DOCUMENT RESUME

ED 342 646 SE 052 500

AUTHOR Resnick, Lauren B.; And Others
TITLE Thinking in Arithmetic.
INSTITUTION Pittsburgh Univ., Pa. Learning Research and Development Center.
PUB DATE 90
NOTE 26p.; In: Means, Barbara, Ed., Knapp, Michael S., Ed. Teaching Advanced Skills to Educationally Disadvantaged Students. Final Report. p135-157, March 1991.
PUB TYPE Viewpoints (Opinion/Position Papers, Essays, etc.) (120)

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS *Arithmetic; Classroom Environment; *Educationally Disadvantaged; Grade 1; Grade 2; Instructional Design; Instructional Innovation, Intuition; Mathematical Logic; Mathematics Achievement; Mathematics Education; *Mathematics Instruction; Mathematics Skills; Models; Primary Education; *Problem Solving; *Teaching Methods; *Thinking Skills
IDENTIFIERS Learning Research and Development Center; Mathematical Thinking; Problem Posing

ABSTRACT

This paper discusses a radically different set of assumptions to improve educational outcomes for disadvantaged students. It is argued that disadvantaged children, when exposed to carefully organized thinking-oriented instruction, can acquire the traditional basic skills in the process of reasoning and solving problems. The paper is presented in three sections. The first discusses the intuitive basis for early mathematical reasoning, describing the reasoning about amounts and sizes of material that preschool children do without measurement or exact numerical quantification as "protoquantitive" reasoning. The integration of protoquantitative schema and counting, the first step in making quantitative judgements, is discussed. The second section discusses six principles for a reason-based arithmetic program: (1) develop children's trust in their own knowledge; (2) draw children's informal knowledge, developed outside school, into the classroom; (3) use formal notations as a public record of discussions and conclusions; (4) introduce key mathematical structures as quickly as possible; (5) encourage everyday problem finding; and (6) talk about mathematics, don't just do arithmetic. The final section presents the results of the program. Two cohorts of first- and second-graders were evaluated by the California Achievement Test (CAT) on reading and mathematics achievement with an experimental group and a control group at each level. The intervention group in mathematics rose from about the 25th percentile to the 70th percentile and maintained that level into the second year of the program. The paper concludes that an interpretation- and discussion-oriented mathematics program is an effective instructional approach, suitable for children not socially favored, and provides mathematics classroom activities that exercise reasoning skills. (MDH)

Thinking in Arithmetic Class

Lauren B. Resnick
Victoria Bill
Sharon Lesgold
Mary Leer

Learning Research and Development Center
University of Pittsburgh
Pittsburgh, PA 15260

Center for the Study of Learning

LEARNING RESEARCH AND DEVELOPMENT CENTER

University of Pittsburgh

Thinking in Arithmetic Class

Lauren B. Resnick
Victoria Bill
Sharon Lesgold
Mary Leer

Learning Research and Development Center
University of Pittsburgh
Pittsburgh, PA 15260

THINKING IN ARITHMETIC CLASS

For many years now, most efforts to improve educational outcomes for disadvantaged students have been based on the premise that what such children need is higher expectations for learning coupled with intensified and careful application of traditional classroom methods. Thus, what is typically prescribed is more careful explanations, more practice, and more frequent testing to monitor progress. Such methods seem to work—up to a point. That is, they produce gains on basic skills tests. But they are not designed to teach children to reason and solve problems. Today, such abilities are fundamental for participation in the economy and society in general.

The nearly exclusive focus on the kinds of "basic skills" that can be taught by repetitive drill does not necessarily derive from a lack of ambition for disadvantaged students or from a belief that the children are inherently incapable of thinking and problem solving. Rather, it is rooted in an assumption that most educators share about all learning by nearly all children (some would except the "gifted"): that successful learning means working step by step through a hierarchical sequence of skills and concepts. The common view is that skills and concepts are ordered in rather strict hierarchies, and that asking children to perform complex skills before they master the prerequisite, simpler ones is to doom them to failure, or at least to frustration, in the course of learning. This hierarchical mastery learning approach dictates that children who have trouble learning some of the simpler skills practice them longer. But in practice this turns out to deny disadvantaged children the opportunity to learn higher-order abilities. Because many disadvantaged are among those who learn slowly at the outset, they are doomed to more and more supervised practice on the "basics." They never get to graduate to the more demanding and interesting problems that constitute the "higher-order" part of the curriculum.

The work we describe in this paper is premised on a radically different set of assumptions. We argue that disadvantaged children, like all children, can begin their educational life by engaging in active thinking and problem solving. We argue further that, when thinking-oriented instruction is carefully organized for this purpose, children can acquire the traditional basic skills in the process of reasoning and solving problems. As a result, they can acquire not only the fundamentals of a discipline, but also the ability to apply those fundamentals, and—critically—a belief in their own capacities as learners and thinkers.

Reviewing research and practical efforts to teach higher-order thinking skills a few years ago, Resnick (1937) concluded that shaping a disposition to critical thought is as important in developing higher-order cognitive abilities in students as is teaching particular skills of reasoning and thinking. Acquiring such dispositions, she proposed, requires regular participation in activities that exercise reasoning skills within social environments that value thinking and judgment and that communicate to children a sense of their own competence in reasoning and thinking. This, in turn, calls for educational programs suffused with thinking and reasoning, programs in which basic subject matter instruction serves as the daily occasion for exercising and extending cognitive abilities. Explicit attention to thinking and reasoning seems particularly important for children who are not experiencing daily practice in such reasoning in their homes or who do not trust their own out-of-school experience as being relevant to school success. Such children often fail to learn the "hidden curriculum" of thinking and reasoning that more favored children acquire without much explicit help from teachers.

We report here on the early results of an effort to apply these ideas to early mathematics teaching for disadvantaged children. To embed basic mathematics learning in a thinking curriculum, we had to design a new set of practices for the mathematics classroom. We wanted to create an environment in which children would practice mathematics as a field in which there are open questions and arguments, in which interpretation, reasoning, and debate—all key components of critical thought—play a legitimate and expected role. To do this, we needed to revise mathematics teaching in the direction of treating mathematics as if it were an ill-structured discipline. That is, we needed to take seriously, with and for young learners, the propositions that mathematical statements can have more than one interpretation, that interpretation is the responsibility of every individual using mathematical expressions, and that argument and debate about interpretations and their implications are a normal part of mathematical activity. Participating in such an environment, we thought, would develop capabilities and dispositions for finding relationships among mathematical ideas and between mathematical statements and problem situations. It would develop skill not only in applying mathematics but also in thinking mathematically. In short, it would socialize children into a developmentally appropriate form of the practice of mathematics as a mode of thought, reasoning, and problem solving.

This goal, however, seemed at first to pose an insurmountable problem for school beginners—especially, perhaps, those we label *disadvantaged*. To engage in the kind of mathematical discussions we were aiming for, children would have to know *some* mathematics at the outset. They would need something to think *about* if the exercise was not to be an empty one. A first question, then, was whether children entering school knew enough about numbers and quantities to permit a reasoning- and discussion-oriented program from the outset. Fortunately, a large body of research accumulated over the past decade suggests that almost all children come to school with a substantial body of knowledge about quantity relations and that children are capable of using this knowledge as a foundation for understanding numbers and arithmetic (see Resnick, 1989; Resnick & Greeno, 1990, for interpretive reviews). This knowledge, we thought, could provide the initial foundations for children's participation in a reasoning-based mathematics program.

