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ABSTRACT

The theme of the material contained in this annotated resource list is the relationship between teaching and learning mathematics in the specific content area of graphing functions. The list contains 30 articles, papers, and unpublished manuscripts written from 1979 through 1989. The articles treat various aspects of concept formation, misconceptions, representational links, and technology for graphing functions. Purpose, sample, method, and findings are presented for research studies. Summaries are provided for papers included on the list. (MDH)

Annotated Bibliography of Selected Articles on Graphing and Functions

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Technical Report No. CLIP-90-01
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LEARNING RESEARCH AND DEVELOPMENT CENTER

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This manuscript was prepared as part of a study of the teaching and learning of graphing and functions. A talk based on this work was presented at the annual meeting of the American Educational Research Association, San Francisco, March 1989. Most of the articles annotated in this collection are referenced, discussed, or analyzed in more depth in Leinhardt, G., Zaslavsky, O., & Stein, M.K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. Review of Educational Research, 60(1), 1-64.

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Classroom Learning and Instruction Project (CLIP)

Technical Report Series

The Classroom Learning and Instruction Project (CLIP) reports consist of a series of technical reports describing a program of research at the Learning Research and Development Center, University of Pittsburgh. This research is supported by a number of private and public non-military agencies and is under the general direction of Gaea Leinhardt. The theme of the research included in this series is the relationship between teaching and learning in particular subject-matter areas such as mathematics and history. Some papers focus on teachers and how their understanding of specific content (e.g., graphing functions) impacts on their teaching; some papers focus on new assessment instruments that are attempting to measure the complexity of the interrelationship between content knowledge and pedagogy; others focus on the students and how their learning is influenced by their own prior knowledge in a content area and by the teacher's instruction. It is hoped that the cumulative findings of these studies will contribute to our understanding of learning and teaching. Particularly they will contribute to those aspects that are unique to particular topics and may in turn enrich our understanding of the field of teaching and learning as a whole. A list of CLIP reports appear at the end of this report.

Table of Contents

Clement, J. (1985). Misconceptions in graphing. In L. Streefland (Ed.), Proceedings of the ninth international conference of the International Group for the Psychology of Mathematics Education, (Vol. 1, pp. 369-375). Utrecht, The Netherlands: IGPME.

A competence model is proposed which then serves as a theoretical framework for providing cognitive explanations for the errors that students make when interpreting graphs. (The errors are both errors that have been reported in the literature and errors that have been observed in the author's own research; most of the graphs are qualitative).

Two types of common misconceptions are then discussed in terms of the model:
1) treating the graph as a picture, and
2) confusing slope and height.

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. Focus on Learning Problems in Mathematics, 11(1-2), 77-87.

This paper is very similar to Clement (1985)*. The competence model proposed has four levels of representation of a situation:

1) practical representation
2) isolated value correspondence
3) length model
4) graph model

This static model has a more dynamic version that emphasizes representations of variation. Both models could be useful in understanding historical developments of the notion of function as well as new instructional strategies.

Two types of graphing misconceptions (reported by others and also observed by the author) are explained through this model:

1) slope-height confusion--can be due to a misplaced link between a successfully isolated variable and an incorrect feature of the graph (or unsuccessful isolating of the two variables).
2) treating a graph as a picture--in which a student appears to be making a figurative correspondence between the shape of the graph and some characteristics of the problem scene. There could be a global correspondence error or a local correspondence error.

Oresme's diagram (1361) is discussed and shown to use vertical lines in the same way as they are used in level 4 of the static model. In modern notation the vertical lines are omitted, even though there is some evidence suggesting that vertical line graphs are fairly natural and intuitive symbolizations for children (10-year- olds)--thus such line graphs may be useful for introducing graphs and functions.

9

* See above

Clements, K. (1981, 1982). Visual imagery and school mathematics. <u>For the Learning of Mathematics</u>, <u>2</u>(2), 2-9; and <u>2</u>(3), 33-39.

This article covers the following:

1) Work in psychology on difficulties associated with decoding images.
2) Sternberg--how pupils' strategies could flip-flop between verbal and visual representations.
3) Math education section is general.

Aims to investigate the role of visual or spatial imagery in mathematics learning.

Covers 7 major areas:

1) <u>Review of theoretical discussions</u> (most from psychology) regarding the best way to define or conceptualize imagery--
   - -Pictures in the mind
   - -Propositional theories of imagery (i.e., knowledge can be represented by sets of propositions. from which both verbal and visual images can be generated)

   Concludes, at present, there is no strong reason to discard picture theory.
2) <u>Problems associated with the externalization of imagery</u>--
   - -Finding an appropriate research methodology
   - -Introspective reporting--pros and cons
3) <u>Measurement of imagery ability</u>--not far along.
4) <u>Attempts to improve visual imagery skills</u>--Soviets have done more in this realm.

   -Twyman (1972). "The creation of an image can introduce difficulties associated with decoding the image." For example, images might possess irrelevant details that distract pupils from the main elements in the original problem stimulus, and make it more difficult for them to formulate necessary abstractions.

   -Gagne & White suggest that teachers might require students to "paraphrase" diagrams to ensure that students have digested diagrams.
5) <u>Verbalizer-Visualizer Hypothesis</u>--
   - -Sternberg--Flowchart for representation

     Strategies that may be adopted for Verbalizer, Visualizer & mixed
6) <u>Relationship between visual and spatial ability</u>
7) <u>Research into the role of visual image in mathematics learning</u>
   - -ATI--mixed
   - -Marriott (1978)--Children taught with manipulatives tended to use them more in problem solving (Fractions)
   - -Fary--Even though 3 methods of instruction were used to teach operations on integers, all kids wanted algorithms

Davidson, P.M. (in press). Early function concepts: Their development and relation to certain mathematical and logical abilities. Child Development.

Purpose: The purpose of the study was to investigate the relationship between functional thinking and performance on Piagetian-like operation tasks.

Sample: 72 children in the age range of 5 - 7 were administered 3 categories of tasks.

Tasks: 1) functions defined as exchanges of properties (e.g. colors)
2) functions defined as rotations of regular polygons
3) morphisms, or functions that preserve an operation or relation.

Findings: Consistent with other findings (Case et al., 1986; Piaget et al., 1968, 1977)*, the subjects accurately solved problems involving nonnumerical functional relations by age 5 or 6. Further, the results suggest a possible process by which developing function notions may contribute to the emergence an understanding of reversible operations. The principal developmental finding was a shift from trial-and-error strategies to anticipatory or inferential strategies. Finally, the results also suggest that functional reasoning may develop within separate domains, with functions involving *extensive* content having implications for quantitative operations, and those involving *intensive* content having implications for logical operations.

