1989) for some very preliminary results concerning dynamic spatial reasoning.

developed a short taxonomy of systems, which I describe as types of mental models, that have complementary structures, strengths and weaknesses, different learning trajectories, and to some extent also complementary functions. Learning a programming language is viewed as building and articulating properly all of these systems. Designing a comprehensible system is creating one that has good properties with respect to all of these systems.

In this area of mental models, I believe it is important to understand not only the structure of the systems involved, but their properties in several different modes in which they may be used. That is to say, the system may be complicated enough that it may configure itself in several rather different patterns.[14]

Most recently, I have tried to extend knowledge system analysis into a general view of the *evolution* of knowledge systems. I've tried to define a general scheme of causality by which one system may transform into a different one. This work is, at present, very speculative. While it might prove to be very general and possibly powerful theoretically, connections to empirical work are weak. In contrast to the work with intuitive physics and mental models of computational systems where the knowledge system analysis followed as a systematizing phase of a "characterize, systematize, re-examine loop" (see text, in the section on Cultivating a Theoretical Turn of Mind), I am attempting this work more top down. Thus, I've tried to "build the theoretical system" first, to some extent, rather than doing a more bottom up first pass through data relating to an approachable example.[15]

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14. diSessa (1986) gives a brief introduction to this kind of analysis. diSessa (1991-b) gives a thorough treatment.

15. diSessa (in press-b) presents the program describe briefly here. diSessa (1991-b) tries to bring it a step closer to empirical development and test.

Getting Ahead: With Theories*

I Have a Theory About This

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> "Don't paint the material of the sleeve. Become the arm! Get your love into it." (Newell Wyeth to his daughter, Carolyn. In [Meryman, 1991, p. 100].)

Andy has developed a formidable challenge. He wishes mathematics and science educators to develop a predilection toward theory and theory building. To understand what Andy means by this is not simple. I suspect that the more familiar you become with his work the deeper will be your appreciation of what Andy has in mind. I encourage all to become concretely intimate with Andy's point of view. There is much to be gained.

When Andy spoke of theories he referred to theories in the social sciences. I feel uncomfortable speaking so generally, so I will confine myself to theories of learning mathematics.[1] This is not overly restrictive if we take broad views of learning and of mathematics. To learn mathematics is to learn ways of reasoning, so we automatically include mathematical reasoning. Children do not learn mathematics in isolation of a social context, so automatically we include teachers and teaching. Teachers learn (and often re-learn) the mathematics they teach, so automatically we include teachers' learning. Explication is part of mathematical reasoning, so automatically we include communication, and thus we include teaching. This is the context in which I frame my remarks.

I will address three questions in discussing Andy's paper: 1) What is theory for? 2) What is theory about? and 3) When is theory useful? In many respects these questions cut across the issues Andy raised instead of building on them. My defense is that I hope addressing them increases the dimension of the discourse instead of being irrelevant to it.

I want to make clear that my first paragraph is more than laudatory. It opens a theme I want to develop. Andy's ideas about theorizing in mathematics and physics education stem from his strong, personal image of doing mathematics and physics creatively. On one hand

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[1] From here on I will say "mathematics" when I in fact I mean "mathematics and science".

this is hardly surprising. *Anyone's* theorizing about understanding stems from their images of what understanding is like. On the other hand, Andy's thinking about the goals of theory are textured by his highly principled knowledge of mathematics and physics.[2] I admit that technical knowledge of a subject matter is insufficient to guarantee insight into matters of understanding. But I do not hesitate to claim that studying a subject deeply and conceptually[3] provides an experiential basis for studying what it means to understand. If Andy's wish is realized, I suspect that the theorizing he envisions will be done by people who have built deep conceptualizations of the subject matter of which the theories pertain.

Finally, I will follow one of Andy's heuristics: Take a line and push it until it breaks. I will state my thinking about theory and theory building forcibly and await the crash of hammers.

What is theory for?

We are in the business of improving people's learning of mathematics. We focus sometimes on the learner, sometimes on the teacher, and sometimes on both. But our ultimate aim is to improve learning. This is the activity from which we draw our problems. It is an article of faith that insightful solutions to problems begin with understanding the problem. Principled understandings are the most productive, for they allow us to solve problems larger than the one we faced. We become theoreticians once we orient ourselves to developing principled understandings of learning and understanding.

Here I make my radical constructivism explicit. When we theorize about mathematics learning and understanding, our theories must aim to account for mathematics learning and understanding—*including our own (mine and yours, whether pedagogue, researcher, or practicing mathematician).* If they apply only to children, then the mathematics of our theories is impoverished, and is probably the mathematics of schools (at least as they exist now). Skemp (1979) made the observation that his model of intelligence was more powerful than Skinner's behaviorism, for it had the potential to account for Skinner's and Skemp's activities as theoreticians, whereas Skinner's behaviorism did not. Children grow up. They become adults. They become us. We are *never* blank slates, and our theories must be sensitive to this. Here I address the mathematics education community. Our school mathematics curriculum is conceptually incoherent, and so is mathematics instruction in the majority of school classrooms. A minority of students do learn something of value, but it is

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[2] I especially encourage you to read Abelson and diSessa (1981). Here you will not see theorizing on mathematics or physics learning. Rather, you will gain insights into Andy's image of *doing* mathematics and *doing* physics.

[3] Here I must remain vague. By "studying a subject conceptually" I mean at least that one comes to envision techniques, conventions, and methods in relation to goals and motivations.

not because of any systematicity in the curriculum. A practical aim of our theory-building is to re-conceptualize the curriculum so that it is at least conceivable that someone can learn it. To re-conceptualize the curriculum, however, we need to have principled understandings of the learning we wish to happen in the children experiencing it.

Andy doesn't say so, but in reading his work it seems evident that he operates under the constraint that adult science must be explainable as an outgrowth of children's science. He operates under a strong constraint of coherence in his theorizing about learning physics. We must also operate under the constraint that our mini-theories (to use Andy's phrase) of learning mathematics be coherent with each other and with what we personally understand about mathematics. If we make this coherence operative in our theorizing, we might make disconfirmable theories.

What is theory about?

Andy alluded to Alan Newell's article "You can't play 20 questions with nature and win" (Newell, 1973b) when speaking of the necessity of theories. In that same year Newell published an article on distinctions between process and structure (Newell, 1973a), noting that whether we consider something to be process or structure depends on our grain of analysis.[4] In this same regard, the texture of our theories of mathematics learning will be dependent upon our grain of analysis. Our grain of analysis will be influenced heavily by two considerations: the learning we wish to explain and the community with which we wish to communicate. Learning as a neurological phenomenon is at one extreme; learning as exhibited behavior is at the other. The chasm between gives ample room for widely varying grains of analysis. I won't pretend to know why, in principle, anyone might choose a particular grain, but I suspect it has something to do with the community to which we make a commitment. If we commit ourselves to a community that values detailed functional explanations, then we should find value in Andy's orientation to computational theories. If we commit ourselves to a community that values imagery and metaphor, then Andy's orientation might feel too constraining. If we commit ourselves to a community that values immediate, practical action, then Andy's orientation might seem irrelevant.

What a theory of learning is about is also dependent on our vision of what is to be learned. If we think that mathematics is applying rules for making marks on paper, then we will end up with Buggy-like theories of learning (Brown & Burton, 1978; Brown & VanLehn, 1981; Lewis, 1981). I have said enough about the small educational value of

[4] For example, from one perspective teeth are structures; from another perspective teeth are processes of calcium formation. The two views differ by whether we take time into account. Even then, if we take time into account our unit of measure will affect how we think of teeth.

such theories (Thompson, 1989). I must say, however, that there is a cultural heritage in the United States of which we must become reflectively aware, and this is the heritage that elementary mathematics is ultimately about calculating. Our theories of mathematics learning will be hamstrung if we incorporate this heritage unthinkingly.

Finally, I take issue with one matter in Andy's paper. Theory-building in the physical sciences differs categorically from theory-building in education, and the difference has implications for how we respond to Andy's call for determining "what we know for sure." Physicists don't ever suspect that nature acts intentionally. Mathematics educators almost always assume learners act intentionally. We could say that intention is a natural characteristic of self-regulating systems, and thus kids differ from atoms only in their magnitude of complexity. We could, but it doesn't help. I see no way to theorize about learning without somehow framing the activity within personal experience. The trick is to reflect on where personal experience frames one's theories. Andy's suggestion to try finding why and when our propositions are true and false seems a promising mechanism for such reflection.

When is theory useful?

Andy alluded to how we often hear "theory" denigrated as if it has nothing to do with practical life. This may be most true of school teachers and undergraduate education majors, and it may be true of a larger number of our colleagues than we would like to admit. I have asked more than a few generalists who teach *Theories of Learning* courses to prospective teachers if they (the generalists) could teach algebra, or calculus, or differential equations given what they know about learning. "Algebra, perhaps, but not calculus and what is differential equations?" The teacher must rely on personal expertise in the subject. But what of the students sitting in this course, who do not have subject-matter expertise? Can we expect them to have a high sense of relevance of the course's content for their future lives as mathematics teachers?

In many respects I fail to see how theory can be useful to one who views "theory" as something out there, to be studied as an object in and of itself. If theory is to be productive for you, it must be *your* theory. This does not mean that you must construct it from scratch, or in absence of conversation. It means that the principles by which you observe and reflect are of necessity your principles. They cannot be propositions outside of your thinking. The distinction I have in mind is the same as the distinction between simile and metaphor. To think 'simile-ly' you have two things in mind, relating them analogically. To think metaphorically, you have one thing in mind, and you see it having characteristics which under other circumstances you wouldn't see. Theoretical thinking is metaphorical.

Put differently, you have a theory when you assimilate the domain of interest to it. That's the way you see the world. *Useful* theory is "a light to the eye and a lamp to the feet ... an organ of personal illumination and liberation .... [it's value] consists in provision of *intellectual instrumentalities* to be used by an educator" (Dewey, 1929, p. 29).

Perhaps it is a matter of orientation as to what makes a theory useful. My orientation has been influenced by Les Steffe, who makes a strong distinction between mathematics *for* the learner and mathematics *of* the learner (Steffe, 1988). If one of our axioms is that we start where the learner is and build from there, then it follows that we must be able to think as if we were the learner. Thus, a theory of mathematics learning is useful to me when I can follow a paraphrase of Wyeth's exhortation: "Don't describe the child. Become the child!" This act of becoming, this attainment of coherent empathy, is only possible through theory. Without theory we are constrained to see children only as we *see* them; without theory we are constrained to hearing them only as we *hear* them. We can reflect on our mathematics to make it coherent, but without theory we cannot reflect on nor make sense of the coherence of children's mathematics. Reflective empathy is theoretical; theory building in mathematics education is the construction of reflective, analytic empathy.

Whence theory?

We sometimes hold the counterproductive view that theory comes from theoreticians. We all make theory. But of what do we make theory? Not from data, as Andy has already said. We have the freedom not only to build theory *of* practice, but to build theory *from* practice. Here I defer to John Dewey:

> The sources of educational science are any portions of ascertained knowledge that enter into the heart, head and hands of educators, and which, by entering in, render the performance of the educational function more enlightened, more humane, more truly educational than it was before. (Dewey, 1929, p. 76)

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Considering the "Pragmatical Consequences" of Constructivist Approaches:

Three Cases of "Constructivist" Mathematics Teaching

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Considering the "Pragmatical Consequences" of Constructivist Approaches:

Three Cases of "Constructivist" Mathematics Teaching

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Introduction

Although recent calls for reforming mathematics curriculum and teaching in the United States hold forth a fairly consistent vision of desired changes in mathematics instruction--less emphasis on practice of isolated computational skills, more emphasis on understanding, problem solving, and flexible mathematical reasoning--they fall considerably short of providing descriptions of what successful mathematics instruction might actually look like in elementary school classrooms (e.g., National Council of Teachers of Mathematics (NCTM), 1989; National Research Council, 1989). Teachers are being encouraged to shift their teaching from an approach based on "transmission of knowledge" to a student-centered practice featuring "stimulation of learning" (National Research Council, 1989a). The *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) describes the changes needed in instructional practices in elementary mathematics as involving *decreased emphasis on* rote practice, one answer and one method, written practice, and teaching by telling; and *increased emphasis on* use of manipulative materials, discussion of mathematics, justification of thinking, a problem-solving approach to instruction, and writing about mathematics. However, these desired changes might be enacted in multiple ways by teachers in their mathematics teaching practice. For example, in a recent survey of self-reported goals and practices of elementary mathematics teachers in three states in the United States, Peterson, Putnam, Vredevoogd, and Reineke (in press) found five completely different clusters of teachers, based on their patterns of responses regarding these instructional practices.

How and why are elementary mathematics teachers making certain called-for changes in their instructional practices and not others? Through case analyses of the thinking and practice of five elementary teachers, a group of us at Michigan State University have come to recognize the complexity and diversity of ways that teachers are interpreting ongoing mathematics education reform efforts (Ball, 1990; Cohen, 1990; Peterson, 1990; Wiemers, 1990; Wilson, 1990). We saw multiple ways in which teachers enacted in their practice the changes that were hoped-for by reformers and writers of the state-level *Mathematics Curriculum Framework* in California (California State Department, 1985), a document that extolls many of the same themes as the NCTM

Standards. In addition, our cases of teachers reveal the important roles that teachers' thinking, beliefs, and understandings play in how teachers interpret the called-for mathematics education reforms and how these reforms influence their practice. An important finding was that although some teachers had made slight surface level changes in their mathematics practice, such as using of cooperative learning, employing manipulatives, and incorporating word problems into their teaching, none of the teachers had changed or even reconsidered their views of how children learn mathematics or what it means to "know" mathematics. These findings are particularly important in light of the fact that, although not always stated explicitly, most of the rhetoric of the current mathematics education reform has as its basis a shift in assumptions away from traditional views of learning and knowledge toward "constructivist" views of learning and non-traditional views of mathematical knowledge. Although obvious to most scholars in the mathematics education research community, such assumptions are not always apparent to teachers who have their own theories, beliefs, and frames within which they work. More often than not, in the United States elementary mathematics teachers tend to fall into one of two groups--either they have do not have a coherent, consistent view of mathematical learning and knowledge that is reflected in their practice, or their practice reflects an implicit behavioral view of learning and a "transmission" view of knowledge.

One problem is that teachers often see the "surface level" features of the reforms being advocated, such as the changes in instructional practices, without seeing the assumptions and theoretical frames of the persons who have constructed the hoped-for changes in instructional practices including the researcher, the reformer, the textbook writer, or the expert teacher. Thus, the teacher interprets the new instructional practices in terms of his or her own assumptions, beliefs, and understandings.

A second possibly more serious problem is that although much of the current mathematics reform rhetoric in the United States is couched in terms of "constructed knowledge" (National Research Council, 1990) or learners constructing their own mathematical understandings (National Research Council, 1989, pp. 58-59), visible differences continue to emerge between scholars in the mathematics education community with regard to what these terms mean and the underlying views of mathematics, knowledge, learning and teaching that are being promulgated (Sowder, 1989; Peterson and Fennema, 1991). In a recent study by our Center of university experts' and teacher experts' view of the ideal curriculum in each of six elementary subjects, we found that virtually every expert premised their statements on curriculum and teaching by asserting that they took a "constructivist" view of learning. Similarly, Bauersfeld (1991) has suggested that "The initial statement 'I am a constructivist' has become a kind of academic lip service" (p.3). If so, one might ask, Is "constructivism" destined to join the ranks of education reform terms such as "school restructuring" that , as Kirst (1988) pointed out, "means everything and nothing simultaneously" (p.7)?

A third problem that is exacerbated by the first two is what Bauersfeld (1991) has called the "pragmatical consequences" of constructivist approaches for mathematics education. What would a constructivist approach to mathematics education look like in practice? In this paper we address this question by examining three actual cases of "constructivist" mathematics teaching in practice. Through examination of these cases, we attempt to identify similar themes as well as to pinpoint what seem to be important differences. We conclude by using these cases to raise questions about possible issues for indepth analyses and further discussion as well as points for departure and further exploration.

Three Cases of Constructivist Mathematics Teaching

We consider first, the case of teachers who have been involved for five years in an approach called Cognitively Guided Instruction (CGI). The approach derives from the findings of Thomas Carpenter and other researchers on how individual children construct mathematical knowledge (see, for example, Carpenter and Moser, 1983; Riley and Greeno, 1988) and from a constructivist view of learning that takes a cognitive perspective and focuses on the individual learner (Hiebert and Carpenter, in press). Then we move to consider the typical Japanese mathematics teacher as portrayed by researchers, James Stigler and Harold Stevenson who have spent more than a decade studying Japanese, Chinese and American elementary classrooms (see, for example, Stigler and Stevenson, 1991). Finally, we consider the case of Deborah Ball who, as a researcher and professor of teacher education at Michigan State University, attempts to reflect her own continuing developing knowledge of and thinking about mathematics and children's mathematics learning in her practice as a third-grade mathematics teacher, (see, for example, Ball, in press; Ball and Lampert, 1991).

The Expert CGI Teacher--A Cognitive, Individual Constructivist View

In a year-long experimental study, Thomas Carpenter, Elizabeth Fenneme, and I recruited forty-two experienced first-grade teachers to work with us. The teachers spent twenty hours per week for four weeks with us during the summer of 1986 learning about children's thinking in addition and subtraction. During the summer workshop, we shared with teachers a framework for addition/subtraction problem types and related children's solution strategies derived from Thomas Carpenter's findings from interviews with young children on their solving of addition/subtraction word problems (Carpenter and Moser, 1983). We pre- and posttested children in the CGI and control teachers' classes, and we observed these teachers' mathematics teaching on a regular basis during the 1986-87 school year. We also assessed the teachers' knowledge and beliefs about teaching mathematics both before the workshop and at the end of the school year We compared the instructional practices, beliefs, and knowledge of the CGI teachers and the learning of CGI students with the control group of teachers and their students. (For a complete description of results, see Carpenter, Fennema, Peterson, Chiang, and Loef, 1989; Peterson, Carpenter, and Fennema, 1989).

When compared to control teachers, the CGI teachers spent significantly more time on word problem solving in addition and subtraction, and they spent significantly less time drilling on addition and subtraction number facts. In comparison to control teachers, CGI teachers encouraged their students to solve problems in many different ways, listened more to their students' verbalizations of ways they solved problems, and CGI teachers knew more about their individual student's problem solving strategies. CGI students outperformed control students on written and interview measures of problem solving and number fact knowledge, including a measure of complex word problem solving on the Iowa Test of Basic Skills, and they reported greater understanding and confidence in their problem solving abilities. Although CGI teachers spent only half as much time as control teachers did in teaching number fact skills explicitly, CGI students demonstrated greater recall of number facts than did control students. Those teachers who believed more in the ideas of CGI and had more knowledge about their children listened more to their children's verbalizations of their thinking, and they implemented CGI more than did those teachers who had lesser knowledge and weaker beliefs. These latter teachers can be characterized as the more expert CGI teachers (i.e., the "case study teachers" described by Carpenter and Fennema, in press; and the Group 1 teachers described by Knapp and Peterson, 1991).

Although each CGI classroom is, in some sense, unique, Peterson, Fennema, and Carpenter (in press) have pointed out three key themes that all CGI classrooms have in common. First, problem solving is the focus of all CGI classrooms. Teachers carefully write or select problems to be appropriate for their children, and they have children construct their own problems and pose them to each other. Problems are constructed to be relevant to the children's real lives in school and out and to integrate mathematics with different subject areas including literature, science, and social studies. A second important element is that multiple solution strategies to problems are recognized, encouraged, and explored as children describe their solution strategies and make their thinking visible within the context of solving problems. A third key element element of CGI classrooms is that teachers have an expansive view of children's mathematical knowledge and thinking. CGI teachers believe that all children know something about mathematics and that as teachers, they need to figure out continually what children know about mathematics and then use this knowledge to plan and adapt their mathematics instruction.

Carpenter and Fennema (in press) provide an analysis of the cases of expert CGI teachers. An excerpt from their analysis is provided in Appendix A including an example of one teacher's (Ms. M's) discourse with a small group of children about a mathematics problem.

The "Typical" Japanese Mathematics Teacher--Constructing the Crafted, Polished Lesson

In summarizing what they have learned from their research over the last decade on elementary mathematics classrooms in Japan, China, Taiwan, and the United States, Stigler and Stevenson (1991) assert as important that "Asian teachers subscribe to what would be considered in the West to be a 'constructivist' view of learning. According to this view, knowledge is regarded as something that must be constructed by the child rather that as a set of facts and skills that can be imparted by the teacher" (Stigler and Stevenson, 1991, p. 20). Stigler and Stevenson describe the typical elementary mathematics classroom as characterized by several features. The lessons have a coherence and are typically organized around one or two interesting problems that the teacher poses to students and follows up throughout the lesson with provocative questions. Japanese teachers routinely make use of real-world problems and objects and concrete representations. During the course of classroom discussion, students construct multiple solutions to the problem posed by the teacher. Japanese teachers handle diversity of students' mathematical knowledge and abilities by making effective use of students' errors.

Stigler and Stevenson(1991) described a fifth-grade Japanese teacher's "effective use of errors" in introducing the problem of adding fractions with unequal denominators. This example also appears on the videotape entitled, "The Polished Stones" by Stevenson and Lee (1989). I have transcribed the videotaped excerpt from this Japanese teacher's classroom, and the excerpt is presented in Appendix B. In this excerpt, the teacher poses the problem to the whole class and writes it on the board: 1/3 + 1/2 = . Then the teacher calls on three different students one a time to give their solutions to the problem she has posed. Students raise their hands to respond. As each student is called upon by the teacher, he stands and states his solution to the problem. The teacher writes the solution on the board as the student states it. The following different solutions are proposed by three different students:

1/3 + 1/2 = 2/5
3.1 + 2.1 = 5.2
1/3 = 2/6 and 2/3 = 3/6 "The answer is five sixths."

Interestingly, here my interpretation of what happens on the videotape differs from the interpretation offered by Stigler and Stevenson (1991). In the tape the announcer's voice suggests that the teacher then calls on the first student to explain his solution. The student begins an explanation, then pauses looking puzzled. With little wait time and no further questioning of the student, the teacher launches into an explanation, which according to the announcer, was intended to clarify what was wrong with the first method given by this student. In contrast, Stigler and Stevenson (1991) gave the following interpretation:

> The teacher returned to the first solution. "How many of you think this solution is correct?" Most agreed that it was not. She used the opportunity to direct the children's attention to reasons why the solution was incorrect. "Which is larger, two-fifths or one-half?" The class agreed that it was one-half. "It is strange, isn't it, that you could add a number to one-half and get a number that is smaller than one-half?" She went on to explain how the procedure the child used would result in the odd situation where, when one-half was added to one-half, the answer yielded is one-half. In a similarly careful, interactive manner, she discussed how the second boy had confused fractions with decimals to come up with his surprising answer (Stigler and Stevenson, 1991, pp. 44-45).

In contrast, I failed to hear or see the teacher pose the above questions to the class nor did I observe the teacher attempt to determine whether or not the students agreed or disagreed with this solution. Questioning of the students and seeking to determine how they think about the solutions (in terms of their agreement or disagreement with other students' solutions) is a critical feature of expert CGI teachers' practices and also, as we shall see, of Deborah Ball's teaching. Similarly, on the tape the question, "Which is larger, two-fifths or one-half?" appears to be asked as a rhetorical question which the teacher then goes on to answer herself in her subsequent explanation. All in all, the overwhelming impression that one gets from viewing this segment is not so much one of an "interactive" classroom experience, but rather one of a smoothly orchestrated and planned, teacher-directed lesson in which the teacher plans to and does surface several student misconceptions through her questions so that she can proceed to use these to demonstrate what is wrong with certain methods and to show what is correct about the right method.

The Case of Deborah Ball--A Social Constructivist View

As a third-grade teacher who has taught for twelve years at an elementary school near Michigan State University, Deborah Loewenberg Ball aims to develop a "practice that respects both the integrity of mathematics as a discipline *and* of children as mathematical thinkers" (Ball, in press, p.3). Like her colleague, Magdalene Lampert (1990a; 1990b), Ball strives to create a classroom environment in which the norms of discourse are informed by patterns of discourse in the mathematics community as well as by the culture of the classroom. Further, she strives to shift authority for mathematical knowledge from the teacher and the "text" to the community of knowers and learners of mathematics in her classroom. While Ball and her students engage in extensive discourse in the whole-class setting, they also work in small groups. She tries to select and create mathematics tasks that engage students in learning the content of mathematics as they learn the ways of knowing. Ball (in press) provides an example of discourse from her third-grade mathematics class in which students discuss the problem: 6 + (-6). Ball and her students spent over thirty minutes discussing solutions for that problem. At one point, a student gave the correct answer, but the student's explanation was problematic. Students gave two other solutions that received "equal air time." Ball explains that she did not "tell or lead the students to conclude that 6 + (-6) equals zero--by pointing them at the commutativity of addition or at the need for the system of operations on integers to be sensibly consistent. Ball thinks that the time that students spend "unpacking ideas" is time well spent. Too often she has seen evidence of students who fail to understand even though they have been "taught" the mathematical procedure.

Like the expert CGI teachers, Deborah Ball's practice reflects a coherent point of view. However, while expert CGI teachers' practice reflects a cognitive, individual constructivist view, Deborah Ball's practice reflects a social constructivist perspective. Bauersfeld (1991) has argued that, taking a social constructivist perspective, the following would clearly exist in a teachers' classroom practice:

> 1. Periods in the classroom designed for self-organized problem solving, for small group work on "new" tasks, for eliciting children's inventions...There will be also intensive 'negotiation' of different ways and solutions, of how to come across different ideas, of argumentation and defending.
>
> 2. Polishing of the students' verbal production and taking care of adequate descriptions, drill and rehearsal, even under self-controlled time limits..careful furthering of the process of constructing itself, promoting reflection on just finished tasks, discussing alternatives...
>
> 3. Written tests, homework and "debugging" procedures related to results..taking 'mistakes' and 'errors' as necessary concomitant phenomena of an active participation and engaged construction--a positive sign for "being on the way"--rather than as accidents which have to become erased promptly.
>
> 4. The teacher's inescapable role of an expert, of an agent of the society. Teachers also have to be exemplary, a living

> model of the culture wanted with transparent modi of thinking, reflecting, and self-controlling...the teachers themselves have to 'live' the relevant norms (Bauersfeld, 1991, pp. 19-20).

In Deborah Ball's teaching, we see clearly these four themes, including invention, argumentation, active construction and reflection. For example, in the attached excerpt from Ball's own analyses of her thinking and practice, we can see how she strives to make transparent her own thinking and reflecting (See Appendix C). In her work Ball tries to provide some perspective on the "tensions" inherent in "constructivist"-based pedagogies. We see how Ms. Ball allows Shea and the class to pursue a new mathematical idea and subsequently invent a new number, but she also expresses the uncertainties and tensions she felt during the discussion. Through thoughtful orchestration of the classroom discourse, Deborah Ball facilitates Shea's invention of a new kind of number--which the class names "Shea numbers"--"numbers that have an odd number of groups of two" (Ball, in press; Ball, 1991). In reading through some history of mathematics after her class had discussed and invented this new kind of number, Deborah later discovered that mathematicians in ancient Greece had also discovered and played around with this same kind of number. Further, Ball (in press) reports that when she later gave her students a quiz on odd and even numbers, "the results were reassuring. Everyone was able to give a sound definition of odd numbers, and to correctly identify and justify even and odd numbers. And, interestingly, in a problem that involved placing some numbers into a string picture (Venn diagram), no one placed 90 (a Shea number) into the intersection between even and odd numbers. If they were confused about these classifications of number, the quizzes did not reveal it" (p. 25).

Comparisons Across These Cases of Constructivist Mathematics Teaching

What do we observe about the instructional practices in the three cases? Do we see any similarities? One striking similarity between the cases of the mathematics teaching of the expert CGI teachers and Deborah Ball is that, in each case, the teacher has a coherent view of mathematics learning that is reflected in her mathematics practice. But do typical Japanese teachers actually have a coherent constructivist point of view that is consistently reflected in their practice as Stigler and Stevenson (1991) suggest or, in fact, do wide variations exist in Japanese teachers' knowledge and beliefs about mathematics learning as well as in the ways in which these views are represented in practice? Indeed, we have found the latter to be the case for CGI teachers, and that is why we focus here only on those we refer to as "expert" CGI teachers (Peterson, et al., 1989; Knapp and Peterson, 1991). Yet if the typical Japanese elementary teacher does have a coherent view of learning and knowledge that is reflected in her practice, then in addition to the observed differences in instructional practices, such coherence may be the most important way that the typical Japanese teacher differs from the typical American teacher (cf., Richardson, 1990; Sosniak, Ethington, and Varelas, 1991; Peterson and McCarthey, 1991). In addition, if Japanese teachers do have a coherent view of learning and knowledge that is reflected in their practice, that coherence constitutes an important commonality that they share with the expert CGI teachers and with researcher/teacher, Deborah Ball.

Second, in each of the three cases we see three common features in the teachers' instructional practices. In contrast to traditional elementary mathematics teaching in the United States, the instructional practices in these cases show greater emphasis by the teacher on: (1) posing mathematical problems; (2) expecting and exploring a wide variety of students' solutions for mathematical problems; and (3) listening to students describe their thinking and problem solving processes. Similarly, in our initial year-long experimental study comparing CGI and control teachers, our behavioral observations of teachers' classrooms revealed these to be the significant features that distinguished CGI teachers instructional practices from control teachers' instructional practices. In addition, CGI teachers knew more about individual students' problem solving processes, and CGI teachers' students showed greater problem solving achievement than did control teachers' students (Carpenter, Fennema, Peterson, Chiang, and Loef, 1989). From these findings, we might infer that all three cases of constructivist teaching described in this paper embody these three instructional themes that have been found to be significantly related to the development of students' abilities to solve mathematical problems.

Do we see any differences? Indeed, getting clearer about the differences between the cases we have described may be more crucial to advancing researchers' knowledge and understandings of "constructivist" mathematics teaching than extolling the virtues or the similarities. One important difference lies in the perspective that the teacher takes on children's mathematical knowledge. Deborah Ball and the expert CGI teachers assume that children bring to a mathematics lesson significant mathematical knowledge and understanding, and the role of the teacher is to understand how children are thinking about a mathematical problem, and to build on, encourage, and facilitate that thinking. Thus, these teachers ask questions and probe students' thinking in order to figure out and make visible how students, individually and as a group, are making sense of a mathematics problem. The assumption is that students *are* making sense. In contrast, Asian teachers are portrayed as surfacing students' problem solving methods and listening to students' thinking on the assumption that students have major misconceptions that need t o be corrected. In the example of the addition of fractions with unlike denominators, the Japanese teacher herself attempted to corrected the two student "misconceptions" that she had uncovered by explaining the correct way and thereby reinforcing the correct solution given by the third student. In this way, tha Japanese teacher seemed striking reminiscent of our less expert CGI teachers. In a within group analysis of our CGI teachers, we examined the relationship between teachers' knowledge of their students' mathematical understanding to teachers' mathematics instruction and to their students' mathematics problem solving achievement (Peterson, Carpenter, and Fennema, 1989). We conducted correlational analyses of the data of twenty teachers supplemented by case analyses of the teacher whose students did best on problem solving and the teacher whose students did worst. We found that teachers who students had higher problem solving achievement were those who were more knowledgeable about their students' problem solving knowledge. Teachers with more knowledge about their students' mathematical understanding tended to question students about problem solving processes and listen to their responses, while teachers with less knowledge were more likely to explain problem solving processes to students or to merely observe students' solutions. From tha classroom practices of tha Asian teachers represented on tha "Polished Stones" videotape, we are given to think that the typical Asian elementary teacher is like tha less knowledgeable CGI teachers than the more knowledgeable ones in terms of their understanding of their individual children's mathematics understanding.

However, although Deborah Ball and the expert CGI teachers share a common "positive" view that children have mathematical knowledge and understanding rather than lack it, they differ in the extent to which teachers' knowledge of children's mathematical knowledge is specified and constrained. (See also, Lampert, 1988, for an analysis of this issue). Working within a research-based framework which they were given to interpret and think about children's addition/subtraction problem solving, CGI teachers are inclined to think within that framework for children's mathematical knowledge while Ball thinks within the frame she has developed for children's mathematical knowledge, and she continues to expand and develop her knowledge of children's mathematical understanding. Yet expert CGI teachers speak often about their continuous amazement at what their children can do, how their children think, and how their students are capable of solving complex mathematical problems that the teachers hadn't previously thought first graders could solve. As a result, soma CGI teachers hava ventured to give their first-grade students multiplication and division problems as a result of "learning" from their students, and as a result, they have expanded their knowledge of children's mathematical knowledge (Knapp and Peterson, 1991; Peterson, Fennema, and Carpenter, 1991).

A second important difference among the three cases lies in the teacher's view of mathematical knowledge. In her teaching, Ball strives to give her students a sense of the dynamic nature of the way mathematical knowledge develops, grows, and changes. She has her students, conjecture, experiment, invent and make arguments, justify and defend them. In contrast, expert CGI teachers have a more constrained and bounded view of mathematical knowledge. While expert CGI teachers challenge students to explain and justify their thinking, and they attempt to shift authority for what is "right" to the individual student himself or herself or to the students in the class, they affirm that in mathematics there are "right" answers and solutions. Expert CGI teachers would be unlikely to let students leave a particular day's class sessions without it having become apparent which of the children's solutions that were discussed are right or wrong. On a continuum moving from mathematical knowledge as changing/unbounded toward mathematical knowledge as fixed/constrained, the Japanese teachers seem to be furthest toward the fixed end with Deborah Ball more toward the changing/unbounded end, and the CGI teachers in the middle. Further, for Asian teachers not only does mathematics knowledge seem fixed, but the authority for knowing seems to still rest with the teacher and not with the student or students. The very metaphor of Asian teachers "polishing each lesson to perfection" like a polished stone implies the notion of knowledge as fixed and determined--like a stone--incapable of undergoing fundamental change, invention, or reconstruction. In contrast, constructivist teachers, Ball and Lampert, have used the metaphor of teaching a mathematics lesson as "traversing a territory" with an "eye to the mathematical horizon." In the Ball and Lampert metaphor, there exists no fixed path through the territory, and in a sense, each trip will be one in which tha travelers learn something new.

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Appendix A: The Expert CGI Teacher (as portrayed by Carpenter and Fennema (in press)

Note: The following was excerpted directly from Carpenter, T. C. & Fennema, E. (in press). Cognitively guided instruction: Building on the knowledge of students and teachers. In W.G. Secada (Ed.), Researching educational reform: The case of school mathematics in the United States. (Special issue of the International Journal of Educational Research). The words in italics are those of Carpenter and Fennema.

The following protocol of a teacher working with a group of five students illustrates how a CGI teacher gave children an opportunity to discuss alternative solutions.

Ms. M: "The African elephant ate 37 peanuts. The Indian elephant ate 43 peanuts. How many fewer peanuts did the African elephant eat than the Indian elephant.

The children worked on the problem for about two and a half minutes. Some of the children used stacking cubes that had been joined together in stacks of ten cubes. Others did not use any materials. After a minute and a half several of the children had raised their hands. After two minutes, only one child, Ubank, was still working on the problem. Ms. M asked him if he was done. When he shook his head, she told him to keep working. After another half minute, he raised his hand.

Ms. M: "Got it? How many fewer did the African elephant eat, Ubank?"

Ubank: "Six."

Ms. M: "Does everyone agree with that? . . . How did you figure it out, Ubank?"

Ubank: "Well, I had 43 here (pushing out 4 stacks of ten cubes and 3 additional cubes joined together), and I had 37 here (pushing out 3 stacks of ten cubes and a stack of 7). I put 30 on top of these 30. I took 3, and I put them here. There were 4 left, so I took 4 off, and there were 6 left." As he described what he did, he took 3 of the ten stacks from the collection of 43 and put them on top of the 3 ten stacks in the collection of 37. Then he took the 3 single cubes from the original set of 43 and put them on top of the 7 cubes in the set of 37. Then he took the remaining stack of ten cubes from the original 43 and broke off 4 cubes. He put these 4 cubes on the 4 cubes in the set of 37 that were not covered. He was left with 6 cubes from the set of 43 that did not match up with cubes in the set of 37.

Ms. M: "Did he do it a good way? . . . Did anyone do it a different way?"

Merci: "I took 37, and I needed 43. So I counted up 3 more. That was 40. Then I took 3 more to 43."

Ms. M: "Good. Does her way work well? . . . (Children nod in agreement.) It sure does. Did anybody do it differently?"

Linda: "Well first I got 37. Then I got 43 (pushes out collections of 37 and 43 cubes joined together in stacks of ten, with the extra cubes also connected together). See, I know it couldn't be 10, because if you had 10 it would be 47 instead of 43. So I realized that it had to be less than 10. So what I did was I imagined 3 more cubes here (points to the top of the stack of 7 cubes in the set of 37), and I imagined 3 more right here (pointing to a space next to the collection of 37 that corresponds to where the 3 cubes are in the collection of 43)."

Ms. M gave each child in the group time to complete the problem, and she gave each child who had a different solution an opportunity to explain his or her solution. The children all listened attentively to other children's solutions, so the children had the chance to learn from each other. Ms. M also learned what each child could do, and she learned more than whether a child got the correct answer. The different solution strategies reflected quite different levels of understanding. Ubank had to model the problem directly, whereas the solutions of Marci and Linda reflect more flexibility in operating with numbers. While the children were working on the problem, Ms. M made notes about the solution processes she observed. This is how Ms. M gains information about her students. In this classroom assessment is an ongoing part of instruction.

Because the research base the teachers (including Ms. M.) studied provided a coherent framework for organizing problems and the processes that children use to solve them, the teachers had a rationale for selecting problems and a context for interpreting the students' responses. Consequently they knew what questions to ask and what to listen for. They could attend to important variations in students responses and did not have to keep track of a vast array of unrelated details.

Appendix B: A "Typical" Asian Teacher's Mathematics lesson (from Stevenson and Lee, 1989)

Note: The following excerpt was transcribed verbatim by Penelope Peterson from the videotape entitled, " The Polished Stones: Mathematics Achievement Among Chinese and Japanese Elementary School Students" by Harold W. Stevenson and Shin-ying Lee, 1989. The words in italics were added by me to provide additional information on the context as it appears on the videotape.

The setting: Students are sitting at desks lined up in rows and columns facing a blackboard at the front of the room. During this whole-class lesson, the teacher calls on students one a a time to give their solutions to the problem she has posed. Students raise their hands to respond. As each student is called upon by the teacher, he stands and states his solution to the problem. The teacher writes the solution on the board as the student states it.

Announcer: Rather than using errors as an index of failure, errors are used as an indication of a need for more understanding and practice. In this classroom, students are working for the first time with fractions that have different denominators. In the course of solving one problem, the students suggest several incorrect methods. The teacher puts these errors to good use in clarifying the meaning of fractions. She presents the problem: to add one half and one third. *The teacher has written on the board: 1/3 + 1/2 =* .

First boy: One third plus one half equals two fifths. *The teacher writes: 1/3 + 1/2 = 2/5 under the problem that she has written on the board.*

Second boy: Three point one plus two point one equals five point two. *The teachers writes: 3.1 + 2.1 = 5.2 under the first solution.*

Teacher: Please listen to him until he finishes.

Second boy: If I change it into a fraction, it's two fifths. *The teacher writes: = 2/5 after 5.2 in the student's solution that she has written on the board.*

Teacher: Now I understand how you get this answer. OK, how about someone else who solved it in a different way? *The teacher calls on a third boy by name.*

Third boy: I reduced the numbers to the least common denominator--six. One third equals two sixths; one half equals three sixths. The answer is five sixths. *The teacher writes 1/3 = 2/6 and 2/3 = 3/6 on the board as the third solution.*

Announcer: The teacher points out that they now have three ways to solve the problem. She asks for an explanation of the first method.

First boy: One third means one whole is divided into three parts. One half means that something is divided into two parts. The denominator is different so if you add....*The boy pauses.*

Announcer: Seeing that the child can't explain the method, the teacher clarifies what was wrong:

Teacher: Which is bigger--one half or two fifths? *As she asks this, the teacher circles 1/2 and 2/5 in the first solution on the board.* One half, isn't it? Even so, some of you added one third and one half and mysteriously got an answer that was smaller. Let me explain. *She*

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writes on the board: 1/2 + 1/2 = . Two plus two equals four. *Teacher writes 4 in the denominator of the sum.* One plus one equals two. You get two fourths. *Teacher writes 2 in the numerator of the sum so she now has 1/2 + 1/2 = 2/4 written on the board.* Now lets reduce this. *Teacher crosses out 2/4 and writes = 1/2.* One half--is that correct? Can I use this equals sign here? No, that's wrong. Takihitokun, you're a little mixed up here. You confused three point one and one third, didn't you? *She writes 1/3 = 3.1 on the board.* Well, *(she calls the student by name)*, one third is a part of one whole, and three point one means that there would be three whole things so the second solution is confused, isn't it? I will find the least common denominator. After you get the least common denominator of three and two, you must multiply the numerator by the same number as the denominator so that you get two sixths and three sixths. *She writes 1/3 + 1/2 = /6 + / 6.* If you add, you get five sixths. *She fills in 2/6 + 3/6 = 5/6 as her solution.* And now, you've successfully solved the problem.

Announcer: By allowing students to come up with their own solutions and then having the students explain them, a teacher can clarify some common misunderstandings and show why these solutions will not work. *The announcer concludes by referring to these "interactive classroom experiences" as one of the reasons why Japanese and Chinese students achieve at higher levels.*

Appendix C: The Case of Deborah Ball's Teaching (as portrayed by Ball, 1991)

Note: The following was taken from Ball, D. L. (1991, May). Materials from presentation at the annual meeting of the National Council of Teachers of Mathematics (NCTM), New Orleans, LA. The words in italics are Ball's description of her thinking and her teaching.

At the beginning of class on this particular day, I was trying to have a brief discussion with the students about a meeting we had had the day before. After this I was going to have them work a little more in their groups on the conjecture about adding even and odd numbers (such as the ones I told you about above)--and then, hopefully, begin having some discussion about some of these. The point at which the tape begins, class has been underway for about 7 minutes. A boy named Benny has just made the observation that even numbers can be "made" from two other even numbers--e.g., 8 can be "made from" 4 + 4; 12 can be made from 6 + 6. The segment opens with my asking if anyone has any other comments and I call on a boy named Shea, who figures prominently in the events of this particular day.

I think it is important for you to know that Shea is a student who, on some days, seemed to be totally tuned out. On some days, he would write nothing in his notebook, say little or nothing in class, and would not work in a small group. Sometimes he say under his desk instead of at it. But Shea was unpredictable. Sometimes when he seemed most tuned out, he would suddenly burst into a discussion with an important point. Shea, despite this, was making reasonably good progress in mathematics. Buy my ongoing concern for finding ways to engage him productively was undoubtedly a factor in my thinking on this particular day.

So, you will see, when I call on Shea, he says he has no comments about the meeting we had yesterday, but he has noticed something special about the number six. He claims that it could be even or it could be odd. The segment you will see centers on my efforts to understand what he is thinking and my struggles in deciding what to do.

Lin	*I* think I know what he's saying.
Kip	Which is *even*, Shea.
Ball	Lin? *(to Shea)* Could you stay there? People have some questions for you.
Lin	I think what he is saying is that it's almost, see, I think what he's saying is that you have three groups of two. And three is a odd number so six can be an odd number *and* a even number.
Ball	Do other people agree with that? *Is* that what you're saying Shea?
Shea	Yeah.
Ball	Okay, do other people agree with him? *(pause)* Lin, you disagree with that?
Lin	Yeah, I disagree with that because it's not according to like . . . here, can I show it on the board?
Ball	Um hm.
Lin	*(She comes up to the board.)* It's not according to like. . .
Ball	Rania, can you watch what Lin's doing?
Lin	. . . how many groups it is. Let's say that I have *(pauses)* Let's see. If you call six an odd number, why don't *(pause)* let's see *(pause)* let's see--ten. One, two . . . *(draws circles on board)* and here are ten circles. And then you would split them, let's say I wanted to split, spit them, split them by twos. . . One, two, three, four, five . . . *(she draws)*  then why do you not call *ten* a, like-
Shea	I disagree with myself.
Lin	. . . a, an odd nu- ber and an even number, or why don't you call *other* numbers an odd number and an even number?
Shea	I didn't think of it that way. Thank you for bringing it up, *so* --I say it's--ten can be an odd and an even.
Lin	Yeah, but what about ...
Liz	Ohhh!!
Lin	What about *other* numbers?! Like, if you keep on going on like that and you say that other numbers are odd and even, maybe we'll end it up with *all* numbers are odd and even. Then it won't make sense that all numbers should be odd and even, because if all numbers were odd *and* even, we wouldn't be even having this *discussion!*
	(1:15:40)

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At this point I thought that Shea was just confused about the definition for even numbers. I thought that if we just reviewed that, he would see that six fit the definition and was therefore even. I assumed that after this we would be able to get on with our discussion.

Within a couple of minutes, we had settled on a definition of even numbers. Jillian said:

> If you have a number that you can split up evenly without having to split one in half, then it's an even number.

So I turned to Shea in order to make the connection and clarify things:

Ball Can you do that with six, Shea? Can you split six in half without having to use halves?

Shea Yeah.

Ball So then it would fit our working definition, the-¹ it would be even. Okay?

There was a pause.

Shea *And* it could be odd. Three twos could make it.

Ball Okay. One of the points here is that if it's fits the definition then we would call it even. If it fits our working definition, then we would call it even.

Shea It fits the definition for odd, too.

I began to see that the issue was more complicated than I had thought.

Ball What is the definition for odd? Maybe we need to talk about that?

We discussed a definition for odd numbers--before this we had had an explicit definition for even numbers only. I had assumed that this was sufficient. I think now that I was wrong. So we agreed that odd numbers were numbers that you could not split up fairly into two groups. But this still did not satisfy Shea yet. He persisted with the observation he had made about what made six special:

(1:11:24)

Shea You could split six fairly, *and* you can split six not fairly. You can like cut six in half, um .... there's like, say there's two of you and you had, and you had, um, six cookies and you didn't want to split it in *half* and so that each person would get three and you wanted to split it by twos. Each person would get um, two and there would be two left.

Ball For which number now? For six?

Shea Uh huh.

Ball So, are you saying *all* numbers are odd then?

Shea No, I'm not saying all numbers are odd, but . . .

Ball Which numbers are *not* odd then?

Shea Um. . . Two, four, six . . . um, six can be odd or even . . . eight

Students *No*. . . I

Kip I don't know how. Show us.

Shea *Because* there's three twos. One, two. Three, four. Five, six.

Kip Prove it to us that it can be odd. *Prove* it to us.

Shea Okay. *(He rises and comes up to the board.)*

Ball Does everybody understand what Shea's trying to argue? He's saying six could be even or it could be odd.

Students I disagree . . .I don't think so . . .

Ball Well, watch what he's going to prove and then you can ask him a question about it.

Shea Well, see, there's two, *(he draws)* number two over here, put that there. Put this here. There's two, two, and two. And that would make six.

Kip I know, which is even!

(1:06:31)

Ball More comments about the meeting? I'd really like to hear from as many people as possible what comments you had or reactions you had to being in that meeting yesterday. Shea?

Shea Um, I don't have anything about the meeting yesterday, but I was just thinking about six, that it's a . . . I'm just thinking. I'm just thinking it can be an odd number, too, 'cause there could be two, four, six, and two, three twos, that'd make six ...

Ball Uh-huh . . .

Shea And two *threes*, that it could be an odd and an *even* number. Both! *Three* things to make it and there could be *two* things to make it.

Ball And the two things that you put together to make it were odd, right? Three and three are each odd?

Shea Uh huh, and the other, the twos were even.

Ball So you're kind of--I think Benny said then that he wasn't talking about *every* even number, right, Benny? Were you saying that? Some of the even numbers, like six, are made up of two odds, like you just suggested.

Other people's comments?

(1:06:33)

I was assuming that this clarified things and showed that a number was not both even and odd. I interpreted that Shea was saying something that connected with what Benny had said a few minutes earlier when he pointed out that two even numbers combine to make other even numbers. I thought that Shea's point was that two odd numbers could also make an even number. I assumed that we could now move on with our discussion of the meeting and then move from there to what we were supposed to be working on today. But I was wrong. The class pursued Shea's point:

(1:06:38)

Ball Tina?

Tina I disagree with Shea when he says that six can be an odd number. I think six can't be an odd number because . . . look--*(she gets up and comes up to the board)*

Ball *(Interrupting)* Jillen, Bernadette?

Tina Six can be an odd number because this is *(she points to the number line, starting with zero)* even, odd, even, odd, even, odd, even, how can it be an odd number because *(startingwith zero again)* that's odd, even, odd, even, odd, even, odd. Because zero's not a odd *number.*

Shea Because six, because there can be three of something to make six, and three of something is like *odd,* like see, um, you can make two, four, six . Three twos to make that and two threes make it .

Kevin But that doesn't--

Ball Kevin?

Kevin That doesn't necessarily mean that six is *odd.*

Students Yeah.

Ball Why not, Kevin?

Kevin Just because two odd numbers add up to an even number doesn't mean it has to be odd.

Ball What's the definition--Shea?-- what's our working--

Shea Two odd numbers make--

Ball Shea? What's our working definition of an even number? Do you remember from the other day the working definition we're using? What is it?

Shea It's, um, that *(pause)* . . . I forgot.

Ball Could somebody help us out with this? Because we need in the group to have an idea that we're working with. What's the working definition we're using? *(pause)* Do other people know it besides Liz and Shekira? *(pause)* I think other people do. Maria, do you know what the definition is that we've been using for an even number?

(1:09:18)

INTEGRATING THEORIES FOR MATHEMATICS EDUCATION

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Author's Note

The present text is a preliminary version; the formatted view misleads. At many spots I have to give more details, replace examples, some paragraphs will have to become erased etc. Anyway, it gives an idea...

1. An actual historical perspective

> "It is not at all true that concepts, even when constructed according to the rules of science, get their authority uniquely from their objective value. It is not enough that they be true to get believed. If they are not in harmony with the other beliefs and opinions, or, in a word, with the mass of the other collective representations (the concepts taken for granted by most people in a given time and place), they will be denied; minds will be closed to them; consequently it will be as though they did not exist."
> E.Durkheim 1912, engl. translation 1965, p.486[1]

It is quite surprising that the growing difficulties with computer simulations of human communication and cognition combined with the strong actual interest in adequate solutions have led to the development of new and challenging models for these processes. Obviously, computer science, education, and philosophical discussions are nearer to each other than ever before Even more interesting I find the relative convergency of these technologically oriented approaches with a few older and more developed theoretical approaches from different disciplines, where thery have been formed mostly aside of the mainstreams, e.g (pragmatic) linguistics, (radical) constructivism, ethnomethodology, social (or earlier symbolic) interactionism, history and theory of sciences, and last not least new perspectives on mathematics itself (see the very detailed overview in [Ernest 1991])

Limited to "the contributing fields to the Science and Technology of Cognition - STC," Francesco Varela has recently described this development as three successive waves ([Varela 1990], p 26/8)[2],.moving

* from "representation" and "symbol processing", where symbol processing is both based upon *sequential* rules and is *located* within the system, "so that the loss or malfunction of a part of the symbols or rules of the system results in a serious tmalfunction" (p 56/27)--"the cognitivist paradigm" (p.27/9)
* towards "emergence alternatives to symbol manipulation" (p.27/9), where meaning is with the function of the whole state of a network rather than

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1 The quote is taken from [Leary 1990], p.359

2 Though the original is written in English, "Cognitive Science - A Cartography of Current Ideas", the text obviously has not been published in English yet. My quotations, e g. p 26/8, refer to both the German translation (p.26) and to the English manuscript (p.8). Varela announces in the English manuscript the publication as "Les Sciences Cognitives: Tendences et Perspective Actueles" with Editions du Seuil, Paris

localizable in certain symbols--the "connectionist paradigm" (p.79/41) or "subsymbolic paradigm" ([Smolensky 1988], quoted on p.79/41)
* towards "enaction: alternatives to representation" (p.27/9), where the phenomenon of interpretation is "understood as the circular activity linking action and knowledge, knower and known in an indissociable circle. ... with the dominance of usage, instead of representations"--the "enactive approach" ([Varela 1990], p.91/49).

Varela has organized his overview in a "polarized map" (p.119/66) in which each following wave includes the preceding one like a set of "Chinese boxes" (see *Figure 1*). "The centrifugal direction is a progressive bracketting of what seems stable and regular," or "one can go from enaction to a standard connectionist view by assuming given regularities of the domain where the system operates." Whilst in the centripetal direction .one goes from emerge to symbolic by working with symbols at face value and bracketing the base from which symbols emerge." Varela insists, the notions in the table "should not be seen as logical (or dialectical) opposites. They represent more the particular and the general, the local and the more encompassing category." (p.120/66)

insert here about *Figure 1*

From a philosophical perspective Richard Rorty has recently[3] pointed at the drastic change which the idea of language and the potential use of it has undergone, from treating language as an limited object and words as carriers of meaning towards a pragmatic "bottomless" stance, a view from which has been abandoned

- the idea of language "as a clear and common structure which users internalize and apply to single cases" (Donald Davidson, quoted by [Rorty, 1991], p.69); it was this idea which Wittgenstein had in mind in his preface of the "Tractatus": "What can be said at all, can be said clearly." ([Wittgenstein 1972], p.3), and which he was to abandon in his later "Philosophical Investigations" [Wittgenstein 1974].

- the concept of "meaning," as there is no chance for to use language as an instrument for transcendental and objective reductions; Rorty quotes

[3] At the symposium on the 100th anniversary of Wittgensteins death at the Universität Frankfurt/Main 1989 Richard Rorty has hold a lecture on "Wittgenstein, Heidegger und die Hypostasierung der Sprache." Published in the German translation only [Rorty 1991]. My quotations are taken from the English original.

Hacking's notion of "the death of the concept of meaning" [Hacking 1975], though he himself prefers to speak of a "naturalization of semantics" ([Rorty 1991], p.69 and 74), pointing at the specifity of "meaning" to the actual situation, and

– the discrimination between "schema" (or "form") and "content"; Varela has called this invention "the bright idea which has created the cognitive paradigm." ([Varela 1990], p.78)--because a manifold of forms of experiences or of forms of consciousness do not appear to be much different from a manifold of realities " ([Rorty 1991], p.74).

On the other side Rorty accepts that

+ "whether a sentence makes sense or not depends upon the truth of another sentence, one about the societal practice of people;" in other words. "Language is not a limited whole with presentable borders, it is merely a boundlessly expandable set of societal practices" ([Rorty 1991], p 79/80), and related to language use he adds "every exactness, in particular, has a social practice as prerequisite" ([Rorty, 1991], p 85)

+ there is access to something 'given' only via something ready at hand, in Heidegger's notion: "das Vorhandene ist nur über Zuhandenes zugänglich", (in order to see or to 'realize' something one has to have a more or less developed expectation of what one might come to see; compare [Rorty 1991], p 85), a conviction which the late Wittgenstein has shared[4]

We come to the core point: These positions indeed are very near to radical constructivist' statements as well as to pragmatic linguists' or social interactionists' theses and to findings from Discourse Analysis. One can not expect to identify clear border lines for the area of convergency at this level of abstractness. But it appears to be possible to enlist a few shared core convictions in this area (The descriptors will present a mixture, just because it is impossible to describe the defecitary parts of an approach with the specific "language game" of this very same approach.)

[4] Rorty discusses a striking similarity between the pragmatism of the young Heidegger ("Time and Being") and the late Wittgenstein ("Philos Investigations")--"there are no final analyses of and through language"-- and, the other way round, between the early Wittgenstein's and the late Heidegger's (over)estimation of language Whereagainst their related ways of theoretical development have taken opposite directions and have crossed half way inbetween.

1.1. *Learning* is a process of personal life forming, a process of an interactive adapting to a culture through active participation (which in parallel also produces and develops the culture itself) rather than a transmissal of norms, knowledge and objectifyed items

1.2. *Meaning* is with the use of words, sentences, or signs and symbols rather than in the related sounds, signs or pictures

1.3. *Languaging* (the French "parole" as different from "langage"/language) is a social practice, serving in communication for pointing at shared experiences in the same culture rather than as an instrument for the direct transportation of sense or as a carrier of attached meanings

1.4. *Knowing something* denotes more an actually fixed and uttered status of a sense producing system rather than a storable, treatable, and retrievable object-like issue, named knowledge

1.5. *Mathematizing* (According to [Davis & Hersh 1980] I could say as well Mathematics) is a practice based on social conventions rather than the applying of an universally applicable set of eternal truths.

