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AUTHOR Connell, Michael L.
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ABSTRACT

Findings from prior research are drawn together to create a learning model for elementary school mathematics in the cognitive-constructivist tradition. A potential teaching/learning process consistent with the model was developed and applied in a longitudinal collaborative arrangement between university personnel and a local elementary school using a conceptually based curriculum that posed problems requiring active student involvement with physical materials to model mathematical situations, defined symbols, and developed solution strategies. As children used these materials, they actively construed the operations and principles of arithmetic. In another phase children sketched the materials and situations in a move toward abstraction. They then constructed mental images through imagining actions on physical materials. Experiences with the mental images allowed for student construction of arithmetic generalizations and problem solving skills. The computer served as another tool for constructing methods of dealing with problems. The conceptual frame of cognitive constructivism appears to provide for an awareness that understanding in elementary mathematics must involve the active search for, creation of, and use of links between the abstractions and generalizations and the world of personal experiences. Eleven figures illustrate the model and the discussion. There is a 38-item list of references. (SLD)

How do they know?
An investigation into student mathematical conceptions and beliefs.

By

Michael L. Connell
University of Utah

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ABSTRACT

Sternberg (1984a, 1984b) defines consequential knowledge as deciding what information is important to learn and then incorporating that information into an already existing knowledge base. Yet, in looking at the elementary educational experiences of students in mathematics Davis (1975), Erlwanger (1973), and more recently Peck, Jencks, and Connell (1989) identified the focus as being upon memorizing facts and rules and not in making sense of the subject. This has come to have a profound impact upon the sense-making efforts of children, their perceptions of what mathematics is, in what their roles should be in learning mathematics, and what their teachers roles should be in teaching mathematics.

Peck, Jencks and Connell (1985) suggest these difficulties in elementary mathematics originate in a rote teaching methodology where students use procedures in isolation, sidestepping the development of a referent base. This results in problems being viewed by the student as always having unique, specific answers which are wholly determined not by the logic of the problem but by the answer book, a neighbor, or a teacher. Peck and Connell (1991) suggest that even when clearly identifiable student conceptual change occurs, it has limited effect due to interference from previously acquired mental structures. Newly acquired information appeared to serve in a superordinate capacity with previously learned procedures or concepts being automatically applied - bugs and all.

This paper draws together these and other findings from prior research in an effort to create a learning model designed in the cognitive-constructivist tradition. This model is then developed, a potential teaching/learning process consistent with the model is developed, and an application example showing the impact of this process upon classroom student is provided.

INTRODUCTION

Ceci and McNellis' (April, 1987) suggest that knowledge, beliefs, and cognitive processing are inseparably linked. They develop symbiotically in very much a chicken-and-egg fashion. Cognitive processes such as encoding, memory-recall, inferencing, and problem solving require a knowledge base upon which to operate. This knowledge base, in turn, develops through the operation of cognitive processes which are directly affected by meta-cognitive considerations such as belief, idealization of task, and perception of progress. Any look at human mathematical abilities should reflect the dynamic nature of such a system. Before looking at mathematics in particular, let us look at knowledge in general and how we might claim to understand.

Although cognitive and meta-cognitive science is a new field, we may turn to a much older tradition for help in our initial efforts. For over 2,000 years philosophers have examined the nature of knowledge in the branch of philosophy known as epistimology. Traditionally epistimologists have viewed knowledge as consisting of justified, true beliefs. Logical arguments then designed to focus attention upon one or more of these elements. Should we choose to follow this lead and think of knowledge as consisting of such justified, true, beliefs then issues concerning the nature of what is to be justified, what is meant by truth, and how we believe play increasingly important roles.

The first step in this regard is to distinguish the knower from that which is to be known. There are at least two basic attributes to any learning experience: that which is to be learned, generally external to the individual; and the learner. Lest we get lost in meta-physics, Feibleman (1976) offers a useful approach to use in examining this goal with metaphysical attributes approached not as abstract structures, but rather as they would appear to an actual knower.

Let us look at a representative attempt to gain or verify knowledge within a given field in the external world from this perspective. Through sensory processes portions of the external world are experienced together with concurrently discernable attributes. Should we attend to the experiences invoked by a field, related experiences are subsequently retrieved together with remembered events and previously successful schemas. These form a network of relations within which to utilize these perceptions. For convenience, this process may be divided into two parts. The first, an unconscious awareness of the immediate individual experience within the field; the second, a growing consciousness and cogitation concerning the observed events and how this experience relates to the observations. Approaching this from the standard definition of knowledge it can be argued that, to the extent knowledge requires belief, one must be aware of an experience before it can be believed and thus known. This has severe implications for a view of knowledge, however. By the time an experience crosses the awareness threshold, Minsky (1986), it has been colored by myriad unconscious processes.

A further complication lies in multiple recursive uses of sensory data over time for differing purposes, a characteristic which seems to be shared by human physiology as well as psychology (Kolb & Whishaw, 1985; Bloom, Lazerson, & Hofstadter, 1985). These findings lead one to question whether a single belief or belief system could offer sufficient grounds for justification. This skepticism is strengthened when one investigates the extent which perceived knowledge is a function of expertise and the mapping of this perceived knowledge into real world experiences. There is a clear trend in the literature indicating that experts organize their efforts differently than novices. Chi (1981, 1985), Schoenfeld (1983) and others have pointed out that there are clear differences concerning what self-reports declare field knowledge consists of when one speaks with novices and experts within a field. A large portion of this difference is due to the presence of links and multiple instantiations of field specific data through application which comes with the development of expertise. As expertise in an area is acquired the nature of the links becomes more complex. Yet, research seems to indicate that the experiences and supporting concepts forming the basis of evidence differ as a function of expertise and sophistication in field knowledge.

What counts as knowledge is clearly contextualized in this case. As an example of this, O'Brien (1974) describes children's thinking as being *atomistic* in nature. By this he means that they have the view that the things, events, and ideas or experience are unrelated to one another. Knowledge becomes of a network of experiences interacting with current goals and sensory experiences. These experiences, irregardless of coherence, compete with one another for belief and justification, with justification itself a function of the field and the individuals perceived progress within the domain.

DEVELOPMENT OF A LEARNING MODEL

How might this epistemic information help us in developing a model of learning? First, it provides a background within which to view varied cognitive and meta-cognitive findings. Secondly, it gives us a broad picture of the nature of interactions which a learning model should possess.

The first level of the model (Figure 1) separates the individual learner from that which is to be known. This separates the physical and metaphysical attributes of the external world from the internal cognitive and meta-cognitive attributes of the individual learner.

INSERT FIGURE ONE ABOUT HERE

Taking a cue from Case's ongoing research with the evolving role of Short Term Storage Space (STSS) (Case, Kurland, Daneman, and Goldberg, 1982; Case, 1984), Osborne and Wittrock's Generative Learning Model (1983), and Davis' discussion of workbench memory partitioning (1984). The next development in the model (Figure 2) divides human cognition along an interesting dichotomy: tremendous storage capability with severely limited attentional resources. Long term memory is viewed as containing different types of memory units such as images, propositions, sensory impressions, intellectual skills, and rules for action. Working memory functions along a frame retrieval model and is partitioned into sections, as in the Davis and Case model, with the partitioning subject to change with development.

INSERT FIGURE TWO ABOUT HERE

It is important to acknowledge that no matter how experience the learner might possess, there will always be more to be known than that which is already known. (This can be shown from the following argument: the real world, if it exists at all, will always contain more than the individual since it contains the individual as a subset.)

As was suggested earlier, we gain knowledge of the external world through our senses - either directly or as aided by devices such as rulers, microscopes, telescopes, cyclotrons, etc., which in turn are percieved by our senses. To the extent that our these devices and our senses accurately reflect the real world we may gain accurate knowledge.

INSERT FIGURE THREE ABOUT HERE

All of our sensory information is not available for our use, however. In their 1980 text Nisbett and Ross identify several additional fundemental screens which are often imposed by individuals. We may attend to some areas of the real world while ignoring others. Our thoughts may stray to other past events. We may stop paying attention to additional evidence once a tentative theory has been reached.

We often tend to forget the degree to which our cognitive acts are governed by our expectations and our beliefs. To help recapture the feeling of power which accompanies an intuitive belief consider the following example from Hewitt (1983).

