DOCUMENT RESUME

ED 340 570 SE 050 764

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TITLE Making Math Mean.
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SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE May 89
CONTRACT NSF-TEI-8751398
NOTE 20p.
PUB TYPE Reports - Research/Technical (143)

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *College Mathematics; *Concept Formation; *Concept Teaching; Higher Education: *Mathematical Concepts; Mathematics Achievement; *Mathematics Instruction; Mathematics Skills; Protocol Analysis; *Teaching Methods
IDENTIFIERS *Constructivism

ABSTRACT

This paper describes the implications of constructivism both for what to teach and for how to teach. The first part discusses three approaches on what to teach: (1) relating mathematical knowledge to other knowledge; (2) checking for self-consistency; and (3) putting thinking before facts. The second part is on how to teach based on constructivism. Skepticism and curiosity are the important characteristics of constructivist teaching. Examples from an introductory physics course and a calculus course are presented to illustrate students' diverse understandings of certain concepts. Some instructional methods, such as lectures, tutoring, and group discussion are critiqued from the constructivist viewpoint. The use of protocol analysis as a tool for reflecting students' informal thinking is introduced. (YP)

MAKING MATH MEAN

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May, 1989

In this paper, I will attempt to describe some of the implications of constructivism that have proved useful in the establishment and maintenance of a moderately large mathematics course. That is, I will not be considering some theoretical ideal, but rather describing what has proved possible under relatively normal conditions. The course in question has over a dozen sections, as many instructors, and 30 - 40 students per class. The course is designed to help students remedy deficiencies in their previous mathematics preparation and therefore attracts a very diverse clientele.

Constructivism has implications both for <u>what</u> we teach and for <u>how</u> we teach it. First, constructivism is a statement about the nature of knowledge and its functional value to us. Mathematical knowledge viewed from this perspective is not the same as when viewed from other perspectives. Second, constructivism implies a mechanism for how we acquire knowledge and, hence, how it is possible to teach. To date constructivist thinking has been more effective in describing what sorts of teaching will not work than in specifying what will. But it does at least define a direction for future exploration.

Finally, constructivism implies new relationships between the teacher, the learner, and the content being studied. These new relationships often do not fit the expectations of the students, the teachers, and the teachers'

This paper was supported by NSF Grant #TEI-8751398.

supervisors. Even without using an extreme fully constructivist approach teachers often provoke reactions such as the following:

> "If teachers would stick more to the facts and do less theorizing, one could get more out of their classes... A certain amount of theory is good, but it should not be dominant... The facts are what's there. And I think that would be... the main thing." (Perry, 1970 p. 67).

In the mathematics classroom the conflict can be even more severe. As Lakatos (1976) and Klein (1980) suggest, mathematics is the last refuge of those who believe in certain knowledge, and to challenge the absolutes of mathematics is to shake the foundations of all truth. No self respecting student can let this go by without at least a whimper.

What to teach.

Knowledge is used to organize past experience and to predict future experiences. There is no unique or even preferred way to do this. Another person's knowledge can be thought of as equivalent to our own only insofar as it seems to induce them to react to experiences in ways we would predict. Thus the teaching of mathematical content needs to be understood in terms of a set of mathematical experiences which the teacher attempts to arrange for the student. When the teacher believes that s/he and the student have negotiated a consensual domain (Maturana, 1978) within the realm of these experiences, then the material can be said to have been taught. The process is every bit as slippery as the description is convoluted. In short, mathematics is taught by having students do mathematics. When students react appropriately to a set of mathematical situations then we can assume they have learned a mathematics for those situations. We cannot automatically assume that their mathematics will lead them to react to new situations in the same way we do.

Therefore it is essential that we are careful to pick a set of mathematical experiences that includes a basis for all the mathematics we want to teach. Since most mathematics is taught for the purpose of enabling students to handle certain specific applications it is important to use those applications in teaching, and to consider very carefully the range of applications that will be explored. None of this is revolutionary although it can be highly controversial. Rather, there is a somewhat subtle shift in emphasis giving greater importance to examples and especially to the thought that goes into their selection.

Constructivism implies a rather convoluted approach to the teaching of mathematical content. It is much more straightforward on the teaching of mathematical thinking. Here the emphasis is on teaching students to be more effective constructors of mathematical concepts. Three aspects are worth special mention.