1.6. *Using visualizations and embodiments* with the related intention as didactical means depends on taken-as-shared social conventions in a classroom culture rather than on a plain reading or discovering of inherent or inbuilt mathematical structures

1.7. *Teaching* is

In the following I shall speak of the "*integrating perspective*" when I refer to these common core convictions

What is all this for? A few months ago S.I.Miller and M.Fredericks stated in the Educational Researcher [Miller & Fredericks 1991]. "The major concepts of the new philosophy of science are, at best, only marginally relevant to many of the issues studied by educational researchers." (p 3). Moreover, and somehow funny to read after the above notes, they lament about "the related problem of ambiguity on how these terms are to be applied *exactly* to the field of educational resea ch" (p.2, my emphasis) My conviction is with the opposite position It seems to be difficult to *over*estimate the importance of fundamental orientations, because they function as part of the researchers' "epoché of the natural attitude" (H.Schütz), i.e. they are implicitly subsumed, they are taken as granted, and are never questioned[5]. It may be useful, therefore, to consider consequences which arise from an *integrating perspective*. In the interest of students and teachers at least such an attempt appears to be as necessary as the violent discussions of compatibilities, of the drawing of border lines, and of the dominance of one model over the other one are.

2. The Culture of a Mathematics Classroom

> "Explicit rules might play a part in learning to think, but (as suggested by the long history of failure of instruction in logic to improve thinking) a very limited one."
> The rule-based "family of instructional theories has produced an abundance of technology on an illusory psychological foundation."
> [Bereiter 1991], p.14

It is neither by chance nor an act of keeping neutrality only, that N.C.T.M.[6] did not produce more detailed criteria for the recommended actions of teachers than: "... in ways that facilitate students' learning," "... providing a context that encourages ...," and "... necessary to explore sound mathematics." But how to decide about the facilitating of learning without a pragmatical theoretical model for it? What is an encouraging context? And what is necessary to explore mathematics? Among mathematics educators, I think, there is an increasing awareness for the need of more developed *theoretical* bases for the teaching and learning of mathematics. And this is an international phenomenon. The thematic orientation of many conferences across the last few years--not at least of this PME/NA meeting--speaks for the assumption. In the following I will try to draw some general inferences from the integrating perspective for mathematics education. Clearly, the outcomes can not represent more than tacit and preliminary working hypotheses.

The radical constructivist principle says in core, that every cognitive construction is not passively received but (a) a person's individual

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5 --with rare exceptions, admittedly. In his "Structure of Scientific Revolutions", 1970, Th. Kuhn has pointed to the fact, that whenever the first doubts come up the natural attitude as a matter of course is already broken and the revolution has begun.

6 In their "Executive Summary" of N.C.T.M.s "Professional Standards for Teaching Mathematics", issued March 1991, which just has reached my hands.

construction, and (b) in adapting to constraints and acting upon challenges [von Glasersfeld 1987 and 1991]. Accordingly, as a prerequiste of schools' adequate functioning the student will have to find him/herself in a situation which inspires and promotes engaged personal activities, which enacts an effective--and not necessarily overt--interactive control over the adaptation, and which potentially goes beyond the already available.

Further let us take into account the key concerns of connectionism. What the "symbolic connectionist" (Holyoak) models can do best is "what people do best—recognize patterns and similarities. They work in the messy, bottom-up way that nature seems bound to They approximate rather than embody rationality. In a very natural way, they model the gradual transition from vagueness to clarity, from uncertainty to decision, that characterizes much of human thought and understanding. Whereas rule-based systems tend to be helpless when presented with situations where their rules do not fit, connectionist models exhibit humanlike abilities to make best guesses and to capitalize on partial information" ([Bereiter 1991], p.13).

Of particular interest for the organisation of the mathematics classroom is the question of *how* rationality develops with student.: "That private thought conforms to public standards of rationality is conventionally conceived of as internalizing a set of rules From a connectionist viewpoint, this concern errs on both sides—in assuming that public rationality is based on rules and that individual cognition is as well. The development of personal rationality is better conceived of as the tuning of a massive network so that its outputs achieve an increasingly fine fit to what is publicly justifiable." ([Bereiter 1991], p.14). From this, it should be clear, that single lessons on the objectivated and isolatedly thematized issues can hardly provide for the necessary support of related learning. Which chances do students have to develop argumenting, inferences and adequate decisions in mathematics if such issues are not an *integrated part* of the *regular* classroom *processes* ?

The general possibility for taking into account the whole affective domain as well as the experience of the own body marks another advantage of connectionism "The ability of connectionist models to incorporate feeling into cognition may eventually prove to be decisive in their competition with rule- based models." ([Bereiter 1991], p 13) This indeed, is an essential extension, since learning is a wholistic process. In each situation all of the senses are involved They cannot be switched off deliberately. One never learns cognitively (or physically or .) only The cognitivists' growing interest in "enbodied cognition" [Johnson 1987] indicates the realized deficits in this direction.

The metaphor of "tuning in" again points at the crucial role of something outside of the person what cognitivists may name "environment", "context" etc., but what can be described more adequately from sociological perspectives. In ethnomethodology and social interactionism concepts are used like "social interaction," "negotiation of meaning," the "accomplishment of taken-as-shared norms", the "emergence of regulations and structures of common actions,".the "reflexivity" and "indexicality" of the interactive processes in the classroom (see e.g. [Mehan, 1975], [Erickson, 1986], and [Cobb, 1990]), and the "language game" which is more or less specific to each classroom, even in mathematics.

With all this we are very near to form an analogy between classroom realities and the functioning of a subculture. Both concern the person as a whole. Both are permanently changing and developing microworlds, intimately interrelated and intertwined with the change and the mutual development of their participants. Both are under the impact of more powerful societal forces, and both are limited in time. Therefore, I like to speak of the *culture of a mathematics classroom*.. This concept of "culture" is very near to the one described by Michelle Rosaldo, a student of the anthropologist Clifford Geertz: "A matter less of artifacts and propositions, rules, schematic programs, or beliefs, than of associative chains and images that tell what can be reasonably linked up with what. ... Its truth resides not in explicit formulations of the rituals of daily life but in the daily practices of persons who in acting take for granted an account of who they are and how to understand their fellows' moves." (in [Bruner and Haste, 1987], p.90).

I do prefer the notion of culture for the processes under discussion, not at least because of the connotatively related dimensions of time and history. Cultures are permanently developing, reproducing and renewing jointly. One can become a member of a culture through active participation only; it is a processual adaption and cooperation. Most of what is learned in terms of acceptability, validity, norms, languaging, and even personal identity is learned on the way, implicitly, emerges in the interaction. The ever historical result is on the person's side something like a "habitus" (see [Bourdieu 1990], specializing his notion I speak of the *school mathematical habitus* of a student), and on the social side the practice of a living culture, the structures and regulations of which a member lives but rarely reflects upon and which only an informed observer can describe.

Finally, school is a place where students learn to *know of* but not to *know about*. School cannot replicate or even substitute ordinary and professional life. School has to use simulations. and all too often simplified or "elementarized" versions only. The most powerful simulation (mis)used in schools is language, is *talking of* things rather than *actively work* on them. And if things are done at school; the structure is different from doing it under everyday life conditions, because failure can pass without the obligation for taking the responsibility, and without having to bear the consequences. With very rare exceptions school simulations suffer from being *as if's*, - though it is a practice or reality of its own. In first grade classes already students begin to understand, that in the classroom real things, actions, words are used for something else, have to be taken as means for different purposes. The crucial point with the qualities of these situations seems to be the nature of the related teacher x student interactions, the "social climate", or the *culture* of the classroom. How serious does the teacher her/himself take the matter taught? To which extent does the teacher "live" the virtues wanted like a model for any other serious member of the (school) mathematics society? Students, I am very sure about this, have a very sensitive perception of the teacher's concerns and thoroughness. It depends upon the teacher's aptitudes and the whole person's engagement to which extent this as-if microworld becomes a culture of prime importance for the mathematical development of the student.

3. Characteristics of Alternative Classroom Cultures

> "I am fully convinced that a mere mechanical facility in manipulating figures, sufficient though it may be for the calculation necessity in everyday life, is in no way conducive to a healthy development of the reasoning faculty."
> [Chakvavarti 1890], p.1, preface to 1st edition)[7]

What are possible particular and more concrete consequences and inferences drawn from the outlined fundamental changes towards a more integrated theoretical basis in Mathematics Education? The following characteristics are noot at all new. But their combination, I think, may mark another design of what mathematics education can be and how alternative approaches may look like. According to the main field of my own empirical work I shall limit the examples to the early years in school:

3.1. Fundamental Attitudes.

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7 I owe this quote to Ernst von Glasersfeld.

If the mental development rests upon the students's repeated and engaged own activities on the one hand, and on the potential power and richness of the culture of the mathematical classroom on the other hand, then the permanent support of an *attitude of curiosity, of inquisitiveness*, of searching for pattern and regularities, of expecting to find surprising issues, appears to be helpful.

Let me give an example of how to challenge such attitudes. My class[8] is accustomed to an opening of each math lesson with mental arithmetic and geometry One day early in grade 3 I started with a series of "number houses" with two given numbers in the first floor of the first house. Their sum is to appear in the bottom and their difference in the roof. And these two results make the first floor of the next house, and so on (see *Figure 2*):

insert here about *Figure 2*

The students soon caught up with the simple procedure, began to fill the houses and a few tried new starting numbers already Once the teacher intervened and asked for predictions about the next house's roof and bottom numbers--w *ithout* doing the required calculations, students swiftly came to see the 4, 6, 8 sequence in the roof and expected 10 to be the next result. The surprise "12" lead to different assumptions, in particular when the bottom numbers were taken into consideration too. In the following, the doubling in each overnext house became obvious the more of the houses were completed Marion Walter's and Stephen Brown's excellent book on the variation of problems [Brown & Walter 1983] made me ask for other types of completion, e.g. (see *Figure 3*):

insert here about *Figure 3*

The students took this idea up and tried other patterns to start. They did a lot of trying out and calculating with these houses, also at home, coming up with different ideas in the next lessons as well: "I got a new one!" etc. Over a week or so they were keen to find new patterns, especially also in the quite different situation of routinely solving the rather boring sets of calculation tasks in their textbook. So we found ourselves tempted to invent other new tasks and to maintain the movement Within each next piece of

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8 If I speak of "my class" then the children are meant, with whom I am working in their math lessons at an Elementary school in a project since August 1988 The final responsibility is with an experienced woman teacher (an excellent specialist in arts and crafts), we 'share' the teaching.

mental arithmetic, in consequence, but soon with geometry also, the students searched for simila. phenomena and pattern. The quicker students were the first ones to try their developing attitude in other situations as well, which we appreciated and encouraged as creative insertions.

What is different in comparison with earlier "discovery" approaches? It is the shift from a rare exceptional situation towards a permanent and integrated process as part of the life.in the mathematics classroom:

Discovery approach	*Integrated (Culture) approach*
In explicitly defined situations the student researcher starts off from an introduction to working on prepared material, and finally to a discussing and clearing of the findings in a whole class session.	In every classroom situation the students are expected to search for pattern, to assume regularities, and to relate developing or contrasting ideas, as well as to give reason and arguments for the issue under discussion.

Aside of that there are fundamental doubts, which are not to be discussued here in detail, about what "discover" can describe at all from a constructivist perspective (see [Bauersfeld 1991] for details).

3.2. Language, languaging, and the teacher. In a narrow interchange with the described attitudes the view on language will have to undergo change also. "Learning how to use language involves both learning the culture and learning how to express intentions in congruence with the culture." ([Bruner & Haste 1987], p.89) And "one has to conclude that the subtle and systematic basis upon which linguistic reference itself rests must reflect a natural organization of mind, one into which we *grow* through experience rather than one we achieve by learning." (Bruner, ibid., p.88; emphasis in the original).

For many teachers the strength and the generalizability of mathematics is inseparable from the strictness and the precision of the related verbal or other symbolic representations. Similar to priests who celebrate the esoteric language game of their caste, many mathematics teachers permanently insist on saying things as sharp as possible. An observer may find, the teacher insists on this technical language. For the students the force functions as to say it exactly "as he said it."

One may suspect that many teachers do not "have it" in any other way. That is to say, they know how to talk about "it" in the terminology of the accepted language game. But there seems to be not much more beyond, as the limited availability in other 'contexts', the difficulties and shakiness with the use in other situations, and the inability of metaphorizing the issues adequately, indicate. Nobody has trained them to speak about the matter meant in everyday language, to "point at" similar issues etc. (Cognitivists may prefer descriptions like: they cannot "translate," or "say it in other words," they cannot "embed" or "visualize," or "refer it to", thus treating the matter meant as an object rather than as something emerging from the actually situated processes). In consequence many mathematics teachers are quite rigid in their *verbal* aspirations and their related evaluations of students' utterances. But they are quite permissive with the *social* organization of their class. Under the changed integrating perspective the other way round appears to be more promising: to accept and encourage students' mathematical utterances within very wide limits for the how it is said, as long as a serious background (reason, argument etc.) can be identified with it; but to be absolutely rigid in the insisting of listening to other's inventions and explanations, in keeping turn-taking order, in taking serious the others' serious contributions, etc.[9]

The analyses of many videotapes have convinced me of the all too general poverty of classroom communication under this view. If the culture the students live in at the classroom is poor in languaging and in presenting models of the wanted, if it is lacking incentives and challenges, if it is more a nontransparent celebration of technical language rather than a participation in a scaffolding[10] culture, and if it is neither providing resistance for the critical mind nor further orientation for the keen minded, what then are we to expect from our schools?

A counter-example may demonstrate what I speak of here: Many years ago, during teaching practices with my teacher students, I observed a lesson in which a young teacher tried to introduce 6th graders into the characteristics of reflections, in particular the relations between original and image elements. He had followed recommendations for to use a vertically fixed glass pane in a dark room and a lighted candle placed in front of it. He asked

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9 The "Executive Summary" (see footnote 6) explicitly recommends an "encouraging (of) students to take intellectual risks ... by formulating conjectures " But there is no mentioning of the intellectual risks which the teacher has to take in classroom communication and, consequently, has to be trained for.

10 See Jerome Bruner's use of the concept of "scaffolding" related to language learning in early childhood [Bruner 1983]. What are possible analogies in terms of little communicative games, which children can take over and which give children the chance to take the active part in mathematical discourse, not only to join it, but also to contribute to it?

two students for support. One to move an second lighted candle behind the pane, following the directions of the other student who was placed in front of the arrangement. The latter had to try to make the second candle move in a position, where both the image of the first candle and the original second candle would come into coincidence.
The pity was, this teacher talked about the arrangement but had not prepared for doing it. The students ran into difficulties. They would not believe, that one can see three candles, the two originals and the image of the candle in front: "This is impossible! Either you can *look* through the glass, then there is no image! Or you *can't* look through the glass, then there is no candle behind!" The poor teacher ended the situation, shrugging his shoulder in desperation and saying: "O.k., mathematicians use to say so in order to visualize the relations!"

3.3. Problems as developing processes. Teachers usually treat mathematical tasks and problems like objects, like carriers of a more or less well defined enigma to be solved Mostly tasks are "given" tasks (with the exception of the few problems the students get a chance to define by their own). The students are expected to understand the text problem, transform it into a mathematically tractable form and solve it. What happens in many cases, is, that in an obscure and weakly controlled process prima facie associations lead directly to calculations and to results, both through following frequently used paths of related activities and applying related procedural skills Consequently the students learn to treat the tasks as "given" ones, everything one needs to know is "in it" Tasks fall into two classes. "known" and "unknown" problems It becomes a case of "application" of methods ready at hand rather than a case of an active production of possible ascriptions of sense, of selecting among possible alternatives, and then of calculating and checking.

If the individual student's adaptation to the approach favoured by the teacher comes to happen only across the frequent "right" or "wrong" evaluations and remains restricted to merely a discussion of the technical solution procedures (operations, order, writing schemes etc.) then students will have no chance to develop better strategies, a more sophisticated self-awareness, and self-control over their fundamental processes of ascribing and formating mathematical meanings. The technical solution procedures dismiss the vulnerable tacit production of helpful ideas and in the end they replace them by the drilled fluency of current solution techniques. But these techniques
- are bound to narrow classes of "problems" and to the specifities of the "presentation" of the tasks

- get lost if not trained permanently, and
- resist transfer and generalization.

The often bemoaned mathematical "inferiority syndrom" (or complex)--"I wasn't good in Math's!" (nobody would admit that for his mother tongue!)--may have parts of its origin here.

The process-oriented integrated view realizes problems as *always problems forming*[11]. "Precisely the greatest ability of all living cognition is ... to *pose* the relevant issues to be addressed at each moment of our life. They are not pre-given, but *enacted* or *brought forth* from a background, and what counts as relevant is what our common-sense sanctions as such." ([Varela 1990], p.90, emphases in the original). According to the radical constructivist' principle the student develops her/his own sense related to the symbols, texts, or pictures offered by the teacher or the textbook during the solution process. And every step and every decision taken in the process of dealing with 'the problem' changes the issue. What in the end the problem has been for the individual is open to an interpretative reconstruction from step to step in retrospect only. It is a kind of a biography of this 'problem' related to this specific 'solver'. (I do not speak here about the set of "number facts" or other routinized operations which everybody has available by heart.)

These tacit and obscure processes of creating and selecting are developed across--or better: *emerge* from-- the related classroom interactions. During the first years at school already students' participation in the classroom culture leads to the emergence of a typical *school mathematical habitus* (by analogy with Bourdieu's concept of "habitus" as a structuring, "structure generating mechanism"), which enables them to produce somehow acceptable solutions. But obviously, in regular classrooms these covert processes are very rarely touched and redeveloped explicitly.

It may be useful to pay more attention to these tacit and usually covert processes. They should undergo more overt demonstration in the classroom, and--as far as possible--discussion and negotiation, thus opening and developing another language game, rich of metaphors and open to a manifold of supportive associations and analogies.To avoid misunderstanding and to withstand the easy reproduction of the usual instructional methods applied to new content only: I do not speak of "teaching" such ideas and alternative constructions here. The *enacting* of principles and decisions, the *living* of a

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[11] The notion is taken from Hugh Mehan's statement: "Forms of life are always forms of life forming. Realities are always realities becoming." ([Mehan & Wood1975], p.205).

culture, appear to be helpful alternative models rather than the talking into. Every communication functions only over a shared practice. Likewise effective enculturation in the classroom functions across active participation, realizing others in doing "it", doing it yourself, and communicating about the "it" and the "doing," negotiating ideas and what one thinks one has just learned. Insight will emerge from a practice only.

Let me give another example from my class. Late in grade 1 the textbooks begin to present pictures from everyday scenes as story problems. They are treated as the first steps of an introduction into the solution of text problems. Usually such "picture problems" appear right after an elaborated section on addition or subtraction. And teachers invest much effort to make students "read" the expected number sentence into such pictures . All students around the world encountering a picture with three birds sitting on a roof and two others approaching them flying will (in a math lesson!) react with 3 + 2 = 5. Or, in case the two birds are flying away the answers will be 5 - 2 = 3.

That this one-to-one relation between a pictorial presentation and a number sentence is merely a social convention (and *not* an objective truth) becomes clear, when, before such 'introductions' come to deform the minds, students get the chance to comment on the pictures (see for more details [Bauersfeld 1991]). With my class I have tried to organize a certain training for the creation of more serious and reflected mathematical interpretations--"mathematizing"--of a picture. And once the floor was open a rich variety of number sentences and related reasons (no acceptance without reason!) came about for the same picture. Most helpful. the students interactively varied each others interpretions. They competed with new relations (number sentences) and new arguments. It became very clear, that each mathematization of a picture, of a text etc depends on the analysing person's actual interests: What do you want to do or to know?" This, I find, has helped a lot with the later interpretation of text problems, when students on the way to produce acceptable solutions also discussed the sense of exotic intepretations and the quality of others' arguments and ideas.
I am happy to realize that my students have begun recently to turn my permanent "How'd you come to think that?!" against myself.

3.4. Taboos and Theory. Classroom taboos are among the least discussed issues in mathematics education. But they belong to the most effective forces in classroom realities. "Never tell a student what he can find by himself" is anchored as a guiding principle in many German syllabi, the state regulations for mathematics instruction at school. Also in-service teacher training institutions disseminate this principle, presumably in the US as well. There is a clear relation to the suspected issue of "discovery" (see 3.1 and [Bauersfeld, 1991]).

A correlative issue is the conviction that students' mathematical errors are mostly caused by a strategy or a rule. (As linguists say: "In human communication nothing comes to happen by chance.") The usual attempt for repair is to replace the faulty rule by the adequate one. From a connectionist perspective, "the computational algorithms, the things that generate thought, are not anything like rules of logic They are, rather, algorithms for constraint satisfaction." ([Bereiter 1991], p 14). In other words: if a student acts in a specific situation as if he followed a rule, then this will be an indicator for a developing network functioning towards the fluency of repeated common action rather than an outcome of the conscious knowing, selecting, and applying of a rule. Since there is no explicit rule in the game, and since there is no chance for a direct adoption or "internalization" of another rule the idea of repair--even if the invention seems to end up successful--appears to be an illusion.

Bereiter points at several examples from research on instruction "indicating that the rules students learn are not the same as the rules they are taught " and he asks consistently: "If rules are useful in teaching but are not what students actually learn, how are we to make sense of their function?" ([Bereiter 1991], p.14). Referring to experiments from Magalene Lampert [Lampert, M., 1988], his answer, is very near to my notion of classroom culture: "Instead of concentrating on getting rules into the minds of the students, the teacher uses *rules as a way of representing* and talking about mathematics and *encourages the students to do likewise.*" ([Bereiter1991], p.15; my emphases).

What functions as hint for a necessary change on the students side mainly will be the experienced constraints and the negative sanction. Only the student's repeated activity under similar but varying conditions *and* the interactive participation in a challenging classroom culture, in which the teacher functions like a model of the wanted may support a process of reformatting the individual's network and lead to a kind of changed productions, that an observer then can describe as a satisfying approximation to the rule wanted.

Related to his earlier writings Carl Bereiter himself has changed his position quite radically: "The classical rule-based view of rationality enjoys

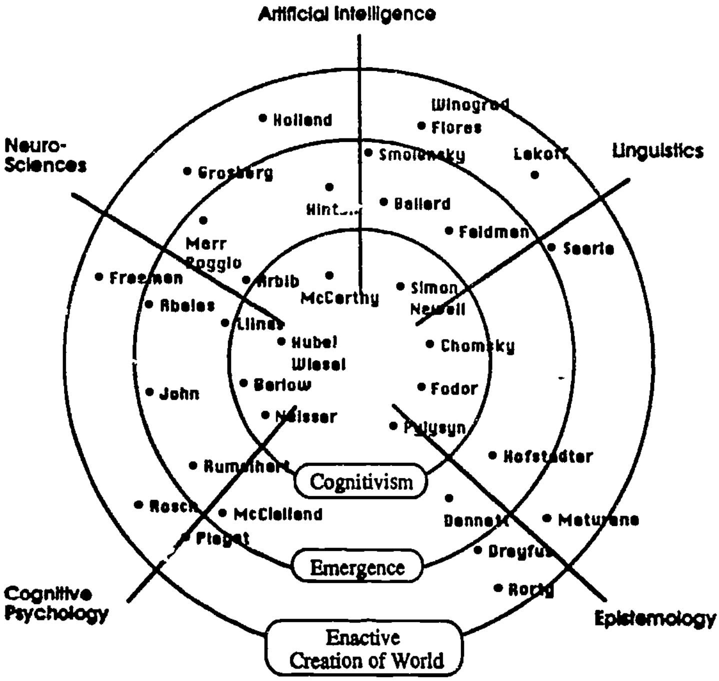
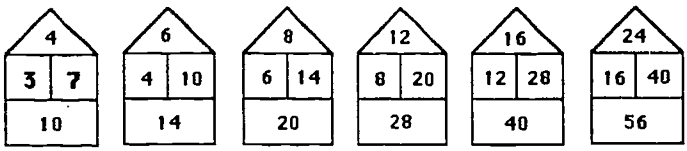


Figure 1: The "polarized map of Science and Technology of Cognition", from F.J.Varela: Cognitive Science, a cartography of current ideas. 1988



etc.

Figure 2 : Sum-and-difference-houses for mental arithmetic

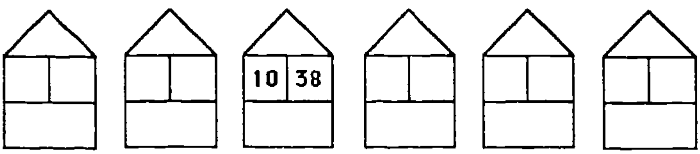


Figure 3: : Can you complete this sequence of houses forward and backwards?

such prestige that when we think of actual thought as an approximation, we tend to assume it is an inferior approximation. Although this is surely true on some counts, the opposite may be true on the whole." He demonstrates the case through an analogy , using the relation between recipes and actual cooking performance: "With a novice cook, Actual performance is an inferior approximation to the recipe; with an expert cook, the recipe (even if written by the expert) is an inferrior approximation to actual performance." ([Bereiter 1991], p.14).

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Age Level: Secondary and College
Identifier #1: Statistics
Identifier #2: Assessment

EVALUATING STUDENTS' UNDERSTANDING OF STATISTICS: DEVELOPMENT OF THE STATISTICAL REASONING ASSESSMENT[1]

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This paper describes the development of the Statistical Reasoning Assessment, an instrument designed to assess students' understanding of probability and statistics for the purpose of evaluating the effectiveness of new curricular programs and materials. A review of the literature related to assessment of statistical knowledge was used to determine the components and framework for this instrument.

Probability and Statistics in the Secondary Mathematics Curriculum

As part of the reform movement in mathematics education, probability and statistics have been given an important place in the K-12 mathematics curriculum. The NCTM Standards (1989) state that students should learn to use probability and statistics to solve problems and evaluate information in the world around them. Additionally, these standards recommend which topics in probability and statistics should be included at different grade levels and how these topics should be taught.

For example, the standards suggest using hands-on activities to teach data collection and organization using technology for representing and modeling data. The standards also emphasize verbal and written communication of statistical ideas (such as distribution, randomness, and bias) and in helping students to gain experience choosing appropriate measures, methods, theoretical distributions in data analysis. The standards for teaching probability include use of simulations to estimate probabilities, creating and interpreting discrete probability distributions, and understanding and applying the idea of a random variable.

Because these topics and techniques are new to the high school mathematics curriculum, several projects were funded to develop curricula and software to help implement the NCTM standards (e.g., the Quantitative Literacy Project, the Reasoning Under Uncertainty Project, and the Chance-Plus Project). These projects offer curriculum materials and flexible, easy-to-use software for representing,

[1]This research was supported by NSF Grant No. MDR-8954626, Chance-Plus: A Computer Based Curriculum for Probability and Statistics, Clifford Konold, Principal Investigator.

exploring, and analyzing data. The new software packages in particular offer unique opportunities for students to explore data and build models, and are also designed to facilitate the development of probability and statistics concepts.

The Need for Common Assessment Methods

One problem that these projects share is lack of appropriate tests to use for determining how well the new materials helps students learn probability and statistics. Although tests were constructed for two of the projects to assess students' knowledge and skills, items were written to reflect the specific skills taught rather than to test for more general conceptual understanding and higher order reasoning skills.

In his forthcoming chapter on research on learning probability and statistics, Shaughnessy (in press) describes the need for some standard, reliable tools to assess students' conceptions of probability and statistics. If fact, he lists the development of assessment instruments as the first item on his "wish list" for future research in this area. Although a variety of items and tasks have been used by researchers or evaluators in the past, it is difficult assemble these items and tasks into a test because of their different purposes and formats (e.g., paper and pencil, clinical interview). Shaughnessy stresses the need for new instruments which incorporate and build on the ideas of previous research but which have greater applicability. Ideally, these instruments will help us determine if the new standards for learning probability and statistics are being achieved.

Acknowledging the lack of a general instrument for assessing students' understanding of statistics and probability, the NSF-funded Chance-Plus project (at the University of Massachusetts, Amherst) is developing such a test. Assessing the reliability and validity of this instrument is crucial in order for the instrument to be used in further research and evaluation studies. To develop the test, a review of the literature related to assessment of statistical understanding was conducted. This review is summarized below, followed by a description of the Statistical Reasoning Assessment and its intended validation plan.

Research Related to Assessing Statistical Understanding

A review of research by Jolliffe (1990) organizes the relevant literature in the categories of classification schemes for assessment tasks, newer methods of assessment, attitude scales, and studies of understanding. A modification of these categories yields five groups of studies relevant to the assessment of statistical understanding: 1) students' attitudes and anxiety towards learning statistics, 2) students' computational skills in using probability and statistics, 3) students' misconceptions of probability and statistics, 4) conceptual frameworks for assessing statistical learning, and 5)methods of assessing mathematical learning and problem solving. Each is described below.

Student Attitudes and Anxiety. The Statistics Attitude Survey (SAS) scale (Roberts & Saxe, 1982), the Attitudes Toward Statistics test (ATS) (Wise, 1985), and the Statistical Anxiety Rating Scale (STARS) (Cruise, Cash, & Bolton, 1985) are Likert-type scales written for college students in statistics courses. The SAS was designed to assess various components of statistical attitudes, such as students' perceptions of their own statistical competence and the usefulness of statistical analysis. The ATS was developed specifically to measure attitude changes during statistics courses and is designed to be given as a pre- and post-test. Two scores are calculated: attitudes towards the course and attitudes toward the field of statistics. The STARS measures students' attitudes towards six areas: worth of statistics, interpretation anxiety, test/class anxiety, computation self-concept, fear of asking for help, and fear of statistics teachers. None of the three instruments assess student understanding of and beliefs about what the field of statistics is, what it means to "do" statistics and solve statistical problems. Instead, they deal with the more specialized attitudes and anxiety faced by college students.

Students' Computational Skills. Tests written to accompany commercial textbooks are the most common form of assessment for measuring students' ability to perform statistical calculations. Items on standardized tests and the National Assessment of Educational Progress (NAEP) tend to be of this type. One example from the NAEP asks students to calculate the mean, median and mode for a set of data consisting of inches of snowfall (Brown & Silver, 1989). Although these items typically test whether or not students can use formulas and come up with a single, correct answer, they do not assess whether or not students understand the concepts and can use them to analyze and interpret data. For example, students may be able to correctly calculate the median and mean but not know when one is a better average to use than another. This type of skill is best assessed in classes through assignments and quizzes, and does not need to be on a general test of statistical understanding and reasoning.

Students' Conceptions and Misconceptions about Probability and Statistics. The only test written and used on a large scale to assess students understanding of probability concepts was developed and administered by Green (1983) to 3000 students in Great Britain. Some of these items and other items appearing in the research literature have been found to be useful in detecting misconceptions and helping researchers to understand how student think about probability and statistics. Research reviews by Garfield and Ahlgren (1989) and Shaughnessy (in press) refer to many of these studies. Items used are often open-ended and many have been used in clinical interviews to probe students' beliefs. Many have been used with adults or college students and involve a substantial amount of reading. Although these items are good at detecting student conceptions and misconceptions, many need to be revised and adapted for high school students.

Frameworks for developing assessment tasks. There have been at least two attempts to design frameworks for developing tasks for assessing statistical learning. Chervaney, et al., (1977) used a model of the problem solving process to develop a three stage model of assessment (comprehension, planning and execution, and evaluation and interpretation). These three stages contain 10 different steps in statistical reasoning which can be used to guide item development. Although this framework was designed to evaluate innovative college courses and was successfully used to design tests for a college level course (Garfield, 1981) it does not appear to have been used in other studies. Nitko and Lane (1990) also designed a framework for generating assessment tasks that provide a richer description of students' thinking and reasoning than just giving them problems to work out. This framework was developed for college and graduate level statistics courses and can be used to assess relationships among knowledge and whether or not important principles and concepts are understood by students. Three interrelated categories are used to classify statistical activities: problem solving, modeling, and statistical argument. Although developed for students at a level higher than secondary school, these models are useful in providing frameworks for organizing statistical knowledge and skills.

Assessment of mathematical learning and problem solving. There are 13 standards for evaluation included in the NCTM curriculum standards. These standards describe the assessment of students' mathematical knowledge, conceptual understanding, procedural knowledge, problem solving, reasoning, and mathematical disposition. Assessment is viewed as the process of understanding the meaning which students give to mathematics; it should be dynamic and involve a variety of approaches (Webb & Romberg, 1988). Recently, more attention has been given to assessment of higher order mathematical thinking (Kulm, 1990). Educators are encouraged to move away from using single number summaries to represent students' knowledge, and using two dimensional frameworks for developing assessment measures, to instead explore alternative models of assessment and ways of building on more recent models of learning mathematics.

Development of the Statistical Reasoning Assessment

The Statistical Reasoning Assessment currently under development by the Chance-Plus project, is designed to assess students' beliefs about statistics, their understanding of basic concepts of probability and statistics, and their ability to use these concepts in interpreting information, reasoning, and solving problems. After reflecting on the the research literature reviewed, previous tests and test items, teaching experience, and much group discussion, the ChancePlus project team of psychologists, educators, and statisticians outlined a framework of important beliefs, ideas, concepts, and reasoning skills. Components of the instrument were then collected, revised, or written from scratch to assess these ideas and skills. Although some parts of the instrument look like traditional test items, others appear unique in their format and ability to capture students' thinking and reasoning. Four parts of the test were created to be used at various times and in various combinations:

Part 1 assesses general beliefs about the nature of statistics and statistical work. Two formats are used for these items. One part contains statements about statistics and statistical work (e.g., there may be more than one way to correctly solve a statistical problem). The other part asks students to rate the skills that someone would need to have in order to analyze and interpret data (e.g., types of communication and mathematical skills).

Parts 2 and 3 assess general ideas about probability and statistics students would have before a course of instruction. These items do not use specialized vocabulary with which a student might be unfamiliar. Items are designed to see how students interpret information and make judgements about different situations. Items are also designed to assess students' intuitions and misconceptions about probability and statistics that interact with or are resistant to instruction. An example of a probability item is:

> Jim, Barb and Rebecca are playing a game and need to decide who should go first. Barb suggests rolling two dice to determine who will start. If both dice come up odd, Jim will go first. If both dice come up even, Barb will go first. If one die is odd and one is even, Rebecca will go first. Do you think this is a fair method of determining who should go first? Why or why not?

Part 4 assesses students' ability to reason about and solve probability and statistics problems. In order to develop realistic contexts for solving these problems, a research study conducted by a high school class is described. All questions are based on the analysis of this project. One version of this test describes results from a survey of how students spend their money. Questions involve a decision, interpretation, or conclusion about some aspect of the data analysis. A sample question is:

> Jack says that because the distribution of money spent for entertainment is skewed, a median is a better measure of average money spent by students on entertainment. Sarah says that a mean is always the best average to use because more people know how to calculate the mean. Do you agree with either Jack or Sarah? Why or why not?

Validation Plan

The four parts of the test were sent to a variety of people for first-stage evaluation. Raters were asked to evaluate how well each item measured the designated concept or skill, to revise items as needed, to indicate whether the item should stay in the test, and to indicate if any additional items should be added. After the indicated revisions are made, the next stage will be to give these items to students, to assess how they interpret them and how able they are to answer the questions. Again, a set of revisions will be made. A third stage will be to code student responses to open-ended questions so that they may be transformed into a multiple-choice format. A fourth stage will be to administer the test to different groups of students, to establish scoring rubrics, and to determine the reliability for different components of the test. A fifth stage will be to identify different measures of student performance, such as tests and class projects, and to correlate these with the instrument. These five stages will be completed by summer of 1992. At that time the Statistical Reasoning Assessment should be available for general use.

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Identifier #2: Teachers' conceptual change

THE DEVELOPMENT OF TEACHERS' CONCEPTIONS OF MATHEMATICS TEACHING*

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A framework of the development of teachers' conceptions of mathematics teaching is proposed for consideration of its viability. The framework is based on reflections from work carried out with twelve preservice and inservice teachers over the past 5 years.

The ideas presented in this paper took shape upon reflecting on work conducted over the past five years in collaboration with preservice and experienced mathematics teachers involved in two research projects. It is on the basis of that work that I offer—more in the spirit of a hypothesis than of a theoretical model—a description of what I have come to see as a fairly consistent pattern of development in teachers' conceptions of mathematics teaching. For lack of a better word and at the risk of being judged pretentious, I use the term *framework* to describe that pattern.

The framework is offered for consideration and investigation of its viability. The issue of its potential usefulness for teacher educators, staff developers, and others—whether used as a frame of reference against which to gauge the progress of their work or as a means of facilitating communication among them—is open for examination.

The development of a given teacher's conception of mathematics teaching is influenced by the personal experiential background of that teacher, including his professional and educational experiences, and how those are interpreted and internalized by the teacher. Insofar as there are commonalities across teacher education programs that are designed to effect change in teachers, one might expect to see patterns of thought and commonality of themes in their development.

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However, exposed to the same experiences, teachers will come away with different conceptions, as our own experiences can attest. Much of what a teacher makes of a particular experience depends on the conceptual schemas available to the teacher into which the experiences are assimilated or on the accommodation of schemas that takes place. It is with these ideas in mind that I propose the framework for examination of its viability.

In the limited space available it is impossible to include descriptions of the experiences afforded the teachers on whom the description is based. Nor is it possible to include anecdotal excerpts from the data to substantiate the statements and claims made. Support for the framework can be found in detailed accounts of the development of individual teachers' conceptions described in case studies by Thompson and Bohn (forthcoming) and Thompson and Boyd (forthcoming).

The paper is organized in two parts. The first is a description of the framework which consists of three levels. The features that characterize teachers' conceptions of mathematics teaching at each level are described. The second part includes a discussion of issues related to the restructuring and development of teachers' conceptual schemas.

The Framework

The proposed framework consists of three levels in the development of teachers' conceptions of mathematics teaching. Each level is characterized by conceptions of:

1. What mathematics is.
2. What it means to learn mathematics.
3. What one teaches when teaching mathematics.
4. What the roles of the teacher and the students should be.
5. What constitutes evidence of student knowledge and criteria for judging correctness, accuracy, or acceptability of mathematical results and conclusions.

Level 0

Conception of mathematics is based on perceptions of common uses of arithmetic skills in daily situations. This translates into instructional practices that focus on developing students' arithmetic skills through memorization of collections of facts, rules, formulas, and procedures with little or no consideration of their origin, validity, or logical relations among them. Mathematics instruction is conceived as progressing through a sequence of topics and skills specified in a textbook. Because of the hierarchical organization of topics common to most textbooks, each of these topics and skills is viewed as prerequisite for the next and all are viewed as equally important. Thus, an undifferentiated view of topics in terms of their relevance or mathematical significance characterizes this level.

The role of the teacher is perceived as that of demonstrator of well-established procedures which are viewed as constituting the core of mathematical knowledge. The students' role is to imitate the demonstrated procedures and to practice them until they become habituated. Obtaining accurate answers via the prescribed procedure is viewed as the goal of mathematics instruction with little or no consideration of mental processes. Authority for correctness or accuracy lies in the teacher or in the book (i.e., it is external to the learner; experts are the ultimate judge).

Problem solving is viewed as tantamount to getting answers to "story problems" by applying a prescribed procedure that the students are presumably adept at using. Thus, instruction in problem solving is construed as helping students identify the procedure or sequence of procedures necessary to get the answer to the problem. To accomplish this, the teacher may resort to any of a number of techniques. One such technique may be to call students' attention to "rules of thumb" or to "key words" in the problem statement that the teacher deems suggestive of the desired procedure. A characteristic of the techniques used at this level is that they all circumvent discussions of the problem's quantitative relationships and of the appropriateness of alternative mathematical operations and procedures in light of those relationships.

Level 1

Conception of what constitutes mathematical knowledge is broadened from rote, procedural proficiency to include an emerging appreciation for understanding the concepts and principles "behind the rules." Rules, however, continue to be perceived as predetermined and as governing all work in mathematics. There is an incipient distinction between "meaning" and "skill" triggered perhaps by exposure to the use of "manipulatives" in teaching mathematics.

Conception of mathematics teaching is characterized by an emerging awareness of the use of instructional representations—physical and pictorial—of mathematical concepts and procedures to help students develop meaning and understanding. But teaching for conceptual understanding is viewed as requiring the teacher to possess a collection of unique pedagogical techniques—typically involving the use of concrete or pictorial representations—for explaining isolated concepts, procedures, algorithms, and formulas. There is little generalization or adaptation of these techniques to teaching other topics for which specific techniques have not been encountered.

The use of manipulatives in instruction is highly valued, but more for their potential in helping achieve attitudinal goals than in achieving cognitive objectives of instruction. A perception that negative attitudes towards mathematics are widespread among students, feeds an overriding concern to engage students in activities that will ensure a view of mathematics as "fun." Because many manipulatives are colorful and can be used to involve students actively in a lesson, they are regarded as ideal for promoting the view that "math is fun." Thus, manipulatives are valued primarily for their potential in helping achieve this attitudinal goal.

Conceptions of teaching at this level are characterized by a rather narrow view of the possible uses of representations for achieving cognitive objectives of instruction. Manipulatives and pictorial representations are viewed as useful in providing some sort of empirical justification for standard mathematical procedures. But the connections between the actions performed on objects or diagrams, the verbalization of those actions, and their representation in mathematical notation are not explicitly discussed in instruction. Rather, connections are typically left for the students to make on their own.

There is an emerging appreciation of complexities in mathematical content previously perceived as unproblematic. This appreciation emerges from conceptual analyses of content domains and from reflecting on the abstract nature of familiar concepts (e.g., number, rate, variable) and the subtleties inherent in them. Instructional implications of such analyses begin to take shape.

Except for attending to the "reasons behind the rules," the role of the teacher is still perceived much as in Level 0. Views on the role of the student are somewhat broadened to include some understanding of the justifications for the standard procedures of the curriculum. Authority for correctness or accuracy still lies with experts.

Problem solving is accepted as important in the mathematics curriculum, but it is viewed as a separate curricular strand to be taught in isolation from the "traditional content." Integrating problem solving into the curriculum is construed as interspersing routine and non-routine problems amidst ordinary lessons.

Problems presented are unrelated to mathematical topics currently being studied and are generally viewed as unrelated to the mainstream curriculum. The dominant view is one of teaching "about" problem solving (i.e., phases and strategies), as distinct from teaching "with" problem solving (i.e., as an instructional approach). This view leads to problem-solving instruction that tends to be prescriptive in nature, focusing primarily on the selection and use of strategies, and bearing little connection to what is regarded as the mainstream curriculum.

A characteristic of this level is the absence of cognitively-based principles that are consciously used to guide instructional decisions or of well-articulated criteria for judging cognitive effects of instructional actions. Pedagogical decisions regarding instructional actions and activities are often based on perceptions of what a community of experts (e.g., staff developers, school district personnel, teacher educators, professional organizations) deem to be desirable practices. Novel instructional ideas are embraced and implemented with little critical consideration of their suitability given the mathematical content of a lesson or of important details concerning their implementation.

Level 2

Conception of how mathematics should be taught is characterized by a view that students must engage in mathematical inquiry if they are to make sense of mathematical ideas. The development of students' mathematical reasoning in the context of investigating and constructing mathematical ideas is viewed as being as important a goal of instruction as their understanding of the ideas themselves. Thus, the view of teaching for understanding that begins to develop at Level 1 is replaced at Level 2 with a view that understanding grows out of engagement in the very processes of doing mathematics. Processes such as specializing, conjecturing, refuting and validating conjectures, and generalizing are viewed as integral to learning and teaching mathematics.

Physical and pictorial representations are regarded as providing contexts in which students can engage in tasks that have been carefully designed by the teacher for exploring ideas and generating procedures. The legitimacy of non-standard procedures generated by students is judged in terms of whether they meet the purpose or need for which they were generated and whether they make sense. Students' competence in making such judgments is viewed as an important cognitive objective.

An understanding on the part of students of how different concepts, procedures, and representations are interconnected in sets of problems and situations, and student recognition that the same or similar mathematical ideas arise from seemingly different situations are viewed as major long-term goals that guide and help shape instruction. Distinctions concerning the relative importance of various topics are based on the centrality of the mathematical ideas inherent in the topic to various areas of mathematics.

The role of the teacher is perceived as steering students' thinking in mathematically productive ways. Questions are posed with the intent of stimulating, guiding, or focusing students' thinking rather than for the sole purpose of eliciting answers. Instructional decisions are informed by concerns about the quality of students' reasoning inferred from their work and discussions. There is an increasing awareness of subtleties inherent in mathematical ideas that pose cognitive obstacles for students and lead to common misconceptions; careful consideration is given to shaping instruction so that it helps students make those subtleties explicit to themselves. Opportunities for students to express their ideas and for the teacher to listen to and assess their reasoning are viewed as essential to the quality of a lesson.

The hallmark of this level is the presence of cognitively-based principles that are explicitly used to guide instructional decisions. Cognitive objectives of instruction are also explicitly used in selecting and designing instructional activities. Criteria for judging the soundness of instruction are stated in terms of student outcomes consistent with broad goals that drive instruction.

Discussion

Of the twelve teachers (five experienced and seven preservice) with whom we have worked over the past five years, none can be said to have developed conceptions of mathematics teaching that fully fit the Level 2 description. All of the preservice and three of the experienced teachers had initial conceptions at Level 0. Only two of the experienced teachers were judged to have initial conceptions at Level 1. All of the teachers starting at Level 0 showed change to Level 1 with some evidence that aspects of Level 2 conceptions were beginning to take shape. The three teachers starting at Level 1 have shown little evidence of growth to Level 2 conceptions over a period of eight months. This is despite a genuine desire on their part to teach in ways that are consistent with such conceptions and efforts on our part to help them do so.

There are plausible explanations for the relative ease with which teachers moved from Level 0 to Level 1 conceptions and the relative difficulty observed in moving from Level 1 to Level 2 conceptions. Growth from a Level 0 to a Level 1 conception can occur without major restructuring of conceptual schemas. The ideas at Level 1 can be assimilated into structures that support Level 0 conceptions by merely expanding or broadening them, but without the need for restructuring those schemas, i.e., without the need for the reconceptualization of fundamental ideas that is necessary to progress from Level 1 to Level 2. The restructuring necessary to advance to Level 2 requires that a teacher experience numerous occasions to become aware of and question his deeply rooted ideas and unexamined assumptions about what it means to know, learn, and teach mathematics. Furthermore such occasions must take place in the context of experiencing alternatives to their pre-conceived notions about mathematics teaching and their second nature instructional habits. The kind of restructuring necessary calls for a concerted and sustained effort. Indeed, our experience cautions us not to underestimate the resilience of teachers' conceptual schemas. This resilience was noted by Skemp (1978) when he stated the following as one of four factors contributing to the difficulty of teachers changing their instructional practices:

> *The great psychological difficulty for teachers of accommodating (re-structuring) their existing and longstanding schemas*, even for the minority who know they need to, want to do so, and have time for study (p.13; emphasis in original).

Studies of teachers' conceptual change that provide detailed and insightful analyses of such changes are necessary to improve our understanding of the mechanisms that bring about the restructuring and development of teachers' conceptual schemas. A better understanding of those mechanisms is critical to the design of strong and truly successful teacher education and enhancement programs—programs that go beyond raising the level of enthusiasm of the participating teachers. The need for such an understanding is particularly critical at a time when federal, state, and local agencies are investing considerable funds in such programs in the United States.

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Age level:	5–9 years
Identifier #1:	Whole numbers
Identifier #2:	Word problems

CHILDREN'S SOLUTION STRATEGIES FOR DIVISION PROBLEMS

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This paper reports some of the effects of a classroom teaching experiment on young students' understanding of division and their ability to solve division-type problems. The experimental curriculum is built around students' construction of their own conceptually-based algorithms as a problem-solving activity, supported by a classroom atmosphere of discussion and negotiation.

Introduction

Verbal problems involving division are generally seen as difficult for children to solve. The difficulty resides mainly in children's predicament in choosing the "correct" operation. This situation may be the direct outcome of conventional school programs that formally *first* teach all the "prerequisites" for solving "division" problems (the meaning of division, the division facts and a standard division algorithm) and *then* require the children to *apply* this knowledge to the solution of word problems.

On the other hand, researchers generally agree that young children enter school with a wide repertoire of informal mathematical problem-solving strategies that reflect and are based partly on their understanding of the problem situation and partly on their existing concepts (Olivier, Murray & Human, 1990; Carpenter & Moser, 1982). Instead of ignoring or even actively suppressing children's informal knowledge, and imposing formal arithmetic on children, instruction should recognize, encourage and build on the base of children's informal knowledge. Steffe and Cobb (1988) state: "In those cases where adult teaching is in harmony with the child's methods, the generative power of the child is extremely exciting and is unchartered (sic)" (p. 26).

Our research group is engaged in an ongoing research and development project on the mathematics curriculum in the first three grades of school, trying to build on children's informal knowledge and studying and facilitating the development of their conceptual and procedural knowledge (Murray & Olivier, 1989; Olivier, Murray & Human, 1990). In this paper we focus on children's construction of increasingly sophisticated meanings of division and solution strategies for "division" problems in a curriculum that is radically different from traditional classroom practice.

Theoretical Orientation

Our theoretical orientation and research base have been outlined elsewhere (Olivier, Murray & Human, 1990), but its main characteristics are summarized briefly.

Our theoretical framework is based on a constructivist theory of knowledge in which children actively build up their knowledge based on their own experience. Our approach is further inspired by socio-constructivism: Learning mathematics is as social activity as well as an individual constructive activity.

Our baseline study indicated that the majority of children invent powerful non-standard algorithms alongside school-taught standard algorithms; that they prefer to use their own algorithms when allowed to (or even when forbidden to!); and that their success rate when using their own algorithms is significantly higher than the success rate of children who use the standard algorithms or when the themselves use standard algorithms. Our research also identified a model specifying the conceptual (and related procedural) advances that children make and the processes by which they make them.

Our theoretical framework, research base and the availability of calculators which necessarily leads to a re-evaluation of objectives for computation, has led us to formulate a teaching approach with the following main features:

- The development of the meanings of operations and solution strategies through true problem solving, i.e. meanings and strategies are *not taught*, but the teacher poses a word problem to a group of students and expects them to solve it in whatever way suits them individually. This is followed by a general discussion and comparison of methods used. The teacher does not suggest a method, and mistakes are identified and corrected by the group.
- A mixture of types of word problems are posed from the very beginning of grade 1, and are not classified as "addition" or "division" problems, since students select those operations that suit their strategies. For example, we believe that presenting students with both partitive and quotitive division-type word problems and requiring them to construct their own solution methods in response to the structure of each particular problem will firstly prevent discontinuities between the student's procedures and his concepts (Steffe & Cobb, 1988), and secondly enable him to construct an integrated meaning of division which makes possible eventual problem transformation. The idea of progressive schematization (Treffers, 1987) is implemented: The teacher starts with a general problem which the students solve by means of crude methods, and then creates a series of situations which will encourage students to refine their methods. Mathematical notation is only introduced when students have trouble in documenting their solutions logically.
- Strong emphasis on number concept development by helping children to construct increasingly sophisticated concepts of different units, especially ten, and to build these concepts on children's counting-based meanings by encouraging increasingly abstract

counting strategies and child-generated computational strategies. There are no number barriers, i.e. a particular teacher and her students can operate in any number range within the students' conceptual development.

Objectives and Methodology

The objectives of the study as it relates to division, includes the description and analysis of children's solution strategies, analysis of the relationships between strategies used and the semantical structure of the problems, the mechanisms of transition to more sophisticated strategies, and analysis of the role of classroom social interaction in the construction and evolution of children's division schemes.

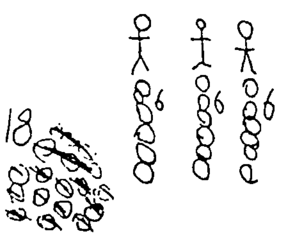
Our data is gathered by qualitative research methodologies, including observation and interaction with small groups of children in the classroom setting and interviews with individual students. The mathematics lessons of 40 project schools were regularly observed by a team of seven researchers and three education department supervisors. Additional data sources include video-taped lessons, protocols of clinical interviews of several case studies with individual children, and copies of all the children's written work.

We describe below some typical strategies for division-type problems in more or less an order of increasing sophistication in terms of its mathematical representation and the (implicit) underlying properties of operations (theorems-in-action).

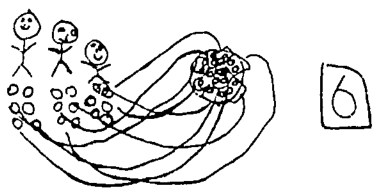
Different Strategies

Direct representation Although informal writing materials as well as counters are always available, it seems that students seldom use counters to model a problem. Rather, the problem context is *drawn* in greater or lesser detail, and then solved by further drawing in the actions needed. For example, Leana (grade 1) divides 18 cookies among three children one at a time, and Conrad (also grade 1) two at a time:

Leana



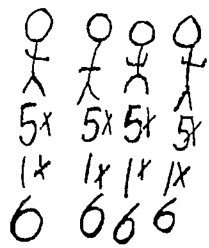
Conrad



The solution is found by a *double count:* counting the number in the dividend and (afterwards) counting the number in each group (partitive division) or the number of groups (quotitive division). This double-count strategy, in increasingly sophisticated form, underlies all the strategies.

Numerical representation In this strategy students model the structure of the problem using *numerals*, without employing arithmetical operations in their representation. For example, Yolande (grade 1), divides 24 balloons among four children as shown.

Yolande



Although children choose the number of objects to be dealt out per round according to the size of the iterable units they are able to cope with in that context (Steffe & Cobb, 1988), this accelerated dealing out strategy is optimized by sound estimation. There are two estimation-based strategies in this particular context: The first is a repeated-estimation strategy (trial-and-error). For example, to share 70 cookies among five children, a first estimate of ten turns out to be too low, a second estimate of 15 is too high but almost there, and the third estimate of 14 is just right. The second estimation strategy may be called an "estimate-and-adjust" strategy, where the first convenient estimate is corrected not by a new estimate, but by dealing out the remainder if the estimate was too low. Moana (grade 2) does the following:

$66 \div 3 = 22$
20 20 20
2 2 2

This estimation dealing strategy is quickly formalized by writing it as subtraction, addition, or multiplication sentences (see the following sections). It also forms a conceptual basis for applying the distributive property as illustrated in the section on transformations, for example, $70 \div 5$ is solved as $50 \div 5 + 20 \div 5$.

Subtraction Subtraction as a strategy for division can represent three different conceptualizations:

- estimation dealing out for partitive problems. For example, Emmerentia (grade 3) divides 81 apples equally among three boxes as follows:

  $80 - 20 - 20 - 20 \rightarrow 20 - 6 - 6 - 6 \rightarrow 2 + 1 \rightarrow 3 - 1 - 1 - 1 \rightarrow 0 \qquad 81 \div 3 = 27$

- subtracting the number of objects dealt out in each round to solve a partitive problem. For example, Estelle (grade 1), divides 18 sweets among three children as follows, explaining that she was "getting rid of" three sweets during every round of dealing out:

  $18 - 3 = 15 \rightarrow 15 - 3 = 12 \rightarrow 12 - 3 = 9$
  $9 - 3 = 6 \rightarrow 6 - 3 = 3 \rightarrow 3 - 3 = 0$

- solving a quotitively-interpreted problem by repeatedly subtracting the divisor. For example, Antoinette (grade 3) finds how many buses are needed to transport 350 children if there are 70 children per bus, as follows:

  $350 - 70 \rightarrow 280 - 70 \rightarrow 210 - 70 \rightarrow 140 - 140 \rightarrow 0 \qquad 350 \div 70 = 5$

Addition and multiplication Double counting occurs in two closely-related forms, e.g. to compute 27 + 3, the student may write down 3 + 3 + 3 + ... until the running total reaches 27, and then count the number of threes he had written down, or he may mentally count in threes, saying the running total or writing it down, and keep track of the number of threes on his fingers (both quotitive interpretations).

Addition and multiplication can be used for both partitive and quotitive interpretations of division. Stephen (grade 2) divides 18 sweets among three children by repeated estimation:

$4 + 4 = 8 \quad \cancel{5+5} \quad 18 = 6 + 6 + 6$

Students progressively formalize such strategies, eventually expressing them as multiplication. An estimation dealing out strategy can also terminate in multiplication, for example $468 \div 12 = 39$:

initial version
$30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30 = 360$
$7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 = 84$
$2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 24$

final version
$12 \times 30 = 360$
$12 \times 7 = 84$
$12 \times 2 = 24$

Henriëtte (grade 3) uses multiplication in the repeated-estimation strategy she employs to solve $278 \div 12$:

6×25
$120 + 120 \rightarrow 240 + 60 = 300$ too many.
6×23
$240 + 36 = 276$ 23 and 2 left over.

