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ABSTRACT

A new approach to teaching and curriculum takes seriously the knowledge children have when they enter school. Teachers use the knowledge each child has to make instructional decisions so that the child learns mathematics with understanding, how to solve problems, and the computational skills. Research concerning the problem-solving strategies actually used by children has led to the development of the Cognitively Guided Instruction (CGI) project, in which the use of such knowledge has been studied. Forty first-grade teachers in Madison (Wisconsin) were randomly assigned to treatment or control groups. The 20 experimental group members received extensive training in children's solution strategies during a training workshop in the summer of 1986, but were allowed to plan for themselves how they would use the knowledge. The other 20 teachers served as a comparison/control group in 1986 and took part in a similar workshop in the summer of 1987. Observations after one year showed that experimental group teachers adapted CGI ideas according to their own styles. However, the following three key elements were recognized: (1) multiple solution strategies were recognized and encouraged; (2) there was a focus on problem solving; and (3) teachers had an expansive view of the children's knowledge and thinking. When teachers know about children's mathematical thinking and problem solving, they can facilitate the development of mathematical abilities for children from disadvantaged backgrounds. Two tables, one graph, and a 24-item list of references are included. The paper's discussant is Judith Johnson Richards in a training section entitled "Appreciating Children's Mathematical Knowledge and Thinking in Ethnically, Linguistically, and Economically Diverse Classrooms". (SLD)

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USING CHILDREN'S MATHEMATICAL KNOWLEDGE

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USING CHILDREN'S MATHEMATICAL KNOWLEDGE

Maria has 4 peanuts. Her mother gave her some more. Now she has 11 peanuts. How many peanuts did her mother give her?

Most first- and second-grade teachers, and probably most adults, see the above problem as a subtraction problem that will be difficult for young children to solve. However, consider what Elissa (a four-year-old) did when asked to solve this problem. First, Elissa counted out four counters. Then she added more counters until she had 11. With her hand she separated out the original four counters, then she pointed to the group she had left and said to the interviewer, "This many." The interviewer asked her, "How many is that?" Elissa counted, "One, two, three, four, five, six, seven." Turning to the interviewer, Elissa announced firmly, "Seven peanuts!"

Elissa did what most young children do. She invented a way to solve the problem that was based on how she thought about the problem, not on any procedure that had been taught to her. She recognized that the problem involved joining some things together, and she did that. Elissa is not unusual. In fact, we have learned from research that all children come to school knowing a great deal about mathematics. If adults take children's mathematical knowledge seriously, they can help children use their knowledge to solve problems and learn more mathematics.

Adults have not always taken children's knowledge seriously. Typically, parents and teachers have assumed that children begin school with little or no knowledge of mathematics. This assumption was not unreasonable when the primary goal of the elementary mathematics curriculum was to develop skill in computation (e.g., to learn the basic facts and the algorithms of addition and subtraction). Children did not come to school with much knowledge of formal algorithms, so it made sense to assume that children did not have much mathematical knowledge. Although most educators knew that computational skills were not sufficient, they presumed that before children could understand the algorithms and use them to solve problems, children needed to have mastered computational skills. Thus, primary school instruction has focused on the practice of these skills to attain mastery. This emphasis is even more pronounced in the instruction of children in less advantaged socioeconomic areas, who spend more time in computational tasks than children in schools with more resources (Zucker, 1990). The tacit assumption is that once children have learned to compute with a reasonable level of facility, they can be taught to understand why the various procedures work and to apply the procedures to solve problems.

Findings from the National Assessment of Educational Progress and other research programs have documented, however, that this heavy emphasis on computation has been misplaced (Dossey, Mullis, Lindquist, & Chambers, 1988). Although children in the

United States can demonstrate computational skills at a reasonable level of proficiency, most children do not appear to understand the mathematics in the skills, and they cannot apply the skills to even simple problem situations. This situation has led the National Council of Teachers of Mathematics (1989) and the Mathematical Sciences Educational Board of the National Research Council (1989) to propose that problem solving and the development of mathematical understanding should be the foci of the mathematics curriculum *for all students* and that problem solving should be integrated throughout the mathematics curriculum rather than tacked on after computational skills are mastered.

Reconsidering Children's Mathematical Knowledge

A new approach to teaching and curriculum, which holds promise for achieving the expanded goals of mathematics instruction, takes seriously the knowledge that children have when they enter school. In this approach, teachers use the knowledge of each child to make instructional decisions so that the child learns mathematics with understanding, learns how to solve problems, and also learns the computation skills. This approach uses knowledge that has been accumulating from research on children's thinking in mathematics.

The Research Base on Children's Thinking

A growing body of research documents that children develop understanding, problem-solving abilities, and skills concurrently as they engage in active problem solving (Fennema, Carpenter, & Peterson, 1989a). This research also shows that children invent ways of solving problems that are not tied to traditional arithmetic solutions (Carpenter, 1985; Ginsburg, 1983; Lave, 1988). In fact, children's problem-solving experiences actually form the basis for their development of basic arithmetic concepts and skills.

Over the last 10 years, an extensive body of research has also accumulated on the development of basic addition and subtraction concepts and skills in primary school children (Carpenter, 1985; Fuson, 1988). This research shows that young children are adept at solving simple word problems, and their solutions often involve relatively sophisticated problem-solving processes. Even before children receive any formal instruction in addition and subtraction, they consistently solve simple addition and subtraction word problems by modeling and counting.