The Intuitive Basis for Early Mathematical Reasoning

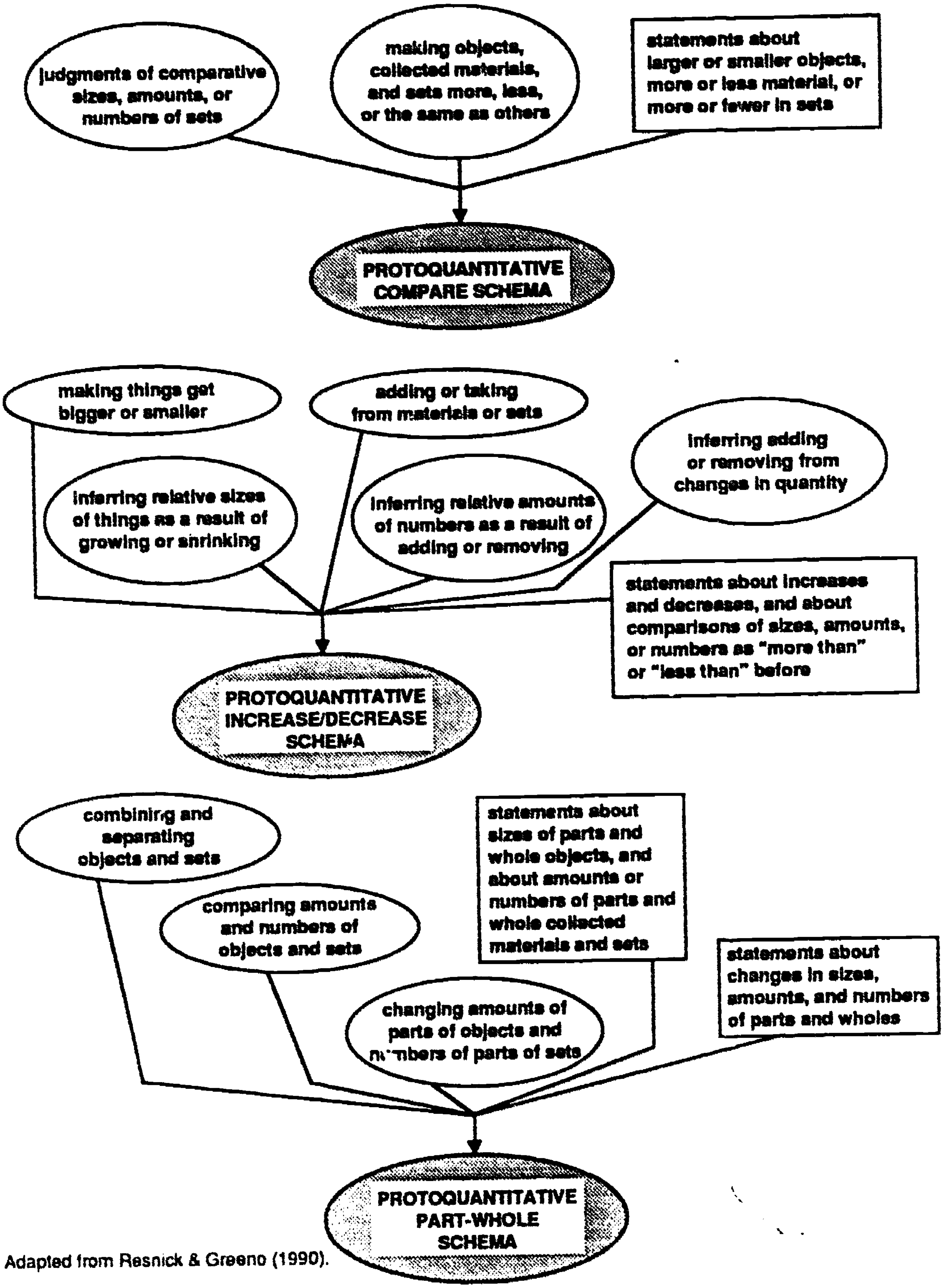
Children come to school with two kinds of intuitively developed knowledge relevant to mathematics learning. First, they know a good deal about amounts of physical material and the relations among these amounts, even though they cannot yet use numbers to describe these relations. Second, most children know the rules for counting sets of objects. This gives them the beginning tool for using numbers to manipulate and describe quantity relations.

Protoquantitative Schemas

During the preschool years, children develop a large store of knowledge about how quantities of physical material behave in the world. This knowledge, acquired from manipulating and talking about physical material, allows children to compare amounts and sizes and to reason about changes in amounts and quantities. Because this early reasoning about amount of material is done without measurement or exact numerical quantification, we refer to it as *proto*quantitative reasoning. We can document development during the preschool years of three sets of protoquantitative schemas: *compare*, *increase/decrease*, and *part-whole* (see Figure 1).

The *protoquantitative compare* schema makes greater-smaller comparative judgments of amounts of material. Before they are two years old, children express quantity judgments in the form of absolute size labels such as *big*, *small*, *lots*, and *little*. Only a little later, they begin to put linguistic labels on the comparisons of sizes they made as infants. Thus, they can look at two circles and declare one bigger than the other, see two trees and declare one taller than the other, examine two glasses of milk and declare that one contains more than the other. These comparisons initially are based on direct perceptual judgments without any measurement process. However, they form a basis for eventual numerical comparisons of quantity.

The *protoquantitative increase/decrease* schema interprets changes as increases or decreases in quantities. This schema allows children as young as three or four years of age to reason about the effects of adding or taking away an amount from a starting amount. Children know, for example, that if they have a certain amount of something and they get another amount of the same thing (perhaps mother adds another cookie to the two already on the child's plate), they have more than before. Or, if some of the original quantity is taken away, they have less than before. Equally important, children know that if nothing has been added or taken away, they have the same amount as before. For example, children show surprise and label as "magic" any change in the number of objects on a plate that occurs out of their sight (Gelman, 1972). This shows that children have the underpinnings of number conservation well before they can pass the standard Piagetian tests. They can be fooled by perceptual cues or language that distracts them from quantity, but they possess a basic understanding of addition,



Adapted from Resnick & Greeno (1990).

FIGURE 1 THE PROTOQUANTITATIVE SCHEMAS

subtraction, and conservation. The protoquantitative increase/decrease schema is also the foundation for eventual understanding of unary addition and subtraction.

The *protoquantitative part-whole* schema is really a set of schemas that organize children's knowledge about the ways in which material around them comes apart and goes together. The schemas specify that material is *additive*. That is, one can cut a quantity into pieces that, taken together, equal the original quantity. One can also put two quantities together to make a bigger quantity and then join that bigger quantity with yet another in a form of hierarchical additivity. Implicitly, children know about this additive property of quantities. This protoquantitative knowledge allows them to make judgments about the relations between parts and wholes, including class inclusion (Markman & Siebert, 1976) and the effects of changes in the size of parts on the size of the whole. The protoquantitative part-whole schema is the foundation for later understanding of binary addition and subtraction and for several fundamental mathematical principles, such as the commutativity and associativity of addition and the complementarity of addition and subtraction. It also provides the framework for a concept of additive composition of number that underlies the place value system.

Counting

Counting is the first step in making quantitative judgments exact. It is a measurement system for sets. Gelman and her colleagues have shown that children as young as three or four years of age implicitly know the key principles that allow counting to serve as a vehicle of quantification (Gelman & Gallistel, 1978). These principles include the knowledge that number names must be matched one-for-one with the objects in a set and that the order of the number names matters, but the order in which the objects are touched does not. Knowledge of these principles is inferred from the ways in which children solve novel counting problems. For example, if asked to make the second object in a row "number 1," children do not neglect the first object entirely but, rather, assign it one of the higher number names in the sequence.