Transformations This method indicates the ability of the student to reconceptualize a number as the sum of multiples of iterable units. The strategy also includes a fair amount of estimation and the use of known number facts. For example

for $51 \div 3$: $30 \div 3 = 10$; $12 \div 3 = 4$; $9 \div 3 = 3$; $10 + 4 + 3 = 17$ (Gerhard, grade 2)
for $70 + 5$: $12 \times 5 = 60$; $2 \times 5 = 10$; $60 + 10$; $70 \div 5 = 14$ (Jean Pierre, grade 2)

Transforming a number in order to apply a multiple as a known number fact is extremely common. Here follows a slightly more complex transformation: To compute $76 \div 4$ the following change and compensate method is frequently used:

$80 \div 4 = 20 \quad 4 \div 4 = 1 \quad 20 - 1 = 19 \quad 76 \div 4 = 19$

Division by four, accomplished by two successive halvings of the dividend, is common. Division by five by doubling the dividend and then dividing by ten is less common, as is Mario's (grade 3) strategy of dividing by 15:

105 + 15: 105 ÷ 5 → 21 ÷ 3 → 7

Both strategies indicate the use of quite advanced theorems-in-action (Vergnaud, 1988). We have found that children easily progress towards using these theorems-in-action to cope with larger numbers. We give two examples from a grade 2 class who had been asked to compute 4158 ÷ 11:

> Bernie: "I first did 300 × 11 in my head. That gives 3300. The I took 80 × 11 because I wanted it to be 880 but then I saw it was too much. So I decided on 78 × 11 which was right. So the answer is 300 + 78 = 378"

> Sheryl: "300 × 11 = 3300
> 50 × 11 = 550
> Then I decided to use 25 × 11 because I have just now used 50 × 11 and I can halve 550. Then I added 3 × 11. So 300 + 50 + 25 + 3 gives 378."

Discussion

Children's strategies to a large extent correspond to those identified by Kouba (1989), but our subjects seem to use additional sophisticated strategies in working with larger numbers. Following we briefly discuss matters related to the frequency of strategies and the evolution of strategies.

Implicit models We find that young children can solve both partitive and quotitive problems at an intuitive level prior to any formal instruction. This refutes Fischbein *et al*'s (1985) conjecture that "initially, there is only one intuitive primitive model for division problems – the partitive model. With instruction, pupils acquire a second intuitive model – the quotitive model" (p. 14).

Contrary to Fischbein *et al*'s notion that the implicit model for quotitive division is repeated subtraction, we find that very few children naturally use subtraction – they rather use *building-up* or addition strategies, and if they use subtraction they quickly change to other strategies.

Evolution of strategies Although we have only anecdotal evidence at this stage, we are beginning to form a clear picture of the interrelated variables affecting students' development towards more sophisticated strategies and are now engaging fine-grained research on each of these variables. First, students' *number concept* development inform their strategies. As children develop increasingly abstract iterable units and can decompose numbers into convenient units, so their strategies evolve. Second, students' solution strategies are initially clearly determined by the *semantic structure* of the problem; they use different strategies for partitive and quotitive problems, illustrating two independent conceptions of division. They gradually develop a unified meaning and strategy as the separate meanings and strategies become more abstract, allowing them to transform between problem types and to divorce their strategies from the semantic structure of the problem ("distance from problem"). Third, students' solu'on strategies are paralleled by their level of awareness of the properties of operations or theorems-in-action (Vergnaud, 1988). Intuitive awareness of the commutative property of multiplication helps them transform between different context (e.g. from 3 groups of 5 to 5 groups of 3) and the distributive property underlies many sophisticated strategies.

On the other hand, we have evidence of how students' strategies are not merely *based* on conceptual understanding, but simultaneously also *inform* their number concepts and awareness of theorems-in-action, showing that conceptual and procedural development go hand in hand.

The Role of Discussion A crucial aspect of the experimental approach is the role of discussion among students to promote reflection (compare, for example, Cobb, Yackel & Wood, 1988), leading to the improvement of strategies by reflecting on one's own and others' strategies, and the prevention of misconceptions taking root and the clarification of errors.

The following serves as an example of how quickly strategies can be improved: A group of ten second-graders were asked to share 27 sweets equally among three children. Eight students drew a direct representation and shared out the sweets one at a time, whilst two students dealt out five each during the first round and two each during the next two rounds. After explaining their thinking to each other, they were asked to divide 37 sweets among three children. This time, oniy one student shared out one at a time, whereas the other nine students used either two rounds of five each or one round of ten each, followed by a round of two each.

Conclusion

Our results show that it is viable to base the teaching of division on word problems and children's own solution strategies in an instructional approach compatible with a socio-constructivist view of learning. Children do not experience the solution of division-type problems as more difficult than any other problem types, but find them interesting and stimulating.

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Age Level: Adult
ID #1: Constructivism
ID #2: Teacher Educat

PROBLEM SOLVING AND THINKING: A REPORT OF FINDINGS

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This paper presents the Problem Solving and Thinking Project's framework and results of researching a constructivist view of learning. The goal of the project was to advance our understanding of the psychological aspects of teaching and learning mathematical problem solving.

The Problem Solving and Thinking Project (PSTP), 1986-1990, sponsored by the National Science Foundation, investigated the relationship between middle school inservice teachers' metacognitive activity and knowledge and their problem-solving ability. We assumed a relationship between metacognitive activity and mathematical problem-solving performance; specifically, that the monitoring and regulation of one's knowledge, beliefs, and strategies could favorably influence problem solving. Improvement of teachers' problem-solving abilities occurred through a metacognitive/constructive teacher education process (Schultz & Hart, 1991). The purpose of this paper is to present the PSTP research experiences and results which are serving the current Atlanta Math Project (AMP) (1990-1994) teacher enhancement grant.

Method

We originally asked if a teacher's metacognitive activity could be increased through instruction if it focused on metacognitive experience and knowledge. And, we asked if problem-solving success could be improved through increased metacognitive experience and knowledge. The research approach was overwhelmingly qualitative, grounded in naturalistic inquiry (Lincoln & Guba, 1985) and relying on the technique of episodic parsing of protocols (Schoenfeld, 1983). The one exception was an exploration of techniques to quantify beliefs to study their directionality and magnitude, where principal component analyses and multiple regression analyses were used (Lee, 1990).

For 10 weeks, 15 teachers from four Atlanta area school systems participated in an Institute on Problem Solving and Thinking, a Georgia S. 'e University graduate course on the teaching of middle school-level mathematical problem solving, developed by and for the project. The Institute centered around modeling and facilitating activities by three groups: the teacher educators (Schultz & Hart, the researchers), the classroom teachers, and the teachers' students. Videotaping was liberally used to reflect on the problem-solving protocols of each group.

Data were collected from the teachers in the form of: (a) pre and post videotaped problem-solving protocols of individuals and small groups, (b) pre and post videotaped problem-solving lessons taught by the teachers to their mathematics students, (c) videotaped segments of teachers' respective individual problem-solving, (d) written reflection logs, (e) pre and post problem-solving tests, and (f) problem-solving sort tasks. Similar data were collected from the teacher educators.

During the data analysis, which occurred side-by-side with data collection, we identified belief systems (Hart, 1987), a highly charged, salient factor associated with metacognition, problem-solving performance, and construction of mathematical knowledge. It was impossible to sort this single factor out completely, so we studied it in relation to others, putting on hold the original question of the study. Nine of the 15 data sets were used for the analysis reported here.

The evolution of our research focus (Schultz, in press), techniques of data collection, data analysis, and data interpretation occurred through a series of negotiations with other people. The project involved five research consultants from other institutions for brief or long-term consultation depending on what stage our work was in; collaborative graduate student research internships; three educational specialist scholarly papers; two doctoral dissertations; and negotiation among the researchers and, very importantly, the teachers in the project. We called the PSTP "constructivist research," where new knowledge was constructed through the interpretive frameworks of all PSTP participants.

Results and Outcomes

Opposite directional beliefs were found in practice in nine teachers, which we called productive and nonproductive (Lee, 1990; Lee & Schultz, in progress). The beliefs found were similar to those reported by others, e.g., Garofalo, Frank, Schoenfeld, Lesh, Lester, Silver. Included are beliefs related to memorization versus thinking, time, exactness of answers, teacher versus student as authority, number of ways to get an answer, number of answers, solving problems alone versus with others, neat versus messy mathematics, and size of numbers. For example, a nonproductive belief expressed by a teacher during a problem-solving protocol, using the think-aloud technique, concerned an inability to recall a formula in order to solve the problem. The alternative to that would have been to rely on thinking the problem through.

In general, it was found that before the Institute teachers exhibited more nonproductive beliefs than productive beliefs during individual problem-solving protocols. Moreover, it was found that in the brief 10-week course, beliefs were reconstructed. Interestingly, all of the commonly-held beliefs were nonproductive before participating in the teacher training with a positive correlation between nonproductive beliefs and problem-solving ability. However, the number of teachers who expressed productive beliefs increased after the Institute with a higher correlation with successful problem-solving ability. It appears that beliefs can be reconstructed through training and that productive beliefs can increase problem-solving performance.

Hart (1987) conducted an in-depth study of Jill, one teacher who repeatedly verbalized references to not having enough time to solve the problems assigned even though no time restriction was ever given. She experienced unsuccessful attempts at the pre and post problem-solving sessions. It appeared that Jill's belief may have incapacitated her to the point of failure. "I could figure it out if I had more time" (p. 240) was Jill's account of her performance. This, we feel, is a classic case of beliefs driving one's problem-solving behaviors.

Two other teachers, Gail and Marsha, were the focus of a study on beliefs of attributions of success and failure related to achievement and performance outcomes (Najee-ullah, 1989; Najee-ullah, Hart, & Schultz, 1989). These two teachers taught high school basic skills mathematics courses to students experiencing repeated failure in mathematics. It was thought to be particularly useful research to facilitate speculation on teacher influence of student beliefs.

The most common attribution offered by Gail and Marsha was ability, which was most often offered when explaining their problem-solving performance failures rather than their successes. Gail's explanations for her successes and failures had to do with internal and stable factors. Gail's internal attributions might suggest that she takes responsibility for her performance. Her stable attributions might suggest that she would expect similar results at problem solving--whether they be successful or not--and in case of failure might avoid similar situations or avoid putting effort into similar situations. For example, when trying to solve a problem while not meeting much success, Gail said, "...I don't feel like, I think I've probably intelligently explored all my options. I haven't really gone crazy on it yet. There is a depth to which I will sink on these things" (Najee-ullah et al., 1989, p. 281). The sense was that this was a typical response to a problem-solving effort gone bad--she knew she was not trying as hard as possible, nevertheless she decided to give up. It was her decision and she took responsibility for it. Monitoring this belief and corresponding behaviors during the Institute gave Gail an opportunity to question whether she might be modeling this belief and behavior in her mathematics classes.

Marsha's attributions of success were primarily external and unstable, suggesting that she does not take responsibility for her successes. Factors attributed to her failures were stable and sometimes external. For example, when she indicated that the camera was distracting her, she said, "If I had longer and the camera...I don't know...I probably could have solved it" (Najee-ullah et al., 1989, p. 282). The implications--that Marsha could have low expectations to succeed and high expectations to fail in mathematical problem solving--are serious for the population she teaches.

Analysis of the teacher educators' beliefs in practice were studied in light of our espoused beliefs, vis-a-vis having designed and implemented the Problem Solving and Thinking Institute. The analysis of Hart and myself (Schultz, 1988) was done through a series of reflective interpretations of data by Hart, Ropp (a PSTP teacher), and myself, where Ropp took the teacher-as-partner-in-research perspective. Ropp (1988) replicated the study renegotiating her perspective to that of student-in-the-class with reflective feedback from Schultz and Hart.

One activity from a four and a half-hour Institute class was chosen for analysis. The purpose of the activity was for Schultz and Hart to model constructivist teaching to be applied in the teachers' classrooms. This particular activity was chosen for its recursive significance in the Institute. We were to model in our teaching the very subject of the lesson. Learner outcomes would determine the subsequent lessons. To conduct the analysis as well as to model it was of particular significance in the spirit of the Institute. The results of this analysis are reported formally here, but were reported informally, as a part of the class instructional experiences in the Institute. "It is our view that telling teachers what they should do is no longer enough... We need[ed] to model this--we need[ed] to model how to reflect, how to listen, how to construct our own understandings of mathematics, of teaching mathematics, and of what mathematics education is all about" (Schultz, 1988, p. 8).

Through review of videotapes, transcriptions, and class notes, it was determined that 70% of the opportunities to be "constructivist" were, and that good judgement was exercised in deciding when to "tell" in contrast to facilitating the teachers' thinking through questioning techniques. However, an alarming number of instances were identified that clearly violated the hallmark of a constructivist classroom. In all 14 instances of non-constructivist teacher educator behaviors were found in a 1 1/2-hour class activity. These instances can be classified according to ourselves as teacher educators disregarding teacher responses, not giving teachers a chance to talk, rewording teacher responses, cutting teachers off, and talking too much. Though some of these behaviors might be classified as infractions of good manners, nevertheless good manners respects the needs of others to speak and to use the choice of words they so choose. It seems that a person (teacher or teacher educator) with good manners (constructivist view) accepts, though may not always agree with, others' ideas, but allows those ideas growth and development in a supportive exchange of thought-provoking and meaningful discourse. We did not anticipate "slipping up" at all, much less, so much. However, it was better to reflect back on this model lesson to see our counterproductive behaviors and to contemplate being more attentive next time, than to not have reflected back at all and missing the rich opportunities for our own growth and development as mathematics teacher educators.

Finally, a teacher education model evolved out of the Institute. In our approach, the metacognitive component of our teaching kept us honest since we reflected weekly with the teachers at the beginning and ending of each class about the general direction of their experiences, our teaching practices, and their sense of how much they were gaining by participating in the Institute. We've described what is now called the Experiential Teacher Education Model for Reflective Mathematics Teaching repeatedly in print, with the most comprehensive description in Schultz and Hart (1991).

Very briefly, the model includes a heterarchy of learners engaging in a recurring cycle of modeling, experiencing, and reflecting. Learners include teachers and teacher educators along with students--they are all "learners;" they are all "teachers." It is in this mode that we serve the mathematics teachers now in nine school systems through the Atlanta Math Project. And so this brings us full circle with our efforts to improve middle school teaching and learning. But now our circle is bigger.

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TRANSLATING FROM NATURAL LANGUAGE TO THE MATHEMATICAL SYSTEM OF ALGEBRAIC SIGNS AND VICEVERSA

(A CLINICAL STUDY WITH CHILDREN IN THE PRE-ALGEBRAIC STAGE)

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This paper deals with the translation, in both directions, of natural language (NL) into the mathematical system of signs (MSS_1) generated by previous learning during the arithmetical and pre-algebraic training of the pupils in primary school and the first grade of high school (12-13 years of age). This translation between NL and MSS_1 is one of the central features of classical teaching strategies for the solution of word problems by means of algebraic Mathematical Sign System (MSS_2).

BACKGROUND TO THE STUDY

This study is part of a more extensive research project entitled "*The Acquisition of Algebraic Language*". All the other studies previously reported under the overall title "*Operation of the Unknown*" (Filloy and Rojano, 1984-85-89; Filloy, 1986; Rojano, 1986) also form part of this same research project. The study deals with the same issues found in the work of Leslie Booth (Booth, 1984) and D. Küchemann (1981), concerning the different meanings of literal symbols and, in that of Matz (1982) regarding errors in the interpretation of syntactic rules when algebraic expressions are used. The study is also related to the research on learning strategies in which the solution of algebraic problems is one of the principal components (see Rojano, 1986; Trujillo, 1987). Other studies which have bearing on the subject can be found in Gallardo and Rojano (1988), where an analysis of difficulties in reading algebraic expressions among children with low pre-algebraic performance levels and of the same age as those studied here (13-14 years).

THEORETICAL FRAMEWORK

The overall theoretical framework of this article car be found in Filloy (1990). In this case, the model of teaching developed during the clinical interview is composed of a sequence of questions related to the translation NL $\longleftrightarrow$ MSS_1. The majority of the questions are taken from the usual text books. The moment of observation (13-14 years) and the grade are both selected so as to observe the tensions existing between the meanings attributed to the elementary algebraic concepts that are on the point of developing (greater operativity in linear equations, solution of word problems, introduction of algebraic expression, etc.), based on the arithmetic and pre-algebraic conceptual field constructed up to that moment (MSS_1). The tension is a result of the need to give the new operations and concepts a new sense (given by the new uses) that, in turn, will attribute new meanings to algebraic expressions represented by the same signs or more elaborated versions of them.

READING AND WRITING ALGEBRAIC EXPRESSIONS: OBSERVATION

The observation of the way that children read and write algebraic phrases was carried out by means of videotaped clinical interviews in a school called "*Centro Escolar Hermanos Revueltas*" in Mexico City. The children interviewed (12 in all) had already received instruction in pre-algebra an had been introduced to elementary algebra with the theme of solving linear equations and the corresponding word problems. But the teaching they had received had not yet been systematic regarding the use of open expressions and on the equivalence of algebraic expressions (tautologies like $(a + b)^2 = a^2 + 2ab + b^2$), nor had they dealt with the solution of simultaneous equations.

Children of three different levels of performance in mathematics were selected for the interviews: high, medium and low, that worked with a basic sequence of four blocks of items:

Block 1. The reading of equalities corresponding to geometric formulae, expressed in algebraic symbols, like $A = \pi r^2$, $A = l^2$, etc.

Block 2. The reading of open algebraic expressions like $(a+b)/2$, ab, $3ab$, a^2.

Block 3. The reading of algebraic equivalencies (tautologies) like $(a + b)^2 = a^2 + 2ab + b^2$.

Block 4. The interpretation of sentences expressed in natural language and their translation to mathematical symbols. For example, "the double of *a*", "*a* increased from two". "*a* decreased from two". Only in some cases of children with high and medium performance did we apply a fifth block consisting of systems of simultaneous equations of the type

$$\begin{array}{c} x = a \\ y = bx + c \end{array} \quad \text{and} \quad \begin{array}{c} x + y = c \\ ax + by = d \end{array}$$

with a, b, c and d particular whole numbers.

SOME RESULTS

I.- In Block 1 we found three levels of the interpretation of the formulae:

a) Textual Reading in NL of the expression without reference to any context.

b) Reading as in a), accompanied by a verbal reference of the elements of the expression to dimensions of a geometric figure, without specification of the latter by the subject.

c) Reading as in b), accompanied also by the association of a specific geometric figure (circle, square) and of the corresponding attribute (area, perimeter); this was not always done in a correct way. These three interpretative levels appeared both in a partial and in a total manner, depending on the level of pre-algebraic performance of the subject.

II.- A) With respect to Block 2, the textual reading in NL of expressions like (a+b)/2 was accompanied by: a) a reference to the dimensions of "ideal" geometric figures (heights, bases); b) the need to assign specific values to the letters in order to obtain a result and "close" the expression thought up by the subjects themselves (a not quite so complete behaviour has been reported by Booth (1984) and Collis (1975); c) the elaboration of an equation or equality starting from the expression and the numerical substitution for some of the literals.

B) In some cases in Block 2, in the numerical substitution, the election of the values by the subject appeared to be arbitrary; however, in expressions such as a-b, identical values for **a** and b are not immediately accepted, since the association of different values with different letters and viceversa is present (Collis, 1975). In children with a low pre-algebraic performance, it was observed a resistance to assigning a higher numerical value to b than to **a**, given the imminence of a negative result.

C) Furthermore, within the same Block 2, it was observed a tendency to give meanings to the open expressions in the context of word problems. This was found very clearly in the case of a mid-level girl in the following way:

Open Expression ⟶ Posing a Problem ⟶ Formulating an "Equation" (the expression is closed) ⟶ Obtention of **a** Result.

III.- The interpretation of "composite" expressions like $(a + b)^2$ and of algebraic tautologies like the development of the squared binomial (Blocks 2 and 3) presented a high level of difficulty and the majority of the subjects did not get beyond the most primitive level of reading in NL. The reading in NL of $(a + b)^2$ gave the typical error of $a^2 + b^2$, this being a counter-example of the explanation given by Matz (1982).

IV.- A) In the translation NL ⟶ MSS_1 we observed difficulties in elaborating the corresponding expressions in MSS_1, especially because in the direction NL ⟶ MSS_1 the meanings attributed to the terms used in NL predominate and terms like to increase and to decrease are not spontaneously identified with operations like + and -.

B) Here also it was found the need to create expressions by means of assigning specific values to the letters mentioned in the initial text.

V.- It was also observed the creation of personal representations corresponding to the verbal sentences given and the need to create conventions for a unique reading of these invented "personal" representations. In these cases, symbols from arithmetic (numbers and operational signs) are accepted as unambiguous expressions of the verbal sentences involved.

FINAL DISCUSSION

Eleven Cognitive Tendencies which are present when learning more abstract concepts are described in "Cognitive Tendencies and Abstraction Processes in Algebra Learning" (Filloy, E. 1991). Results I to V of this study which are related to the passage from NL and MSS_1 to MSS_2, are below analyzed in terms of such tendencies:

I.- The three different interpretation levels detected in Block 1 (Reading Formulae) can be described through <u>Tendency eleven</u>: <u>The need to confer senses to the networks of ever more abstract actions, until converting them into operations</u>. But, the obstruction to pass from one level to the other can be described through <u>Tendency FIVE</u>: <u>Focusing on readings made in language strata that will not allow solving the problem situation</u>.

II.A.- A reproduction of the interpretative levels mentioned in the results of Block 1 was observed in Block 2, when students were asked to read open expressions. In this case, such levels can be described in terms of <u>Tendency THREE</u>: <u>Returning to more concrete situations upon the occurrence of an analysis situation</u>, since these kinds of expressions belong to the algebraic realm and interpretation levels (where geometric referents are associated with them) correspond to the so called "more concrete situations".

II.B.- When children don't accept that a and b in a-b can have the same numerical value, it can be noticed the manifestation of <u>Tendency SEVEN</u>: <u>The presence of appellative mechanisms</u>

which focus attention on the unchainment of wrong solving processes and Tendency EIGHT: The presence of inhibitory mechanisms (in this case, the avoidance of zero or negative results or the unacceptability of assigning the same value to different letters inhibits the possibility of dealing with this sort of expressions at an upper level of generality, where a and b can have arbitrary values).

II.C.- Connecting open expression with word problems or equations is a manifestation of Tendency NINE: The presence of obstructions arising from the influence of semantics on syntax, and viceversa.

III.- Behaviours described in part III of the Results have a description based on Tendency FIVE (mentioned above) and not in Tendency SIX: The articulation of erroneous generalizations, which would correspond to Matz's explanation.

IV.A.- Tendencies THREE and FIVE (mentioned above) are present when predominance of meanings attributed to words in the Natural Language obstructs the translation to MSS_2.

IV.B.- The same kind of behaviour reported in the reading section (Block 1, 2 and 3) was detected in the writing section (Block 4), when children tried to assign specific values to the letters mentioned in the text written in NL (Tendency THREE).

V.- When children displayed the creation of conventions to read their own symbolic representations (translating a text written in NL to symbols), Tendencies THREE, SEVEN, NINE and TEN: The generation of syntactical errors due to the production of intermediate personal codes in order to confer senses to intermediate concrete actions are present, not necessarily related to syntactical errors but to the need of producing unambiguous expressions.

It seams that future studies should be focused on the use of more elaborated Cognitive Models for the learner in which the cognitive tendencies described in this paper and in [7] will be natural manifestations of the abstractions processes produced when the subject is learning to be a competent user of the different strata that can be (theoretically) recognized in the different MSS (NL, MSS_1, MSS_2, ...) socially produced, whose codes the subject under observation have to master through the usual Teaching Models.

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Age Level: Teachers
Identifier 1: Frameworks
Identifier 2: Epistemology

EPISTEMOLOGICAL FOUNDATIONS FOR FRAMEWORKS WHICH STIMULATE NOTICING

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ABSTRACT

This paper outlines succinctly[1] the epistemological foundations of the Discipline of Noticing[2], which is concerned with change: not with researchers and educators changing teachers, which I believe to be ethically unsound and doomed to failure, but with attracting teachers to the enterprise of changing themselves. It focuses on the development and use of frameworks (Freudenthal (1988) called them *condensation kernals*.) for helping people to notice opportunities in the moment. The paper ends by suggesting that noticing is the core of everyone's functional (as distinct from theoretical) epistemology.

BACKGROUND

Most of the time, students, teachers, educators and researchers react to stimuli. Their actions are the working out or unfolding of decisions made hours, months or years earlier. Occasionally there is a moment of awakening, a moment when an opportunity presents itself, or more accurately, when there is fleeting awareness of alternative possible behaviour. Such a moment is a *moment of noticing*. The word *noticing* has its etymological roots in the making of a distinction, in stressing some perceived features and consequently ignoring others.

For an opportunity to exist in the moment, there has to be a convergence of recognition of some typical situation, with awareness of alternative action or

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[1] The page restriction makes it impossible to do full justice, or to refer explicitly to the many parallel ideas to be found in the Upanishads, Plato, Chuang Tsu, St. Augustine, Da Vinci, Montaigne, Dewey, James, Mead, Vygotsky and Gattegno, not to say Von Glasersfeld, Kilpatrick, Schoenfeld, Cohen, Boud, Schön, Gates, and Jaworski, to name but a few.

[2] Drawing particularly on a vivid expression of it by my late colleague, Joy Davis (1990).

behaviour. Thus there is no opportunity when you become starkly aware of the unstoppable flow of events in which you are embroiled (awareness alone is not enough); there is no opportunity if you have prepared a dozen different strategies, but none of them occur to you in the midst of an event (alternative actions are not sufficient). The two have to come together.

The Discipline of Noticing is an attempt to outline a disciplined form of personal and collective enquiry into how to sharpen moments of noticing so that they shift

> from the retrospective "I could have . . ." or "I should have . . ."
>
> to the present 'spective' "I could . . .",
>
> by means of the descriptive but non-judgemental postspective review: "I did . . .", and the prospective preparation of "I will . . . ",

by imagining oneself in a similar situation in the future, entering that moment as vividly as possible, and mentally carrying out each of the actions amongst which you wish to choose in the moment.

UNDERLYING ASSUMPTIONS

Construal

People try to make sense of events in which they become involved. That sense making is partly cognitive, partly affective, and partly enactive, and all three feed and support each other. Thus the practices of a group produce behaviour, emotion and belief, which in turn sensitise people to particular practices.

Each person has a multiplicity of selves constructed or formed in response to social and psychological forces in order to defend the core emptiness from exposure to the outside world. Personality is the many layered shell which shields that emptiness form everyday awareness. Multiplicity evolves because each social context involves a collection of practices which define participation.

Experience is not enough to promote learning. One thing that people fail to learn from experience is that they do not often learn from experience alone. Some action is necessary, whether it be based in cultural practices, in cognitive reconstruction or in some combination.

Attention

When you think back to past events, they come to mind in fragments, not in complete detail. Experience recalled is fragmentary. Fragments begin with moments of noticing, and slide into unaware, mechanical behaviour which is eventually overlaid by a fresh moment of noticing initiating a new fragment.

At any given time there are aspects of our world to which we are sensitised, and aspects of which we are oblivious. Certain aspects of an event or situation stand out and are attended to, while other details are not even noticed. The aspects of an event or situation which make it stand out are aspects resonant or dissonant with past experience and present expectation.

DISCIPLINED NOTICING

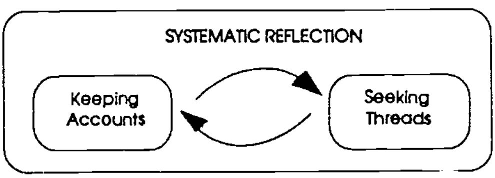
> . . . *is it not rather what we expect in men, that they should have numerous strands of experience lying side by side and never compare them with each other?* (George Eliot, Middlemarch).

The full discipline of noticing involves a number of overlapping phases. More than one phase can be operative at the same time, and there is a great deal of recycling and revisiting, not just a simple linear development.

Systematic Reflection

Systematic postspective review is essential. Brief but vividly descriptive *accounts of* salient moments that are readily recalled are collected and paralleled by memories of similar moments from other lessons, all without judgement or criticism, that is without *accounting for* them.

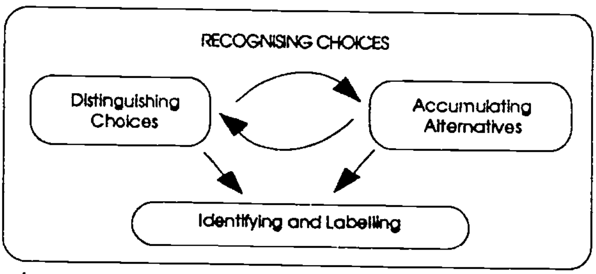
The accounts of moments of noticing provide a record of the threads of experience which are most salient for that teacher at that time. Once a corpus of accounts begins to accumulate, it is possible to look back over the accounts, say once a week or so, and to seek common threads.



There is a back and forth process of keeping accounts of moments, and looking for threads in those moments. Accounts provide data for detecting threads, and threads detected will tend to influence the subsequent moments which are noticed. In this way, the reflective practitioner begins to work on strands of their experience.

Recognising Choices

Decisions are about choices. By laying the strands of recent experience alongside the strands of past experience, you gain access to some possible ways of behaving in similar situations. You can choose to respond in a different way. You can also get ideas about alternative responses from reading other people's accounts, both practical and theoretical, and by watching colleagues in action, or imaginatively entering their descriptions of moments of noticing.



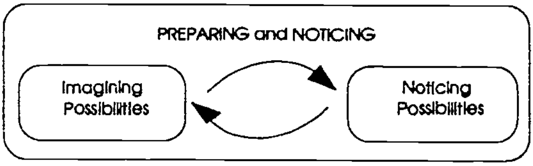
New behaviour patterns will only be attractive if they conform with attitudes and beliefs, and their manifestation will be guided by those attitudes and beliefs. That is why you cannot *deliver* a strategy to a colleague, since any strategy is intimately bound up with the user's own awareness of possibility.

In this phase of the discipline, your own strands are being juxtaposed with those of colleagues. By describing significant moments to each other, and then following up with moments jogged into memory as the result of someone else's descriptions, a collection of related accounts can be accumulated and *labelled* as a short hand for such incidents. The labels act as a framework to stimulate further noticing.

The search for resonance, for others to recognise similar moments to your own, part of the validation process. What is being validated is the incident, the vividness of description, and the similarity of what several people are attuned to notice. Comparing accounts of moments with colleagues enables negotiation of precision in what is noticed and what is possible.

Preparing and Noticing

When an alternative behaviour pattern is identified postspectively for a typical situation, mental imagery is very useful for prospectively preparing to use that strategy in the future.

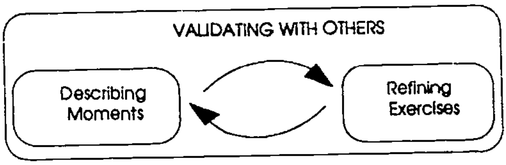


Validating With Others

As with any group endeavour, there are dangers of self-complacency and idiosyncraticity. A disciplined mode of enquiry, and an effective epistemology must provide safeguards against being carried away by individual or corporate delusion. If you do find something which seems to inform your teaching, which seems to offer you opportunities to behave in fresh ways, then you have a personal tool. But you could also be deluding yourself. Validation in disciplined noticing is twofold. There is personal validation, laying the strands of your own experience alongside each other and comparing them, and there is validating with colleagues, by juxtaposing your own and their experience.

You cannot get someone else to have your experience. All you can do is look for resonance in their experience. You can describe briefly-but-vividly some salient moments which illustrate your new found awareness in action and your use of particular strategies in that moment, and you can construct activities and exercises for colleagues which you think will highlight and focus attention on what is for you a potent distinction. In this way you hope to help others to notice, to distinguish, what you notice. Whether they do, and whether they find it possible and helpful to act upon such noticing is a matter for them to explore. The whole cycle goes around again.

Popper's fallibilism requires theories to be falsifiable. The Discipline of Noticing requires theories to be sufficiently resonant to inform future practice, but this may take work. Lack of resonance falsifies the theory only at the given time and under the given conditions, since resonance may arise later. By maintaining a questioning attitude, seeking recognition of salient moments and resonance with the experience of others, you act against a tendency to become fixed in your new ways.



THE PRACTICE OF FRAMEWORKS

To develop professionally, as student, teacher, educator or researcher, it behooves us to extend our sensitivities and to extend our moments of metacognitive awareness. One way to do this is through frameworks. On the surface, a framework is a collection of words, such as

Do, Talk, & Record;	See, Say and Record;	Know and Want;
Conjecturing Atmosphere;	Specialising and Generalising;	Manipulating, Getting a Sense of, Articulating
Directed, Prompted, Spontaneous;	Noticing and Marking;	Account Of and Account For.

These are just words. They become frameworks **only** when they come to summarise, conjure up, and label recent experience which itself is resonant of past experiences. With effort, frameworks can become triggers to sharpen awareness and to release moments in which real choices can be made. A framework has validity for you, for a time, it it serves to inform your practice by stimulating moments of noticing in which choices can be made.

Epistemological Roots

How does sensitivity to notice certain things arise? When in the midst of some event, whether reading, talking, dreaming, or listening, it may happen that something suddenly strikes a chord. There is resonance with some aspect of past experience. A latent sensitivity has been activated. It needs strengthening and modification, and especially useful is some sort of tag or label which in turn will trigger the sensitivity in the future.

How then do latent sensitivities arise in the first place? This question is critical to educators and teachers who want to awaken sensitivities in their clients. Undifferentiated experience provides the subsoil: systematic reflection (through making distinctions) generates soil, and intention waters that soil and makes it fertile.

The combination of *framework labels* richly resonant of past experience, preparation for the future by imagining yourself in a typical situation activating some fresh response that you wish to try, and a supportive network of colleagues dedicated to building a disciplined approach to professional development, can lead to real growth. Furthermore, exactly the same process is what a teacher aims for in their pupils: real growth in awareness, construal, comprehension and understanding.

CONCLUSION

Ultimately validity is an individual matter, supported by a community practice, to the extent of resonance of description with experience, moderated by predisposition to other epistemological positions, and mediated by the strength of intention brought to bear upon the enterprise.

The extent to which you found resonance with your own practice provides a measure of validation for you, now. Resonance may arise later through the development of a corresponding bed of experience. I suggest that *noticing* describes how most people actually operate, whatever their avowed epistemological and methodological stance.

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Age Level: primary / secondary
Identifier 1: symbols
Identifier 2: language

METAPHORIC AND METONYMIC DISCOURSE IN MATHEMATICS CLASSROOMS

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ABSTRACT

This paper explores a tension between an erstwhile concern with 'meaning' on the one hand (often confused with simple reference) and the development of fluency and automaticity of symbol manipulation on the other, as desirable goals of mathematics education. The theoretical discussion is illustrated with a brief account of a secondary algebra lesson where the focus is more on metonymic rather than metaphoric discourse as an approach to symbol generation.

> The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of 'meaning', of the 'existence' of mathematical objects. *René Thom*

> Civilisation advances by extending the number of important operations we can perform without thinking about them.
> *Alfred Whitehead*

There is a current tension in discussions of the teaching of mathematics, which has been polarised into a conflict ostensibly between 'understanding' (adopting a so-called 'meaningful' approach) on the one hand, and automation and fluency at 'doing' on the other (usually perjoratively labelled 'rote' methods). One current view is that pupils should always understand *before* being asked to do a task or carry out a calculation, for instance.

Such a reaction is very understandable, in response to a history of mathematics teaching in schools which resulted in distressed, sometimes frantic pupils on the one hand, and in concerns about the level of mathematics learned on the other. However, one question which this view leaves unanswered is how to work on gaining fluency in handling mathematical symbols, in automating and consequently 'forgetting' what you are doing, so that conscious attention can be freed up for places where it will be needed in the future. The following quotation expresses a similar view.

> We do not pay enough attention to the actual techniques involved in helping people gain facility in the handling of mathematical symbols. ... in some contexts what is required – eventually – is a fluency with mathematical symbols that is independent of any awareness of current 'external' meaning. In linguistic jargon, 'signifiers' can sometimes gain more meaning from their connection with other signifiers than from what is being signified.
>
> Linguists have called the movement 'along the chain of signifiers' metonymic whereas 'the descent to the signified' is metaphoric. ... The important point is that there are two sharply distinguished aspects (metonymic relations along the chain of signifiers and metaphoric ones which descend into meaning) which may be stressed at different times and for different purposes. Dick Tahta, 1985, p. 49

We can choose to offer our pupils Cuisenaire rods, for instance, in order to supply a more tangible referent for number. We can also choose to offer pupils number-word games and rhymes, where there is no such appeal to physical materials, where the activity is almost entirely linguistic. Classroom decisions need not be an either / or, and both activities alluded to above contribute to a sense of number. to the meaning of number. Neither is it a transparent question as to which should come before the other.

Meaning comes about from associations and connections (for example, the play on words which links 'pie' charts to 'pi'), as well as a more direct sense of reference, of knowing 'what the fraction 2/7 refers to' in some particular context. Meaning also comes from images, and from the creative use of language. Why do we use the same word, *multiplication*, for different operations between whole numbers, negative numbers, fractions and matrices? Why do we call the first three entities 'numbers' and not the last? Naming is an important aspect of doing and learning mathematics. and far from being 'arbitrary' as some people would have us believe it, is directly concerned with the creation and expression of mathematical meaning.

The Tahta quotation above employed Roman Jacobson's fundamental distinction between the metaphoric and metonymic axes of language in his account of how language 'works'. To the extent that it is productive to explore the linguistic aspects of mathematical discourse as a means of gaining greater insight into the phenomenon known as 'doing mathematics', what perspectives does Jacobson's distinction offer? (For further discussion of metonymy, see Pimm, 1990.)

In The *Mastery of Reason*,Valerie Walkerdine (1988) explores the complex significations that occur on relatively commonplace mathematical terms and draws attention to the creation of meaning *within* practices. She comments: (pp. 93-6) "The same signifiers may exist across practices, but this does not mean that the same signs are created. ... Formal academic mathematics, as an axiomatic system, is built precisely on a bounded discourse, in which the practice operates by means of suppression of all aspects of multiple signification". (For further discussion of this book, see Pimm, in press.)

Walkerdine also poses a key (general) question right at the beginning of her book: "How do children come to read the myriad of arbitrary signifiers – the words, gestures, objects, etc. – with which they are surrounded, such that their arbitrariness is banished and they appear to have the meaning that is conventional?" (p. 3) It is this question, with regard to beginning algebraic symbolism, that I start to address in this paper.

In a chapter entitled The *achievement of mastery*, Walkerdine offers an episode from a top infant class (6–7-year-olds) where one pupil, Michael, comes to grips with the possibility of working with signifiers (the numerals) alone when doing two-digit additions, despite the teacher using bundles of matchsticks as erstwhile signifieds for the procedure. What sort of discovery has Michael made? It is not about action with objects. His discovery is a linguistic one about the mathematical writing system, which allows him to operate with the symbols *as if* they were the objects of mathematics. This acting 'as if' is one of the powerful practices of mathematicians.

Walkerdine's minutely-detailed analysis of this episode illustrates the complexity and mutability of signification in the area of mathematics, as well as giving the lie to much of what is claimed about the 'concreteness' or 'transparency' of so-called concrete materials. She attempts to document some of the subtle, linguistic ways in which the teacher (through a combination of talk and gesture) *creates* mathematical meanings in classroom settings. She is also pointing to experience with symbols as a necessary part of learning mathematics, even with the youngest children: for them, mathematical signifiers form part of a system whose properties can be explored, as signifiers *per se* rather than as signifiers of something.

AN ALGEBRA LESSON

> Algebra is not what we write on paper, but is something that goes on inside us. So, as a teacher, I must realise that notation is only a way of representing algebra, not algebra itself. Dave Hewitt, 1985, p. 15

I illustrate this theoretical discussion with an instances from a secondary mathematics classroom, one where the teacher is concerned with pupils gaining facility with the manipulation of algebraic symbols. Specifically, I shall describe and present some transcripts of a lesson (involving use of algebraic expressions as forms, both naming and generating the referents) where there is considerable metonymic focus, although at certain points in the lesson, a switch into metaphor ('the descent into meaning') occurs. I am particularly interested in those moments of transition.

Before going on, here is an brief instance where such shifts can be seen at work. In working on 'Think of a number' problems, the focus can be on 'undoing' equations like $11(6(x + 4) - 5) = 100$ (perhaps built up by a rigmarole of "I think of a number, add four, multiply by six, ... and my answer is a hundred"). By staying with the form of matched pairs of operations, each undoing the other, a solution of $x = ((100/11) + 5) / 6 - 4$ can be obtained. At this point, a choice occurs: the teacher may decide to evaluate this number (say, offering calculators to the class) and then check that the number obtained does indeed satisfy the conditions of the problem. This would be a descent to the signified. Or the teacher can stop at the previous solution, and work on discussing with the class why this has to be a solution, focusing on the links between sets of symbols. Even descending to the signified can loosen the sense that x is a number that the teacher has thought of, when computation produces recurring decimals (as well as raising the problem that back-substitution may not exactly solve the equation).

Dave Hewitt has also developed a class activity which he calls 'Rulers'. It involves developing a relative positional number line on a blackboard which leads to the generation of equivalent arithmetic or algebraic expressions, documenting different sequences of moves ending up in the same place. He uses a ruler to focus attention. Movement (indicated by banging the ruler) to the left moves down a number and to the right moves up a number, and the starting number is announced anew at the start of each sequence. So, "This is

16, [three bangs to the right] 19". Tasks can be set concerning moves and start or finishing positions and a notation developed first to record (the language is descriptive) sequences of moves and later produce them (the language is generative). Further complexity can be introduced by adding a second row where each number is the double of the one above it. (Further rows for higher multpiles may be added too – for more detail, see PM647H, 1991.)

Because the link between number and position is relative, it is possible to generalise and not work with particular numbers, but rather some name for the number in order to act as a trace. "Did I tell you what number I started with? No. Let's call it something." Now, moves can be made and the notation develops to record anew at each stage. It is possible to take different routes and end up at the same box, thereby generating different expressions which must be equivalent (because they 'name' the same box). This is similar to the fact that finding multiple expressions (arising from different 'seeings' of certain situations) suggests the possibility of algebraic transformations because they all represent 'the number of ...'. One aim of the activity is to give pupils experience of generating equivalent expressions, yet where the overt focus is not the expressions.

> At one time, I introduced the equals sign and I didn't actually say anything about it and explain it at the time. Because at the time, we understood that this and that have the same effect. With both of them, I started here and ended up there, so there was a sense of the equivalence between the two. Since we were talking about equivalence at the time, by me putting the equals sign whilst we're talking, the two get associated. The implicit meaning gets taken up in this way and it becomes the written language through which we record what we've just said to each other.

There is a deliberate blurring in the activity as to whether the algebraic expressions developed are names of locations or instructions as to how to move. (When going from expressions to moves, they clearly have to be interpretable as moves.) There is a similar structure of state then operation giving rise to a new state to ordinary arithmetic expressions (e.g. $6 + 7 = 13$), interpreted as state (start with 6), then do something (add 7) to give a new state (13). So it is possible, for instance, to write $y - 2 = y - 2$ as a result worth recording: namely, start at the box labelled y, go back two squares, resulting in being at the box labelled y - 2. The teacher commented in an interview afterwards on how he works on symbols by avowedly *not* attending to them:

> I deliberately don't want to have the emphasis on the symbols because then they become an issue and I can imagine people getting quite worked up about what is x, what is y, which they do to some extent anyway, but with the attention on the operations, then you realise that it doesn't matter what's being said and they can be introduced and taken on board because that is not what I am attending to. ... So very deliberately I shift their attention to the process that's involved and not on the symbol. At one time I introduced the equals sign and didn't actually say anything about it and explain it at that time. ... There was a sense of equivalence between these two [expressions], and so since we were talking about equivalence at the time, by me putting the equal sign while we are talking, the two get associated and the implicit meaning of the equal sign gets taken up in this way and it becomes the written language through which we record what we've just said to each other.

As for images supporting the development of meaning, he commented:

> Offering images is a very important part of teaching mathematics. I have to say though that really I don't ... I can't *give* them images, I can only offer an activity. It will be up to each individual pupil to create their own image for what is actually taking place ... and the activity contains implicitly the mathematics I want to work on.

With regard to one of the metonymic foci of the lesson, namely the uses of certain noises consistently for different operations, as well as body images related to the symbolic forms (e.g. large hugging gestures with his arms when indicating the scope of brackets), Dave added:

> Well, I use noises at times because they're a form of getting attention to a particular thing, so I might use a particular noise as I'm going along a dividing line, and the fact that I'm using a noise means that I want some attention here, and they can provide an image for whatever it is. ... Sometimes the image stays with them and when they're writing, they have been known at time s to be making noises and the [aural] image helps them to recall how things are written.

CONCLUSION

In my talk, I hope to illustrate through video and discussion of this lesson how a particular mathematical practice can establish certain broad meanings for working on symbols and give particularity to Dick Tahta's comment (cited earlier) about how to work on fluency and automaticity of manipulation of mathematics symbols.

With regard to the purposes of mathematics teaching, the following discussion was offered as part of the second world conference on Islamic education (1980): it offers quite a different view from those customarily couched in terms of pragmatic societal 'usefulness'.

> The objective [of teaching mathematics] is to make the pupils implicitly able to formulate and understand abstractions and be steeped in the area of symbols. It is a good training for the mind so that they [pupils] may move from the concrete to the abstract, from sense experience to ideation, and from matter-of-factness to symbolisation. It makes them prepare for a much better understanding of how the Universe, which appears to be concrete and matter-of-fact, is actually *ayutullah*: signs of God – a symbol of reality.

A mathematician, David Henderson, has said, "I do mathematics to find out about myself". One reason for teaching mathematics may be so that our pupils may develop this way of finding out about themselves, as well as offering them access to their shared inheritance of mathematical images and ideas, language and symbolism, and the uses to which humans so far have put it.

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Adults
Teacher Empowerment
Professional Development

TEACHER EMPOWERMENT IN MATHEMATICS: NEGOTIATING MULTIPLE AGENDAS[1]

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This paper uses a theoretical framework developed from the literature on teacher empowerment to describe changes in the discourse process engaged in by six elementary teachers and four researchers participating in a mathematics study group. The purpose of this collaborative effort is to try to move toward a more conceptually based approach to mathematics instruction, curriculum development, and student assessment.

For the past two years, participants in a Math Study Group (MSG) have been meeting on a weekly or biweekly basis to explore various aspects of teaching for understanding in mathematics. This project, which involves four researchers and six teachers, is part of a larger effort initiated by the Michigan Partnership for New Education to prepare students for the 21st century. The partnership sponsors Professional Development Schools (PDS); this effort is consistent with recent attempts to redefine the nature of university/school relationships in the context of a restructured school environment (Holmes Group, 1990). Teachers and other practitioners collaborate with university faculty to improve teaching and learning for K-12 students, improve the education of new teachers and other educators, and make supporting changes in both the schools and the College as organizations. The MSG is one of several projects at the Elliott•Michigan State University PDS. The purpose of this collaborative effort is to try to move toward a more conceptually based approach to mathematics instruction, curriculum development, and student assessment.

During weekly meetings the MSG has examined several important elements of teaching for understanding including a wide range of curricular and instructional options have been examined. The assumptions that underlie this collaborative work can be summarized as follows: First, the vision of mathematics instruction outlined by reformers (National Research Council, 1990; National Council of Teachers of Mathematics, 1989, 1991; Mathematical Sciences Education Board, 1990), is at variance with what one is likely to observe in most classrooms. As Stodolsky (1988) explains, "While many math educators are proponents of problem solving and analysis, most instruction is geared to algorithmic learning" (p. 7). Second, changing existing practice involves more than adopting better curriculum materials, or increasing teacher accountability. Increasingly, teachers are being viewed as the agents of school policy rather than as its "dumb instruments" (McDonald, 1988, p. 471). Thus, there is a shift in reform strategy from top-down control to approaches that are more consistent with the notion of teacher "empowerment" (Porter, Archibald, & Tyree, in press).

Teacher empowerment is a popular but difficult concept. It encompasses both political and epistemological agendas (Prawat, 1991). From a political perspective, the most important empowerment goal is to increase teachers' professional authority, particularly as it relates to issues of curriculum and instruction. From an epistemological perspective, the purpose is to stimulate teachers' thinking about teaching and learning. By focusing on the possibilities for empowerment inherent in the school setting one creates new opportunities for teacher growth and change. A further complication in the empowerment concept is the fact that some theorists stress the importance of the personal context (i.e., "conversations with self"), while others assume more of an outward perspective, attempting to change the nature of the "conversations" teachers have with their settings. The press here is to expose teachers to new and more effective ways of construing and structuring the classroom and school environment (Prawat, 1991). These various aspects of empowerment are depicted in Figure 1 (see p. 7).

The framework presented in this figure informs the present study in two ways: First, it suggests that participants in any empowerment-oriented, educational reform effort are apt to be pursuing multiple--and often conflicting--goals. University participants, for example, may focus on the epistemological agenda; the intent, from this perspective, might be to get teachers to reexamine their traditional, hierarchical views of mathematics (Cell 3 in Figure 1). Teachers, on the other hand, may be pursuing a type of political "conversation with setting" (Cell 4), seeking more autonomy and control as a way to better address the needs of the "clients" they serve. These two purposes connect in various ways, but they also diverge--at least at the strategic level.

The framework presented in Figure 1 has been useful in another way as well. It has helped the Math Study Group evaluate past progress and plan more intelligently for the future. A look backward in time is the focus in the present paper. We will present case study data illustrating how the nature of discourse in the Math Study Group has changed over time. Hopefully, this study will demonstrate the usefulness of bringing a broad conceptual framework to bear in thinking about issues of teacher empowerment. It is our contention that math educators often fail to attend to the sorts of issues discussed in this paper. There is a tendency to overlook "process" related concerns in our work in schools--considering such issues to be tangential to the main thrust of changing teachers views about the teaching and learning of mathematics. This may reflect the fact, in part, that we lack the conceptual tools necessary to judge the quality of discourse in collaborative efforts. The present study attempts to address this need.

METHODS

Each MSG meeting was tape recorded and minutes were taken. Data sources for this paper include transcripts of two MSG sessions and additional documents relevant to the focus of discourse throughout the project's existence (i.e., meeting agendas and minutes). The meeting minutes were subjected to an informal analysis. These documents were carefully reviewed, noting separate issues raised on each occasion, and grouping them into two broad categories: resource and implementation issues. Within each of these general categories more refined distinctions were made as necessary. Complete transcripts of the two sessions were coded. Total lines of transcript as well as a range of lines per turn were determined for university and teacher participants. Categories were constructed which reflected the variations in the focus of discourse during the meeting. Turn-taking counts for university and teacher participants were determined within each category.

RESULTS

Meeting Minutes

According to these data sources, a host of issues were raised during the two years that the MSG has been deliberating. Analysis of meeting minutes reveals an early preoccupation with "resource" issues (i.e., availability of calculators for students, overhead projects, manipulatives). Included under this rubric were a host of more complex concerns relating to teacher time. While teachers grew more appreciative of the need for additional planning time as they became more committed to the concept of teaching for understanding , they also became more dissatisfied with the typical way of dealing with absence from the classroom. That is, a reliance on "busy work" administered by substitute teachers was increasingly viewed as being contradictory to project's overall thrust. Interpreting this in terms of the framework, it appears that teachers were wrestling with issues that reflected the interaction between epistemological and political issues (i.e., cells 3 and 4). Over time, "implementation" issues predominated in the meeting minutes. An examination of this data source reveals that the implementation category includes a wide range of issues. It encompasses practical issues such as how one manages cooperative groups, as well as complex instructional and curricular issues having to do with problem selection and topic focus (i.e., being able to appreciate the power of certain mathematical ideas).

Meeting Transcripts

The second main data source used in the study is the verbatim transcripts of two particularly important meetings. The first, which took place March 14, 1990, represented an initial effort to "map" the conceptual domain for a measurement unit. Although measurement topics are commonly dealt with at the elementary school level, instruction frequently occurs in a way that is inconsistent with the reform recommendations put forth by various educational groups . Thus, one of the goals of the MSG was to develop creative new ways to conceptualize and teach this content. The intent was to use the ideas identified during this meeting as the basis for planning a common unit, which would include similar activities and that would span grades K-5.

The second meeting took place nine months later, on November 13, 1990. The agenda was similar to that of the March meeting in that the purpose was to identify key ideas in an important mathematical domain--namely that of place value. This conceptual map guided lesson planning but did not result in a common unit. In this sense, the goal was different from that pursued in the earlier set of deliberations. The intent was to draw on the key ideas in place value and apply them to teaching grade related content (e.g., subtraction, division, fractions, numeration). These key ideas underlie much of the elementary curriculum. In this sense, planning for teaching place value became a broader, more ambitious agenda for teachers than a self-contained unit such as the measurement unit.

One of the most important assumptions underlying the current school reform effort is that teachers will be equal participants in the dialogic process that promotes innovative practice. Therefore, the quantity and quality of individual contributions to this process should be of concern to the reform community. In the present study, the former has been operationalized in a global way: "Turn taking" represents occasions in which participants both initiate and respond to discourse. To provide further insight about the nature of this discourse, researchers have developed a coding scheme which attempts to characterize the various purposes that may underlie a turn taking occasion. A complete analysis of both transcripts indicates that turn-taking purposes can be categorized in the following ten ways: (1) sharing information (e.g., building wide professional development school issues, upcoming PDS events); (2) facilitating group process (e.g., offers to schedule meeting rooms, taking minutes, fostering a climate for discussion); (3) discussing dissemination (e.g., negotiation of MSG products, planning presentations, reflecting on dissemination efforts); (4) discussing mathematical ideas; (5) characterizing and connecting important ideas; (6) raising questions about the mathematical ideas (e.g., questions to generate discussion and to probe mathematical understanding, questions to clarify individual mathematical knowledge); (7) referring to students as a context for discussing ideas; (8) formulating student interview questions; (9) referring to outside authority (e.g., textbooks, standardized tests, district guidelines, NCTM Standards, research articles, university participants); and (10) requesting resources.

In presenting results on changes in the nature of the discourse process in the collaborative project, two further distinctions had to be made in the data source. Both meetings began with a consideration of procedural issues. Included in this general category is discussion of dissemination issues, updates from representatives concerning building wide issues, and information about upcoming PDS events. Both in the March, 1990, and November, 1990 meetings, procedural issues occupied approximately 30% of the total meeting time. During the procedural portion of both meetings, most turns were taken in the first three of the ten categories and were distributed equally between university and teacher participants. Other concerns--those centered on curricular and instructional issues--constitute the second major component of each meeting. Approximately 49% of the March, 1990 meeting was devoted to this category of concern. In contrast, 69% of the November, 1990 meeting time dealt with this set of issues. In the

March, 1990 meeting, a third, closely related issue was addressed: how can teachers assess student knowledge in the measurement domain? Due to time constraints, this issue was not addressed in the November, 1990 meeting.

The focus in the remainder of this section will be on the curricular/instructional portion of each meeting. Here, some interesting contrasts between the measurement and place value discussions are evident. For example, in the earlier meeting 62% of the discourse turns were coded into categories 4, 5, or 6. These categories deal with the raising and questioning of mathematical ideas and the development of connections among different ideas. Thus, these three discourse categories lie at the heart of the collaborative exchange about mathematics among MSG participants. In the second meeting, dealing with place value, a higher percentage (81%) of the contributions were of this sort. While it may be useful to consider these three categories as a whole for purposes of characterizing the quality of discourse in this portion of each meeting, there are some important distinctions within these categories--particularly as they relate to differences between the university and teacher participants. In both the early and late meetings, the majority of "connecting type" contributions (category 5) were made by university participants. Given the backgrounds and experiences of the university participants, this is not surprising.

There was an interesting shift between the early and late meetings in who took primary responsibility for raising questions about the mathematical issues being discussed. 16 of the 19 question-raising comments in the March, 1990 meeting were initiated by teachers compared to only 13 of 34 in the November, 1990 meeting. While there is a decrease in the number of questions asked by teachers, further analysis reveals that this reflects an increase rather than a decrease in teachers' willingness to examine knowledge claims (cell 1). Even though the teachers asked more questions during the first meeting, most of these questions were directed at university "experts" in order to clarify individual mathematical knowledge. For example, during the March, 1990 meeting, one teacher pressed a university participant to tell her whether or not she should teach the metric system. Other teachers joined in the effort to push the university participant to make a judgment about this.

In contrast, during the November, 1990 meeting, when a teacher raised a question about the position of zero in relation to the decimal point, it was directed to the group. Both university and teacher participants took part in a lengthy discussion to generate ideas in response to this question. During the second meeting, the responsibility for question asking was more evenly distributed between university and teacher participants; furthermore, there was less reliance on the "university source" of expertise in answering these questions It is our contention that the MSG itself has increasingly become the source of knowledge for participants.

The hypothesis that there has been an important shift in the basis of authority within the group is supported by the pattern of turn taking responses coded as category 9. Nearly 10% of the curricular/instructional comments in the March, 1990 made reference to outside authority (i.e., district guidelines, NCTM Standards, research articles) as the source for claims about the mathematics curriculum. The overwhelming majority (i.e., 86%) of these responses were made by teachers. In contrast, only 1% of the contributions made by participants of the MSG during the November, 1990 meeting were of the "outside" authority variety. The insignificance of this amount is a testament to the group's willingness to test ideas among themselves.

One final aspect of the data is worth mentioning. During both meetings, university and teacher participants made reference to students. Teachers frequently referred to students as a means for both making sense of and contributing to the mathematical discourse. Interestingly, the teachers discourse turns tended to be longer when they were talking about students than their turns during other portions of the mathematical discussions. In much of the literature on teacher "voice" (cell 2), reliance on personal experience is regarded as the key to gaining entry into unfamiliar discourse domains. It is not surprising, therefore, that teachers rely heavily upon their experience with students in contributing to mathematical discourse. The fact that there was less of that on the latter occasion may suggest that teachers feel more comfortable dealing with the mathematical content on its own terms.