Relating mathematical knowledge to other knowledge

At the level of pure mathematics, as experienced by pure mathematicians, there is no need for reference to the nonmathematical world. But the experience of these individuals is based on a complex mathematical machinery which they have constructed over a period of years. The mathematical novice must construct his/her knowledge using conceptual structures and experiences which have already been constructed from their own, largely nonmathematical, experience. Students need to learn to evaluate their mathematical constructions in relation to their other knowledge. Since normal school mathematics often has the effect of discouraging such comparisons, it can be quite difficult to convince university students of the relevance of such activity.

It is also difficult to pick examples that really fit the students' experience. For example, one may try to evaluate knowledge of the weighted mean by comparison to the typical American college and high school grading system. A simple question would be to calculate the grade point average of 6 credits of A in French with 3 credits of F in German. Unfortunately research has shown (Hardiman, 1983) that although most American students are subjected to this sort of grading system from at least the age of 13, few college students actually understand it. In one study a student who had proposed the unweighted mean, A (4.0) + F (0.0) / 2 = 2.0, was asked if it was fair to count the 3 credit course as much as the 6 credit. He replied "it is not fair but that's how they do it."

Checking for "self-consistency"

A more traditional method of evaluating mathematical constructions is checking for self-consistency. However, consistency really is only in the eye of the beholder, Lochhead (1988). A striking example comes from the extension of the concept of exponent beyond whole numbers. In deciding what meaning to associate with a negative exponent one can investigate what interpretation would be consistent with previous conventions, and in particular the observation that

$$x^a/x^b = x^{a-b}.$$

Here self-consistency implies that a negative exponent should be interpreted as the multiplicative inverse, $x^{-a} = 1/x^a$. But this interpretation comes from a selective notion of consistency because it ignores the original definition in which the superscript was the number of times the base number was to be multiplied by itself, and there is no way to multiply a negative number of times. It is even worse with fractional exponents, as in $x^{1/2}$. Students therefore, often find such definitions by consistency to be very suspect. To overcome such concerns it is desirable to expose students to quite a few examples of this kind of reasoning and to point out the inconsistency in the consistencies sought.

Think before you fact.

Perhaps the most poisonous piece of realist dogma corrupting current instruction is the notion that students need to be taught some set of basic facts before they can be asked to think. This misconception springs from a failure to understand the dynamic nature of facts. Neither facts nor the process of learning them are static. A fact is the output of a thinking process and the fact production system is itself constructed via thinking. Thinking must come first. When a set of "facts" are memorized prior to serious work, that merely means that little thought has gone into their construction and that they are therefore ill-defined and poorly constructed. Knowledge composed of such facts tends to be fragile, disorganized and difficult to apply. While it is certainly possible to begin learning by acquiring large numbers of fragile facts which are later laboriously reshaped into coherent knowledge structures, it is not obvious that this is the most efficient way to proceed.

The reason that the dogma of putting facts before thinking survived as long as it has, probably stems from the difficulty teachers have in conceiving thinking activities that are appropriate to the learner's situation. To better understand that difficulty it is useful to consider the child's construction of number as described by Steffe, von Glasersfeld, Richards and Cobb (1983). Here it is shown that children first learn to recite the number words in sequence prior to understanding counting or number. It is tempting to assume that before mastery of the number word sequence there is little opportunity for numerical thought. That is not the case. Tasks such as distinguishing number patterns or matching up items as in setting a dinner table can be quite thought provoking. Instruction that emphasizes such activities can be remarkably effective, e.g. see Paul Cobb's chapter in this book. Thus even the most mundane knowledge should be seen not as a set of facts to be learned, but as a set of situations to explore.

How we teach

The most important characteristics of constructivist teaching are skepticism and curiosity. One should always remain skeptical about the effects of one's own efforts as a teacher and curious about the efforts of one's students. In general the teacher should spend more time listening to students than the students spend listening to the teacher. But before a teacher can be convinced of the need to reverse the usual ratio of listening times s/he must recognize just how ineffective certain traditional forms of telling tend to be.

Lectures can be a great deal of fun, especially for the lecturer; they can be inspiring and occasionally thought provoking, but they are rarely effective for producing polished knowledge. One way to convince oneself of the limits of lectures is to try the following type of experiment. Deliver a lecture on some non trivial but relatively simple concept. Leave enough time at the end to ask students to describe the concept in their own words; do this by asking a question which makes sure students will not respond by simply repeating your own words.