Other research has challenged Gelman's assessment of the ages at which children can be said to have acquired all of the counting principles. Some of the challenges are really arguments about the criteria for applying certain terms. For example, Gelman has attributed knowledge of *cardinality*, a key mathematical principle, to children as soon as they know that the last number in a counting sequence names the quantity in the whole set; others would reserve the term for a more advanced stage in which children reliably conserve quantity under perceptual transformations. A challenge that goes beyond matters of terminology comes from research showing that, although children may know all the principles of counting and be able to use counting to quantify given sets of objects or to create sets of specified sizes, they may not, at a certain point, have fully integrated their counting knowledge with their protoquantitative knowledge. Several investigators (e.g., Sophian, 1987) have shown that many children who know how to count sets do not spontaneously count in order to compare sets. This means that counting and the protoquantitative schemas exist initially as separate knowledge systems, isolated from each other.

Integrating counting with the protoquantitative schemas. Such findings make it clear that, even after knowledge of counting principles is established, there is substantially more growth in number concepts still to be attained. A first major step in this growth is integration of the number-name sequence with the protoquantitative comparison schema. This seems to happen as young as about four years of age. At this point, children behave as if the counting word sequence constitutes a kind of "mental number line" (Resnick, 1983). They can quickly identify which of a pair of numbers is more by mentally consulting this number line, without actually stepping through the sequence to determine which number comes later.

In the child's subsequent development, counting as a means of quantifying sets is integrated with the protoquantitative part-whole and increase/decrease schemas. This integration seems to develop as a result of participating in situations in which changes and combinations of quantity are called for and there is a cultural mandate for exact quantification. Out of school, this can occur in various play or household activities—particularly when age segregation is not strict so that young children engage freely with older children and adults. School settings can mimic the conditions of everyday life to some extent. However, a principal resource for promoting quantification of the schemas in school is the story problem. Several researchers (e.g., De Corte & Verschaffel, 1987; Riley & Greeno, 1988) have shown that children entering school can solve many simple story problems by applying their counting skills to sets they create as they build physical models of the story situations. Because the stories involve the same basic relationships among quantities as the protoquantitative schemas extensive practice in solving problems via counting should help children quantify their original schemas. Such practice should not only develop children's ability to solve problems using exact numerical measures, but also lead them to interpret numbers themselves in terms of the relations specified by the protoquantitative schemas. Eventually, children should be able to construct an enriched meaning for numbers—treating numbers (rather than measured quantities of material) as the entities that are mentally compared, increased and decreased, or organized into parts and wholes.

Principles for a Reasoning-Based Arithmetic Program

With this research base as a grounding for our efforts, we set out to develop a primary arithmetic program (for grades 1 through 3) that would engage children from the outset in invention, reasoning, and verbal justification of mathematical ideas. The school in which we worked served mainly minority (94% were African-Americans), low-income (69% were eligible for free or reduced-price lunches) children. Our goal was to use as little traditional school drill material as possible in order to provide for children a consistent environment in which they would be socialized to think of themselves as mathematical reasoners and to behave accordingly. This meant that we needed a program in which children would successfully learn the traditional basics of arithmetic calculation as well as more complex forms of reasoning and argumentation. The program evolved gradually over a period of months. We describe it here in somewhat schematized form as the instantiation of a set of six principles that guided our thinking and experimentation.

1. ***Develop children's trust in their own knowledge.*** Traditional instruction, by focusing on specific procedures and on special mathematical notations and vocabulary, tends to teach children that what they already know is not legitimately mathematics. To develop children's trust in their own knowledge as mathematics, our program stresses the possibility of multiple procedures for solving any problem, invites children's invention of these multiple procedures, and asks that children explain and justify their procedures using everyday language. In addition, the use of manipulatives and finger counting ensures that children have a way of establishing for themselves the truth or falsity of their proposed solutions. Figure 2 provides examples of multiple procedures used by second-grade children to solve the same addition problem, *158 + 74*. The examples are copied from six different children's homework papers. Child A used a procedure of adding the value of the leftmost digits, first *100 + 70*, then *50 + 4*. This unusual decomposition left the *8* of *158* still to be added, which the child added to the already accumulated *54*. To add the resulting *62* to *170*, the child decomposed it to *60* and *2*. He added to *60* first, yielding *230*, and then the *2*, to yield the final answer. Child F used a more conventional place value decomposition, first adding up the hundreds (note that she indicates that there are *0* hundreds in *74*), then the tens, then the units, and finally combining the three partial sums. Child E also used a place value decomposition but worked initially on the hundreds and tens combined (*150 + 70*). These and the other solutions in the figure illustrate the ways in which written notation and mental arithmetic are combined in the children's procedures.

2. ***Draw children's informal knowledge, developed outside school, into the classroom.*** An important early goal of the program is to stimulate the use of counting in the context of the compare, increase/decrease, and part-whole schemas to promote children's construction of the quantified versions of those schemas. This is done through extensive problem-solving practice, using both story problems and acted-out situations. Counting (including counting on one's fingers) is actively encouraged. Figure 3 gives an example of a typical class problem, showing how it can generate several solutions; the notations shown are copied from the notebook in which a child recorded the solutions proposed by several teams who had worked on the problem.

