DOCUMENT RESUME

ED 341 560 SE 052 488

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TITLE Workshop in Al Kalfus: History of Mathematics, Problem Solving, Enrichment. Dissemination Packet--Summer 1989: Booklet #7.
INSTITUTION Hofstra Univ., Hempstead, NY. Dept. of Mathematics.; Hofstra Univ., Hempstead, NY. School of Secondary Education.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 89
CONTRACT TEI8550088,8741127
NOTE 53p.; For related documents, see SE 052 482-490.
PUB TYPE Guides - Classroom Use - Teaching Guides (For Teacher) (052) -- Tests/Evaluation Instruments (160)

EDRS PRICE MF01/PC03 Plus Postage.
DESCRIPTORS Higher Education; High Schools; *Inservice Teacher Education; *Mathematical Concepts; *Mathematics Education; *Mathematics Teachers; Program Descriptions; Secondary School Mathematics; Secondary School Teachers; Teacher Education Programs; Teacher Workshops
IDENTIFIERS *Hofstra University NY

ABSTRACT

This booklet is the seventh in a series of nine from the Teacher Training Institute at Hofstra University (New York) and synthesizes the contribution of the late mathematics educator, Alfred Kalfus, to the institute as adviser, guest lecturer, and instructor. Descriptions of the enrichment topics for secondary school mathematics included in his course are presented with problems and some solutions, namely: infinity, non-Euclidean geometry, mathematical induction, graphing and curve sketching, geometrical transformations, the Cantor set, Zeno's paradox, and more. A bibliography and a starter list of topics for math fair presentations are also included. (JJK)

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HOFSTRA UNIVERSITY

TEACHER TRAINING INSTITUTE

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DISSEMINATION PACKET – SUMMER 1989

Booklet #7

JANET BARBERA, DAN DRANCE, AL KALFUS AND DAVID KNEE
WORKSHOP IN AL KALFUS:
HISTORY OF MATHEMATICS, PROBLEM SOLVING, ENRICHMENT

NSF Grant # TEI8550088, 8741127

This booklet is the seventh in a series of nine booklets which constitute the Hofstra University Teacher Training Institute (TTI) packet. The Institute was a National Science Foundation supported three-year program for exemplary secondary school mathematics teachers. Its purpose was to broaden and update the backgrounds its participants with courses and special events and to train and support them in preparing and delivering dissemination activities among their peers so that the Institute's effects would be multiplied.

This packet of booklets describes the goals, development, structure, content, successes and failures of the Institute. We expect it to be of interest and use to mathematics educators preparing their own teacher training programs and to teachers and students of mathematics exploring the many content areas described.

The late mathematics educator, Alfred Kalfus, contributed to the TTI as advisor, guest lecturer, and instructor. He lectured in the "History of Mathematics" course on his favorite overarching themes in the development of mathematics and gave his own course on "Enrichment Topics in High School Mathematics" in Cycle II. This booklet presents descriptions of topics from these courses, such as Non-Euclidean Geometry, Mathematical Induction, Graphing, Geometrical Transformations, Infinity and many others. It also presents many of Al's favorite problems along with some solutions and a reminiscence by one of his former students and coworkers.

TEACHER TRAINING INSTITUTE

Workshop in Al Kalfus: History of Mathematics, Problem Solving, Enrichment Topics

Booklet #7

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Contents

1. Introduction

Al contributed to the Teacher Training Institute as an advisor, guest lecturer and instructor. He lectured in The History of Math course in Cycle I and in Cycle II expanded those lectures (summer of 1987) and continued with an Enrichment Topics course in the fall. We present Al's description of these classes, and a summary from the notes of participant Janet Barbera. We are glad that Al had a chance to start work with Janet on these notes before his untimely death this past spring.

The reader will notice the great amount of ground covered by Al. Al loved teaching mathematics, he loved his colleagues and his students and held nothing back. He thought highly of those he taught. Of the enrichment topics he exposed his high school students to he told us, "We always assume this is too hard. But you'd be surprised, if you get the kids interested." and again, "The kids are smarter than us, let's face it. Don't feel bad - we're here to motivate them."

He liked the fact that in his high school classes he had the kids for a whole year because more of them were likely to start sharing his excitement and start working hard and achieve a deeper understanding of mathematics. He went beyond the given curriculum in content and in ways of presenting and solving problems; in the fall the going was slow, but by spring it usually paid off. "I want to be original and you should be too. When a kid does something a different way, I make a big deal out of it. It's the different thinkers who may discover something new. These are the imaginative ones, the creatives ones. Encourage creativity...Go beyond the text. To heck with the text!"

He wanted his students to question, to work it out themselves, and not just accept something on the teacher's say-so. "Sometimes I make a false statement and continue as if it's true. This teaches kids to question and trust their insights...It's great fun when students disagree. Wake them up! The kids are fascinated by these things."

Al felt it was important to encourage all math students to do a research paper. Once he had a know-it-all in his class who challenged Al's statement that there are different infinities, some larger than others. "Whadaya mean? That's baloney!", the youngster shouted out indignantly. Al told him to check it out, maybe Cantor <u>was</u> wrong and he suggested some books to look at. The student became interested and eventually gave an excellent Math Fair presentation on Set Theory.

One of Al's favorite topics was graphing the easy way through the use of transformations and the study of families of curves. Included here is Dan Drance's essay on some of these techniques and also Dan's personal reminiscenses of Al.

2. An Outline of Al's Workshop.

The workshop lectures offer creative approaches to topics in three primary areas:

First, key developments in the history of mathematics are analyzed in detail and then related to appropriate topics in the secondary school curriculum. This approach reveals the pertinence of the history of mathematics to the teacher's classroom pedagogy, and at the same time presents the kind of material that may be used to make the subject matter more appealing to the student. Topics such as Euclid's Fifth Postulate and its consequent impact on the development of non-Euclidean geometry, Infinite Sets and Transfinite Numbers, and Zeno's Paradoxes offer entertaining reliefs from the routine rituals of typical mathematics classrooms.

Second, challenging problems based on material covered in the lectures are propsoed. These problems are both within the scope of the curriculum and tangential to it. The teachers are given the opportunity to solve the problems, at home or in class, and then various solutions are discussed. Non-routine techniques are presented and emphasized. The objective here is to include the idea that there are many ways to approach a problem, some more elegant than others, and whatever way the student manages to find a correct solution is to be commended. Of course, the simple, elegant solution should always be encouraged. Here, once again, is an area where mathematics can be fun. We must promote and exploit its potential for drawing the students into a personal involvement with mathematics.

Third, innovative graphing and curve-sketching techniques are presented. Stress is placed on the importance of methods which are too often overlooked or underemphasized in the usual curriculum. Thus, the importance of transformations is demonstrated, especially the use of translations and the effects of changing the frame of reference. Standardizing the format of difficult or strange equations is shown to reveal patterns which make the equations easy to recognize and graph. Sketching polynomial and rational functions is simplified by analyzing multiplicities of roots and multiplicities of the factors which produce vertical asymptotes. Symmetry, especially in the lines $y=x$ and $y=-x$, is studied as another powerful tool which must be used more effectively in secondary school classrooms. Rotations using complex numbers are also presented as a graphing technique. These various methods, as well as other non-routine approaches, are used to solve some unusual problems. The purpose of this analysis is to demonstate that these concepts and techniques are useful not only to help visualize the assocaited functions and relations by sketching their graphs, but also to gain a better understanding of the abstract notions they represent.

Course Outline

Many topics also included historical material and suggestions for classroom presentations and student research.

A. Real Number System

1. rationals/irrationals, algebraics/transcendentals

2. historical development of the number system

3. problems: proving given expressions to be irrational

4. 3 impossible constructions of antiquity

*B. Cantor's Set Theory

1. 1-1 correspondence, cardinality

2. $\aleph_0$, c

*C. Zeno's Paradoxes

*1. Achilles and the tortoise

2. limits, summing infinite series

*D. Non-Euclidean Geometry

1. Euclid's Elements

2. deductive systems: axioms, definitions, logic, proof, theorems

3. what is truth? Bertrand Russel's definition of 'mathematics'.

4. consistency, independence, completeness

5. Euclid's fifth postulate

6. Lobachevsky & Hyperbolic Geometry, Reimann & Elliptic Geometry; Felix Klein, Einstein

E. A Unifying Thread: Sets, Logic, Pascal's Triangle, Powers of 2, the Tower of Hanoi, and Mathematical Induction

1. demonstrate a unifying thread in the high school mathematics syllabus, show the similar structures of logic and set theory, and the nature of isomorphism, emphasize the importance of patterns and the utility of Pascal's Triangle, and introduce a variety of proof techniques including mathematical induction

2. the number of subsets of a set

3. tree diagrams, Pascal's triangle, subsets of a set, and powers of 2

4. truth tables, membershp tables, Venn diagrams, tautologies

*5. The Tower of Hanoi and powers of 2

F. Figurate Numbers and Mathematical Induction

1. the search for patterns in mathematics, non-routine problem solving

2. elementary derivations of:
$1+2+3+...+n = n(n+1)/2$ and

$1+3+5+...+(2n-1) = n^2$

3. more advanced derivations of:

$1^2 + 2^2 + 3^2 +...+ n^2 = n(n+1)(2n+1)/6$ and

$1^3 + 2^3 + 3^3 +...+ n^3 = [n(n+1)/2]^2$

and utility of this in demonstrating Archimedes' approach to finding the area under a curve

G. Graphs and Transformations

1. minimizing arithmetical procedures, avoiding tables

2. standard equations and cannonical forms:
linear functions, absolute value, quadratics, the conics, polynomials, rational functions, irrational and other messy equations

3. curve sketching techniques: transformations, graphs and their inverses, symmetry, asymptotes and limits

H. Graphs and Problems Solving

*1. using graphs to help visualize a problem, Diophantine equations

2. rotations of graphs, complex numbers

3. translations and multiple roots, finding maximum and minimum of a cubic without calculus

4. symmetry

5. inequalities

*I. Problem Solving

1. non-routine problems used to motivate students

2. the search for patterns

3. solving geometrical problems by "extending the figure"

4. combinatorics

J. Mathematical Systems

1. groups, rings, fields

2. finite and infinite systems, familiar and abstract operations

3. application to polynomials - solving equations

4. isomorphism

5. permutation group, transformations of an equilateral triangle, complex numbers, 2 X 2 matrices

K. Geometric Transformations

1. transformations may be used to simplify proofs and clarify theorems in geometry: the sum of the angles of a triangle, the base angles of an isosceles triangle, a line joining the midpoints of two sides of a triangle

*Capsule lessons or problems provided in the following pages.

3. A Sampling of Topics.

Cantor's Set Theory

DEFINITIONS:

1-1 CORRESPONDENCE: A correspondence between two sets for which each member of either set is paired with exactly one member of the other set.

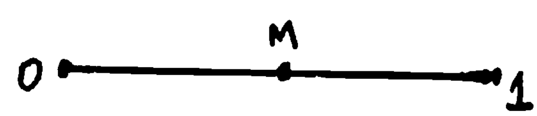
CARDINALITY OF A SET: A number which designates the manyness of a set of things, the number of units, but not the order in which they are arranged.

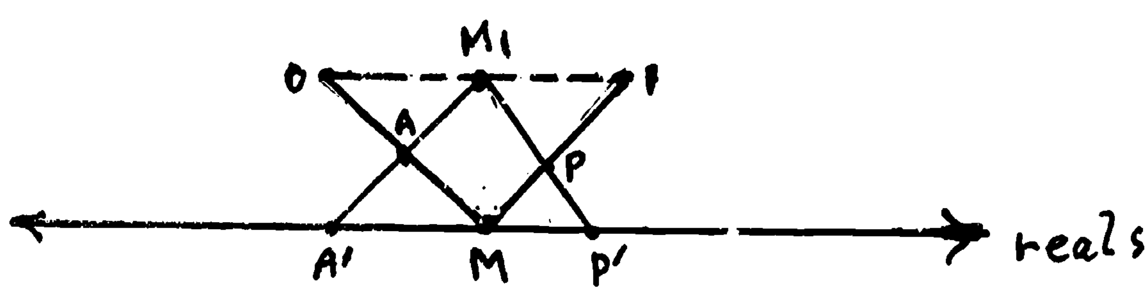
DENUMERABLE SET: An infinite set whose elements can be put into 1-1 correspondence with the positive integers: a countable set.

INFINITE SET: A set which can be put into a 1-1 correspondence with a proper subset of itself.

Problems and Solutions:

1. Prove the segment of reals 0 - 1 has the cardinality of 2 more than the entire real number line.

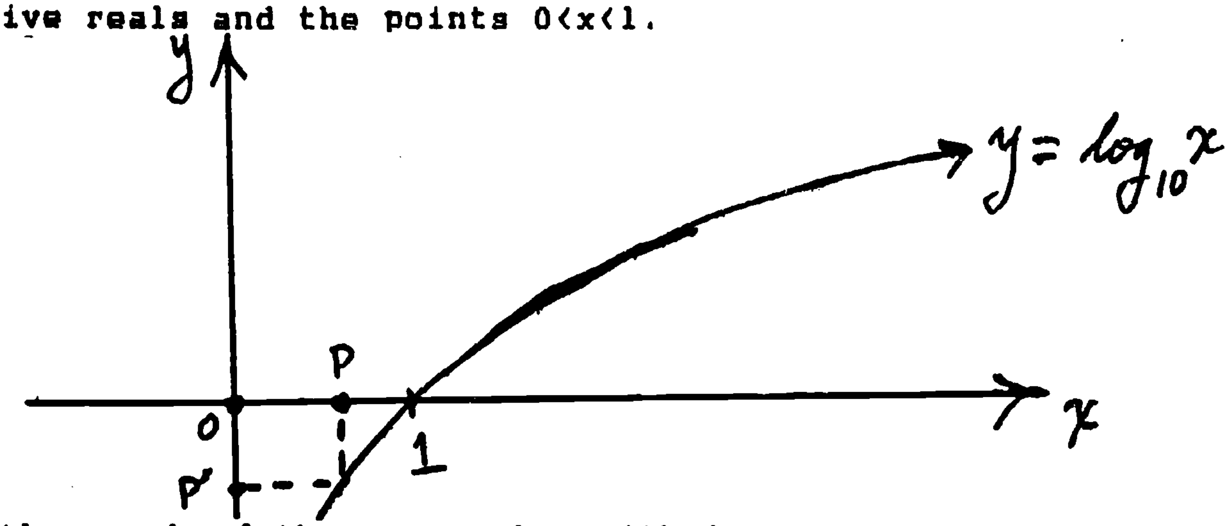




Bisect the segment 0 - 1 at point M. Construct the triangle 0-M-1 by bending at point M. Construct the midpoint M1 of segment 0-1. Draw the real number line parallel to the segment 0-1 and passing through point M.