* Case, R., Marini, Z., McKeough, A., Dennis, S., & Goldberg, J. (1986). Horizontal structure in middle childhood: The acquisition of dimensional operations. In I. Levin (Ed.), Stage and structure: Reopening the debate (pp. 1-39). Norwood, NJ: Ablex.

Piaget, J., Grize, J.B., Szeminska, A., & Vinh-Bang. (1988). Epistemology and psychology of functions (F. Castellanos & V. Anderson, Trans.). Dordrecht, The Netherlands: D. Reidel. (Original work published in 1968.)

Dreyfus, T., & Eisenberg, T. (1982). Intuitive functional concepts: A baseline study on intuitions. Journal for Research in Mathematics Education, 13(5), 360-380.

Purpose: This paper examines junior high students' mathematical intuitions in the particular area of functional concepts and considers four hypotheses.

- H_1: Intuitions on functional concepts grow with pupils' progress through grades.
- H_2: Intuitions are independent of sex.
- H_3: Intuitions of high-level students are more often correct than those of low- level students.
- H_4: Intuitions are more often correct in concrete situations than in abstract ones.

The authors introduce a 3 dimensional block for thinking about function knowledge, where the x axis displayed settings (or representations), the y axis displayed concepts, and the z axis displays levels of abstraction or generalization.

Sample: The sample is drawn from students in grades 6 through 9 in classrooms defined as either High or Low Absolv (Absolv = combined ability/social disadvantage ranking). 24 classrooms from 12 schools in Israel were equally distributed across grade and Absolv.

Methods: All subjects were tested at the beginning of the school year with one of three questionnaires on function concepts. In each classroom students randomly received either the diagram, the graph or the table form as the setting on which they were tested on 5 concepts, each at 2 levels of abstraction.

Findings: Significant differences in performance were found for Absolv (High did better), Grade (overall improvement from 6 to 9), and Setting (Diagram most difficult). Within Grade, there was a dip from 7th to 8th for Low Absolv subjects. Within Setting, graphs were preferred by High Absolv subjects & tables by Low Absolv subjects. No overall difference by Sex, except that boys did better in grades 6 & 7, and girls did better in grades 8 & 9. May be related to girls' earlier maturation or boys' diminishing seriousness. Previous studies of mathematics ability of girls this age show them to have poor attitudes and poor performance. Girls' superior intuitions in these results demonstrate the need to consider a teaching approach that exploits their intuitions.

Considering levels of abstraction, all factors that were significant on the whole test carried over to the concrete part. All factors except setting and 3-way grade-by-Absolv-by-sex carried over to the the abstract part.

Among concept, image was answered best and slope worst. Within setting-by-Absolv, Low Absolv students preferred tables and High Absolv students preferred graphs, across almost all concepts.

There was a noticeable drop between 7th and 8th grade performance on Cartesian coordinates despite 7th grade instruction on the topic.

Conclusions:

1) Pupils' intuitions on functional concepts do grow with pupils' progress through the grades.
2) Boys and girls have equal intuitions although they develop at different rates.
3) High Absolv pupils have more correct intuitions than Low Absolv pupils.
4) Reject the 4th hypothesis. Intuitions are no more correct in concrete situations than abstract.

Dufour-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), <u>Problems of representation in the teaching and learning of mathematics</u> (pp. 109-122). Hillsdale NJ: Erlbaum.

Makes distinction between internal & external representations:
- Internal--mental formulations of reality (the signified)
- External--all external symbolic organizations that attempt to represent a certain mathematical reality (the signifier)

The chapter is mainly concerned with external representations, but also examines how internal representations can be linked to external representations in learning. The chapter attempts to answer five questions about the use of representations by presenting examples and summaries of the authors' research, most of which has been conducted with elementary students. The five questions and answers appear below.

1). <u>What are the motives for using external representations in math teaching</u>?

-Representations are an inherent part of math (i.e., representations are used as tools for treating concepts) Certain representations are so closely associated to a concept that it is hard to see how the concept can be conceived without them.

-Representations are multiple concretizations of a concept. We want students to grasp the common properties and extract the intended structure.

-Representations are used locally to mitigate certain difficulties.
1) a task is given along with several representations hoping that the student will find one useful
2) in the course of learning a concept, sporadic recourse is made to representations on which the student may lean
3) teacher draws attention momentarily to a difficulty.

-Representations are intended to make mathematics more attractive and interesting. Representations are used in textbooks to motivate the child or to present analogies to the real world.

Side point: Some representations have as their primary concern to be as accessible as possible to students, whereas others have as their primary concern the math object itself.

2) <u>What are the expected outcomes of the use of such a wide variety of representations in the learning of math</u>?

- Learner perceives the representations as tools.
- Learner appropriately selects one representation and rejects another and knows why
- Learner passes (translates) from one representation to another
- Learner knows possibilities, limits, and effectiveness of each representation
- Learner will be able to grasp the common properties of diverse representations and be able to construct the intended concept
- Learner will solve a given task using the suggested representation
- Learner will approach tasks with the attitude that if one representation does not work he can try another

3) <u>Are these expected outcomes achieved in current teaching of mathematics?</u>

-Are conventional math representations perceived as tools for solving problems? Example given from their research of students not using graphs to help them solve an equation.

-How do concrete multiple representations used in current teaching contribute to the child's construction of the concept?
According to their research, children often do not realize that different representations embody the same situation. Example of subtracting on an abacus and with paper and pencil.

-To what point can a child, who is experiencing difficulty in solving a problem. select a representation that will bring him to resolve this same problem?
Does a child really "select" a representation?--No, they simply retain the one that was provided or use the one which is most familiar to them (even though it may not be the most appropriate). Does the child see the same task in each of the representations given?--Children say that there are as many different problems as there are representations. They will say that two representations concern the same problem only if the same numbers are in both.

Is the child convinced that regardless of the particular representation he uses he will necessarily arrive at the same result?--No. They give an example of a child correctly working through a problem using one representation and then shown the incorrect answer of another child who worked through the same problem using a different representation. The child shows no distress over the fact that the same problem has two different answers.

How do children develop the attitude of having recourse to representations in case they encounter difficulties?--by acquiring the conviction that representations can be a help in unscrambling a problem situation. According to their research, most students do not have this conviction.

4) <u>How accessible to children are representations</u>?

-Examples of representations introduced prematurely: Children being introduced to ">" and 2-dimensional tables before they can comprehend them. Teaching the representation becomes the goal rather than using it as a tool for attaining the "real" goal of increased math understanding.

-Examples of representations used inappropriately: The numberline representation has several misconceptions associated with it: the stepping-stone analogy, the lack of appreciation for equal distances, and the "arrow." Authors suggest that these potential problems can lead to problems in later learning. Teachers often use external representations for a problem situation without realizing the discrepancy that can exist between those used and the one envisioned by the child.