> ..extend your left hand upward full length, and your right hand halfway between your left hand and your eyes. Judge the relative sizes of your hands. Aren't they about the same size? What happened to the inverse square law? The image of the closer hand covers four times the area on your retina, yet your belief that your hands are the same size is so strong that your mind shows them to be about equally sized. You can check this if you look with one eye and compare the relative sizes of your hands against a reference in the background. pp. 308.

Lester and Garofalo (1985, 1987) have postulated that an individuals failure to solve a problem when the individual possesses the necessary knowledge; where knowledge refers to both formal and informal mathematical knowledge, knowledge of heuristics, and knowledge of contextual information; stems from the presence of non-cognitive and metacognitive factors that inhibit the appropriate utilization of one or more pieces of necessary knowledge. The factors defined by Lester and Garofalo include: *affects, beliefs, control,* and *socio-cultural conditions.*

Of particular interest from a mathematical perspective Jencks, Peck, and Chatterley (1980), Peck, Jencks, and Connell (1985), Peck, Jencks, and Connell (1989), and Peck and Jencks (1979) describe commonly held student beliefs. Most notably students had no meaningful referents for the symbols and rules they were using. In addition, they were convinced that their teachers had taught them their (incorrect) methods.

In short, in addition to the limitations imposed by our sensory bandwidths, our view of reality is filtered by past experiences, percieved successes or failures, habits of attention, and the actions we may take.

INSERT FIGURE FOUR ABOUT HERE

A fundamental assumption is that the brain is not a passive consumer of information (Kolb and Whishaw, 1985). Rather, the brain actively constructs meanings and uses these meaning to justify further inferences. This is done through an interaction of stored memories, the perceived task, and the incoming sensory information, while attending to some information and selectively ignoring other data sources Figure 4). The stored memories and information processing strategies of the brain interact with the sensory information received from the environment to actively select and attend to the information and to actively construct meaning. Cobb (April, 1987) goes as far as to describe knowledge as being based upon knowledge-in-action. This type of knowledge construction is active, often finding the meaning in the activity itself. These findings are reflected in McCloskey (1983) where it is likewise suggested that the mind of today's student is not empty. It is a jungle of ideas about nature.

INSERT FIGURE FIVE ABOUT HERE

The next two figures finish the development of the model. The processing of Long Term memory is partitioned into two types of activities: a mostly passive (automatic) storage operation and a more active (subject to conscious control) retrieval operation (Figure 6), possible linkages and interactions among the components are sketched in and the circle is completed in (Figure 7) with actions of the individual, based upon ongoing constructions of meaning, effecting the real world and in turn effecting future efforts at understanding.

INSERT FIGURE SIX ABOUT HERE

INSERT FIGURE SEVEN ABOUT HERE

A feel for the operation of the model can be gathered form this example, one experiences a real world situation leading to the construction of a problematic, this leads to the retrieval and execution of a procedure; the execution of the procedure yields a modified visual input, which leads to the retrieval and execution of the next segment of procedures and so on. With experience multiple sequences become developed into holistic entities which can be contemplated without the necessity to go through in a step wise fashion as evidenced by the well-documented chunking phenomena. In this model, in order for understanding to take place the learner must be an active participant in constructing meaning. To fully comprehend, each individual must invent a model, an assimilation paradigm if you will, that organizes the information selected from the experience in a manner that fits our unique experiences and perceptions of the situation.

An implication of this model is that one does not come to a full understanding from any single experience. Noddings (1986) observes that proponents of various cognitive processing models get their problem spaces and representations from the finished solutions and then seek a reasonable approach toward its reconstruction. A stymied thinker is not allowed this option, however. They must build up a space that contains noise and junk before they can select items for representation. Ideas develop and may be described at intermediate levels of development. Even in individual problems there may be a need to try out ideas and cluster them before deciding on an algorithmic solution.

Full understanding comes after selective attention to that experience, attention which is influenced and directed by previous experiences and habits of thinking. This selective attention results in selective perception in which the events we experience are viewed from within a preexistent mental framework which influences the sensory information available. To construct meaning from this sensory information, it is necessary to generate links to and among what are perceived to be relevant aspects of information in Long Term Memory.

IMPLICATIONS FOR INSTRUCTION

An immediate implication of this model of learning is the need for problems leading later mathematical abstractions to initially come from the real world experience of the child and to be firmly anchored in actual experience. These experiences must lead to the creation of a commonly defined problem space within which the problem exists Mayer (1983). For a problem to accomplish this within this framework, it is important that the solution not be obviousand relatively open ended. When multiple 'right' answers are present it requires a re-examination and evaluation of the solution process to verify each result. Problems are only effective when they are at an appropriate level for the child. Should problems be given which are too hard or too easy there can be either no growth, or trivial growth. Finally, a good problem will have a tendency to generate other problems.

To facilitate this goal the problems which must be developed should reveal the central concerns of curriculum through the usage of ordinary elements familiar to the child. This allows problem solving episodes to be made less artificial and more easily mappable into an internal structure in the mind of the child (Case, 1984). Gains from a motivational standpoint are also made by relating problem solving to the natural curiosity of the child concerning the immediate world and tying in with the ongoing experience of the child. Growth in this learning model consists of internalizing events into a storage system, or conceptual structure, that corresponds to the real world. In the problem situations designed to accomplish this goal it is important that we attempt to provide experience in aspects of logical thought. To be successful we must establish in the child's mind the problem situation and a individually useful representation of the desired end state.

Manipulatives are carefully selected to serve as tools to internalize the concepts and ideas of the real world. It is apparent that these manipulative referents play a pivotal role in the conceptual development of the child. Because of the prime role these materials play great care must be taken to provide referents lending themselves to as many different structuring techniques and problem solving applications as possible. The more elementary a course and the younger its students, the more care which must be used in this selection.

One method of internalizing the experiences and creation of abstractions and linkages among abstractions proceeds from initial use of manipulative items through four transitional problem types. These problem types are closely interlinked and are designed to aid in internalizing the problem situation reflected in the real world into an internal structure for use by the student. In their useage these problem types roughly parallel, and support, the development from manipulation of real world objects to abstraction.

In the presentation of these problem types it is helpful to observe two trends which occur as children gather experience in problem solving. The first trend is that they become more nearly exhaustive in their processing of information presented in the problem, and consider all or almost all of the information presented (Sternberg, 1984a, 1984b). The second trend is that they spend relatively more time in planning how to go about solving a problem, and less time in actually solving it (Chi and Glaser, 1985). This suggest that in skilled problem solvers more time is spent in higher order structural processing, and less time on lower order processing. In order for this increase in processing efficiency to be accomplished, however, the lower order structures need to be firmly in place.

Consider one approach toward enabling such student construction of meaning in which four problem types involving the use of manipulatives, sketches, visual imagery, and abstraction are presented in three phases requiring memory/recall, instructor posed problems, and self posed problems.

The inital phase consists of committing to memory the symbolism of the referent and assorted terms with which it may be labeled. In a very real sense we are providing a common 'language' which can be used by both students and teachers to talk about problem situations at this point. In terminology, every effort should be made to keep terms to a minimum with essential terms in the natural language of the child. It is of prime importance that the language be clearly presented, defined, and understood. It is equally important that the child is comfortable with the symbolism being suggested. When initially presented at the physical object level it is often possible to tie in sketches to reinforce terminology. In general, the earlier such a tie in to a recording scheme can be made, the more successful will be the approach.

Once the teacher is sure that the basic terminology and symbolism is clear to the students the second phase of instructor pose problems is entered. Provided the students have beer properly prepared, the instructor should now try to pose problems which relate to the referent provided and lead to internalization of the concepts presented in the problem situation. There is great peril, as well as great potential, for the teacher in this phase of problem solving. Teachers often tend to provide too much guidance and instruction in their presentation of problems to the students.

If a great deal of explanation is required prior to problem solving, perhaps the referent selected is not apprepriate. When the referent has been chosen properly the teacher can easily suggest problems to the students. These problems have the added virtue of being able to be solved by the student's useage of the referent itself. In these cases, the referent itself becomes the gauge of correctness of the child's work. The teacher must still correct the student, it is true; but only in a manner that will enable the learner to assume ownership of correctness. This ownership is assumed by the student's reliance upon and use of the structures created through use of the referent. If the teacher, a peer, the textbook, or any other source becomes the source of correctness the purpose of the entire approach is defeated (Peck, Jencks, & Connell, 1985).