A 1-1 correspondence is established between the points of the segments 0-M and M-1, such as A, and the points on the real number line, such as A', by drawing lines from point M1 through point A to its image at A'. Similarly the image of point P is found to be point P'. No image points are possible for the two endpoints 0 and 1, however, so the cardinality of the segment 0-1 is equal to the cardinality of the real number line + 2.

2. Prove there exists a 1-1 correspondence between the negative reals and the points 0<x<1.



Consider the graph of the common logarithmic function. The logarithm of any value 0<x<1 will be the desired negative number image.

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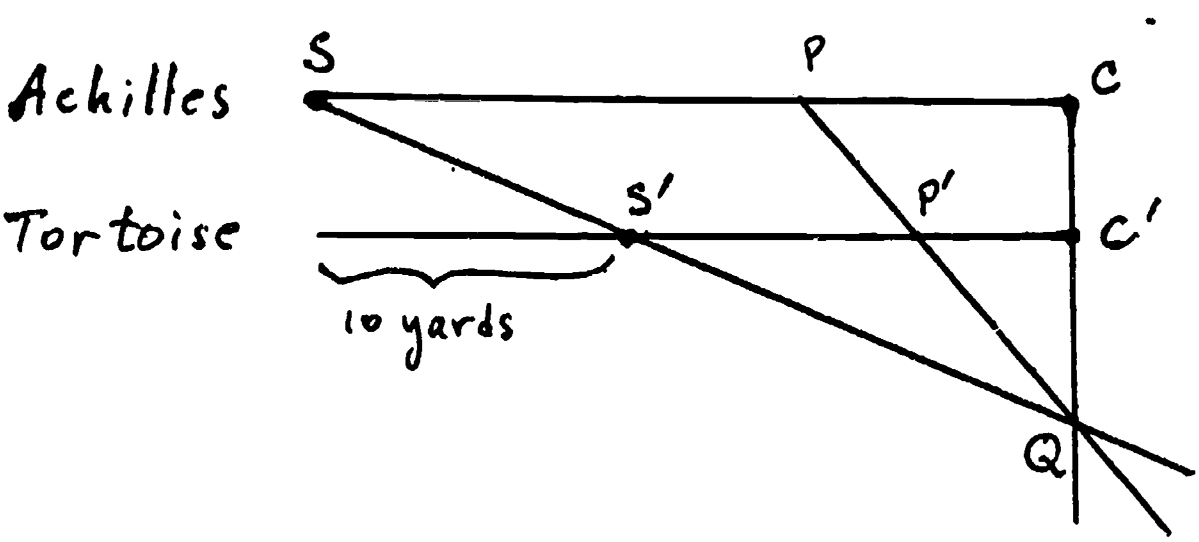
Zeno's Paradox of Achilles and the Tortoise

In one version of this paradox, Achilles and the tortoise run a race in which the slow tortoise is allowed to start from a position that is 10 yards ahead of Achilles' starting point. It is agreed that the race is to end when Achilles overtakes the tortoise. At each instant during the race Achilles and the tortoise are at some point of their paths, and neither is twice at the same point. Then, since they run for the same number of instants, the tortoise runs through as many distinct points as does Achilles. On the other hand, if Achilles is to catch up with the tortoise he must run through more points than the tortoise since he has to travel a greater distance. Hence, Achilles can never overtake the tortoise.

Resolution of the Paradox:

Draw segments representing the distance traveled by Achilles and the tortoise, allowing for a 10 yard head start for the tortoise.



It is really no contruction for Achilles to catch up to the tortoise at, say, C (and C'). To establish a 1-1 correspondence between points traversed by Achilles and points traversed by the tortoise, intersect SS', the line connecting their starting points with CC', he line connecting their catch-up positions. Call this intersection Q. Lines through Q match up points of SC with those of S'C'. Any sample point P has its image point P'. Thus although the distances they have traversed are unequal, the number of points each has passed through is the same - bizarre but no contradiction.

It is also illumintating to use infinite series. Let us assume that Achilles runs 10 times as fast as the tortoise. Using the formula for the sum of an infinite geometric series, $S = a/(1-r)$, the distances traversed by the two competitors can be computed.

$$S_A = 10 + 1 + 1/10 + 1/10^2 + 1/10^3 + \ldots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$S_T = 1 + 1/10 + 1/10^2 + 1/0^3 + \ldots$$

$$S_A = \frac{10}{1-1/10} = \frac{100}{9} = 11.1 \text{ yards}$$

$$S_T = \frac{1}{1-1/10} = \frac{10}{9} = 1.1 \text{ yards}$$

$$S_T + 10 = S_A$$

Another version of the paradox points out that at each "stage" in the race, Achilles will still lag behind the tortoise and so never catches up. More precisely, at the n^{th} stage Achilles has only traveled

$$1 + 1/10 + 10^2 + \ldots + 1/10^{(n-2)}$$

yards beyond the tortoise's starting point while the tortoise has already gone

$$1 + 1/10 + 1/10^2 + \ldots + 1/10^{(n-1)}$$

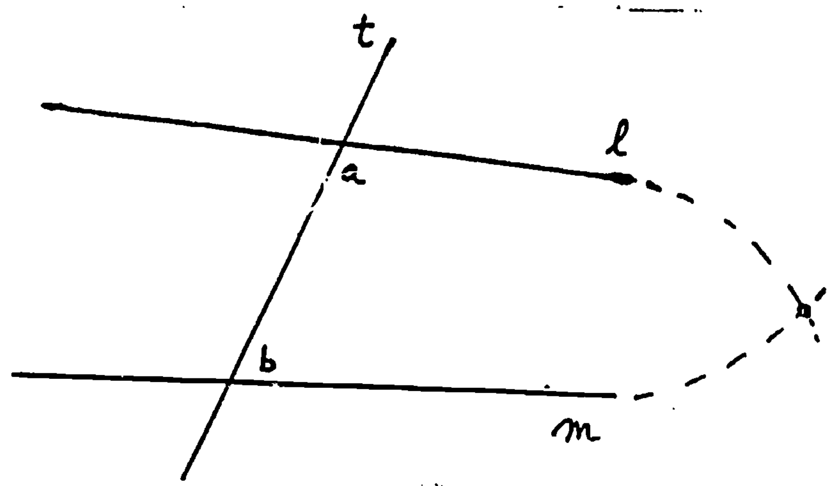
yards. So at the n^{th} stage the tortoise is still $1/10^{n-1}$ yards ahead. This is true forever (i.e. for any $n \geq 2$), and so it seems that the tortoise is always ahead.

* * * *

Euclidean Geometry as a Deductive System

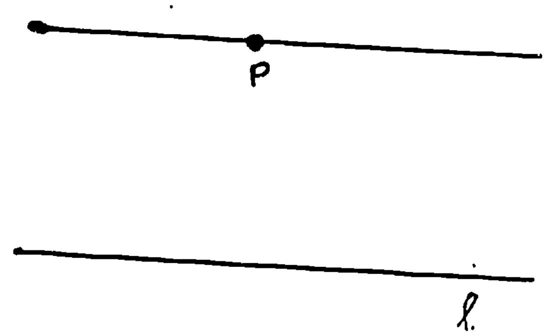
Euclid's Fifth Postulate:

If two lines l and m are cut by a transveral t so that the interior angles a and b, on one side of t, add up to less than two right angles, then l and m will meet on that side of t on which these angles lie.



Playfair's version of this Postulate (John Playfair, 1748-1819):

Through a point P outside a given line l one and only one line can be drawn parallel to the given line.

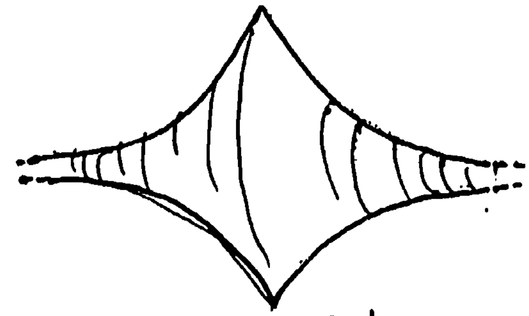


Mathematicians questioned whether or not this statement had to be accepted as a postuate. Could it be proved as a theorem? If so, then assuming the negation of the postulate should have led to a contradiction. No contradiction was found, however, and the mathematicians investigating the issue were amazed and rewarded by entirely new geometries. The negated postulate produced models consistent with the rest of the logical system. The existence of consistent models verified that the new geometries were free of contradictions.

Hyperbolic Geometry was developed by Girolamo Saccheri (1667-1733), Carl Friedrich Gauss (1777-1855), Nikolai Lobachevsky (1793-1856), and John Bolyai (1802-1860).

New Postulate: There exist at least two lines parallel to a given line through a given point not on the line. The result of this postulate is to re-define a "line" and there develops an infinite bundle of lines hyper-parallel to the given line. The lines are asymptotic but not intersecting so they are in fact parallel. A visual model of this geomery is the surface of a pseudosphere formed by revolving a tractrix about its asymptote.

Tractrix



Pseudo-Sphere

The resulting negative curvature causes many differences between hyperbolic geometry and the traditional Euclidean geometry. For instance, the sum of the angles of a triangle in hyperbolic geometry is < 180 degrees.

Spherical (Elliptic) Geometry was developed by Bernhard Riemann (1826-1866) and Felix Klein (1849-1925).

New Postulate: There exist no lines parallel to a given line through a given point not on the line. A simple model is the surface of a sphere where a "line" is defined as a great circle. Since all great circles intersect there are no parallel lines possible.

The resulting positive curvature causes many differences between spherical geometry and the traditional Euclidean geometry. For instance:

1. The sum of the angles of a triangle is > 180 degrees.

2. All straight lines have the same finite length.

3. All perpendiculars to a straight line must meet in a point.

* * * *

The Tower of Hanoi and Other Special Patterns

1. The Tower of Hanoi problem follows a pattern using powers of two. The Tower of Hanoi is actually three pegs. There are 64 disks of graduated size on one peg (smallest to largest from top to bottom). The object is to transfer the disks to a different peg moving them one at a time and never placing a larger disk on a smaller one. The third peg is for holding disks temporarily during the transfer process. According to legend, if the disks are moved at the rate of one per second and no mistakes are made the universe will end at the moment that the task is completed.

Consider the chart:

# disks	# moves	formula
1	1	$2^1 - 1$
2	2(1)+1	$2^2 - 1$
3	2(2(1)+1)+1	$2^3 - 1$
4	2(2(2(1)+1)+1	$2^4 - 1$
n	pattern continues	$2^n - 1$

2. Another pattern with powers of 2 comes from Pascal's triangle:

Pascal's Triangle	Sum of the Row
1	2^0
1 1	2^1
1 2 1	2^2
1 3 3 1	2^3

Each value is used twice in generating the next row so each row sum is twice the preceding one.

$$2^n = 2(2^{n-1}) = 2^{n-1} + 2^{n-1} \quad \text{, but } \quad 2n^{-1} = 2^{n-2} + 2^{n-2}$$

$$= 2^{n-1} + 2^{n-2} + 2n-2$$

$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + 2^{n-3}$$

...

$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2 + 1 + 1$$

$$2^n - 1 = 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2 + 1$$

Mathematical induction can also be used to verify this last pattern:

Prove: $1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$

Let n = 0: $2^0 = 2^1 - 1$

$1 = 1$

Assume: $1 + 2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1$. Then.

$$
\begin{aligned}
1 + 2 + 2^2 + 2^3 + \ldots + 2^k + (2^{k+1}) &= 2^{k+1} - 1 + (2^{k+1}) \\
&= 2(2^{k+1}) - 1 \\
&= 2^{k+2} - 1 .
\end{aligned}
$$

which is of the desired form.

* * * *

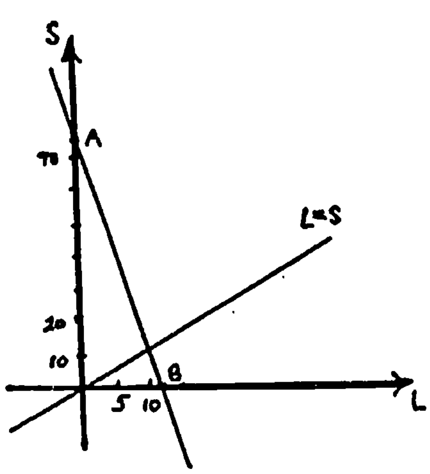
DIOPHANTINE EQUATIONS

1. Problem: A man sells marbles. He orders 19 large packets of marbles and 3 small packets. He receives 224 loose marbles. How can be restore the right number of marbles to the packets (both large and small)?

$$19L + 3S = 224$$

To help visualize the solution, rewrite this equation as

$S = (-19/3)L + 224/3$ and graph by slope-intercept.



Any solutions (l,s) must lie on line segment AB. A is (0,74 2/3) and B is (11 15/19,0).

Rewrite the slope-intercept equation in a new form:

$S = (-6\ 1/3)L + 74\ 2/3$

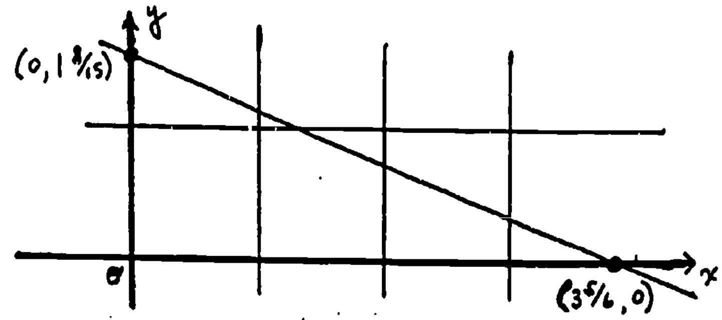
$S = -6L - (1/3)L + 74 + 2/3$

$S = 74-6L + (1/3)(2-L)$

From the graph we know $0 \leq L \leq 11$ and from the last equation 3 must divide (2-L) evenly. L=2 is one obvious such number and the rest are found by adding three's: L=2, 5, 8, 11. But S<L since it S is the number of marbles in a small packet and L the number in a large packet. So, the solutions are points with integer coordinates on line segment AB which lie below the line L=S. The only such is L=11 and S=5.

2. Problem: Solve $6X + 15Y = 23$ for all integral X and Y.

There can be no solution since 3 is a factor of the coefficients 6 and 15 but 3 is not a factor of 23. Alternatively, consider the graph:



The slope intercept equation is $Y = (-2/5)X + 23/15$. Starting at the Y-intercept 23/15 we count off the slope but never land on a lattice point (no integer solutions).

3. Problem: A benefit is attended by 100 people. The men are charged 5 cents. the women are charged 2 cents and the children are charged at the rate of 10 for 1 cent. The gross proceeds are $1. Find the number of men, women, and children who attended.