5) <u>How can we organize instruction to maximize the contribution of children's use of their own representations in their learning</u>?

In traditional teaching, representations are imposed from the outside; children are not encouraged to construct or exploit their own mathematical representations. The only times that they may be encouraged to "think up" a representation is in word problems. Examples are provided of children not spontaneously using representations to explain or solve a problem.

Even, R., Lappan, G., & Fitzgerald, W. (1988, November). Pre-service teachers' conceptions of the relationship between functions and equations. In M.J. Behr, C.B. Lacampagne, & W.W. Wheeler (Eds.), Proceedings of the tenth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 283-289). DeKalb IL: IGPME-NA.

Purpose: The purpose of this paper was to report on a study of prospective high-school teachers' understandings of the relationship between functions and equations.

Subjects: 152 prospective secondary math teachers, in their last phase of professional education.

Tasks were designed to get at subject-matter content knowledge and pedagogical content knowledge about the notion of a function, in particular the relation between graphical and algebraic representation of a function.

Data collection: An open-ended questionnaire was administered to all subjects and 10 subjects were interviewed on abstract functions.

Results: (based mainly on the questionnaire):

1) The role of equations in the definition and image of function:
    a) close to 20% identified the notion of a function with its algebraic representation;
    b) almost 50% related functions directly to an equation in at least 1 out of 2 of their answers to a pair of questions.
2) The values of functions as solutions to equations:
    a) 80% did not make the connection between the two representations.

Goldenberg, E.P. (1988). **Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions.** Journal of Mathematical Behavior, 7, 135-173.

This paper deals with implications of graphical technology on the learning of graphing and functions. The author raises the issue that even though children work with visual forms (e.g. drawings) for many years, way before studying graphs in general and graphs of functions in particular, graphs are not more accessible to students than other symbolic representations. Graphs have their own conventions and ambiguities, which earlier experience does not necessarily make easier. This can mean that graphical representations are in a way unique kinds of symbolic representations. On the one hand one tends to rely on what one sees in a graph (more than in other less visual symbols), yet the conventions are often quite different than expected, so relying on what one actually sees may turn out misleading in many instances. There is some evidence in Goldenberg's work that suggests that perception may be more dominant in some cases over logical thinking, when a conflicting situation arises.

There are several sources of graphical illusions that might be created by a computer:
1) finitude of the "window"
2) shape of "window"
3) position (comparative)
4) scaling ("picture" changes)
5) difference between a dot and a point

Learning through a computer allows the development of different notions of a function machine. These notions are:
1) number in/number out machine (variable)
2) number in/function out machine (parameter)
3) number in/pair out machine (point)
4) function in/function out machine (transformation)
5) function in/graph out machine (object)

The computer also may create a "magic" effect that should be taken into account.

Greeno, J. G. (1988, November). The situated activities of learning and knowing mathematics. In M.J. Behr, C.B. Lacampagne, & M.M. Wheeler (Eds.), Paper presented at the tenth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, (pp. 481-521). DeKalb IL: IGPME-NA

Conceptual Framework. The author begins by laying out a context for his research on the concept of function. He first develops ideas related to the active and the situated nature of mathematical learning and knowing. Given these characteristics of mathematical practice, he notes that the normative mode of classroom instruction, which emphasizes passive watching and mimicking, cannot be expected to produce individuals who are capable of actively generating mathematical meaning. Rather students too often view mathematics as the rote manipulation of meaningless symbols and view themselves as possessing a low level of mathematical ability.

Research Questions. The research focused on 3 general questions:

1) How one might characterize the implicit understanding that students have regarding functions and variables;

2) Identifying characteristics of situations in which students reason effectively about functional relations among quantities;

3) Describing the language that students used as they referred to quantities that were properties or relations in the situation that was used to exemplify functions.

Method. Ten pairs of students (grades 7-11) were asked to reason about the operation of a simple machine that embodies linear functions. Interviews occurred in 5 phases: (1) students gained familiarity with the system by manipulating the components and talking about it; (2) students were asked questions about specific situations set up by the experimenter; (3) students were asked to make situations happen (on the machine) that were verbally described by the experimenter; (4) students were asked a series of questions requiring inferences based on the linear functions in some situations; and (5) students were asked a question designed to elicit discussion in general terms.

Preliminary Findings. Based on the students' responses to questions in the fourth section of the interview, the author drew the following tentative conclusions:

1) Students did have an implicit understanding of the concepts of function and variable. Evidence included the following:

- A clear sense that the number of turns and the final position are functionally dependent.
- An appreciation that a variable (spool size) could take on different values.
- A realization that one variable (spool size) could be compensated by a larger value of another variable (number of turns).
- An appreciation that the intercepts of functions (starting points) could be varied.
- An understanding that, for any of 4 variables, the value could be inferred from values of the other 3 variables.
- An understanding of the ratio of values of 2 functions.

2) In answer to the question, "What was there about the machine situation that enabled individuals to reason effectively?" The following was suggested: The quantities and their relations could be observed as structural features of events occurring in the machine situation. In other words, understanding relationships was supported by the evident causal relations between turning the handle, winding the string around the spool, and pulling blocks along the path. Thus concrete objects and events played an important role in reasoning about complex properties such as quantitative relations.

3) Language serves a crucial role in connecting between individuals and to situations. In particular, the level of generality may be a key feature to examine when attempting to understand math principles.

The author concludes by proposing that instruction treat students as active participants in the social process of mathematical learning and suggests that Lampert's (1986, 1988)* teaching illustrates that this is possible.

* Lampert, M. (1986). Knowing, doing, and teaching multiplication. Cognition and Instruction, 3(4), 305-342.

Lampert, M. (1988). The teacher's role in reinventing the meaning of mathematical knowing in the classroom. In M.J. Behr, C.B. Lacampagne, & W.W. Wheeler (Eds.), Proceedings of the tenth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. DeKalb, IL: IGPME-NA.

Janvier, C. (1981). Use of situations in mathematics education. Educational Studies in Mathematics, 12, 113-122.

Purpose: The study questions a common assumption that situations guarantee "concretization" of abstract notions. Thus it examines the effect a situation may have in the abstraction process.

Sample: Twenty secondary level pupils (ages 11-15) were interviewed: 7 in 1st year, 7 in 2nd year, and 6 in 4th year. In addition, forty pupils of the 1st year received a written form of the task.