CONCLUSIONS

The empowerment framework presented in this paper is a useful heuristic for examining changes in the nature of collaborative discourse over time. The need to evaluate the success of such a process will grow in importance as reformers and teachers become increasingly involved in long-term, school-based projects. This, however, will require a more complex repertoire of assessment techniques. Short term projects, in contrast, will continue to lend themselves to more traditional, outcome-oriented modes of evaluation. Broad conceptual frameworks of the type used here will undoubtedly play a role in helping to generate these new techniques.

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[2]Other members of the Mathematics Study Group include Bart Crider, Karen Dalton, David Rentfrow, and Karen Ostermiller. We appreciate their feedback on earlier drafts.

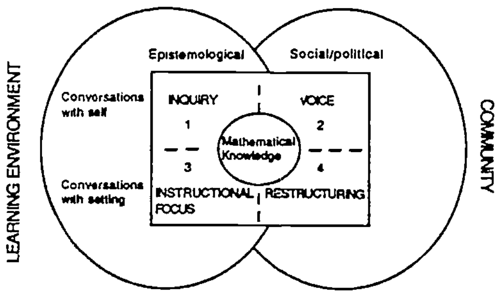

Figure 1

Age Level: College
Identifier #1: Rational Numbers
Identifier #2: Multiplicative Structures

MULTIDIMENSIONAL SCALING OF RATIONAL NUMBER CONCEPTS

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The purpose of this study was to investigate students' ability to perceive similarities among basic rational number problems. Multi-dimensional scaling was used to determine how the students organized their thinking about the problems. This was done by estimating the parameters and assessing the fit of various spatial distance models for proximity. Results show that college students enrolled in developmental mathematics are unable to determine the relatedness of basic rational number problems.

One goal of mathematics education should be the integration of mathematical concepts. This is particularly important in the learning of rational numbers. Mathematicians and mathematics educators have been able to form their own structure of rational number concepts, but this structure eludes many students.

Research in the area of rational numbers has shown that students invent strategies for dealing with rational numbers (Post, Behr and Lesh, 1986 and Koch, 1987). Furthermore, Post, Behr and Lesh suggested that it would be inappropriate to tell learners when certain schemata are appropriate: "children need to learn how to make such a determination on their own" (p. 345). Vergnaud (1983) suggested that multiplication, division, fractions, ratios, rational numbers and linear functions are not mathematically independent of one another and referred to this unified framework as the conceptual field of multiplicative structures. He did not, however, consider how this conceptual field is viewed by individual learners.

Current national reports are critical of students' performance in mathematics at all levels. Most attention is devoted to the pre-college curriculum, with reforms being mandated and research being designed to revolutionize the way mathematics has traditionally been taught. Less attention, however, has been paid to college-level students whom the K-12 system has failed. Currently, about 25 percent of all students entering public and private colleges and universities enroll in remedial courses in arithmetic and algebra (Hall, 1985). However, few of these college-age students achieve levels of higher competence in mathematics or gain the analytical thinking skills required for success in college-level mathematics. Little is known about why many of these students do not succeed and others do.

Lochhead (1981) stated that for students who have had difficulty in learning mathematics, particularly older students, poor learning and thinking habits can lead to misconceptions. This may be the result of memorizing procedures and focusing on the answer, as opposed to trying to comprehend the problem situation. Confrey and Lanier (1980), in their study of ninth grade general mathematics students, found evidence that the students' primary objective was the answer. They further posited that students lack of flexibility and their persistence of a single method often led to incorrect generalizations. The inability to see beyond the answer and make correct generalizations regarding arithmetic procedures may get students through a course in arithmetic, but can be detrimental to them in future courses such as algebra and calculus. This is especially true with problems relating to rational numbers. Difficulties that students have in algebra are often related directly to problems with arithmetic (Kieran, 1982).

It appears that students have been attending to the development of isolated skills rather than focusing on building the of concepts, relationships and understanding. The distinction between skills and understanding, or conceptual knowledge and procedural knowledge, is germane to problems related to those who have had difficulty learning rational number concepts. Results of the Fourth National Assessment of

Educational Progress (Kouba, Carpenter and Swafford, 1989) found that "many students appear to have learned fraction computations as procedures without developing the underlying conceptual knowledge about fractions" (p.79). Furthermore, approximately 33% of the seventh graders and 25% of the eleventh graders had limited knowledge of these procedures. Unfortunately, mathematics can be, and often is, taught without the necessary conceptual foundation. Silver (1986) suggested that procedural knowledge is limited unless it is accompanied by the appropriate underlying conceptual knowledge. He contented that researchers can increase their understanding of "complex relationships among elements of a student's conceptual and procedural knowledge base" by investigating students' perceptions of problem relatedness (p.186).

Multi-dimensional scaling is one technique that can be used to determine students' perceptions of problem relatedness by making similarity judgements among specific concepts and/or problems (Diekhoff, 1983; Fenker, 1975). This study used multi-dimensional scaling to analyze students' ratings of similarities to determine their perceptions of rational number problem relatedness.

Method

Subjects. The subjects involved in this study were twenty students enrolled in a developmental arithmetic course at a large mid-western university.

Instruments. Two instruments were developed for this study. The first was a ten-item rational number test (RNT). Each item on this test was chosen from the rational number concept of order, the concept of equivalence or operations with rational numbers. The second instrument, the Rational Number Rating Scale (RNRS), developed for this study was based on the same ten problems. Each problem was paired with every other problem. The problem pairs were each placed on a five-point scale. The final instrument consisted of these forty-five randomly ordered items.

Procedures. Prior to the administration of the instruments, all subjects received instruction in rational number concepts and operations. The subjects were then given the RNT to determine whether or not they had procedural knowledge of rational numbers. All subjects received greater than or equal to 80 percent on the RNT. The subjects were then given the RNRS.

Results

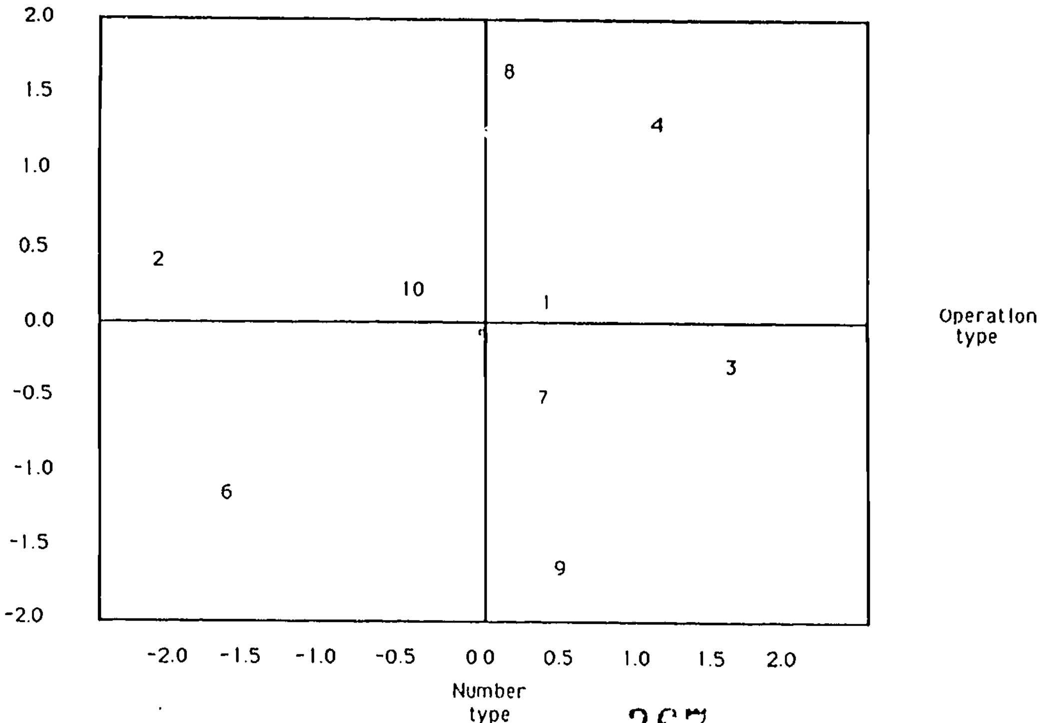
Analyses. The results of the RNRS were analyzed using ALSCAL to obtain the dimensional solutions. The measure of fit, both stress-by-dimension and RSQ (R^2)-by-dimension show that the group model represents a good fit for this group. Table 1 shows the Stress-by-dimension and the RSQ.

Table 1

DIMENSION	STRESS	R^2
4	0.027	0.987
3	0.117	0.852
2	0.206	0.723
1	0.382	0.565

Although the fit measure-by-dimensionality in Table 1 above suggests that the appropriate solution would be a four-dimensional solution, it was impossible to make meaning of each of those four dimensions. The fit for the two-dimensional solution is not as good as the fit of the four-dimensional solution, but it is still acceptable and the only solution in which meaning could be attributed to the dimensions. Figure 1 shows the configuration of the two-dimensional solution. The two-dimensions identified in this configuration are operation type: simple to more complex (horizontal axis) and number type: whole numbers to fractions (vertical axis). In this configuration, there appears to be some agreement between the ordering of the points around a helix line and the ordering of the corresponding items with respect to "face' item difficulty. Face item difficulty refers to the item difficulty superficially assigned by students.

Figure 1: Two-Dimensional Solution for Students



Discussion

When students try to make connections or generalizations on their own they often focus on irrelevant information such as units of measurement, problem format and problem context (Gliner, 1989). Students' attention to the surface structure of the problem and not the mathematical structure, can lead to the development of misconceptions and prohibit students from future success in mathematics. It is evident that this study supports and extends the notion that students focus heavily on surface structure and not on mathematical structure and are unable to see the relatedness among rational number problems. When the RNRS instrument was administered to fourteen university mathematics instructors, the clustering around the concept of order, the concept of equivalence and operations with rational numbers was evident. The results indicate that the students viewed each item as though they were unrelated to each other. This implies that although the students have developed the procedural knowledge, as demonstrated by their success on the RNT, they have not developed the underlying conceptual knowledge that is necessary to carry them through college level mathematics.

This study provides evidence that some students who have taken eleven or twelve years of mathematics, including two years of algebra, before enrolling in college, are still unable to grasp the mathematical relatedness among basic rational number problem types that is necessary in the learning of higher levels of mathematics.

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Appendix A: Sample Items

Rational Numbers Test

1. 3/5 = ?/20

5. Which is larger? 4.29713 or 4.297129

10. 3/5 + 6/7

Rational Number Rating Scale

	very different				very similar	
1. 3/5 = ?/20	1	2	3	4	5	3/5 + 6/7

Age level: 16-18 years and teachers.
Identifier #1: spatial visualization/ imagery.
Identifier #2: Krutetskii.

KRUTETSKII: A VIABLE THEORETICAL FRAMEWORK FOR RESEARCH ON IMAGERY IN MATHEMATICS EDUCATION.

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The paper describes part of the theoretical framework underlying a comprehensive study which involved three years of full-time research. Krutetskii (1969 and 1976) distinguished between ***level*** *of mathematical abilities in schoolchildren, determined largely by a verbal-logical component of thinking, and* ***type****, determined largely by a visual-pictorial component. This distinction was confirmed in the study of 54 grade 12 visualizers and 13 teachers in interaction in mathematics classrooms.*

Although studies of spatial **ability** abounded in the psychological literature (and these were mainly factor-analytic in nature), prior to the research the theoretical foundations of which are the subject of this paper, very few studies had examined the **preference** for using visual imagery when learning mathematics, and none had focused on the psychological implications in the high school mathematics classroom of various preferences of teachers and learners in this regard, as they interact in high school mathematics classrooms. Krutetskii's writings, published in English in 1969 and 1976, provided a viable theoretical and cognitive basis for this research. Researchers such as Lean and Clements (1981) had worked on the assumption that methods of solution of mathematical problems could be placed on a continuum between the poles **analytical** and **visual**. But as Krutetskii argued strongly, these are separate dimensions: an individual may be strong or weak on either or both of these independent dimensions.

The division of schoolchildren into three types with respect to mathematical abilities ("analytic", "geometric" and two types of "harmonic" in Krutetskii's case) antedates Krutetskii's work. Moses (1977) quoted such a study by Haeker and Zienen in 1931. The importance of Krutetskii's research, however, lies in its distinction between **level** of mathematical abilities, determined largely by a verbal-logical component of thinking, and **type** of mathematical abilities, determined largely by a visual-pictorial component. It is to the latter, the type of abilities, that preference for using visual imagery in mathematics relates.

Moses (1977) in the U.S.A. and Suwarsono (1982) in Australia had both done psychometric research which embraced Krutetskii's framework, and both studies confirmed Krutetskii's distinction between ability and preference. For instance, Suwarsono (1982) concluded from his data, "Individuals who have the ability to generate and manipulate visual images if required to do so might not prefer to use visual imagery in solving problems if the use of such imagery is voluntary" (p.261). All three studies (i.e., Krutetskii's, Moses' and Suwarsono's) provided the groundwork for the research described in this paper. The following sections describe how Krutetskii's theoretical framework grounded each of the stages in the research. Some of the results of the research have been published elsewhere (Presmeg, 1986 a and b); this paper will focus on some aspects of its theoretical framework. Key terms such as visual image, "a mental scheme depicting visual or spatial information", and mathematical visuality, "the extent to which a person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods" have also been described in these publications. The research was carried out between 1982 and 1985 in Cambridge, England, and Natal, South Africa.

Rationale for the methodology.

Suwarsono (1982) had developed a test and questionnaire which he used to measure preference for diagrams and imagery in the mathematical thinking of grade 7 students in Victoria, Australia. Moses (1977), on the other hand, was concerned with spatial ability rather than preference, and this was reflected in her test, which was designed for grade 5 students in the U.S. Krutetskii (1976) devoted one series of his experimental problems to spatial ability, and he wrote, "The ability to visualize abstract mathematical relationships and the ability for spatial geometric concepts showed a very high intercorrelation in our experiments. In every instance we observed a correspondence of the one with the other" (p.315). One gains the impression, however, that spatial ability as such lay on the periphery of his interests, whereas use of visual imagery was a central construct in his analysis of types of mathematical abilities. Perhaps as a result of the fact that the research of Suwarsono and Moses was grounded in the "Western tradition" as exemplified by Macfarlane Smith (1964), spatial ability played a central role in the psychometric research designs of both these investigators, although both made use of Krutetskii's constructs, experimental problems and results, and built on these.

The writer worked with students approaching the end of their high school mathematics career, and with their mathematics teachers. (For a full rationale for this choice, see Presmeg, 1985.) Thus the tests of Suwarsono and Moses were not suitable: amongst other considerations, the perceived level of difficulty of problems might have influenced the need for visual imagery (Kaufmann, 1979; Paivio, 1971). It was realized that for depth of understanding of the thought processes of students and teachers, Krutetskii's case study methodology based on task-based interviews involving think-aloud procedures over an extended period (months or years rather than weeks) was a viable methodological framework. The research study was thus hermeneutic in nature (Eisenhart, 1988). Case studies were based on observations in classrooms and clinical interviews conducted over an eight month period with students and teachers. Transcripts from 188 audiotaped interviews and 108 lessons provided the database for the study, which involved progressive focusing rather than pre-ordinate design (Hartnett, 1982). A further year was spent analyzing these data. Krutetskii's case study methodology was found to be an effective vehicle in the quest for depth of understanding.

However, there are difficulties associated with think-aloud procedures, as Krutetskii (1976) noted and as the present study confirmed. He wrote that "the study of problem solving is greatly complicated because the process is not always expressed objectively enough; many links in the mental process of solving a problem escape the investigator" (p.92). Two basic disadvantages may be summed up as follows:

(1) students may not be able to verbalize, and in fact may not even be aware of their solution processes;
(2) the presence of an observer might unsettle and distort the process of solution (ibid.).

In a recent study of imagery used by children in primary school, Owens (1991) found similar difficulties.

These considerations influenced the writer's decision to use a test and questionnaire for the initial choice of teachers and students for the study, as, too, did the practical advantage of being able to administer such an instrument in group format to a relatively large number of students. The instrument provided a rough measure of mathematical visuality which was adequate for the initial choice of both teachers and students. More refined categories and constructs emerged in the case studies later (Presmeg, 1985). The tests and questionnaires which followed Suwarsono's (1982) format and extended it for teachers, involved three sections of progressively more difficult word problems, some of which were taken from Krutetskii's (1976) series XXIII. No diagrams were provided, since it was realized that the presence of a diagram or any instructions to use imagery or diagrams might distort the natural preferences of teachers or students. Sections B (12 problems) and C (6 problems) were done by teachers and sections A (6 problems) and B by students. Comparison of scores of teachers and students on section B suggested that teachers have far less need for visual processing than do students on problems such as these: a median test yielded a significant difference between teachers and students. There was no significant difference between median scores of boys and girls (sections A and B).

Data collection and analysis.

After validation of the instrument described in the previous section, 13 high school mathematics teachers of a range of mathematical visuality scores were chosen and the appropriate sections were then administered to students in their grade 11 classes. (In South Africa there is one integrated mathematics curriculum for grades 11 and 12 and these years are usually taught by the same teacher. It would then be possible to follow these grade 11 students into their final year with the same teacher.) Although Krutetskii (1969 and 1976) had classified students into types with regard to their "use of visual supports in problem solving" (1976, p.318), it is clear from his writings that he considered this dimension to be a continuum; cutoff points are thus an arbitrary decision. In this fieldwork, then, visualizers were taken to be those individuals whose mathematical visuality (MV) scores exceeded the median MV scores of the sample from which they were drawn. 54 visualizers of a range of mathematical aptitudes, from low to high, were chosen in the classes of these 13 teachers.

During the eight months of intensive interviewing with these 54 visualizers (now in grade 12), one of the many surprizes was that the difficulties experienced by these students could be classified according to the structure of mathematical abilities which Krutetskii (1976) worked out from his studies of students who were mathematically gifted. Briefly, his categories were as follows.

1. Obtaining mathematical information: grasping the formal structure of a problem.
2. Processing mathematical information: logical thought, generalization, curtailment, flexibility, economy and reversibility.

3. Retaining mathematical information: generalized mathematical memory.

4. General synthetic component: mathematical cast of mind (1976, p.351).

In addition to confirmation that the nine categories represented above are viable not only for gifted students, evidence was also found that Krutetskii's "non-obligatory" categories are not essential for high achievement in school mathematics. Whether this finding applies at the of university or research mathematics is another story. Krutetskii's non-obligatory components were as follows.

1. Swiftness of mental processing.
2. Computational ability.
3. A memory for symbols, numbers and formulas.
4. An ability for spatial concepts.
5. An ability to visualize abstract mathematical relationships and dependencies (1976, p.351).

Conclusions.

Amongst other areas, the research project yielded conclusions about the difficulties and the strengths associated with visual processing in high school mathematics, about teachers' mathematical visuality in relation to the way they actually teach in the classroom (not necessarily consonant), about teachers' and students' attitudes towards and beliefs about mathematical visuality, and about the intricacies of interactions between teachers and students with regard to these matters. "Visual" teaching was not in all cases optimal for these visualizers! Suffice it to say here that Krutetskii's structures and categories, and his analysis of types of mathematical abilities in schoolchildren, provided a very viable theoretical framework on which to build and extend this research into visualization in high school mathematics, in all these areas.

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Age level: NA

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EPISTEMOLOGICAL UNDERPINNINGS OF PSYCHOLOGICAL APPROACHES TO MATHEMATICS INSTRUCTION

by

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ABSTRACT: A review of representative major works in mathematics instruction written over the last 20 years from a psychological perspective reveals a shift from behaviorist to cognitive to constructivist views of learning. Current research employs the terminology of constructivism quite loosely. Mathematics is implicitly viewed as a body of true knowledge that students can discover. Learning mathematics is treated as constructing the "right" knowledge.

Psychological research into learning and teaching has a longstanding tradition of using mathematics as the content domain for its investigations. Ultimately, the findings of this research seem to make their way into the practice of mathematics instruction through the design of curricula and textbooks, and as prescriptions for how mathematics should be taught. However, prior to the question of how mathematics should be taught is the question of what it means to know mathematics. This question is actually an amalgam of two separate categories of epistemological questions - questions about the nature of mathematics itself and questions about the nature of knowing and learning. Positions on these questions form the epistemological underpinnings of research into mathematics instruction. They are only sometimes explicitly stated, and are not uniform, as we will establish later in this paper. The goals of mathematics instruction have their roots in epistemological positions, as do the prescriptions of instructional techniques to meet those goals. Rene Thom points out "all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (Thom, 1973, p. 204). It rests also on conceptions of learning.

In this paper, we first survey the most common distinct positions about the nature of mathematics and the nature of learning and knowing. We then examine, in relation to this scheme, the epistemological positions contained in a sample of representative major works in mathematics education written from a psychological perspective and specifically concerned with instructional issues. We consider in particular Lauren Resnick's *The Psychology of Mathematics Instruction* (1981), Richard Skemp's *The Psychology of Learning Mathematics* (1971), and Alan Schoenfeld's *Cognitive Science and Mathematics Education* (1987).

Three views of the philosophical basis of mathematics are extant today. From a Platonic viewpoint, mathematical objects, like the line, the circle, the triangle, are real. They exist independently of our knowledge of them, although their existence is not physical. They are Plato's perfect immutable forms. Mathematical knowledge consists of truths about these abstract structures. Mathematicians do not invent mathematical knowledge; they discover it intuitively. Nevertheless, insight alone is not enough;

mathematical truths must be demonstrated to be true by a formal proof—by deduction from a set of definitions, axioms and postulates (Cornford, 1945, p. 223).

According to formalism, mathematical objects are not real; they have no existence either in the physical world, or any other. Formalism turns mathematics into a game of playing with symbols and formulae. Everything is invented. The starting assumptions are inventions and make no claim to match any external reality. Doing mathematics is manipulating symbols according to some set of invented rules (Davis and Hersh, 1981, p. 319). Any notions of truth or falsity pertain to the physical interpretation, not to the mathematical formula.

Both positions outlined so far have connotations of infallibility. In Platonism, mathematical knowledge is infallible because it is a true description of real mathematical objects. In formalism, the infallibility comes from the reliance on internal consistency and the inapplicability of truth value. But, whether the objects are real or not, whether the rules are real or not, as long as one defines the objects, specifies the rules, and follows them faithfully, one cannot go wrong. Critical fallibilism removes this sense of certainty from mathematical knowledge. In this view, first described by Lakatos, mathematics is quasi-empirical (Dawson, 1971; Lakatos, 1976). It starts from a problem or conjecture and proceeds by the same kind of criticism and correction that scientific theories are subject to. Mathematicians seek both proofs and refutations. The proofs are "explanations, justifications, elaborations which make the conjecture more plausible, more convincing, while it is being made more detailed and accurate under the pressure of counterexamples" (Davis & Hersh, 1981, p. 347). The strongest claim that one could make about a mathematical system would be that it is well-corroborated, but not that it is true. However, the missing element in Lakatos' view of mathematics is an answer to the question of what mathematics is about.

These three perspectives of the nature of mathematics differ along two interrelated dimensions. One is the polarity of mathematical knowledge as "discovered" vs "invented". The second is the polarity of "infallible" vs "uncertain". Let us see how these questions are played out in the psychology of mathematics education.

The classic behaviorist view entirely avoids the question of mind and thought. All human actions are interpreted in terms of behavior —a particular stimulus evokes a particular response. A person can be taught to produce the desired response by conditioning, that is, through a pattern of giving positive reinforcement for desired behavior and/or negative reinforcement for undesired behavior. A striking feature of the behaviorist approach is the reductionist view of knowledge. Learning tasks are treated as discrete and atomic stimulus-response pairs.

Where behaviorism is reductionist, Gestaltism is holistic. In many respects, Gestaltism is the antithesis of behaviorism. Gestaltists believe in rich mental structures that allow one to understand a situation as an indivisible whole. They hypothesize the existence of organizing principles in the human mind according to which all incoming sensations and experiences are interpreted. Thus there is no such thing as pure reception of information. Rather, there is recognition of pattern or structure —insight. Gestaltists urge instruction to promote the development of insight, but do not provide guidance as to how this goal might be accomplished.

Cognitive models of learning and knowing invoke a computer metaphor. The information-processing model of the mind describes cognitive processes in terms of flow and storage of information. Incoming data are initially stored in a short-term working memory which has a limited capacity both in quantity and time. Items of previously learned knowledge are activated and brought into short-term memory as well. Old and new information are compared and linked (or not, if learning is unsuccessful) and the new relations are stored in long-term memory. Th learning involves the formation or extension or change of cognitive structures in long-term memory. The. cognitive structures are typically imagined as networks of relations and may consist of both declarative (propositional) knowledge and procedural knowledge. One could view the formation of cognitive structures as essentially a process of accretion, where new nodes are added to existing branches in a deterministic fashion. This image is consistent with a "transmission" model of teaching, one in which information is believed to be received and stored by the student just as it was sent. The student's role is passive.

This view of the student as a passive recipient of information has been replaced by a constructivist view that assigns an active role to the learner in gaining knowledge. Knowledge is not transmitted. Each person constructs an understanding of his/her experiences. Thus, two people who participate in the same lesson will construct their own unique cognitive representations or schemata . Nevertheless, the underlying metaphor of the mind is still a computer metaphor.

References to learning as construction of knowledge are found in psychology, and also in philosophy (e.g., radical constructivism) and sociology (e.g., social construction of knowledge). The same words are used in different senses in the three contexts. As a result the surface similarity of positions that one infers from the use of common terminology can dissipate when the positions are examined more closely.

The constructivism that is coupled with the information-processing model seems to be essentially the position that the learner is an active agent in the process of learning. The learner makes interpretations and connections to build or transform cognitive structures into new forms. In this context, the claim that knowledge is "constructed" does not commit one to any particular claim about the nature or the truth of the knowledge. It seems to refer entirely to process, and more specifically, to the distinction between passive storage and active "construction".

Social constructionism denies that the warrant for what is commonly accepted as knowledge is observation, but rather that knowledge is a social artifact, resulting from the interactions of people in relationship with each other. What counts as intelligible is the product of negotiation and agreement within the operative community. Thus the warrants for "truth" (if the term can even be used in this context) are constantly evolving and free to change. Both radical constructivism and social constructionism deny the possibility of attaining objective knowledge of the real world. A critical aspect in which they differ is in the role that each position assigns to experience of the world. Von Glasersfeld (1984), in his view that the test of the viability of knowledge is its fit with the world, takes an empirical stance. Social constructionism is more extreme, positing that the very notion of what "fitting the world" entails is culturally determined, that the ways in which people experience the world are not innate or stable, but socially constructed and variable.

We proceed now to look at the interplay of epistemological positions about the nature of mathematics and psychological traditions of learning. In her book, *The Psychology of Mathematics for Instruction*, Lauren Resnick surveys the predominant bodies of psychological research in mathematics education, from Thorndike's associationism with its emphasis on drill and practice, through Gagne's cumulative learning theory with its emphasis on rational analysis of skills into component subskills, and Dienes' learning cycle model with its emphasis on free play and manipulatives, to cognitive psychology with its emphasis on constructed knowledge structures. While the cognitive paradigm is now a dominant one in research, the others are still evident in practice.

Thorndike (1922) applied behaviorist principles (he called it associationism) to mathematics instruction. Associationism holds that knowledge is built of simple connections or associations between stimuli and responses. Learning consists of establishing and strengthening these bonds through positive reinforcement. The fundamental instructional prescription arising out of Thorndike's associationism is drill and practice. For teaching arithmetic, for example, one would begin by analyzing and breaking down the subject matter into its most primitive processes. Next, one would draw up well-organized lists of all possible bonds or associations that constituted the subject matter to be taught. The goal is to habituate the child's mind to carry out the procedures quickly and accurately.

Gagne's cumulative learning theory is another model based on a reductionist view of mathematics. He proposed the existence of learning hierarchies which consist of skills which can be decomposed to simpler component subskills. In order to learn a complex skill, a child would first have to learn, in the hierarchical sequence, all the component subskills of that task. The instructional principles that arise out of Gagne's work are the necessity of rational analysis to develop the learning hierarchy prior to instruction, and then providing experiences for the child through which (s)he can learn the component subskills. Although empirical studies have generally supported Gagne's theory, they have also indicated that the hierarchies are not quite as precise or deterministic as they appear (Resnick, 1981, p. 48).

Both of these models contain the implicit belief in mathematical structure as a property of the discipline itself. Resnick argues that these views do not claim absolute truth for mathematics; the truth could be decreed by consensus. For all practical purposes, however, mathematics could be treated as a fixed, true body of knowledge (Resnick, 1981, p. 98).

The curriculum reform movements of the 1960's led to a change of stated emphasis from learning as performance of procedures to learning as understanding of mathematical concepts. This in turn, raised the question of what it means to "understand" mathematics. The common elements in the way that the term is used are insight (note the Gestaltist influence), recognition of interrelationships and reorganization of elements to see them in a new way. (Resnick, 1981, p. 105 & p.132)

The structural view is based on a view of mathematics as a conceptual and evolving discipline (Resnick, 1981, p. 105). This view of mathematics seems to be an eclectic blend of Platonism and formalism with a just a touch of empiricism thrown in. Understanding the structures of mathematics means understanding both the interrelationships among mathematical concepts and the rules by which they may be manipulated. Developing this kind of understanding cannot be accomplished by the drill and practice approach that focuses on disconnected individual mathematical procedures. Structural teaching approaches emphasize the use of concrete materials and multiple representations from which children can abstract mathematical concepts. The works of Montessori, Bruner, and Dienes are all examples of structure-oriented approaches to teaching mathematics.

Another constructivist approach to knowledge and learning is Piaget's. Piaget is "a realist of a rather special kind" (Sinclair, 1987, p. 29). While absolute knowledge of reality is impossible, theories can be constructed that are successive approximations of this reality. In *The Psychology of Learning Mathematics*, Richard Skemp outlines his psychological theory of children's learning of mathematical concepts. Skemp holds a Piagetian constructivist view of learning. Learning is the acquisition of schemata. By "schema", Skemp seems to mean the cognitive structure that represents a mathematical concept or set of concepts (Skemp, 1971, p. 25). Implied in Skemp's work, is the notion that there really is mathematical truth. He generally refers to mathematical knowledge as a hierarchy of concepts, similar in tone to Gagne's learning hierarchies. There is a presupposition that mathematical knowledge can be mapped out in a way that reflects its innate structure, with connotations that this structure is fixed, unchanging and true. Whether it is truth by matching an objective reality, or truth by consensus among mathematicians, Skemp clearly holds the position that mathematical knowledge is true and has some sort of existence outside the mind of the individual thinking about it.

Cognitive psychology has become the dominant paradigm in the psychology of instruction. In 1984, two conferences were held in Rochester, New York to examine the implications of cognitive science for mathematics instruction. The papers generated as a result of these conferences represent a shift in views of learning mathematics (Schoenfeld, 1987, p. xiv-xv). Learning and understanding are interpreted as the construction of particular kinds of representations of information. Greater stress is placed on the organization of knowledge and the role of metacognition, on the acquisition of problem-solving schemata and strategic knowledge. Silver (1987, p. 52-3) emphasizes the constructive nature of learning—that new information is not just added to an existing store, but is actively connected to "old" knowledge, and that entirely new relationships are invented. The information-processing model of the mind is the fundamental principle underpinning theories of how mathematics is learned. Learning however is interpreted much more broadly than before. The view of knowledge in general, and therefore, of mathematical knowledge in particular, presented by Silver is most like Lakatos' quasi-empirical view of mathematics. It encompasses a view of mathematical knowledge as dynamic, as having multiple "answers", and emphasizes making sense of the world. A similar view is contained in Kilpatrick's writing about problem-solving. He introduces the importance of problem formulation as well as problem solution and sees problem formulation as both a goal and a means of mathematics instruction (Kilpatrick, 1987a, p. 123). Statements like these illustrate a constructivist perspective of learning and imply a quasi-empirical view of mathematics.

In the writings examined, there is a close tie between the views of learning held by the authors and the instructional advice that they offer. There is less evidence of an explicit connection between their views of mathematics and the instructional prescriptions. Positions about the nature of mathematics are rarely made explicit and are not treated as central to instructional issues.

In one of the few exceptions to this pattern, Nesher specifically addresses the relationship between the nature of mathematical knowledge and approaches to the teaching of mathematics. She takes a pragmatic approach to the epistemological questions (Nesher, 1989, p. 188). Her approach is to avoid the controversy between different philosophical views of mathematics, and to adopt an epistemological position for the purpose of instruction. She opts for treating mathematics as real and mathematical statements as verifiable in terms of their correspondence with this real mathematical world. The teacher's task is to set up learning experiences for the child that will ensure "that the child's knowledge of mathematics will converge on the standard conventions" (Nesher, 1989, p. 197). Here is yet another example of the predominant pattern - a view of learning as an active, interpretive, constructive experience, coupled with a view of mathematics as a true body of knowledge. In other words, *learning mathematics means constructing the right knowledge.*

We see the changes over the past twenty years as three stages in the evolution of psychological theories of mathematics education. The first stage is behaviorist - viewing mathematics as a set of algorithmic procedures, mathematical knowledge as the ability to perform these procedures quickly and accurately, and learning as reinforcing particular stimulus-response pairs through drill and practice. The second stage is cognitive - viewing mathematics as an accepted set of concepts or structures, mathematical knowledge as conceptual understanding and problem-solving, and learning as construction of knowledge representations, with an emphasis on the role of metacognition. Both stages are evident in current practice in mathematics education.

A third stage, emerging in research and prescriptive literature like the NCTM Standards documents, is the situated cognitive view. Cognitive psychologists are paying increasing attention to the importance of context, of belief systems that learners have as they come to learn mathematics, and of the actual situation in which the mathematics is to be practiced. With this increasing awareness of the influence of context and social factors, as in the work of Jean Lave (1988), for example, new models of learning and instruction are likely to emerge. Some models, like "collaborative knowledge construction" (Brown & Palincsar, 1989) and "cognitive apprenticeship" (Collins, Brown, & Newman, 1989) from other content domains may be transferable to mathematics instruction.

Simple constructivism seems now to be firmly established in psychological views of mathematics instruction. What seems much less clear is the extent to which radical constructivism is an epistemological position held by either psychologists or mathematics educators. In the reviewed literature, the distinction between simple and radical constructivism is very rarely explicitly made. The term "constructivism" is used quite loosely and seems to function more as a legitimizing label than as any helpful description of an epistemological position. Moreover, the ways in which the term, "constructivism", is used imply that constructivism is compatible with both Platonic and quasi-empirical (and maybe even formalist) views of mathematical knowledge. The three views of mathematics considered here appear to be mutually exclusive positions. It would therefore be remarkable for all of them to be compatible with a constructivist view of learning. There appears to be at least a surface similarity between the radical constructivist view of knowledge and Lakatos' (Dawson, 1971; Lakatos, 1976) quasi-empirical view of mathematics. This apparent relationship needs to be explored (Kilpatrick, 1987, p. 20).

Three underlying issues in further psychological research in mathematics instruction must be attended to. The first is an explicit statement of researchers' view of mathematical knowledge. The second is a more responsible use of the term "constructivism". Authors need to specify the particular sense in which they use the term so that readers have a better chance to interpret their intended meaning accurately. The third is an examination of the relationship between constructivism and the prevailing views of mathematics. Productive research into mathematics learning and teaching must be based on coherent and compatible views of both learning and mathematics.

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Age level: Inservice teachers
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Identifier #2: teacher education

ASSESSING TEACHER CHANGE IN THE ATLANTA MATH PROJECT

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The Atlanta Math Project (AMP) is implementing a constructivist pedagogical model that supports the NCTM (1991) teaching standards. This paper presents the theoretical and conceptual frameworks for assessing teacher change over the four years of the project and briefly presents data collection sources.

The complex environment of the mathematics classroom provides a tangled web of factors that interact and impede an easy explanation of why the mathematical performance of school children in the United States is not noticeably improving. Recommendations for change come in the form of standards for curriculum and evaluation (National Council of Teachers of Mathematics, 1989) and standards for teaching (National Council of Teachers of Mathematics, 1991). These documents suggest learning environments that are quite different from the lecture dominated mathematics classroom that many teachers and students have experienced.

The Atlanta Math Project (AMP), a four-year National Science Foundation sponsored project in the first year of operation, is implementing a teacher education model which assists teachers in constructing new knowledge about the teaching and learning of mathematics which is consistent with the recommendations mentioned above and will study how these teachers change their instructional practices over the four years of the project. The project involves 3 teacher educator/researchers, 11 mathematics supervisors, and 22 teachers (in year one--more teachers will be added each year) in nine school systems in the metro-Atlanta area.

The teacher education model was developed as part of an earlier research project (Metacognition, Teachers and Problem Solving; Schultz & Hart; NSF, MDR 865-0008). It is based on the belief that researchers, teacher educators, mathematics supervisors, classroom teachers, and students are all teachers and are all learners in the mathematics education process. There are times when each individual will need to communicate or teach his or her mathematical understandings to another and when each individual will need to learn the mathematical understandings and thinking of another. For the purpose of this paper, however, teacher will be used to mean researcher, teacher educator, mathematics supervisor, or classroom teacher. The project is attempting to facilitate and study change in all teachers--ourselves as researchers included. This paper will describe the framework for studying teacher change.

Theoretical Orientation

The theoretical orientation of the Atlanta Math Project is framed by our view of learning and our understanding of cognitive processes. The following discussion will describe our perspective.

Constructivism and Teacher Learning

Our work is informed by a constructivist theory of learning (Van Glaserfeld, 1983) that suggests that acquiring knowledge is a process of providing structure and organization to the world in an effort to "make sense" of experience. Since several interpretations and organizations of an experience may be possible, it follows that the knowledge acquired by any one individual is unique and compatable with his or her pre-exisiting framework. Knowledge is modified in the face of problematic situations in order to remain viable, i.e., learning occurs. When applied to learning to teach mathematics or the acquisition of new or different pedagogical knowledge, this theoretical perspective suggests that teachers will reconstruct or modify their currently held knowledge and beliefs about learning and teaching if it is problematic. Further, what is learned by an individual teacher about alternative pedagogical practices will be unique to that teacher. How that knowledge is applied into practice in individual classrooms could look quite different.

Metacognition and Teacher Learning

Additionally, the theory of metacognition (Flavell, 1979; Schoenfeld, 1987) provides a perspective for our research. Metacognition theory consists of (at least) two components, metacognitive activity and metacognitive knowledge. The former consists of the monitoring and subsequent regulation of what you know, and of what you do with what you know. This "ability of the mind to observe its own operations" (Van Glaserfeld, 1983) is a critical component in productive mathematical thinking. It takes the individual beyond rote or algorithmic behavior to rationally controlled choice. The second component of metacognition, metacognitive knowledge--often referred to as beliefs (Flavell, 1979)--exists as information about ones cognitive processes and knowledge. Individuals hold beliefs about such things as the mathematics, about learning mathematics, about teaching mathematics and about the mathematical strengths and weaknesses of themselves and others. These beliefs motivate much of mathematical behavior. Both aspects of metacognition, metacognitive activity (monitoring and regulation) and metacognitive knowledge (beliefs) are important components for studying change.

Change in metacognitive activity. Individuals often do not monitor and regulate their cognitive processes during mathematical experiences or they do so infrequently . Change, however, may not follow the "more is better" philosophy. The frequency of monitoring behavior does not necessarily correspond to productive mathematical experiences. On the contrary, excessive monitoring may hinder productive processing. The quality and substance of what is monitored appears to have more impact.

Change in beliefs. In order to acquire new beliefs or change existing beliefs learners need to have problematic experiences that are contrary to what they believe about themselves as problem solvers or the nature of mathematics. For example, if learners believe there is usually one right solution to a mathematical problem and only one way to arrive at that solution, then in order to alter that knowledge they need experience with problems having more than one right answer and numerous solution paths. Applied to learning to teach in a manner consistent with the current recommendations, a teacher who believes learning occurs best when material is presented through lecture in an orderly, careful way, must encounter a problem with that approach. In order to motivate change in teaching behaviors, beliefs that are consistent with a traditional lecture-dominated rote-learning environment must be replaced with beliefs that are consistent with current recomendations for reform.

The Role of Reflection in Change

Since much of metacognition is unconscious, the process of becoming aware is a critical component of change. Learners are frequently not aware of the beliefs they hold that are motivating their mathematical behavior or their teaching behaviors and they are not aware of when or if they are monitoring and regulating their thinking. The coordination of new knowledge, whether it be cognitive or metacognitive, with already existing knowledge structures is facilitated through reflection. When learners are conscious of and get control over their beliefs, understandings and procedures they are more likely to change or alter their knowledge.

In addition, the process of constructing new knowledge, whether by students or teachers, is facilitated through reflection on the experiences that are motivating the change. Teachers need the opportunity to look back on their teaching strategies--to reflect on the outcome of their behaviors--and to learn from the experience. In turn, teachers need to assist students in reflecting on their mathematical experiences so that they also can learn from them.

The Teacher Education Model

In an effort to influence change in teacher knowledge we have designed experiences that are consistent with our theoretical perspective and with the recommendations mentioned at the beginning of this paper. In particular we provide experiences where the teachers role is to facilitate the conceptual organization of experience in their students rather than to provide information to them. We have designed experiences (1) that provide opportunities for teachers to develop beliefs that are productive for mathematical learning from a constructivist perspective; and (2) that model and encourage monitoring and regulation of behavior.

The most powerful vehicle for facilitating this change has been through the use of videotaping. It not only allows teachers to reflect on their personal problem-solving performance, it also allows them to observe how they teach and model problem solving, how students think about the mathematics they are learning, and how other teachers teach. In addition, it allows us as teacher educator/researchers to reflect on our mathematical performance and on how we model the teaching process.

Studying Teacher Change - The Conceptual Framework

For many, teaching from a perspective consistent with the NCTM (1989, 1991) recommendations requires a new set of assumptions about learning and requires the acquisition of new knowledge. This paradigm shift requires that the learning environment be radically altered. AMP is studying this change within the following conceptual framework which has three primary components: (a) teacher knowledge, (b) the learning environment, and (c) the project. Each will be discussed below.

Teacher knowledge. Our thinking in the area of teacher knowledge has been impacted by Shulman (1987). In particular we are interested in change in teacher metacognitive knowledge, pedagogical content knowledge and mathematical content knowledge. Our thinking in this area has been influenced heavily by our own work on metacognitive knowledge or beliefs as well as that of others (Cooney, 1985; Thompson, 1984). As teachers make a paradigm shift toward a constructivist view of learning, we raise the following questions:

(1) teacher beliefs

How do teacher beliefs about learning mathematics change over time?

How do teacher beliefs about teaching mathematics change over time?

How do teacher beliefs about the classroom environment change over time?

How do teachers beliefs about mathematical tasks and content change over time?

To what do teachers attribute success and failure in mathematical performance?

(2) pedagogical knowledge

How do teachers plan for instruction?

What do teachers consider as they do lesson planning?

What influences instructional choices during a lesson?

What do they "see" as they observe teachers teach?

(3) content knowledge

What problems do teachers have with the content they are teaching?

How does teaching from a constructivist perspective effect teacher content knowledge?

The learning environment. Our research on the nature of the learning environment as well as that of others (Cobb, Wood Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991; Lampert, 1988), has assisted in the development of our thinking in this area. The following questions are raised.

(1) the nature of classroom discourse

Whose ideas are being explored?

What types of questions are being asked?

How is conflict resolved?

How is student thinking encouraged?

Is mathematical thinking modeled?

Who are students talking to?

(2) the nature of the mathematical tasks

Are the tasks problematic for the learners?

Is the mathematics sound?

What types of representational systems are employed?

The project. A difficult aspect of this project is the role participation in the project has on teacher change beyond the obvious and intentional experiences we are providing. It is clear that interaction with the researchers and participation in the project will have an impact as well as exposure to other sources of knowledge such as professional meetings and journals. We are concerned about such things as

(1) what changes do teachers see about themselves professionally as a result of participating in the project?

(2) what "ripples in the pond" are apparent?

It is important to note that we are making every attempt to not let our conceptual framework restrict what we find. We will remain open to the data so that we are not limited to our expectations.

Methods

Data are currently being collected from all participants in the project. The data are in the form of videotapes of teachers teaching in their own classrooms, of project directors and mathematics supervisors teaching in classrooms and in staff development sessions, of oral reflections, of unstructured interviews, and of lesson planning sessions. Written data in the form of written lesson plans, of field notes made during classroom observations, samples of student work and of teacher made tests, of framing questions and of written logs are also being collected.

Since we are only six months into the project, the analysis and assessment is in the preliminary stages. It is being facilitated by our project participants--our partners in research. The participants in the project reflect over their own teaching experiences as well as the teaching experience of other project participants, be they supervisors, teachers or researchers. They begin to identify change for us--frequently in areas we have not considered. For example, a teacher at a school where all the teachers participated in a pilot study for the Atlanta Math Project recently commented that for her one of the most valuable aspects of change had been in the communication established between teachers in her department. Teachers are now talking about mathematical ideas to each other. They are talking about how these ideas could be explored in the classroom. They are planning and reflecting on lessons together. They are observing each other teach. She has made our project aware that communication outside the classroom may be just as powerful toward teacher change as the communication within the classroom! Are we seeing ripples in the pond? This type of information is critical as we assess the impact of our work in the school systems and begin to describe the nature of change in all the teachers who are participating.

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Elementary
teaching, decimals, classroom discourse

TEACHING AND LEARNING ABOUT DECIMALS IN A FIFTH-GRADE CLASSROOM

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This paper examines the teaching and learning in one fifth-grade classroom during a two-week unit on decimal fractions. The teacher holds goals of wanting her students to enjoy and understand mathematics, but student learning data show mixed success in reaching these goals. Analysis of classroom instruction and interaction point to possible reasons for students' limited learning.

Objectives

This study is part of a series of case studies of mathematics teaching and learning being conducted by researchers of the Center for the Learning and Teaching of Elementary Subjects, focusing on classrooms in which teachers are consciously trying to make their instruction focus less on isolated computational skills and more on students' understanding, mathematical thinking, and problem solving. This paper focuses on the teaching and learning of decimals in the classroom of one teacher, Elaine Hugo, providing insights on the difficulties of teaching mathematics for understanding and on relationships between instruction and learning.

Conceptual Framework

Although recent calls for reforming mathematics curriculum and teaching in the United States (e.g., NCTM, 1989) hold forth a fairly consistent vision of desired changes in mathematics instruction--less emphasis on practice of isolated computations skills, more emphasis on understanding, problem solving, and flexible mathematical reasoning--they fall considerably short of providing descriptions of what successful mathematics instruction might actually look like in elementary school classrooms and providing a research basis for understanding the learning that takes place as various changes are made in instructional strategies and materials. We have taken as our goal in this research to better understand the teaching and learning in particular classrooms in which teachers are trying to move toward the kinds of teaching described in various reform documents, bringing to bear a variety of research perspectives. We combine the focus on teachers and instruction in classrooms that has been the hallmark of research on teaching (e.g., Wittrock, 1986) with cognitive psychological perspectives on the learning of individuals (e.g., Resnick, 1985). In addition, like many psychologically oriented researchers, we are increasingly trying to pay more serious attention to the social contexts of teaching and learning, both in considering mathematical knowledge to be a social and cultural construction (e.g., Stigler, 1988) and in viewing the social structures and interaction patterns of the classroom as key aspects of the learning environment. We seek to develop rich and useful understandings of classroom teaching and learning of particular mathematical topics, including analysis of the structure of classroom lessons, the role of the teachers' knowledge and beliefs--of mathematics, of teaching, of learning--and of the understandings that students construct or acquire as a result of instruction.

Methodology & Data Sources

Data for the study come from a variety of sources collected over the course of the school year. Providing evidence for the general context for the teaching and learning in this classroom are: fieldnotes and audiotapes from weekly observations of mathematics lessons; interviews with students (conducted at the beginning and end of the school year) dealing with key mathematical content, knowledge, beliefs, dispositions, and problem solving; and ongoing interviews throughout the year with the teacher concerning her goals, strategies, and reflections on her mathematics teaching and the learning of her students. The more intensive analysis of the teaching and learning about decimals draws on: videotapes of all the mathematics lessons during a two-week unit focusing on decimals; student work samples; and in-depth interviews about decimals conducted with six target students before and after the unit. The target students included males and females identified by the teacher as being high, middle, or low achievers in mathematics.

Results

In describing this classroom, we first discuss Hugo's goals for instruction and her pedagogical and subject-matter knowledge. We then briefly characterize what target students learned during the decimals unit. Finally, we consider features of Hugo's instruction that might help account for student learning.

Hugo's Instructional Goal and Knowledge

Through interviews focusing on what she wants students to learn, Hugo revealed her instructional goals. We provide only a sketch of these goals here; they are explored in greater detail in a previous paper (Putnam & Reineke, 1991). Hugo's goals for her students are in many ways consistent with the NCTM *Curriculum Standards* and other reform documents. She wants students to learn arithmetic with understanding, not through rote memorization and drill. She wants to instill an enjoyment of mathematics and a desire to learn more, as well as understanding of arithmetic concepts and when to use them. This entails not ignoring important computational skills, but placing the emphasis on understanding and being able to construct or reconstruct reasonable procedures for solving problems, rather than carrying out memorized computational procedures.

Elaine's relatively strong subject-matter knowledge enabled her to think critically and creatively about the instructional representations and activities she used to teach particular mathematical concepts and procedures. She read research and other writing dealing with the teaching and learning of decimals and thought carefully about what representations to use to best capture important aspects of decimal fractions and building on representations with which students were already familiar, settling on a variety of activities using base-ten blocks. Elaine also worked hard to incorporate research-based techniques, including cooperative learning and writing about mathematical ideas and solutions.

Student Learning

Evidence from student interviews suggests that students' learning during the decimals unit fell short of Elaine's expectations. In general, two students, John and Melody (both identified by Hugo as strong students), showed evidence of learning a fair amount about decimal numbers as a result of Hugo's instruction. In contrast to their relatively poor performance on the early interview, both of these students were able to identify the larger of two decimal numbers, identify, draw a picture of, and explain a decimal number, write a decimal and fraction that corresponded to a rectangular representation of the number, correctly line up like parts and add decimal numbers, and order decimals, fractions and whole numbers. The other four students, Janet, Nancy, Richard, and Rob, performed much like they did on the early interview. Aside from improvement in correctly identifying a shaded rectangle as .3 and knowledge that .7 means 7 out of ten parts, these students showed little improvement. They continued to be unsuccessful at choosing the larger of two decimals or making a coherent ordering of the set of numbers. The errors these students made on the end-of-year interviews were often the same errors they made on the early interview.

Thus Hugo's instruction seems to be working for some students but not others. In the following section, we take a closer look at the instruction during a lesson on decimals to explore why this might be.

Classroom Instruction

Several noteworthy features of Hugo's instruction emerged during the decimals unit. These features both supported the importance Hugo placed on fostering student understanding and illustrate ways in which the instruction hindered--or at least failed to support--students' developing the desired understandings.

Focus on understanding. In her interviews, Hugo talked a lot about the importance of students understanding the mathematics they were learning, not just memorizing rules and procedures. This emphasis on understanding was evident in her classroom, both through explicit statements that Hugo made in framing classroom tasks for students and through the content of the lessons. When introducing small-group tasks, for example, Hugo said, "I need these worked for understanding." She talked at length in some lessons about making sure that everyone in the small group understood the solutions of the group and could explain them. One requirement of the group task was that each student in the group sign their jointly completed paper to indicate their understanding of what was done. Throughout the unit, Hugo focused on having students represent and talk about decimals with base-ten blocks. Comments in a stimulated-recall interview support the notion that her goal was to focus on student understanding. For example, referring to having students show .8 she said she was trying to see "who's making sense of eight tenths and isn't at that point" (stim. recall 3/5/90).

Opportunities for students to express their thinking. Consistent with her beliefs about the importance of verbalization--of giving students opportunities to talk about mathematics--Hugo structured the class to allow this to happen. She often structured lessons to include small-group activities, in which students were encouraged to talk through their solutions to the problems being solved. She evaluated the groups performance publicly in terms of how well they were interacting with one another and working together on the tasks.

During whole-group instructions, Hugo elicited explanations and justifications from students. For example, in a lesson focusing on having students represent various decimal fractions with base-ten blocks, she asked students to explain how their physical representations did or did not show the desired decimal. In another part of the same lesson, Hugo wrote the words *Fractions* and *Decimals* on the board and had students talk about the similarities and differences between these two kinds of numbers, providing them with the opportunity to talk about what they knew and understood about fractions and decimals.

In other lessons, Hugo had students write explanations to make their thinking visible and explicit. For example, on March 12, groups were to write a description of the procedure for adding decimals along with an explanation of why it is important to do it that way (i.e., why it is important to line up the decimal points).

Addresses important ideas. Through the questions she asked and tasks she posed, Hugo clearly brought up important ideas related to understanding decimals. For example, during the lesson in which students represented decimals with base-ten blocks and discussed fractions and decimals, the following potentially important ideas were at least touched on:

- equivalence of ten tenths to one whole;
- equivalence of one tenth to ten hundredths;
- discussion of whether 2 is a decimal and/or a fraction;
- whether one can have negative decimals (raised by a student) and if so, how to represent them with base-ten blocks;
- whether all decimals can be expressed as fractions and vice versa;
- with fractions, thinking of shaded circle as 1/8 or 7/8 depending on whether you are talking about the shaded or unshaded portion;
- with decimals, must divide the whole into 10 parts, or 100 parts....; this is *powers* of ten;
- the difference between 5 tenths and .5 tenths.

Interaction routines. In spite of Elaine's attempts to focus on important ideas and to get students thinking about mathematics through the use of such pedagogical devices as cooperative groups and writing, the interaction routines and lesson structures in her classroom often failed to make students' solution strategies and mathematical thinking sufficiently public to allow many opportunities for teacher and students to reflect upon and discuss students' ideas. Much of the discourse in class continued to be highly structured and focused on the production of single correct answers and solutions, rather than on more open-ended reflection and discussion that might make students' understandings and misunderstandings more focal. In addition, Elaine tried to cover much material in a given lesson, resulting in a rapid-fire succession of problems for students, often precluding much thoughtful reflection on any given problem. Here is one episode from the class to illustrate some of these features of Hugo's interactions with students around potentially important ideas. The students had just displayed .8 with the base-ten blocks by placing eight longs on their mats and on the overhead and talked about this being eight of the ten parts that would make up a whole. Hugo then asked what eight of the small squares would be:

T: What are the little ones, if you had 8 little white ones, what would that be? Mitch?

Mitch: eight tenths?

T: No, eight tenths is what I have up here (points to 8 longs on overhead)

Mitch: No, I mean eight, uh ones

T: eight--?

(some one says, "Oh, I know, that would be hundredths")

T: Sandra?

Sandra: one hundredths

T: eight hundredths. It would take one hundred of those little white ones to make one flat

T: Ok, there's *eight* tenths. Would you show me *nine* tenths.

The problem here, which seems endemic to Hugo's interaction with students, is that the important ideas are brought up or touched on, but it is the *correct* understandings that get most of the attention. Hugo is fairly convergent about where she is going; she wants students to say a particular thing, in this case "one hundredths." When Mitch does not say that, Hugo keeps asking, finally turning to someone else. She seemingly ignored Mitch's response that the little squares would be "eight ones," and, although Mitch's response might be interpreted as "fishing for the correct answer," eight little squares *was* eight ones when the students were using base-ten blocks to represent whole numbers. Once the "correct" interpretation is brought to the table by Sandra, Hugo essentially repeats it and goes on. Where Mitch is left at this point--what sense he's making of it--is not clear and possible explanations of his responses are left unexplored. This kind of episode repeats itself again and again in Hugo's class: An important idea comes up (often raised by a student, so it is not the case that Hugo ignores students' contributions or is unwilling to take time to explore ideas that students raise). But once the idea comes up, the discussion is fairly convergent toward the way Hugo is thinking of it, so students do not seem to have the opportunity to connect the "correct" idea to what they are thinking.

Instructional representations. Although Hugo thought carefully about what instructional representations to use, she may have underestimated the complexity and difficulty for students in working productively with these representations. Analysis of instructional discourse revealed aspects of teacher and student talk about the representations that may have been confusing. For example, in the episode described above, students represented .8 with 8 longs from the base-ten blocks. This is the same way the students earlier in the year represented 8 *tens*, or 80, when using the blocks to represent whole numbers. There is nothing in this use of the base-ten blocks or the way the teacher and student talked about them to draw attention to the longs representing *tenths* rather than *tens*. That students may have been confusing the use of the base-ten blocks to represent decimals with the way they had been used with whole numbers was supported by Mitch's response that eight of the smallest blocks would represent "8 ones." Hugo seemed unaware of this confusion, even though in a stimulated-recall interview for this lesson she pointed out that she was concerned that students might find the use of base-ten blocks for decimals confusing after having used them in different ways previously.