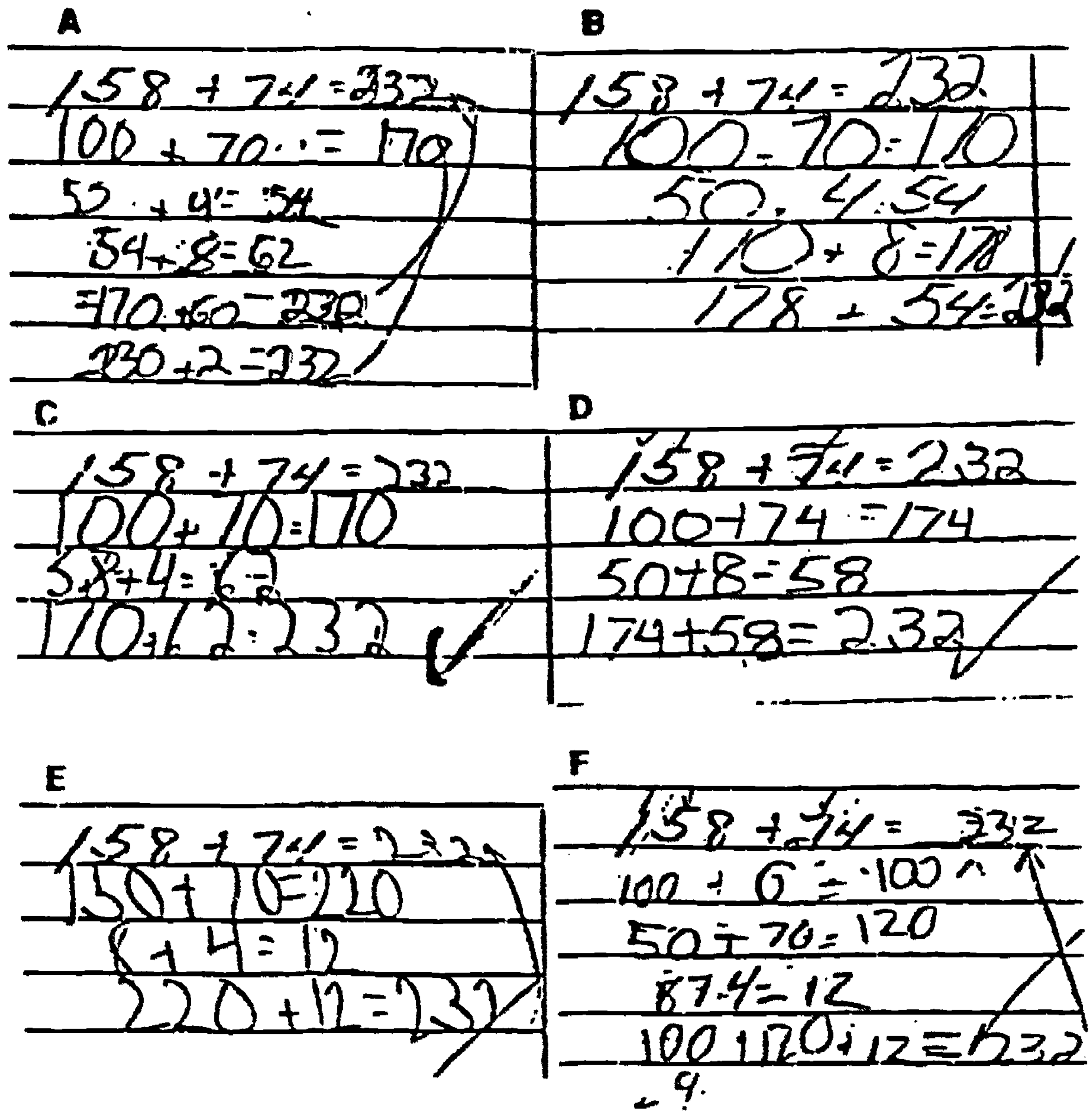


FIGURE 2 EXAMPLES OF SEVERAL SECOND-GRADERS' SOLUTIONS TO A COMPUTATIONAL PROBLEM

Mary told her friend Tonya that she would give her 95 barrettes. Mary had 4 bags of barrettes and each bag had 9 barrettes. Does Mary have enough barrettes?

The class first developed an estimated answer. Then they were asked, "How many more does she need?" The solutions below were generated by different class groups.

Group 1 first solved for the number of barrettes by repeated addition. Then they decomposed 4 x 9 into 2 x 9 plus 2 x 9. Then they set up a missing addend problem, 36 + 59, which they solved by a combination of estimation and correction.

Group 2 set up a subtraction equation and then developed a solution that used a negative partial result.

Group 4 began with total number of barrettes needed and subtracted out the successive bags of 9.

Est. 4x10=40 NO

1-29-90

#1 9+9+9+9=36 → 4x9=36
2x9=18
2x9=18
18+18=36

60
36 + 59 = 95
36 + 60 = 96
96 - 1 = 95
60 - 1 = 59

#2 95 - 36 = 59
90 - 30 = 60
5 - 6 = -1
60 - 1 = 59

#4 95 - 9 = 86
86 - 9 = 77
77 - 9 = 68
68 - 9 = 59

FIGURE 3 A SECOND-GRADE PROBLEM AND SEVERAL SOLUTIONS

3. Use formal notations (identity sentences and equations) as a public record of discussions and conclusions. Children's intuitive knowledge must be linked to the formal language of mathematics. By using a standard mathematical notation to record conversations carried out in ordinary language and rooted in well-understood problem situations, the formalisms take on a meaning directly linked to children's mathematical intuitions. First used by the teacher as a way of displaying for the class what a child had proposed, equations quickly became common currency in the classroom. Most of the children began to write equations themselves only a few weeks into the school year. Figure 4 shows part of a typical teacher-led sequence in which children propose a solution to a story problem. The teacher carefully linked elements of the proposed solution to the actual physical material involved in the story (the tray of cupcakes) and an overhead schematic of the material. Only after the referential meaning of each number had been carefully established was the number written into the equation. The total sequence shown took about 1 minute 20 seconds.

4. Introduce key mathematical structures as quickly as possible. Children's protoquantitative schemas already allow them to think reasonably powerfully about how amounts of material compare, increase and decrease, come apart and go together. In other words, they already know, in nonnumerically quantified form, something about properties such as commutativity, associativity, and additive inverse. A major goal of the first year or two of school mathematics is to "mathematize" this knowledge—that is, quantify it and link it to formal expressions and operations. It was our conjecture that this could best be done by laying out the additive structures (e.g., for first grade: addition and subtraction problem situations, the composition of large numbers, regrouping as a special application of the part-whole schemas) as quickly as possible and then allowing full mastery (speed, flexibility of procedures, articulate explanations) of elements of the system to develop over an extended time. Guided by this principle, we found it possible to introduce addition and subtraction with regrouping in February of first grade. However, no specific procedures were taught; rather, children were encouraged to invent (and explain) ways of solving multidigit addition and subtraction problems, using appropriate manipulatives and/or expanded notation formats that they developed.

It is important to note that a program built around this principle constitutes a major challenge to an idea that has been widely accepted in the past twenty or thirty years of educational research and practice. This is the notion of learning hierarchies—specifically, that it is necessary for learners to master simpler components before they try to learn complex skills. According to theories of hierarchical and mastery learning, children should thoroughly master single-digit addition and subtraction, for example, before attempting multidigit procedures, and they should be able to perform multidigit arithmetic without regrouping smoothly before they tackle the complexities of regrouping. We propose instead a *distributed* curriculum in which multiple topics are developed all year long, with increasing levels of sophistication and demand, rather than a strictly sequential curriculum.