It is unfortunate that in many classrooms the instruction cycle is complete with the teacher's presentation of sample problem types. If we are to be successful in teaching problem solving, we must allow the students to pose problems. Bruner puts this very well when he states

> A body of knowledge, enshrined in a university faculty and embodied in a series of authoritative volumes, is the result of much intellectual activity. To instruct someone in these disciplines is not a matter of getting him to commit results to mind. Rather, it is to teach him to participate in the process that makes possible the establishment of knowledge. (1968).

In this model of instruction we would try to allow for this by the formal inclusion of the third phase which allows the students the opportunity to use the developing referent to pose and investigate problems of their own. This is an extremely important that this be allowed, as it is at this point the children develop the essential linkages which later serve to tie their data into useful problem solving structures. It is during these independent investigations that we can best promote the development of self accounting. This self accounting then enables the student to progress beyond adaptive behaviour to the conscious application of logic and reasoning (Campione, Brown, and Connell, 1989). Furthermore, it is in independent investigation that the child begins to develop a sense of ownership over their problem solving strategies. This ownership leads to the establishment of self-rewarding sequences, as previously mentioned, and becomes an incentive towards further learning. When a student finds that he is capable of posing and solving problems this becomes a reinforcement for further problem solving attempts in the future.

These three phases can occur in a single instructional period. In a workshop held at the University of Utah in 1979, Robert Wirtz reported that:

> At a single setting children can move from one cognitive level to another -- from remembering experiences, to solving problems, to making independent investigations. (Wirtz)

In internalizing the problems from the real world we would apply these three phases as we progress through a series of four distinct problem types. These begin with usage of the manipulative referent itself and proceed to the abstraction which we hope to develop. These four basic problem types will be referred to as manipulatives, sketches, mental pictures, and abstraction.

Manipulative will be used here in its broadest senseand refer to any physical construct using materials familier to the child. This gives us a great deal of latitude in our discussion. More importantly it allows us to include materials in our instructional model that would otherwise fall outside of our classification scheme. When we discuss manipulatives and their significance it is important to realize that in this model the true power of a manipulative lies in the structures which can be built upon it, the linkages it enables in the mind of the student, and its power in explaining concepts. Within this framework the merit of a manipulative lies in its power for simplifying information, for generating new propositions, and for increasing the manipulability of a body of knowledge. It has been observed by many sources that those manipulatives that possess great structuring power tend to be economical and to have application in many varied settings apart from those for which they were originally constructed.

It is very important in the selection of manipulatives that these criterion are met. It seems to be very easy for many teachers to fall into the trap of using physical objects for their own sake, without considering their pedagogical effectiveness. Not only is this ineffective in building concepts for the students, but actually causes blockages to occur should similar manipulatives later be used in an appropriate manner.

In creation of manipulatives it is important to remember that many problems relevent to children have their origins in the real world about us. The symbolism, which can the reality base of a problem from its formal presentation, is adopted as a result of formal attempts to solve those problems. These formal efforts often result in an alogorithm which is then used in attempts to generalize those problems. This process of generalization is indeed a worthy goal, but often tends to divorce the concept being utilized from the symbolism used to record the process. If we are to use manipulatives successfully we must look beyond the symbolic representations of process presented by our textbooks into the underlying physical world problem. When this is done we may construct our manipulatives to reflect this underlying problem.

The next stage in the presentation consists of problems utilizing sketches of the underlying manipulative. For sketches to be effective in our model they must follow the form of the original manipulative as closely as possible. The mapping from manipulative to sketch, then sketch to mental picture, and later to abstraction must be carried out as smoothly as possible. By maximizing the the amount of commonality between these forms and holding the amount of divergent information to a minimum it is possible to ease this transition. If we select an appropriate manipulative, the subsequent sketch will draw much of its descriptive power from the underlying manipulative.

This serves to re-emphasize the care with which manipulative must be selected. One way of insuring that mapping from manipulative to sketch will occur naturally is to tie the presence of a recording scheme reflecting the real world nature of the manipulative, in sketch form, at the earliest levels of manipulative problem solving.

In continuing this process of internalizing the real world into the mind of the student we attempt to develop a mental construct corresponding to that of the sketch. Based upon current research mental pictures developed in the course of problem solving efforts share many of the properties of sketches, pictures, and diagrams. The power of the mental image can be considerable. In a quote attributed to Albert Einstein it is said that he arrived at the theory of relativity by "visualizing.. effects, consequences, and possibilities" through "more or less clear images which can be 'voluntarily' reproduced and combined." (Cooper and Shepard, 1984).

These characteristic of mental pictures enables the transfer of information contained in the developed sketch into mental imagery to take place with comparative ease. It is important that we not let this aspect of mental pictures blind, however. We must remember that, despite there many aspects of correlation, mental pictures are not pictures or sketches and proper care must be exercised to assure that we do not confuse familiarity with a sketch with possession of the underlying mental picture.

In the classroom setting there is no necessary reason why a mental image would have to share any of the properties of the preceding sketches. It is quite possible for the student to develop a working mental image of the concept underlying the problem situation having absolutely nothing in common with the sketches presented in the course of class work. In my experience this has proven to be a very rare occurrence, however. What is more often found is the child's initial mental pictures, as described in use of the placement test, are nearly identical with previously derived sketches of the problem situation.

This proves to be highly beneficial from an educational standpoint. It is often possible to stimulate the creation of mental pictures by selective manipulations of the sketch being worked with. One technique which seems very productive consists of covering up sections of the sketch. When questioned about the problem situation the child will seem to mentally reconstruct the hidden information in the sketch from his/her mental picture. By this process the mental picture is not only utilized in a problem solving setting, but strengthened for future use.

At this time there are many conflicting theories concerning the mechanisms behind the creation and utilization of mental imagery (Cooper and Shepard, 1984; Gardner, 1983; Jencks and Peck, 1973). Each of the different theories seem to agree, however, in that whatever is going on in the brain when we have an image, it produces a representation that has certain useful functional properties in structuring and organizing information. These are the properties which we attempt to utilize in our work with the students.

The final step lies in the mental structuring of the real world into an abstract structure. For any given problem set once abstracted the student is in full control of the concepts underlying the problem situation. The sequence of internalizing the real world problem into understood processes of solution has been completed. This structure can then be used in future problems, and as a stepping stone towards independent investigations.

One clue to gauge when the abstraction level is reached is that the child does not need the actual physical referent to be present, yet can utilize data that only familiarization with the manipulative could give. Mental pictures will come to replace many of the simpler sketches, with the number of sketches required per problem being reduced dramatically. What sketches are made reflect more complex variations of the problem situation. At this point the child seems to have full access to previous forms of problem solving techniques, yet does not require them to solve the problem.

If we are successful in following the steps outlined in this teaching approach the student will possess not just a single answer schema, but an entire structural linkage which can be utilized by the student in varied circumstances. The student has developed a sound conceptual building block which can be used in later, more complex, endeavors in problem solving.

AN APPLICATION EXAMPLE

In an attempt to address concerns such as those outlined in the introduction a longitudinal collaborative research arrangement was made between university personnel and a local elementary school. In this project a significantly different perspective was taken as regards to the curriculum, the instructional focus, and the evaluation methods.

Curriculum focus. The curriculum used in this project was conceptually based and utilized the approach outlined in the earlier section. Rather than using manipulatives to demonstrate procedures or rules, problems were posed which required active student involvement with physical materials to model mathematical situations, define symbols, and develop solution strategies via actions with the materials. As the children used these physical materials to solve problems, they actively constructed the operations and principles of arithmetic. The third phase required sketches of the physical materials and situations experienced by the students to encourage a move toward abstraction. The sketches then served as the basis for additional problems and as tools for thinking. In the fourth phase, the children constructed mental images through imagining actions on physical materials. The experiences with mental images allowed for students construction of strong arithmetic generalizations and problem solving skills.

The computer in this project was just another "tool" available to the students in their ongoing efforts to construct meaningful methods of dealing with the problems they encountered. The nature of this "tool", which was provided for the students to "think-with", came to shape their performance and cognitive styles. When a computer was available for the students use the problem solving situation shifted toward the identification and selection of what data to include in the problem, identification of the problem goals, and choice of appropriate procedures and control statements to obtain and verify the desired results. As a consequence of the instructional sequence outlined above the children constructed a series of related mathematical concepts. When these concepts and applications were overlearned the students instructed a MacIntosh via Hypertalk to carry out the necessary instructions and operations which they had derived (Peck, 1989).