Conclusions

The teaching and learning in this fifth-grade classroom represent an important case of a teacher trying very hard to move instruction beyond the mechanical and computational focus that dominates much of current elementary school mathematics teaching. Hugo has goals that are consistent with current reform efforts--she clearly wants students to enjoy and understand mathematics. But, as this case illustrates, having the right goals is not enough; teaching mathematics for understanding can be elusive and difficult. This case illustrates how difficult teaching for understanding can be, even for a knowledgeable and committed teacher. In particular, it seems that teachers like Hugo need to continue to work toward classroom interaction routines that make students' mathematical thinking--how they are making sense of the instruction and the instructional representations--play a more prominent role in instruction.

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Elementary
Classroom discourse
Mathematics teaching

CLASSROOM NORMS AND EXPECTATIONS: DO THEY HINDER MATHEMATICAL COMMUNICATION?

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Recent calls for reform in mathematics education suggest students should assess situations mathematically, apply appropriate tools, and justify their responses. Students, however, bring with them a well developed set of interactional norms from their previous classroom experience. The research presented here investigates some of the difficulties encountered when trying to change how students talk about mathematics in one elementary classroom.

Recently mathematics educators and researchers have called for radical revisions in how mathematics is taught in elementary school classrooms. Reformers argue that, rather than learning isolated computational skills, students should learn to recognize the mathematical elements in situations, flexibly apply appropriate mathematical tools, and engage in mathematical reasoning such as conjecturing and justifying. All these goals suggest the importance of making students' mathematical thinking more prominent in instruction—making students' thinking public. Students need opportunities to communicate, either orally or through writing, their thoughts about particular mathematical situations or problems and develop a willingness to reflect upon and discuss their own thinking and that of others. According to Polya (1954), students involved in such communication need to develop three traits: Intellectual courage—being willing to revise one's thoughts; Intellectual honesty—being willing to change their thoughts when it is warranted; and, Wise restraint—refraining from changing one's beliefs when it is uncalled for or prior to serious inspection. One portrait of mathematical communication, then, has students sharing and commenting on alternative ways of solving rich mathematical situations in an attempt to clarify their mathematical knowledge.

Students, however, bring to school well developed motivational sets (Dweck, 1989); norms of interaction (Heath, 1982); and, ways of knowing what problems are worth solving and what constitutes a good solution in various out-of-school situations (Goodnow, 1990). Furthermore, as students gain experience in a school setting they grow accustom to certain patterns of interaction.

The research presented here focuses on how the beliefs, norms, and expectations brought to the classroom by the students or the teacher facilitate or hinder attempts to change mathematics instruction in one classroom. In doing so we attempt to address the following questions: What interactional norms existed in this classroom? How do these norms either facilitate or hinder the development of mathematical discussion? What might teachers be able to do in order to promote the development of interactional norms that reflect Polya's three dispositions?

We investigated these questions in a mathematics class of 25 fourth- and fifth-grade students during the 1990-91 school year. Alice Smith, the teacher in this class, was an experienced teacher. Throughout our investigation, we observed and audiotaped ongoing classroom instruction weekly and interviewed the students and the teacher. With Alice, we discussed and developed problems and pedagogical techniques that might facilitate mathematical discussion in the classroom. Each of us—the teacher and the two researchers— tried these activities and problems with a small group of students. Following each small group session we met with Alice and engaged in lengthy discussions to identify problematic aspects of the problems and difficulties enlisting the participation of the students. These discussions, too, were audiotaped and became a part of our data.

<u>Existing classroom norms</u>

During our early visits to this classroom, the interactional norms we observed fit with traditional views of classroom instruction; that is, the content of the lesson was presented by the teacher at the front chalk board and the students worked quietly at their desks. During the presentation Alice asked "teacher questions" (Edwards & Mercer, 1987) and her students responded with what they believed to be the right answer. If, by chance, their answer was not correct the teacher would inform them of its incorrectness and tell them what they had done wrong. The students would repeat the problem at their desks until they solved it correctly. Once the right answer was announced the other students would look to see if they had computed the problem correctly. Following the presentation, the students would be given an assignment which often included many problems of the same type. This usually occurred twice during each lesson—once for the fifth-grade students and once for the fourth-grade students. While the teacher was addressing students in one of the two grade levels, the other students would work independently at their desks.

Alice began one lesson by drawing a series of examples on the overhead projector at the front of the room. Each drawing consisted of a row of ten boxes with some of the boxes shaded in to represent a specific decimal number. For example three tenths was drawn:

For each drawing Alice asked her students how the number being represented was written and spoken. For three tenths, one student suggested that it should be written .3 and spoken "three tenths." Alice responded "good' and tried to go on. Another student, however, thought he knew another way to write that number. He suggested it could be written $\frac{3}{10}$. Alice responded that the class was talking about decimal numbers, not fractions, so $\frac{3}{10}$ would not be correct—at least not in this situation.

The next example, five tenths, was drawn on the overhead and the teacher asked a student to come to the front and write and say the number. The student wrote 51.0 and the class objected to what the student had written. Alice stopped the class from commenting on the student's work saying "Just be quiet please. This is a learning experience. . . everybody gets a chance to show how they are understanding and if you don't understand, that's quite all right." The student told Alice that the number should be read "fifty-one and zero tenths." Alice asked if he had shown 51 wholes in the drawing and the student said "No." Alice told the student he was reading it right when he said "zero tenths" and used that as a way to help the student with the problem. She wrote __ . __ __ on the overhead and asked the student to fill in the blanks. She asked him what place in the drawing represented the tenths place. When the student had difficulty identifying the tenths place, Alice turned back to the numeral the student had written earlier and said "you said this was 'zero tenths' so how could you write 'five tenths' in these spaces?" After a short discussion the student wrote "5" in the blank just to the right of the decimal point and Alice summarized saying "Good, whatever number is just to the right of the decimal point is the number of tenths."

The lesson continued with a series of these examples. For each example Alice drew a picture to represent a specified decimal number. For each picture she called on a student to write and say the number. After the student responded, she would evaluate the student's response by either praising them or by walking the student through the problem until he or she could answer the problem.

Students like those in Alice's classroom are faced with a difficult task. Along with trying to make sense of the content being presented, students need to determine what actions the teacher deems appropriate in specific situations (Leinhardt & Putnam, 1987). These "rules of conduct" then become the norms of interaction in the classroom. But knowing what behaviors are appropriate covers only part of what needs to be considered. Students need to understand acceptable ways of interacting among themselves and with the teacher. In the example presented above, the students in Alice's class easily participated in the discussions Alice initiated. They seemed to understand when it was appropriate to speak and when they should listen. When her students spoke at an inappropriate time, Alice reminded them of what is and isn't an acceptable way of talking.

Edwards and Mercer have suggested that classroom conversation is "an instance of talk in general" (Edwards & Mercer, 1987, p. 42). As such, classroom interaction is framed by local versions or instantiations of the co-operative principle (Grice, 1975). This principle holds that people involved in a conversation will (a) contribute only what they have evidence for and believe to be true, (b) provide only the amount of information that is necessary, (c) make their contribution relevant to the conversation, and (d) make their contribution intelligible. What each of these maxims actually mean in practice is dependent on the particular social situation in which they are used; that is, what these maxims look like in a given classroom emerges through participation in classroom discourse.

The norms of interaction Alice and her students had constructed in her classroom reflected the *I-R-E* (Initiation, response, evaluation) pattern identified by educational researchers (Cazden, 1988; Edwards & Mercer, 1987; Mehan, 1979). In this pattern the teacher presents the class with a problem and elicits a response from one or more students. Following the students response, the teacher evaluates what they have said, either praising them for being correct or pointing out a mistake and working to correct the error. After learning this pattern of interaction, the students, it would seem, would construct an instantiation of the co-operative principle that reflects the pattern and anyone attempting to restructure the norms of interaction would be seen as violating this principle.

Speaking Mathematically

Interrupting the students patterns of interaction was exactly what we intended to do. All three of us brought to this project the goal of getting elementary students talking and thinking about mathematics. The students had grown accustom to interacting in specific ways and we were asking them to change those ways. The existing instantiation of the co-operative principle had students providing only a numerical answer for which they did not need to provide evidence. We were asking them to tell us how they had solved the problem and why they thought their solution worked. In the existing version or the co-operative principle, the intelligibility of the students' responses was not problematic, we wanted them to convince their classmates that their solution worked. The instantiation the class constructed, we hoped, would be informed by the discipline of mathematics; that is, students would develop an intuitive understanding of the problems we posed, make conjectures about the mathematics involved in the problems, and attempt to refute the solutions presented by the group members. Trying to develop norms of interaction where students were actively engaged in assessing mathematical situations and possible solution strategies, however, proved difficult.

At the beginning of this project, the students seemed to expect similar interaction when working on the problems we developed. They had difficulty attending to what was being said by other members of their group. They did not see this as a necessary part of doing mathematics for a couple of reasons. First, in the past the teacher had decided which response was correct and there was only "one right way." They were not familiar with the responsibility of assessing a solution for its value in solving the problem at hand. Second, they were not used to talking among themselves. During their previous classroom instruction, interaction occurred between the teacher and the student responding to the problem posed. The only interaction between students was surreptitious discussions of things not associated with the mathematics being discussed. Furthermore, some students were rather unwilling to think about problems in a way that was different than what they had done in the past. Students who could compute mathematical algorithms with little or no difficulty saw little utility in drawing a picture or deriving a way of convincing other people in their group that their solution worked.

During one of the small group sessions the students were discussing the number of sundaes a store owner could make with a specified number of ice cream flavors and toppings. In the first part of the problem, the store owners had four flavors of ice cream and three toppings. The students were asked to find out how many different types of sundaes the store owners could make and to formulate a way of convincing their fellow group members. Many students immediately said the store owners could make 12 different types of sundaes because 4 x 3 = 12. Once the students agreed that 12 was the correct response, some of them no longer attended to what was being discussed. Any further discussion violated the co-operative principle the students had constructed during their previous mathematics instruction. To provide a justification for their interpretation of the problem was not part of the normal interaction routines developed in this class. In the past the students became accustom to a set of ground rules where they needed only to tell what answer they had calculated for a given problem. The teacher then told them if their answer was right of wrong and, if wrong how they could correct it. As evidence for their solution, needed only a record of their computation. Discussing the problem beyond deciding what was the correct answer seemed to be providing information that was not, to the students, relevant to the current situation. They saw no reason to provide an intelligible explanation of their interpretation of the problem.

Difficulties associated with mathematical discussion in the classroom are not limited to the students. The traditional forms of teaching that characterized Alice's classroom have recently been characterized as being authoritarian and impoverished (Putnam, in press; Romberg & Carpenter, 1986). Traditional teaching methods, it is argued, over emphasize isolated computational skills. Getting students to discuss mathematical ideas in the classroom, it is hoped, will provide a more thorough understanding of mathematical concepts. The recent calls for reform suggest that teachers need to transfer the authority for assessing what is right or wrong to the students. But, like other teachers we have talked with (Peterson, Putnam, Vredevoogd, & Reineke, in press), Alice was concerned about the importance of covering the curriculum. She felt that getting students involved in discussions of mathematical concepts might hurt the algorithmic competence they would need for the district wide mathematics test that was administered each fall. Alice felt that she could ensure the students familiarity with the algorithms if she continued to systematically stress computational skill during her mathematics instruction. As a consequence of this belief, Alice, at times, reverted to direct instruction of algorithms. At other times, however, Alice followed the ideas brought out by students. Her reaction to these conversations was mixed. On one hand, Alice expressed interest in what her students were thinking and, consequently, enjoyed these discussions. On the other hand, Alice was often concerned that the conversations were wasting valuable instructional time. Indeed, the conversations we, as researchers, found exciting, Alice often found problematic.

Summary and Implications

The goal of our project was to get students to take part in "serious mathematical discourse." But what constitues mathematical discourse, we found, is difficult to discern. While classroom discourse ought to, in some way, be informed by the discipline of mathematics, teachers also must remain sensitive to the communication norms already established in the classroom. In large part, getting students involved in mathematical discussions is directed toward developing their ability to assess mathematical situations and solutions. In traditional forms of teaching, the teacher has been charged with the responsibility for assessing their students' responses. Transferring this authority to the students was problematic for Alice. Her concerns for efficiently covering the curriculum need to be taken seriously. The notion that teachers need to give authority to students in constructing and assessing mathematical knowledge. like the development of mathematical discourse, needs to be thought of within the constraints of classroom teaching (Putnam, in press). Consequently, it seems suspect to say that teachers can simply believe that getting students to think and talk among themselves about mathematics is a good idea and start teaching in a way that reflects this belief. To hold to this position would require that teachers change their entire belief system and teaching practice. Rather, teaching in a way that reflects the calls for reform does not mean giving up everything that has been done in the past (Reineke, 1991, April). Teachers need to strike a balance between existing classroom norms, their professional responsibilities, and the discipline of mathematics and where this balance can be found, it seems, will vary from classroom to classroom.

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Third Grade
Mathematics discussion
Classroom interactions

NEGOTIATION OF SOCIAL NORMS IN MATHEMATICS LEARNING

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Abstract
Theoretically, engaging students in mathematics disc sion is a powerful way to learn mathematics. The analysis of a discussion episc e underscores the essential role of negotiating social norms in making possible the negotiation of meaning.

In his book 'Proofs and Refutations" Lakatos (1976) demonstrates that a mathematics theorem does not carry objective truth beyond a set of assumptions and methods which a group of mathematicians agree upon. A mathematics "truth" is established through conjecturing, investigating, proving, refuting, and revising/rejecting, and the truth status of an assertion is always subject to further scrutiny. This view of mathematics is compatible with a constructivist view of knowledge. From a constructivist perspective, knowledge originates in a learner's activity as they attempt to give meaning to their experiences (Wheatley, 1991; Johnson, 1989). Accepting Lakatos' view that mathematics knowledge is created through proofs and refutations, engaging students in mathematics discussions should promote learning. But the conditions for these intellectual exchanges described by Lakatos requires that a consensual domain (Richards, in press) be established. This paper describes an attempt to establish an atmosphere for negotiation of mathematical meaning in a grade three classroom.

Many potential learning opportunities exist in a mathematics class discussion conducted after small group problem solving (Lo, Wheatley, Smith, 1990). The main goal of class discussion in problem centered learning (Wheatley, 1991) is to provide students opportunities to present their solution methods to their peers and to compare/contrast different mathematical ideas. There are at least two tasks a presenter must accomplish in order to make an effective presentation. First, she needs to reflect on her problem solving activity and reorganize it for verbal presentation. The act of reflecting increases the individual's awareness of their thought process, thus creating the potential for revision and elaboration (Barnes, 1976; Bruner, 1986; Duckworth, 1987). Second, in order to make a mathematics explanation communicable, the presenter has to 'create a reader' (Sless, 1986) which means she has to consider peers' expectations. Cobb and his colleagues (Cobb, Wood, Yackel & McNeal, in press) illustrate the complex nature of mathematics communication by identifying the negotiation necessary for two students to agree on what counts as an explanation. Problematic situations are more likely to occur in achieving these two tasks, thus creating the potential for learning.

Potential learning opportunities also exist in a class discussion for non-presenters. In this instructional model, when a solution is presented, all students have either solved or attempted to solve the task. Thus it is natural for them to compare and contrast their ideas with the presenter's explanation, or more accurately, their construction of the presenter's explanation. They also have the task of 'creating an author' (Sless, 1986). That is, they reflect on the speaker's beliefs, intentions and stance. The situation becomes even more complex when there is a disagreement among students. Not only what counts as a justification must be negotiated, but a set of social norms must be negotiated for students to communicate mathematics ideas.

Technically speaking, a 'discussion' occurs when the need arises to negotiate differing opinions. The mode of justifying and refuting is different from the presenting mode. In order to respond to a request for negotiating mathematics meaning, a presenter has to understand the reason why others are voicing disagreement or lack of understanding. She must then fashion an explanation based on her interpretation of the queries. This is certainly not an easy task. Quite often other students will join the discussion to help negotiate meaning. The need for negotiating social norms is greater when there is more than two or three persons involved.

The focus of this paper is an analysis of one episode in which students' attempted to communicate mathematical ideas in a classroom discussion setting. This episode occurred in a third grade classroom on September 6 which was the fourth mathematics lesson of the school year. These students had experienced problem centered learning as a primary form of mathematics instruction during the previous school year when they were in two different second grade classes. The third grade teacher also used this type of instructional model for a year with another group of students. The goal of this study was to analyze the social dynamics and the potential learning opportunities in this particular type of mathematics class discussion. No attempt is made to evaluate the effectiveness of this instructional strategy. Even though having the learner become actively engaged in mathematics learning has been called for (National Research Council, 1989), having students justify their assertions has not become a regular feature of mathematics instruction.

A classroom episode and analyses

Becoming a presenter was the most common way to participate verbally in class discussion. At the beginning of the school year, students appeared to have difficulty presenting their ideas and to understand other student's explanations. Even though students had previously participated in problem centered learning, the social norms of these 27 students and teacher had to be negotiated. The teacher recognized the need to negotiate the social norms of the class. Prior to September 5, the class had two discussions about what presenters could do to help other students understand. Various suggestions had been offered by students. Our initial analysis of these suggestions indicated that they were of two types. The first type dealt with issues like, speaking clearly, talking to the class, drawing a picture if it would help, letting other students see the board, all of which were related to the <u>form</u> of the presentation. The second type dealt with issues like, explaining to the class, recording what your group did, planning together before raising your hand, all of which were related to the preparation and the <u>substance</u> of the presentation.

On September 6, after three groups of presenters failed to provide explanations which the majority of students felt they could understand, the teacher had the third discussion about what a presenter could do to help other students understand. Items similar to the previous list were again interpreted and elaborated by various students. Then the teacher asked, "Does anyone believe at this point that they can do a strong, sharp, effective way of explaining the problem to us?" Peter raised his hand and was confident he could give a clear explanation. He and his partner was asked to explain their method. Peter, a bright student, had formatted with his partner a sophisticated solution method. He understood what they had done and was thus quite confident.

After some confusion about which problem Peter and his partner Jeff were going to explain, Peter gave a lengthy and complex description of what he and his partner did to solve the following task.

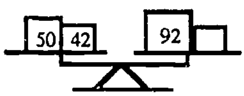

Figure 1. Task explained by Peter and Jeff

Peter's explanation included not only how they solved this task but also two unsuccessful attempts he made before a satisfactory solution was obtained. First he thought 40 and 52 was 62, then they thought the sum of 40 and 52 was the answer to this task. In his explanation, he tried to re-construct his problem solving process for other students. Because Peter recognized that the two in forty-two could 'cancel out' the two in sixty-two, the only thing he needed to do was to figure out the difference between ninety (sum of forty and fifty) and sixty which he then solved by counting tens. Therefore, after he said *"50 and 40 was 90.. and then um you add 2 on um that 60, um em, 2 takes care of the 2 and it takes care of some 60 part."*, he spoke as if this task was about fifty, forty and sixty; rather than fifty, forty-two and sixty-two. He said, *"I counted, I got 60, no I got 9, no, yeah, I got 60 and I um and I, I then I went, then I went from went 70, 80, 90. There was 3, and then, um em, just put a 30 there."* Most of the students were unable to give meaning to his explanation.

Peter's speech was full of repetitions and hesitations and the solution was indeed complex and difficult for some students to understand. Our data showed that at the beginning of the school year, many students had not constructed ten as an abstract composite unit (Steffe and Cobb, 1988). These students might expect to see counting-up-by-one, or using unifix cubes to solve this task. Even though for those students who had constructed ten as an abstract composite unit, few of them could mentally transform "62+?=92" to the equivalent sentence "60+?=90." For example, Jenny, one of the better students, insisted that Peter had to do something about those twos. The unusually critical attitude toward Peter may have been influenced by the conceptual gap between Peter and the other students but other factors were likely more influential.

As Peter finished his explanation, he was surprised when some students began raising a series of type one issue about his presentation. The student responses to Peter's presentation are shown below.

1 *Casey:* *Peter, I don't understand.*
2 *Jenny:* *I don't understand it at all.*
3 *Brad:* *He went from 60 to 90, that was 30.*
4 *Ann:* *But he didn't explain it right.*
5 *Jenny:* *What about these (...)*
(Jenny's question could not be discerned because of the talking in the classroom)
6 *James:* *Raise your hand if you want to talk.*
(Students were quiet down.)

Initially, Casey indicated to Peter that she could not understand his explanation, there was an implicit request for Peter to clarify his explanation. Jenny concurred with Casey and emphasized that she could not understand it at all. However, it became clear on line 5 and Jenny's later statement, "*You are not saying anything about the two in the sixty. You are not talking about the 2 in this side. You are just talking about the sixty.*" that she was able to understand a larger portion of Peter's explanation then she indicated. Brad tried to help Peter clarify his explanation. Ann made five statements during this episode. Her first statement on line 4 and other statements which followed, indicated that she was not pleased with the form of Peter's presentation. It seemed as if she had an image of a "good presenter" in her mind and any presentation which did not fit this image would be considered unacceptable. James took a legalistic stance and tried to maintain the turn taking.

Even in such a short exchange, it was evident that some students tried to negotiate meaning while others felt that it was more important to negotiate the rules of discussion and the correct way to make a presentation. Not all students in the latter group had the negotiating of mathematics meaning as their goal. For example, Ann had decided that there was nothing Peter could do to make his explanation understandable to her. There were also students who had no intention to negotiate social norms or mathematics meanings, they made one or two statements but did not expect them to be taken seriously. Certainly, there were other students who were confused by these rapid exchanges.

How did Peter react to these comments? Because of the transcript limitations, we will include only those parts which relate to the above questions. The number in front of each statement is the number in the complete transcript.

Responding to James' insistence on hands being raised before speaking, Peter called for hand to be shown but was clearly insincere.

7 *Peter:* *(in a funny voice) Raise your hands. You have a hand.*

(Some students laughed)

James' comment was seen by Peter as a distraction. He wanted to talk about the method and was not concerned about the teacher imposed rules. This intention was clear by his next statement.

9 *Peter:* *Okay, this is how we did it, see. Do you want us to start from the beginning?*

Peter attempted to direct attention back to his method. Because some had indicated that they could not understand, it was natural for Peter to ask which part of his explanation needed to be clarified. Peter was upset by this accusation, because he felt he was communicating his ideas but other students were overly critical.

12 *Peter:* *I am mumbling. Um.. Oa... ummmmmmm*

Even though Peter was disturbed by Ann's accusation, he still made two more attempts to explain his method.

16 *Peter:* *Okay. (pause). Be quiet... I'll explain if you'll be quiet, okay?*

19 *Peter:* *Okay. I'm trying to share. (Pause) I'll explain how we did it. Okay?*

Peter was pleading with the class to forget the bickering about rules, so they could continue talking mathematics. Unfortunately, both attempts were interrupted. Casey suggested that Jeff should explain, and Ann concurred with that suggestion. However, Jeff refused.

23 *Peter:* *(. . .) want me explain it. It's not my fault.*

24 *June:* *You raised your hand. You raised your hand (. . . not right) and Mrs. Smith picked you (. . .).*

25 *Peter:* *I can't see to the future, June,*

When Peter raised his hand, he believed his explanation would be a good one. He never anticipated the possibility that students would not understand what he said. He felt it was unfair when he was of accused of being dishonest in volunteering.

26 *Hillary:* *June, if you let them speak, I think they will.*

27 *Peter:* *Okay, this is how we did it.. First of all we thought that it was um. . (spoke to Jeff) what did I say it was?*

28 *Ann:* *A zero. You just said the same thing you had last time.*

It was clear from the complete transcript that this was the first time Peter was able to talk about his method. After the numerous distractions, it was not unreasonable for Peter to forget where he began. Ann's response further supported our interpretation of her position. Ann did not feel Peter was capable of making an understandable explanation. Her accusation distracted other students. It was even more difficult for Peter to explain when he was accused of repeating himself. Yet remember that on line 9, Peter did not get any response to his question. It was natural for him to assume that other students wanted him to start from the beginning. He made two more attempts before he finally gave up and went back to his seat in frustration.

Conclusions

Our analysis showed that a "clear" explanation should not be taken for granted. According to Richards (in press) the first step toward communication is to establish a "consensual domain" which means an agreement on how participants will interact with each other. A consensual domain was not established in this instance. The class responded negatively to Peter because he was over confident and the explanation he gave was beyond their comprehension and judged to be presented in an unclear matter. Because of the previous attention to "clear explanations," the class was particularly sensitive to violations of their inferred rules of discourse. The students were anticipating a "clear" explanation (Type one) and felt Peter's explanation was not clear. Peter failed to consider whether his explanation was appropriate for the class; it was clear to him so he assumed it would be clear to everyone else. Nonnegotiatory positions were being taken by both Peter and students.

Our analysis showed that a fruitful mathematics class discussion requires the negotiation of social norms. In any group it is not unusual for there to be a transition period during which the discussions does not proceed smoothly. A supportive atmosphere with a facilitative set of social norms is crucial for the development of student-to-student exchanges. Students learn how to discuss mathematics in a group as they negotiate the classroom social norms and have opportunities to discuss and make sense of different expectations.

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Grade Three
Units
Imagery

THE CONSTRUCTION OF ABSTRACT GEOMETRIC AND NUMERICAL UNITS

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ABSTRACT

The identification of a unitizing operation in a geometric setting suggests that the operation of constructing abstract units may play an important role in mathematics learning. Classroom activities which encourage the construction of units in a variety of settings are likely to be useful to students in coming to act mathematically.

As we build models of children's mathematical activity, it is useful to identify the cognitive operations used. Piaget (1980) has contributed greatly to our understanding of the epistemologies of children. Particularly important here is the insight that learning involves a series of reflective abstractions rather than being a process of empirical abstraction or imprinting. Children's actions are seen as meaningful to them in their attempts to make sense of their world, regardless of how those same actions might appear from the adult perspective.

Cobb and Wheatley, (1988), investigated children's initial understandings of ten in second grade students as they completed tasks involving increments and decrements of tens and ones. There was evidence, which agreed with the findings of Steffe (1983), that the unitizing operation was central in children's construction of ten as a mathematical object. Students constructed ten as an abstract unit at several levels. At the first level ten was constructed as a numerical composite in which the meaning given to ten was no different from the meaning given to other number words at the abstract stage - as ten ones, or sometimes as a single entity that can be called "ten", but it is not both simultaneously. The child may be able to distinguish those items which can be counted using the sequence 10, 20, 30,..... (ten as an abstract singleton) from those to be counted using the standard number word sequence but does not see one ten as composed of ten ones. At the second level ten was constructed as an abstract composite unit when ten can be taken as a single entity while maintaining its tenness. Finally ten became for the child an iterable unit where the unit of ten could be used to "measure out" tens as in adding ten to 37 to get 47.

Unitizing

A thesis of this paper is that the unitizing operation is not restricted to numerical settings but can be observed in geometric activity as well. In this paper we describe the mathematical activity of grade three children as they use a given shape to make a tiling of the plane. Particular attention is given to the construction of abstract units in this activity. In conjunction with this we describe the same children's methods and thinking strategies as they engaged in tasks requiring addition and subtraction of whole numbers.

The unitizing operation is an important mathematical activity. Much of mathematics involves the construction of abstract units, whether it be with whole numbers (taking six as an abstract unit), fractions (units of one-third), decimals (units of one tenth) or measuring (using a liter as a unit of capacity). In this study, we noted a relationship between a child's ability to construct abstract gestalt units from nonrectangular shapes and their use of ten as an abstract unit in adding and subtracting whole numbers. The parallel between constructing units of geometric shapes and using these to make a tiling closely parallels the construction of ten as an abstract unit and using it in computation (Cobb and Wheatley, 1988). We conjecture that constructing abstract units (Steffe and Cobb, 1988) is a quite general and significant mathematical operation which transcends number.

The tiling task

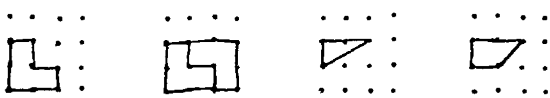
Square dot paper was provided with a particular shape drawn on it. The child was asked to draw a plan for tiling with that particular shape so that a pattern was developed using only that shape. The activity presented allowed for the construction of abstract units as students tiled with the shape and attempted to form an extendable pattern. In presenting the task, it was important to negotiate the conventions of interpretation so that the child attempted the intended task. The child was asked to draw a plan for a tile master to use in laying the tiles in her kitchen using only the given shaped tile.

Data

The tiling task was first presented to a class of grade three students over a period of several days. Field notes were made during and immediately after each class. Each day their work was collected and analyzed. Subsequently, six students from this class were identified for clinical interviews which investigated their tiling activity as well as their number constructions. Each session of one hour each was video recorded for later analysis. A second interview was conducted three months later with two of the students to test conjectures and generate additional data. Detailed analyses of three of these students' tiling and number activity are presented below.

Analysis

In analyzing the responses of the 27 grade three students to the tiling task, it became apparent that the task involves the coordination of several components. The child must construct an image of the shape, develop a production plan for drawing the shape on dot paper, make a covering, and plan a pattern using that shape. In attempting to draw a tiling with a given shape it was possible for the child to construct a larger unit that would facilitate the coordination of each of these components. For example, in tiling with a given shape (Figure 1a), Betty immediately constructed a rectangular unit with two of these shapes and proceeded to fill the page with this new unit, subdividing it to form the given shape (Figure 1b).



(a) (b) (c) (d)

Figure 1. Tiling shapes.

Prior to drawing the tiling, Betty described her action plan, indicating that she could anticipate the use of the composite shape in tiling. Thus Betty performed a unitizing operation in constructing a rectangle by combining two of the given shapes; she constructed the rectangle as an abstract composite unit. The rectangle was a mathematical object of her creation. Betty's intention in making this shape was to form a gestalt which would be easy to work with, that is, a shape which would fit together nicely in covering the plane. This action was possible only because she could distance herself from drawing the shapes and reflect on her activity.

Betty's construction of number also reflected this unitizing action. Betty has constructed both tens and hundreds as <u>iterable</u> units. For example, in computing 536 - 258, she changed the problem to 500 - 258, setting aside the 36 to be added back later while indicating that 500 was an "easier" number to work with. She then set aside the 58, subtracted 200 from 500, then subtracted the 50 followed by the 8, and finally added back the 36. Her geometric and numeric activity showed evidence of using iterable units flexibly.

In contrast to Betty's sophisticated activity, is the way Laura completed the same tiling task. In the process of drawing the shapes (see Figure 1a) to form a tiling which repeated, Laura made rectangles from two of the shapes but showed no evidence that she formed the intention of constructing a composite shape. At no time did she draw the rectangle first and subdivide it as did Betty. The rectangles *resulted* from her production plan for making the given shape rather than being intentionally drawn.

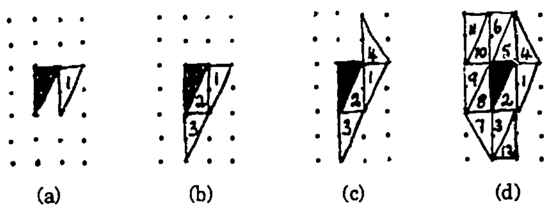


Figure 2. Laura's tiling activity.

In attempting to tile with a right triangle (Figure 1c), Laura experimented with a variety of positions before developing some regularity in her placement of the triangles. Her placement formed rectangles but they were unplanned. There was no evidence she formed the intention of drawing a rectangle as a composite of shapes. She began by reproducing the shape as in Figure 2a. At this point she closed the space between these two triangles and immediately drew another triangle below this shape (see Figure 2b). She had now constructed a large triangular shape which formed a closed gestalt. Laura paused for some time deciding where to place the next triangle, finally deciding to draw a triangle back to back with her first triangle (as in Figure 2c). This activity of drawing triangles back to back was repeated twice more before she began to form rectangular patterns (Figure 2d). Once again the rectangle resulted from her action - an action which she subsequently repeated. However she did not at this stage reflect on her actions. She appeared to be "in the action" and unable to reflect "on her action." (Schon, 1983).

In her number activities, Laura appeared to be in the process of constructing ten as an abstract composite unit. To find the sum of 37 and 48 she made 37 strokes on paper and counted by ones. On occasion these strokes were grouped in tens but her counting procedures did not suggest she had constructed ten as an abstract unit. She did not spontaneously use ten as a counting unit. When I suggested to Laura that it might be possible for her to use these units of tens that she had made, she was able to do so effectively. However, consistency in the use of units of tens was not evidenced. Here, as in the tiling activity, Laura appeared to be in the action of creating units but not yet able to take her construction of tens as an object of reflection.

A third child we interviewed had not yet constructed ten as an abstract composite unit. Her attempt at tiling was characterized by difficulty in drawing a copy of the shape (Figure 1d). In order to make a shape the child must develop a production system, that is a series of actions which result in the desired shape. Donna first looked at the shape drawn, made exploratory motions with her finger and then slowly and hesitantly drew the shape segment by segment, stopping after each move to plan the next. She could not anticipate where to draw the next segment. As she repeated this act she created a sequence of actions which she came to use more easily. This is not unlike a child making four from one, two, three, four. For each shape, a new sequence of actions must be constructed.

Donna drew the shape only in its given orientation. To draw the shape in an inverted position, she turned the paper. Donna was not able to draw the shape in a rotated position. In one case a 90 degree rotated position of the shape was needed to fill a space. Donna made several attempts at drawing the shape to fill the space but was unsuccessful. Each of her trials resulted in a shape quite different from the given tile. Also Donna was so intent on drawing the shape that she did not always achieve a covering and at times drew non-congruent shapes.

Drawing a right triangle on the dot paper also proved challenging for Donna. In attempting to tile with a one by two right triangle (Figure 1c), Donna paused for 15 seconds before beginning to draw the shape. When she began to draw the shape she paused after drawing each segment, deciding on the next move only after completing the previous one. She could not anticipate the sequence of actions before beginning.

Donna's numerical activity was characterized by the same hesitancy, proceeding one step at a time. When asked to find 23-6 she sat thinking for 50 seconds and said she could not do it. I asked, "Tell me what you were trying." She said she was thinking 22, 21, . . 19, 18, 17, 16. In repeating the numbers backwards she had to figure out what came next after she had said the previous number name. As she was saying the number word sequence backwards, she kept track of the counts on her fingers. That is, she counted to six on her fingers as she repeated 22, 21, ... She obtained 16 as the answer because she skipped over 20 in the number word sequence. There was a close correspondence between her drawing of shapes on dot paper and computing the result of a subtraction task. She could not anticipate which segment to draw next just as she could not anticipate which number to say next in counting backwards. One might say that she did not construct the shape as a composite of segments but "counted by ones" (drew one segment at a time). In another instance Donna quickly gave the answer to 6+6 as 12 . However, when next asked 6+7, she used her fingers to keep track as she counted on in ones from 6 to determine the sum. Donna did not use her previous knowledge (6+6=12) to determine 6+7; the second problem was a new experience for her, which she solved by counting on, just as the tiling task had been for her one in which each tile needed to be constructed segment by segment. Once again, the relation between the construction of units in geometric and numerical settings was striking. Instances from other students' tiling activities, when compared with their construction of ten in number activities, also reflected this relationship between constructing an abstract geometrical unit and an abstract numerical unit.

The tiling task was presented to a group of sixth grade students and the construction of units was a common occurrence. The students designed many ways of tiling with the given shape, many of which were composed of units of units. Thus the formation of abstract composite units facilitated their tiling just as construction of units contributes to abstract thought in many other areas of mathematical reasoning.

Conclusion

The identification of unitizing in a geometric setting suggests that the operation of constructing abstract units may play an important role in many mathematical settings. Classroom activities which encourage the construction of units in a variety of settings are likely to be useful to students in coming to act mathematically. Tiling is a rich source for developing the unitizing operation. Students are likely to benefit greatly in their mathematical development from opportunities to construct tilings of geometric shapes. Further investigations of the use of units in mathematical reasoning are planned.

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Age level: 6-10
Identifier #1: Measurement
Identifier #2: Length

CHILDREN'S READINESS FOR MEASUREMENT OF LENGTH

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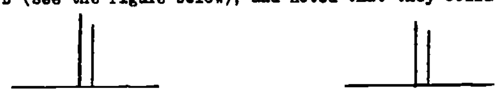
To determine the grade level at which instruction in measurement of length should be introduced, 383 children in grades 1-5 were individually interviewed with a transitivity task and a unit-iteration task. Seventy-two percent of the second graders demonstrated transitive reasoning, and 55% of the third graders engaged in unit iteration. It was, therefore, concluded that children are ready for measurement of length late in third grade. The children were also asked to measure lines with a ruler and were found not to understand the meaning of zero on the ruler even in fifth grade.

Piaget, Inhelder, and Szeminska (1948/1960) gave blocks to young children and asked them to build a tower as tall as a model. The model was on a table 3 feet higher than the one for the copy, and it was built with larger blocks so that neither direct comparison of the two towers nor one-to-one correspondence of the blocks would be possible. The children were given long sticks and a ruler, but young children before the age of approximately 7 found them useless. They used perceptual estimation or their body parts to decide how tall the copy should be. The reason for this preference is that preoperational children do not have transitivity.

Transitivity refers to the ability to deduce a third relationship from two other relationships. For example, Piaget presented young children with two sticks, A and B (see the figure below), and noted that they could all say that



AB BC

A > B. He then hid A, brought out stick C, and asked children whether or not B and C were the same. Upon ascertaining that they could say that B > C, he asked whether A (which could not be seen) was just as long as C, or one was longer than the other. Preoperational children replied, "I can't know because I didn't see them together." When children become able to deduce that A is longer than C, they are said to have constructed transitivity.

In the "towers" task, children who had transitivity (and could, therefore, use a long stick to determine the equality of the two heights) could not always think of a way to use a short block that was offered as a possible tool for comparing the height of the two towers. Around age 8, however, they became able to think about the length of the block as part of the total length and to use the block as a unit to iterate.

Piaget's research and theory thus demonstrated that two logico-mathematical abilities are necessary for children to become able to measure length—transitivity and unit iteration. However, he did not provide norms about the ages at which children construct these abilities. Therefore, this study was undertaken to collect normative data to know the grade level at which measurement of length becomes developmentally appropriate in the curriculum. Measurement of length is now introduced in textbooks in first grade (and sometimes in kindergarten). Knowing when to introduce this topic should make it possible to correct at least in part phenomena such as those reported in Table 1 by the National Assessment of Educational Progress (Lindquist, 1989, p. 39).

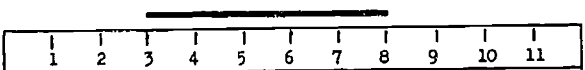


Table 1. Responses to an NAEP Item

	Percent Responding[a]	
	Grade 3	Grade 7
How long is this line segment?[b]		
3 cm	4	1
5 cm	14	49
6 cm	31	37
8 cm	30	9
11 cm	6	2
I don't know.	15	2

[a] The response rate was .80 for grade 3 and .97 for grade 7.

[b] An actual centimeter ruler was pictured.

Method

A total of 383 children in grades 1-5 attending two public schools participated in this study. The schools served a lower-middle to middle-middle-class community near Birmingham, Alabama, and the number at each grade level varied from 75 to 79.

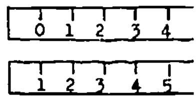
The materials used were the following:

Sheets of paper, 11 inches x 17 inches, each with an inverted T (⊥) photocopied on it. Both lines of the inverted T were 8 inches long but created the illusion that the vertical line was longer.

A strip of tagboard 0.5 inch wide and 12 inches long

5 plastic blocks (1.75 inches long, 0.9 inch wide, and 0.25 inch thick)

A yardstick cut down to 27 inches, with the units numbered 0-25 on one side, and 1-26 on the other side as shown in the figure



A pencil

The following five questions were asked in individual interviews in January and February, 1990. All the interviews were videotaped.

1. Perceptual judgment. The interviewer asked, "Do you think this line (vertical line) is as long as this line (horizontal line), or is this one (vertical) longer, or is this one (horizontal) longer? The purpose of this question was to motivate the child to be involved in the task and to give him or her reasons for answering the subsequent questions.
2. Transitivity. With the tagboard (12 inches long) in hand, the interviewer asked, "Can you use this to prove (or show) that this line is longer than the other (or whatever the child had said)?" This question was asked to determine if the child could demonstrate transitivity with a third term that was longer than the 8-inch-long lines being compared.
3. Unit iteration. Offering one of the blocks (1.75 inches long) to the child, the interviewer asked, "Can you use this to prove (or show) that this line is longer (or whatever the child thought at that time)?" The purpose of this question was to determine if the child was able to compare the two lengths by using a small third term as a unit to iterate. Each 8-inch line was about 4.5 blocks long.
4. Transitivity (second attempt). This question was posed only to the children who were unsuccessful with the strip. The remaining four blocks were introduced, and the child was asked, "Can you use these to prove (or show) that this line is longer (or whatever the child thought at that time)?" This question was asked to determine if the child could demonstrate transitivity after having had an opportunity to think about the preceding questions.
5. The use of the ruler. The child was given the ruler with the end that showed

the 0 and asked, "Can you tell me in inches how long this line is (vertical line)?" The other end that began with 1 was then offered, and the child was asked the same question about the horizontal line. Follow-up questions were asked depending on the child's responses. For example, if the child knew that the two lines were the same length but got 7 inches for the vertical line and 8 inches for the horizontal one, he or she was asked why a difference was found.

Results

The findings concerning transitivity are summarized in Table 2. In columns 3-5 (labeled "with strip"), the data show the degree to which children were able to use the strip to compare the length of the vertical and horizontal lines. Column 3 (labeled "-") includes children who said that the strip could not be used to prove that one line was longer than the other. Column 4 (labeled "±") indicates children who partially demonstrated transitivity, for example, by using the strip without precision. Column 5 (labeled "+") indicates children who clearly demonstrated transitivity with the strip. The sixth column (labeled "with blocks") shows the percentages of children who did not use the strip but utilized the five blocks with transitivity. It can be seen from the last column that children construct transitivity gradually and that most (72%) have constructed it by second grade.

Table 2

Percentages of Student Responses on Transitivity Tasks

Grade	n	With strip			With blocks[a]	With strip or blocks[b]
		-	±	+	+	+
1	78	73	6	21	8	29
2	79	40	4	56	16	72
3	75	21	1	77	8	85
4	75	17	0	83	1	84
5	76	8	3	89	3	92

[a]Percentages of children who were unsuccessful (- or ±) with the strip but successful (+) with the blocks

[b]Percentages of children who were successful (+) either with the strip or with the blocks

Table 3 summarizes the findings related to unit iteration. As can be seen in the last column of this table, children construct unit iteration gradually, too, and the majority (55%) have constructed it by third grade.

Table 3

Percentages of Student Responses on Unit Iteration Task

Grade	n	-	±	+
1	78	90	0	10
2	79	60	7	33
3	75	40	5	55
4	75	16	8	76
5	76	17	5	78

Children's ways of using the ruler are summarized in Table 4. The third colum (labeled "0") indicates the percentages who used the 0 on the ruler correctly and said the vertical and horizontal lines were both 8 inches long. It is disappointin to note that only 11% of the oldest group, fifth graders, could use the 0 on the ruler correctly. The percentage of 9 in third grade is about the same as the 14% reported by NAEP (see Table 1). The most frequently found response (column 4 labeled "end of ruler") was to align the end of the ruler with the end of the line and to say that the two lines were, respectively, 7 and 8 inches long.

Table 4

Percentages of Students Following Rules in Using the Ruler

Grade	n	0[a]	End of ruler[b]	1[c]	First number[d]	Total following rules
1	78	2	34	16	14	66
2	79	1	63	8	5	77
3	75	9	53	13	3	78
4	75	7	75	9	0	91
5	76	11	74	5	7	97

[a]These students reported that both lines were 8 inches long.

[b]These students reported that one line was 7 inches long and that the other line was 8 inches long.

[c]These students reported that both lines were 9 inches long.

[d]These students reported that one line was 8 inches long and that the other line was 9 inches long.

Discussion

Since transitivity was found in 72% of the second graders, and unit iteration, in 55% of the third graders, it can be concluded that measurement of length should be introduced late in third grade. Measurement of length is now introduced prematurely and taught merely as a technique. Authors of textbooks recommend that teachers make children go through certain behaviors such as laying out paperclips in a line along a pencil and counting them to know how many paperclips long the pencil is. This procedure is neither measurement nor unit iteration. Unit iteration requires being able to make part-whole relationships _mentally_ and to be able to _think_ about the length of a paperclip as part of the length of the pencil. When authors of textbooks introduce a ruler, they also give advice under headings such as "Correcting Common Errors." Their advice is that teachers correct children who do not place the end of the ruler on the edge of the object being measured. Such an error is a manifestation of the absence of transitivity.

Authors of textbooks should make the distinction Piaget made among three kinds of knowledge according to their ultimate sources--physical knowledge (such as the fact that a paperclip is metalic and shiny), logico-mathematical knowledge (such as transitivity and unit iteration), and social (conventional) knowledge (such as inches and centimeters). Children construct logico-mathematical knowledge from within, and instruction becomes successful only when it meshes with and extends this development from within.

The conclusion from this study cannot be generalized to groups belonging to higher and lower socioeconomic strata. Further research is necessary to determine when measurement of length should be introduced to children in other socioeconomic groups.

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Age Level: Elementary teachers
Identifier #1: teacher beliefs
Identifier #2: beliefs-in-action

TEACHER BELIEFS AND PRACTICES: A SQUARE PEG IN A SQUARE HOLE

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This study examined the consistency between beliefs and practices of two elementary mathematics teachers through an analysis of interviews and classroom behaviors. Beliefs and practices were described on two levels and each level was coded as empiricist, maturationist, or constructivist. Findings suggest that when defined in these terms, beliefs are generally consistent with practice, but that surface beliefs tend to be more consistent with superficial practices while deep beliefs tend to be more consistent with pervasive behaviors. Implications of the findings for teacher education are discussed.

Introduction

The issue of whether teacher's beliefs are reflected in their practices is a critical one for evaluating teacher education programs and is of particular concern for those who try to promote change in the use of constructivist teaching practices in the mathematics classroom. The literature that exists on relationship of beliefs to practices, however, has been inconsistent. On the one hand, it suggests that teachers' beliefs and self-report statements of behavior are not necessarily reflected in their practices (Clark & Peterson, 1986; Good, Grouws, & Mason, 1990) and on the other, it reports that teachers' beliefs are observable in teaching practices, particularly when the beliefs studied focus on pedagogical content (Bolin, 1990; Brickhouse, 1990; Kidder, 1990) or examined as belief clusters (Pearson, 1985).

These inconsistencies may in part be due to the fact that beliefs are often assessed in terms of relatively superficial self-report measures that lack a theoretical framework for linking behaviors to belief systems. In addition, belief data are rarely collected <u>along with</u> direct observations of teaching performance, but rather with behavioral data gleaned from self-reports describing future plans or retrospective events. Even when behavior is measured directly, it usually is analyzed in terms of predetermined outcome categories rather than based on observations of teachers' spontaneous behaviors.

The case studies described here are an attempt to offer one approach to resolving the apparent differences in the research findings by setting up a model for investigation that includes a) a method for evaluating both stated beliefs and observed practices and b) a dual-level structure for identifying beliefs and practices within three theoretical frameworks.

Theoretical Perspective and Definition of Terms

This study utilizes the three main philosophical orientations associated with the study of human development and learning and assumes that these perspectives can be used to define both teachers' beliefs and practices in the mathematics classroom. These orientations include a) empiricism (E) which implies a passive learner, b) maturationism (M) which suggests biologically determined abilities, and c) constructivism (C) which regards learning as an active process.

The dual levels of beliefs and practices described in this study refer to:

o Surface beliefs. These are defined as self-reports of seemingly objective statements from teachers about their philosophies of learning and instruction. They represent isolated relatively decontextualized samples of teacher's viewpoints.

o Deep beliefs. These can be described as a personal philosophy of education to which a teacher is both intellectually and affectively committed. This kind of belief would be strongly defended if challenged and not easily shaken even in the face of discorroborating evidence. Any particular deep belief would be embedded in a structural whole or system of related beliefs.

o Superficial practice structures. These refer to the organization or form of the learning task consciously planned by the teacher. It includes aspects of the instructional setting such as whether students work in small groups or sit in rows or whether students use concrete hands-on materials suitable for problem solving and self discovery explorations or work only with pencil and paper.

o Pervasive behaviors in the classroom. These refer primarily to the verbal communications the teacher directs toward the students. These communications define the classroom by prescribing the roles that are acceptable for both teacher and students. They include how teachers give instructions, how they ask and answer questions, and how they communicate their formative assessment of instructional goals to students.

Hypotheses

The thesis of this study was that surface and deep beliefs often reflect different orientations and that these different orientations will manifest themselves in particular aspects of teachers' educational practices. It was expected that teachers' surface beliefs would be more closely associated with the superficial aspects of classroom structures, rather than with teachers' pervasive verbal communications. Conversely, it was expected that deep beliefs would impact on pervasive classroom behaviors and that, therefore, they would be more closely associated with the kinds of questions and comments made by the teachers to their students than with the particular structure employed for classroom activities.

Subjects and Procedures

To develop this thesis, two elementary suburban, private school teachers, one from third grade and one from second grade, were clinically interviewed about their views on a) children's conceptions and misconceptions of school mathematics content, b) how children develop mathematical knowledge, and c) the best methods for teaching mathematics. All areas covered were discussed in the context of the grades they taught. Each interview lasted for about 45 minutes. After the interviews, the teachers were videotaped conducting a mathematics lesson utilizing one of the methods they recommended for instruction. The interviews and lessons were then transcribed and coded according to their compatibility with one of the philosophical orientations mentioned above and outlined in detail for coding purposes.

Indicators of surface beliefs included all general statements made during the interviews about overall pedagogical philosophy or principles of mathematics education. Deep beliefs expressed by each teacher were identified through the holistic evaluation of the interviews. In this process, responses were looked at not in isolation from one another, but as a group of comments taken in the context of others relating to a single issue or question.

The classroom behavior data obtained from transcriptions of the teachers' lessons were examined in terms of superficial practice structures based on an analysis of how the overall lesson was organized and the role that students were supposed to have had in the context of this structure. The teachers' pervasive classroom behaviors were identified by examining the questions, comments, and directions the teachers provided for their students.

Results

After coding teachers' responses, the results of these codings were represented as the percent of responses in each philosophical category by task condition for each of the teachers.

Joan's Surface Beliefs: In her interview, Joan, the second grade teacher, made 24 isolated statements that reflected her surface beliefs about mathematics education and learning in general. These were about evenly distributed with approximately one third of these responses in each philosophical category.

Joan's Superficial Practices: Joan presented a lesson in which her students were divided into two homogeneous groups, a high group and a low group, with the low group given what the teacher considered a simpler task to do. This division was categorized as maturationis . because it was based on a notion of fixed ability. The groups were engaged in parallel tasks using chip trading materials in an activity that was supposed to be run by the student groups. This setup was categorized as constructivist (C). Based on this assessment, the approximate percentage values attributed to the structure of Joan's superficial practices was C = 50%; M = 50%; E = 0%.

Joan's Deep Beliefs: When Joan's interview statements were taken in context, the empiricist position seemed to dominate her deep beliefs (C = 12.%; M = 7.3%; E = 80.5%). Typically she made statements such as,

> "OK but I can honestly say that she has no understanding whatsoever of the whole process of addition or subtraction. I mean the whole idea - why do we have to know that the bottom - why does does a child have to be told if that number is bigger than this number, then we have to do something?" The whole idea is to get them to understand if you have six, you're taking away nine."

In this example, we see that Joan insists that the child has absolutely no understanding "whatsoever" and that it is the teacher's job to make the child understand. In this vignet, an active teacher structures the experience for passive learners and conveys the case for an empiricist orientation (E).

Joan's Pervasive Practices: In the analysis of Joan's verbal communications to the students only 15 percent of her comments were of a constructivist variety, while 85 percent were categorized as empiricist. They tended to be directive and even though the children were engaged with manipulative materials, they were often not allowed to work with them independently. Rather, they were instructed very specifically about what they were to do with the materials. Typically her comments to the children were statements such as,

> "Give him one blue chip and he gives you four yellow." "No, you're going to throw it over again and you're going to throw it right this time."

Nomi's Surface Beliefs: In terms of the results of her interview, Nomi made 43 statements that reflected her surface beliefs about mathematics education and learning in general. Her statements, like Joan's were fairly evenly distributed in each category, although they tended to be dominated more by maturationist and empiricist orientations than a constructivist one (25.6% = C;

39.5% = M; 34.9% = E).

Nomi's Superficial Practices: The desks in Nomi's class were set up in four long rows with the teacher in the front of the room. The type of problem presented was a simple two-step translation problem that had a single correct answer. The children worked only with pencil and paper. This aspect of Nomi's classroom structure was considered to be consistent with an empiricist (E) approach to mathematics education. The setup, however, is not indicative of the type of activity in which students were supposed to engage, i.e., working independently using any solution strategy that they thought would be appropriate for the problem. This aspect of the lesson focused on process, individual approaches, and active personal engagement and was categorized as constructivist (C). Therefore, in total, the approximate percentage values attributed to the structure of Nomi's superficial practices was C = 50%; E = 50%; M = 0%.

Nomi's Deep Beliefs: Unlike Joan, when Nomi's interview statements were taken in context, they did not look very different from when they were viewed as isolated statements. She maintained relatively equal proportions of comments in each philosophical orientation, although in the measure of deep beliefs, comments of an empiricist nature tended to dominate (C = 25.4%; M = 30.2%; E = 44.4%).

Nomi's Pervasive Practices: In terms of her verbal communications to the students representing her pervasive practices, 20 percent of Nomi's comments were of a constructivist variety, while 80 percent were categorized as empiricist. Like Joan, but to a lesser extent, Nomi's comments to the class tended to be directive and even though the children were instructed to be thoughtful and come up with unique solutions, she seemed to have in mind the kinds of solutions and answers that would be acceptable. Typically her comments to the children were statements such as,

> "Let's focus our attention on number 93 which is the third problem in the set, which is a multiple-step problem." (Tells students what kind of problem it is) or "How many steps did you need to figure out this problem?" (Here there was a single two-step correct method to use).

Discussion and Conclusions

In general, the analysis of the two case studies presented here tend to support the expectation that when they differ, teacher's deep belief systems are better predictors of pervasive classroom practices than are surface beliefs, but that surface beliefs may be better predictors of superficial classroom practices than of more pervasive classroom practices.

Beliefs and practices

In the case of Joan, the teacher whose surface and deep beliefs sharply differed, her isolated interview statements and the basic structure of her lesson presented a picture of a more constructivist-oriented teacher than did her interview statements taken in context or her verbal communication with students. In both her deep beliefs and her pervasive classroom practices she was clearly presenting herself as an empiricist. Thus, her communication with the children in class demonstrates how her deep-rooted empiricist core and not her surface constructivism is linked to pervasive practices. This was as predicted and suggests that surface beliefs are more easily modified to accomodate new ideas although they are not necessarily consistent with classroom practices.

Nomi's case presents a somewhat different picture of the relationship of beliefs to practices. Her profile, in general, was more consistent across all conditions. For Nomi, the trend in the interview data toward empiricism was consistent with her superficial classroom practices in the way she physically set up her classroom as well as with her deep beliefs as reflected in her communications with students. In addition, the relative proportion of observable constructivist behaviors was consistent with both her statements of deep and surface beliefs. In fact, Nomi did not seem to be that much of a proponent of constructivist learning, even though her choice of a divergent problem solving process was designed for this framework. The result was that the task she proposed was not carried out in the spirit of constructivism and instead took on the flavor of an empiricist classroom. In general, then, it can be concluded that when surface and deep beliefs are consistent with one another, as they were with Nomi, both types of beliefs are essentially equivalent, if not completely accurate, predictors of practice.

The findings of these case studies have some important implications for planning and evaluating the impact of teacher education programs. Both teachers tended to be empiricists in their pervasive classroom practices. This finding is not terribly surprising given the general tendency of traditional educational practices in this country (O'Laughlin & Campbell, 1988). However, both teachers were also trying to infuse constructivist approaches into their students' mathematics experiences. These approaches, however, were not consistent with their deep beliefs about learning and education and so they simply did not succeed, but rather were played out as empiricist lessons. These teachers, though, like many others, are under the mistaken impression that they have made significant adaptations in their standard curriculum and teaching methods and that they are actually executing construcivist kinds of lessons.

Beliefs and practices

Given this situation, the question then is what kind of experiences would teachers need in order to alter their deep beliefs so as to bring about real changes in their teaching practices? Consistent with Pearson's (1985) findings about belief clusters and McLaughlin and Campbell's (1988) work on reflective inquiry in teacher education, it is suggested that the first steps in bringing about real and consistent changes toward constructivist educational practices, are a) to help teachers become aware of their own deep beliefs about learning and instruction and b) then to examine the role of their own philosophies on their pervasive educational practices - prior to any intervention procedures. With this awareness, it is possible to begin to make some changes. Without this awareness, we can only find that our courses and workshops at best are preaching to the converted and at worst are perpetuating exactly the kind of education, albeit under the guise of a new name, to which so many of us are opposed.