Task: The task given to the pupils was a racing car problem. The problem comprises three parts: 1) the presentation includes a verbal description of the situation represented by the graph (speed [km/hr] of a racing car along a 3 km track in relation to the distance along the track [km]), and the graph itself. The presentation is followed by 2 central questions (and a few probes in case the pupil does not answer correctly). The questions are basically: a) How many bends are there along the track? b) which bend is the worst? the easiest? the "second worst"? 2) The graph is replaced by a sheet of paper on which three racing tracks are drawn. For each track there is a grid on which the student is asked to sketch a speed graph representing a racing car driving that track; 3) The first graph is brought back and a sheet of paper with a selection of tracks is presented. The pupil is asked to choose the one track that is described in the graph.

Results: The main mistake pupils committed was confusing the graph with the track. This mistake was more common among the 1st year students. Students appeared to use the graph simultaneously at a symbolic and a pictorial level. Interviews indicated that familiarity with the situation, namely car racing, was supportive to some of the pupils in using the graph at the proper symbolic level in the first part of the task. For these students, the third part became more difficult.

Janvier, C. (1983). Teaching the concept of function. Mathematical Education for Teaching, 4(2), 48-60.

The paper lays out some theoretical considerations regarding the notion of function that lead to a proposed spiral didactical approach. This approach comprises three main phases:

1) the concept of variation (staircase analysis)
2) introducing some equation
3) the functional notation

The paper deals with different meanings of a function, for example, rule, dependent variable, correspondence, set theory definition, Cartesian product, and particular types of relations.

The set approach to functions leaves out the rule, which is the basic notion--didactically and historically. The relation approach is not biased towards the co-domain.

There are three equivalent notions of a function that are discussed: "the way it varies", rules defining types of variations, and functions as mathematical models in situations.

A classification of functions is suggested:

1) non-ordered finite domain
2) ordered finite domain
3) continuous with ordered, dense domain

The claim is that functions of types 1 and 3 are fundamentally different.

Functions differ along another dimension too--whether there are time dependent variables (explicit or implicit) or not time dependent.

Functions are usually represented by equations, tables, graphs, or verbal descriptions. Different types of translations can be carried out from each mode to another.

Janvier, C. (1987). Translation processes in mathematics education. In C. Janvier (Ed.), Problems of representation in mathematics learning and problem solving (pp.27-31). Hillsdale NJ: Erlbaum.

The author claims that particular uses of representations are overlooked, specifically the translation processes. By translation processes he means the psychological processes involved in going from one mode of representation to another e.g., from an equation to a graph. It is hard to identify the literature on translations because often it is dealt with indirectly/implicitly.

Seven Points

1) Comprehensive 4x4 table of translation processes in the graphing domain is provided. The 4 representations are table, graph, equation, and verbal description.
--Illustrates the problem of "naming" a translation.

2.) Direct and indirect processes are alternate ways to achieve a translation.
--Example: moving from a table to an equation is often carried out as moving from table to graph to equation.
--Indirect processes are substantially different from direct processes.
--Math programs develop exclusively "indirect" vers s of translations.

3) Translations (a) involve two representations, the source & the target; and (b) have the property of directionality, (i.e., going from a graph to an equation is not the same as going from an equation to a graph.)
--His research suggests that translation processes are best developed in symmetrical pairs.

4) The role of language: words play a central role in translations.
--Verbal tags are given to relevant elements and the translation is carried out through the efficient handling of those tags.

5) Application to the teaching of music.

6) In his opinion, the study of more complex math topics can be made more meaningful if approached from the translation framework.

7) Translation processes and curriculum design.

Kaput, J. J. (1987a, July). Linking representation in the symbol systems of algebra. Unpublished manuscript, Southeastern Massachusetts University, Department of Mathematics, North Dartmouth.

The author distinguishes between two notions of representations:

1) The notion of mental representation as the means by which an individual organizes and manages the flow of experience.

2.) The notion of representation system as a materially realizable cultural or linguistic artifact shared by a cultural or language community.

It should be noted that:

1) A mathematical representation system and a mathematical symbolic system are used synonymously.

2.) A distinction is made between a mathematical representation and a "natural" nonmathematical representation (i.e. language).

3) There is a direct analogy between the notions of symbol systems as a specifiable organizing structure and the idea of grammatical structure in natural language.

According to the author, there are 4 sources of meaning in mathematics, which are grouped into 2 complementary categories:

A) Referential Extension:

1. Via translations between mathematical representation systems.

2. Via translations between mathematical and non-mathematical representations (such as natural language, visual images, etc.)

B) Consolidation:

3. Via pattern and syntax learning through transformations within and operations on particular representaticn system.

4. Via mental entity building through the reification of actions, procedures and concepts into phenomenological objects which can then serve as the basis for new actions, procedures and concepts at a higher level of organization ("reflected abstraction")

According to the author, the third source is the most dominating in school mathematics.

The author does not assume an existence of an absolute meaning. For example, there are several different meanings for the mathematical term "function", and each family of meanings has its more congenial representations.

Kaput, J. J. (1987b). Representation and mathematics. In C. Janvier (Ed.), Problems of representation in mathematics learning and problem solving (pp. 19-26). Hillsdale NJ: Erlbaum..

The author argues that even though the idea of representation is continuous with the mathematics itself, the role of representations in the math curriculum is often underestimated. It seems as if the main focus in the first 8 years is on numbers, while we really only work with particular representations of numbers. Some of the properties are sensitive to the representation while others are independent of it (e.g. primeness of a number is an independent property while the simplicity of a graph is sensitive to the kind of scaling we choose to use).

The author describes a representation system as consisting of 2 worlds and an explicitly describable correspondence between them under which one is the representing world and the other is the represented world. There are aspects of the represented world that are being represented and aspects of the representing world that are doing the representing. He mentions 4 broad and interacting types of representations:

1) Cognitive and perceptual representation.
2) Explanatory representation involving models.
3) Representation within mathematics.
4) External symbolic representation.

He restricts his discussion to cases in which the worlds embody some kinds of mathematical structures (such as various number systems, vector space, sets of functions between number systems, etc.).

A *symbol system* is a symbol scheme S together with a field of reference F, and a systematic rule of correspondence c between them. A mathematical symbol system is a special symbol system in which F is a mathematical structure. In most cases, F itself is also a symbol scheme that can be taken to be the representing world in another symbol system. The *medium* - a physical carrier in which the symbol scheme, hence symbol system, can be concretely instantiated - plays a role in possibly creating confusion. *In practice, the syntax of a given symbol scheme is coordinated with its field of reference in order that the correspondence between them preserves certain attributes. Nonetheless, a given set of characters, and even certain rules for combining them, may participate in several different symbol systems and hence have different "meanings", i.e., referents. This situation is responsible for a whole class of symbol-use errors in mathematics, e.g., those having to do with the "=" character, which participates in several different, but related mathematical symbol systems.*
The author goes into a detailed analysis of a specific example of several symbolic systems involved in graphing algebraic equations.