$M + W + C = 100$
$5M + 2W + C/10 = 100$

We can see that 10 is a factor of C and that $C < 100$. Through trial and error we find that $C = 70$ is a solution ($M = 11$, $W = 19$).

Inmproved solution:

$M + W + C = 100$ so $M + W = 100 - C$ or ** $-2M - 2W = -200 + 2C$
$5M + 2W + C/10 = 100$ so ** $5M + 2W = 100 - C/10$.
Add the starred equations,

$$\begin{aligned} -2M - 2W &= -200 + 2C \\ 5M + 2W &= 100 - C/10 \\ \hline 3M \quad &= -100 + (19/10)C \\ M &= (-1000 + 19C)/30 \end{aligned}$$

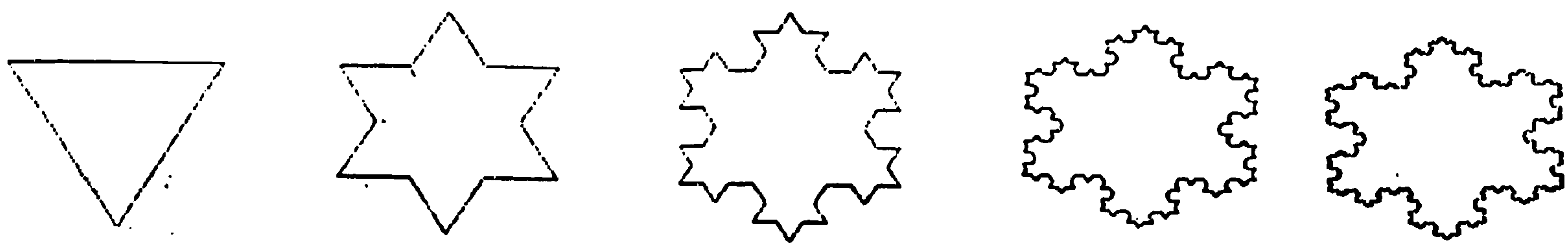
C is a multiple of 10 that must be ≥ 60 to keep $M > 0$. but also $C < 100$ since there are only 100 people. So the possibilities are narrowed to C = 60, 70, 80, 90. Guessing is still needed but the first try yields the correct solution, C = 70.

4. Problems and Some Solutions

1. Graph $X^2 + 2XY + Y^2 - 2X - 2Y = 0$.

2. Derive a formula for the perimeter of the Koch Snowlfake. This curve starts as an equilateral triangle. Centered on its sides we form new equilateral triangles with side lengths equal to 1/3 the side lengths of the original triangle. Further equilateral triangles are centered on every exterior segment in the current figure with side lengths equal to 1/3 of the previous the side length (therefore, 1/9 the original side length). This process continues indefinitely.

The first five stages of the SNOWFLAKE are shown below:



3. Find the endless product $(\sqrt{2})(\sqrt[4]{2})(\sqrt[8]{2})(\sqrt[16]{2})\ldots$

4. Solve $3X^2 - 16X + 3\sqrt{(3X^2 - 16X + 21)} = 7$

* * * *

Solutions

1. Graph $X^2 + 2XY + Y^2 - 2X - 2Y = 0$.

   The expression factors as a trinomial and a binomial:

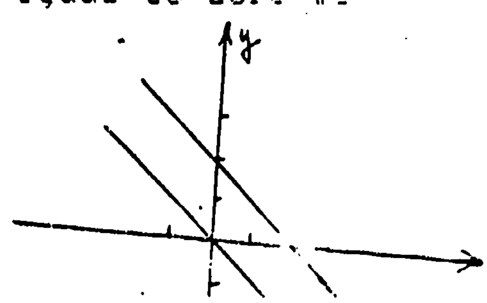
   $(X^2+2XY+Y^2) - (2X+2Y) = 0$

   $(X+Y)^2 - 2(X+Y) = 0$. Factor again:

   $(X+Y)(X+Y-2) = 0$. Setting each factor equal to zero we have two linear equations.

   $X+Y=0$ $X+Y-2=0$
   $Y=-X$ $Y=-X+2$

   and so the graph consists of 2 parallel lines.



2. The perimeter of the Koch Snowflake is infinite. Consider the creation of the first set of extra equilateral triangles. The original length of a side, s, is now

transformed into the length 4(1/3)s. The other sides are similarly increased so the perimeter has changed from 3s to 4s. The same factor of increase, 4/3, occurs at every level of the growth patten so the perimeter is infinite, since $(4/3)^n \to \infty$ as $n \to \infty$.

3. Rewrite the problem with fractional exponents as:

$(2^{1/2})(2^{1/4})(2^{1/8})(2^{1/16})\ldots$ which becomes

$2^{1/2\ +\ 1/4\ +\ 1/8\ +\ 1/16\ +\ \ldots}$ The exponent is an infinite geometric series which can be evaluated by a formula.

$S=a/(1-r)$.

$S = (1/2)/(1-1/2)$

$=(1/2)/(1/2)$

$= 1$

So the final answer is 2^1, that is, 2.

4. Solve $3X^2 - 16X + 3\sqrt{(3X^2 - 16X + 21)} = 7$

The correspondence between the terms $3X^2 - 16X$ and the

radicand $3X^2 - 16X + 21$ suggests making modifications to

achieve a factorable trinomial of the form $ax^2 + bx + c$. Add 21 to both sides of the equation.

$3X^2 - 16X + \underline{21} + 3\sqrt{(3X^2 - 16X + 21)} = 7 + \underline{21}$

$(3X^2 - 16X + 21) + 3\sqrt{(3X^2 - 16X + 21)} - 28 = 0$

$(\sqrt{(3X^2 - 16X + 21)})^2 + 3\sqrt{(3X^2 - 16X + 21)} - 28 = 0$

Now call $\sqrt{(3X^2 - 16X + 21)} = Z$ to see that this equation has
the form $Z^2 + 3Z - 28 = 0$ which factors as $(Z+7)(Z-4)=0$. So $Z=4$ or $Z=-7$. But we cannot have $Z = -7$ since Z is a positive square root.

So $\sqrt{(3X^2 - 16X + 21)} = 4$ and therefore $3X^2 - 16X + 21 = 16$. Thus,

$3X^2 - 16X + 5 = 0$

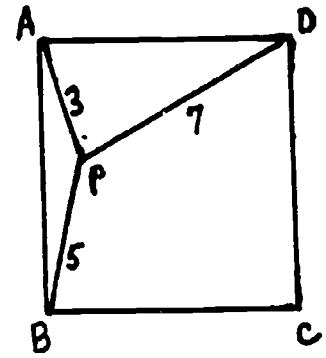
$(3X - 1)(X - 5) = 0$

$X = 1/3$, $X = 5$ and both check

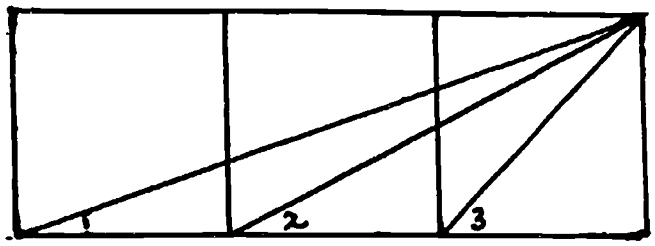
* * * *

EXTENDING THE DIAGRAM: Problems and Solutions

5. Point P is inside square ABCD.
PA = 3, PB = 5, and PD = 7.
Find the area of the square ABCD.

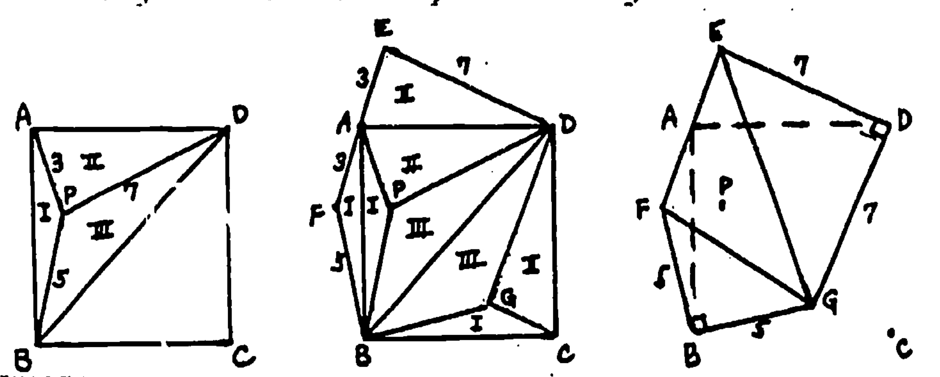


6. Given three congruent squares. Angles 1, 2, and 3 are labeled as shown. Show $m\angle 1 + m\angle 2 = m\angle 3$.



Solutions

5. Construct a diagonal BD and reflect Δ PBD about this diagonal. Also reflect triangles PAD and PAB outward along the sides of the original square. At vertices D and B we have right angles because the reflections have doubled the angles of 45 degrees formed by the diagonal of the original square.



The area of the third figure is the same as that of the original square. Triangle EDG is therefore a right isosceles triangle whose area is .5(7)(7) or 24.5. Triangle FBG is also a right isosceles triangle whose area is then .5(5)(5) or 12.5. We have only to find area triangle FEG (the segment FE is straight at point A because the reflections have doubled the original right angle at A to a straight angle of 180°). From the isosceles right triangle ΔEDG we have $EG = 7\sqrt{2}$ and from isosceles right triangle ΔFBG we have $FG = 5\sqrt{2}$. We also know FE = FA + AE = 6. Use Hero's formula to find area of ΔFEG:

$$p = \text{perimeter} = 6 + 7\sqrt{2} + 5\sqrt{2}$$

$$s = \text{semi-perimerter} = \frac{p}{2} = 3 + 6\sqrt{2}$$

$$A = \sqrt{[(3 + 6\sqrt{2})(3 - \sqrt{3})(3 + \sqrt{2})(-3 + 6\sqrt{2})]}$$

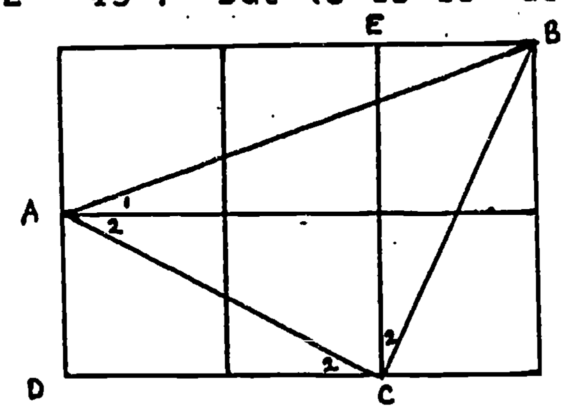
$$A = \sqrt{[(-9 + 72)(9 - 2)]}$$

$$A = \sqrt{[(63)(7)]}$$

$$A = 21$$

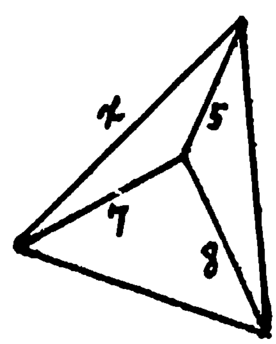
Area square ABCD = Area ΔEDG + Area ΔFBG + Area ΔFEG
=24.5 + 12.5 + 21
= 58

6. Extend the diagram with a second set of congruent squares. Construct triangle ABC which is isoscles because AC and BC are both diagonals across two squares. Rotate right <DCE about vertex C to the position of <ACB to see that <ACB is a right angle also, and so ΔACB is an isosceles right Δ. Therefore <1+ <2 = 45°. But <3 is 45° also.



MORE PROBLEMS (solutions left for the reader):

7. Find x, the side of this equilateral triangle:



8. 12 pennies all look alike but one is phony and its weight is different. Using a balance and three weighings, find the phony coin and whether it weighs more or less than the real ones.

9. Find all rectangles whose perimeter equals area. Find all such with integer sides.

10. Find all integer solutions to $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$

11. Graph $|x-3| - |y+4| = 5$

12. Graph $y = \sqrt{\left(\frac{2x^2 - 4}{x + 4}\right)}$

13. a) In how many ways can 11 people shake hands?
    b) How many line segments are determined by 11 dots on a page?
    c) How many diagonals does an 11-gon have?

14. 11 people - 6 women and 5 men:

    a) how many man-woman handsakes are possible?
    b) how many bridge games are possible where each of the 2 teams consists of a man and a woman?
    c) how many such games can be formed if Mr. & Mrs. Smith

insist on playing as partners only?

d) ...if Mr. & Mrs. Smith refuse to be partners?

e) ...if Mr. & Mrs. Smith won't play in the same game?

5. Reminiscences of Al

It was May of 1975 and I was just finishing up my first year of teaching at Great Neck South Junior High School. I was very fortunate to have begun my teaching career in such a wonderful district which put such a strong emphasis on education. However, I was a sabbatical replacement at Great Neck and the gentleman who was on leave was returning, so I was facing the Fall without knowing where (or if) I would be working.

The mid-seventies was a tough time to get a teaching job, as you may recall. I was lucky to get the sabbatical replacement coming right out of college in the first place. So now, the resumes had to go out again. I was fortunate enough to get a call from the Babylon District and went in for an interview. I was first interviewed by the Principal, then I met with the Math Department Chairman. This gentleman was in his mid-fifties and looked to be a cross between Rumplestilskin and Mahatma Ghandi. Though small in stature and with a curious appearance, it didn't take me long to see that this was no ordinary teacher or person. This was when I first met Al Kalfus.

I got the job at Babylon and while I was not aware of Al's established reputation at that time, I still knew that I was in the midst of a "guru." It was with great anticipation that I began my tenure at Babylon.

I think it would be accurate to say that Al took me under his wing. And if I was not his protégé, he most certainly was <u>my</u> mentor. He taught me as much about myself as he did about mathematics. I truly must be one of the most fortunate of individuals in the education field to have had daily contact with someone of Al's stature. It would be like some small businessman being Donald Trump's understudy, or some aspiring ballet dancer working with Nureyev every day.

Yet my relationship with Al Kalfus was not purely professional. I did get to know the real Al over the course of my fourteen year relationship him. He, at times, was the "absent minded professor" he appeared to be. His mind was always "on." Even while driving, he would often be thinking of some unresolved math problem or of some challenging question he would pose to his students. Of course during these mental exercises, his driving was affected. I often drove behind him during what must have been one of these "thought sessions." He would weave from lane to lane, run red lights and exceed the speed limit by about 40 to 50% . I had my heart in my throat watching all this, but Al would be oblivious and cool as a cucumber as the complex mathematical thoughts filled his head.