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Age Level: Adult
Identifier #1: Spatial Skill
Identifier #2: Computer Games

A STUDY OF COMPUTER GAMES AND PROBLEM SOLVING SKILLS

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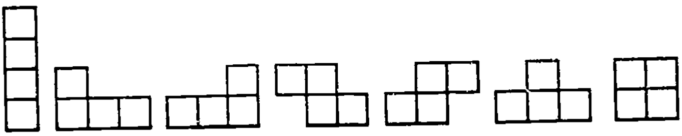
This study examined the problem solving and mathematical skills involved in playing the computer game, TETRIS. Because it involves a dynamic activity which includes characteristic problem solving strategies such as Guess and Check, TETRIS provides a different aspect of spatial skill. This "dynamic spatial skill" was described and compared by gender and high vs. low scores.

Spatial ability has traditionally been distinguished as either spatial visualization or spatial orientation. In spatial visualization tasks the student is expected to imagine moving a geometric object by rotating it or transforming it in some way. The second category, spatial orientation, is where the geometric object remains stationary and the student's orientation changes. The process of playing TETRIS is a dynamic representation of spatial visualization; the player sees the next "piece", visualizes where it might fit, manipulates the piece by turning or sliding it, constantly evaluating its position and continuing this process until it "falls" to the bottom. This use of a computer model of spatial visualization includes visualization *and* manipulation *and* systematic guess and check. We call this type of spatial exercise *dynamic spatial skill.*

The computer game TETRIS involves manipulating geometric shapes to fill a rectangular area. The shapes are all composed of four squares in different configurations. See Figure 1. The player is able to slide or turn the shape as it "falls" into place. This is a powerful tool for developing spatial skills and other geometry concepts as well as problem solving skills. It is a dynamic exercise utilizing geometric shapes in a transformational context; the student visualizes where the "piece" will fit, and then places it there and sees the result.

Figure 1. TETRIS Shapes



Several studies, including Battista, Wheatley & Talsma (1989), Fennema & Sherman (1977), and Flake (1990), have noted the correlation between spatial ability and problem solving skill. The process of manipulation, whether mental or with the computer model, is a problem-solving exercise. The student is using the "Guess and Check" strategy when he or she moves the geometric figure, either by visualizing it or by actually manipulating it on the computer screen.

Many students, particularly females, are weak in spatial skills (Battista, 1990; Fennema & Carpenter, 1981; Ben-Chaim, Lappan & Houang, 1988). Spatial visualization and spatial orientation have both been found to be differentially related to mathematics and/or problem solving performance for males and females (Fennema & Tarte, 1985; Linn & Petersen , 1985; Tarte, 1990). This means that it is likely that there is an overall difference in how males and females think and solve problems. The importance of spatial skills may be different in students who think in these different ways, or who have a different approach to a problem solving task. While there is considerable evidence that gender differences in spatial ability do exist and do have a strong relationship to problem solving skill and to overall mathematics achievement, it is less clear which specific spatial skills are involved, and how they relate to gender and to skill level.

Some studies have found evidence that experience playing video games may improve spatial skills. After treatment consisting of playing video/computer games, two studies found evidence of improvement in spatial skills (Lowrey & Knirk, 1982; McClurg & Chaille, 1987). In a recent survey of fourth and fifth grade students, Flake (1990) found a significant positive relationship between spatial ability and experience playing video games.

Another important facet of this picture is motivation. We need to carefully study students who are willing to concentrate on an out-of-school activity such as TETRIS or other video/computer game, and determine a way to transfer this motivation to the classroom. It is very possible that these media utilize some learning facility that we, as educators, have overlooked. Even though we often categorize the knowledge about video games as useless (or worse), those students are *learning*. This experience may well have a strong impact on those students' school learning. There is a definite need for more information about learning and video/computer games.

The problem-solving literature indicates that novice and expert problem solvers have different internal cognitive structures for knowledge and procedure regarding specific domains of problem solving. Silver (1979) noted that there was differentiation among novices. That is, good and poor novice problem solvers could be identified and classified.

Methodology

Twelve college student volunteers (six male and six female) who had no prior experience with the game participated in the study. Each student was individually given a brief introduction to the game and permitted to practice for ten minutes. A videotape was then recorded as each participant played the game on an Apple IIGS computer for 30 minutes.

The tapes were analyzed and coded by the two researchers. In addition to total game score, four spatial activity scores for each participant were obtained from the coded transcripts: (1) number of slide moves, (2) number of turns, (3) frequency of "dropping" pieces, and (4) frequency that pieces were fit exactly into the board. An additional variable called "manipulations" indicated the total number of manipulations (the sum of the slides, turns, and drops). Comparisons were made by gender. Further comparisons examined the specific playing strategies of the good and poor novice participants (top half vs. the bottom half of the scores).

Participants were also asked to complete a questionnaire at the end of their session. They were asked a series of open-ended questions about their TETRIS performance and their attitudes toward the game.

Results

Males and females were not significantly different on total scores. For further analysis, the twelve participants were also divided into six high and six low scores (three male and three female in each). Means and standard deviations for all variables are included in Table 1.

Table 1. Means and Standard Deviations of Average Manipulation Variables for Each Player per Piece Played*

	TURNS	SLIDES	DROPS	MANIPS	EXACT FITS
Mean	1.448	2.754	0.426	4.201	0.608
St. Dev.	0.487	0.325	0.261	0.610	0.142

* n = 12

An ANOVA was computed for each of the five spatial activity scores (turns, slides, drops, total manipulations, exact fits). Results revealed that there were no significant effects (all $p > .05$) on any of these scores from gender, high/low score groups, or an interaction of the two.

While these "negative" results are at first disconcerting, they actually confirm our expectations. Dynamic spatial skill as measured by TETRIS activities in a group of novice players was not different for males and females. Further, when the participants were grouped by high and low scores, there were no significant differences in playing actions, as would be expected for novices.

Data from the questionnaire revealed that there were few differences in the groups' responses, so the data for the entire group was examined together. When asked about playing strategies, participants stated that their performance depended "much" on planning where the next piece would fit, visualizing the whole game board, and motor skill in manipulating the keys. They also reported "some" influence of luck in which piece would come next and artistic ability in putting the pieces together.

All participants said that they thought they could improve their scores with practice. While they did not mention schema, they said practice would make them "more familiar with how the pieces fit", "better able to visualize the possibilities", and "more confident". They said that they were more concerned with correct placement than with speed, and that their performance depended more on planning than on motor skill.

All participants except one said they enjoyed playing the game, and felt that it was educational. Many mentioned geometry and spatial skills. One person mentioned that this eye-hand coordination activity would be good for preschool and/or special education students. Another participant said that to win you have to "think ahead and see the potential of playing the pieces in patterns." One person said playing this game made them think in a "spatial, geometric way." Their reported feelings while playing included the following: challenged, elated, excited, stimulated, and intrigued.

Conclusions

While we are not great supporters of computer/video games in general, it may be that this particular game, TETRIS, may have educational benefit in improving mathematics achievement through improving dynamic spatial skill for certain groups of students. In attempting to improve instruction in mathematics, we are constantly looking for an interesting and exciting model of mathematical concepts. The computer is attractive to students; they like almost any computer endeavor. Therefore, we must take advantage of that interest, and use the computer as a model whenever it is appropriate. There is a need for students to *experience* mathematics. Constructivism is based on this experience and individual cognitive concept building. For spatial visualization, it appears that TETRIS experience may have a possible positive influence on development of dynamic spatial skills.

In the area of problem solving, we know that one of the most valuable activities in developing good problem solving skills in students is practice. The more problems students solve, the better they are at solving problems. This is due to the development of schema with information and procedural information. Playing TETRIS is a good context for practice of problem solving. While students are motivated and interested in this game, are using the Guess and Check problem solving heuristic, as well as experiencing dynamic spatial visualization.

While this study did not find evidence of gender differences at the novice level of playing TETRIS, further research should examine gender differences in dynamic spatial skill as experience and expertise increase. Dynamic spatial skill and its relation to problem solving and mathematics ability should also be further studied.

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Age level: 9-12

Identifier #1: Mental models

Identifier #2: Change problems

MENTAL MODELS AND PROBLEM SOLVING: AN ILLUSTRATION WITH COMPLEX ARITHMETICAL PROBLEMS.

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Our study focuses on implicit mental models used by children in solving certain complex arithmetic problems, involving the reconstruction of a change. The study was structured in two phases. First, we used a written test to identify different stable resolution procedures. (198 students) in a set of complex change problems. Then, at each level, students, representing all procedures of resolution, were interviewed in order to make the models more explicit.

Modelling plays an important role in science and mathematics where one develops models in order to illustrate observed phenomena. A model is a tool used to capture the fundamentals of a complex situation; it provides descriptions, interpretations and predictions; it is a common and useful tool for scientists in problem solving. However, modelling is not an activity restricted only to scientists. Children when solving problems will also develop, and build their own models which will enable them to analyze, to interpret the data, to make relations between the data and then act. The models they are using are an internal representation that will guide their actions in solving the problems.

Our study focuses on implicit mental models used by children in solving certain complex arithmetical problems. In the following, the role and importance of mental models in problem solving will be discussed, after which the results of our study will be presented.

1- Mental models.

The notion of mental models plays a central role in the process of problem solving. Brousseau (1972) defines "implicit mental models" as follows:

> "When a child in a series of comparable situations (same structure) shows a series of comparable behaviours (same reaction), one can conclude that this child has perceived a certain number of elements and relations of this strucure. He, therefore, has a certain mental model of this situation." (1972, p. 58)

Rouse and Morris (1985) have produced a synthesis of various definitions of mental model and they have shown that they share a series of functions and goals:

"The common themes are describing, explaning and predicting, regardless of whether the human is performing internal experiments, scanning displays or executing control actions" '(1985, p. 11)

A mental model has therefore a heuristic function. It represents a structured entity and its structure must relate to the reality it represents. Fischbein, 1989, is defining a model as follows:

"Given two systems A and B, B may be considered a model of A, if, on the basis of a certain isomorphism between A and B, a description or a solution produced in terms of A may be reflected, consistenly, in terms of B and vice versa. (1989, p. 9)

This definition emphasizes certain aspects of a model. First of all, Fischbein mentions that the model must be able to become a substitute of the original. Then, the relation between the original and the model must be based on some type of structural correspondance. Finally the model must be au:onomous from the original. Fischbein, Tirosh, Staby and Oster (1990) have studied this last feature:

"Being structurally unitary and autonomous, the model often imposes its constraints on the original and not vice versa! Consequently, a model is not simply a substitute, an auxiliary device (more simple, more familiar, more accessible)." (1990, p. 24)

Fischbein states that this autonomy of the models is a condition to their heuristic efficiency. Even though a mental model must be a substitute to the original, it cannot just be a mere reflection of the original but rather a structured governed by its very own rules and parameters. In conjunction with its autonomy, Fischbein mentions that the model must also be stable:

"The autonomy and stability of mental models seem to suggest that they are not mere products, mere reflections of the originals. They belong to the mental structure of the individual, well integrated into this structure, reflecting its requirements, its particularities, its schemata, its laws." (1990, p. 29)

This ties in very well with Brousseau's definition presented above. The autonomy of the model with respect to the original and its stability mean that the mental model originates from the mental structure of the subject. As the mental model guides the child's action when solving problems, it will bring about stable procedures, sometimes erroneous, which will be a reflection of his own mental structure.

2- The study of mental models

The importance of better defining the implicit models that lead the children's action in a series of situations is of paramount importance to understand how knowledge is constructed. This analysis can be done in several ways. Stewart and Hafner (1989) have identified three types of research about mental models in problem solving: model-using, model-elaborating and model-revising.

In the model-using perspective, researchers have studied how children use existing and often erroneous models to solve problems perceived to be solvable by using those models. The second type of studies, model-elaborating, looks at what the children learn during problem solving and what metacognitive strategies they use. Finally, according to Stewart and Hafner, another type of research is also needed in problem solving: "This research would focus on the thought processes of solvers who encounter data that is inconsistent with their existing models." (1989. p. 13) This is what they call "model-revising". In this type of study, the child confronted to problems different from the usual ones cannot always ignore the initial model, which becomes an obstacle to solving this new type of problem. Our study is in accordance with this perspective.

3. The actual study.

During the first years in elementary school, the students are confronted with change problems in which the change is unknown:

> "Mary has four marbles. Her father gives her some more. She now has twelve marbles. How many marbles did her father give her?"

Several researchers (Carpenter and Moser, 1982; Vergnaud, 1982; DeCorte and Verschaffel, 1985...) have illustrated the difficulties that these problems create. Furthermore, Riley, Greeno and Heller (1983) have developed models that explain the succes and failure of young children in solving change problems.

Although similar in certain aspects to these studies, our work differs by concentrating on the mental models older children (aged from nine to twelve) use in solving more complex change problems:

> "John plays with marbles. In the first game, he lost 7 marbles. He plays a second game; we are not telling you what happened during it. If, after the two games, he has won 5 marbles, has he won or lost during the second game and how many?

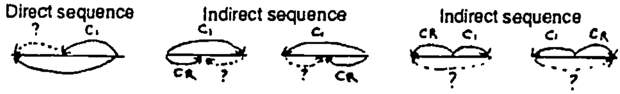
Objective.

The objective of this study is to analyse the implicit mental models used by children in complex change problems.

Method.

This study is structured in two phases. First, we used a written test to identify different stable resolution patterns used by children in a set of problems involving the reconstruction of a change (see table 1). This written test was given to three groups of each level (4th, 5th and 6th grade, ages from 9 to 12) for a total of 198 students. Then, at each level, fifteen students, representing all patterns of resolution, were individually interviewed to gather additional data in order to make the different underlying mental models explicit.

Inter-subject variable school level	Intra-subject variable Problems involving a reconstruction				
		5 types of increasing complexity			
	Type2	Type 3	Type 4	Type 5	Type 6
		2 different contexts for each type			
4th (3 groups)	XX	XX	XX		
5th (3 groups)		XX	XX	XX	XX
6th (3 groups)		XX	XX	XX	XX
	Direct sequence	Indirect sequence		Indirect sequence	



Written test's structure at each level
Table 1

Results analysis.
The statistical analysis ("analyse classificatoire", Lebart) of the written test has permit the identification of distinct groups of procedures leading to erroneous solutions. These results are in agreement with reports from other studies (Vergnaud, Conne, Bednarz et al...).

The interviews were then analyzed to substantiate and better understand how each procedure used by the children in solving the problems functions. This analysis has shown that specific erroneous procedures identified in the written test relate to the same reasoning or mental model. The following three general models have been identified (they will be described with the problem: "John.." presented above):

1- Linear model

The child considers the first change as an initial state on which the resulting change operates thus producing a final state. Generally, the subject treats "lost 7 marbles" as an initial state "had 7 marbles"; he then operates the resulting change "has won 5 marbles" thus obtaining a final state:

$$7 \xrightarrow{+5} 12$$

The child's answer will vary depending if he answers in terms of the final state "He now has 12 marbles" or in terms of the resulting change which has become the second game (the crux of the question) "He has won 5 marbles". Some show signs of requiring an initial state to solve the problem and, when given one, they are using a linear model. Fundamentally, the child does not perceive that he has to reconstruct a change.

2- Comparison model

The child is still treating the changes as states but here he understands that there is a reconstruction involved; he thus compares the two states to find their difference. When confronted with the same problem as above, the child will compare 7 marbles to 5 marbles finding a difference of two marbles. He thus simplifies the problem by treating it as a reconstruction of a change from states.

$$7 \overset{?}{\frown} 5$$

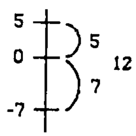
3- Success model

Not only does the child deal with the problems in terms of reconstruction, as in the previous model, but he also think in terms of changes. This model is however used by very few children. When solving the same problem as above, these children will treat the data correctly:

(Copy of a sixth grader)

first game: 7 marbles lost
second game: ?
alltogether: 5 marbles won

Some of them have used a number line or a thermometer to find the difference between -7 and +5:



Conclusion

Three models have been identified and they depend on the child's perception of the underlying structure "a + ? = c" and his concept of the number. The first model (linear model), normally used more often by fourth graders, is characterized by the structure "a + c = ?" and numbers are considered as states. Children using the second model (comparison model) perceive the structure "a + ? = c" but they treat numbers as states; this model will work for simple problems but will fail with more complex problems as it cannot be generalized. Only a minority have understood the problem structure as well as that numbers should be considered as changes; this will enable them to successfully solve the more complex problems. Being capable of making the transition of considering numbers as states to considering them as changes constitutes a considerable conceptual evolution.

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Age level: 6-7 years old
Identifier #1: External representations
Identifier #2: Change problems

A STUDY OF EXTERNAL REPRESENTATIONS OF CHANGE DEVELOPED BY YOUNG CHILDREN

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External representations generally proposed in current teaching materials, with the intention of evoking change process, are not decoded as such by the children. Our knowledge of difficulties encountered by young children in solving change problems, on one hand, and in using representations meant to give a picture of this dynamic concept, on the other hand, led us to an investigation aimed to a better knowledge of representations developed by children in change contexts. An experimentation conducted with 173 children (6-7 years old) revealed different graphic codes used by them to represent these situations.

The area of the representation of dynamic situations is particularly intriguing and important in mathematics (in reference to concepts such as transformation, displacement, fonction, variable...). Researches, conducted in different fields, reveal difficulties encountered by children in solving situations involving these concepts, and corroborate the problems experienced by children in interpreting dynamic representations. So, the situations, involving mental reconstruction of a dynamic process, are often perceived as statics by children The same statics conception is revealed by the interpretation given by children to the external representations used, in current teaching materials, to express these concepts.

Our previous research, on the difficulties encountered by children whilst solving certain complex change problems and on the interpretation given by children to external representations meant to recreate this process of change, corresponds in many ways to the results previously mentioned.

In a first study (Bednarz & Janvier, 1987), we considered additive problems which consist either of a transformation or of a displacement, in other words of a single change where an entity passes from a given initial state to a final one. We considered also more complex problems involving a sequence of changes whose effect can be combined and replaced by a single change, called resultant. In such situations, requiring the reconstruction of a change, our research shows that children (from 6 to 12 years old) operate with the same erroneous procedures at one level or another, revealing implicit models underlying their solution (Poirier & Bednarz, 1991). These procedures originate from the children's excessive centration on the states, and reveal a static conception of the relations underlying these situations. Everything occurs as if the thinking process of the child acted only on states without being able to reconstruct the changes.

On the other hand, our research on problem of representation of dynamism in mathematics (Bednarz & Janvier, 1985a) shows how difficult it is for children to perceive change in the external representations (illustrations, drawings, diagrams...), generally proposed in the teaching of mathematics to give a picture of dynamic concepts (transformation, displacement...). Graphic codes and conventions which are in use, with the intention of recreating change, are mostly interpreted in a static way by the children.

The problem of representation of "change"

Several studies corroborate our observations in the difficulty experienced by children when they must read the graphic codes presented to them and exhibiting a certain dynamism (Mary, 1983; Campbell, 1981; Newton, 1984; Friedman & Stevenson, 1980; Girardon-Morand & Janvier, 1987). So, Claudine Mary (1983), in her work on "the film and the teaching of mathematics: theoretical analysis and experimentation" showed that most secondary school students are not able to describe movement presented in the film. They are sometimes attracted by an isolated movement, and are not able to situate the movement of an element in the global system.

The study of Patricia Campbell (1981) on the interpretation given by children to pictures used in the teaching of mathematics to illustrate a transformation (action to substract or add), described by posture signs or conventional graphic codes, shows that representation of action is difficult to interpret by young children. Newton (1984), Girardon-Morand and Janvier (1987) corroborate these observations in the context of science. They show that movement lines (called pictural metaphors) in the first case, or arrows in the second case, used to illustrate transformation, are not understood by children.

Our study (Janvier, Bednarz & Bélanger, 1987) indicated the large gap between the intentions of the authors (for the point of view of textbooks conception) and the interpretation given by children in the use of these representations. Graphic codes are mostly interpreted in a static way. For example, arrows on a numerical line (that illustrate displacements) are interpreted as something which designates a number, or as a set of points. As a consequence of these results, serious doubts need to be raised on the pertinence of these representations as aids to learning.

Our knowledge of difficulties encountered by young children in situations, on one hand, involving mental representation of a change, transformation or displacement (Bednarz & Janvier,

1987) and, on the other hand, in situations involving external representations, used with the intention of recreating change (Bednarz & Janvier, 1985a) led us to the object of this study, centered on a better knowledge of representations developed by children in change contexts. These external representations can be informative of how children perceive change problems, and thereby can provide guidelines for designing pedagogical interventions. A study of these representations might also furnish suggestions for formulating alternative representations on which a learning strategy could be articulated.

Aims of the project

In a constructivist perspective, where complex significants are strongly articulated on the symbolic representations built by children (Bednarz & Dufour-Janvier, 1985b), our study intends chiefly to improve our knowledge of the external representations developed by children in change contexts. More precisely, the project's objectives can be stated as the following:

- to elaborate a typology of representations developed by children to illustrate change (and so characterize the representations developed);
- to catalogue the graphic codes and conventions used by children for this purpose (these codes can then be compared with the representations in use in mathematics teaching);
- to put in light how children modify their external representations of change from one grade to another (1st to 2nd grade);
- to reveal, by the study of these external representations, the conceptions children have of the relations underlying situations involving reconstruction of a change.

Method

173 children in 1st grade (91 children) and 2nd grade (82 children) were invited to illustrate, on one hand, situations where qualitative change (transformation of a collection, or displacement) were involved, and on the other hand, to solve problems involving reconstruction of a change.

In the problems proposed, an entity (collection, measure, position...) is submitted to a change (transformation or displacement), passing from a given initial state to a final one. The process needed to solve these situations requires the reconstruction of the change (the complexity of these situations for young children have been shown by several researchers – Vergnaud, 1976; Carpenter & Moser, 1982; Riley & Greeno, 1983; Resnick, 1982; Bednarz & Janvier 1987).

In the situations requiring illustration of qualitative change, two kinds of situations were considered: temporal change and procedural one. This important distinction, proposed by the researchers Girardon-Morand and Janvier (1987), was considered because it can influence the process of representation by children. In the first case, the process of change is characterized by a temporal and continuous proceeding. In the second case, the process of change is more characterized by a punctual procedure defined on a state.

An example of items of each type, experimented in grades 1 and 2, are given here:

- Problem to solve (involving reconstruction of a change):
  "I had 8 marbles in my pocket. A friend brought me some more. I have now 17 marbles. How many marbles did he bring me?"
- Situation to illustrate (involving qualitative change):
  Procedural change (Mary): "Mary has almost finished placing playing cards into the box. Her little brother comes in and throws some..."
  Temporal change (balloon): "Your mother inflates a balloon. It increases, increases..."

Results

- The analysis of representations (cf. figures 1 and 2) developed by children puts in light four types of representations involved and the codes used.

Representations developed by young children to illustrate change situations

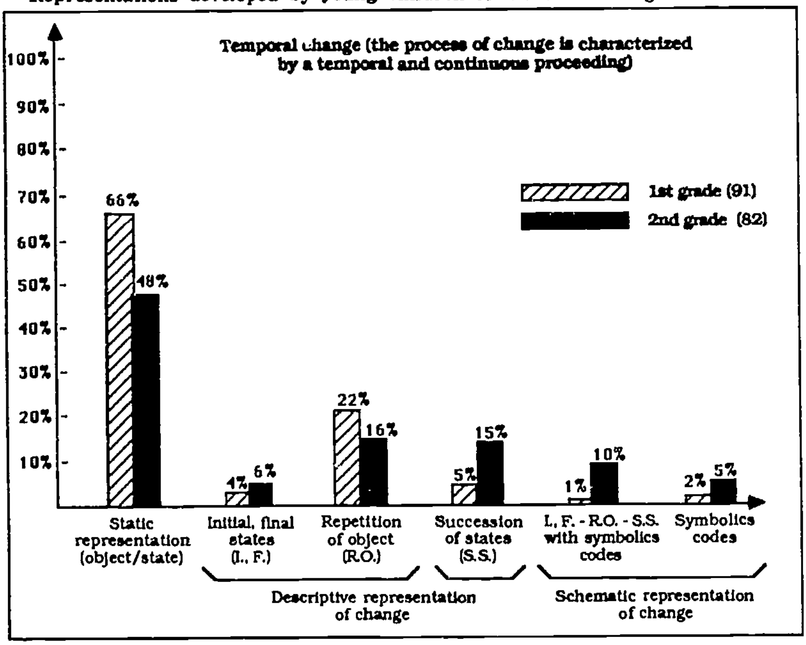

Figure 1

A first class, mostly developed in 1st grade, is formed by **static external representations**: children illustrate only the elements of the situation (we call this type of illustration: **object**) or they illustrate a given state, initial, final or intermediate one (**state** illustration).

A second group is constituted by **dynamic descriptive representations of change**: children then illustrate the initial and final states, a repetition of the transformed object, or an action in progress, with in both cases the use of descriptive codes outcome from the situation (for example, change in size, in the case of the "balloon" situation, change of location, code of disequilibrium, in the case of "Mary" situation).

In a third category, less important at these grade levels, representations illustrate a temporal progression by a "wordless" story (or cartoon) (**illustration of successive states**).

Finally, more schematic **representations** are used by some children to illustrate change (by the use of codes borrowed from comic strips, symbolic codes, codes of trajectories...).

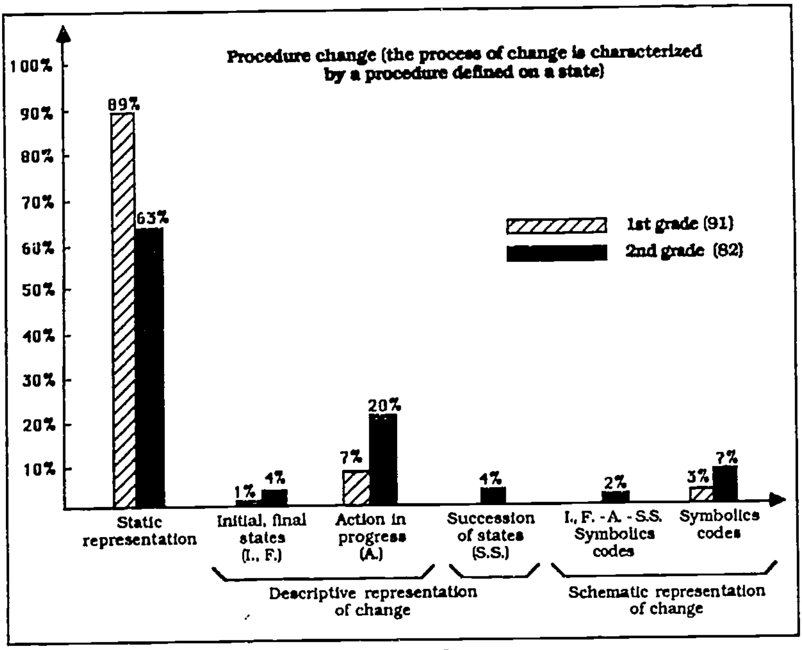

Figure 2

In some cases, use of external codes which reinforce the idea of change also appear (participation of the actor, context...)

- From first grade to second grade, external representations evolve from a static illustration to a dynamic one. In the second grade, most of the dynamic representations are descriptive of the situation.
- Finally, results reveal interesting links between external representations developed to illustrate change and the performance in solving problems involving the reconstruction of a change: children who fail in these problems mostly elaborate a static illustration (they illustrate a state of the situation). On the other hand, in the majority, the "reconstructors" illustrate the process of change.

Conclusion

The results of this research put in light the richness of graphic codes and representations developed by young children to illustrate change situations. These graphic codes go beyond symbolic conventions generally used in mathematics teaching. We can realize there the large gap that can exist between representations used in teaching materials submitted to children and the representation envisaged by the child to illustrate the problem situation. Young children want to find the characteristics in a representation that they perceive as essential to the situation studied. So, their first representations are descriptive of the situations, and even if they finally abandonned these first very descriptive representations elaborated to illustrate the process of change, they still remained attached to it for a long period of time. These representations enlighten us on graphic codes useful for formulating transitory representations on which a learning strategy could be articulated. Moreover, the external representations developed are informative of how children perceive change problems. So, the majority of children in first grade and a great part in second grade are centered on states, they illustrate a given state of the problem (initial, final or intermediate one). We find there the conceptions revealed in our previous research on difficulties encountered by children in solving change problems.

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Metacognition during Problem Solving

Age Level: Teachers
Identifier #1: Metacognition
Identifier #2: Problem Solving

METACOGNITION DURING PROBLEM SOLVING: ADVANCED STAGES OF ITS DEVELOPMENT

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Abstract: The present paper presents two case studies (Galileo and Simplicius) which offer a glimpse into the metacognitive processes of two subjects with some previous experience in problem solving and with extensive and successful backgrounds in mathematics and the teaching of mathematics. The case study of Simplicius also gives clear evidence of the automatization of metacognitive processes.

Metacognition during problem solving has been the topic of much discussion and research in mathematics education in recent years (e.g., Lester, Garofalo, & Kroll, 1989; Schoenfeld, 1985; Schultz & Hart, 1989; Silver, 1985). All of that research has been done with subjects of limited backgrounds in mathematics, that is, less than an undergraduate major in mathematics. The present paper offers a rare glimpse into the problem-solving processes of subjects with extensive and successful backgrounds in mathematics and in teaching mathematics--Galileo and Simplicius (the code names they chose for the project). Specifically, their case studies were examined to see whether such subjects would externalize metacognition during problem solving before an intervention treatment, whether they would exhibit development of or change in such activity and awareness of such activity during and after an intervention treatment, what kinds of metacognitive activity they would exhibit, whether they would exhibit any evidence of automatization of metacognitive processes, and whether they would experience any change in their teaching of mathematics. The case studies were selected from a data set collected to study the development of metacognition over time. Two other case studies from that data set were reported in DeGuire, 1987, 1990.

Method

A variety of techniques--journal entries, written problem solutions with explicit "metacognitive reveries," videotapes of talking aloud while solving problems, and general observation of the subjects--were used to try to capture the development of metacognition from several viewpoints over time. The data were gathered throughout a semester-long course on the teaching of problem solving in the mathematics classroom. The course progressed from fairly easy problem-solving experiences to quite complex and rich problem-solving experiences, gradually introducing discussions of and experiences with the teaching of and through problem solving and the integration of problem solving into one's approach to teaching. Throughout the course, participants discussed and engaged in reflection and metacognition.

During the course, subjects were given six problem sets of 2 to 5 problems each. They were encouraged to work together to solve the problems. The written solution was to include all work on the problem, including blind alleys, and was to include a separate column for "metacognitive reveries." Subjects also wrote a journal entry each week on topics chosen to encourage reflection upon their own problem solving processes and their own development of confidence, strategies, and metacognition during problem solving.

The subjects in the entire data set were 18 graduate and undergraduate students, all inservice and preservice teachers of mathematics, mostly on the middle-school level, but with some teachers on the intermediate level and some on the high school level. The students had registered for the course as part of their programs of study. The subjects of the case studies presented below were chosen because they were the only two participants in the course who had extensive and successful backgrounds in mathematics and in teaching mathematics. They each had an undergraduate major in mathematics and at least 30 semester hours of graduate-level mathematics. They each had extensive experience teaching mathematics on the high school and/or middle school levels. As the course began, each expressed considerable confidence in his or her own problem-solving skills. Throughout the following descriptions, direct quotes are from the subjects' written problem solutions, journal entries, or videotape transcript.

Galileo

Galileo had taught mathematics in high school for about 15 years. His schedule did not permit him to participate in any of the videotaped problem-solving sessions, a fact that is likely to have affected the view we have of Galileo's problem solving, particularly of blind alleys and incomplete attempts he may have pursued in his solutions. From the first problem set, Galileo was a highly successful problem solver. His solutions were direct with no blind alleys. His metacognitive reveries were mainly of the experience kind. For example, he commented "I'll do this problem first. It looks easy! . . . [after solving it] I was right!" On another problem, he concluded, "I liked the other 2 problems better."

As the course progressed, Galileo's metacognitive reveries became more detailed and included a greater variety of metacognitive statements. He drew pictures, made charts, looked for patterns, thought of simpler or similar problems, used trial and error, looked for more than one way to solve a problem, listed and eliminated all possibilities, tested extreme cases, etc., etc. He continued to express many metacognitive experiences such as "I'm think I'm getting close to the solution," and "What a surprise! . . . Actually, amazing [upon finding the Fibonacci sequence in a problem situation]." The most interesting developments in Galileo was his increasing use of intermediary checks of his progress and looking back after solving a problem, his increasing awareness of his use of such checks, and the variety of means he used to check his solutions. He frequently found a simpler solution method for a problem upon looking back and commented that the fact that both solution methods gave the same solution meant that they each served to confirm the other. Often he generalized or extended problems and then solved the extensions. Throughout, he frequently expressed his enjoyment of problem solving and evaluated the problems for classroom use.

By the end of the course, Galileo was successfully tackling, solving, and extending the most complex problems given in the course. He exhibited a wide variety of problem-solving strategies, monitored his progress in the solution effectively, and looked back over his solutions consistently. When asked to reflect in his journa' ı whether he had developed a greater awareness of his cognitive processes or just a vocabulary with which to express them, he responded,

> I can see the involvement of many of the problem solving principles and strategies studied in this class in my cognitive processes throughout my years of taking math courses. In this sense I can certainly say that this course has given me a greater awareness of what is going on in my head, as well as a vocabulary with which to express it.

Did the problem-solving guidelines actually influence Galileo in the classroom? He states, "Without a doubt, I believe this course has helped me improve my overall methodology of teaching as well as become a better problem solver." Later, in reflecting on implementing a problem-solving approach to teaching, he observes, "This almost becomes contagious to the student. I have noticed students beginning to imitate the very same processes which I utilize in confronting problems." Thus, it would appear that the intervention (i.e., the problem-solving course) had indeed influenced Galileo's teaching in the classroom.

Simplicius

Simplicius had taught mathematics in the high school for several years but in recent years had been teaching mathematics in middle school. Even in her first written solutions of problems, Simplicius was very successful. Her solutions were direct with no blind alleys and included some explicit references and use of problem-solving strategies. For example, "This problem screams to be drawn out," and "My first thought is to try all possibilities." Even at this early stage, she checked solutions carefully and even solved one problem a second way and recognized the second solution as a way to "reinforce my first answer."

As the course progressed, Simplicius' written solutions became more detailed and richer in the variety of problem-solving strategies used and the kinds of metacognitive statements made. She also began to include metacognitive experience statements. She combined elegant mathematical solutions (for example, to a number theory problem) with an elementary approach to the solution, again letting one form of solution confirm the other. She generalized and extended problems and sometimes proved generalizations with the method of finite differences. Her solutions continued to be clear and direct with few blind alleys. She frequently commented about especially enjoying certain problems.

By the end of the course, Simplicius was successfully tackling and solving the most complex problems in the course. She exhibited a wide variety of problem-solving strategies, frequent monitoring and checking statements, and gave rich, full discussions of the problem solutions. Her explanations of solutions were exceptionally clear, well-organized, and easy to follow. When asked to reflect in her journal on whether her cognitive processes had changed or she had merely developed a new vocabulary, she responded, "I do not really feel that my cognitive processes or my vocabulary have been significantly improved. I do, however, feel that my awareness. . .has blossomed."

Simplicius did make time to participate in the third videotaped session, a source of data that adds greatly to the richness of her case study. She was the only student in the data set to correctly solve the problem. The problem for that videotaped session was: How many different rectangles are on an 8-by-8 checkerboard? (Note, rectangles are considered different if they are different in position or size. So, a 2-by-1 rectangle is considered different than a 1-by-2 rectangle.) She began by relating it to a similar problem we had done (i.e., how many squares are on the 8 x 8 checkerboard?). Very soon (within 5 seconds), she decisively rejected that approach and chose another. "I'm not going to do that. I'm going to try and make a simpler problem" From there, she counted the rectangles in a 1 x 1 checkerboard, then in a 2 x 2 board, then in a 3 x 3 board. At that point, she recounted her 2 x 2 and 3 x 3 boards "just to be sure." Then she counted (incorrectly, missing the large 4 x 4 square) the rectangles on a 4 x 4 board. At this point, she said, "I think I'm going to find a pattern here." She examined the numbers, looking for a pattern. Suddenly, she corrected her error on the 4 x 4 board. "Stupid. I didn't do 4 by 4. There are 100." Next, she recognized the triangular number pattern but observed "I'm squaring." Then she wrote the general form (i.e., $[(n^2 + n)/2]^2$) and checked it for the 1 x 1, 2 x 2, 3 x 3, and 4 x 4 boards. Finally she evaluated her formula for n = 8 and declared "I figure there are 1,296 rectangles in that thing. . . . I'm real sure of my pattern, finally." The transcription of her talking aloud during the solution was less than 2 pages long! Her solution was devoid of blind alleys and contained only one very minor error that was quickly corrected. Her solution path was unusually clear and easy to follow. Her solution in this videotaped session served to add much credibility to the directness of her written solutions.

Did the course experiences influence Simplicius' teaching in the classroom? As she expressed in her journal, "I feel that . . my ability as a teacher has blossomed. I have definitely made more effort to incorporate problem solving into the curriculum. . . . I feel that what I have learned in this course goes far beyond my own cognitive processes or vocabulary. . . . I feel that this course has fundamentally changed my attitude toward teaching and what the focus of my teaching should be. It is difficult to change after so many years, but I believe that an effort to change is essential."

Discussion

The case studies of Galileo and Simplicius represent development of metacognition that is far along the possible continuum of such development. Both subjects externalized at least some metacognitive activity at the beginning of the course. Yet both subjects stated that their levels of awareness of their cognitive and metacognitive activity increased substantially during the problem-solving course. Both subjects exhibited a wide variety of problem-solving strategies and consistent and effective use of various metacognitive strategies during problem solving. Both subjects also seemed to exhibit a growth in appropriate and effective metacognitive activity before and after problem solving. Also, both subjects stated that the course experiences had or would influence their teaching in the classroom. No attempt was made to observe their classrooms to verify such self-report information. However, this carry-over into their own teaching practices confirms the results of Schultz and Hart (1989).

The contrast between Simplicius' written solutions and videotaped solution offers an interesting insight on another question—the possible automatization of metacognition. Like the written solutions of both subjects, Simplicius' videotaped solution was very direct and devoid of blind alleys; however, it also contained few references to metacognitive activity. Her metacognitive references during the videotape were very brief but very effective. They occurred so rapidly and so naturally as to appear to be automatic. Thus, it seems possible that at least some of the metacognitive reveries in Galileo's and Simplicius' written solutions were added as retrospective reconstructions of what might have occurred rather than reports of what did occur during the problem-solving process. More importantly, it appears that Simplicius' metacognitive activity (and perhaps Galileo's, also) had become automatic. This possibility of automatization of metacognition has been hypothesized previously by the present author and by Lester, Garofalo, and Kroll (1989), but no such clear evidence for it has been seen before.

The possibility of automatization of metacognition has implications for teaching in the classroom. In order to develop metacognitive and problem-solving abilities in their students in the classroom, teachers must verbalize their own metacognitive and problem-solving activities. They cannot do so if they are unaware of them. It would appear that an intervention, even a short one such as the problem-solving course in this study, can effectively bring to consciousness for the teacher important aspects of their own metacognitive and problem-solving activity. It is probable that such an intervention will have a positive effect on their teaching in the classroom.

As with all self-report data, one must assume that, to a certain extent, the subjects reported what they feel the researcher wanted to hear or read. Further, there is no way to know to what extent the subjects may have been aware of metacognitive processes without making explicit verbal references to them. However, these case studies offer compelling evidence that, even in subjects with extensive and successful mathematics and teaching backgrounds, the development of metacognition can be sparked through an intervention such as the problem-solving course in this study and can positively influence the subjects' teaching in their own classrooms. They also offer clear evidence of the automatization of metacognitive processes in successful problem solvers.

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Age level: NA
Identifier #1: *Humanistic mathematics*
Identifier #2: *Framework development*

RECONCEIVING MATHEMATICS EDUCATION AS HUMANISTIC INQUIRY: A FRAMEWORK INFORMED BY THE ANALYSIS OF PRACTICE

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Both the product and the process of creating a framework for mathematics education emphasizing student inquiry and the humanistic nature of mathematics are reported here. The paper also calls more generally for explicating the pedagogical assumptions informing proposed instructional innovations, so as to better appreciate their implications for classroom practice and teacher education.

The assumptive base of instructional innovation

A new wave of reform in school mathematics has recently been called for by many constituencies (e.g., NCTM, 1989, 1991; NRC, 1989). As new goals for school mathematics, new curriculum guidelines and new teaching standards are proposed, mathematics educators need to seriously consider what can be done to insure that these efforts towards improving instruction in everyday mathematics classes succeed better than those which came before.

As we examine the failure of past attempts at school mathematics reform, it is important that we recognize the lack of appreciation for the *paradigmatic shift* that the proposed innovations required and, consequently, the lack of support provided to teachers to deal with such a shift. Research on teachers' beliefs and practices (e.g., Brown and Cooney, 1988; Thompson, 1988) has made us aware that teachers' instructional decisions about curriculum, teaching strategies, classroom organization and management are informed by the system of beliefs about mathematics, learning and teaching that each teacher holds, whether explicitly or implicitly. Consequently, a real change in instructional practices is not likely to occur unless a compatible shift in pedagogical beliefs is achieved at the same time.

Consider the teaching practices prevalent in today's school mathematics. With few exceptions, a typical mathematics class will consist of review of homework, followed by teacher's presentation of new material and sample exercises, and then students' practice on similar exercises. Though widely criticized in the mathematics education research literature, these practices are quite reasonable if one accepts the following set of assumptions:

- a view of *mathematical knowledge* as a body of established facts and techniques, which are hierarchically organized, context-free and value- free, and thus can be broken down and passed along by experts to novices (*logical positivistic view of knowledge*);
- a view of *learning* as the successive accumulation of isolated bits of information and skills, which are achieved mainly by listening/observing, memorizing and practising (*behaviorist view of learning*);
- a view of *teaching* as the direct transmission of knowledge, which can be achieved effectively as long as the teacher provides clear explanations and the students pay attention to them and follow them with memorization and practice (*transmission view of teaching*).

It is obvious, therefore, that attempts at changing the way mathematics classes are currently taught are not likely to succeed unless the *transmission* paradigm characterized by these pedagogical assumptions is challenged at the same time. This does not mean, however, that mathematics teachers have simply to be presented with and "converted" to a new set of pedagogical assumptions, from which the proposed innovations in terms of curriculum and teaching approaches would logically follow. Rather, teachers and researchers alike need to continually engage in a critical analysis of their pedagogical beliefs and practices, as well as of possible alternatives, and through this process try to articulate, question and refine their own system of beliefs, its theoretical and/or empirical justifications, and its implications for mathematics instruction.

Developing a framework grounded in practice: an illustration

In what follows, I would like to share my own experience in engaging in the process of examining my belief system as an integral part of my work as a researcher in mathematics education committed to improving the state of mathematics instruction in schools. My objective here is not only to communicate the results of my efforts to date – i.e., the preliminary articulation of an alternative framework for mathematics education informed by the notion of *humanistic inquiry* – but also to illustrate how the process of re-examining one's practice and making connections with theoretical contributions coming from various areas of educations may develop.

In the past six years, I have conducted several research projects[1] with the goal of

[1] Among these research projects, I would like to mention two that have been made possible by grants from

developing instructional strategies that would help students appreciate the "real" nature of mathematics. These projects were initially motivated by the belief that a logical positivistic view of mathematics is likely to convey to students a dysfunctional perception of mathematics as a cut-and-dried, impersonal and non-creative domain (Borasi, 1990) and, furthermore, that such as view does not accurately describe the nature of mathematics. As several mathematicians and philosophers (e.g., Kline, 1980; Lakatos, 1976) have argued, mathematical knowledge is in fact neither predetermined nor absolute, but rather the result of human construction. The debates and controversies that punctuated the development of some fundamental topics (such as infinity), the existence and legitimacy of geometries alternative to Euclidean geometry, the doubts raised by Godel's work on the absolute truth of even the most fundamental mathematical results, are all evidence that mathematical knowledge, as any other product of human activity, is contingent and localized and may therefore admit alternative conceptions and organizations. This more relativistic view of mathematics also means that cultural and personal values, context and purposes, and affective elements all play an important role in the creation and application of mathematical knowledge – as various supporters of a *humanistic* view of mathematics have helped us appreciate (e.g., Brown, 1982; Lerman, 1989).

One of the instructional strategies that I have tried to develop so as to enable students to appreciate the more humanistic dimensions of mathematics involves the use of errors as "springboards for inquiry" – that is, the analysis *on the part of the students themselves* of the causes and potential consequences of specific mathematical errors, together with the generation and exploration of mathematical questions that arises from such errors (e.g., Borasi, 1987). Both the way mathematicians themselves often take advantage of errors in their work and the powerful role played by "debugging" in learning computer programming languages had suggested this use of errors to provide students with opportunities for genuine mathematical problem solving and problem posing.

In order to explore how mathematics students could take advantage of errors in similar ways and what benefits could be derived by their constructive use of errors, I designed and taught a few units where errors were consciously exploited as "springboards for inquiry". Yet, because I took on a teaching role in these situations, I soon felt responsible for providing my students with the best possible opportunities for learning mathematics in my classroom. This, in turn, forced me to make more explicit the vision for mathematics instruction that I was implicitly trying to realize in my teaching.

the National Science Foundation – "Using errors as springboards for inquiry in mathematics instruction" (award no. MDR-8651582) and "Reading to learn mathematics for critical thinking" (award no. MDR-8850548).

I decided that the in depth analysis of a teaching experience representing a somewhat successful attempt at implementing such a "vision" would be very important to help me identify my overall goals for teaching mathematics, their rationale and their implications for classroom practice. At the same time, I also tried to articulate and examine my pedagogical assumptions by seeking theoretical contributions from the mathematics education literature as well as areas such as philosophy of education, cognitive science, everyday cognition and curriculum studies in subject matters other than mathematics.

The specific instructional experience I chose to analyze was a ten-lesson "mini-course" on the topic of mathematical definitions, conducted as a teaching experiment with two math-avoidant female students in an instructional setting relatively free of constraints and quite supportive of innovation[3]. The experience consisted of a series of thought-provoking mathematical activities designed to enable the students to become aware of specific characteristics of mathematical definitions, recognize their various roles and uses within mathematics, and come to realize that mathematical definitions are by no means absolute or pre-determined as many perceive. These activities often took advantage of the opportunities provided by errors to engage the students in genuine mathematical problem solving and explorations, and thus act as real mathematicians.

As I examined closely the nature of the students' mathematical activity and learning in this unusual instructional experience, it became evident that the notion of *problem solving* was too limited to describe what the students were doing. The focus of our lessons was in fact not so much on finding the solution to isolated problems set by the teacher, but rather on engaging more broadly in an *inquiry* around the notion of mathematical definition – a process that brought us to formulate and address specific mathematical questions and problems, but more as means to improve our understanding of this fundamental notion rather than end in themselves.

Contributions to my thinking about the nature and role of this process of *inquiry* were provided by the literature in critical thinking – more specifically, the interpretation of critical thinking as an attitude of inquiry and informed skepticism (e.g., McPeck, 1981; Siegel and Carey, 1989). This interpretation is explicitly based on a view of knowledge, initially suggested by Dewey and Peirce, as a process of inquiry motivated by uncertainty. Learning, as well as any other form of knowing, is thus seen as a generative process consisting in the continuous creation, evaluation and refinement of hypotheses, and involving the negotiation of goals, strategies and solutions in consideration of the context in which one is operating. This view of knowledge is compatible with and further supports both the

[3] The "story" of this experience has been reconstructed in detail in Borasi (in press).

constructivist perspective that has informed many recent studies of mathematical learning and problem solving (e.g., Charles and Silver, 1988; Davis, Mahen and Noddings, 1990) and the results of anthropological studies of mathematical problem solving in everyday situations (e.g., Lave, 1988). At the same time, its emphasis on the tentativeness of established results, on one hand, as well as on the motivational role played by conflict, ambiguity and anomalies in the continuous search for understanding, on the other, supports and generalizes other important elements of the view of mathematics as a humanistic discipline I had started with.

As a result of these theoretical contributions, I was now able to characterize the learning activity experienced by my students in the mini-course on mathematical definitions as *humanistic inquiry* – that is, mathematical inquiry that was led by the desire to resolve doubts and anomalies, rather than by the instructor's plan to reach some pre-determined results, and was informed by the belief that mathematical results can be constructed to respond to specific needs and purposes. I could also articulate the innovation of mathematics education that I was trying to achieve as *reconceiving mathematics teaching as supporting students' humanistic inquiry in school mathematics.*

The in-depth analysis of the mini-course on definitions became crucial as I tried to spell out the implications of such an interpretation of teaching mathematics and become aware of the challenges that it may present for mathematics teachers in today's schools. For example, as I carefully reviewed the transcripts of the lessons, I became aware that despite my professed beliefs in the value of encouraging students to pose and solve their own problems, as a teacher I had not always been willing to relinquish my control of the class agenda. Consequently, a number of times I chose (without even realizing it!) to follow the original plan for my lesson rather than let the students explore a question they had raise and for which I did not know the answer. This made me appreciate how difficult it is for teachers to give up control, especially given the way schools are organized today and, consequently, become better aware of the radical nature of reconceiving mathematics education as *humanistic inquiry.*

A "humanistic inquiry" framework for mathematics education

As I described the process of analyzing my pedagogical beliefs in light of my practice in the previous section, I have already implicitly identified key dimensions of the *humanistic inquiry* framework that such a process has allowed me to begin to articulate. I would like now to characterize such a framework in a more succint and organized fashion.

First of all, the major pedagogical assumptions behind the proposed framework can be summarized as follows:

- a view of *mathematics* as a humanistic discipline, where results are not absolute and immutable but rather socially constructed and thus fallible, shaped by the purposes and context that motivate their development and use, and affected by cultural as well as personal values;
- a view of *knowledge* more generally not as a stable body of established results, but rather as a process of inquiry, where uncertainty, conflict and doubt provide the motivation for the continuous search for a more and more refined understanding of the world;
- a view of *learning* as a generative process of meaning-making, requiring personal construction as well as the support of a community of learners, and informed by the context and purposes of the learning activity itself;
- a view of *teaching* as supporting the students' own search for understanding and construction of meaning, by creating classrooms that act as communities of learners and a rich learning environment to stimulate students' inquiries.

Among the major instructional implications of these assumptions, I would like to highlight the following ones: a shift of instructional focus from product to process; mathematics curricula that are flexible enough to accommodate the unexpected directions students' inquiries may lead to and to give the students' themselves some control on what they are learning; classroom dynamics that make possible the continuous negotiation of instructional goals and activities; the creation and adoption of evaluation standards and procedure that reward risk-taking and initiative over the production of right answers; and, finally, the need for developing a variety of instructional strategies and learning activities to stimulate and support student inquiry in mathematics.

The *humanistic inquiry* framework sketched in this paper presents a comprehensive and coherent alternative to the *transmission* model that informs most of today's mathematics classrooms and is compatible with the goals and recommendations put forth in the most recent call for school mathematics reform (e.g., NCTM, 1989, 1991; NRC, 1989), while at the same time it highlights some elements that I feel have been neglected in the current debate. Most notably, in contrast to the "focus on problem solving" that has characterized many of the innovations proposed for school mathematics in the last decade, a *humanistic inquiry* framework calls for a new emphasis on highlighting ambiguity and uncertainty in the mathematical content studied so as to generate genuine doubt or conflict and, consequently, the need to pursue inquiry, and on students' initiative and ownership in their learning of mathematics (involving, for example, experiences in which the students themselves formulate the problems and questions they want to study and learn to evaluate their worthiness).

I would like to conclude by emphasizing that the notion of *humanistic inquiry* as well as the pedagogical assumptions articulated in this paper are not presented as a "declaration of faith" that needs to be absolutely accepted or rejected. Rather, they are offered as a starting point of discussion for all mathematics educators engaged in improving the current state of mathematics instruction, with the hope that these notions will be further examined, elaborated and modified so as to contribute to our increasing understanding of the processes of mathematics learning, teaching and instructional change – in a true spirit of inquiry.

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Age Level: NA
Identifier #1: Teacher Education
Identifier #2: Teacher Change

A COGNITIVE FRAMEWORK FOR TEACHER CHANGE

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Essential psychological components of how teachers change emerged from an ongoing research project with elementary, middle, and high school teachers. The interwoven network of perturbation, commitment, vision, cultural environment, and reflection make up the cognitive framework. This framework applies to teacher education programs as prospective and practicing teachers make changes in their epistemologies and instructional practices.

The purpose of this paper is to elaborate on a grounded theoretical framework of how teachers change. This framework emerged from an on-going research project focusing on enhancing mathematics and science teaching. Recently, mathematics educators have encountered a significant number of reports on the crisis in mathematics education (e.g., McKnight et al., 1987; National Research Council, 1989) and a plethora of recommendations to improve the teaching and learning of mathematics (e.g., Blackwell & Henkin, 1989; Mathematical Sciences Educational Board & National Research Council, 1990; National Council of Teachers of Mathematics [NCTM], 1989; NCTM, 1991). The proposed cognitive framework will illustrate how recommendations can be personalized by teachers in today's schools. We have learned from research that taking recommendations and mandating that teachers use them is counterproductive (Wirt & Kirst, 1989). Before worthwhile change can occur, teachers must first desire to make changes within their own classrooms. It is with this premise that we address the cognitive framework of teacher change.

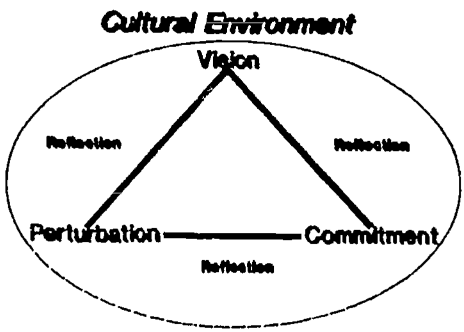
METHODOLOGY AND PROCEDURE

The case study method was used to establish and explicate the framework of teacher change. An in-depth investigation of three elementary teachers, two middle school teachers, and two high school teachers occurred from fall 1989 to the present. These teachers are collaborating together as part of a project to improve mathematics and science teaching and learning (Tobin, Davis, Shaw, & Jakubowski, in press). The project is rooted in a constructivist epistemology (von Glasersfeld, 1989; Yackel, Cobb, Wood, Wheatley, & Merkel, 1988) and is designed to assist teachers in becoming active change agents in their own classrooms and within their own schools. The methodology included (a) observing teachers during instruction and during their bi-weekly collaborative meetings, (b) writing rich thick descriptions (Geertz, 1975) based on interviews, fieldnotes, and teachers' journals, (c) constructing meaning from the context and through negotiated meaning with the teachers, (d) being inductive about the data by making and testing conjectures about how teachers change, and (e) gaining a holistic perspective of teacher change by comparing individual cases (Reichardt & Cook, 1979; Bogdan & Biklen, 1982).

FRAMEWORK

After analyzing and comparing the different case studies at the end of the first year, we developed a framework of teacher change. The results of explicating the framework during the second year illustrated the complexity of the teacher change process. We found that teachers change in different ways as a result of (a) their cultural environment, (b) the quality of perturbations they experience, (c) their commitment to change, and (d) their vision of what changes they want to make.

Framework for Teacher Change



Cultural Environment

The cultural environment for each teacher is different. Geertz (1973) stated that a "man is an animal suspended in webs of significance he himself has spun" (p. 5). For each teacher in the project, distinctive elements of the culture were noticed that affect the change process. These are support, time, money, resources, taboos, customs, and common beliefs.

The quality of support to make changes from administrators, colleagues, researchers, parents, and students has had a direct impact on how, what, and why teachers change. Rogers (1969) stressed that the support given should be genuine, with respect and unconditional acceptance, and with sensitivity and understanding toward the teacher. This level of support for the teachers, fortunately, can be attributed to administrators at the county office, principals, and researchers who realized that changes are needed in the mathematics and science curricula. However, support from students has not come as easy. Students are steeped in the traditional format of learning mathematics, that is, listen to the teacher present instructions about the assignment, then complete the work individually. When teachers incorporate a new teaching strategy, for example, cooperative learning, students' routines are abruptly changed and dissonance is often the result. This classroom dissonance can easily influence a teacher to return to the traditional way of teaching if they have received no other support. A supportive climate is vital if effective change is to occur.

The support needs to be corroborated with time, money, and resources. Teachers need the time to learn about alternative epistemologies and methods to teach mathematics, time to reflect and re-evaluate their roles, time to observe and collaborate with colleagues. It is important that teachers have access to resources that will provide alternatives (e.g., books, articles, manipulatives, people with differing ideas). Without alternatives, teachers will not be as effective as they could be in improving their classrooms.

Within each culture, people hold certain beliefs about existing taboos and customs. An example of a taboo is a teacher's reluctance to question an administrative decision. To question a decision was something she never thought was allowed. Some common customs and beliefs found in the schools include teachers' beliefs concerning the curriculum and the environment for learning. For effective learning to occur, teachers may believe they should strictly follow their textbooks and that students should be quiet. Cultural taboos and customs of the particular school culture can often influence the change process of teachers. The collaboration among teachers from different school cultures and different grade levels unveiled many of the customs and taboos, causing teachers to question and re-evaluate their beliefs about customs and taboos in their school.