It must be noted that although the computer played a pivotal role in this project, it is a much different role than that usually associated with CAI. For rather than using the computer for it's incredible speed, the computer's infinite patience and need for exactness of logic and clarity of expression was utilized. Such use of the computer allowed the individual student to use a variety of techniques and representations to share developed knowledge and expertise effectively. The computer assumed the role of an active listener that would do exactly what it was told, as opposed to a pre-programmed instructor requiring a specific type of answer.

Throughout the project, a major goal of the curriculum was to enable the successive internalization and abstraction of the preliminary physical experiences the children shared. Each of the outlined phases was viewed as a step along the path toward eventual mathematical abstraction. For example, the sketches drew much of their power from earlier experiences with objects. In a similar fashion, the mental images reflected the sketches and manipulations performed by the students. The interrelated nature of these experiences set the stage for abstractions and the intuitive foundation upon which the abstractions could safely rest. These abstractions, rather than being based upon a single demonstration of rules, rested upon a tightly woven network of understandings.

Instructional focus. An explicit instructional objective was to help each child find a way to answer the question, "How can you tell for yourself?" for all portions of the mathematics they were learning. The instructors shared the common belief that children must be allowed to figure things out and be responsible to themselves, not a teacher or answer key, for their results. It was felt that if children are to engage in thinking about and solving problems for themselves, then they must have a "place" to go in order to be able to determine if they are making sense. Physical objects in this instructional model served this purpose. These beliefs, coupled with the earlier described curriculum focus, led to the following principles:

1. The instructor did not explain. The instructor served as a problem poser, skeptic and question asker focussing upon student explanations.
2. Manipulations with physical materials defined meanings which were associated with arithmetic symbols and operations. Problems were developed requiring an appeal to those objects and meanings.

3. The instructor attempted to enable the children to internalize and abstract their experiences by requiring them to work problems in the absence of the physical materials.

4. The instructor used a meaning-centered evaluation scheme (Peck, Jencks, & Connell, 1989).

The following illustrates of the use of these principles with the fifth grade class. Fraction symbols were defined from physical materials in two steps. First, a meaning for denominator was developed by asking the children to take some objects and share them between two people including themselves as one of the two. A bar was drawn over the symbol "2" and defined to mean "share (fairly) with two". Once this meaning was clear, the instructor began posing problems. For instance, an egg carton was used as a model with the following "share" instructions:

INSERT FIGURE EIGHT ABOUT HERE

The instructions "share with two and one half", "share with eight", "share with five", and "share with thirteen" posed problem situations which required the children to expand their understandings through active involvement with the physical materials. The question of how to share with two and one half required some extended experimentation and discussion. The students finally agreed that they could think of it as sharing with two older children and a small child, where the small child would get exactly half as much as a "big-child" share. Figure 9 shows a few examples of how children solved the problem of "share with eight".

INSERT FIGURE NINE ABOUT HERE

Such activities as illustrated in Figure 9 and "share with five", or "share with thirteen", etc., helped the children overlearn meanings in a problem-centered environment as opposed to overlearning manipulations of symbols in an abstract setting. The children learned that there are many ways of solving a problem and were encouraged to use sketches provided they could justify their thinking and approaches.

The meaning of the fraction symbol was completed by writing a numeral above the bar and suggesting it meant to do something with that many shares. For example, 3/4 could mean to "share with four" and "color in three of the shares" as is illustrated in Figure 10:

INSERT FIGURE TEN ABOUT HERE

This completed the development of meaning for the fraction symbol. Again complicated problems were posed as needed to insure as broad a base for fractions as possible, to motivate a constant attention to meanings, and to foster a willingness to work with unfamiliar problems.

Students were frequently asked to visualize appropriate objects when working examples. The students in this group universally selected a "cake" model for dealing with fractions because they seemed to sense it's general applicability. This visualization helped the children form a "mental image" which enabled them to generalize algorithmic procedures. For instance, these fifth grade students developed and used the cross multiplication rule as a computational convenience for comparing fractions, as shown in Figure 11:

INSERT FIGURE ELEVEN ABOUT HERE

The strategies and perceptions the children developed in this effort transferred to "story problems' and other "real world" situations. The meanings these children had mastered for fractions allowed them to address a variety of problems without discussing in advance a precise method for doing them. These students developed the following characteristics during the course of this work:

1. The children had meanings for the symbols that guided their thinking.
2. The students were active as opposed to passive in their attempts to learn.

3. The students developed rules as conveniences, not as binding procedures.

4. The students had confidence in their own thinking and could decide whether they were making sense.

5. The students were able to readily make interpretations and work toward solving unfamiliar problems.

SUMMARY

This research has implications for helping educators address some of the burning issues facing mathematics education. The conceptual frame of cognitive constructivism appears to provide the means for a continuing and deepening awareness that understanding is more than an iterative procedure done without meaning. The author is reminded that many of the fractals, commonly described in terms of chaos theory, are generated in just this fashion.... an iterative procedure, done thousands of times without meaning in and of itself - leading to chaos (Gleick, 1987). Understanding in elementary mathematics must involve the active search for, creation of, and use of links between the powerful abstractions and generalizations of mathematics and the world of personal experiences from which they derive their application and utility. Cognitive constructivism provides a valuable set of perceptual lenses through which to look at the problems and potentials of learning in mathematics.