Yet as a passenger in a car, Al took on a totally different persona. He was very relaxed and placid. I drove up to the Nevele for the annual AMTNYS convention one year, and Al came with me. Predictable small talk gave way to vivid recollections of his growing up in Brooklyn, and how he was able to get maximum production as a student with minimal effort (perhaps this is why so few students could "pull the wool over his eyes," and why he was impatient with the lazy, though bright, student). Al mentioned, with only a hint of regret, that he had been a real "oddball" as a kid. The unusual thing about it was that he recalls being aware of his "uniqueness" yet not feeling compelled to conform or be like the others. He also spoke fondly of the wonder of his uncle's fish market and pranks he was involved with at summer camp. Despite his passion for academia, Al coached soccer at Amityville and, believe it or not, had a short lived interest in, of all things, boxing!

As brilliant as he was, Al was terrible with names. In my first few years at Babylon, Al would take me to conferences with him so that he could introduce me to people and get me involved in the mathematics education community. But when it came to the introductions, he often would forget the name of the person we were talking to, or worse, forget my name!

If Al had any professional flaws, it may have been that he didn't realize that some (or many) of his students and colleagues simply did not share the same zeal and passion for Mathematics in particular, and for learning in general, as he did. If a student was having difficulty with a certain topic, Al figured it could only be that the students was not trying. He rarely entertained the possibility that the material (or his own presentation of it) was simply over his or her head. This was true even when he spoke to teachers at conferences and workshops. Often his topics were very subtle, advanced and "esoteric" (one of Al's favorite words). Sometimes, he was literally over the audience's heads. When those listening to him did not respond to his prompts or questions, he couldn't understand it. Weren't they paying attention? Didn't they care? If it was so clear and obvious to him, why wasn't it to everyone else?

Al dealt with Mathematics as an art. He pursued elegance in proofs and processes (He would say, "Tables are for barbarians!"); efficiency in computations ("Arithmetic is for peasants!"); crystal clarity in terminology (when working with fractions, if a students referred to the numerator and denominator as "top" and "bottom", Al would say, "Are we talking about fractions or pajamas?") and detail in graphs ("You must make your graphs 'beeyoootiful!'").

Al Kalfus was an animated, often lovable, sometimes ornery man. I was fortunate enough to have benefitted from his vast experiences. He nurtured me as a professional and counselled me as a friend. Yet his mark was left in a far larger and more significant arena.

He founded the Suffolk County Math League in 1955, and established the "Owen Bradford Trophy", presented annually to the highest scoring team in Suffolk County. Owen Bradford was a former student of Al's who was killed in a boating accident in the late 1950's.

Al began the New York Mathematics League (NYML) in 1972. His desire to promote mathematics excellence and to encourage the solutions to non-routine problems were the motivating force behind this. This was the first statewide mathematics competition. In 1976, he expanded this competition in the founding of the Atlantic Region Mathematics League (ARML). In 1983, the "A" in ARML was "expanded" to now signify "American". A national mathematics competition was born. It was Al Kalfus who coined the term "mathlete."

Al started the Long Island Math Fair in 1960. He was the president and driving force behind this prestigious mathematics competition from 1960 to 1971. Fittingly, in his honor, this competition shall forever more be called the Al Kalfus Long Island Math Fair.

Despite the energies he shared in Amityville and Babylon High Schools, he was also an adjunct Professor of Mathematics at Nassau Community College and at C. W. Post, Polytechnic and Hofstra Universities. Even at the university level, Al was recognized as a giant in mathematics education.

Al was the recipient of many awards and citations, far too many to mention. His most prominent awards were from the Long Island Society of Professional Engineers in 1974, and the crowning achievement to his career- receiving the Presidential Award for Mathematics Teaching Excellence in 1983. Especially noteworthy is that Al was the first such recipient from New York State. Having been the first certainly puts him among the best.

He retired from Babylon High School in 1985 at age 63- eight years later than he was eligible to leave the field. I succeeded Al as the Mathematics Department Chairman at Babylon. His were shoes I was eager to step into, but ones I certainly did not expect to fill.

Health problems dictated that he pursue a less taxing schedule. He did "slow down," but only for a while. His work with the State Education Department, joining the staff at Hofstra and attending speaking engagements returned him to a demanding schedule.

Al died in March of 1989 at the age of 67. Though his loss is mourned by thousands, his legacy remains. His numerous contributions to large scale mathematics organizations as well as to individuals, such as myself, are eternal.

Thanks for everything Al. We miss you.

Dan Drance
-Babylon High School
-NSF-Hofstra Institute
Cycle I
June, 1989

6. Curve Sketching and Transformational Geometry

PREFACE

One day some twelve years ago, in only my third or forth year of teaching, I asked Al Kalfus something about some graphical characteristics of a certain third degree polynomial that I was to discuss with my Trig. class later that day. Al began to "think out loud." What I mean is, there was no "rough draft" to his thoughts. Out on a piece of paper came flowing such beautiful and precise mathematics. A small percentage of it did not pertain directly to my question, but it was fascinating to see such a knowledgeable and confident mind at work. I dared not interrupt or attempt to redirect Al's energies. Anyway, at the time most of what he was saying went right over my head. I remember him saying something like, "Well, you know what the graph of a basic third degree polynomial looks like, so ..." and he proceeded from there.

Of course, I did not know what the graph a basic third degree polynomial looked like.

I let him continue what sounded like a beautiful and magnificently comprehensive discourse on my question, not having the heart to tell him that he had lost me by about the eigth sentence. I was, at once, humiliated and inspired.

I even have a feeling that Al knew I wasn't following all that he was saying. He was just hoping that I would grasp on to enough to motivate me to do some digging on my own and/or to come back to him with more questions.

I did both!

Starting from that fateful day, I have learned a far greater appreciation for my profession and for the material I teach. Underlying that appreciation was always a deep-seated respect for mathematics. But what combined and solidified these two virtues was the beginnings of mastery of what I was teaching. I, like so many teachers, would simply stay a few lessons ahead of the class; look over the answer key or teachers manual for solutions to problems- and if I didn't understand the solution, I just didn't assign that particular problem; and if there ever were any really tough questions from my students, I would either "fake it" or stall by saying something like, "...that's a good question, Sally, and I'll work on it for you and have an answer for you tomorrow" and hope that Sally would forget she asked such a clever question. It was a sad fact- and continues to be for some of us- that we just really don't know enough about all that we teach.

I was lucky, very lucky! While I think I probably do possess above average intelligence, I was fortunate enough to be in daily contact with a master; with someone who would not only nurture me and take me under his proverbial wing, but would also provide relentless challenges; with someone with the highest of academic and personal standards; with someone who could bring the best out of me. This last aspect of my good fortune is most noteworthy, since most people must motivate and challenge themselves. While I know that I did experience much personal and individual struggle and growth, I also had the enormous benefit of enlightened guidance along the way from a mentor that was a giant in our field.

In the pages that follow, I will attempt to share some of this enlightenment that I received from Al Kalfus as well as some personal insights that I developed or came to realize on my own. This will be in the area of "curve sketching." Curve sketching was one of Al's prizes, but it is only a very small part of his contributions to mathematics education. If what follows is even a small help in the never-ending growth process of a teacher, my efforts here will be well worth while and the memory and contributions of Mr. Alfred Kalfus will continue to live on.

June, 1989

Dan Drance
NSF-Hofstra Institute
Cycle I

INTRODUCTION

Curve Sketching is indeed an art form. Al, whose standards for his students, as well as for himself, were extremely high, would penalize a students if his or her graph was not symmetrical, properly rounded... if it were not, as he would say, "Beeyoootiful!"

While we all should share similar high standards for our students when it comes to graphing, the "cost" of such beauty is not high, nor must it be at all tedious or difficult. Our students must possess a fundamental background regarding basic graphs (like $y=x$, $y=x^2$, $x^2+y^2=r$, $y=\sqrt{x}$, $y= 1/x$, $y=\log x$,etc.) and then master the relatively simple tenets of transformational geometry and apply them to these basic graphs to come up with a satisfactory graph.

Our challenge as teachers is to convince our students (and maybe ourselves) of the validity and, indeed, common sense of these principles involved in curve sketching. Al Kalfus succeeded in this conveyance of knowledge by persistence, patience (although more persistence than patience, I think) and total student involvement. He wouldn't let anyone "hide" or avoid his challenges. While our styles vary and are unique to each of us, we all can have our students truly understand the mathematics that we teach them. This, of course, is every teachers ultimate goal. Al challenges us all to this end. If we can have a student master (not remember- memorizing is the lowest form of thinking, and memorized material is most easily forgotten) a few bits of basic information and apply this to related situations, true learning has taken place. This is at the heart of all problem solving. When we graph any parabola, it should be closely related to the basic $y=x^2$ graph, with only a few small alterations. If we graph $y=x^2+2x-10$ one way and then graph $y=x^2-3$ differently (as if starting over), we are failing to see the vital relationships that exist between these and all parabolas. Oh, and by-the-way, tables are out! Al used to call students who made tables for their graphs "barbarians". Tables are tedious, sources of needless errors and unnecessary.

Al Kalfus often commented on the rich presence of patterns in mathemactics. Patterns are abundant in all branches of mathematics- from arithmetic to algebra; from group theory to curve sketching. These patters should be studied and should be viewed as built-in aids to the broadening of our background in Math. Patterns are especially present in curve sketching. All circles are round. All parabolas have a "U" shape to them. All sine and cosine curves are wavy. If students are permitted to make tables for their graphs, then they avoid the essential knowledge of being aware of what a graph should look like before it is sketched. A student must know that the graphs of all first degree equations are lines and all second degree polynomial equations are parabolas, etc.. According to Al, "there is only one parabola" ($y=x^2$) and, according to me, we just do a bunch of stuff to it.

Identifying what to do and how to do it is the focus of this paper.

BASIC TRANSFORMATIONS

Translations:

-slides (horizontal and/or vertical shifts); no effect on shape of the curve (isometry)

-achieved by adding or subracting to/from the x-value/coordinate (horizontal translation) or the y-value/coordinate (vertical translation) as a quantity (in parentheses)

-refer to diagrams on page 6

Ex. $(y-5)= (x+2)$ slides $y=x$ 2 units left and 5 units up

Dilations:

-stretches, shrinks, expands or contracts a basic curve; does not result in a congruent figure (non-isometry) unless dilation factor (multiplier) is ± 1

-achieved by multipying the x-and/or y-value/coordinate, or the quantity involving x and/or y

-refer to diagrams on page 6

Ex. $y= 3x$ is line $y=x$ but 3 times as steep (slope of 3)

Ex. $y= (1/2)x$ is broader or wider (not as steep) as $y=x$

Ex. $4x^2 +4y^2 =16$ is a smaller circle than $x^2 +y^2 =16$

Reflections:

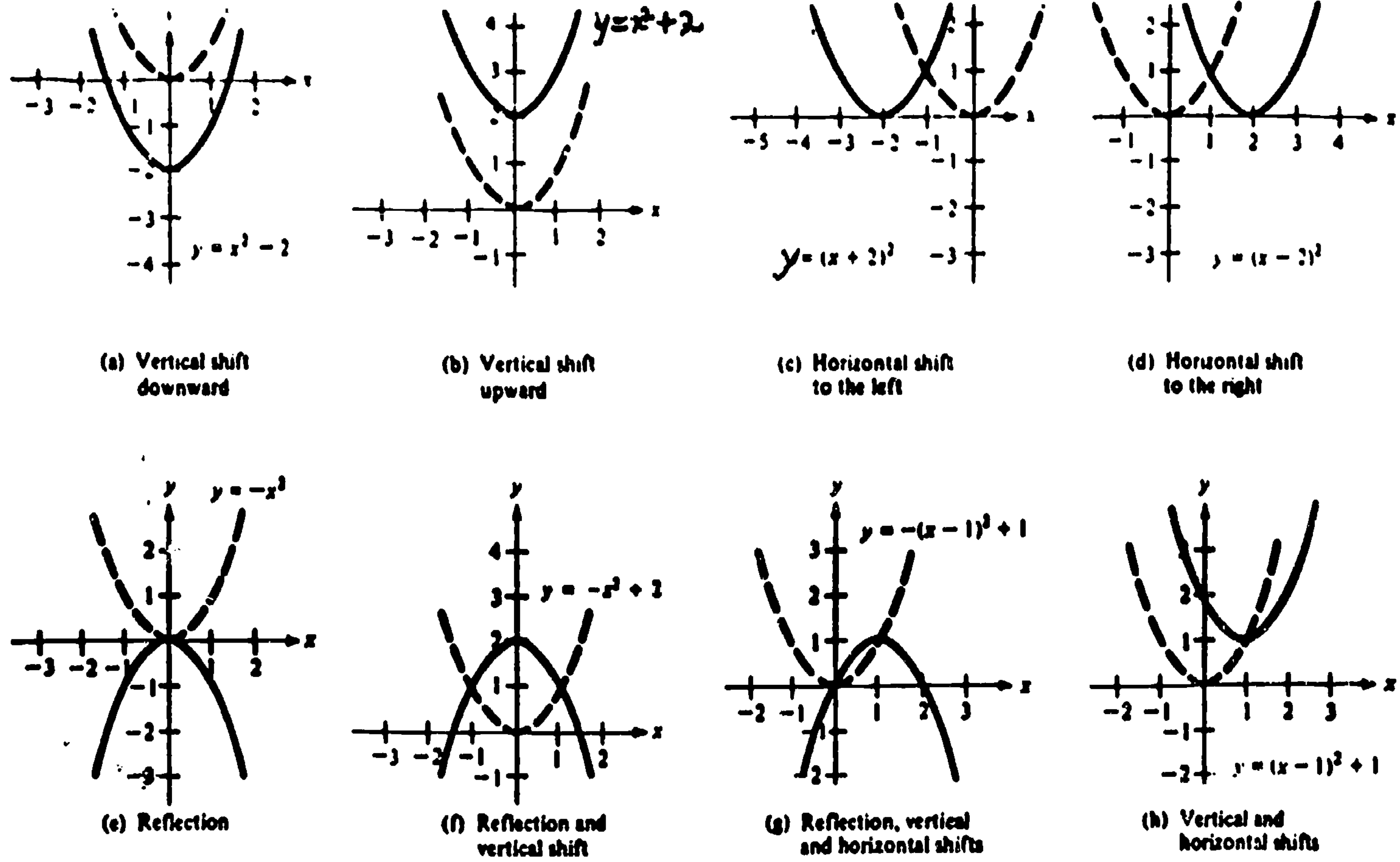
-reflections are basically performed through lines (such as the x-axis, y-axis, line y=x, etc.), although a composition of line reflections can result in a point reflection and/or a rotation in a point (usually the origin)

-achieved by negating an x- and/or y-value/coordinate, or the quantity involving x and/or y. If the negation can be made to precede the y-value or the y-quantity, it is a reflection in the x-axis. If only the x-value or x-quantity is negated, it is a reflection in the y-axis (see chart on p.6)

-refer to diagrams on pages 6 and 16

Ex. $y= -x$ is $y= x$ reflected in the x-axis
[NOTE: This equation is the same as $-y= x$, so negation can be made to precede the y-value]

Ex. $y= 2^{-x}$ is $y= 2^x$ reflected in the y-axis
[NOTE: The negation here cannot be made to precede the the y-value, as in example above]



(a) Vertical shift downward

(b) Vertical shift upward

(c) Horizontal shift to the left

(d) Horizontal shift to the right

(e) Reflection

(f) Reflection and vertical shift

(g) Reflection, vertical and horizontal shifts

(h) Vertical and horizontal shifts

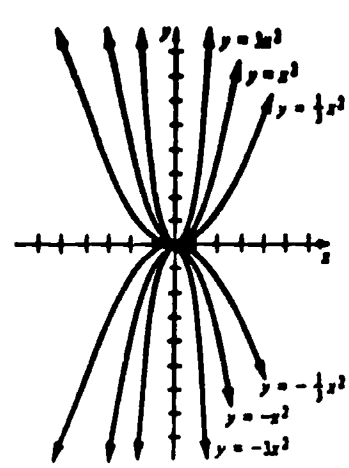
FIGURE 2.11 *Transformations of the Graph of* $y = x^2$

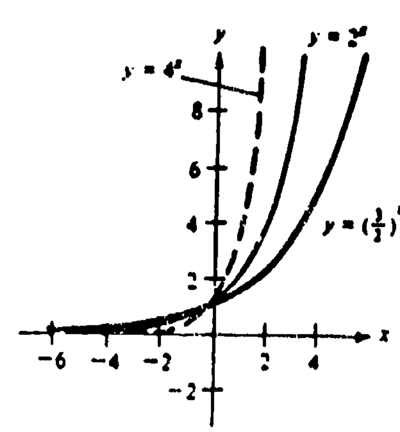
Each of the graphs in Figure 2.11 is a transformation of the graph of $y = x^2$. The three basic types of transformations involved in these eight graphs are (1) horizontal shifts, (2) vertical shifts, and (3) reflections.