Let me share with you two of my own attempts. The first comes from an introductory physics class and was given about midway through the semester.

The following are student answers to a quiz question:

The question was asked directly after a lecture on force in which the "force causes velocity" misconception was addressed directly! Students had been explicitly told that the correct equation was Force = mass times the change in velocity per unit time ($F = m\,dv/dt$), and that force is not proportional to velocity (i.e. $F = kv$, is false). These equations were explained with examples and informal verbal descriptions as well as the standard formal explanations.

The question I asked was: "What is the cause-and-effect relationship between force and velocity?"

I got the following student answers:

> "As one increases, the other increases, and as one decreases, the other decreases."
>
> "A change in force causes a change in velocity."
>
> "Force causes velocity to occur. So if force increases, so does velocity, be it negative or positive."
>
> "The stronger the force on an object, usually the velocity is greater."
>
> "When the force increases, velocity increases, and when force decreases, velocity decreases. In other words, force is the cause and velocity the effect."
>
> "If the force is constant, the velocity will remain constant. If the force is increasing, velocity will increase. If the force is decreasing, the velocity will decrease."
>
> "Force is the cause which gives the object its velocity."
>
> "Force causes velocity, but when force stops, the effect may go two different ways: it may either speed up or slow down."

To the nonphysicist it may not be apparent that these answers miss the mark.

In fact, making a reasonable guess as to what each answer meant to its author is far from trivial. This is an illustration of why it is important for the constructivist teacher to know her/his subject well, even though he/she may not choose to expound upon it.

Another example comes from a course in calculus.

I asked the following question halfway through a semester:

> "During the past 6 months, the rate of inflation has dropped from 18% to 10%. Explain what this change means in terms of the value of a dollar."

Of the 26 responses, 16 stated that the value of the dollar would increase. The ten that were not clearly wrong are given below.

* * * * *

> "In terms of the dollar if inflation has dropped the value of the dollar had increased or at least its decrease in value has slowed down."

* * * * *

> "The value of the dollar is decreasing with inflation, but if the rate of inflation drops, then the change in the value of the dollar is increasing."

* * * * *

> "Dollar increasing 18%
> 10% increase Decrease of 8%
> * The value decreases by 8%."

* * * * *

> "When the rate of inflation is large, the value of the dollar goes down because what you used to be able to buy with a dollar you can't anymore because they cost more. If the rate drops, then you should be able to buy more with your dollar after. Everything will cost more, but it will just take a longer time to get to the price, it would have with a higher rate of inflation."

* * * * *

"t = 6 months
value has gone from 18% to 10%, .18 on a dollar to .10 on a dollar. di/dx = dropped 8%.
Dollar will be worth less as the months go on."

(This answer was accompanied by a large graph showing a downward curve of dollar value, with inflation on the y axis and months on the x axis.)

* * * * *

"This change means the value of the dollar is still being inflated, but not as fast - the slope of the curve measuring the rate of inflation is still positive but decreasing."

* * * * *

"The rate of inflation has dropped from 18% to 10%, a decrease in 8%. All this means is that the rate of inflation has slowed down, not that there is no inflation. The value of a dollar will not really change."

* * * * *

"6 months
rate of inflation - 18% - 10%
Value of a dollar = ?"

The change of the inflation rate from 18% to 10% means that the value of the dollar is not going to decrease so rapidly. This means that the increase of the inflation rate has slowed down.

This can be seen by plotting the two percent's as slopes of .18 and .10. The slope of .10 does not increase as rapidly, rate of inflation.

The 2nd derivative would tell how the slope (1st derivative) is changing - change in the value of the dollar."

* * * * *

"t = time
v = value of $ I = inflation

Change in value = dv/dh   rate of inflation = dI/dH

This change means that as the rate of inflation drops, the value of the dollar is dropping, as time increases."

(This answer also has a graph plotting time along the x axis and value of the dollar on the y axis.)

* * * * *

Once again it is apparent that questions of this type produce answers which can only be accurately interpreted by a teacher with a solid understanding of his/her subject. Of course struggling with such answers is a very good way to sharpen one's own understanding and is invigorating as well as difficult.