Perturbations

Change cannot occur without a perturbation, that is, a mental dissonance. This is analogous to Newton's first law of motion, "A body at rest or in uniform motion will remain at rest or in uniform motion unless some external force is applied to it." A teacher will continue teaching a similar way unless perturbed by something or someone. Perturbations often cause frustration, discomfort, and a great deal of reflection. Perturbations can come from many sources (e.g., students, colleagues, parents, administrators, teacher educators, books, articles, self-reflection). The types of perturbations that will influence teachers to change are as varied as there are teachers. We have clearly seen that the collaboration among teachers and researchers in the bi-weekly meetings have caused both teachers and researchers to be constantly perturbed by comments about each others' experiences. We have learned that when perturbations are evoked in teachers in a genuine, caring and supportive way teachers are more likely to make a commitment to change.

Commitment

Commitment is a personal decision to make a change as a result of one or more perturbations. One teacher described commitment as "an inner feeling that there's a need . . . and that you are going to do something about it." Many of the teachers in the project saw a need to change, but were initially reluctant to change their own practices. During the past 2 years teachers are beginning to question what is happening in their classrooms, why is it happening that way, and whether or not the learning environment in their classrooms is what they wanted. Through self-reflection, teachers are willing to take more risks in implementing new strategies to improve the students' learning of mathematics.

Vision

For teachers to change, they need to construct a personalized vision of what mathematics teaching and learning should be like in their classroom. We found that teachers need alternative images to replace the traditional views of teaching. The teachers initially negotiated and developed several components which made up their vision for the ideal classroom. They each contributed to the vision and felt ownership that this was where they wanted to concentrate their efforts. Some of the teachers mentioned that they did not know how to obtain the goals but wanted to try. Having high ideals and traditional classrooms were common during the first part of the project. For example, some teachers were verbalizing very positive aspects about a particular instructional approach, cooperative learning. Yet, when we observed their classroom teaching, we found them either teaching in a very traditional manner or grouping students together to work independently on some assignment. However, through negotiation among teachers and between teachers and researchers, alternative images were created. This led to an increased emphasis in problem solving, cooperative learning, communication, and making connections. Working together as a family also caused the teachers to be much more reflective about obtaining their vision. As they tried new ideas in their classroom, they would report the progress or frustrations of trying to make the change in their own instruction. As the project has continued and their experiences with new teaching and learning strategies has increased, teachers beliefs have changed in terms of how students learn. They firmly believe that students actively construct their own knowledge and are not mere receptors of the teacher's knowledge. This belief has made them more cognizant of the importance of creating a supportive learning environment for their students. Hence, more cooperative learning and negotiation of meaning and priorities are taking place within their classrooms.

EDUCATIONAL IMPLICATIONS

For successful and positive change to occur, teachers need to be perturbed, they need to be committed to do something about the perturbation, they need to establish a vision of what they would like to see in their classroom, and develop a plan to establish this vision. We found teachers want to improve their instructional strategies and they want to enhance their students learning with understanding. However, we also found the process of change to be a very complex endeavor. Change is a slow arduous process which requires patience, persistence, and respect. Respect is shown when the teachers are given the ownership of what they are changing. We found that teachers must first believe change is necessary and that this change will make a significant impact on student learning.

The implication for educational reform is that teachers must be actively involved in the planning phases of the innovation. However, teachers who are even in on the first levels of planning may hold deeply-rooted beliefs that may cause them much mental dissonance when they return to their classroom. But through collaboration with other teachers, they can discuss and deal with their personal problems. The cultural environment that developed as teachers worked collaboratively made a significant contribution to our thinking of teacher change. Teachers are now more empowered; they take charge of what happens in their classroom and help students realize that they too should take full responsibility for their own learning.

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age 15-16
rate of change
physical models

CARS, COMPUTERS, AND AIR PUMPS: THOUGHTS ON THE ROLES OF PHYSICAL AND COMPUTER MODELS IN LEARNING THE CENTRAL CONCEPTS OF CALCULUS[1]

Andee Rubin and Ricardo Nemirovsky, TERC

Six students explored problems involving "rate of change" in teaching interviews using one of three computer-enhanced learning environments we had designed. In analyzing protocols of their sessions, we focused how students used the three environments differently to support their construction of tentative, yet vital, webs of related concepts from prior intuitions, implicit knowledge, and the experimental situation.

Core mathematical notions of a conceptual field are acquired and constructed through a long-term process which involves the gradual articulation of a great diversity of situations, interrelated symbol systems, and varying levels of complexity. In calculus, two of the most important central concepts are rate of change and accumulation. For the past year, we have been studying environments in which we believe students can construct, based on their intuition and implicit knowledge (Fischbein, 1987) and their activity in the experimental situation, these basic concepts of calculus. Our goal is not to study their mastery of calculus notation, but rather to explore how different experimental situations may contribute to the acquisition of knowledge that underlies and provides meaning for notational expressions of calculus. In other words, we are studying how students construct webs of conceptual mathematical relationships in contexts where they have the opportunity to combine intuition and experimental data.

We have developed three prototype computer-enhanced environments. Our design goals were to construct environments that mapped into core calculus constructs in different ways. In retrospect, we see that the three situations also embody certain tradeoffs in the resources they make available to students. For example, one environment provides flexible symbolic representation but no physical model, while the other involve a physical model, but no symbolic expressions with which to operate. The three environments we created are the following:

[1]This research was supported by Grant MDR-8855644 from the National Science Foundation. Opinions expressed here are those of the authors and not necessarily those of the Foundation.

-a motion environment, in which the student manipulates a small Lego car in front of a motion detector that can record the car's relative position many times a second. The computer which is connected to the motion detector displays graphs of either position and velocity vs. time. The software provides students with the capability of finding out the value of points of the two graphs and of comparing two different "runs" of the car.

-an air pump environment, which uses a hand-driven air pump instead of a car and motion detector as the physical world analogue, with very similar supporting software. Stude s control the flow of air into and out of a transparent, calibrated bag using a hand pu..p and a series of valves; the computer records air flow several times a second and can display both volume and air flow vs. time.

-a spreadsheet environment, which allows students to define functions in terms of first and second differences and initial values. A spreadsheet representation of the functions' values is derived and the corresponding graphs are drawn. The function is labelled A, the first differences B and the second differences C.

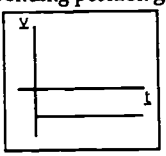
We have tried out each environment with two high school students who had already taken algebra, but who had not yet taken calculus. With each student, we carried out a three-hour teaching experiment broken into two hour and a half sessions. The sessions were based on a structured interview in which we posed a pre-selected set of problems, e.g. "Try to create (using the experimental environment) a graph of velocity (or air flow or B) that is a non-horizontal straight line. What do you think the graph of position (or volume or A) will look like?" Students would first try to generate the velocity (or air flow or B) graph using the car, air pump, or spreadsheet. They would then predict the position graph. We video-and audio-taped all sessions. The analysis underlying this paper focused on important differences among the environments, with a goal of understanding what each environment made available to students as fodder for their conceptual constructions and what mathematical concepts it made difficult to access.

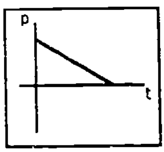
This paper will focus salient differences among the three environments, both in this specific application and, more generally, as examples of concrete situations which may function as sources of mathematical meaning for students. In this focus, we follow Monk's (1990) work on college students studying calculus and Greeno's (1988) and Meira's (1991) work on simple machines that embody linear relationships. We will examine three categories of differences among the environments. The first concerns mappings between the environment and the mathematical model of the situation. The second comprises vocabulary that students used to describe changes. The third is the different kinds of strategies students used, given the tools and metaphors each environment made available.

<u>Mapping into Mathematics</u> A large part of the task for students working with any environment is to understand how to relate components and actions of the situation to mathematical entities. Each of our environments had particular characteristics that affected how students constructed such mappings. We shall analyze below students working on the same problem in two different environments: motion and airflow.

The problem on which students worked was the following.

If this is a velocity vs. time graph, what would the corresponding position vs. time graph look like? (Below is both the velocity graph she worked with, and a possible corresponding position graph.)





This problem was preceded by several similar ones involving only positive velocity. Students' task was to generate the velocity graph and predict the position graph.

The maps students constructed from the car environment to a mathematical model were at times inconsistent. One of the sources of this difficulty was that there was no natural zero or natural positive or negative direction in the environment; both choices were strictly conventional. We arbitrarily set 0 at the motion detector and movement <u>away</u> from the motion detector as positive motion, but any other choices would have been as consistent and made as much mathematical sense. While students were doing the first few problems, which involved only positive velocity, this convention caused no trouble. But when students began to work on problems involving motion <u>toward</u> the motion detector, represented on the graph as negative velocity, they all faltered. S., who had a relatively easy time with the early problems, assimilated the convention after only a few runs. Still, when he was asked to review what he had learned in the first session, he considered the directionality of the motion detector one of the key points he had learned: "Well, what I figured out so far is that at a positive velocity the car, that little man, will be going away from the motion detector...Where here is the decreasing or reverse velocity going towards the motion detector" S. was careful, in fact, to face the little Lego man in the car away from the detector to reify the positive direction.

N., on the other hand, never quite constructed a consistent meaning for negative velocity. Her primary confusion had to do with an ambiguity in the interpretation of the position vs. time graphs that existed only in the car environment. N. often thought of position as "distance traveled," rather than "relative position, " as if she were referring to the odometer reading of the car. She commented when analyzing the position graph above, "I don't understand what the difference is if you <i.e. the car> go the other way." And later, when reflecting on her interview, "I still don't quite get the concept of negative velocity...if you're going you're going, what difference does it make if you're going forward or backwards." The "distance travelled" metaphor was so strong for N. that even when she had glimmerings of understanding, she wanted to keep her intuitions, "Well, just in terms of like looking at the two graphs, the distance between them, it makes sense...But..in reality...(laughs)"

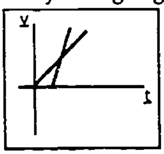
The air pump environment had fewer of these semantic pitfalls; both zero and negative first derivatives were more naturally depicted. Rather than being a conventional value, zero had a natural correspondence in the air flow environment with the empty bag. The lack of negative volume was also not conventional, but based on a property of the environment; the bag couldn't be emptier than empty. The confusing "distance traveled" interpretation of "position" was unnatural in this environment, as there was no natural "odometer" reading that added together both negative and positive air flow into "quantity of air exchanged." And the analog of negative velocity - negative air flow - was easily identified by students as air flowing out of the bag. In the following comments, F. grew to understand the concept of negative flow rate in the course of a relatively short conversation. When he first saw the flow rate graph (corresponding to the velocity graph above), he commented, "How can you get less than zero flow rate?....How are you going to do that?" The interviewer asked him to move the bellows to generate any pattern of air flow. In doing this, F. naturally made air go both in and out of the bag. He almost immediately understood the contrast between positive and negative flow. "Oh, now I see, all right. Cause my, when I pulled it down it goes, it was the, the amount of air going in was on the positive, it was going up. Then I pull it down and it came back down to negative...It can never, the volume can never go down to be less than zero. The flow rate can."

Environment-Specific Language The motion, air flow, and computer environments all differed in the language they evoked in students' descriptions. In many respects, the car environment suggested the most specific language. For example, the motion environment was the only one that had a specific word for second derivative: acceleration. In the airflow environment, students had to describe second differences as changes in air flow. F., for example, said, "the only reason could be that flow rate increased, but it didn't increase so, that so much per second that it increased more rapidly." This specificity in the motion world, however, was not always helpful to students. (Nemirovsky and Rubin, 1991b). One confusion arose from the fact that students referred to negative acceleration with different words, depending on the sign of the velocity. A car with positive velocity experiencing negative acceleration was "slowing down;" if it had negative velocity and was accelerating negatively, it was "speeding up" (but in the opposite direction). The fact that students' language is based on the absolute value of the velocity (i.e. the speed) made such problems about negative acceleration especially challenging in the motion situation

In the computer environment, neither zero, positive or negative numbers had a special real-world meaning or connotation that spilled over into students' mathematical modelling. But the use of the word "difference" was confusing to at least one student. M. was modelling the situation, "The cost of home computers is still decreasing, but more slowly than it was last year." He understood that the difference between consecutive values of A (cost of computers) was changing by a smaller and smaller amount over the course of the graph. He arranged his spreadsheet so that B took on the values 50, 48, 46 etc. He was then surprised to see that the graph of A increased. The interviewer asked, "The first month is $1500. What happened the second month?" In answering this question, M. understood his confusion, "Well, we added $50. Well, I guess I'm using this wrong...So we want all of these <values of B> negative." In order to understand this situation, M. had to disambiguate two meanings of the word "difference." The first is as in "B is the difference between two consecutive A's," e.g. 50 is the difference between 1450 and 1500. The second is the number used in the operation of subtracting to generate the next A, as in 1450 = 1500 - 50.

The motion and airflow environments also differed in the kind of language students used to refer to changes in velocity or airflow. In the motion environment, students used words from their everyday vocabulary for cars: speeding up, slowing down, getting slower. None of these phrases contained any indication of directionality; negative and positive velocity were described in the same phrases. In the airflow environment, on the other hand, there was always an indication of the direction of change in volume corresponding to the direction the bag moved: went up slowly; go up slower,slower,slower; let it back up quickly, it has to go down, but slow. Thus, in the very language they used to describe the action of the air pump, students made the distinction between positive and negative airflow. This specificity helped them understand negative velocity more easily than in the motion environment.

Problem-solving and explanation strategies. The character of each of the three environments lent itself naturally to different problem-solving strategies. Here again, the motion and spreadsheet environments provided the greatest contrast. Because the car environment was so familiar, students often used real-world memories to guide their thinking. Using these mental pictures, students were able to carry out thought experiments in the motion environment. S., faced with the following velocity graph of two different cars that started in the same position, was trying to figure out if they were ever in the same position again and, if so, if it were before, after, or simultaneous with the time they were going the same velocity.



S. first guessed that the cars would meet at the same time they were going the same velocity. He supported his opinion by describing a familiar situation in which two cars are going the same speed down the road, next to one another, "...they would end up meeting there [pointed to the velocity graph at the intersection]. That's where they would end up being side by side."

After a while, S. figured out, through a series of attempts to make the velocity graph and look at the corresponding position graph, that the cars actually meet after their velocities are equal. Both his problem-solving and his explanation involved heavy use of real-world language, thought experiments and simulations with the cars. He moved his hands numerous times to show how one car would catch up to the other and finally pass it. Finally, he was able to articulate a rationale for why the velocities would be equal before the distances were equal in general. His story was accompanied by hand motions that recalled his experiences with the motion detector and his real-world experience with cars.

> "The reason is that they're starting at the same place; however, this one --I'll call this number one --is starting at a much faster speed so while this one is sort of (rrr noises indicating slow velocity) this one is (vroom noise indicating fast velocity). And it's accelerating at a steady pace of whatever, although it's accelerating a small amount while this one is accelerating a lot. So that when they're going equal speed, say this one <#1> would be this far behind this one <#2> and then suddenly it's going to catch up a little later cause this one <#2>just isn't accelerating as fast as this one<#1>."

The spreadsheet environment, on the other hand, supports a very different kind of problem-solving strategy. Because there are no real-world semantics associated with the variables, students generally do their thinking in an entirely mathematical domain. Since the relationship between A and B is described by the interviewer in terms of differences (B is the difference between adjacent values of A), students' talk in general mirrors this way of thinking. M. worked on the crossing lines problem in the spreadsheet environment. One of his first statements about the problem was, "The difference <between adjacent values of A in the steeper line> is big at first...We have 500, then we have a big difference of say 10." M. then generated lists of numbers for both lines before he used the spreadsheet to see the graphs. Through working with the numbers, M. could see patterns that convinced him that the A lines would cross after the B lines. By trying several combinations of numbers he was convinced of the generality of his result. He even generated the conjecture (which turned out to be correct) that if the B lines crossed at x=t, the A lines crossed at x=2t. He could not, however, describe his result in the kind of coherent story S. generated because there were no real-world referents available in the spreadsheet environment.

These two categories are representative of the kinds of distinctions we are finding among these environments. In future analyses, we intend to compare the three in terms of the vocabulary students use to describe the shape and properties of the graphs, how they describe the process that generates the graphs, and the relationship between their qualitative and quantitative reasoning.

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K-8, teachers
Mathematical discourse
Teacher beliefs

TALKING MATHEMATICS: "GOING SLOW" AND "LETTING GO"[1]

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Teachers attempting to develop better mathematical discourse in their classroom engaged in a period of "going slow" characterized by gradual changes in amount of time devoted to mathematics, types of questions, and the nature of mathematical problems presented. A more complex and difficult phase of change, "letting go," involved giving up planned goals or topics to pursue ideas arising from the students' mathematical work.

For the past year, we have been working with a group of 12 elementary grade teachers (spanning grades kindergarten through seventh grade) who are investigating ways to develop mathematical discourse in their classrooms. The goals of this project (Talking Mathematics) are to: 1) work with a group of master teachers to explore techniques, principles, and models of mathematical talk in the elementary grades, 2) identify the difficulties teachers experience in supporting mathematical discourse in their classrooms, and 3) document the project's effects on teachers' beliefs about mathematics and on the nature of mathematical discussions in their classrooms.

After initial interviews and observations of all participating teachers, the project began with a 3-week seminar in the summer of 1990. During this time the teachers did mathematics together and began to explore the project's research questions with project staff. The team of two mathematics educators and a mathematician[2] deliberately focused the first two weeks of the seminar on mathematical investigations. We further decided that we would involve teachers in doing mathematics for their own development (Simon & Schifter, in press), regardless of whether the particular mathematics content and problems we chose could be used directly with their students. This approach contrasted sharply with much of the teacher training they had encountered, in which the "activities" from a workshop on Monday could be used in their classrooms on Tuesday. At first, teachers' discussion of mathematical investigations centered on "how I would do this with my students" or "how I would simplify this so my students would understand it." This classroom focus acted as a barrier--and perhaps as a safety

[1] This research was supported in part by a grant from the National ScienceFoundation TPE-8954631. The views expressed in this paper are those of the authors and do not necessarily represent the views of the Foundation.
[2] The authors and Harriet Pollatsek, Mt. Holyoke College.

valve--to teachers' grappling with mathematics *for themselves*. By the second week, however, teachers were engaging eagerly in mathematical investigations for their own intellectual development. A single mathematical investigation might require several hours or even several sessions because they insisted on continuing their work. By the end of the third week, we laughed together each morning about the "unagenda" for the day, since we all knew it would change as teachers became immersed in mathematics. As they gradually let go of immediate classroom application, they began to be captured by the pleasure of deep involvement in mathematics. As one teacher wrote in her journal, after an investigation that involved geometric relationships, "I loved doing the polyhedra problem . . . I don't want to leave it. I wish I didn't have other plans, a house to clean, a husband, so I could work on it. O boy, would I like to engage children the way I am engaged."

Starting the School Year: Going Slow

Feeling legitimately daunted by the nature of the task ahead, the teachers agreed at the end of the summer that "going slow" in the face of the complexity of change was the only way they could proceed. They recognized that they would be returning from their intense summer experience to a school culture of mathematics in which the expectations of students, parents, and administrators, the constraints of "the curriculum" and the tests, and even their own well-established routines might act as barriers to the changes they envisioned. They understood that if they demanded fast, radical change of themselves, they would end up feeling discouraged. As one teacher remarked: "We all have changed and I'm afraid of . . . what will happen to us once we're back in the system. . . I am afraid to walk in in September happy and with beautiful ideas and after three weeks all will be shattered."

As the school year began, teachers made gradual shifts in their approaches to teaching and learning mathematics in their classrooms. Analyses of teacher interviews, teacher journals, documentation of semimonthly seminar sessions, field notes and videotapes from classroom observations suggest five major shifts. We will list the first four here briefly, then discuss at greater length the fifth, which we view as central: a shift from "going slow" to "letting go." The first four indicators of change we documented are:

1. **Teachers planned and scheduled more time for mathematics.** This increase in time reflected a new balance between doing mathematics and reflecting on mathematics. At the beginning of one class session, a first grade teacher told our observer it would be a brief lesson. At the end of almost 40 minutes, with students

still eager to pursue a new discovery, she remarked to the observer, "This isn't going to die today! I didn't think this would last this long!" Teachers increasingly found that the complexity in apparently straightforward mathematical ideas led students to longer, deeper immersion in mathematics periods.

2. **Teachers asked different kinds of questions and refrained from accepting the first right answer offered by students.** Many of the classroom conversations we had observed which led us to undertake this project were dominated by teacher questions which generated single right/wrong responses. These tended to shut down conversation about mathematics. Teachers in the Talking Mathematics group are finding ways to word their questions so that they open up the conversation: "I think there is more of an attempt on my part to slow down and give kids a chance to use their own words to express things. I'm not as quick to take what I think kids are saying and make it the 'right thing.'" A good example of this kind of more prolonged exchange, provoked by a teacher's question, "what do you mean by *even*?," is described later in this paper.

3. **Teachers required students to share their thinking, and students became more able to do so.** In our field notes, we see teachers gently insisting that students try to explain their clarity or confusion ("you're thinking something--explain it to me"). In this context, mistakes become fuel for mathematics, rather than an evil to be avoided. Even the youngest students, in grades K and 1, are participating in mathematical discussions. A kindergarten teacher remarked, "I've noticed a decline in 'I don't know' [or] 'I just knew it,' and a second grade teacher echoed her, "Instead of 'I just know,' they can little by little define what they are thinking . . . Instead of telling, telling, telling, and rote, rote, rote, it's . . . their own information."

4. **Teachers structured mathematics experiences to focus on finding patterns, describing and analyzing those patterns, and devising conjectures, generalizations, formulas, and rules about how mathematical objects behave.** A good example of this trend occurred in a first grade classroom in which the students invented and investigated "rules" for addition and subtraction. Unlike the rules which usually come from an expert authority--teacher or textbook--to be learned and memorized, these rules were generated by the students. The beginnings of this discussion arose spontaneously and the teacher built the children's observations into a search for conjectures. As a result, these 6-year-old students have delved deeply, at their own level, into commutativity and the relation between addition and subtraction.

Changes in Beliefs: The Nature of Mathematical Authority

Even while "going slow," teachers' explorations of their own beliefs about the nature of mathematics and of mathematical authority (see Lampert, 1988) led some to more radical departures from previous classroom practice. Supported and challenged by the group, some teachers profoundly changed their views of the nature of mathematics as an endeavor. These changes are not uniform nor are they incremental, yet changes are occurring. Many are coming to believe that mathematics is something that is invented and constructed, and that mathematical convention is simply that--an agreed-upon set of definitions and procedures. The shape of these beliefs forms such an important basis for decision-making about teaching that they now enjoy discussions that we believe represent significant internal struggles about the nature and sources of mathematical knowledge.

In particular, teachers grappled with the role of convention in mathematics: To what extent can students participate in constructing and inventing mathematical knowledge? When do you give students a definition, formula, or procedure? What if students' constructions aren't "right?" In a February meeting, one of the teachers asked, "Say this group [of students]... all agreed and they came to some conclusion. But it wasn't a correct mathematical conclusion. Then what would a teacher do?" This question led to a powerful discussion of the nature of mathematical authority. The following excerpts provide a sense of the cognitive dissonance created in the group as they tried to find a legitimate role for mathematical convention while, encouraging students' construction of their *own* mathematical ideas:

- I think that there are some things that we need to confirm for kids. What it is that mathematicians, after having these kind of discussions for the past 5,000 years, agreed upon as universal enough to accept it. ... Let's say there's some practical reason to know the basic facts in base 10. So we don't need to have them debated at the point that it's useful for them to have that information...it's practical at some point to give [students] the conventionally agreed upon thing.
- We're trying to teach kids that they can ... be an inventor and they can create. If we <u>give</u> them convention they're always thinking that there is an answer out there that someone has already predesigned or pre-made. And then there is no inventor within them.
- I think that it's important to always leave kids with the feeling that <u>at this time</u> this is what we know about it. And there may be more information that

you'll get later that might make you think differently... The same thing happens in science all the time, that information changes pretty quickly. But I think if that seed of doubt or ability to change is part of discussions, then you kind of cover for it... Leaving kids with the feeling that nothing really is certain is all right, even though that's a little unnerving.

A Qualitative Change: "Letting Go"

As teachers began to change their pedagogy to reflect their changing beliefs, **their classroom work was characterized by a series of attempts to "let go" of the planned goal or subject matter *in order to pursue important mathematical ideas.*** Just as we had "let go" of our agenda during the summer, some teachers "let go" of their previous goals in mathematics teaching, of their control of the course of mathematical activity in their classrooms, of the shape and structure of the mathematics lesson, and, perhaps most difficult, of "getting through" all the subject matter they are expected to "cover". "Letting go" involved allowing more time for reflection and analysis, for students to articulate their own approaches. It required listening harder to students and probing beyond surface understanding. This process was accompanied by a loss of the comfortable feeling of closure and tidiness mathematics once seemed to embody. As teachers spent more time listening to their students, they were shocked to find that their students truly did not understand ideas which they had thought were straightforward. They sometimes became discouraged that their students "knew so little." They faced difficult questions: how do I treat a wrong answer? when do I introduce a definition? what do I do when a child does not "discover" what I hoped she would discover? how long do I wait until I "just tell them how to do it?"

For example, Martha, a fourth grade teacher, noticed as she watched her students count by twos on a hundred board that many of them did not seem to be using the terms *even* and *odd* comfortably. Taken by surprise that her fourth graders might not have a thorough understanding of even- and odd-ness, she asked, "What do you mean by *even*?" A long conversation ensued; this is a small excerpt:

M: What do you mean by *even*?
S1: You add them up and they add up to even.
M: What is even?
S2: On one chart you got 1, 3, 5, 7, 9, and on the other, 2, 4, 6, 8, 10.
M: What are they, 1, 3, 5, 7, 9?
S3: Multiples of 1.
M: Is that true?
S4: Odd is if you have three apples. You couldn't split it with a friend.
M: Split how? Why can't I get two and you get one?

S4: No, it's not even. If we had four, you could have two and I could have two.
M: Is 5 even?
Students: No . . . yes . . . I'm not sure . . .
S5: I couldn't get the same amount as you.
S6: If you had five apples, we each could get two and split the other half . . .

Martha "let go" both of her previous plan for the lesson and of her assumptions about her students' knowledge. However, this "letting go" was both difficult and uncomfortable. By asking a different kind of question and by not leading students hurriedly to a correct definition, she found that her students did not truly understand something which seemed so basic. While it is easy for researchers to be delighted and intrigued by the diversity of children's understanding, it may not be so easy for teachers, who feel responsible for their students' learning (Ball, 1990), to feel the same kind of delight when they begin to let go and uncover the complexity and confusion of apparently simple ideas .

Anna's fifth grade class was exploring relationships among the faces, edges, and vertices of various pyramids. As they compared their results, they began to realize there were disagreements about some of the conjectures they were positing. They gradually realized that the disagreement hinged on their definitions of *points* and *corners*. Some students had defined *corners* to include all the vertices of the pyramid, while others insisted that corners were <u>only the corners of the base</u> while the vertex "on top was "a different kind of thing." Anna attempted to follow the students' thinking, asking questions designed to challenge and clarify their definitions. A wonderful roaring discussion ensued, and the students pursued their ideas vigorously. As she continued to prod students' ideas and ask for elaboration, Anna made several attempts to bring closure to the discussion. As she became fully aware that they would not stop, and that closure was not likely, she completely let go of control of the discussion by turning it over to the class, saying: "All right. Argue. Or tell me. Or talk!" They did! As the class ended, she pointed to the formulas generated by the class and listed on the board: "Now these are <u>questionable</u> . . . tomorrow this is what you're going to have to do: decide if it [your definition] really matters." In letting go of time frame and lesson plan, this teacher was both following the interests of her students and allowing serious investigation of the nature and purposes of mathematical definition. There was no "summary" of the lesson, no closure--instead, the students left the room in full uproar, still talking about whether points and corners were the same thing.

The process of deep change is very difficult. The teachers with whom we work are extremely thoughtful and committed people; they have enjoyed the mathematics they've done, and they enjoy working together to generate their own theories of how mathematical discourse can best be supported in their classrooms. We see them as outstanding practitioners who are struggling to find a balance between two worlds of teaching--one with tidy little segments of ideas presented in a more-or-less linear manner, the other a much less tidy world of interests, enthusiasms, and uncharted teaching territory where much of their teacher-behavior must be let go.

Because of our deep and growing awareness of the difficulties (and the joys) of such deep epistemological change, we can make some statements about some of the conditions that seem to support them during this process. First, this is not a linear process. Teachers are at many places in a spectrum of possibilities, and any individual moves "forward" and "back" often, depending on many circumstances. "Going slow" and "letting go" are not neat progressive states; they alternate, overlap, and interact. Second, it isn't possible for teachers to disregard convention. Any conscientious teacher cannot simply shrug away all of the mores surrounding "the usual thing"--mathematical algorithms, tests, parental concerns. It is important that the conflict between old practice and new beliefs not be minimized; that it be understood and worked with. Third, just as we expect teachers to respect their students, those involved in teacher enhancement and research must feel deep respect for the teachers they work with as we ask them to take tremendous risks.

In the final analysis, it is in the complexity of the task that our enjoyment lies, for it is truly here that we see the wide range of individuals, teaching practices, changes in classroom practice, changes in belief. Here, in the midst of complexity, we find that the work of supporting teachers is, as always, based on respect, support, and taking time. We too, like the teachers, need to learn to let go of "certain" outcomes. We too need to enjoy and welcome uncertainty.

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age 8-9
representations
changes over time

YOUNG CHILDREN'S SPONTANEOUS REPRESENTATIONS OF CHANGES IN POPULATION AND SPEED

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This study investigates spontaneous representations developed by fourth grade children to depict changes over time in number of people in a place and in motion of cars. In the typical graphing introduced in school, children are taught to begin with a systematic format for the graph. In this study, when no particular format was required, children created idiosyncratic representational systems guided by the need to express actualized data and whatever they found relevant in the situation to be communicated.

Studies of spontaneous representations invented by children for different phenomena and situations, and the contexts in which the children engage to create these representations are emerging as a rich field of research. The creation of representational systems by children is being considered from several points of view:

-as a learning environment.

-as a research technique to investigate cognitive aspects.

-as a way to explore social construction of meaning through classroom interaction.

-as a medium to examine the impact of conventional representational systems in our culture.

Investigations of children's spontaneous representations are described in Bamberger (1988) on development of representations for musical rhythm, diSessa et al. (1991) on development of representations for the changing speed of a car, Ferreiro and Teberosky (1982), and Ferreiro (1988) on the development of reading and writing, and Karmiloff-Smith (1979) on children's representations for directions along a bifurcating path.

The study reported in this paper is most similar to the diSessa et al. study. It explores children's invention and discussion of graphical representations for situations of change. In a teaching experiment, we worked with children's spontaneous representations of changes in the motion of cars and changes in population. It is part of a curriculum project and a research project funded by the National Science Foundation[1]. Rather than teach certain forms of graphing--bar graphs in the early grades, line graphs later--we want to design learning ramps which connect the natural representations of children with more formal graphing techniques.

The teaching experiment was developed in six 45-60 minute sessions at a private school in Cambridge, Massachusetts. The group consisted of eight fourth grade students. One or two observers attended each session. The observers' notes and the children's graphical productions were the main data for our analysis.

[1]MDR-9050210 (Elementary math curric.); MDR-8855644 (Measuring and Modeling). This paper reflects the authors' ideas which are not necessarily those of the grantors.

We asked the children to make "pictures or graphs or charts" to show changing events. The sequence of phenomena we asked them to represent were:

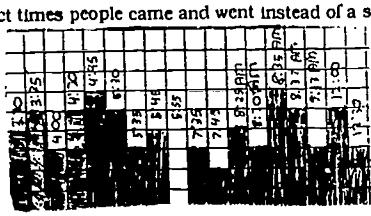
1. Changing population in a restaurant and in their classroom over a day;
2. Changing number of people in their homes over a day;
3. Sequence of changes (adding and taking away) of objects in a bag;
4. Changes in speed of a car described by a story;
5. Motions of a toy car moving across a table, presented in a video.

The representations were shared and discussed by the group. Students tested their understanding of each other's representations by acting them out. To act out the graphs of people in their houses, students moved small blocks "into" or "out of" a drawing of a house. To act out the motion graphs, students moved a toy car along the floor.

After we submitted the proposal for this paper, we observed 45 fourth grade students in an inner city and a suburban public school developing representations for the changing number of people in their homes over a day. Working with a large. number of children in different circumstances reinforced the major patterns we observed in the earlier study.

Patterns in the children's approaches to making graphs to communicate

One of the main attributes of a representational *system* is to delineate a universe of possibilities regardless of their "actuality". However, the children generally limited their representations to what was. All the children in this study had been introduced to the making of bar graphs in school prior to this experiment and some had experience with line graphs. A tension existed for the children between communicating data and using a system that fitted the criteria shown by teachers. The children's inventions incorporated some conventional graph features, but often had room only for the actual data, and they included illustrative features that were external to the system. For example, many children made graphs showing exact times people came and went instead of a scale of equal intervals:



When the teacher in one of the classrooms asked students to combine their data on a graph with equal hourly intervals, the children were concerned about how to do it--is a person who goes out at 7:15 shown to leave at 7:00 or at 8:00?--and they were puzzled at losing the specific data.

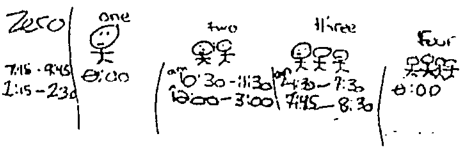
We will describe some typical characteristics of the children's productions that illuminate this tension between data driven and system driven representations: dealing with zero values, representing all data as discrete, and adding figural elements to the graphs.

<u>1. Dealing with no instances and with zero values</u>

In a systematic context, an empty cell is meaningful as a zero value. The children did not understand the bar graph as a system, but as isolated symbols requiring a key. They dealt with zero values in two ways. They explicitly marked the zero, usually in a manner distinctive from the markings for other values, or they omitted categories that had no members.

Children who made bar graphs to show populations through the day, more often than not, put a block for zero people home. One boy put one block for both zero and one person; he then used a color key with a different colors to distinguish them. Another child used one block to show no one home, two blocks to show one person home, etc.

A child who made charts, had trouble symbolizing "never". Since she specified time in terms of intervals delimited by extreme times (e.g. 10:30-11:30), what are the extreme times for an interval of zero duration? She resolved this first by using 0:00 to show that at no time was there only one person home. In her second draft of the chart she further resolved it by altogether omitting the category of one person home:



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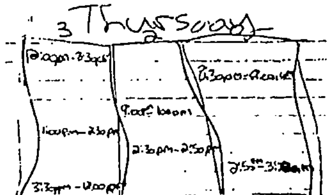
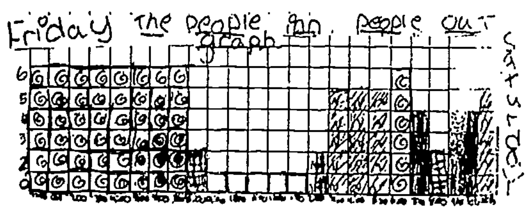


second draft

Another girl both omitted the possibility of one person home and used a special symbol--one empty block--to show zero people home. She then had two blocks represent two people home:



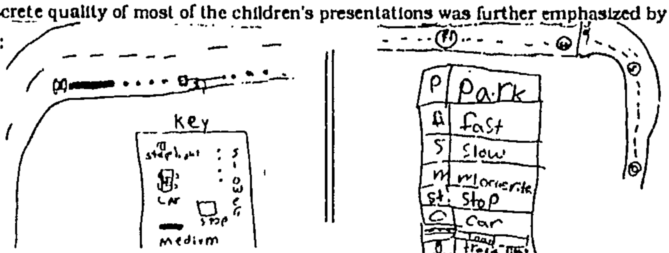
When the children did include zero values as in stopping in velocity and nobody in population, they tended to represent them as exceptional points. Even students whose representations showed some continuity of motion treated a stop as a special kind of situation, not a particular value on the continuum. Thus a child who increased the frequency of a wavy line to show slowing down, drew a dot in a box to show a stop. Others used spaces, vertical lines, or illustrations of stoplights to depict a stop.

Many children left no place for more than the maximum number of people present or the maximum speed of the car in the particular situation they were illustrating, and most omitted evening and night hours on the graph of population of their classroom when no one would be there, but included them on the graph of population of their homes. When we asked some students why they had not listed the evening hours, they assumed we meant there would be some people in the classroom then. One child went to ask her teacher how late she stayed at school and another asked whether the janitor would be present.

2.Tendency to think of change discretely.

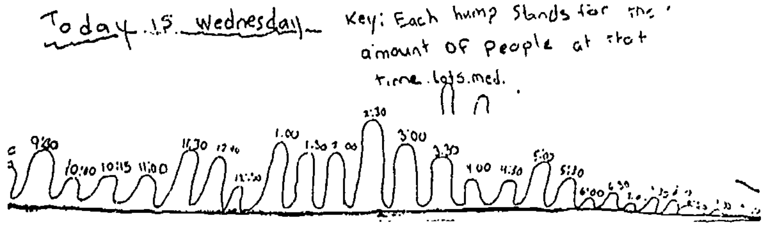
The changing speed of the car is a continuous phenomenon. The car cannot go from motion to a stop without going through every speed in between. However, the children reflected the data they actually had, and that data is always discrete. Instead of showing a continuum of speeds, they made categories such as "slow", "stop" and "fast". When they watched a video of a toy car moving, they jotted down their observations in terms of these categories. Similarly, for the population of a restaurant and of their classroom, they used categories such as "nobody", "few people", and "many people". When data fell between these categories, students did not know how to include them. One student, when trying to represent a situation when one third of the students were out of the classroom, said "I don't know how to put lots [of students in the room] but not as many as lots."

The discrete quality of most of the children's presentations was further emphasized by the use of keys:

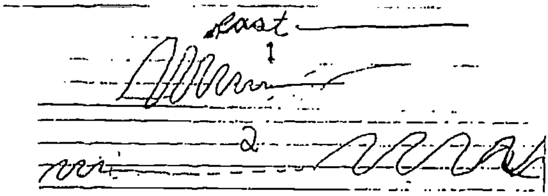


Some children used key codes such as color that could have been adjusted to show gradations, but they kept to the distinct categories.

Two girls made a graph of the changing population of the classroom that illustrated the tension between various conventions and between continuous and discrete presentations. They called their graph a line graph but they discretized it by providing a key and bringing the line down to zero to separate the hours:



All of diSessa et. al.'s sixth grade students developed continuous representations for the motion of a car, but only one of our fourth graders represented the possible speeds of a car as a continuum. He used a very wavy line for slow gradually becoming less wavy to a straight line for very fast, similar to the representations developed by students in the diSessa et. al. study:



3. Tendency to include figural elements.

The keys were a special example of the children's use of figural elements. In representing motion, the students often included drawings of perceptual elements of the situation (a toy car, a table, a street, etc.) even if they were not informative with respect to the motion itself. For changes in population, drawings (usually of stick figures) were used more informatively--as headings in a chart or icons in a pictogram.

Figural elements are not systematic. A system is based on internal consistency, but the figural elements are not. The children want to replicate the context they are representing. They make graphs to represent what is. A graph is complete if it illustrates all the data to be communicated. They are not concerned that the graph include all possible options.

In spite of the variation among their representations, the children were able to interpret each others graphs and to act them out as the author expected. They had difficulty only if the times were out of order

Discussion

The development of any representation integrates three aspects:

1. The information to be conveyed
2. The rules of the representational system
3. The intended use of the representation

These three aspects correspond to the traditional distinction betweeen the semantic, syntactic, and pragmatic aspects of language.

When the fidelity to the information to be conveyed is the most determinant factor in the representation, we refer to it as data -driven. Most the the children's examples we have described fall into this category. There are elements of a representational system--a well defined set of symbols, the ordering of the symbols, and so forth--but the system is simple: it has only one layer of meaning (each symbol means only one thing), and it is full of external elements.

On the other hand, when the emphasis of the representation is on using rules to be coherent, we refer to this as a system-driven representation. In this type of representation, efforts are made to eliminate elements that are not part of the symbolic system or that are not consistent with the system's internal rules.

Each emphasis (data or system) has its own advantages.

A data-driven representation:

• Facilitates communication when the system is unfamiliar to the reader. If we are not sure that the reader masters the representational system, it is helpful to resort to external elements to provide additional cues.

• Preserves the information. A data-driven system keeps all information thought to be relevant, whereas one of the implications of a system-driven representation is that some pieces of information may be lost.

On the other hand, a system-driven representation:

• Helps us to envision new possibilities and to open new questions. A data-driven system tends to reflect no more than what we already know. In contrast with a data-driven representation that tends to be idiosyncratic, a system-driven representation allows us to perceive patterns and generalizations. A well-known example is the Periodic table where the system of listing elements allowed scientists to recognize missing possibilities.

• Conveys more information with fewer symbols. A sophisticated representational system is a network of tacit relationships that become part of the message.

In developing a representation we have to decide how to come up with a good combination of data- and system-driven elements. This is the point where the third aspect, the pragmatic side of the representation, plays a critical role. If the representation is meant primarily to communicate to others, we need to fit it to the readers knowledge. A collaborative process like the one reported by diSessa et al. allows participants to assume a common understanding of their system-driven representations. The children in our study, developing their own representation for their own data, could not assume that someone else knew their personalized systems. This may account for the predominance of data-driven representations among these children. On the other hand, a representation for the purpose of keeping track for ourselves poses a different problem. We know what we need to include in the representation. When the children kept track for themselves (in collecting data or in putting objects in and out of a bag) they did not use figural elements because they did not need them. They knew the context of the problem.

In any case, the bias toward a system- or data-driven representation is an issue of weighing the pros and cons of each. This is why we talk about the tension between them. In the literature, a developmental trend is reported (Bamberger 1988, Ferreiro 1988, Karmiloff-Smith 1979): from data driven to system driven. It is also stressed that this transition does not respond simply to failure to communicate using less systematic representational systems. Apparently children tend to adopt a more systematic approach even when their former practices were successful. There is a spontaneous perception that systematicity, internal consistency, "cleanness" or elimination of external elements, avoidance of redundancies, and so forth are advantageous.

From a pedagogical point of view, we have observed that students are typically introduced to a system-driven representation while they perceive it as a wrong or meaningless way of conveying information. They appropriate it in terms of a data-driven system. We hope that a better understanding of spontaneous representations by children will help us to bridge this gap.

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Fourth grade through adult
Statistics
Average

TOWARD AN UNDERSTANDING OF MEAN AS "BALANCE POINT"[1]

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Twenty-nine children and adults were given problems in which they constructed data sets that could be represented by a given mean. Many of them felt that the notion of "balance" was an important one. As they attempted to construct a data set which "balanced," they explored sym[2] metrical balancing, balancing the sum of the data on each side of the mean, and finally, balancing deviations around the mean.

While most children are introduced to the averaging algorithm in fourth grade, recent research has shown that children and even adults do not understand the way in which the mean represents the data set (Gal et al, 1989; Mokros and Russell, 1990; Strauss and Bichler, 1989.) Our research focuses on how children and teachers think about the mathematical relationship in which the data "balance" around the mean in a particular way.

Twenty-one children (seven 4th, seven 6th, and seven 8th graders) and eight teachers were individually interviewed, using a series of open-ended tasks. Three types of problems were used: 1) Construction problems, in which participants were asked to construct a set of data which would result in a given average; 2) interpretation problems, which involved describing, summarizing, comparing, and reasoning about given sets of data, and 3) traditional averaging problems, solvable through the use of the algorithm. Data construction problems yielded the most interesting results, because these demanded deep thinking about the way that the mean represents the data.

Results showed that at the early grade levels (4th and 6th), children had a sense of the average as representing a modal amount or what is "typical" about a particular set of data. By 8th grade, most of the children thought about average as the midpoint of the data or the point of symmetry in a distribution which looks identical on each side of the mean. In some cases, this was the midpoint of the X axis, or the midpoint of the range.

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When given the opportunity to construct a data set for a particular average, about half of the 8th graders and adults constructed symmetrical data sets around the mean. In the symmetrical approach, people placed an equal number of data points above and below the mean, frequently using a pair-wise approach where they placed one data point a certain distance above the mean, and a second data point an equal distance below the mean. This approach resulted in a construction in which the mean, median, and mode were identical. In these symmetrical constructions, the average was clearly seen as a visual balance point, but this balance could only be maintained for symmetrical distributions: It was a significant problem for these people to deal with skewed distributions. When symmetry was not possible (e.g., when we moved a data point high enough so that the compensating piece of data on the other side of the mean would be below zero), these people had to switch strategies. Their middle point was no longer possible, and their framework for thinking about average no longer worked.

A problem that intrigued us was what strategies people would employ when they were no longer able to use the simple balance of symmetry, and were confronted with a problem where they needed a more complex notion of mathematical balance. In the section which follows, we describe the strategies employed by two individuals—a sixth grade boy and an elementary mathematics specialist—when they sense that balancing involves more than just symmetry.

1. Fred: Balancing symmetrically and balancing totals

Fred is the only sixth grader (and one of only two students) in the group who has both an understanding of symmetry and glimmers of a mathematical balancing strategy that goes beyond symmetry. In the Allowances problem, he was asked to build a distribution of allowance data around a mean of $1.50, by placing scrabble tiles on a large graph in which the horizontal axis was labeled in 25 cent increments. Fred first approaches this problem in a qualitative way, putting a few tiles out fairly close to the average. When asked what he's doing, he replies, "Well, I'm just making some of the numbers close to $1.50. So on average it would be close to $1.50." In fact, his data are quite symmetrical. As he continues to build his distribution, it appears that he is balancing each data point over $1.50 with one under $1.50. When asked whether he thinks the average would be $1.50, he replies that his data are "balanced", and when asked to say more about what he means, he notices something fishy:

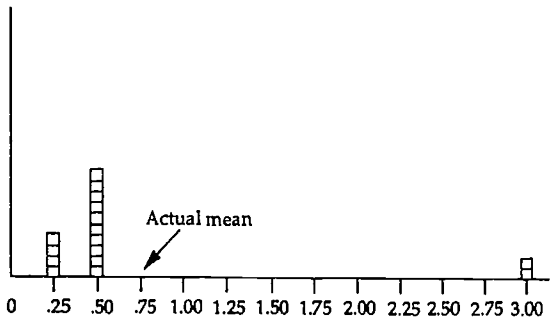
> Well, on either side of $1.50 there are about the same amount of pieces... wait a second... but umm... I think I made a mistake. But there should be more numbers on this side [points to the lower side of the distribution] because these are a lot more numbers. This [points to the upper side of the distribution] is a lot more money than these [lower side].

Fred is becoming aware that there may be another kind of balance—one which is based on the *sum of the data* on either side side of the mean. In the next part of the interview, he explores this new kind of balance. He begins by moving several of his data points from the high side of the distribution to the low side. Now, he has lots more on the lower side of $1.50 than on the upper side. He explains:

> Fred: So there would be some that would be more money [points to the data on the right side], but there would be more that were less than $1.50 to make it even.
>
> Interviewer: OK, and how would that make it even?
>
> Fred: Well, maybe if some of these were added up [values lower than the mean] they would equal what these would add [values higher than the mean].

Fred subsequently constructs another distribution of allowances which uses this principle of "balancing totals", where the sum of the data on each side of the mean is the same. In his second construction, he demonstrates that this technique "works" even if you don't have any data on $1.50.

Of course, the technique does not work, but even many adults who are presented with this strategy are intrigued with it and cannot immediately determine its flaw. The problem becomes clear when one takes an extreme example: Given an average allowance of $1.50, the "balancing totals" strategy might yield a distribution with 2 data points at $3.00 (total of $6), 10 at $.50 and 4 at $.25 (again, a total of $6). As can be seen below, this very skewed distribution appears to have a much lower mean than $1.50, as indeed it does—the actual mean is $.75. The "weight" of the data is at the lower end, between $.25 and $.50.



But for someone struggling with trying to develop a mathematical balance which works, this balancing strategy is appealing: it takes into account the values of all of the data; it applies to nonsymmetrical distributions; it is a kind of balance which works in other familiar mathematical situations, as in balancing equations or using a pan balance. However, what it fails to do is recognize deviation from the mean as the quantity that must be balanced. It is not the value of each piece of data which is important, but how far away it is from the balance point.

2. Evelyn: Scaffolding a flexible theory of balance

Evelyn is one of only two of the teachers in the study who develops a more general and flexible theory of balance, tied neither to symmetry nor to the algorithm. She appears to develop and solidify her ideas about the kind of balance represented by the mean in the course of the interview. In the Potato Chips problem, she is asked to construct the prices for 9 bags of chips so that the average is $1.38. A follow-up probe involves doing this problem without using $1.38 as a data point. After using an algorithmic approach successfully to solve both problems, she muses, "how else could you do it in an organized way, without actually hitting $1.38?" Encouraged by the interviewer, she thinks for a long time, admitting at one point, "I'm getting hung up on the unevenness of it," referring to the odd number of bags which prevents her from using only balanced pairs of values on either side of the mean. Then, she wonders out loud, "what would happen if four of them, yeah, if four of them were five cents under $1.38 and then if five of them were four cents over $1.38, that would work out," and, finally, more confidently, "any combinations like that, if you took six of them and made them three cents under [and] three of them at six cents over, I mean anything like that." With this solution, Evelyn has abandoned both the algorithm with its demand to consider the intermediate total and the midpoint with its demand for symmetry. She is the only person in our study, student or adult, who comes close to devising a general solution to this problem. However, in the Allowance problem, involving a larger and more complex data set, we see that her idea is not completely developed and she must reconstruct the way this balance works.

Evelyn begins the Allowance problem with an image of a normal distribution. She describes how to make a completely symmetrical distribution around the mean of $1.50, but decides that "it's not very interesting." To give herself more of a challenge, she decides to start with two tiles at $4.00, out of the range of a purely symmetrical distribution, which in this case must be bounded by $0.00 and $3.00. She balances the two tiles at $4.00 with 8 at $1.00, creating a balance of the *total amount of money* on each side of $1.50. Like Fred, she balances by using totals.

She explains:

Evelyn: Balance [means] that if you add something to this side of the $1.50, you have to have an equal addition to the other, I mean, the equation idea.

Interviewer: So, for instance, if you added, let's say you added six [tiles] nere at $1.00, then how would you think about ...

Evelyn: Then I'd have to put something over here [greater than $1.50] that would equal $6.00.

She is clearly struggling with these ideas during the interview, interrupting h own actions frequently with questions and reflective statements indicating uncert Although not completely content with her strategy, she does complete a distributi which the total of the data values greater than $1.50 is equivalent to the total of th values less than $1.50. She explains, "The average is $1.50 because the way it's set right now is, the amounts over here would balance with the amounts that are over here." A new strategy of "balancing totals" has replaced the strategy of balancing deviations which she articulated so clearly on the Potato Chips Problem. At this p in the interview, the interviewer introduces two new pieces of data, at $0.00, and a Evelyn how she would balance these. This question quickly places Evelyn in a sta disequilibrium as her "balancing totals" strategy cannot accommodate values of z

Evelyn: My first impulse is that you'd need two kids to go right there [at $1.50]. Is that right? Let me think. OK, we have two more kids, and that way the average is down ... a little bit. Oh, no, we need to go up a little bit. ... You need to up them ... they're $1.50 under the average. [She places two tiles at $3.00.] That doesn't seem right, though, that's too high.

Interviewer: To make them $1.50 above?

Evelyn: [pause] You know, what's holding me up here is if it's $3.00 each, you're actually adding $6.00 over here, whereas down here you're subtracting $3.00, I mean, net... So, I need, let me see, I've taken away a total of $3.00 here. Is that right? Yeah. I need to add a total of three dollars over here, which I could do by just putting one more on the three dollars. And then I would think it should balance... I'm still with the same idea of balancing the value above the $1.50 with the value under $1.50. [pause] There's still something not right with that, though.

Evelyn is actually balancing *the sum of the deviations* below the mean (the two tiles $0.00 are a total of $3.00 away from the mean) with the *sum of the data values* above mean (one tile at $3.00). As the interview progresses, the interviewer asks her to compare the strategy she used on the Potato Chips problem with her current strat In the course of this comparison, she decides to simplify her allowance distributio using only three tiles and explores the possibilities with this small number of valu With this small number of tiles, she quickly returns to a deviations strategy, finall deciding that only the deviations are critical. Unlike a seesaw, where both weight distance from the fulcrum matter, on this "balance beam" only distance matters, a

Evelyn concludes: "See, I had to do it with real numbers ... before I was kind of working it as an equation ... and I was getting stuck on that ... it would depend upon where I put them in the distance away from $1.50 ... what you need to do is look at the distance from the average."

Teaching Interventions

While it was very unusual for children or adults to make use of the concept of deviation in finding the mean, it appears that their own notions of balance are an important foundation for building the concepts of mean and deviation. We have discovered some key teaching strategies for building this concept. All of them involve data construction problems.

First, a simple method of calling attention to different kinds of balance is to have people build a small distribution with these stipulations: 1) it has an odd number of data points, and 2) no data can be placed on the mean. For example, in the Potato Chip problem discussed above, there were nine bags of chips to be priced. People frequently attempted pairwise balancing (e.g., one bag > $1.38, one bag <$1.38) and priced the leftover bag at $1.38. We then asked, "Can you do it without using $1.38?", which prompted people to think about different kinds of balances. Occasionally, we saw older students and adults come up with a "triad" strategy in response—a strategy in which three prices were balanced so that the mean was $1.38. Recognizing triads as a form of balancing is an important step toward understanding deviations.

A second method of moving people toward a more sophisticated view of balance involves the teacher or interviewer in making "adjustments" to the data distributions which children have already constructed. This strategy is a helpful intervention for people who balance in a strict symmetrical fashion as well as those who balance by totals. For symmetry users, simply moving or placing one data point at the extreme right end of the distribution can prompt disequilibrium. For example, in a distribution of the number of children's cavities (with a mean of 3), placing a new data point at 10 cannot be compensated for by placing a data point at -4! A new, nonsymmetrical strategy must be developed. For people who balance by totals, an appropriate intervention might be to add several pieces of data at zero, as was done with Evelyn. This addition has no effect on the total, but has an important impact on the mean.

Finally, we are experimenting with a teaching strategy that starts with the mean, and involves "unpacking" it. We start with all data piled on the mean (for example, data piled on 4 to represent the average family size). We then ask children to move the data points so that they look more realistic but still represent a mean family size of 4.

Most children initially construct the data set by moving pairs of data away from the mean in a symmetrical fashion. Gradually, they see that a large move on one side (e.g., a move from 4 to 8) can be compensated by smaller moves on the other side (e.g., moving two data points from 4 to 2). The fact that a family of 8 cannot be compensated for with a family of zero provides a strong motivation to move beyond symmetry! Our work has been primarily with concrete materials, but we are exploring a software environment which will allow children to manipulate data around a selected mean, and see the results of their moves.

Conclusions

The older children and teachers in our study usually had a concept of average that involved a balance point in the data. What they didn't yet understand was average as a mathematical point of balance that is usually different from the midpoint or the point of symmetry, and which is quite different from the kind of balance which they have encountered in equations or pan balances, in which each "side" has an equivalent amount. In order to have a solid understanding of what the mean represents, and how it relates to the data, the concept of balancing deviations is essential.

We believe that the notion of balancing is central not only to the understanding of mean, but to the understanding of a range of central mathematical ideas. However, there are many different kinds of mathematical balance embodied in a variety of relationships: equations, proportion, balancing weights on a seesaw or in a pan balance. Children and adults need plenty of opportunities to construct balances—in the realms of statistics, geometry, and number—in order for the rules of mathematical balance to make sense.

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Age: 12-13 yrs.
Pre-algebra

Crossing the didactic cut in algebra: grouping like terms in an equation

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The objective of the teaching experiment reported here was to overcome the "didactic cut", that is the student's inability to operate spontaneously with or on the unknown in an algebraic equation. In order to teach the procedure of grouping like terms, we treated a multiple of the unknown as a string of additions (e.g. 3n=n+n+n). This proved to be adequate for the grouping of terms involving multiples of the unknown. A persistent difficulty was observed in the grouping of a multiple and a singleton (i.e. 5n+n=192). A different problem of an arithmetic nature occurred in 5 of the 6 case studies. In jumping over terms in order to group like ones, students tended to be influenced by the operation following their take off term and ignore the operation preceding the term on which they were landing.