BASIC TYPES OF TRANSFORMATIONS ($c > 0$)

Original Graph: $y = f(x)$
Horizontal Shift c units to the right: $y = f(x - c)$
Horizontal Shift c units to the left: $y = f(x + c)$
Vertical Shift c units downward: $y = f(x) - c$
Vertical Shift c units upward: $y = f(x) + c$
Reflection (about the x-axis): $y = -f(x)$
Reflection (about the y-axis): $y = f(-x)$

Dilations





WHERE TO START: Finding the "New Origin"

All basic curves are "centered" around the origin. That is, (0,0) is where we "begin" the graph.

If a curve does not "begin" at (0,0), a translation of the basic curve has taken place. To find where the "new origin" is (that is the place to start, or the point which "simulates" the origin), try to express the equation as a single quantity in x and a single quantity in y, i.e.: in the form $(y-k) = f(x-h)$. [NOTE: This is not always possible or practical. However, curves that cannot be written in this form are not part of the standard high school curriculum prior to pre-Calculus.] The values of x and y that will make these quantities equal to zero will be the coordinates of your new origin. They, therefore, result in translations in the x and y directions, respectively. So it will be the basic $y=f(x)$ curve, simply starting at the point (h,k) instead of (0,0).

What Al would encourage students to do would be to pencil in a new set of coordinate axes at this new origin-call it, say, the x'- and y'-axes. Then all coordinates of the actual graph could easily be adjusted to (or from) the new coordinate system.

Ex. 1: $x^2+y^2=4$ is a basic circle, centered at (0,0) with radius of 2

$(x+4)^2+(y-3)^2=4$ is the same circle, translated to a new starting point, (-4,3). That is because if $x=-4$, the x-quantity is zero, hence a translation of four units to left and if $y=3$, the y-quantity is zero, hence a translation of 3 units up.
[NOTE: In $(x+4)^2+(y-3)^2=4$, students will often think that the translation of $x^2+y^2=4$ is 4 units to the right (due to the "+"4) and 3 units down. This common error can be corrected by emphasizing the need to find the "new origin", as discussed above.]

Ex. 2: $y=2x-5$ can be written as $(y+5)=2x$. This is just the line $y=2x$ where the new origin is (0,-5).

Observe the x- and y-quantities which are negated to identify reflections, as discussed on p.5 . Reflections should be performed before translations.

Ex. 3: $y= -|x-3|+2$ can be written as $(y-2)= -|x-3|$. This is just $y= -|x|$ moved to a new origin of (3,2). (Perform reflections first.) [NOTE: You can actually sketch in a new coordinated axes system (call it the x' and y' axes). Then just sketch the graph of $y= - x$ from there.]

See graphs of examples 1, 2 and 3 (above) on page 9.

The importance of finding this new origin cannot be overemphasized. Not only does it fascilitate the curve sketching, but it provides for excellent review and reinforcement of several algebraic concepts such as equation solving and completing the square. This has been an extra blessing for me. First of all, the Sequential program is deficient in time allotted for development and re-inforcement of algebraic skills. I have also found it extremoly difficult to teach the Sequential program to my satisfaction without watering down the material (especially in Course III.). So this process of finding the "new origin" saves valuable time while providing much needed work in algebra.

****

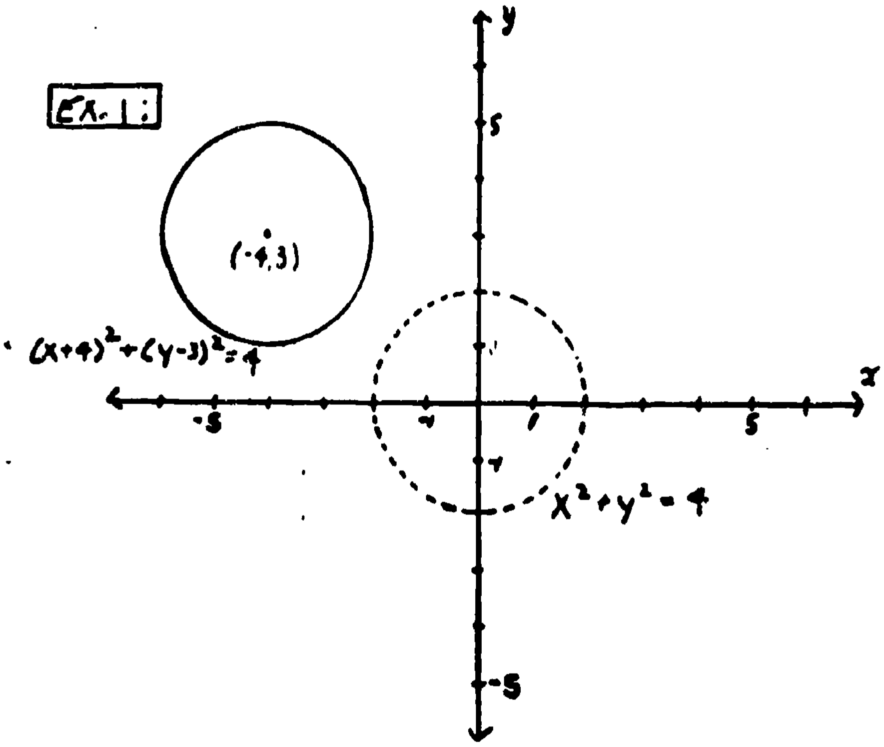
SKETCHING GRAPHS OF TRIGONOMETRIC FUNCTIONS

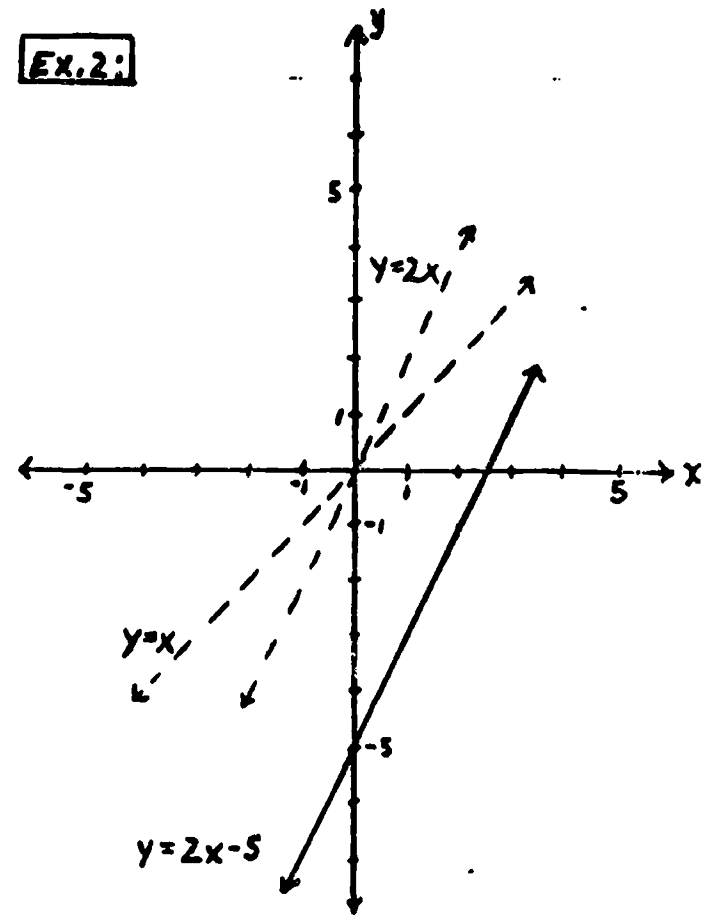
Consider a general form of trigonometric equations:

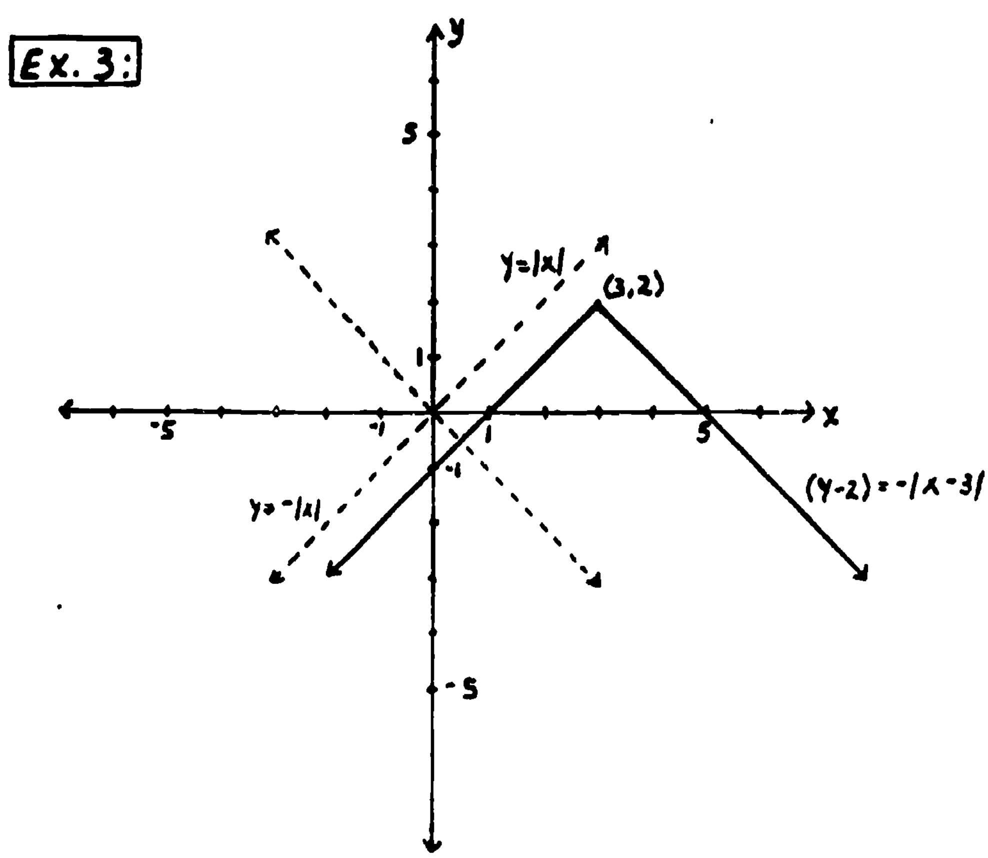
$$y = a \text{ TRIG } b(x \pm c) \pm d$$

where TRIG represents any of the six trigonometric functions (sin, cos, tan, cot, sec or csc) and a,b,c and d are real numbers.

Be able to discuss and identify the effects of a,b,c and d on the basic related trig. graph. (see a sample of basic curves students should know, begining on p. 17, to review the basic trigonometric cruves.)







-9-

Fundamentals:

- "a" effects the amplitude. A change in amplitude will result in a vertical stretching or shrinking (dilation) If $|a|>1$, the curve is stretched. If $|a|<1$, the curve shrinks.

  [If $a<0$, this will result in a reflection in the x-axis (see "Reflections", page 5). Observ that the negative can (and should) be tran   erred to the y-value by dividing or multiplying both sides of the equation by -1.]

- "b" effects the period. A change in period will result in a horizontal stretching or shrinking (dilation). If $|b|>1$, the curve shrinks. If $|b|<1$, the curve is stretched.

  [If $b<0$, this will result in a reflection in the y-axis (see "Reflections", page 5). Note that this negative cannot be algebraically transferred to the y-value.]

- "c" effects horizontal sliding (translation).

- "d" effects vertical sliding (translation). However, "d" should be moved/joined to the y-value so that the general form of a trigonometric equation becomes:

$$(y \pm d) = a \text{ TRIG } b(x \pm c)$$

In this form, "c" and "d" can now be used to locate the "new origin", as discussed on page 7.

It should be noted that the effects that a,b,c and d have on a basic trig graph are identical to the effects that these numbers, appearing in the same positions, would have on a basic polynomial or rational function graph. It just gets back to being able to find that "new origin" and applying the fundamental properties of the basic related graph.

"<u>TRICKS OF THE TRADE</u>"

Every trigonometric curve of the form

$$(y \pm d) = a \text{ TRIG } b(x \pm c)$$

has a high point, low point, x-intercept and/or a vertical asymptote periodically. I call the angles where these special things occur "important angles". Some crucial notes regarding important angles:

-Important angles occur every 90° ($\pi/2$ radians) for "<u>normal</u>" curves ($|b|=1$, or when period is uneffected).