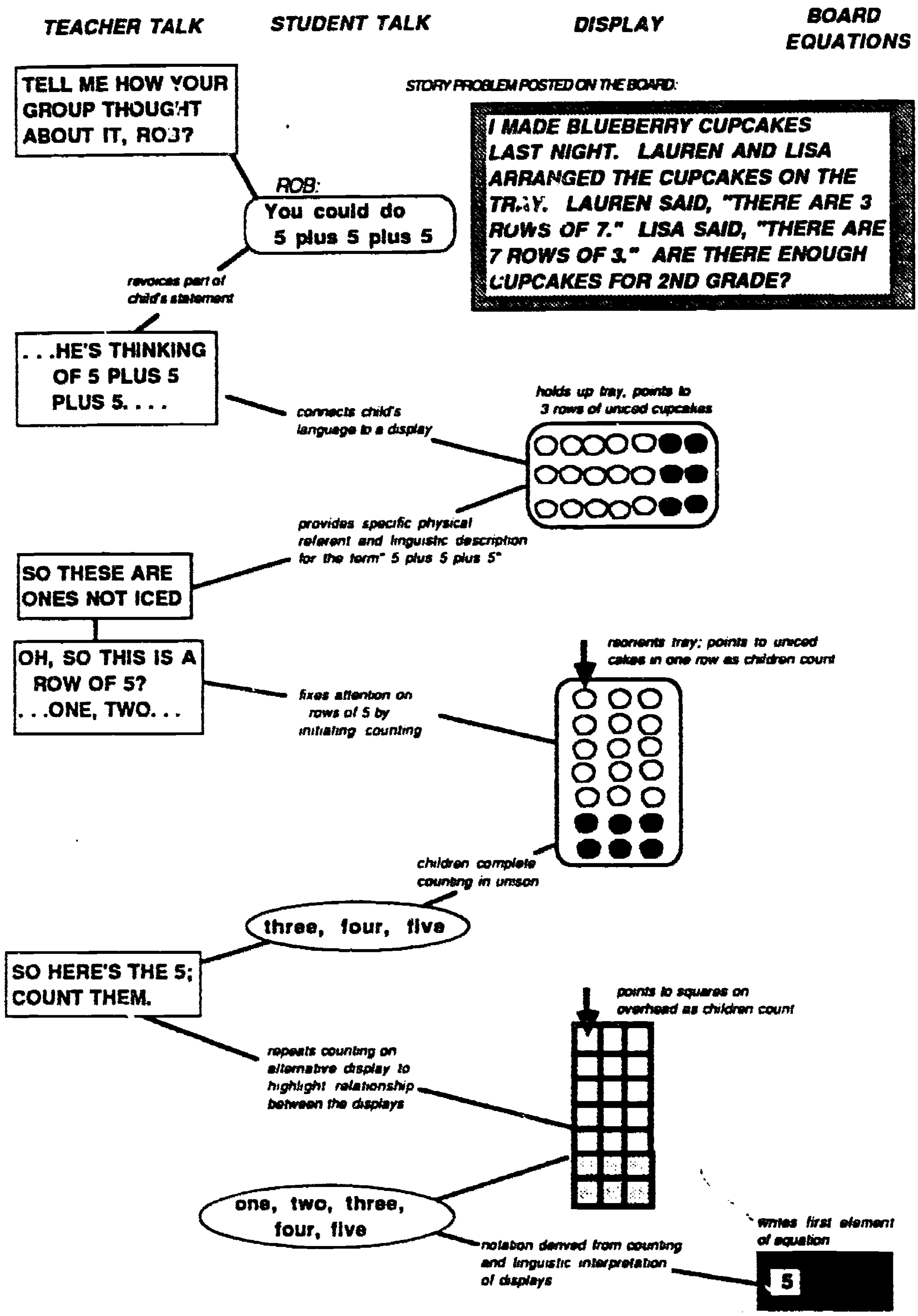


FIGURE 4 PART OF A WHOLE-CLASS DISCUSSION OF A STORY PROBLEM

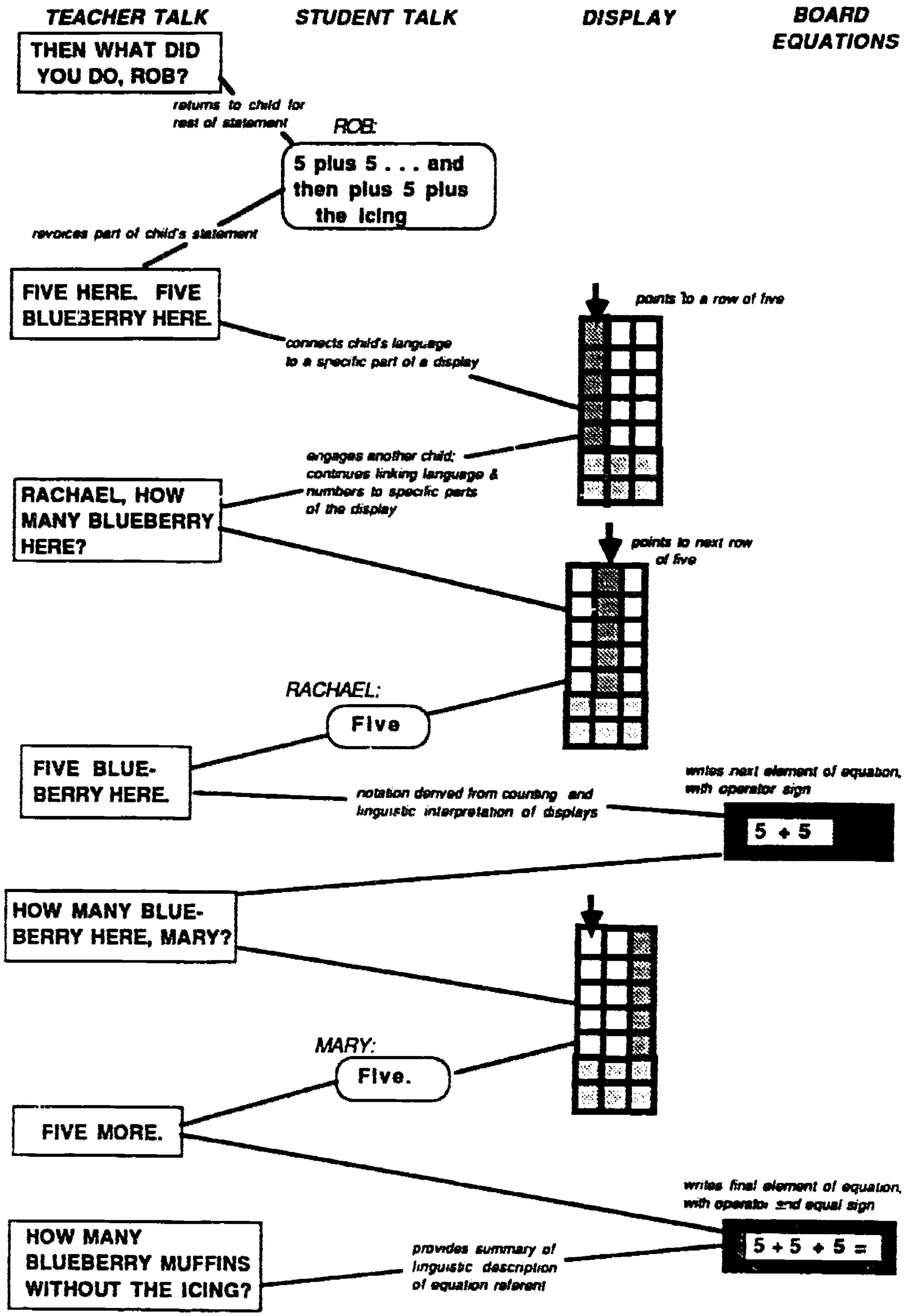


FIGURE 4 PART OF A WHOLE-CLASS DISCUSSION OF A STORY PROBLEM (Concluded)

To convey the flavor of the process, Figure 5 shows the range of topics planned for a single month of the second-grade program. All topics shown are treated at changing levels of sophistication and demand throughout the school year. This distributed curriculum discourages decontextualized teaching of components of arithmetic skill. It encourages children to draw on their existing knowledge framework (the protoquantitative schemas) to interpret advanced material, while gradually building computational fluency.