Tutoring

Tutoring is about as effective as lecture unless a great deal of effort is made by the teacher to find out what the student is really thinking and the teacher is also able to formulate situations that will induce the student to question inadequate concepts. Rosnick and Clement (1980) describe a series of tutoring studies that illustrate just how ineffective even carefully planned tutoring can be. One particularly successful student (in terms of right answers), later explained his strategy as figuring out what made sense and then doing the opposite. This case high-lights the point that both lecture and tutoring can appear to be highly effective if they are evaluated in terms of narrowly defined behavioral objectives. It is only after one has probed for a deeper level of understanding that these modes of teaching become suspect.

Telling It Like It Is

We often tend to operate under the naive belief that notation carries its own interpretation. The careful and consistent use of conventional notation is an important part of good instruction, but it is by no means an adequate substitute for dealing with the underlying mathematical concepts. It is nearly impossible for a person familiar with the conventions of mathematics to see the extent to which the system is inconsistent and illogical. Careful attention to the struggles of neophite students quickly reveals numerous quirks. The operation of dividing 4 by 3 is written as 4/3 or 4 ÷ 3 but also as 3 √4 , so much for the preservation of order. Most mathematicians would interpret f(x+1) as meaning some function of x+1 but would recognize c(x+1) as c times x+1. One of the most dramatic demonstrations of the inability of notation to convey its own meaning was a study by Rosnick (1981) in which he asked the following question:

> At this university, there are six times as many students as professors. This fact is represented by the equation S = 6P.
>
> A) In this equation, what does the letter P stand for?
>
> i) Professors
> ii) Professor
> iii) Number of Professors
> iv) None of the above
> v) More than one of the above (if so, indicate which ones)
> vi) Don't know
>
> B) What does the letter S stand for?
>
> i) Professor
> ii) Student
> iii) Students
> iv) Number of students
> v) None of the above
> vi) More than one of the above (if so, indicate which ones)
> vii) Don't know

Over 20% of calculus graduates chose S stands for professor, and this was not a random guess since every one of these students picked "none of the above" in answer to part A.

In spite of all the evidence against it, mathematics teachers keep searching for the self interpretive notation. Many believe that the confusion in the above problem would vanish if N_s and N_p were used instead of the variables s and p (e.g. Fischer, 1988). The letter N is believed to convey the idea that variables are numbers, but there seems to be no need to tell students how or why it is being used.

The above examples of the failures of lecturing tutoring and careful notation were given not to belittle these important elements of an effective instructional system but rather to place them in fairer context with several other more intuitively suspect components. These other components involve situations in which students are in some degree left to their own devices and in which teachers are unable to keep track of all of the ideas which the students are constructing. (This of course is always the case but in conventional instruction it is easier for the teacher to ignore the students' ideas). To remain sane in these situations, teachers need to cultivate the attitude of trusting the students' own efforts to understand the material, not in the sense of believing that the students' creations will be reflections of the "accepted truth" but in the sense that they are all there is to work with. The more carefully the students think about their ideas the better those ideas will become; it is of little direct help if the teacher thinks carefully about them. (Of course it is enormously important for the teacher to have thought carefully about the topic in question, otherwise it will be impossible to make much sense out of the students creations.)

One powerfully constructivist mode of instruction is student discussion, often in the form of some type of cooperative problem solving, with or without the supervision of an instructor. The primary goal of such discussions is to help students develop skills of constructing, evaluating and modifying concepts in the domain of interest (in our case mathematics). The teacher's role therefore is to work to improve the quality of the discussions rather than to focus from the beginning on the "correct" mathematical answer.

I recently worked with a student who was confronted with the problem of finding a number which, when multiplied by 6, would yield 3. He began by asking himself whether that number would be greater or less than 10. On discovering that 6 times 10 yielded 60 he decided to try numbers less than 10 and gradually worked his way down to 6 times 1. Since this was still too big he was puzzled as to what to try next. Telling him that the answer was one half provided no transfer to finding a number which when multiplied by 9 would yield 3. While I was concerned about the student's lack of number sense, I was more impressed by his analytical reasoning. A short time later I found this same student playing a significant role in student discussion of a very difficult word problem involving fractions. By the end of the semester he was one of the best students in his class. The point is that the facts will straighten themselves out once the tools for analyzing them are developed and refined.