IMPORTANT: Just as the value of "b" effects the period (actual period = [normal period]/$|b|$), "b" also effects the <u>important angles</u> in the same way. So if the period is tripled, the frequency of the important angles are tripled. If the period is cut in half, so are the important angles. That is, when b=1, the important angles occur every 90°($\pi/2$). But if b=3 (which tells us that 3 full cycles of the trig. curve will occur in the space of the basic one, i.e.: things are happening 3 times more frequently than normal) important angles will now occur every 90°/3 or 30° ($\pi/6$). If b=1/2, important angles will occur every 90°/(1/2) or 180° (π), resulting in the "stretching" discussed on page 10. The important angle frequency should be your "unit" on the horizontal (x) axis, and every multiple of the important angle should also be indicated on this axis.

-High and low points (usually 1 and -1) are effected by the value of "a". [<u>NOTE</u>: While the tangent and cotangent graphs do not have high and low points, functional (y) values of 1 and -1 occur at odd multiples of $\pi/4$ (when b=1). So the value of "a" would change those functional values to +a and -a .]

-<u>Reflections</u>, determined by negative values of "a" and "b", should be identified and performed first (before translations and dilations).

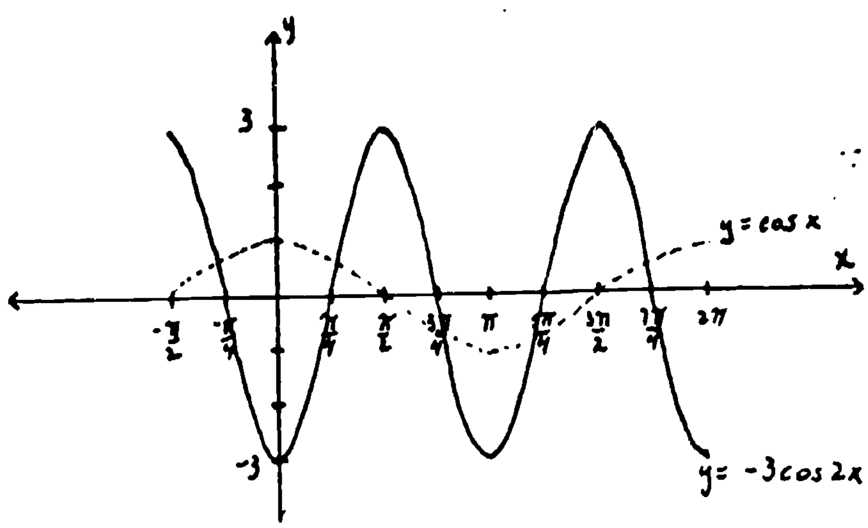
Ex. y = -3COS(2x)

This is just a transformed version of y=COS(x). The "3" stretches the graph vertically. Since the 3 is <u>negative</u>, the curve is reflected in the x-axis. The "2" divides all important angles [as well as the normal period for y=COS(x)] by 2. So, instead of having important angles every 90° , they will occur every 45° . 45° ($\pi/4$) becomes your unit for the horizontal (x) axis.

For y=COS(x), we start (with respect to the origin) at the maximum value ("1" on the y-axis). The max. is (always) followed by an x-intercept at the next important angle (90° or $\pi/2$), followed by a minimum value (-1) at the next important angle (180° or π), followed by another x-intercept at 270° ($3\pi/2$), etc., etc..

So, for y= -3COS(2x), all we need to know is how and where to start and where the important angles are. Therefore, in summary:

- Since nothing is added to the x- or y-value, we "start" at the origin (no translation).
- The negative represents a reflection in the x-axis. So where y=COS(x) would start at a maximum, y= -3COS(2x) will start at a MINIMUM (reflecting the max. in the x-axis).
- The "3" makes all maximum values 3 and all minimums -3 (vertical dilation).
- The "2" divides all important angles by 2, so now, important angles occur every 45° ($\pi/4$) and the period is now 180° (π) (horizontal dilation).
- Since we start with a minimum at 0 , this is followed by an x-intercept, then a maximum, then another x-intercept, then a minimum, and so on, each occuring at the subsequent important angle. This gives us the following, over the interval $-\pi/2 \leq x \leq 2\pi$ *



y= COS(x) is shown for reference and comparison

With this technique, graphs can (<u>and</u> <u>should</u>!) be drawn in <u>one</u> <u>draft</u>. We no longer have to first graph y= COS(x), then y= -3COS(x), then y= -3COS(2x) . We use our knowledge of transformational geometry and properties of the basic trigonometric curves to get one neat, accurate and meaningful picture of the relation. Al (and I) would have his students take extra care that the symmetries are visually correct, that high and low portions of the graph are rounded (not pointy) and that important angles are clearly designated. This is all fascilitated by locating the "new origin" and by carefully and properly employing the "important angles."

Once again, a table is not only avoided, but its use is discouraged. We all know how tedious and how many errors evolve from making any kind of table. These problems are exacerbated when the table is for a transformed trigonometric function. If we sketch all graphs by making a table, then we fail to see any distinction between the different types of graphs. If we just make tables, then lines, parabolas and trig. curves are essentially the same. By teaching, first, what a cosine curve looks like, or what a parabola looks like, then students can learn to draw these pictures in the correct place, using the correct scale. This is a far more efficient and, as Al would say, "elegant" manner of working with and developing this topic of mathematics.

*<u>NOTE</u>: Students should always be encouraged to graph a trig. curve over more than just the interval $0 \leq x \leq 2\pi$. Otherwise, some students erroneously think that trig. curves are finite.

Ex. $y= -2SIN(3x-\pi)-1$

-First, re-write the equation into the general form as seen on top of p. 11. Do this by factoring out the "3" from the parentheses and adding "1" to both sides:

$$(y+1)= -2SIN\ 3(x-\pi/3)$$

-Our "new origin" is $(\pi/3,-1)$. (These are the values that make the respective x- and y-quantities equal to zero.) That is, our origin is translated $\pi/3$ units to the right and 1 unit down. We will set up a new set of axes (call it the x' and y' axes) with $(\pi/3,-1)$ as its "origin."

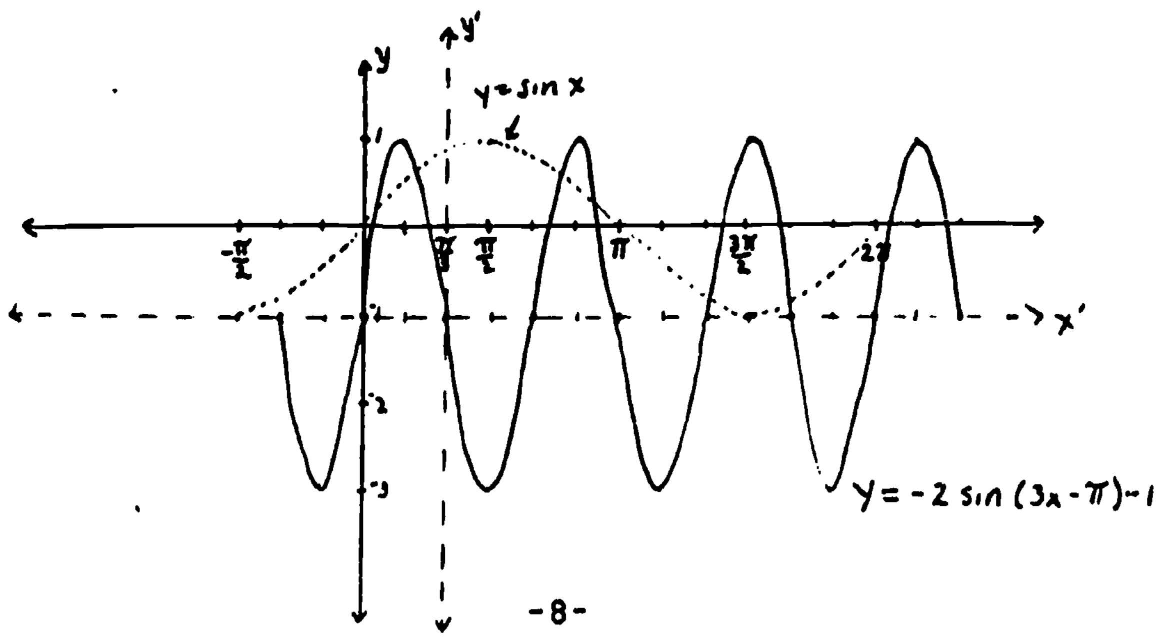
-The "2" stetches the curve vertically.

-Since the 2 is negative, reflect in the x' (horizontal) axis first.

-The "3" divides all important angles by 3. So, important angles will occur every 90°/3 or 30° ($\pi/6$).

-Draw your axes with maximum and minimum values of 1 and -3 (this takes into account the translation <u>down</u> of 1 unit). The horizontal unit is 30° ($\pi/6$) and all multiples of 30° should also be included on the horizontal (x') axis.

Recall that the graph of y= SIN(x) "begins" at zero, followed by a maximum value at 90° ($\pi/2$)- the first important angle. Since we have a reflection in the horizontal axis, we still begin with a zero value (with respect to ($\pi/3$,-1)- the new origin), but we will have a <u>minimum</u> at the first important angle past the new origin. This will be followed by an x-intercept, then a max., then another x-intercept, etc., etc..



Sample graphs illustrating these principals of transformational geometry, as well as other graphs of interest, appear on pages 15, 20, 21, 22 and 23.

SAMPLE GRAPHS ILLUSTRATING APPLICATIONS OF TRANSFORMATIONAL GEOMETRY

$\boxed{Y = -2x^2 + 8x - 5}$

AXIS OF SYMMETRY OR COMPLETING THE SQUARE

$x = -b/2a = \frac{-8}{2(-2)} = 2$

$y = -2(2^2) + 8(2) - 5$

$y = -2(4) + 16 - 5$

$y = -8 + 16 - 5$

$y = 3$

VERTEX: (2, 3)

$y = -2(x^2 - 4x \quad) - 5$

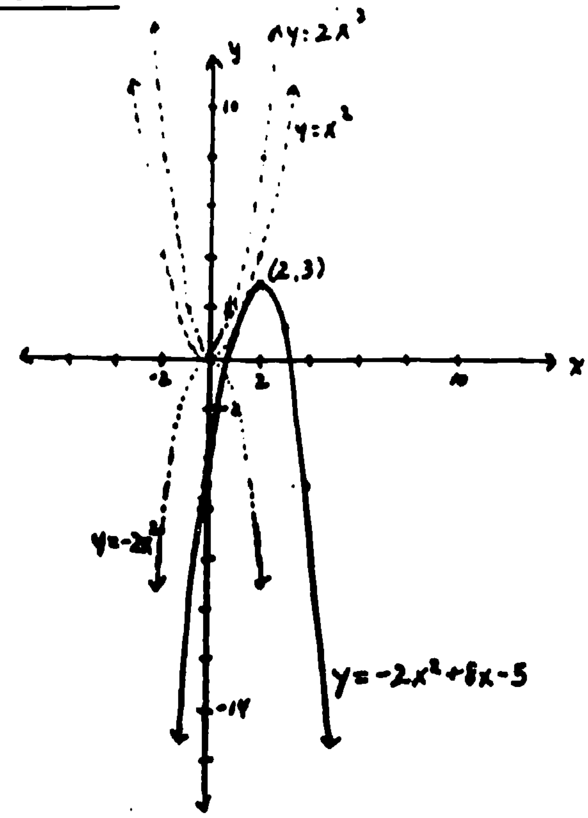
$y = -2(x^2 - 4x + 4) - 5 + 8$

$y = -2(x-2)^2 + 3$

$y - 3 = -2(x-2)^2$

"NEW ORIGIN" or

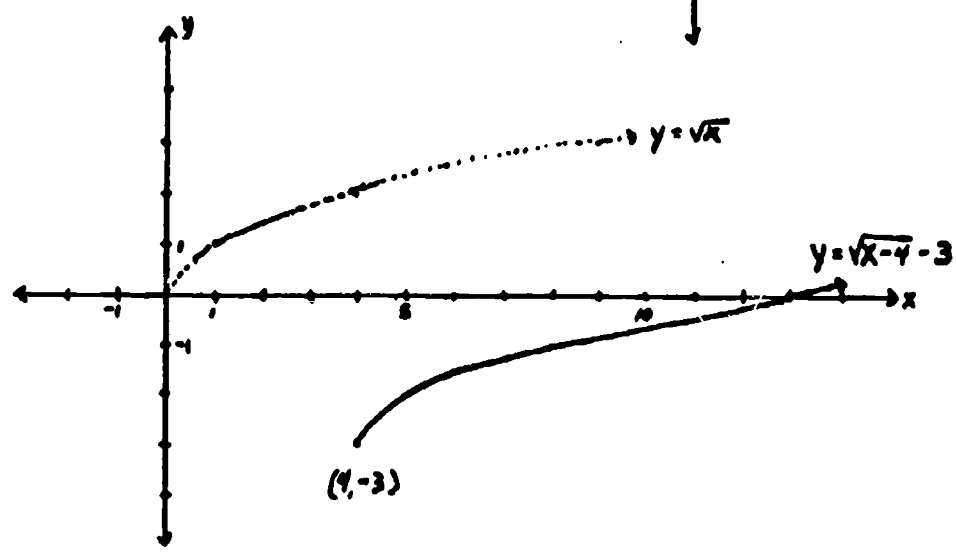
VERTEX: (2, 3)



$\boxed{Y = \sqrt{x-4} - 3}$

$Y + 3 = \sqrt{x-4}$

"NEW ORIGIN": (4, -3)



$\boxed{y = \tan\frac{1}{2}(x - \frac{\pi}{4}) - 2}$

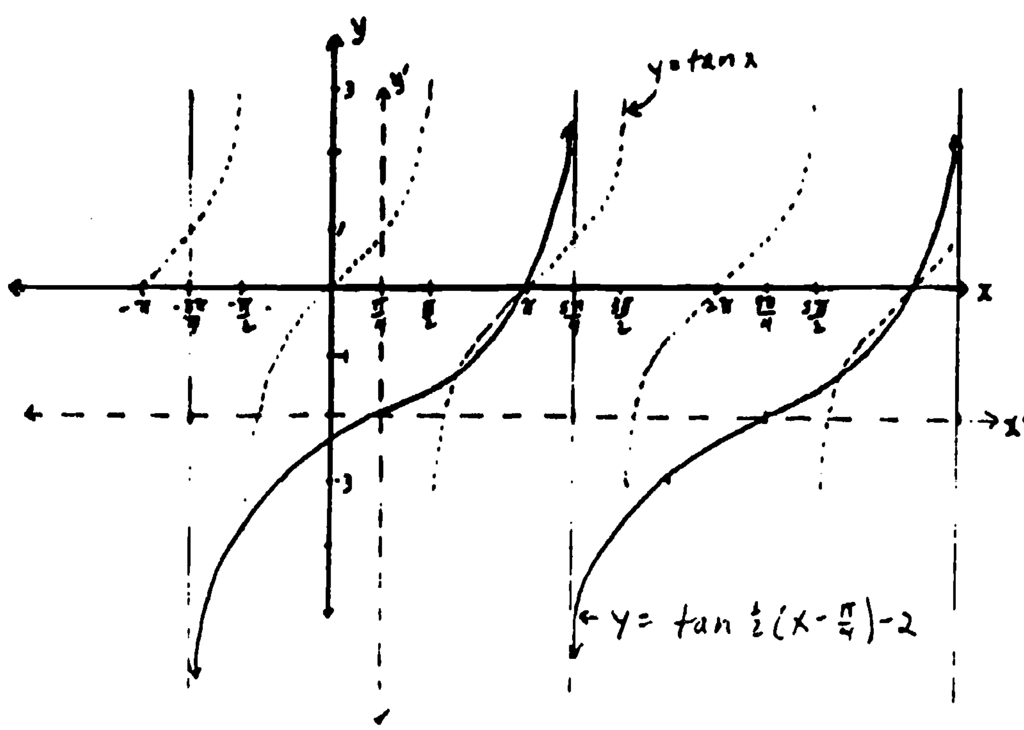
$y + 2 = \tan\frac{1}{2}(x - \frac{\pi}{4})$