Domain	Specific Content
Reading/Writing Numerals	0-9,999
Set Counting	0-9,999
Addition	2- and 3-digit regrouping, Basic Facts 20
Subtraction	2-digit renaming, Basic Facts 20
Word Problems	Addition, Subtraction, Multiplication
Problem Solving	Work backward, Solve an easier problem, Patterns
Estimation	Quantities, Strategies, Length
Ratio/Proportion	Scaling up, Scaling down
Statistics/Probability	Scaling up, Scaling down, Spinner (1/4), Dice (1/16), 3 graphs
Multiplication	Array (2, 4 tables), Allocation, Equal groupings
Division	Oral problems involving sharing sets equally
Measurement	Arbitrary units
Decimals	Money
Fractions	Parts of whole, Parts of set, Equivalent pieces
Telling Time	To hour, To half hour
Geometry	Rectangle, square (properties)
Negative Integers	Ones, tens

FIGURE 5 TOPIC COVERAGE PLANNED FOR A SINGLE MONTH OF GRADE 2

5. Encourage everyday problem finding. In stating this principle, we deliberately use the term *everyday* in two senses. First, it means literally doing arithmetic every day, not only in school but also at home and in other informal settings. Children need massive practice in applying arithmetic ideas, far more than the classroom itself can provide. For this reason, we thought it important to encourage children to find problems for themselves that would keep them practicing number facts and mathematical reasoning. Second, *everyday* means nonformal, situated in the activities of everyday life. It is important that children come to view mathematics as something that can be found everywhere, not just in school, not just in formal notations, not just in problems posed by a teacher. We wanted to get children in the habit of noticing quantitative and other pattern relationships wherever they are and of posing questions for themselves about those relationships. Two aspects of the program represent efforts to instantiate this principle. First, the problems posed in class are drawn from things children know about and are actually involved in. Second, homework projects are designed so that they use the events and objects of children's home lives: for example, finding as many sets of four things as possible in the home; counting fingers and toes of family members; recording numbers and types of things removed from a grocery bag after a shopping trip. From child and parent reports, there is good, although informal, evidence that this strategy works. Children in the program are noticing numbers and relationships and setting problems for themselves in the course of their everyday activities. Figure 6 shows part of a letter from a parent to the teacher, sharing a story of a child's everyday math engagement.

6. Talk about mathematics, don't just do arithmetic. Discussion and argument are essential to creating a culture of critical thought. To encourage this talk, our program uses a combination of whole-class, teacher-led discussion and structured small-group activity by the children. In a typical daily lesson, a single relatively complex problem is presented on the chalkboard. The first phase is a class discussion of what the problem means—what kind of information is given, what is to be discovered, what possible methods of solution there are, and the like. In the second phase, teams of children work together on solving the problem, using drawings, manipulatives, and role playing to support their discussions and solutions. The teams are responsible not only for developing a solution to the problem, but also for being able to explain why their solution is mathematically and practically appropriate.

Dear Mrs Bill,

As a parent, I must share this with you.

On Saturday, Raymond, Jamella and I were having lunch at Monroeville Mall. Raymond and Jamella love to spend their own money, so I told both of them that I would pay for half of their lunch. I asked Ray if he remembered what he had paid for his hot dog, fries and drink. He said "Yes, $4.33." My reply was "O.K., tell me what I owe you. What is half of $4.33?"

He leaned back in his chair and you could just look into his eyes and see him concentrating. On the other hand, Jamella was doing the calculations on a make-believe piece of paper on the table. (This is what I usually do). Not Raymond Mrs. Bill, he was feeling and thinking the math problem. His answer was, "$2.15, Mommy." Well, I almost fell over into my Chinese food. When I told him that he was correct, he just beamed.

FIGURE 6 EXCERPT OF A LETTER FROM A PARENT

The following transcript of a four-minute segment of a third-grade team's conversation as they work independently on a problem, shows how linguistic interpretation and development of manipulative displays interact in the children's work.

Mick, Joe, Anna, and Ms. B. were working on the following story problem:

Mr. Bill bought 3 boxes of Ninja Turtle cookies for $3.79. One box costs $1.50 at other stores. Which is the better buy?

How much are the $3.79 Ninja Turtles per box?

Ms. B.: I want to discuss it with your groups. I want you to show how you figured it out. And when you have it, raise your hand. I'll let you put it on. If you need manipulatives, you may just get them.

Ms. B. circulates around the room while children work at solving the problem in their respective discussion teams.

Joe: Four dollars and that's automatically over.

Anna: So here's the three boxes. [Anna puts three pieces of colored paper on the desk]

Joe: Now it's time to . . . now it's time to . . . Wait, wait a minute.

Mick: What . . .

Anna: What kind of problem could we do?

Mick: We could say, we could say three dollars and seventy-nine cents. Okay, three dollars and seventy-nine cents divided by the three boxes, because we're taking the three seventy-nine and trying to see how much each box would cost if it wasn't in a bulk. [Ms. B. appears at group table carrying the three-box unit of Ninja Turtle cookies]

Joe: All right.

Anna: I agree, I agree because we have three seventy-nine in three boxes . . . . Ms. B. brought it for second grade. Third grade will divided it up . . . in into and divided it up for second grade and third grade class.

Joe: All right, now.

Anna: So I agree.

Joe: All right, now. [inaudible] What's over three dollars [writing in notebook]

Mick: I agree.

Anna: I agree.

Joe: I agree with myself. [all three students writing in notebooks] We have to show [three dollars divided by three]. We have to put the date.

Anna: I agree. I agree . . . three dollars divided by three.

Joe: We have to show this [Joe stands and reaches into the manipulatives bin which contains bundles of 10 and 100 popsicle sticks, as well as single popsicle sticks]

Anna: How can we show this, Joe?

Mick: You could say . . . .

Joe: Three dollars. These are our three dollars. [puts down three bundles of hundred and writes something in his notebook]

Mick: So what is this, Anna, three dollars or three pennies?

Anna: Three pennies.

Mick: Okay, so three, so what do we do with this three dollars?

Anna: We divide it three hundred. [Anna picks up a bundled of one-hundred and begins to take off the rubber band]

Mick: Wait a minute . . .

Joe: We have the other two hundred.

Mick: Yeah, so . . . but are we taking off the rubber band? [addressing Anna]

Anna: Yeah, we have to.

Joe: No, we don't. Here are two more. One, two, three. [picks up and puts down the three bundles]

Anna: One goes here, one goes here, and one goes there. [puts bundles of one hundred, one at a time, on top of the pieces of colored paper]

In the third phase of the lesson, teams successively present their solutions and justifications to the whole class, and the teacher records these on the chalkboard. The teacher presses for explanations and challenges those that are incomplete or incorrect; other children join in the challenges or attempt to help by expanding the presented argument. By the end of the class period, multiple solutions to the problem, along with their justifications (as in Figures 2 and 3), have been considered, and there is frequently discussion of why several different solutions could all work, or why certain ones are better than others. In all these discussions, children are permitted to express themselves in ordinary language. Mathematical language and precision are deliberately not demanded in the oral discussion. However, the equation representations that the teacher and children write to summarize oral arguments provide a mathematically precise public record, thus linking everyday language to mathematical language (as in Figure 4).