We have found that our best instructors spend a great deal of time listening to student discussions (which are the major component of our classes) but very little time actively participating; but when they do contribute it is in such a way as to leave the students in control and to insure that all students remain active participants. To restrain one's natural enthusiasm to such a limited (but highly effective) role, it is necessary to have developed a great deal of humility concerning the probable impact of one's words of wisdom. It is also critical to appreciate the value in having a well developed sense of how the students conceive the mathematics.

One method for obtaining insight into the student's mathematical thinking, while at the same time giving students the opportunity to reflect on their own ideas, is to assign thought process protocols. In such an assignment, students are to write down everything they are thinking as they work on a problem. This is far from easy to do. An example is given below:

PROBLEM STATEMENT:

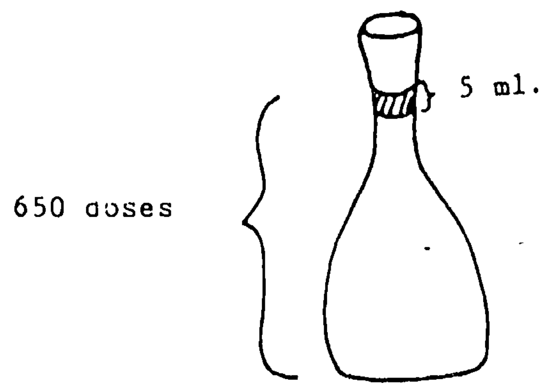
Flask Problem

A flask containing 650 doses of medicine had 5 ml. of the medicine removed for experimentation. Because of an emergency, another clinic was sent one-fifth of the remainder. Afterward, only 512 doses remained. Calculate the dosage in milliliters (ml.).

Hint: Be careful to recognize that there's a difference between number of doses, and number of milliliters per dose.

PROBLEM SOLUTION:

I've read the problem a couple of times, and I'm not sure where to start. "A flask containing 650 doses of medicine..." -- why don't I start by drawing a flask?



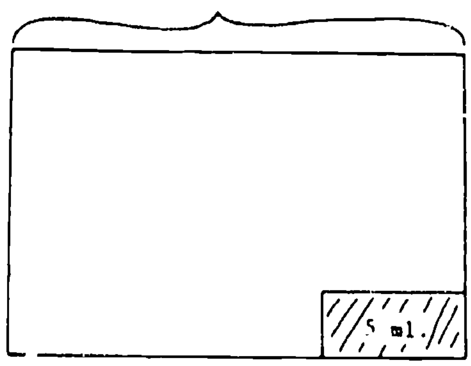
Here is a full flask and it contains 650 doses.

Next, 5 ml. are removed. I don't know how much that is in proportion to the 650 doses, so I will just shade in a little bit at the top which means it has been taken out.

Now, another clinic was sent one-fifth of the remainder. That means that I will have to divide what is left in the flask into 5 equal parts. But, how do I do that? The flask is a funny shape, and if I draw lines to separate it into 5 parts, I won't be sure that they will be 5 equal parts. Is there another way I can draw the picture? Does the flask have to look like the kind in the chemistry lab, or can I just use a box to represent 650 doses? I don't see why not. After all, 650 doses is 650 doses--right? I think so.

O.K. I'm starting over again. I'm going to draw a box which represents 650 doses of medicine. And, I'm going to take 5 ml. out of one corner.

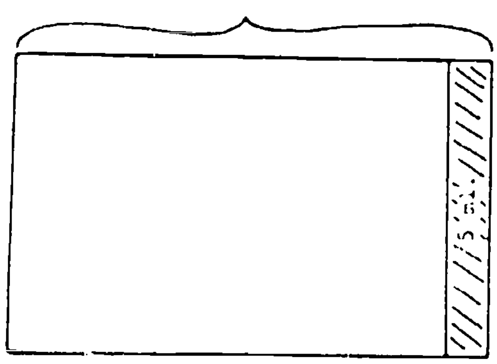
650 doses



I shaded it to show that it was taken out.

Oh, no. I have the same problem as before. I can't separate the remainder (after I take the 5 ml. out) into 5 equal parts. I have to do it another way. I will take the 5 ml. out of one end.