"NEW ORIGIN": $(\frac{\pi}{4}, -2)$

IMPORTANT ANGLES:

EVERY $\frac{90}{1/2} = 180°$ (π)

PERIOD = $\frac{\pi}{1/2} = 2\pi$

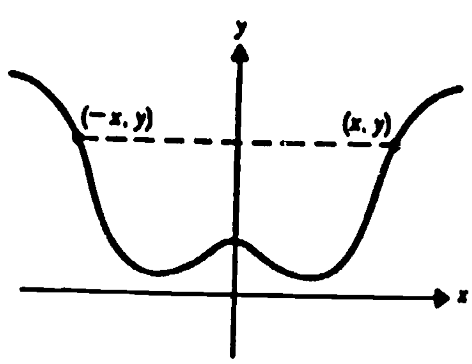


ALGEBRAIC PROPERTIES OF SYMMETRIC GRAPHS

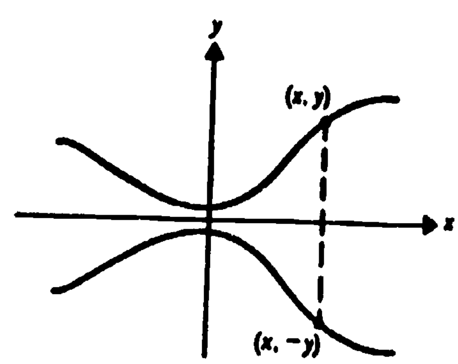
$(x,y) \longrightarrow (-x,y.$	symmetric in y-axis	[even function]
$(x,y) \longrightarrow (x,-y)$	symmetric in x-axis	
$(x,y) \longrightarrow (y,x)$	symmetric in line y=x	
$(x,y) \longrightarrow (-x,-y)$	symmetric in origin	[odd function]
$(x,y) \longrightarrow (-y,-x)$	symmetric in line y=-x	

ALGEBRAIC PROPERTIES OF OTHER TRANSFORMATIONS

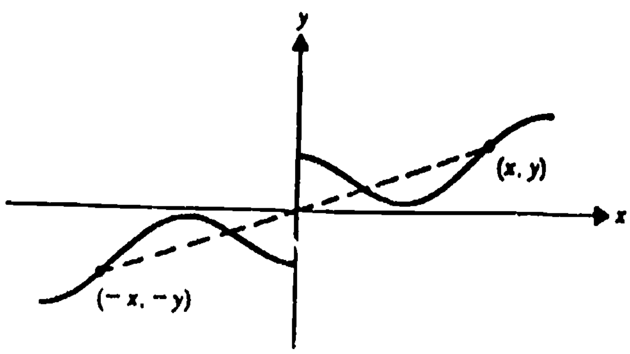
$(x,y) \longrightarrow (x+a,y+b)$	**translation**	[$a,b \in$ Reals]
$(x,y) \longrightarrow (ax, ay)$	**dilation**	[$a \in$ Reals]



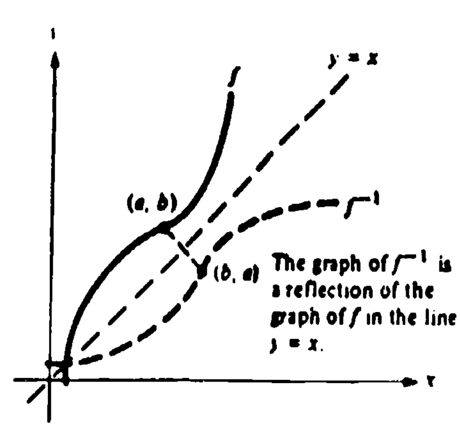
Symmetry about the y-axis



Symmetry about the x-axis

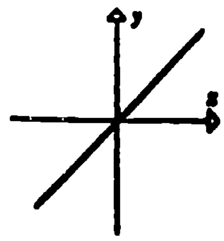


Symmetry about the origin

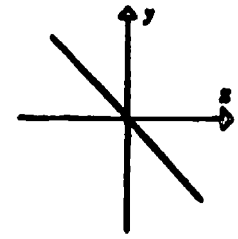


The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

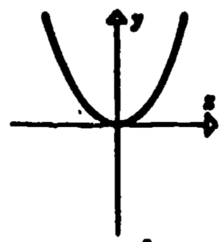
GRAPHS STUDENTS <u>MUST</u> KNOW



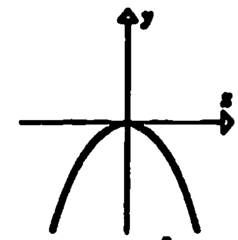
$y = x$



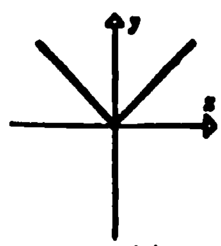
$y = -x$



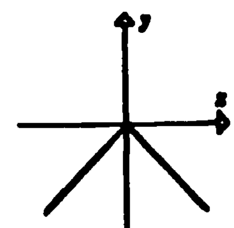
$y = x^2$



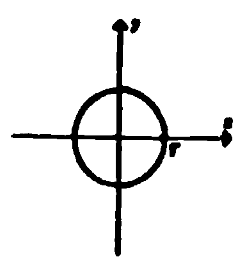
$y = -x^2$



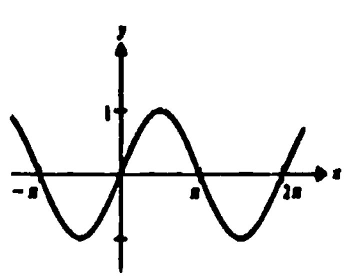
$y = |x|$



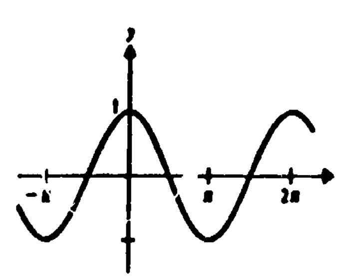
$y = -|x|$



$x^2 + y^2 = r^2$

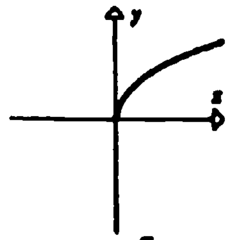


$y = \sin x$

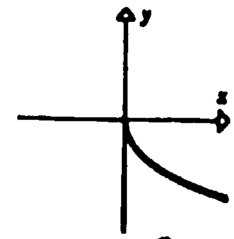


$y = \cos x$

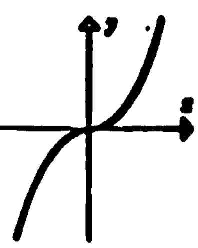
GRAPHS STUDENTS <u>SHOULD</u> KNOW



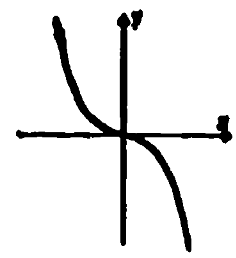
$y = \sqrt{x}$



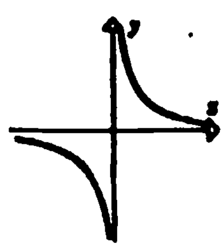
$y = -\sqrt{x}$



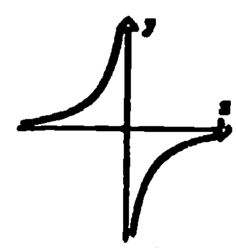
$y = x^3$



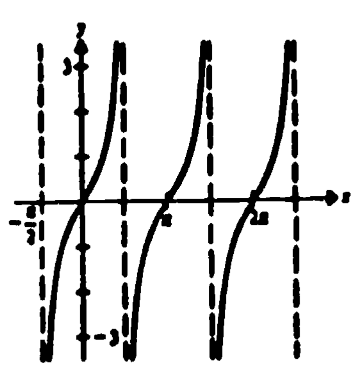
$y = -x^3$



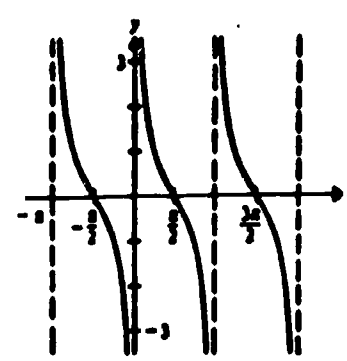
$y = \frac{1}{x}$



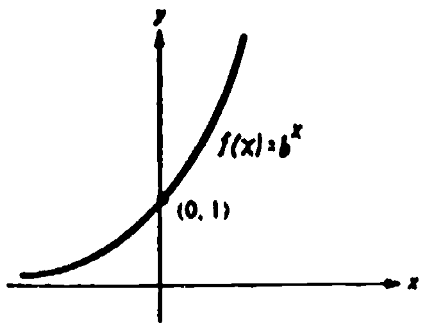
$y = -\frac{1}{x}$

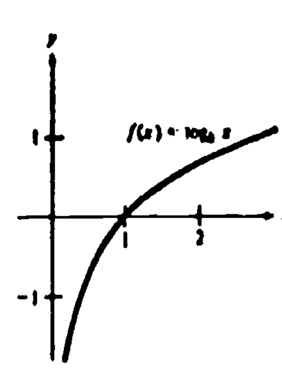


$y = \tan x$

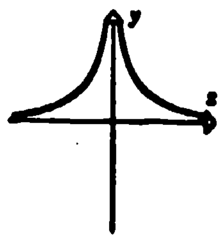


$y = \cot x$

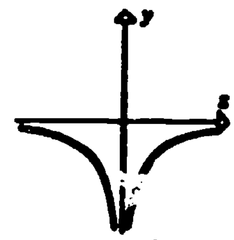




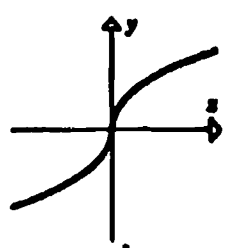
OTHER GRAPHS STUDENTS SHOULD KNOW



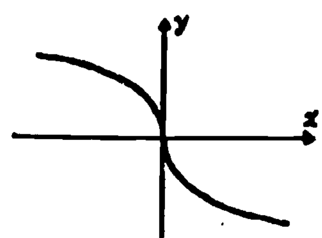
$y = \frac{1}{x^2}$



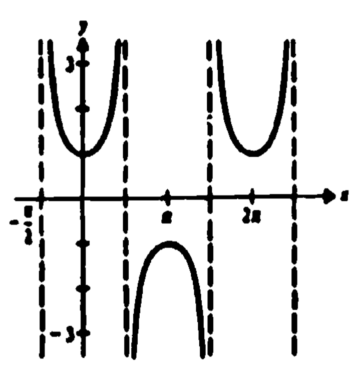
$y = -\frac{1}{x^2}$



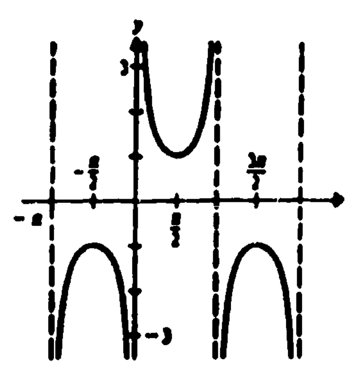
$y = \sqrt[3]{x}$



$y = -\sqrt[3]{x}$



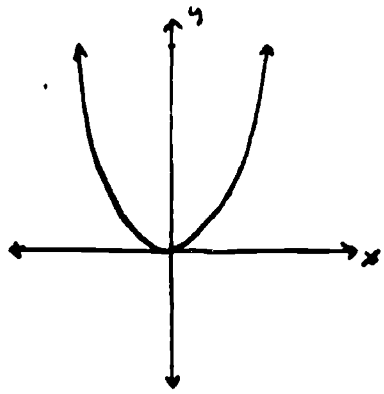
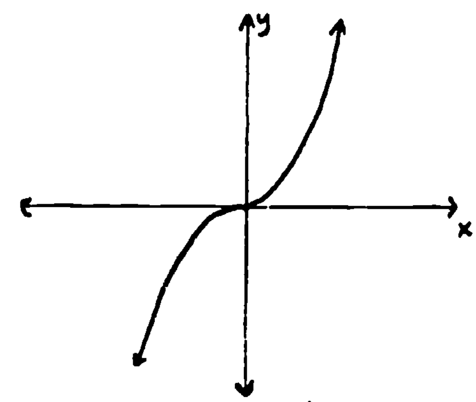
$y = \sec x$



$y = \csc x$

SAMPLE GRAPHS

Graphs of the form $y=x^n$ $n=1,2,3....$ resemble the:

parabola when n is **even**	cubic function when n is **odd**
	

This should be clear since any even exponent will produce positve y-values, therefore the entire graph is above the x-axis (except, of course, the vertex at [0,0]). Similarly, an odd exponent returns positve y-values when $x>0$ (graph is above x-axis to the right of the origin) and negative when $x<0$.

For $y=x^{a/b}$, there are several cases to be considered. First, let us require that the fraction a/b is simplified.

We examine this from a common sense point of view. If $a>b$, then $(a/b)>1$, so $y=x^{a/b}$ must be steeper than $y=x$ (i.e.: $y=x^1$), when $x>1$. And for $x>0$ (first quadrant), the graph will be parabolic in shape, as each succeeding x produces an even larger y-value.

> [NOTE: Between 0 and 1 are proper fractions, which are affected by exponents in the opposite way than numbers greater than 1. That is, while numbers greater than one grow when raised to increasing positive powers, fractions will decrease.]

Similarly, if $a<b$, then $(a/b)<1$, so $y=x$ will be not as steep (i.e.: it will be broader, or below) as $y=x$, and will resemble the graph of $y=\sqrt{x}$ (see top p. 18) when $x>0$.