Karplus, R. (1979). Continuous functions: Students' viewpoints. European Journal of Science Education, 1(4), 397-413.

Purpose: The purpose of this study was to identify the procedures by which secondary-school students deal spontaneously with data pairs that describe continuous functional relationships. Based on the hypothesis that functionality is a reasoning process, answers to the following three questions were sought:

1) What categories are needed for classifying students' responses to tasks requiring interpolation in a continuous non-linear relationship.

2) How is the distribution of student responses among the above categories dependent on the students' grade level and enrollment in math classes?

3) How effective is a brief demonstration of graphical interpolation in changing the students' behavior?

Sample: The sample consisted of 377 students from 6th to 12th grade, in a middle to upper-middle class community.

Method: The subjects were administered tasks by the author, during a regularly scheduled class. The students were given a booklet with written questions. The first was a Bacteria Puzzle. The students answered it, and handed in their solutions. A brief demonstration of a graphical solution to that problem by curvilinear interpolation was given and then the students were asked to answer the next two questions (Cylinder Puzzle and Spacecraft Puzzle).

In the first and third items, functionality was a dominant process. The second item was based on proportions.

Interviews with 37 high school students were conducted to explore the thinking and rationale behind the written answers.

Findings: The main categories identified for items 1 and 3 are:

Category I--Intuitive
Category SL Straight line
Category SC--Straight/curved line
Category C--Curved line.

The categories identified for item 2 are:

Category I--Intuitive
Category A--Additive
Category Tr--Transitional
Category R--Ratio

In the Functionality items, students seemed to tend to complete the graph by connecting two adjacent points with a straight line (SC). This tendency was smaller for older students.

The student responses distribution by categorizes are analyzed in the paper, to each of the three tasks, followed by a discussion of implications for instruction.

Kerslake, D. (1981). Graphs. In K.M. Hart (Ed.), <u>Children's understanding of mathematics concepts: 11-16</u> (pp. 120-136). Oxford, England: Alden Press.

<u>Purpose</u>: The purpose of the study (part of CSMS) was to investigate the important underlying ideas which are necessary components of the understanding of graphs in schools. These ideas were investigated at various levels of difficulty.

<u>Sample</u>: The sample consisted of 459 second year pupils (13 years old), 755 third year pupils (14 years old) and 584 fourth year pupils (15 years old) from a number of different secondary schools in England.

<u>Method</u>: A test was constructed that included items focusing on: block graphs and coordinates, continuous graphs, scattergrams (in which a decision has to be made on whether the plotted points should be connected and whether there is a meaning to points between those that are plotted), choice of axes and scales, distance-time graphs, gradient, parallel lines, and equations of straight lines. Some items were presented in unfamiliar forms, to prevent students from relying always on learned techniques

From all the items in the graphs test three groups of items have been identified (same method, as Kuchemann, 1984). Each group is called a level, and a child is assigned to the highest level in which s/he is successful on about two-thirds of the items.

The three levels are described below according to the tasks that were involved in the items included in them:

Level 1--plotting points, interpreting block graphs, recognition that a straight line represents a constant rate, and simple interpretation of scattergrams.

Level 2--simple interpolation from a graph, recognition of the connection between rate of growth and gradient, use of scales shown on a graph, interpretation of simple travel graphs and awareness of the effect of changing the scale of a graph.

Level 3--understanding of the relation between a graph and its algebraic expression.

<u>Findings</u>: The fourth year had more students at Level 3 than did the two younger age groups, and fewer at level 1. The easiest items were almost equally well done by all age groups. On the most difficult items there was little difference between the two younger age groups.

The data suggests that many aspects of graphs are well within the capability of secondary students. However, there appears to be a large gap between the relatively simple reading of information from a graph and the appreciation of an algebraic relationship.

Some implications for instruction are discussed.

Kuchemann, D. (1984). Stages in understanding algebra. Journal of Structured Learning, 8, 113-124.

Purpose: The purpose of the study (part of CSMS) was to help teachers and curriculum developers improve the match between the cognitive demand of what children are taught in the area of generalized arithmetic (the use of letters) and individual children's levels of understanding.

A Piagetian framework was adopted in developing tests that served as research and diagnostic instruments.

Six categories for describing children's interpretations were formulated, before the final version of the test was designed:

1) letter evaluated
2) letter not used
3) letter as object
4) letter as specific unknown
5) letter as generalized number
6) letter as variable

Sample: The sample consisted of 1128 13-year-olds, 961 14-year-olds, and 731 15-year-olds.

Method: The research instrument was a written test containing items in each category. The first method that was tried was based on item characteristic curves. The measure of association chosen was the phi coefficient, which was used in two ways: "spider diagrams" and factor analysis. The consistency of levels was assessed by using Guttman scalogram analysis.

The remaining items, after rejecting those with a relative lack of consistency, were classified into four levels (stages).

Results: The findings suggest that the algebra test measures a .airly unified aspect of children's understanding of generalized arithmetic. The four algebra levels appear to have some correspondence to Piaget's descriptions of concrete and operational thinking. (see Kuchemann, 1981, for more detail).

Leinhardt, G., Stein, M.K., & Baxter, J. (1988). What a difference a class makes. Unpublished manuscript, University of Pittsburgh, Learning Research and Development Center, Pittsburgh PA.

The research reported in this paper compares and contrasts the instruction of one teacher in two different, back-to-back, fifth-grade mathematics classes. Focused on the presentations of eight lessons on the topic of functions and graphing, this study examines how Mr. Gene taught the same new material under two slightly different sets of circumstances. This research aims to identify similarities and differences in teaching that occurred between the two groups, to understand why differences occurred, and to determine what effects the differences might have had. In addition, the researchers attempt to detect when the teacher made adjustments in instruction for one group or the other and to explore the nature of those adjustments.

The teacher divided his class overall mathematics class of 27 students into two ability groups, with 14 in one and 13 in the other. The former group, considered to be comparatively brighter, had class immediately before the latter group. The groups are referred to as the higher group and the lower group, respectively. The lessons were videotaped and then transcribed in a format that separately shows the teacher's talk, the teacher's action/demonstration, and the students' talk and action. Pre- and post-class interviews with the teacher were also conducted and transcribed, providing supplementary data. These data were subjected to four levels of analysis in an attempt to find a level at which each difference could be most effectively measured. All variables that might indicate differences between the two presentations were identified and categorized by level.