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FIGURES

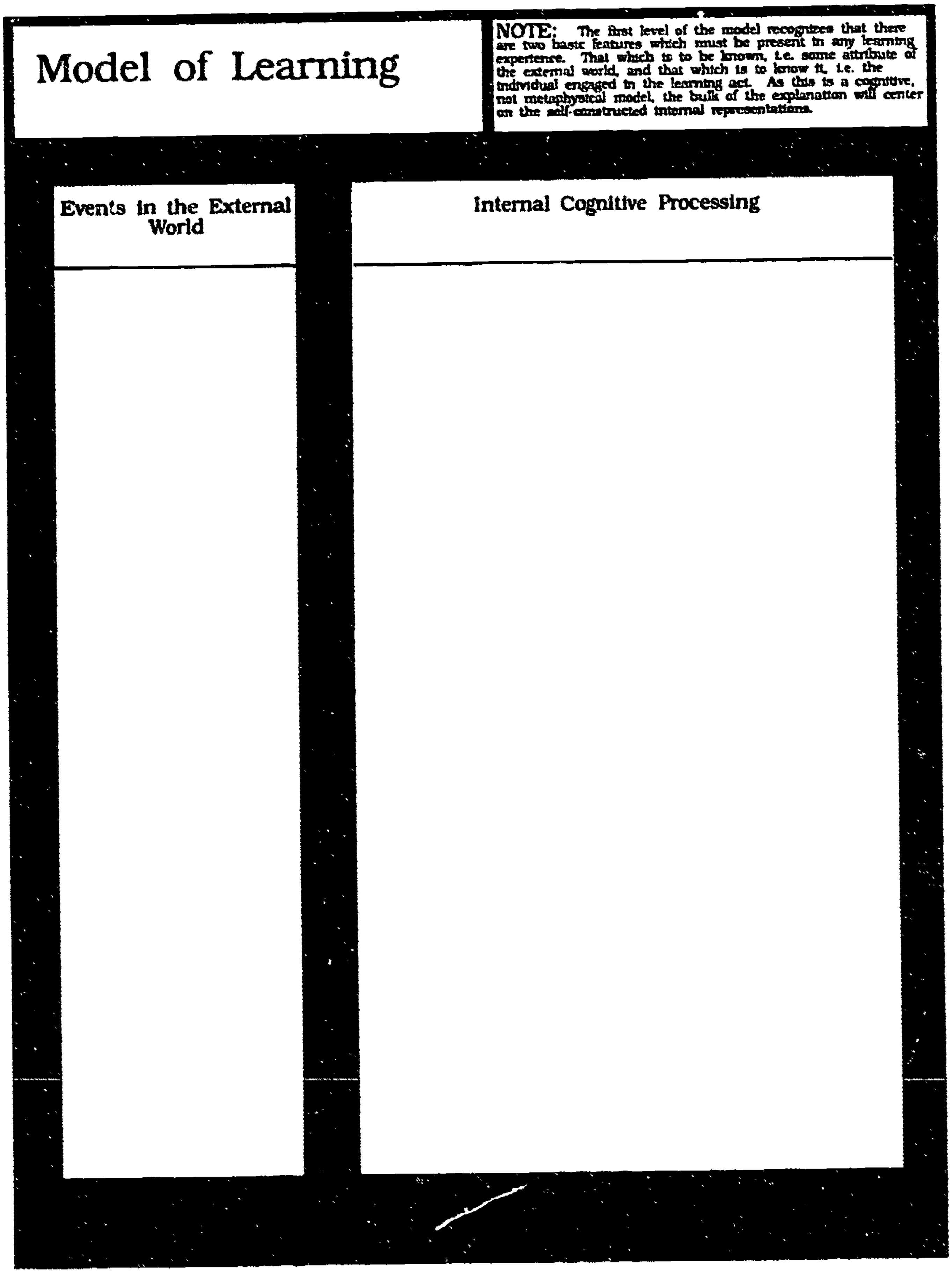

Model of Learning
NOTE: The first level of the model recognizes that there are two basic features which must be present in any learning experience. That which is to be known, i.e. some attribute of the external world, and that which is to know it, i.e. the individual engaged in the learning act. As this is a cognitive, not metaphysical model, the bulk of the explanation will center on the self-constructed internal representations.
Events in the External World
Internal Cognitive Processing

Figure 1.

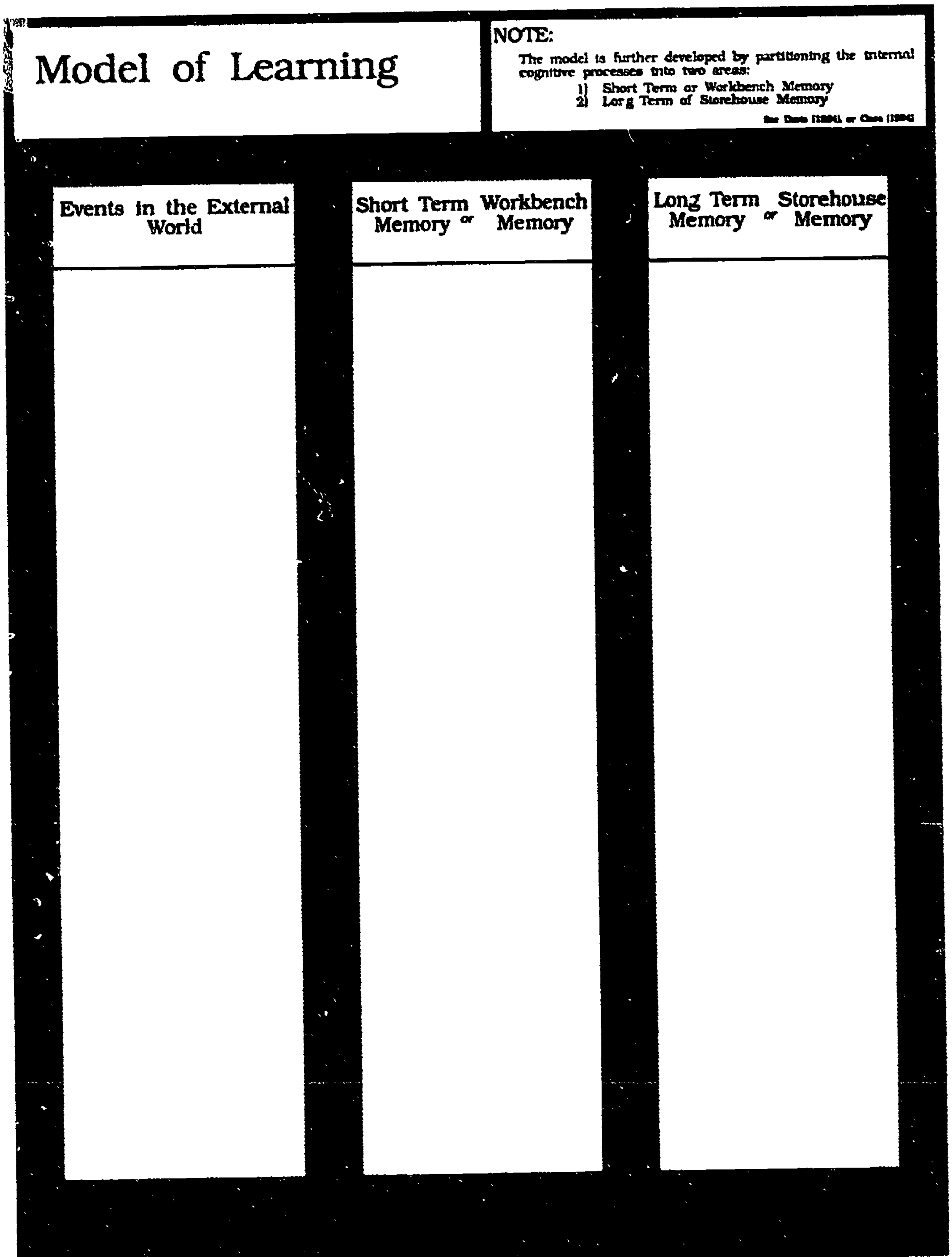
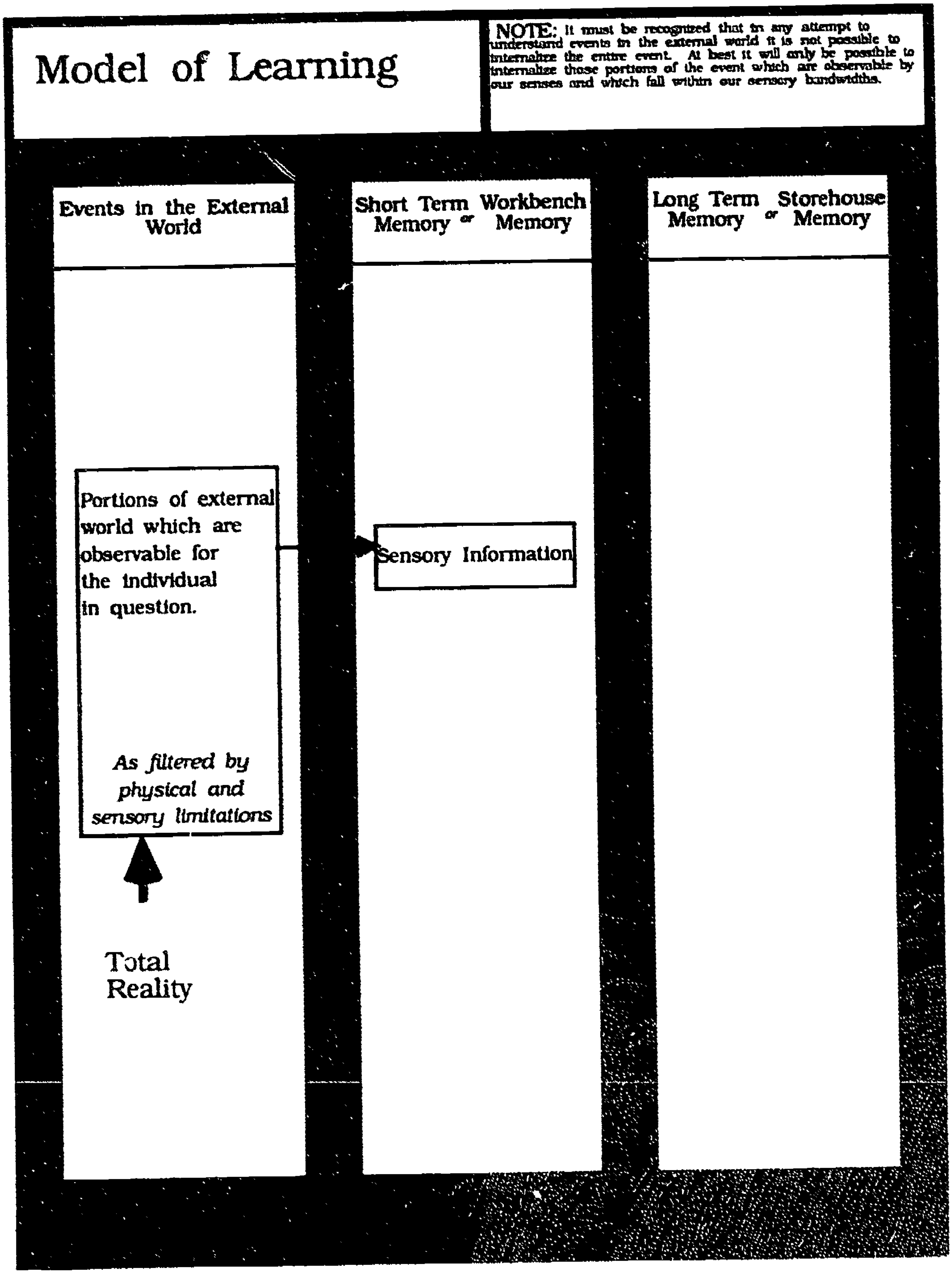

Model of Learning
NOTE:
The model is further developed by partitioning the internal cognitive processes into two areas:
1) Short Term or Workbench Memory
2) Lorg Term of Storehouse Memory
Events in the External World
Short Term Memory or Workbench Memory
Long Term Memory or Storehouse Memory

Figure 2.