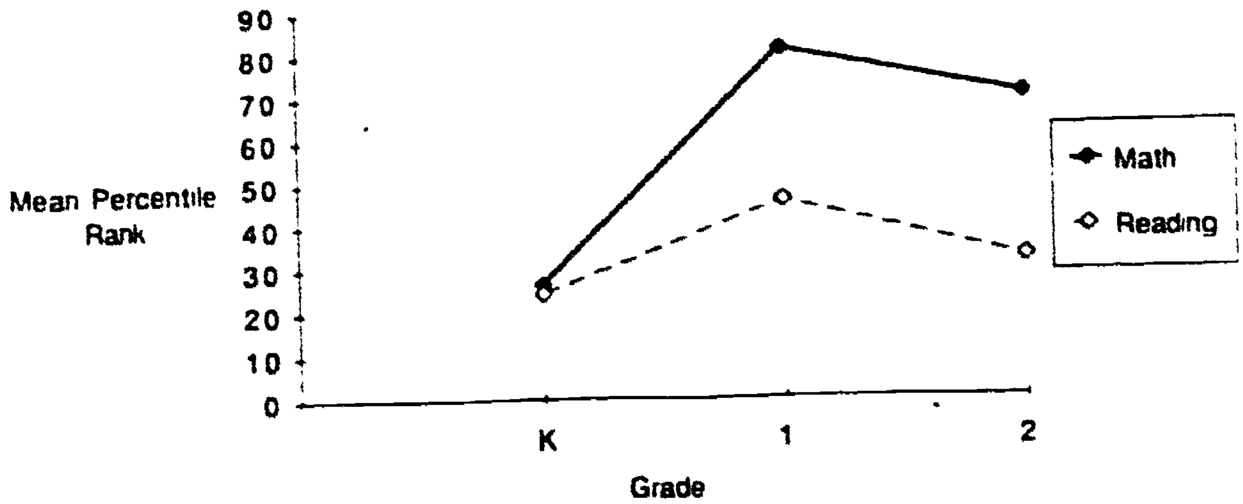
Results of the Program

We are describing here a program that has been under development for a little over two years. The project began not as a research project but as an effort to help an ambitious teacher apply research findings to improve her teaching. During the developmental period, we did not want to impose testing programs beyond those that the school regularly administered. We are thus limited, in this period of the project's life, to data from the school's standardized testing program and from clinical interviews that we conducted with some of the children, along with some impressionistic reports of child and parent reactions to the overall program.

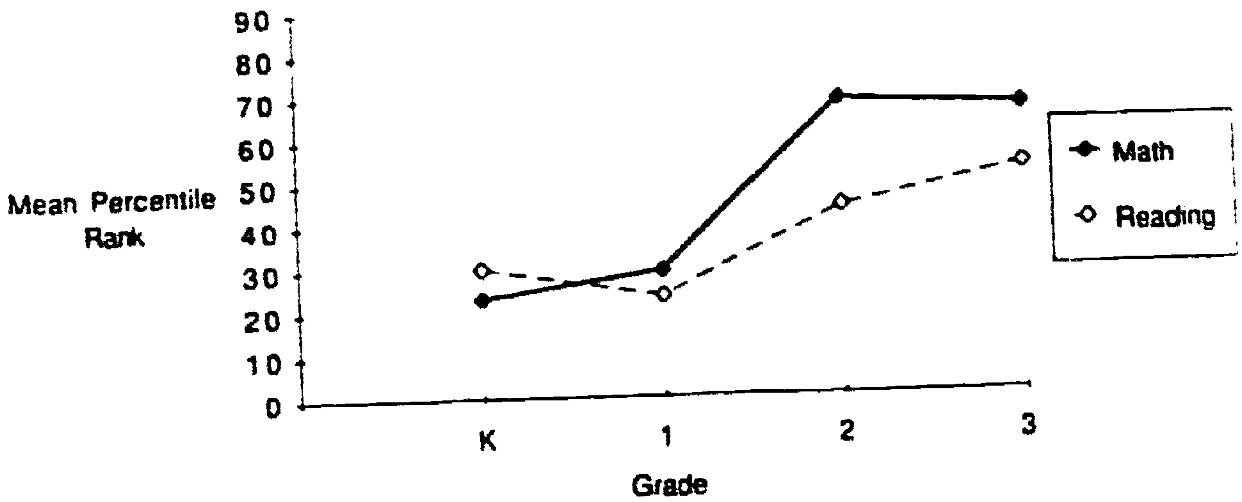
Formal evaluation data consist of scores from the California Achievement Test (CAT), which is administered in the school each September. First-graders were tested at the beginning of second grade, second-graders at the beginning of third grade. Scores on the Metropolitan Reading Readiness Test, administered by the school in March of the kindergarten year, provide data on children's general academic level before entering first grade. We have data on two cohorts of children who participated in the program, one beginning in first grade, one beginning in second grade. Figure 7a shows three years of reading and math data for Cohort A, who began the program in first grade. The children were low performers (about the 25th percentile) in both math and reading in kindergarten and remained quite low in reading in grades 1 and 2. However, their math scores rose dramatically, to a mean of the 80th percentile and stayed high (mean of 70th percentile) during the second year of the program. Figure 7b shows four years of data for Cohort B, who began the program during second grade. Like Cohort A, they were low scorers before the program. When the program was introduced in second grade, their math scores jumped to nearly the 70th percentile on average and stayed in that range through third grade. For this cohort, reading scores also rose somewhat. Reading was taught by a different teacher in the school. We are now investigating what might have been responsible for this gain. For comparison, Figure 7c shows three years of data for a cohort of children taught by the intervention program teacher before she adopted the new program. Throughout the period, mean scores remained at a low 40th to 45th percentile. An important point, one that cannot be seen in the means of the graphs, is that the math gains were not limited to only a few of the children. In Cohort A, for example, the *lowest*-scoring child at the end of the first grade was at the 66th percentile. Thus, the program appeared effective for children of all ability levels.

These global data tell only part of the story, of course. We would like to know much more for which systematic data are not yet available. Nevertheless, we can point to some indicators based on our interviews, class observations, and reports from the school. We interviewed all first-graders three times during the year, focusing on their knowledge of counting and addition and subtraction facts, along with their methods for calculating and their understanding of the principles of commutativity, conservation, and the complementarity of addition and subtraction. At the outset, these children, as

7a. Intervention Cohort A



7b. Intervention Cohort B



7c. Control Cohort

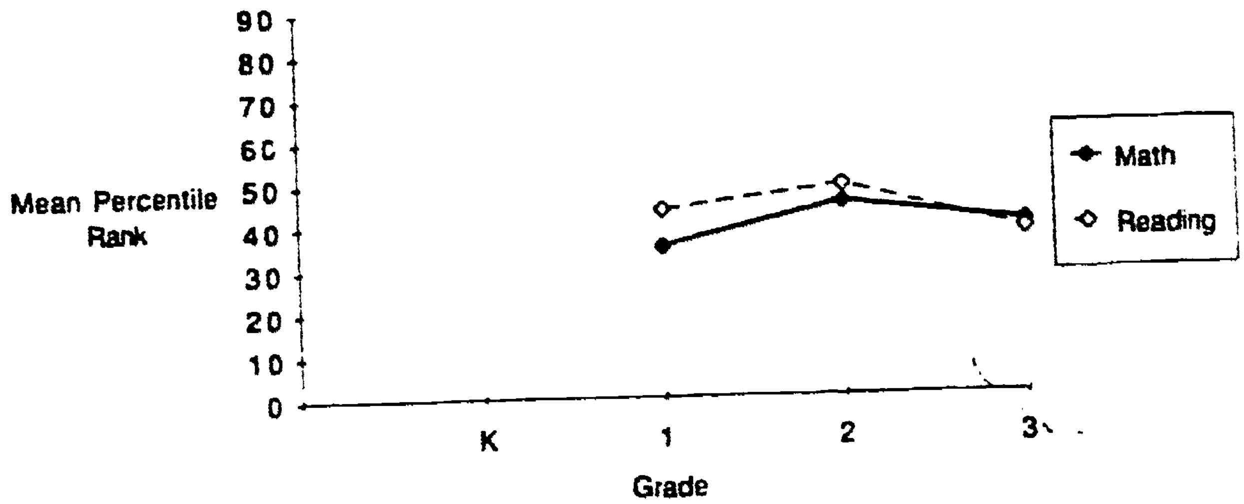


FIGURE 7 CALIFORNIA ACHIEVEMENT TEST SCORES

might be expected given their socioeconomic status and their parents' generally low educational background, were not highly proficient. Only one-third of them could count orally to 100 or beyond, and most were unable to count reliably across decade boundaries (e.g., 29-30, 59-60). The size of the sets that they could quantify by counting ranged from 6 to 20. About one-third of these children could not solve small-number addition problems, even with manipulatives or finger counting and plenty of encouraging support from the interviewer. Only about six appeared able to perform simple subtractions using counting procedures. Thus, these children seemed very weak in entering arithmetic knowledge, especially compared with data presented by a number of investigators for middle-class and educationally favored populations. By December the picture was sharply different. All but a handful of children were performing both addition and subtraction problems successfully, and all of these demonstrated knowledge of the commutativity of addition. At least half also were using invented procedures, such as counting on from the larger of two addends, or using procedures that showed that they understood principles of complementarity of addition and subtraction. By the end of the school year, all children were performing in this way, and many were successfully solving and explaining multidigit problems.