Based on analyses of the data at the more global levels, the investigators conclude that there were more similarities between the classes than there were differences. In those areas where differences were found (e.g., the low group received more lines of protocol whereas the high group received higher numbers of subject-matter-based exchanges), questions were raised regarding the actual content of the instructional episodes. The more fine-grained analyses of the instructional content suggest that Mr. Gene made presentations of greater precision and intellectual clarity to the higher group. Although these differences were subtle, the authors argue that if a pattern is established (whereby, for example, the high group is consistently exposed to more key concepts which are connected to core material in more precise ways), then differences in student learning might be expected. Possible rationales for these behavior patterns are being explored.

Mansfield, H. (1985). Points, lines, and their representations. For the Learning of Mathematics, 5(3), 2-6.

This article deals with misconceptions. It is primarily a study of language confusion, rather than one of mathematical thinking. The author points out properties of graphs that make them unique as a symbol system. She discusses points, lines, and their representations. She explores, through discussion, the ways in which 7th graders and college seniors (teachers) think about points and lines.

Findings:

-Both groups think of points and lines in significantly different ways than do mathematicians

-In general, did not distinguish between the abstract concepts and the physical representations of points and lines.

-Misconceptions were manifested in following ways:

*Lines have width

*Points are entities added to lines

*Points have a definite size and shape

-Their mistaken ideas are based on:

1) the common representation of points and lines in textbooks and instruction

2) everyday uses of the terms

Conclusion:

If students do not distinguish between abstract concepts and their physical representations, then their concepts will include features of those physical representations that are not part of the concepts as used by mathematicians. Instruction needs to take this into account.

Markovits, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. For the Learning of Mathematics, 6(2), 18-28.

Purpose: The purpose of the study was to investigate junior high-school (9th grade) students' understanding of the concept of function as well as their difficulties and misconceptions. The concept of function was learned through the set definition.
Three "sub-concepts" that comprise a function, domain, range, and rule of correspondence (many to one or one to many), were addressed in the study.
Of the four common forms of representation of a function (verbal, arrow diagram, algebraic, and graphical) the study is restricted to two: algebraic and graphical.

Subjects: The subjects were about 400 9th grade students (ages 14-15) in Israel.

Method: Items (problems) were designed to assess eight components of understanding the concept of function. All functions were "abstract," involving no context or situations.

Results: The results were based on written solutions to the problems. The main findings were:

1) Three types of functions caused difficulty: the constant one, piecewise, and discrete.
2) Translation from graphical representation to algebraic was more difficult than vice versa.
3) The variety of examples in the students' repertoire was limited. There was an excessive adherence to linearity.
4) In both representations the concepts of preimage and image were only partly understood.

Preece, J. (1983). Graphs are not straightforward. In T.R.G. Green & S.J. Payne (Eds.), The psychology of computer use: A European Perspective (pp. 41-56). London: Academic Press.

Purpose:: The purpose of the study was to investigate how much of the information presented in a graph pupils can interpret in practice.

Sample: The sample consisted of 122 pupils, 14 & 15 years old

Method: Pupils were given specific tasks that required them to interpret multiple curve graphs, all representing a situation (changes in populations of pond organisms). The graphs are presented on a computer display. A computer simulation program was used for this purpose. Think-aloud protocols were collected for a few of the subjects.

An analysis of what interpreting such a graph involves was carried out, involving the following features:

1) Domain specific concepts (about the situation)
2) Graph concepts
3) The number of graphs, of grouping of the curves, of dependent variables (and type of variable)
4) The absence of scales
5) The axes--what do they represent?
6) The actual syntax of the graph

Results: Findings point to 12 different sources of errors:

1) Reading and plotting points
2) Relative reading
3) An interval is interpreted as a point (answers to "when?")
4) Concepts arising from the variables were not understood
5) World knowledge interferes
6) A pronounced graphical feature distracts the pupil
7) Misinterpretation of a symbol
8) Confusion caused by too many variables
9) Only one curve on a multiple curve display is interpreted
10) Failure to transfer information between graphs
11) Gradients are confused with maximum and minimum
12) The graph is interpreted iconically

There are three categories of error:

1) Cues--errors caused by language or graph (#3, 5, 6, 12)
2) Reading errors--(# 1, 2, 8, 10, 9, 12)
3) Conceptual errors--(# 3, 4, 7, 11, 12)

More details on this can be found in Preece, 1984b.

Preece, J. (1984a). A study of pupils' graph concepts with a qualitative interactive graph sketching program. Computer Education, 8(1), 159-163.

The paper discusses a pilot study in England which investigated 4 pupils (3 fourth-year boys aged 14 and 1 fifth-year girl aged 14) working with an interactive computer program called SKETCH. This program has been developed to research into how pupils develop concepts of gradient (slope) through sketching and interpreting graphs.

Four kinds of data were collected:

1) Pupils' answers to a written test (about a time/temperature graph)
2) An audio cassette recording of the complete session (about 50 minutes) with each pupil
3) The pupils' sketches which were stored on disc
4) Notes of interesting observations and pupils' comments

The data suggest that the program provides a motivating and flexible environment in which pupils gain an intuitive understanding of gradient by becoming involved in a series of simple sketching (modelling) and interpreting activities.

Preece, J. (1984b). Some theoretical issues related to interpreting graphs. (CAL Research Group Tech. Rep. No. 51). Institute of Educational Technology, The Open University, Walton Hall, Milton Keynes, England.

This is a theoretical overview based on research findings that have been reported, most of which include studies in which the author was involved.

Main findings across studies:
1) Students tend to interpret graphs representing situations iconically (context interferes)
2) Students have 2-3 different notions of a gradient, and apply the one which fits best the context and the graph
3) Simulated display influenced the development of pupils' notions (some physically based)
4) Students brought a lot of irrelevant contextual knowledge

In all findings, context appears to strongly influence students' interpretations.

The author discusses how the above findings relate to the following theories:
1) Piagetian theory
2) Alternative conceptions and frameworks (in science education)
3) Mental models

Schoenfeld, A.H. (1987). On having and using geometric knowledge. In J. Hiebert (Ed.), Conceptual and procedural knowledge: The case of mathematics (pp. 225-264). Hillside NJ: Erlbaum.

The generic issue addressed in this chapter is: "*What, if anything, will assure that a student (a) becomes fluent in performing the symbolic manipulations in that domain, and (b) comes to understand the deep meaning of the mathematical notions represented by the symbols and the procedures that are used to operate on them?*" (page 227).

In general there are two phases:

Phase 1 - *Concrete representations serve as a vehicle for making sense of abstract symbolic operations.*

Phase 2 - *Abstracting over the isomorphic aspects of the two symbolic systems leads to an understanding of the underlying mathematical notions and procedures.*

A concrete representation, which serves as a *reference world* (W) has a *dual nature*: The objects in W are meaningful entities and their attributes and the operations on them are embedded in that broad meaning structure. But W can be thought of as a symbolic system, because it is a representation of mathematical entities and procedures (those of the symbolic world, w).