In a recent paper (Herscovics & Linchevski, 1991), we have reported the results of a study assessing the range of first degree equations in one unknown that could be solved by seventh graders prior to any formal instruction in algebra. For equations in which the unknown appeared only in one term, the overwhelming majority of students used inverse arithmetic operation(s) as a solution procedure. For equations involving two occurrences of the unknown, we have to distinguish cases in which the unknown appears on the same side of the equal sign (e.g. 3n + 4n = 35) and others in which it appears on both sides (e.g. 4n + 9 = 7n). Since our subjects had never seen a double occurrence of the unknown, they had to be informed that solving the equation meant finding a number that could be substituted for n in both terms. Practically all our students were able to solve all these equations and about 90%(20 out of 22) used systematic substitutions as a solution procedure. These results are complementary to those obtained by Filloy & Rojano (1984) who pointed out that a sharp demarcation exists between arithmetic and algebra as evidenced by what they called the **didactic cut**, that is, the occurrence of the unknown on both sides of the equality symbol. Our investigation has shown that students use the same solution method (systematic substitution) whether the unknown appears twice on the same side of the equation or when it appears on both sides. This leads us to view the didactic cut not in terms of a mathematical form but in terms of a cognitive obstacle (Herscovics, 1989). We define the **didactic cut** as **the student's inability to operate spontaneously with** or on **the unknown.**

A teaching experiment

In order to study the cognitive potential of some pedagogical interventions aimed at overcoming the didactic cut, we designed an individualized teaching experiment involving six case studies. We opted for this methodology in order to study the student's thoughts in a dynamical state while instruction was taking place (Menchinskaya,1969). We prepared a sequence of three 45' lessons. Each lesson was semi-standardized in the sense that a script had been prepared with the exact wording of each problem as well as some related questions to be raised by the instructor-interviewer. The lessons were semi-standardized so that the interviewer had the freedom to adjust the wording of the questions to make sure that they were

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understood by each student and also in the sense that the researcher could pursue any interesting avenue unforeseen in the preparation of the experiment. A second person acting as an observer was present during all the lessons. The observer came prepared with a detailed outline used to record all the students' responses. The lessons, as well as the pre-test and the post-test, were videotaped so that further analysis was always possible.

The six subjects that were chosen reflected three levels of mathematical ability as determined by the classroom teacher through her regular assessment of their schoolwork performance. Our two top students were Andrew and Daniel, our average students were Andrea and Robyn, our two weaker students were Joel and Audrey. The three lessons planned in this experiment dealt respectively with grouping like terms, cancellation of additive terms, cancellation of subtractive terms. This paper will be restricted to a detailed account of lesson 1. the teaching and learning related to the grouping of like terms.

Pre-test

Since our initial assessment had taken place during Oct.-Nov.1990 and our teaching experiment was scheduled for March-April, '91, we first had to ascertain if any changes had occurred during this time interval. We used the following equations to verify this:

1) $13n + 196 = 391$ 2) $16n - 215 = 265$ 3) $12n - 156 = 0$	4) $11n + 14n = 175$ 5) $17n - 13n = 32$	6) $4n + 39 = 7n$ 7) $5n + 12 = 3n + 24$

By and large, we observed very little change between the solution procedures used in November and in March. Students used inverse operations in reverse order to solve equations 1 and 2. Equation 3 was not part of the comparison for it had been added to assess the student's response when "the answer" was zero. All students used one single inverse operation except for Robyn who first converted it to $12n = 156$. Four of the six students solved the remaining equations by systematic substitution. Two students showed a somewhat different behavior. Robyn was our big surprise: after several attempts to find some numerical pattern, she used grouping and an inverse operations to solve equations 4 and 5, and a transposition followed by an inverse operation to solve equation 6. For equation 7 she reverted to systematic substitution. Audrey had surprised us in Nov. when she spontaneously grouped the terms in equations 4 and 5.and used substitution for equations 6 and 7. In March she used grouping in equations 4 and 5 but continued doing so with equations 6 and 7, thereby indicating that it was a meaningless operation.

Another part of the pre-test which we report here deals with the students' ability to perceive an equation globally. We felt that in grouping like terms, the students had to be able to distance themselves and view the equation as a whole in order to re-arrange the various terms. We used their perception of the possible cancellations in a string of arithmetic operations as some guideline of their global perception. We asked them to evaluate the following two sums: (a) 17+59–59+18–18=? (b) 237+89–89+67–92+92=? In Nov. only 2 students had observed the two possible cancellations in (a), but by March all of them could point them out. Regarding string (b) the second cancellation had been noted by 3 of the subjects. In March, this number had changed to 4 . It is in the second cancellation (–92+92) that we had observed a behavior which we qualified as "**a detachment of the minus sign**": 10 of the 22 students in our initial assessment had first added 92+92, then went on to subtract 184 ! Only one of the students participating in our teaching experiment , Audrey, was among them and she ceased to do so in March.

Lesson 1

In our earlier work, we had presented 7 equations with several numerical terms on either side of the equal sign and found that the students had no difficulty in grouping them spontaneously. The only equation where a problem did occur was 4+n−2+5=11+3−5 . We again had here evidence of the detachment of the negative sign for eleven of the 22 subjects first added 2 and 5 and then tried to solve the equation 4+n−7=9 . At the time we thought of this phenomenon as an aberration or an artifact. Since in the other equations students grouped numerical terms spontaneously we decided that this procedure did not have to be taught.

Part 1: Grouping terms involving the unknown

In our earlier investigation we found that students did not spontaneously group terms involving the unknown on the same side of the equal symbol. This is not too surprising since similar results were found in the context of algebraic expressions (Chalouh & Herscovics, 1988; Herscovics & Chalouh, 1985). However, in our study of the range of solvable equations, we had also included the equation n + n = 76 . Out of 22 subjects,15 (68%) immediately divided 76 by 2, one used number facts and only 6 students used systematic substitution. These results suggest that when the terms involve the unknown without any coefficient, for a majority of students there is a **natural tendency to group** these in the solution process. We used this idea at the beginning of lesson 1 by asking students to solve n+n=178 and immediately thereafter to solve 2n=178 . We repeated this problem with 3n=126 and 5n=155 and then pointed out that "when we collect all the terms in n, we call this **grouping all the n's**". At this point we asked our 6 subjects **Can you group the sum on the left of 3n+5n=126?** Three of our students spontaneously grouped the two terms. The other subjects were requested to expand each term into additions: n+n+n + n+n+n+n+n =136 which they easily regrouped as 8n. After solving 8n=136 they were asked if the answer they had found would also be a solution of the initial equation. None of them had any doubts about it thereby indicating that they accepted the two equations as equivalent. Two more questions were raised:- **Do you think that we can add 3n and 5n even if we don't know what the number is? - Is 3n + 5n = 8n true for every number n?** Our six students seemed somewhat surprised by the questions but answered affirmatively. The first of these questions was to make them aware that they could operate with the unknown and treat it as a **generalized number** by which a symbol such as a letter can be regarded "as an entity in its own right but having the same properties as any number with which they had previous experience" (Collis,1975) The purpose of the second question was to prevent a purely mechanical approach to grouping and to point out its general validity. Another 8 equations were presented:

1. 4n + 3n = 119	3. 5n + n = 192	5. 23n − 14n = 135	7. 3n+9n + 7n = 342
2. 13n+18n= 217	4. 9n−3n =174	6. 7n − n = 174	8. 10n−5n + 3n = 96

In solving equation 1, 4 students grouped the two terms immediately and 2 subjects first expanded each term into a string of additions. For the second equations, all 6 grouped immediately. These results show that our teaching intervention was sufficient for the students to overcome their initial difficulty in operating on the unknown in this specific context. Our sequence of equations was adequately incremental, with the first problem occurring in equation 3 due to the presence of the unknown as a singleton. Three of the 6 subjects were perplexed by this situation but overcame it when asked to expand 5n into a string of additions and then group all the n's. Given another singleton situation in equation 6, two of these three students had overcome their initial difficulty. The results on equation 4 and 5 indicate that the introduction of subtraction was handled with ease, all 6 students grouping the terms without any hesitation.

Equations 7 and 8 required expanding the grouping procedure to three terms. As long as only addition was involved, no problem could be detected. But it is with equation 8 that we observed some interesting failures, that of both the strongest and the weakest students, Andrew and Audrey. Both failures were due to a detachment of 5n from the minus sign preceding it. Both students ignored the subtraction and instead added 5n+3n to get 8n which they subtracted from 10n. In both cases , it was sufficient to question the students with an analogous arithmetic string: "Is 20 – 10 + 5 the same as 20 – 15?" The two students then corrected themselves.

Part 2:Grouping terms in the unknown in the presence of numerical terms

In order to verify if addition of mixed terms occurs spontaneously or not, we asked our students: **Can you show me how you would solve 7n + 6n + 21 = 203?** Four of the 6 students (Andrea, Robyn, Joel, and Audrey) grouped the terms in the unknown with the numerical term (7+6+21=34). For these students we had prepared the following intervention. We asked **"If I have a simpler equation like 9n+17=116, can we add 9 and 17?"** After they had solved the equation and found the answer to be 11, we asked them to look at the arithmetic equation **9x11+17=116** repeating the question about the addition of 9 and 17 .This discussion proved to be sufficient to prevent any further grouping of mixed terms as evidenced in their solution of the next three equation:
1). 17n + 12n + 36 = 210   2) 4n + 12n – 17 = 127   3) 27n – 41 – 19n = 87

Until now we had proceeded as in our assessment experiment. The interviewer wrote down on the student's worksheet each of the operations that was suggested by the subject; the student was strongly encouraged to use a calculator close at hand, and furthermore, he or she could look at the interviewer's notes thus eliminating the need to keep track of past operations. It is while working on the first equation above that we showed the students how to write out all the operations performed in the solution of the equation. After grouping the terms in the unknown, we re-wrote it as 29n+36=210, and then to indicate the last two steps we wrote n=(210–36):29 ., n=6 . Andrew, Daniel and Robyn used this notation in the solution of the next two equations. On the other hand, Andrea, Joel, and Audrey preferred more detailed notation in their solution requiring one equation for each step.

Part 3 : Grouping like terms

It is only when the possibility of adding terms in the unknown as well as numerical terms is present that the concept of "grouping like terms" becomes meaningful. We introduced this notion without any instruction in order to verify if the prior work preventing the addition of mixed terms had been sufficient. We asked our students to solve **9n+13n+15+21=278.** All six subjects solved this through the appropriate grouping of like terms and then the use of inverse operations. It is at this point that we introduced the terminology.

A second equation of this type was presented: **17n+36+8n+51=262**. All students, except one, solved it immediately using the procedure described above. One subject, Robyn, felt the need to re-write the initial equation in order to reduce the "distance" between like terms. She re-wrote it as 17n+8n+36+51=262 , and proceeded to solve it as had done the others.

A third equation, **102=22n–17n+49–12** ,verified if the "direction" of the equation might affect the students' responses. Three of our students did provide some indication of being affected by this new form. Andrea wrote =102 on the right hand side of the equation, but then proceeded to solve it with this number on the left. Robyn re-wrote the whole equation with 102 on the right. Audrey, after grouping like terms, re-wrote the grouped form with 102 on the right.

The grouping of separated like terms becomes far more difficult in the presence of both additions and subtractions as evidenced in our students' solution of

$$19n + 67 - 11n - 48 = 131$$

Three of our students (Andrew, Daniel, and Joel) tried to solve it by grouping 19n and 11n, but the addition sign following 19n seemed to take precedence over the minus sign in front of 11n since they **added** the two terms. The same pattern was used by Andrew and Daniel in grouping the numerical terms.: 48 was subtracted from 67, but they ignored the addition sign preceding 67 and **focused on the subtraction sign following it** and wrote their result as **–19** . We provided two specific interventions. We used the arithmetic string 20 + 5 – 10 and compared it with 20 – 10 + 5, bringing out the fact that the indicated operations had to be carried with the number in any change in the sequence of the string. We then suggested that the equation be re-written and the terms be re-ordered so that the like ones would be consecutive This was sufficient to generate the correct grouping and inverse operations needed to solve the equation. Regarding the other three subjects, Andrea was the only student who did not experience any difficulty with this equation. Robyn spontaneously re-wrote the initial equation in order to gather like terms closer together and proceeded to solve the re-arranged form. Audrey grouped correctly the numerical terms but experienced difficulties in grouping the terms in the unknown. She received the same instruction as the others regarding changes in the sequence of operations.

From these detailed descriptions, it appears that the need to develop the ability to perform operations non-sequentially must overcome the hurdle created by the presence of different arithmetic operations. Clearly, this is one area of pre-algebra that causes widespread problems. However, that the simple interventions seemed sufficient to remedy the situation was verified through the last equation presented in lesson 1:

$$7n + 29 + 16n - 12 - 9n + 49 = 37 + 295$$

Three of our 6 students re-wrote the equation as 7n + 16n – 9n + 29 – 12 + 49 = 37 + 295 and then proceeded to group like terms and use inverse operations. The other three students grouped like terms without any prior need to re-write the equation.

Post-test

A post-test was administered one month after the teaching experiment. One o students, Joel, showed signs of detaching a number from the indicated operatio evaluating 189−50+50 as 189−100 . To evaluate their performance on groupin had prepared a list of 8 equations. However, they had not done any algebra sinc 3 lessons given more than 4 weeks ago, and hence we thought that some o procedures we had taught would not come to their mind spontaneously. We thus prepared three tasks that were to be used as triggers that we hoped would jolt memory and replace them in the framework needed for the solution of equations the grouping procedure, one such trigger was required by only one student, Ro who simply did not know what to do when faced with the first equation, 11n+14n .We had prepared a worksheet used at the end of the three lessons listing the diff procedures we had taught and asking the students to indicate which one was t used on the four equations that were presented. For Robyn, this review pr sufficient to trigger the grouping procedure.

The following equations involving grouping were used in the post-test:

1)11n+14n=175 2) 17n−13n=32	3) 5n+n=192 4)10n−5n+3n=96	5) 7n+6n+21=203 6)39+12n+47+21n=383	7)19n+67−11n−4 =
		8)7n+29+16n−12−9n+49=37+29	

The first difficulty appeared in the solution of equation 3. All our students with exception of Andrew and Daniel failed to solve it. The presence of the unknown i form of a singleton was simply not considered as a multiple of n that could be a since it lacked a numerical coefficient. This was expressed very clearly by Robyn stated" five n plus zero n is five n".

The next three equations were solved by all students without any problem. In equ 4, no one added 5n+3n thereby indicating a detachment of the minus sign from 5 equation 5, not a single student grouped mixed terms although four of them had so during the teaching experiment.

It is on equation 7 that we observed the recurrence of a problem noted earlier. D the initial lesson, three students, Andrew, Daniel, and Joel had added 19n and two of them, Andrew and Daniel had subtracted 48 from 67 but then decided tha result ,19, had to be preceded by a minus sign. In the post test, Daniel starte adding 19n to 11n but then corrected himself. When queried about it he stat thought you're supposed to use the sign **after** the first number." In the post-test, repeated the same mistake explaining quite convincingly that "19n goes with the sign." Audrey, failed to solve the same equation for the same reason.

Conclusion

The teaching experiment was successful in overcoming the cognitive obstacle kn as "the didactic cut", that is, the students' initial inability to operate with or o unknown. Of course, this change is restricted to the very limited operation invo that of grouping like terms in an equation. Our subjects' performance indicates they are viewing the literal symbol as a generalized number. However, the fact four of them had problems dealing with the occurrence of the unknown in the form singleton must moderate this statement. One must also remember that with all our equations we remained in the realm of natural numbers.

As in our first investigation we discovered the tendency to **detach the minus sign** preceding a numerical term. In this experiment too, we found a major cognitive problem of an arithmetical nature. During the teaching experiment, Andrew, Daniel, and Joel solved equation 7 by adding 19n+11n. In the post test, Joel repeated this mistake while Audrey was doing it for the first time. Both Daniel and Robyn showed great hesitation, not being sure whether to add or subtract. Thus, 5 of our 6 students showed at one time or another evidence of some this problem. Two of them clearly stated that they were jumping over terms but carrying with them the operation immediately following their take off term. Perhaps one can summarize this behavior as **jumping off with the posterior operation**. That this type of problem is not trivial is shown by the fact that during the lesson this problem was addressed but that a month later, one of the students was repeating the same mistake, a second one was no longer sure about what to do. This seems to indicate that the problem is rather robust and that perhaps, it should not be dealt with incidentally but should be addressed as a serious obstacle in pre-algebra.

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Age Level: Age 4:9-5:8
Identifier: Number Concepts
Identifier: Minority Students

NUMERICAL KNOWLEDGE OF ENTRY-KINDERGARTENERS: AN URBAN STUDY

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Between day 5 and day 12 of the school year, all of the entering kindergarten children in six predominantly minority, urban, public schools were interviewed to characterize the children's knowledge of number and counting prior to formal schooling. Tasks included enumeration, set construction, and counting on. In addition, the children were administered tasks that required them to apply their knowledge of number to solve quantitative problems presented within a real-life context.

Prior to formal schooling, young children construct knowledge of number and counting through real-life events (Ginsburg, 1982). As noted by Leinhardt (1988), circumstantially-based knowledge may be applied by a child when solving problems based in a context that is familiar to that child. Prior studies have characterized older children's use of informal knowledge to solve mathematical problems (e.g., Carpenter & Moser, 1983) and have examined young children's knowledge of counting and early number prior to schooling (e.g., Fuson, 1988). This study extends this work in two ways. First, it simultaneously examines the entering kindergarten child's proficiency in counting, in constructing sets, and in solving contextually-based numeric problems. Second, this study's sample reflects culturally diverse children of varied socio-economic levels. Need for this type of research was noted by Secada (1988) who cautioned that if research on cognition fails to include minority learners, then that research may fail to identify any unique characteristics

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of those underrepresented learners. Secada warned that such a research base may then legitimize the perception that any minority learners' deviation from "the expected" is either marginal or deviant.

This report examines the understanding and application of cardinal number offered by culturally diverse children upon entry into kindergarten. This investigation is part of a larger study that will attempt to periodically reassess these children over a three-year period. Thus, this report is limited to examining specific constructs at a given point in time in order to establish baseline data characterizing the numerical knowledge of culturally diverse kindergarten children. The constructs selected for inclusion in this assessment were identified in light of existing theoretical models that offer a framework for examining the maturation of a young child's construction of number.

METHODOLOGY

Sample. The sample for this study consisted of each of the 469 kindergarten children in 21 classrooms located in six predominantly minority public schools in Maryland, on the outskirts of Washington, D. C. Each of these schools is located in a generally contiguous urban area reflecting culturally diverse neighborhoods. Because of the three distinct multicultural and socioeconomic patterns reflected in the schools, the six schools may be better characterized as three pairs of schools (pair A, B, and C) with the schools in each pair reflecting similar economic and racial/ethnic patterns. Table 1 depicts the racial make-up within the kindergarten classrooms within the pairs of schools. An economic characterization of the six schools may be inferred from the percentage of children receiving reduced fee or free school breakfast and/or lunch through a government-supported program. Assistance was deemed appropriate for approximately 70% of the children in pair A, 58% of the children in pair B, and 35% of the children in pair C.

Interview Protocol. Because of the number of children involved, the interview protocol was structured with a scripted format. The script outlined alternative directions and clarifications, as well as describing the placement and use of supporting manipulative materials. Although other items composed the entire

Table 1: Racial Distribution Upon Entry into Kindergarten

	White*	Black	Asian	Hispanic
Pair A				
School 1	11 (9%)	53 (45%)	18 (15%)	36 (31%)
School 2	4 (8%)	16 (38%)	8 (19%)	15 (35%)
Combined	15 (9%)	69 (43%)	26 (16%)	51 (32%)
Pair B				
School 1	22 (21%)	24 (23%)	10 (9%)	49 (47%)
School 2	28 (28%)	26 (36%)	4 (5%)	15 (21%)
Combined	50 (28%)	50 (28%)	14 (8%)	64 (36%)
Pair C				
School 1	34 (46%)	33 (44%)	0 (0%)	7 (10%)
School 2	22 (40%)	16 (28%)	3 (5%)	15 (27%)
Combined	56 (43%)	49 (38%)	3 (2%)	22 (17%)

*The category White includes any child who is not Black, Asian or Hispanic

interview, this report only addresses the items, and item clusters, that focused on numerical knowledge. The protocol was translated into Spanish for administration to Spanish-only speaking children (n = 72) by bi-lingual interviewers. Children who were fluent in Vietnamese (n = 6) or Khmer (n = 9), but not English, were assessed by an English-speaking interviewer who was accompanied by either a Vietnamese or a Cambodian translator.

The numerical knowledge items assessed the following understandings: rote counting (children were stopped if they counted correctly to 35), counting sets (of size 3 and 5), constructing sets (of size 3, 4, 9, and 15), cardinality, unit rule (4 + 1; 4 - 1), repair of sets (modify 1 to 3), identifying larger number as more within a story context (3 vs. 2; 10 vs. 4; 7 vs. 3), solving simple join and separate word problems (3 + 1; 4 + 2; 5 - 2), and counting on (identifying what number comes after a specified number during counting and rote counting on). The actual interview items were either identical to or modified forms of items developed in prior studies conducted by Baroody, Bergeron, Carpenter, Cobb, Fuson, Ginsburg, Herscovics, and Steffe.

Design. As part of a larger study, one school from each pair was randomly selected to participate in a teacher enhancement program. Therefore, the design for the analysis of this baseline data was constrained to determine if there were entry-level differences in numerical understanding between the kindergarten children enrolled in the schools where the teachers are scheduled for inservice enhancement and the kindergarten children in their paired schools (TX effect). The design also examined whether their were differences associated with the three different racial and socioeconomic patterns reflected within the three pairs of schools (SES effect). The inclusion of 21 classrooms yields an unbalanced mixed effects model with two fixed effects (TX, SES) and an interaction term (TX * SES) as well as a random effect due to classroom (CLASS), nested in TX and SES.

ANALYSIS

In order to inhibit frustration, item administration branched at given points during the interview. If a child incorrectly counted a set of 3 beans, then the child was not asked to count 5 beans; rather the more difficult item was coded as "incorrect". A similar branch was made for the sequence of set construction items (construct sets of 3, 4, 9, and 15). If a child could not construct a set of size 3 or 4, then the subsequent unit rule, repair sets, word problems and counting on items were not administered as each of these subsequent items presumed this set construction knowledge. These items were then coded as incorrect for the purpose of this statistical analysis.

Reliabilities. Items within the interview protocol that were designed to assess similar constructs were clustered, and subscale reliability was computed using Cronbach's alpha. Subscale reliabilities were computed for counting sets (.547), constructing sets (.869), adjusting set size (i.e., items reflecting unit rule or set repair) (.612), identifying larger number in context (.78), addition and subtraction word problems (.685), stating what number comes next in a count (.853), rote counting on (.672), and numeral recognition (.912). Reliabilities were also computed for two defined aggregate scales: Number in context (cluster of unit rule, set repair, identification of larger number in context, and addition and subtraction word problems) (.816) and early number concept (cluster of counting sets, constructing sets and cardinality items) (.769).

MANOVAs. A multiple analysis of variance was computed on the subscales using SAS's general linear models procedure as defined for a nested, mixed effects model. To hold any Type 1 error rate constant across the multiple statistical tests made for item clusters, Bonferroni critical values were used to determine statistical significance of resulting ANOVAs, as well as for ANOVAs computed on aggregate scales and rote counting data. The only significant overall effect identified in the MANOVA was SES ($F(20,10) = 5.32$; $p = .0048$). Subsequent examination of the related subscale ANOVAs yielded significant SES effects within two of the subscales: identifying the larger number in context ($F(2,15) = 9.42$; $p = .022$) and stating what number comes next in a count ($F(2,15) = 9.59$; $p = .021$). An examination of means revealed that for both of these subscales, school pair C had significantly greater mean scores as compared to school pairs A and B. No significant effects or interactions were noted on ANOVAs computed on the aggregate scales or rote counting data.

DISCUSSION

Descriptive Analysis. As indicated in Table 2, of the 469 children in the sample, 22 children (4.7%) could do no more than imitate the counting stem offered by the interviewer (WOULD YOU COUNT FOR ME?... COUNT WITH ME.... ONE... TWO... THREE... NOW YOU KEEP GOING.). However, 78.9% of the children could count to 10 or beyond upon entry to kindergarten, while 22% could also count beyond 29. Many children clustered within the 11 through 14 range (25.3%) or stopped counting at 29 ($n = 35$; 7.5%).

Table 2: Enumeration Skills of Entering Kindergarten Children by SES: Frequency

Count:	0-3	4-5	6-9	10	11-14	15-19	20-29	30-35
Pair A	7 (4%)	9 (6%)	19 (12%)	14 (9%)	52 (32%)	18 (11%)	20 (12%)	22 (14%)
Pair B	10 (6%)	16 (9%)	21 (12%)	12 (7%)	43 (24%)	14 (8%)	22 (12%)	40 (22%)
Pair C	5 (4%)	3 (2%)	9 (7%)	7 (5%)	24 (19%)	12 (9%)	29 (22%)	41 (32%)

Table 3: Counting Set Skill of Entering Kindergarteners by SES: Frequency Correct

	Pair A	Pair B	Pair C
Counts 3 beans	145 (90%)	166 (93%)	126 (97%)
Counts 5 beans	128 (79.5%)	145 (81%)	115 (88.5%)

The conceptual demand of constructing a set composed of a specified number of items, as opposed to counting a set consisting of a given number of items, is demonstrated in the data in Tables 3 and 4. Although the distinction was not statistically significant, the children in Pair C, with a, higher economic basis and a lower percentage of minority children, seemed to have constructed a scheme for constructing larger sets than the children in the other four schools. As expected, an increase in the quantity to be considered increased the difficulty associated with both the set counting and the set construction tasks.

Table 5 characterizes the children's facility in using number in applied contexts. Approximately 50-75% of the children sampled could apply their construct of number to familiar contexts prior to formal schooling. Although patterns in the data between the three pairs of schools may be discerned, with the one exception of determining the larger of two numbers being used in context, these distinctions are not statistically significant.

When asked what number is one more than 3 and what number follows 8 when counting 37% of the children in Pair C answered both questions correctly while only 16% (Pair A) and 19% (Pair B) of the other children were successful, yielding the significant SES effect.

Table 4: Constructing Set Skill of Entering Kindergarteners by SES

	Pair A	Pair B	Pair C
Constructs set of 3	108 (67%)	133 (75%)	115 (88.5%)
Constructs set of 4	85 (53%)	109 (61%)	104 (80%)
Constructs set of 9	61 (38%)	78 (44%)	76 (58.5%)
Constructs set of 15	20 (12%)	34 (19%)	36 (28%)

Table 5: Entering Kindergarteners' Understanding of Number in Context

	Pair A	Pair B	Pair C
Unit Rule (4 +1)	74 (46%)	90 (51%)	91 (70%)
Unit Rule (4 - 1)	77 (48%)	98 (55%)	92 (71%)
Repair Set of Size 1 to a Set of Size 3	78 (48%)	102 (57%)	99 (76%)
Who has More? 2 or 3?	80 (50%)	103 (58%)	101 (78%)
Who has More? 3 or 7?	76 (47%)	95 (53%)	95 (73%)
Who has More? 10 or 4	59 (37%)	75 (42%)	90 (69%)
Join Result Unknown (3+1)	75 (47%)	94 (53%)	93 (72%)
Join Result Unknown (4+2)	52 (32%)	66 (37%)	74 (57%)
Separate Result Unknown (5-2)	47 (29%)	53 (30%)	52 (40%)

Although the analysis presented here has been constrained by the lack of individual SES data on each student, this work does offer clarification of the broad and varied understandings possessed by the children in predominantly minority kindergartens. In particular, many of the children are quite adept at using their informal knowledge of number to solve problems situated in real-life contexts.

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Age Level: Undergraduate
Identifier #1: Proof
Identifier #2: Concept understanding

A CONCEPT-UNDERSTANDING SCHEME AND THE LEARNING OF PROOF

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ABSTRACT: This study examined the cognitive difficulties undergraduate students experience in learning to do mathematical proofs. The data were collected primarily through nonparticipant observation and interviewing, and analytic categories were developed inductively from the data. The major sources of the students' difficulties are discussed in terms of a concept-understanding scheme involving concept definitions, concept images, and concept usage.

The present study sought to understand the sources of undergraduate students' cognitive difficulties in learning to read and write proofs as they make the transition from the lower-level mathematics courses emphasizing computations and symbol manipulations to the upper-level courses requiring proofs.

Although few empirical studies have addressed the learning of proof at the undergraduate level, the literature suggests the following areas of potential difficulty for students: (a) perceptions of the nature of proof (Balacheff, 1988; Bell, 1976, 1979; Galbraith, 1981; Schoenfeld, 1985), (b) logic and proof techniques (Bittinger, 1969; Solow, 1982), (c) problem-solving skills (Goldberg, 1973/1975; Schoenfeld, 1985), (d) mathematical language (Leron, 1985; Rin, 1983), and (e) concept understanding (Dubinsky and Lewin, 1986; Hart, 1987). Whereas most of these studies focused on a particular aspect of proving, the present study took a broader perspective by attempting to determine which areas of difficulty are the most salient for capable students who are just learning to do proofs.

Methodology

The purpose of the study was to develop a grounded theory of the students' difficulties in learning to do proofs in a first course that emphasizes proofs. I conducted two

This paper is based on the author's doctoral dissertation completed in 1990 at the University of Georgia under the direction of Jeremy Kilpatrick.

preliminary studies and the main study in undergraduate mathematics courses at the University of Georgia in 1989. The first preliminary study was in a group theory course, and the other two studies were in a transition course designed to teach students how to do proofs and to introduce them to certain mathematical concepts that pervade advanced mathematics courses. The topics included in the transition course were logic and proof techniques, set theory, relations and functions, and the real number system. The assigned proofs were short deductive proofs based largely on definitions.

I conducted the main study during the 10-week fall quarter. The professor was a research mathematician who had taught a wide variety of undergraduate and graduate courses. The class consisted of 16 students: 8 undergraduate mathematics majors, 6 undergraduate mathematics education majors, and 2 graduate mathematics majors. I selected five students as key participants. They represented a variety of mathematics backgrounds and were willing to meet with me for tutorial sessions and interviews outside of class.

The data collection methods included daily nonparticipant observation of class, interviews with the professor and with the students, tutorial sessions conducted with individual and small groups of students, two open-ended questionnaires given to the class, and examination of the key students' tests. As the study progressed, I analyzed the data inductively by writing categories of the students' difficulties in the margins of my fieldnotes and interview transcripts and by looking for properties of the categories and relationships among them.

Findings

The data revealed three major sources of the students' difficulties with proofs: (a) concept understanding, (b) mathematical language and notation, and (c) getting started on a proof. These areas of difficulty, and seven particular difficulties (D1 - D7), are shown in the diagram in Figure 1. The arrows between the boxes indicate that a difficulty in one area led to a difficulty in another. Other sources of difficulty, such as poor problem-solving skills and lack of prerequisite knowledge, were also evident in the data but were not major sources of difficulty for the students in this course.

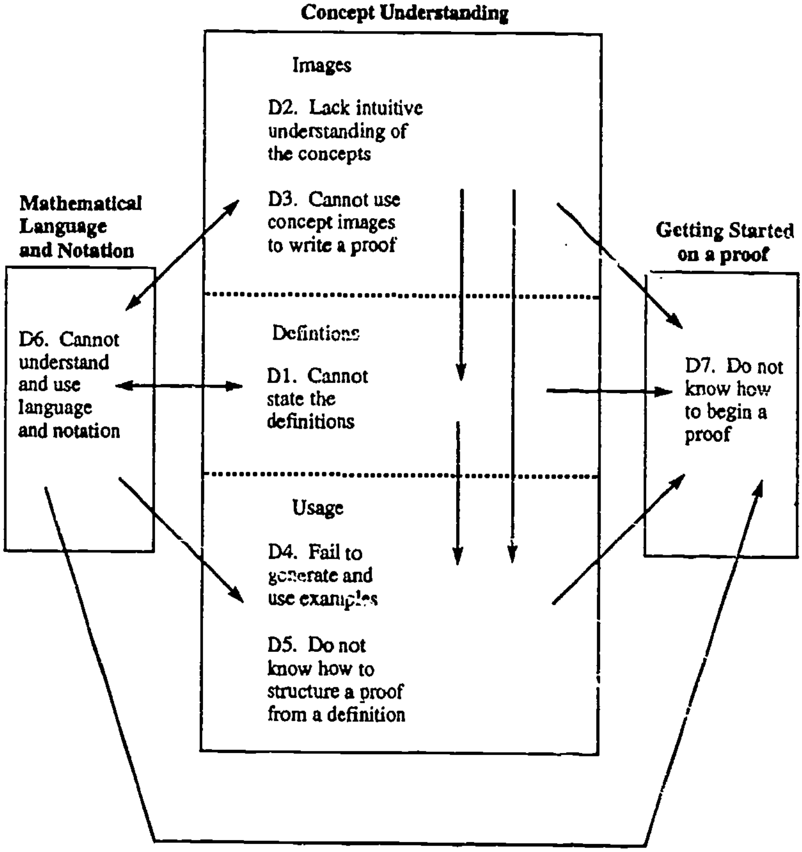


Figure 1. Model of the major sources of the students' difficulties in doing proofs.

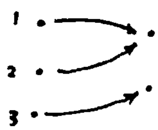
<u>Concept-Understanding Scheme</u>

Tall and Vinner (1981) distinguished between the formal verbal definition of a mathematical concept, the *concept definition,* and the cognitive structure in an individual's mind associated with the concept, the *concept image,* which is derived from examples, diagrams, symbols, and other experiences one has with the concept. The data from the present study revealed a third aspect of concept understanding, *concept usage,* which refers to the ways one operates with the concept in generating or using examples or writing proofs. The term *concept-understanding scheme* refers to these three aspects of a concept: definition, image, and usage.

As an example of the concept-understanding scheme, consider the notion of a one-to-one function. Two definitions are commonly used: (a) A function f is *one-to-one* if no two distinct ordered pairs of f have the same second term, or (b) a function f is *one-to-one* if for all x and y in the domain of f, $f(x) = f(y)$ implies that $x = y$. Although these two definitions are mathematically equivalent, one may be easier to use than the other for a particular task. One's concept image of one-to-one may include specific examples or nonexamples, such as x^3 or x^2, a dynamic mapping of points (drawn on paper with arrows between dots), a graph in the Cartesian plane in which no horizontal line meets the graph more than once, or other representations. A student, Linda, drew the following diagrams when I asked her to explain one-to-one. They reveal aspects of her concept image.

R: { (1,2), (3,4), (2,2) }

not a one-to-one function



not 1-1

Finally, one uses the definition of one-to-one in at least three ways. One way is to generate examples or nonexamples. For this purpose the first definition above may be easier to use than the second. Linda appeared to use the first definition to generate her nonexample involving ordered pairs. Another way to use the definition is to apply it at a particular point in a proof. If one has that $f(x) = f(y)$, then one uses the definition to obtain the next line in the proof, $x = y$. The definition provides both the language and the justification for the statement. A third way is to use the definition for the overall structure of a proof. To prove that f is one-to-one, one begins by supposing that $f(x) = f(y)$ and then shows that $x = y$. The second definition is better for this purpose because it clearly reveals how such a proof should begin and end.

Discussion

As illustrated in Figure 1, the students were unable to do proofs when they did not know the definitions for the concepts involved or did not know the most appropriate definition (D1), when they had little intuitive understanding of the concepts (D2), or when they did not know how to use a definition to structure a proof (D5). When I asked Linda about one-to-one, she stated the first definition and gave correct examples and nonexamples, but she did not know how to write a proof that a function is one-to-one. She chose the wrong definition to work with, and even when I gave her the second definition she did not know how to use it to begin a proof.

The arrows emanating from the Images box indicate that the students often needed to develop an informal understanding of a concept before they could understand its definition, including the language and symbols used in the definition, or know how to use it in a proof. But as shown by the episode with Linda, knowing a definition and having an informal understanding of it are not sufficient to do a proof. Her definition and concept image did not provide the language and logical structure she needed to write a proof (D3).

Mathematical language and notation (D6) was an obstacle for many students. Although most of them overcame most of their difficulties in this area by the end of the course, some students had difficulties throughout the course. In Figure 1, the arrows from the Mathematical Language and Notation box indicate that difficulties in this area prevented the students from understanding concepts and using definitions. In particular, the students could not understand a definition because they did not understand the language or notation used in the definition. On the other hand, they seemed to learn the meaning of the language and notation by learning the definitions; that is, by developing their concept images through examples and diagrams they gained an understanding not only of the definition but also of the symbols, words, and grammar of mathematics. Also, learning to translate a definition into symbolic form in which quantifiers are explicit--for example, f is *one-to-one* if $\forall x \, \forall y \big(f(x) = f(y) \rightarrow x = y\big)$--helped them see the logical structure of a proof based on the definition and facilitated their use of the definition.

In many instances the students were unable begin a proof. Figure 1 shows that this inability was due to deficiencies in all three aspects of concept understanding and in language and notation. Also, the students often began a proof with the wrong hypothesis. Specifically, they tended to begin a proof of an implication "If P then Q" by writing the statement P, rather than using P at the appropriate place in the proof. This error suggests a misconception about the role of hypotheses in a proof.

Finally, the findings suggest differences between the students' cognitive structures and the professor's. His knowledge of a mathematical concept seemed to be organized into a single schema that included multiple definitions for the concept, a well-developed concept image, and an understanding of how the concept is used. His work with the concept was facilitated by his ability to move freely among these different aspects of his schema as demanded by the task at hand. In contrast, the students often knew at most one definition for a concept, had superficial or narrow concept images, and lacked an understanding of how definitions are used. Furthermore, their knowledge of the concept was not orchestrated into a whole but appeared to be organized into separate schemata. In short, the professor had more domain-specific knowledge, more general knowledge, and better knowledge organization.

In conclusion, the study reveals that concept definitions, concept images, and concept usage interact with one another as students learn to do proofs and that these three aspects of concept understanding are linked with other difficulties the students have in reading and writing proofs.

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Age Level: Grade 6
Identifier #1: Fractions
Identifier #2: Measurement

SIXTH-GRADERS' KNOWLEDGE OF FRACTIONAL PARTS OF AN INCH

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Sixth-grade students performed a number of tasks that involved representing fractions and mixed numerals on a ruler scaled to the eighth-inch. Most students successfully measured and identified half-inches and mixed numerals; they had few problems with unit identification similar to those reported in the literature for number lines. Few students consistently responded correctly to fourths and eighths of an inch.

This study investigated sixth-grade students' ability to use a 6-inch ruler scaled to the eighth-inch to measure lengths to the exact half-, fourth- and eighth-inch. In Kieren's (1976) analysis of the rational number concept this would be one application of the measure subconstruct. Previous research (Larson, 1980; Bright et al., 1988) has shown that the measure subconstruct as represented by the number line is very difficult for both elementary and junior high school students. The purpose of this study was to see if students had the same difficulties with representing fractions on a ruler as they did with representing fractions on a number line. In past research, rulers and number lines have been considered to be essentially the same model, with the result that few rational number research studies have investigated the ruler as a model. For example, Lesh, Landau, and Hamilton (1983) classified the ruler as another number line representation on their test, Rational Number Concept Assessment.

A comparison of the features of the number line and the ruler shows that they are similar in that they are both related to linear measurement as they both contain a well-defined unit of length which can be partitioned into fractional parts and a scale. One way in which they differ is that the linear unit used on a number line is a non-standard unit, whereas the unit on the ruler is fixed, has a specific name, and is part of a system of measures (Ohlsson, 1988). The inch was the fixed unit used in this study. Another way in which they differ is the manner in which the points on the scale are indicated. On the number line each rational number, including those that are whole numbers, are marked with identical dots or with vertical lines of the same length. On the ruler, the fixed unit is marked by vertical lines that are longer than any of those used for fractional parts. Also as is customary, the marks on the ruler indicating halves were longer than those for fourths which were longer than those for eighths.

A third way in which the number line and a ruler differ is that rulers are common in the real world of ten- and eleven-year-olds outside the mathematics class both at school and home, whereas the number line as a model for rational numbers is usually only encountered in mathematics lessons. By sixth grade students should have encountered many opportunities to use rulers in and out of school.

Method

Subjects

Sixteen sixth-grade students were selected from 48 sixth-graders who had completed an 84 item Fraction Test that included items testing their ability to associate fractions with area, number line, and ruler models, and to identify equivalent fractions. The pattern of responses on the paper-and-pencil Fraction Test was reported elsewhere (Larson, 1987). The 48 sixth-graders were separated into five quintiles based on their total scores on the Fraction Test. The students who were interviewed in this study were randomly selected from the following quintiles: Quintile 1 (n = 5), Quintile 3 (n = 5) and Quintile 5 (n = 6). All of the students attended the same school which is located in a lower to middle socio-economic area in Tucson, Arizona, and contained many minority students.

Interview

In a clinical interview the 16 sixth-grade students were asked to respond to three types of measurement tasks involving fractions administered in the following order:

Type I Tasks. The interviewer pointed to a mark on an enlarged ruler and asked each student to identify the indicated length from the left edge of the ruler. Lengths indicated were: 4 inches, 3/4 inch, 2 1/4 inch, 3/8 inch, 1 5/8 inches, and 1/2 inch.

Type II Tasks. The interviewer asked each student, "Where would you begin and stop to draw a line *n* inches long?" The following lengths were presented orally and in writing: 3 inches, 1/4 inch, 5/8 inch, 3 1/2 inches, 2 7/8 inches and 15/8 inches.

Type III Tasks. The students were asked to measure strips of cardboard and to write the lengths on a record sheet. The strips measured were the following lengths: 2 inches, 3/8 inches, 3/4 inches, 3 1/2 inches and 2 5/8 inches.

In all tasks a large replica of a 6-inch ruler scaled to the eighth-inch was used so that the students' responses could be adequately recorded on video for later analysis. All of the students identified a 12-inch wooden ruler as being a "ruler" and except for one student, correctly identified the inch as the basic unit on this ruler. The students were then introduced to the enlarged replica of the ruler. None of the students were bothered by agreeing to call the unit on this ruler one inch.

Results

All of the students were proficient with using a ruler to measure a whole number of inches; only one error was made on these tasks. Fourteen of the students were very consistent with using mixed numerals when measuring inches. All of these students' errors related to the fractional part of the mixed numeral. They demonstrated that they knew that a mixed numeral of the form "w a/b" meant "w" inches and a part of the next inch. So, if they could associate halves, fourths, and/or eighths with appropriate parts of an inch, then they could also respond correctly to mixed numerals that included these fractional parts.

In addition to examining the number of correct responses to each individual task, each student's set of responses to all ruler tasks were analyzed for patterns of knowledge and strategies across tasks. Only one student (#88, Q5) responded correctly to all tasks; she was the only student who consistently related improper fractions to the ruler. One other student (#82, Q5) responded correctly to all measuring tasks except for the ones dealing with improper fractions.

At the other end of the spectrum were two students (#108, Q1; & #103, Q1) who were incorrect on almost all tasks involving fractional parts of an inch, including halves. During the interview Student #103 consistently called eighths of an inch halves. He counted the correct number of eighths in Type I and III Tasks but gave the incorrect name to the part; for example for 6/8 inch he said "six-halves of an inch". When he responded "four-halves" for 1/2 inch, the following discussion took place:

I: "Why do you call each one of these a half?"

S: "Cause sometimes in the class that's what I hear her say, half and half. And when it's in the middle, three inches and a half."

I: "Are these in the middle? (I pointed to various eighths.)

S: "Yes."

The other twelve students had some success with the measurement tasks involving fractional parts of an inch. Eleven of these students correctly responded to the three tasks involving half an inch, the twelfth student (#77, Q5) responded correctly to the two tasks involving 3 1/2 inches but not to Type I Task involving identifying 1/2 inch. In all Type I Tasks this student (#77) responded in terms of eighths which she seemed to associate only with the shortest marks on the ruler. Thus she only counted the shortest marks from the left edge of the ruler to the indicated mark or to the shortest mark that preceded the indicated mark if it was longer (indicated fourths or halves). For example, she responded "3/8 of an inch" when 3/4" was indicated and 1 3/8 inches for 1 5/8 inches. After responding incorrectly to two Type II Tasks she correctly showed 3 1/2 inches and 2 7/8 inches on the ruler, in the latter case correctly counting eighths. She then correctly measured all strips in the third part of the interview.

Of the other eleven students, two students (#110, Q3; & #113, Q5) consistently identified all measurements involving fourths and eighths as a correct number of eighths. They were not able to show 1/4 inch on the ruler. One other student (#105, Q5) was successful with all tasks involving fourths of an inch, but she said "I don't know" or responded incorrectly to any measuring tasks involving eighths of an inch. When asked why she had correctly responded "three-fourths of an inch" to a Type I Task, she said, "Because I just know." The other eight students were consistently correct only when half-inches were involved Each of these students responded correctly to only 0 to 3 of the 10 measuring tasks (excluding tasks involving improper fractions) involving fourths or eighths of an inch. When asked for an explanation these students often said, "I don't know." or "I guessed."

Some of the students' explanations seemed to indicate that they identified a half-inch, a fourth-inch (or quarter-inch), and an eighth-inch as smaller units on the ruler. A number of students explained that the line between two inches was called a half because it was "in the middle". Excerpts from interviews with two students shows their reliance on memory for the names of the fractional parts of an inch. Their statements also seem to indicate that these two students' possess a belief in a model of "learning-by-being-told" which VanLehn (1986) claims is an inaccurate model of learning in mathematics classrooms. The remarks of Student #103 above is another example of reliance on memory for the names of the fractional parts and of a student's report that information came from "telling" by the teacher.

> Example 1: Student #81 (Q3) kept commenting during the interview that he did not know what to call the smallest parts (eighths). Early in the interview he asked: "What are these things called?" On the last task he repeated this question. When asked "Is there anything you can do to find out?" He responded, "Ask Mr. W. (his teacher)". This student had no problem identifying proper fractions and mixed numerals when using an area model in a previous interview.

> Example 2: Student #107 (Q3) in trying to identify 3/4 of an inch at the beginning of the interview said, "Couldn't be a half (pointed to a half)...couldn't be a quarter (pointed to a quarter inch)." When asked if there was a way she could figure it out, she replied, "Beats me, I was never told what each line meant." At the end of the interview she said, "I don't know what the small ones are." A possible way for her to find out was "look it up in a dictionary."

Three students' responses to the Type II Task of showing where to stop on the ruler to draw a line 2 7/8 inches long seem to be Van Hiele Level 0 type responses based on a global location (Hoffer, 1983). This was the only task involving eighths to which Student #107 responded correctly; her only explanation was, "My head says yeah, yeah, yeah." Two students (#91, Q1; & #95, Q1), who did not correctly respond to any tasks involving eighths, pointed to the mark associated with 2 3/4 inches. Student #91's explanation was, "Because I have a feeling I would (stop here)." and Student #95 said: "Because seven-eighths is almost a whole inch." This holistic approach to this task seems to be based more on memory associating a specific fraction with a specific location, than with applying a rational number concept. A number of students' correct identification of "two and a quarter inches" on the Type I Task at the beginning of the interview also seemed to be a question of recognizing "a quarter".

Conclusions

Most of the sixth-grade students in this study could not consistently use fractions to measure fourths and/or eighths of an inch on a replica of a 6-inch ruler scaled to the eighth-inch. Students tried to remember the names of the parts rather than use their more general fraction knowledge to figure out the names of fractional parts. Students seldom counted the number of parts in an inch to determine the denominator. They usually either remembered the name of the part or they didn't. Most counting activities were related to the numerator of the fraction.

One error commonly made by students when representing proper fractions on number lines of length greater than one was to disregard the unit and treat the total length of the given number line as the unit (Larson, 1980; Bright et al., 1987). This error was never made by the sixth-graders in this study when associating fractions with marks on the ruler. They consistently identified the inch as being the unit. With a few exceptions, incorrect responses for proper fractions were identified as being between zero and one; and those for mixed numerals of the form "w a/b" were identified as between "w" and "w + 1".

All of the sixth-graders in this study understood the measurement concept of unit iteration and most of them understood the concept of repeated partitioning of the unit to form secondary units (Ohlsson, 1988). They demonstrated the later by treating a half-inch, a fourth-inch, and an eighth-inch as secondary units with arbitrary names that they either recalled or didn't. The equivalent partitions of the inch were not addressed by most of these students by applying their rational number construct. If they had done so then they could have counted the number of eighths in an inch and responded to appropriate tasks in terms of measuring to the eighth-inch. Only four of the students consistently did this, yet 13 of the students in a previous interview could represent proper fractions and mixed numerals with an area model. Most of these students were operating within a measurement construct which they failed to integrate with their rational number construct.

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Poster Title: Using writing to enhance preservice elementary teachers' understanding of mathematical concepts and procedures.

Presenter: Thomas J. Bassarear

Institution: Keene State College, Keene, New Hampshire

The presentation will begin by articulating how writing activities connect with the overall goal of having the students internally construct mathematical knowledge. Connections will also be made to the recent N.C.T.M. Curriculum and Evaluation Standards and the N.C.T.M. Professional Teaching Standards.

The following writing activities will then be presented: (1) thought process protocols to increase students' awareness of their thinking processes as they solve problems and how those processes can help and hinder their ability to solve problems, (2) in-class writing activities to develop understanding of both concepts and procedures, (3) out-of-class writing activities to enable students to learn from their mistakes, (4) assignments to enable students to examine the development of their problem solving abilities, and (5) writing activities at the end of the class which can be used both to present different students' articulation of mathematical concepts and procedures and as feedback for the teacher.

Each of the writing activities discussed will include: (1) an articulation of the purpose of the activity, (2) a description of how it is used in the classroom, (3) directions given to students, (4) examples of students' actual work, and (5) selected students' comments (positive and critical) with respect to the usefulness of these activities.

A bibliography of related articles and books will be included.

Poster Title: Self-Perceptions of Mathematics Learners Derived from Client-Centered Interviews

Presenter: Anne R. Teppo

Institution: Mathematics Education Consultant

The constructivist perception of the relationship between teacher and learner bears similarities with client-centered counseling developed by Carl Rogers, which views the therapist as a facilitator who enables clients to become aware of their beliefs and feelings and take responsibility for personal change. Roger's philosophy can be used to add a psychological dimension to the teaching/learning interaction.

Client-centered techniques were used effectively in extended interviews to identify students' perceptions of themselves as mathematics learners. This information was gathered as part of a qualitative research study of the classroom performance of non-mathematically oriented college students enrolled in a course dealing with the language and structure of mathematics.

The interviewer exhibited the three personal characteristics (identified by Rogers) of genuineness, unconditional positive regard, and accurate empathetic understanding to establish an effective relationship with the students that enabled them to become aware of and articulate their personal feelings about themselves in a mathematics classroom. Analysis of the interviews enabled the researcher to identify components of an effective teaching/learning environment. As an added benefit, students increased their awareness of and reflected on their actions within the mathematics classroom.

Poster Title: The Rutgers-New Brunswick Mathematics Project: An Examination of Children's Mathematical Thinking

Presenter: Roberta Schorr, Sally Weisman, Tom Purdy

Institution: Rutgers University

The Rutgers University-New Brunswick Schools Mathematics Project is a cooperative program between Rutgers University and the public and parochial schools of New Brunswick, New Jersey. It is designed to improve the state of mathematics education in the city. This program has, for the last three years, brought together teachers from the New Brunswick Schools who are committed to a perspective about learning and teaching mathematics that pays close attention to individual children's mathematical thinking.

Project faculty and staff work directly with teachers to design lessons that are built around problem-solving activities, thus allowing children to explore particular mathematical ideas. These activities differ from what is typically found in textbooks in that they allow the child to incorporate a variety of mathematical ideas in a problem-solving setting. The children often work in small groups to construct solutions to problems. Explanation, justification, and comparison of solutions by the children is encouraged.

This poster session is intended to highlight some examples of children's work which exemplify the types of problems which we feel encourage their mathematical thinking. These data will form the basis of a much larger study of children's mathematical thinking which is part of a pending National Science Foundation research proposal.

Poster Title: Learning about Functions: Students and Teachers using Contextual Problems and Multi-Representational Software

Presenter: Erick Smith, Jere Confrey, Karoline Afamasaga-Fuata'i, Susan Piliero, Jan Rizzuti, Mette Vedelsby

Institution: ~~Cornell University, Ithaca, NY~~

This poster presentation is intended to complement the paper presentation, "A Framework for Functions: Prototypes, Multiple Representations, and Transformations" by Jere Confrey and Erick Smith. In that paper we refer to our observations of students working on contextual problems using the multi-representational software tool Function Probe©. This presentation will consist of two parts: First we will illustrate the use of Function Probe, particularly in relation to the use of transformations and multiple representations (table, graph, calculator, algebra) which were described in the paper. Because of the importance of actually using the program to develop an understanding of the dynamic nature of these features, a computer with Function Probe will be available. We will have a few sample problem situations to encourage others to experiment with the program. Second, we will offer specific examples of student work which illustrate both how students have used the program and how we have learned from our students about the ways that functional relationships can be created, modified, transformed, and understood. These examples will be drawn from the work of all the members of the research group.

The use of multiple representations, contextual problems, and results from observing students play a central role in our research. The design of this presentation is intended to foster discussion with other researchers about the importance of these approaches and their relevance to effective mathematics teaching.

Video Presentation Title: The Rutgers-New Brunswick Mathematics Project: Children with Partners

Presenter: Sally Weisman, Roberta Schorr, Tom Purdy

Institution: Rutgers University

The Rutgers University-New Brunswick Schools Mathematics Project is a cooperative program between Rutgers University and the public and parochial schools of New Brunswick, New Jersey. It is designed to improve the state of mathematics education in the city. This program has, for the last three years, brought together teachers from the New Brunswick Schools who are committed to a perspective about learning and teaching mathematics that pays close attention to individual children's mathematical thinking.

An important component of this multi-faceted project for New Brunswick children involved in the project and in grades 3-7 is the involvement of their parents or other adults in Saturday morning workshops. These sessions provide an opportunity for the children to be part of a one-on-one learning experience in mathematics with a parent/partner. Experiences gained in these sessions are intended to carry over into the child's school life as well as the every day life of the child and his/her partner.

Videotaped excerpts that illustrate interesting examples of children and their partners working on mathematical problems are currently being gathered. Some of these excerpts will be presented at this session.

Video Presentation Title:

Lived-In Simulations: A New Genre of Learning Environment?

Presenter: James J. Kaput

Institution: Southeastern Massachusetts University

Traditional quantitative computer-based simulations have taken the form of manipulable models sharing some features of the situations that they are attempting to represent. One's actions on these models typically take the form of choosing and/or setting some quantitative parameters, and then running the model. This is followed by observation of the process and/or the result and then perhaps some analysis. This may then be followed by another round of the same procedures.

We depict in this video an alternative form of simulation that involves continuous, differential feedback. Here the user "lives in" the simulation and modifies its essential parameters continuously, with continuous real-time feedback as the system responds and opportunity to react to differences between intended and actual system behavior. The simulation in this case is entitled "MathCars" - a simulated driving environment, with a crucial addition: time, position and velocity data are collected in graphical form in real time as the driving occurs. There is no delay between user action and system reaction. Here the student has the opportunity to "experience" the quantitative relationships reflected in, for example, the Fundamental Theorem of Calculus. On the other hand, any mathematical representation such as a velocity graph generated via student "driving" can subsequently be treated as a mathematical object in its own right, to be analyzed, modified, etc. We suggest that this genre of simulation offers the chance to build mathematical ideas in a new way, one that is much more closely related to the types of "natural" learning that occur outside school contexts.

Changing Ways of Thinking About Mathematics by Teaching Game Theory

Nin-Ya Gura

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One of the main purposes of teaching mathematics in the schools is to contribute to the enrichment of the mathematical world view of the students. In order to help them to sense the spirit of mathematics, an effort must be made to introduce students to as many kinds of mathematics as possible. It may be done by means of new curricula and new approaches to instruction. In Israel, a mathematics curriculum for high-school upper grades composed of a combination of compulsory courses and 90 hours of elective studies was approved in 1975. The change in curriculum structure gave rise to the idea of creating an elective in game theory. Game theory both satisfies the criteria of the elective mathematics curriculum and exemplifies a branch of the discipline which may contribute to a change in attitudes and approaches to mathematics.

A course in game theory was created such that it is constructed of four topics dissimilar in character and bearing little mathematical relation to each other. The four topics were elected on the basis of their being of special interest beyond their mathematical content, not demanding specific prerequisite knowledge in mathematics and providing general knowledge about game theory and its concerns.

The research was conducted in three different types of classrooms in which I taught the course in game theory. The purpose was to investigate whether it is at all possible to teach game theory at high-school or equivalent level. In addition, there was an attempt to determine the contribution, if any, of this specific course in game theory to the mathematical world-view of the students as well as to their attitudes and approaches towards mathematics in general and to game theory in particular.

As a consequence of the course, the number of students exhibiting an open-minded attitude towards mathematics increased. Students discovered that the world of mathematics is much richer than they had previously thought.

Video Presentation Title: 'Accounting for' versus 'an account of'

Presenter: David Pimm and John Mason

Institution: The Open University, Milton Keynes.

Increasingly, videotape records are being made both as research tools and as in-service education materials in mathematics education. As with any new technology, videotapes are not 'transparent' with respect to what they purport to display, but need to be worked on explicitly with particular techniques in order to highlight stressings and ignorings as well as revealing interpretive projections that any viewer necessarily brings to bear.

We propose to illustrate one framework for working on videotape highlighting a distinction between 'giving an account of' and 'accounting for' what is seen. In the process we hope to exemplify how we work with anyone (teachers or researchers) on videotape of mathematics classrooms.