650 doses

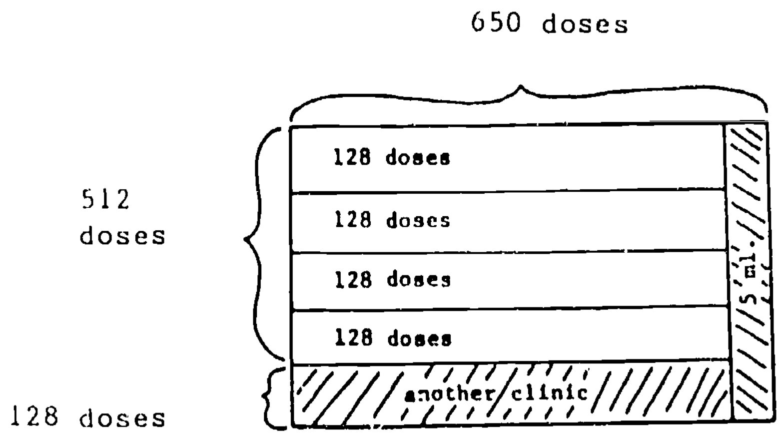


Good. Now I am left with something I can section into 5 equal parts.

I will draw the 5 equal parts and shade in the portion send to another clinic which will mean that it has been taken out.

650 doses

Now, the problem says that only 512 doses remained. That means that there are 4 equal parts remaining, which total 512 doses. Ah ha. Then, there are 512 4 128 doses in each part. That must also mean that the other clinic was sent 128 doses since their portion is the same size as each of the other equal parts. I will write this information in my drawing.



Let me review what the drawing tells me. There are 650 doses total. 512 doses are still in the flask and 128 doses were sent to another clinic. Then 512 + 128 = 640 doses accounted for. What about the 5 ml.?

There are 650 doses in all. 640 of them are accounted for. Then, 10 remain. Those 10 doses must be 5 ml.

If there are 5 ml. in 10 doses, how many ml. in one dose? This reminds me of one of those "miles per gallon" problems. Only now it is "milliliters per dose." With "miles per gallon", I always put the miles on the top and the gallons on the bottom like this:

$$\frac{\text{miles}}{\text{gallons}}$$

(Continued)

I will set up the "milliliters per dose" the same way:

$$\frac{\text{milliliters}}{\text{dose}} = \frac{5}{10}$$

If that is reduced, it comes to 1/2. That means there is 1 milliliter for every 2 doses. So, if there is 1 milliliter in every 2 doses, there must be 1/2 milliliter for every one dose.

My answer for this problem is that there is 1/2 ml. of medicine for every dose of medicine in the flask.

Note: This solution demonstrates one way to solve this problem. There are many other ways to also arrive at the correct answer. The specific operations used in the solution may not apply directly to other problems in this section. This problem solution is intended to be a model thought process protocol. It is not intended to be memorized for future quizzes, tests, etc. Each problem is unique and must be approached as such.

It is important for protocols to include the informal thinking we do when confronting a real problem and often do not do when racing through a well automated exercise. An algebra problem that can be solved algebraically usually involves very little interesting thought. Thus experienced teachers often make poor role models for this sort of assignment. (Note that the above example was written by a highly experienced protocol writer, it should not be anticipated that other undergraduates will write as thoroughly, especially during their first attempts).

It is important not to accept students first attempts as adequate responses. The virtue in writing thought process protocols is that they force a reorganization of thinking strategies. It takes a lot of frustration and struggle to reach a stage where writing protocols becomes a useful tool. Yet for those students who have the patience to master them, protocols can be exam savers. When faced with a problem that seems impossible, writing about the confusion often creates a sudden insight as to what to do. This may be for no deeper reason than that writing allows one to continue to act on the problem with out becoming panicked.

Summary

Constructivist teaching involves giving up the notion that you can do for students what, in practice, they must do for themselves. It demands that you trust your student's minds as much as your own and that you have faith in people's ability to learn. It requires humility concerning one's own ability to explain or expound . But the essence of constructivist teaching is in the realization that you will never know what is going on (in the minds of your students) yet it is fun (and rather useful) to try to find out. An occasional quiz of the sort illustrated for physics and calculus, can provide teachers with a brief glimpse of whom they are teaching. A detailed probing interview offers a somewhat larger picture, but of only one student. The opportunities to explore student understanding are unlimited and unending. The investigation may be infinite, but for those involved with it, teaching need never be dull or routine, no matter how mundane the subject matter.

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