The symbolic system (w) also has a dual nature: It is a symbolic representation of a mathematical idea (perhaps more abstract), and it also is an abstraction of some aspects of W.

Two assumptions:

1) {Understanding meanings, operations and interrelationships within objects in W} + {Understanding the nature of the mapping from W to w} => {Understanding of the symbols and procedures in w}. (phase 1)

2) {Understanding the structures of different symbolic systems that represent the same mathematical idea} + {Identifying and perceiving the underlying structural morphisms between these systems} => {Abstracting the underlying mathematical idea}. (phase 2)

Complicating factors:

1) A dilemma between the importance of having the concrete representation as natural and reasonable as possible and the danger of being too natural to be able to see through it.

2) An isomorphism from W to w at level of syntax may not extend to an isomorphism at the level of attributed meaning and coherence (page 236).

3) Usually there is not one particular representation that captures all aspects of the mathematical idea.

Schoenfeld, A. (1988). GRAPHER: A case study in educational technology, research, and development. Unpublished manuscript, University of California, Berkeley CA.

The paper deals with a computer-based microworld called GRAPHER and the kind of learning and understanding it enhances. The goal of this study was to provide the "cognitive support structure" to help students learn about functions and graphs.

As a starting point, the paper presents an analysis of what it means to understand graphs of straight lines. These aspects of understanding are very complex and include: general geometric knowledge about lines, knowledge of algebra, equation solving, functions and their algebraic representations, knowledge about the Cartesian plane, knowledge of the relationships between functions and their graphs, knowledge about conventions, being able to think simultaneously about different meanings of the graph of y=mx + b, knowing different algebraic representations for straight lines, and having the interconnectedness knowledge of the above.

GRAPHER was designed, keeping the following six principles in mind:

1) To facilitate "conceptual concretizing".
2) To exploit, whenever possible, the dynamic and interactive nature of the computational medium to enable students to operate "directly" on conceptual entities and gain meaning.
3) To make overt the links the students should see.
4) To allow students to focus on conceptual issues without having to worry about technical work.
5) To support "meaning exploration" rather than "knowledge telling".
6) To encourage students' reflection on their understandings.

A learning session with a student working with GRAPHER illustrates the very localized, disconnected, and fragile nature of knowledge structures of a particular subject matter and the ways they grow and change.

Stein, M.K., Baxter, J., & Leinhardt, G. (in press). Subject matter knowledge for elementary instruction: A case from functions and graphing. American Educational Research Journal.

The purpose of this article is to examine the role and function of subject-matter content knowledge in teaching. Until recently, research on the relationship between teachers' subject-matter knowledge and their students' learning has not been very extensive or useful. By focusing specifically on the relationship of one fifth-grade teacher's subject-matter knowledge to several aspects of his classroom instruction, the authors attempt to define and articulate how subject-matter knowledge influences what is taught and how it is taught.

The teacher, called Mr. Gene, had taught elementary mathematics for 18 years. He was interviewed on both his knowledge of the topic and knowledge of how to teach that topic, plus other relevant items. From these interviews, the results of a card sort task, and videotapes of his teaching, the authors developed a characterization of Mr. Gene's knowledge base, a description of the four lessons, and an analysis of their interrelationships.

The results of the interview and card sort task suggest that Mr. Gene's understanding of functions and graphing was less developed than the math experts' understanding. Furthermore, limitations in the teacher's knowledge were found to relate to conceptual holes and missed opportunities in his classroom presentations. For example, Mr. Gene's arrangement of the cards in the card sort task suggested a knowledge base that was not elaborately differentiated or hierarchically organized in terms of mathematically powerful criteria. Most noteworthy, the fact that graphs and equations are alternate representations of functional relationships appeared to be missing from his understanding. Similarly, the authors' analysis of his instruction shows that he failed to develop some important conceptual relationships, including an appreciation of the special relationship between functions and graphs that was a key point of the third and fourth lessons.

For each problem area identified in Mr. Gene's instruction, the authors are able to draw connections between what was or was not presented in the classroom and Mr. Gene's idiosyncrasies or missing pieces in his knowledge base. They conclude that such gaps in teachers' knowledge lead to conceptual gaps in instruction and offer specific suggestions for ameliorating such deficiencies in both the knowledge base and teaching. Their findings also suggest that teachers' beliefs regarding the importance and purpose of a specific mathematical topic have a direct influence on the degree to which aspects of a topic are emphasized and on whether they are included in instruction. This research points to the need for more detailed studies of the levels and kinds of subject-matter knowledge that can support mathematical instruction as a necessary step toward determining what subject-matter understanding new and practicing teachers should possess.

Stein, M.K., & Leinhardt, G., (1989). <u>Interpreting graphs: An analysis of early performance and reasoning</u>. Unpublished manuscript, University of Pittsburgh, Learning Research and Development Center, Pittsburgh PA.

This paper begins with the observation that, although society deems graphing skills to be important, the research community knows surprisingly little about how children come to understand and effectively use graphs. The purpose of the study on which the paper was based was two-fold. First, the authors aimed to identify the areas of graphing knowledge and skills with which elementary students have the most and the least degree of difficulty. Second, they attempt to identify the ways in which children reason about those topics identified as more difficult.

The data base for the study consisted of 10 elementary students' responses to interviews on the topics of functions and graphing. An interviewer administered a series of multi-step problems and asked the students to talk aloud as they were solving the problems or to explain how they had arrived at an answer. The interview problems related to four main areas of graphing knowledge and skills: (1) the plotting and reading of individual points; (2) the construction of Cartesian coordinate systems and graphs; (3) the relative (global) interpretation of graphs; and (4) translating between graphs and other representational systems of functional relationships (i.e., ordered pairs and function rules) . After the interviews were transcribed, two kinds of analyses were performed: a global scoring of the students' performance on the entire set of interview items and a more detailed analysis of the students' reasoning processes as they worked through those problems which the first analysis suggested were the most difficult.

With respect to the students' overall performance, the results suggested that the students had little difficulty with the reading and plotting of points or with the construction of Cartesian coordinate systems. In addition, they could perform translations that required them to move <u>from</u> ordered pairs or function rules <u>to</u> graphs. Along with the relative graph interpretation problems, however, translations that proceeded <u>from</u> graphs <u>to</u> function rules proved to be difficult. Thus, the authors conducted a more detailed examination of those tasks requiring relative graph interpretation skills and those requiring translations that used graphs as their initial input.