Since "b" is the root of x, raised to the "a" power, both a and b will have important effects of the graph. If b is even, the graph will only appear to the right of (and including) the origin. Since we cannot take an odd root of $x<0$, no points will appear in the 2nd or 3rd quadrants (where x is negative).

If b is odd, there are no restrictions of x (i.e.: the Domain is all Reals).

Now, we must examine "a."

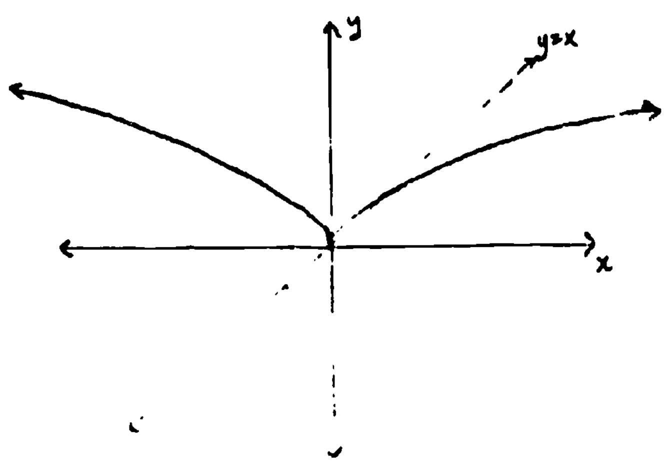
For the values of x defined by the Domain, according to whether "b" is even or odd, if a is even, all x-values will be above the x-axis, as an even power produces only positive y-values. If a is odd, the points to the right of the origin will be above the x-axis while those to the left will be below.

[NOTE: If a<b, the graph of $y=x^{a/b}$ will have a vertical tangent or a "cusp" at the origin. This is an excellent point of departure for a discussion of limits and/or differentiability in a Calculus class.]

EXAMPLES:

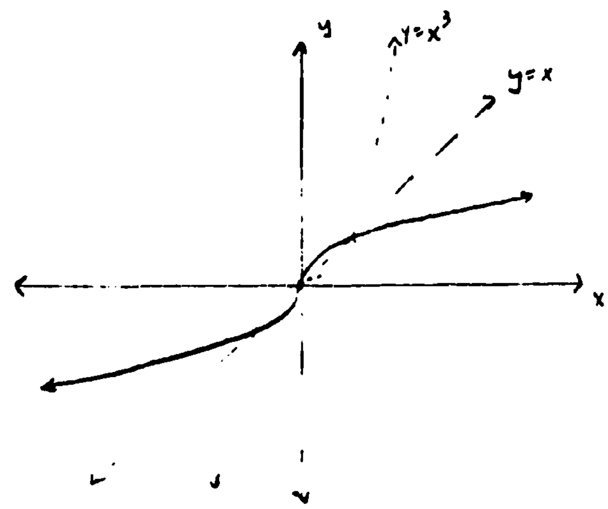
Case I: (a/b)<1, a is even

$y=x^{2/3}$



Case II: (a/b)<1, a is odd

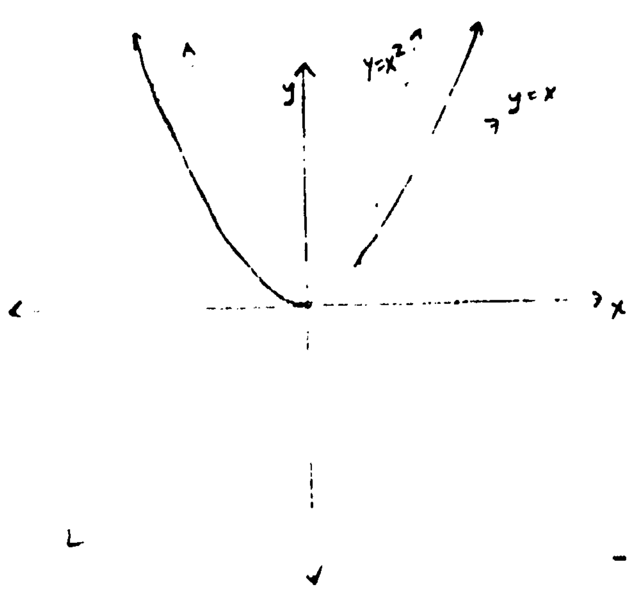
$y=x^{1/3}$ or $y=\sqrt[3]{x}$



NOTE: This is the inverse of $y=x^3$. That is, $y=x^3$ reflected in the line y=x.

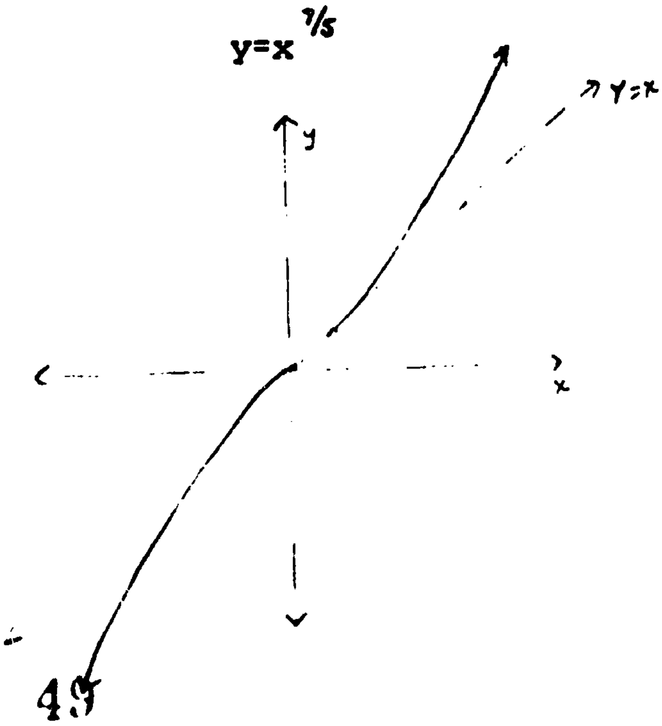
Case III: (a/b)>1, a is even

$y=x^{4/3}$



Case IV: (a/b)>1, a is odd

$y=x^{7/5}$

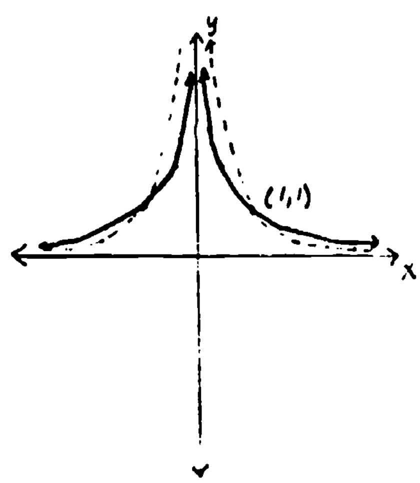


For $y=x^{-a/b}$, there are more considerations, although the same initial considerations will exist as do for $y=x^{a/b}$. We know that a negative exponent tells us to examine the <u>reciprocal</u> of the expression. For reference, we must know what the graph of $y=x^{-1}$ $(y=1/x)$ and $y=x^{-2}$ $(y=1/x^2)$ looks like (see pages 18 and 19).

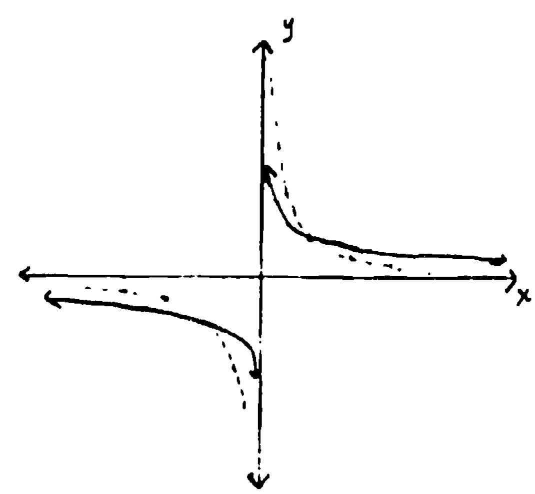
If a is <u>even</u>, the graph will resemble that of $y=x^{-2}$. If a is <u>odd</u>, the graph will resemble that of $y=x^{-1}$. If $a<b$, the vertical asymptotes approach the y-axis faster than the horizontal asymptotes approach the x-axis. If $a>b$, this is reversed. This is easily verified algebraically.

<u>EXAMPLES</u>: Note that $y=x^{-2}$ and $y=x^{-1}$ are sketched in for reference.

$y=x^{-4/3}$



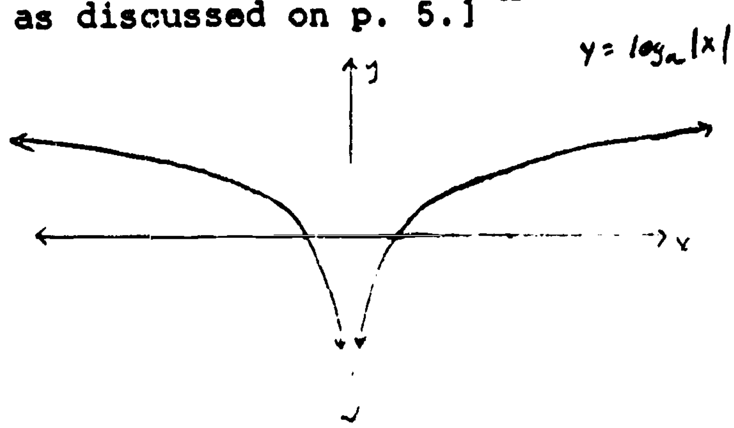
$y=x^{-3/5}$



Finally, let us consider some logarithm graphs. By definition, the Domain of any logarithmic function is $x>0$. Refer to the basic logarithm graph on p. 18.

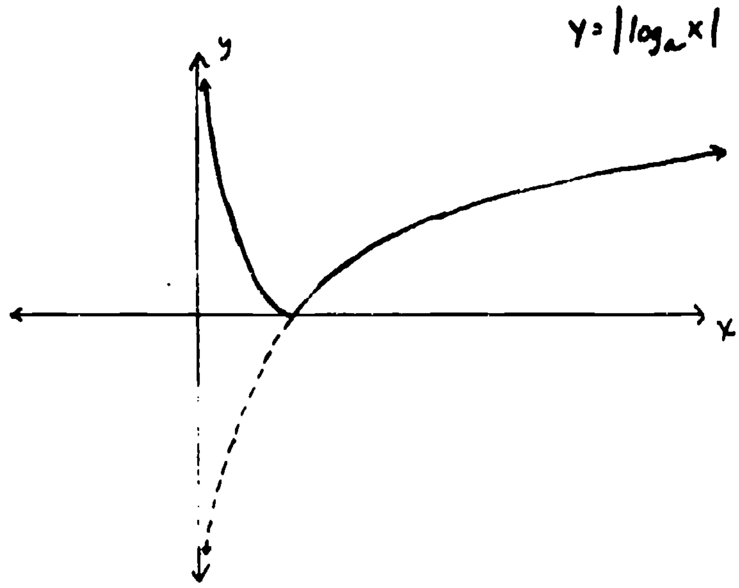
Now suppose we wanted to see what the graph of $y=\log_a|x|$ looks like. Since we are now taking the absolute value of x, we can now consider negative values for x. Yet the y-value for any negative x will be the same as for the corresponding positive x that appears on the graph of $y=\log_a x$. So this graph will be symmetric in the y-axis.

[NOTE: For $x<0$, the graph of $y=\log_a|x|$ is actually $y=\log_a(-x)$. This is simply the reflection of $y=\log_a x$ in the <u>y-axis</u>, as discussed on p. 5.]



For $y=|\log_a x|$, this forces our y-values to be only positive. Therefore there can be no values below the x-axis. As with any absolute value graph, all that needs be done to the related graph <u>without</u> the absolute value bars is to reflect the portions below the x-axis <u>in</u> the x-axis. Simply, this requires that all negative y-values now become their additive inverses.

Recall that y-values are negative for $y=\log_a x$ only when x is a proper fraction. So this will not disturb the original graph for $x\geq 1$, but the y-values generated by $0<x<1$ will be refelcted in the x-axis, resulting in the graph below:



Again, for $\log_a x<0$, the graph becomes $y=-(\log_a x)$, which is equivalent to $-y=\log x$. This is the reflection in the <u>x-axis</u> as discussed on p. 5.

7.

BIBLIOGRAPHY

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Kasner & Newman	Math & The Imagination
Eric Temple Bell	Men of Mathematics
Morris Klein	Math & the Physical World
James R. Newman	The World of Math (4 vols)
Edna Kramer	The Nature & Growth of Math
H.S.M. Coxeter	Introduction to Geometry
W.W. Rouse Ball	Math Recreations & Essays
W.W. Sawyer	Mathematicians Delight
Harold R. Jacobs	Math A Human Endeavor
Allendoerfer & Oakley	Principles of Advanced Math
Allendoerfer & Oakley	Fundamentals of Freshman Math
NCTM	Calculus - Readings from the Math Teacher
H.E. Huntley	The Divine Proportion
NCTM	28th Yearbook - Enrichment Math for High Schol
Burton W. Jones	Elementary Concepts of Math
M.I. Logsdon	A Mathematician Explains
Irving Adler	Groups
Beck, Bleicher, Crowe	Excursions Into Math (Worth)
New Math Librav	MAA Publications - 23 titles
Fehr Fey, Hill, Camp	Unified Math - Course IV
Readings from Scientific American	Math in the Modern World
Court	Math in Fun & Earnest
Rudolph v.B. Rucker	Geometry, Relativity & the 4th Dimension (Dover)

Richard F. Marr	4 Dimensional Geometry (Houghton Mifflin)
R. Courant & H. Robbins	What is Math? (Oxford)
Dan Pedoe	The Gentle Art of Math (Penguin)
H. Rademacher & P. Toeplitz	The Enjoyment of Math
Jagjit Singh	Great Ideas of Modern Math (Dover)

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TOPICS FOR MATH FAIR PRESENTATIONS
Starter List

Topology - Bridges of Koenigsberg
Infinite Sets & Transfinite Numbers
Continued Fractions
Fibonacci Numbers & The Golden Section
Conic Sections
Infinite Series
Math Induction
Limits
Calculus - Slope function & derivative
Greek Math
Archimedes
Zeno's Paradoxes
Non-Euclidean Geometry
Logic & Lewis Carroll
Groups & Fields
Rings & Integral Domains
Non-Routine Graphing Techniques
Transformations
Complex Numbers & DeMoivre's Theorem
$\exp(i\pi)=-1$
Hyerbolic functions
Probability & Pascal's Triangle
The Binomial Theorem
The 4th Dimension & Beyond
Diophantine Equations
Inversive Geometry
Irrational & Transcendental Numbers
Cryptanalysis
Godel's Theorem
Women in Math
Boolean Algebra & Switching Circuits
Math in Art