Model of Learning
NOTE: It must be recognized that in any attempt to understand events in the external world it is not possible to internalize the entire event. At best it will only be possible to internalize those portions of the event which are observable by our senses and which fall within our sensory bandwidths.
Events in the External World
Short Term Memory or Workbench Memory
Long Term Memory or Storehouse Memory
Portions of external world which are observable for the individual in question.
As filtered by physical and sensory limitations
Sensory Information
Total Reality

Figure 3.

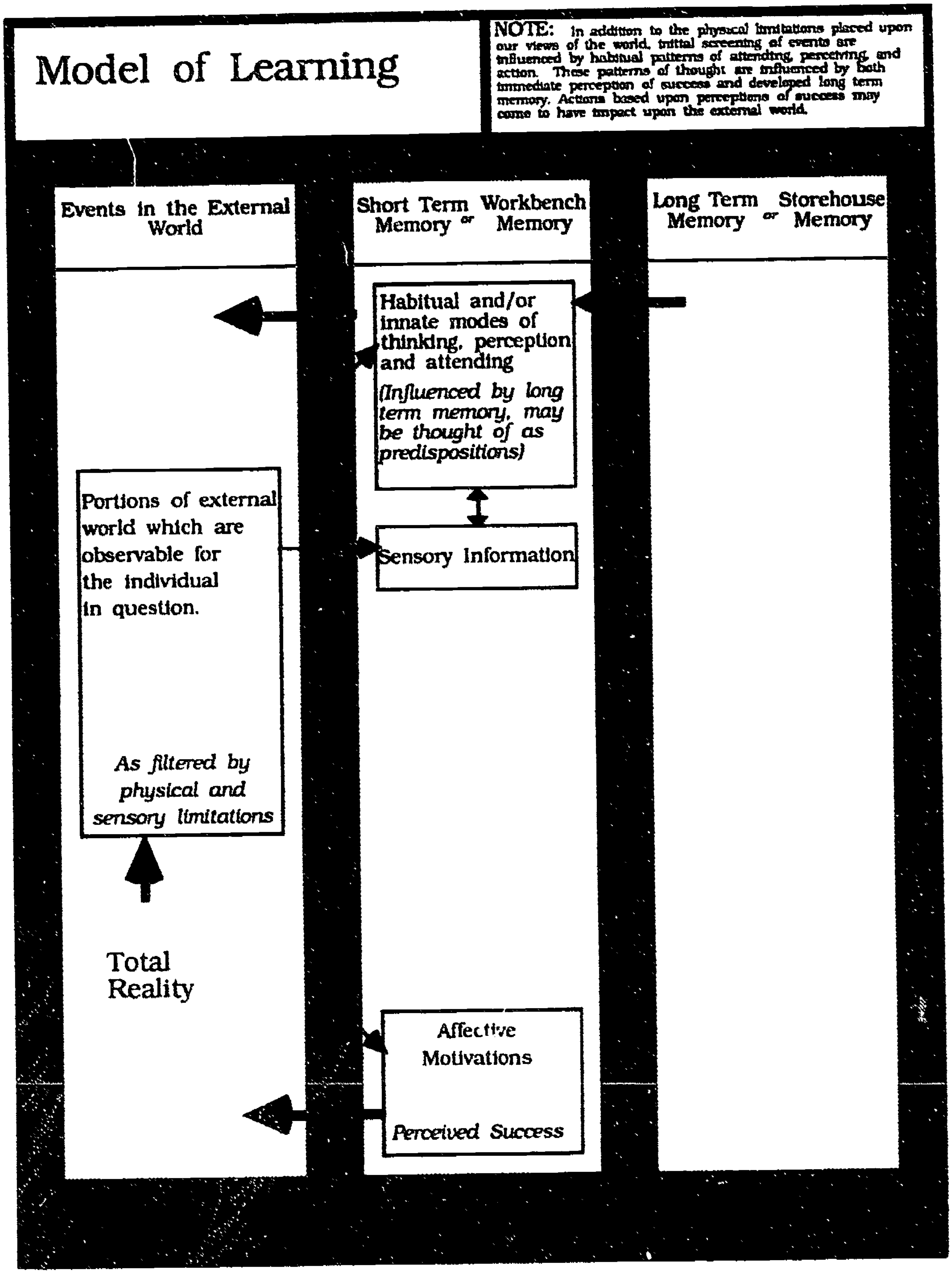

Model of Learning
NOTE: In addition to the physical limitations placed upon our views of the world, initial screening of events are influenced by habitual patterns of attending, perceiving, and action. These patterns of thought are influenced by both immediate perception of success and developed long term memory. Actions based upon perceptions of success may come to have impact upon the external world.
Events in the External World
Short Term Memory or Workbench Memory
Long Term Memory or Storehouse Memory
Habitual and/or innate modes of thinking, perception and attending
(Influenced by long term memory, may be thought of as predispositions)
Portions of external world which are observable for the individual in question.
As filtered by physical and sensory limitations
Sensory Information
Total Reality
Affective Motivations
Perceived Success

Figure 4.