The results of the analysis of data relating to relative graph interpretation skills suggested that students were more likely to be successful when the task explicitly drew their attention to the variables represented on the axes. In addition, two distinct styles of learning to interpret graphs in a relative fashion seemed to emerge. Some students initially focussed on plots of specific x,y pairs which they then "decoupled" (i.e., pulled apart the x coordinate and the y coordinate) in order to separately track first the x values and then the y values. Others began with spatial scanning of the entire line in order to assess its overall shape and direction. The authors termed these two strategies, both of which were successful, bottom-up and top-down, respectively.

The results of students' responses to a task in which they were asked to provide a rule for a graph revealed that the majority of successful responses involved deducing the rule from one specific x,y pair and then checking it against a second ordered pair. A second, less-frequently used method involved comparing the changes in the x variable with the changes in the y variable in a relative fashion. The authors suggest that the first method resembled a bottom-up strategy because it began with a specific point and worked toward an answer that took into account the entire line, whereas the second method resembled the top-down strategy because it began with an examination of the behavior of the two variables and moved toward a specific numeric answer.

The authors interpret their findings about the top-down and bottom-up strategies in light of recent debates regarding optimal methods and sequences for elementary graphing instruction. They argue that both strategies are valid learning strategies and that researchers should focus on designing instructional methods that complement or build on both styles.

Vinner, S. (1983). Concept definition, concept image and the notion of function. International Journal of Mathematics Education in Science and Technology, 14(3), 293-305.

A simple model for cognitive processes is constructed using the notions of concept image and concept definition. The model is used to analyze some phenomena in the process of the learning of the function concept.

Sample: The sample consisted of 65 tenth grade and 81 eleventh grade students in Israel.

Procedure: Students were given a questionnaire with 5 questions. The first four gave a description (3 verbal and 1 graphical) of a set of conditions (properties), and the students were supposed to decide whether or not there exists a function fulfilling those conditions. The students had to choose between *yes* and *no* and to write an explanation for the choice they made. The fifth question was: "In your opinion, what is a function?" The first four questions were constructed to get an idea about students' concept images regarding a function, while the last one examined their concept definition.

Findings: With respect to students' concept images: some concept images were not consistent with the textbook definition (though sometimes consistent with the student's own definition). Some of those concept images are:

1) A function should be given by one rule.
2) A function can be given by several rules relating to disjoint domains providing these are half lines or intervals.
3) Functions (which are not algebraic) exist only if mathematicians officially recognize them (e.g., by labeling them).
4) A graph of a function should be "reasonable."
5) A function is a one-to-one correspondence.

Four main categories were distinguished in respect to students' response to the fifth question:

1) The textbook definition, basically..............(57%)
2) The function is a rule of correspondence.......(14%)
3) The function is an algebraic term..............(14%)
4) Some elements in the mental picture are
taken as a definition for the concepts ....(7%)

Implications for teaching are discussed.

Wagner, S. (1981). Conservation of equation and function under transformations of variable. Journal for Research in Mathematics Education, 12(2), 107-118.

Purpose: The intention of the study was to examine the extension of Piaget's theory of conservation to relational concepts. The study also looked at the "students' ability to conserve equation and function under alphabetic transformations of literal variables." Conservation is defined as the understanding that "the critical attribute, the essence of the concept, is invariant under transformations of certain irrelevant attributes."

Sample: The sample consisted of 30 middle and high school students. Half of the sample was girls and 15 of the students were in middle school. The median age for the middle school sample was 13 and the median age for the high school sample was 16 1/2.

Procedure: Each student was interviewed for about 20 minutes during school hours. The instrument consisted of a series of equations and function charts in which the variables had been interchanged. The students were asked to compare the statements and determine whether or not the change in variable symbol affected the referent of the variable.

Findings: A significant association was not found between age and the ability to conserve either equation or function. However, a significant association was found between the ability to conserve and having completed at least one semester of algebra. Also, "less than half of the students interviewed gave conserving responses to any one of the four tasks."

Conclusions/Implications: First, the study is very limited because of the size of the sample and the nature of the pool. Second, each type of conservation task was only presented once.

The results of the study pointed out two common misconceptions that are often held about functions: "(a) that changing a variable symbol implies changing the referent and (b) that the linear ordering of the alphabet corresponds to the linear ordering of the number system."

The relationship between mathematical background and the ability to conserve suggests that training is an important factor in conservation. The study results also support the theory that one can attain formal operational thought in some areas of cognitive ability but not necessarily all.

Wavering, M.J. (1985, April). The logical reasoning necessary to make line graphs. Paper presented at the annual meeting of the National Association for Research in Science Teaching, French Lick Springs, Indiana.

Purpose: The purpose of the study was to determine the logical reasoning processes necessary to construct line graphs. Three types of line graphs were used: a straight line with a positive slope, a straight line with a negative slope, and an exponentially increasing curve.

Sample: The sample consisted of students in math and science classes in middle and high school.

Method: Three research instruments were used, one for each type of line graph. Each research instrument consisted of a set of instruction, data to make a graph, and two unlined pieces of paper. No time constraints were set for accomplishing the task. Students were required to plot a graph and also to account for their thinking process in writing.

The responses were classified into one of nine categories. The categories ranged from no attempt to make a graph, to a complete graph stating the relationship between the variables. The categories in between represented increasingly more successful attempts at ordering data in one or both variables to correct scaling of the data on the axes.

Results: Middle school subjects exhibited behavior mainly in the first four categories. Ninth and tenth graders overlapped with middle school and highschool subjects, and 11th and 12th graders exhibited behaviors mainly in the last five categories. The author suggests that the reasoning that took place could be characterized in Piagetian terms. The response categories appear to be valid with respect to the three types of graphs.

Classroom Learning and Instruction Project (CLIP)

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CLIP-89-01. Stein, M. K., Baxter, J., & Leinhardt, G. (1989). Teacher subject matter knowledge and its relationship to classroom instruction (Tech. Rep. No. CLIP-89-01). Pittsburgh, PA: University of Pittsburgh, Learning Research and Development Center.

CLIP-89-02. Grover, B. W., Zaslavsky, O., & Leinhardt, G. (1989). An approach to the design and development of a scoring system for a new teacher assessment: The semi-structured interview (Tech. Rep. No. CLIP-89-02). Pittsburgh, PA: University of Pittsburgh, Learning Research and Development Center.

CLIP-89-03 Leinhardt, G., & Ohlsson, S. (1989). Tutorials on the structure of tutoring from teachers (Tech. Rep. No. CLIP-89-03). Pittsburgh, PA: University of Pittsburgh, Learning Research and Development Center.

CLIP-90-01 Leinhardt, G., Zaslavsky, O., & Stein, M.K. (1990). Annotated bibliography of selected articles on graphing and functions. (Tech. Rep. No. CLIP-90-01). Pittsburgh, PA: University of Pittsburgh, Learning Research and Development Center.