The following additional evidence indicates that the program was having many of the desired effects. The children displayed various examples of confidence in doing mathematical work. Many sang to themselves as they took the standardized test. When visitors came to the classroom, they would offer to show off by solving math problems. They frequently asked for harder problems. These displays came from children of almost all ability levels. They had not been typical of any except the most able children the preceding year. Homework was more regularly turned in than in preceding years, without nagging or pressure from the teacher. Children often asked for extra math periods. Many parents reported that their children loved math and wanted to do math all the time. Parents also sent to school examples of problems that children had solved on their own in some everyday family situation. Knowing that the teacher frequently used such problems in class, parents asked that their child's problems be used. It is notable that this kind of parent engagement occurred in a population of parents that is traditionally alienated from the school and tends not to interact with teachers or school officials.

Conclusion

We believe we have made a promising start at reaching our goals. We have shown that an interpretation- and discussion-oriented mathematics program can begin at the outset of school by building on the intuitive mathematical knowledge that children have as they enter school. Our standardized test score data show that this kind of thinking-based program also succeeds in teaching the basic number facts and arithmetic procedures that are the core of the traditional primary mathematics program. It is not necessary to teach facts and skills first and only then go on to thinking and reasoning. The two can be developed simultaneously. Assuming that we can maintain and replicate our results, this means that an interpretation- and discussion-oriented program can serve as the basic program in arithmetic, not just as an adjunct to a more traditional knowledge and skills curriculum.

Moreover, our results show that an interpretation-oriented mathematics program can be suitable even for children who are not socially favored or, initially, educationally able. The children with whom we have worked come disproportionately from among the least favored of American families. Many are considered to be educationally at risk; their educational prognosis, without special interventions or changed educational programs, is poor. Yet these children learned effectively in a type of program that, if present in schools at all, has been reserved for children judged able and talented—most often those from favored social groups.

What is at issue here, as we suggested at the outset, is not only an apparently successful program but also some fundamental challenges to dominant assumptions about learning and schooling. As we worked to develop this program, we realized that a new theoretical direction was increasingly dominating our thinking about the nature of development, learning, and schooling. This is the view, shared by a growing minority of thinkers in the various disciplines that comprise cognitive science, that human mental functioning must be understood as fundamentally situation-specific and context-dependent, rather than as a collection of context-free abilities and knowledge. This apparently simple shift in perspective in fact entails reconsideration of a number of long-held assumptions in both psychology and education.

Until recently, educators and scholars have defined the educational task as one of teaching specific knowledge and skills. As concern has shifted from routine to higher-order or thinking abilities, we have developed more complex definitions of the skills to be acquired and even introduced various concepts of *meta* skill in the search for teachable general abilities. But we have continued to think of our major concern as one of identifying and analyzing particular skills of reasoning and thinking and then finding ways to teach them, on the assumption that successful students then will be able to apply these skills in a wide range of situations.

As we developed our program, we found ourselves less and less asking what constitutes mathematics competence or ability for young schoolchildren, and more and more analyzing the features of the mathematics classroom that provide activities that exercise reasoning skills This meant choosing story problems on the basis of the mathematical principles they might illustrate and developing forms of classroom conversation designed to evoke public reasoning about these principles. Our focus on mathematics as a form of cultural practice did not deny that children engaging in mathematical activity must be knowledgeable and skillful in many ways. However, our emerging perspective led us to focus far less on the design of "lessons" than on the development of a sequence of problem-solving situations in which children could successfully participate. Another way of saying this is that we were trying to create an *apprenticeship* environment for mathematical thinking in which children could participate daily. We expected them to acquire thereby not only the skills and knowledge that expert mathematical reasoners possess, but also a social identity as a person who is able to and expected to engage in such reasoning (see Lave, in press).

Our program constitutes a version of the *cognitive apprenticeship* called for by Collins, Brown, and Newman (1989) in a recent influential paper. Its very success, however, calls into question some aspects of the apprenticeship metaphor as applied to early learning in a school environment. Among these is the nature of the master–apprentice relationship. In traditional apprenticeship, apprentices seek to become like their masters, and masters continually display all elements of skilled productive activity in their field of expertise. Teaching is only a secondary function of the traditional master. This simple—indeed, perhaps oversimplified—relationship does not seem applicable to the school setting, where the teacher's predominant function is not to *do* mathematics but to *teach* it. We will need to work out the particular role of the teacher in designing an environment *specifically for learning purposes.* A second issue surrounding cognitive apprenticeship in school is how to ensure that necessary particular skills will be acquired, even though the daily focus of activity is on problem solving and reasoning. Our first-year standardized test results suggest that we have not done badly on this criterion, but we need to understand better than we do now just *what* it is in our program that has succeeded and what the limits of our methods might be. In short, we offer this paper as only a very preliminary report on what we expect to be a long-term effort to revise instructional practice in ways that will bring educators closer to being able to meet the goal of shaping dispositions and skills for thinking through a form of socialization into cultural environments that value and practice thinking.

References

Collins, A., Brown, J. S., & Newman, S. E. (1989). Cognitive apprenticeship: Teaching the crafts of reading, writing, and mathematics. In L. B. Resnick (Ed.), *Knowing, learning, and instruction: Essays in honor of Robert Glaser* (pp. 453-494). Hillsdale, NJ: Erlbaum.

De Corte, E., & Verschaffel, L. (1987). The effect of semantic structure on first graders' strategies for solving addition and subtraction word problems. *Journal for Research in Mathematics Education, 18*, 363-381.

Gelman, R. (1972). Logical capacity of very young children: Number invariance rules. *Child Development, 43*, 75-90.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number.* Cambridge, MA: Harvard University Press.

Lave, J. (in press). Situated learning in communities of practice. In L. B. Resnick, J. Levine, & S. D. Behrend (Eds.), *Perspectives on socially shared cognition.* Washington, DC: American Psychological Association.

Markman, E. M., & Siebert, J. (1976). Classes and collections: Internal organization and resulting holistic properties. *Cognitive Psychology, 8,* 516-577.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 109-151). New York: Academic Press.

Resnick, L. B. (1987). *Education and learning to think.* Washington, DC: National Academy Press.

Resnick, L. B. (1989). Developing mathematical knowledge. *American Psychologist, 44,* 162-169.

Resnick, L. B., & Greeno, J. G. (1990). *Conceptual growth of number and quantity.* Unpublished manuscript.

Riley, M. S., & Greeno, J. G. (1988). Developmental analysis of understanding language about quantities and of solving problems. *Cognition and Instruction, 5,* 49-101.

Sophian, C. (1987). Early developments in children's use of counting to solve quantitative problems. *Cognition and Instruction, 4,* 61-90.