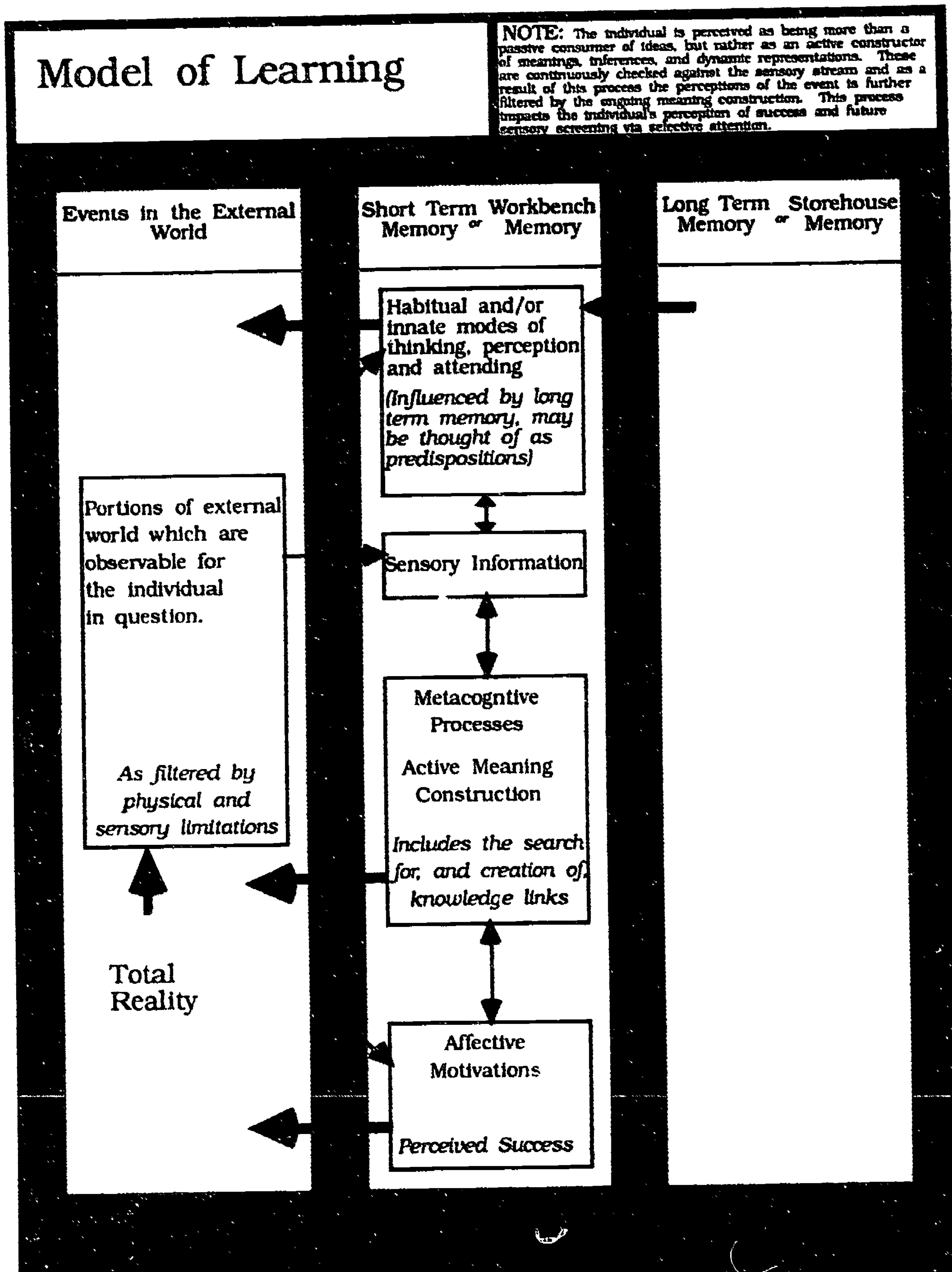

Model of Learning
NOTE: The individual is perceived as being more than a passive consumer of ideas, but rather as an active constructor of meanings, inferences, and dynamic representations. These are continuously checked against the sensory stream and as a result of this process the perceptions of the event is further filtered by the ongoing meaning construction. This process impacts the individual's perception of success and future sensory screening via selective attention.
Events in the External World
Short Term Memory or Workbench Memory
Long Term Memory or Storehouse Memory
Habitual and/or innate modes of thinking, perception and attending
(Influenced by long term memory, may be thought of as predispositions)
Portions of external world which are observable for the individual in question.
As filtered by physical and sensory limitations
Sensory Information
Metacogntive Processes
Active Meaning Construction
Includes the search for, and creation of knowledge links
Total Reality
Affective Motivations
Perceived Success

Figure 5.

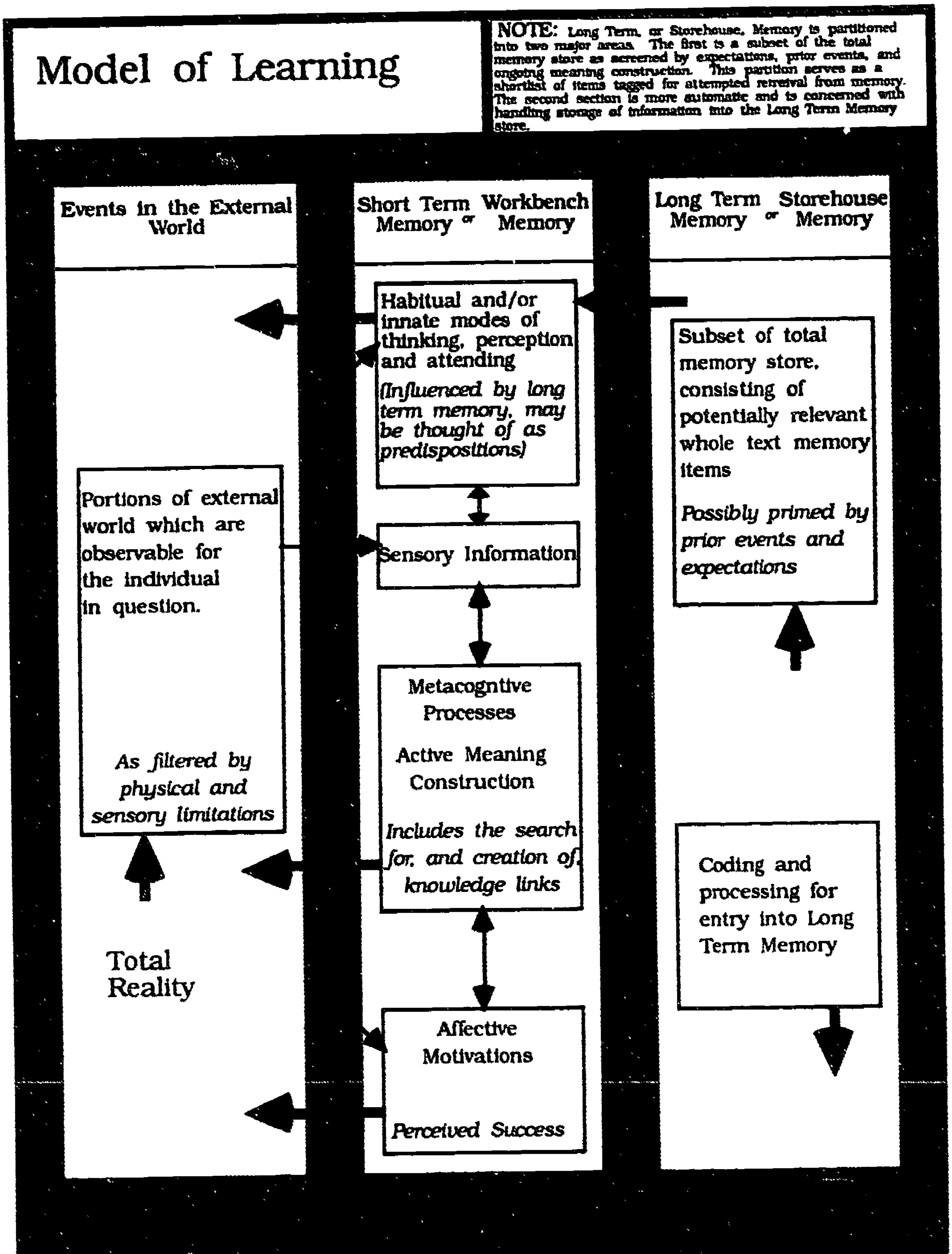

Model of Learning
NOTE: Long Term, or Storehouse, Memory is partitioned into two major areas. The first is a subset of the total memory store as screened by expectations, prior events, and ongoing meaning construction. This partition serves as a shortlist of items tagged for attempted retrieval from memory. The second section is more automatic and is concerned with handling storage of information into the Long Term Memory store.
Events in the External World
Short Term Memory or Workbench Memory
Long Term Memory or Storehouse Memory
Habitual and/or innate modes of thinking, perception and attending
(Influenced by long term memory, may be thought of as predispositions)
Subset of total memory store, consisting of potentially relevant whole text memory items
Possibly primed by prior events and expectations
Portions of external world which are observable for the individual in question.
As filtered by physical and sensory limitations
Sensory Information
Metacognitive Processes
Active Meaning Construction
Includes the search for, and creation of knowledge links
Coding and processing for entry into Long Term Memory
Total Reality
Affective Motivations
Perceived Success

Figure 6.

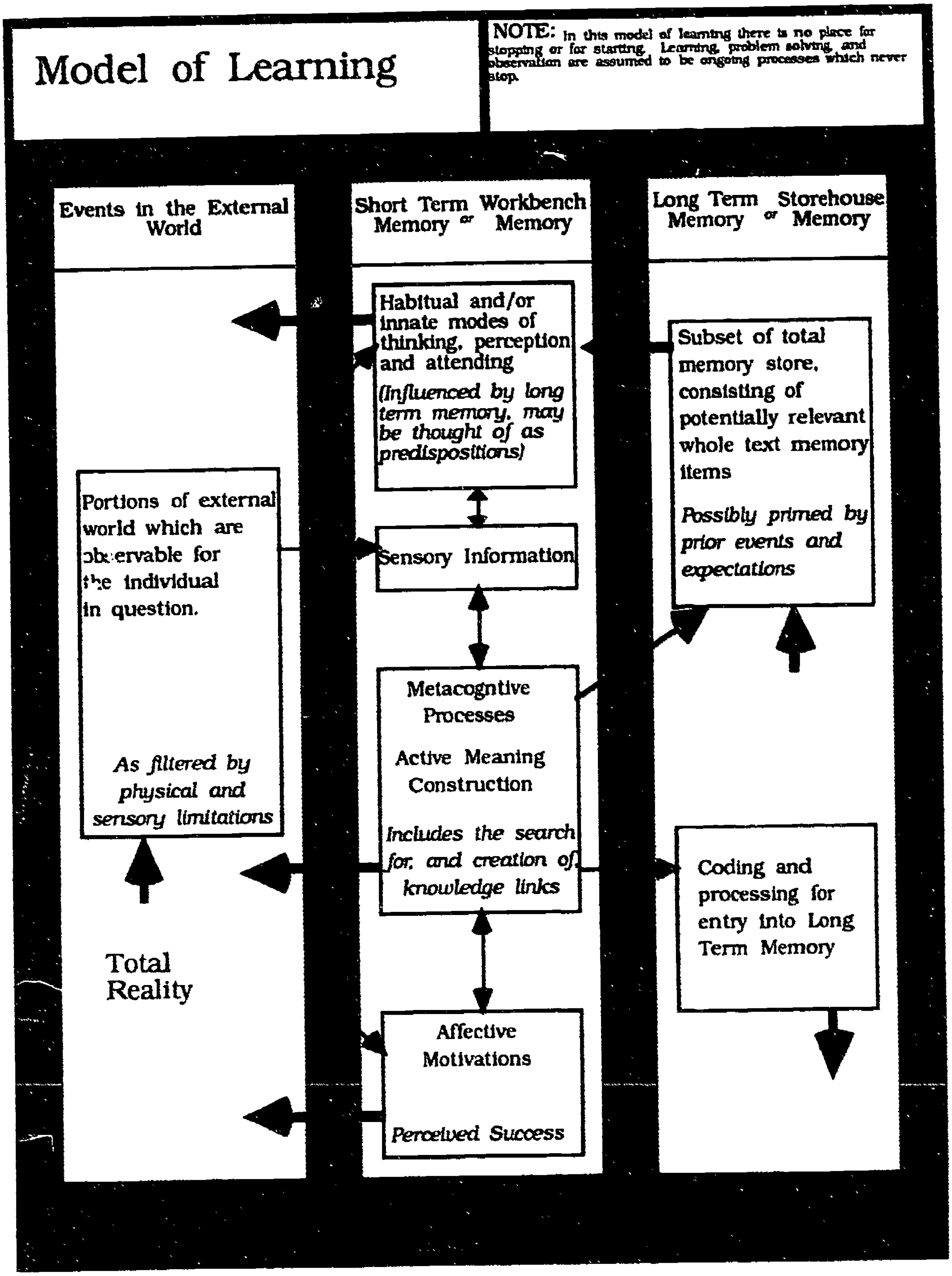

Model of Learning
NOTE: In this model of learning there is no place for stopping or for starting. Learning, problem solving, and observation are assumed to be ongoing processes which never stop.
Events in the External World
Short Term Memory or Workbench Memory
Long Term Memory or Storehouse Memory
Habitual and/or innate modes of thinking, perception and attending
(Influenced by long term memory, may be thought of as predispositions)
Subset of total memory store, consisting of potentially relevant whole text memory items
Possibly primed by prior events and expectations
Portions of external world which are observable for the individual in question.
As filtered by physical and sensory limitations
Sensory Information
Metacogntive Processes
Active Meaning Construction
Includes the search for, and creation of knowledge links
Coding and processing for entry into Long Term Memory
Total Reality
Affective Motivations
Perceived Success

Figure 7.

4	$2\frac{1}{2}$	6	3	8	12	5	13	1

Figure 8. Introduction of "share with"

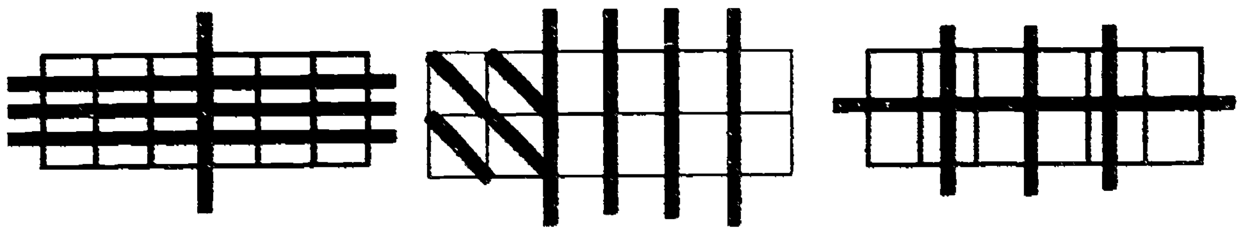


Figure 9. Variations on "Sharing among eight"

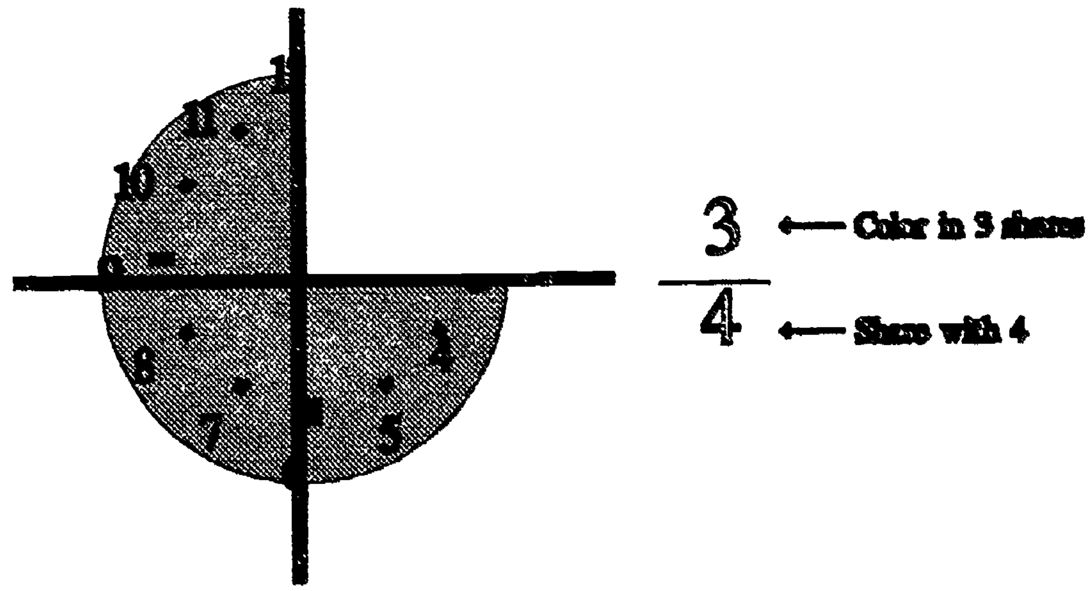


Figure 10. Establishing meaning for fraction symbols.

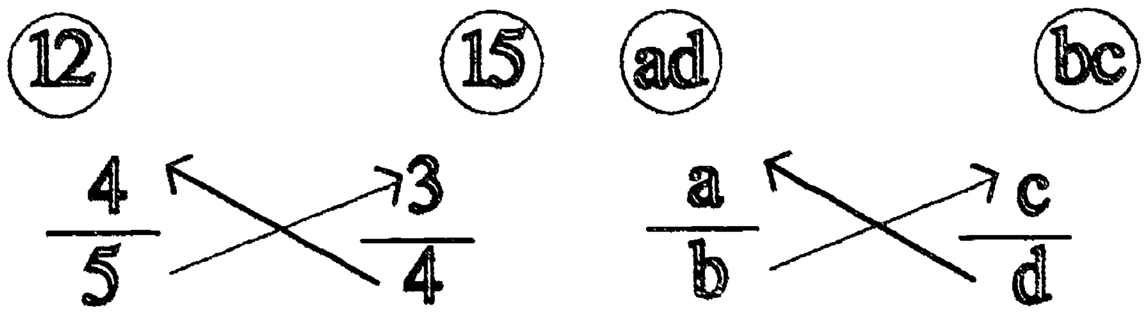


Figure 11. Spontaneous student development of cross multiplication for fraction comparison.