Consider, for example, the problem that Elissa solved at the beginning of the paper. Maria had 4 peanuts. Her mother gave her some more. Now she has 11 peanuts. How many peanuts did her mother give her? Most adults solve this problem by subtracting 4 from 11, but it is not easy to explain why to subtract. Subtraction is usually taught as representing a separating action like the situation in the following problem: Angelica had 14 dollars. She spent 6 dollars on a kitten. How many dollars does she have left? The

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problem about the peanuts, however, might be perceived as asking how much needs to be added to the 4 to get 11 peanuts. Accordingly, young children do not think of this as a subtraction or take-away problem. They solve the problem by modeling the additive action. If children have counters, they make a set of 4 counters and then add counters to this initial set until there are a total of 11 counters. By counting the counters that have been added on, children find the number of peanuts that Maria's mother gave her.

The above example illustrates two features that are important for understanding children's thinking and how elementary mathematics instruction might be designed to build on it. First, different problem situations exist that represent different conceptions of addition and subtraction, not just the simple joining and separating situations that are used to define addition and subtraction in most standard elementary mathematics textbooks. Second, children do not interpret all addition and subtraction problems in terms of pluses and minuses; they attempt to model the action and the relationships described in the problem.

Current research on children's thinking about addition and subtraction problems is based on a detailed analysis of the problem space (Carpenter, 1985). Addition and subtraction word problems are partitioned into several basic classes that distinguish among different types of actions and relationships. Distinctions are made among problems involving joining action, separating action, part-part-whole relationships, and comparison situations. Examples of each of these basic problem types are presented in Table 1. (For a complete description of this problem space and the related solution strategies, see Carpenter, 1985, or Fennema & Carpenter, 1989.) Within each class, three distinct problem types can be generated by systematically varying the unknown in the problem. For example Elissa's problem is a joining problem with the change unknown, while the joining problem in Table 1 is one with the result unknown. The first separating problem in Table 1 is a result-unknown problem, while the second one is a start-unknown problem. This classification scheme provides a highly principled analysis of problem types such that knowledge of a few general rules is sufficient to generate a complete range of problems.

The power of this analysis is that it is consistent with the way children think about problems and solve them. When young children initially solve word problems, they directly model the action or relationships in the problem using counters, fingers, or counting patterns. For example, a young child would solve the first separating problem in Table 1 by making a collection of 12 counters and removing 5 of them. A young child would solve the comparing problem most readily by making two sets and matching them to find out how many are left over. Elissa's solution to the problem at the beginning of this paper illustrates how one type of joining problem is solved.

Table 1

Basic Classes of Addition and Subtraction Problems

Problem Type	Example Problem
Joining	There were 7 birds on a wire. Five more birds joined them. How many birds were on the wire then?
Separating	Twelve frogs were in the pond. Five frogs hopped out. How many frogs were left?
	There were some frogs in the pond. Five hopped out. Then there were 7 frogs left. How many were there to start with?
Comparing	Charles picked 7 flowers. Penelope picked 12 flowers. How many more flowers did Penelope pick than Charles?
Part-Part Whole	There are 7 boys and 5 girls on the playground. How many children are on the playground?

Children at this level generally have difficulty with problems that cannot be easily modeled. For example, the second separating problem in Table 1 is difficult to model because the initial quantity is the unknown, so children have no place to start in attempting to model the action in the problem.

Over time, children invent more efficient strategies for solving these problems. They base these strategies on their growing understanding of number concepts. For example, a child might solve the problem about the peanuts by counting up from 4 and saying, "5, 6, 7, 8, 9, 10, 11. The answer is 7." In this case, the child does not exactly model any of the quantities described in the problem but simply keeps track of the number of steps in the counting sequence using fingers or some other device. Similarly, a child might solve the first separating problem in Table 1 by counting back from 12.

Even when first- and second-grade children appear to be using recall of number facts to solve problems, many children are actually using these modeling and counting strategies. Gradually, they begin to learn the number facts first by using a core of facts they know in order to derive or generate unknown facts. For example, to solve the first joining problem, six-year old Juan might respond: "Well, I know that 5 and 5 is 10, so since 7 is 2 more than 5, the answer is 12 because 12 is 2 more than 10."

Although for many teachers and adults this kind of thinking seems abstract, for Juan it makes perfect sense because he is building on and using what he knows in order to solve a mathematics problem. Derived facts are not used only by a handful of very bright students. Even without specific instruction, many children use derived facts before they have mastered all their number facts at a recall level. In a three-year

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longitudinal study in Madison, Wisconsin, over 80% of the children used derived facts at least occasionally at some time in grades 1 to 3, and 40% of the children used derived facts as their primary strategy at some time during the three years.

Indeed, all these kinds of thinking by children in the primary grades are typical. Almost all children spontaneously use the kinds of solution strategies that we have discussed. By the middle of the first grade, most children can solve many different types of addition and subtraction problems, and they are beginning to use more efficient counting strategies as well as direct modeling.

Children's solutions demonstrate in two ways the kind of mathematical thinking that we want to encourage. First, children can solve a variety of problems by attending carefully to the information given in the problem, not by looking for key words or using other tricks to bypass understanding. Second, the procedures they invent to find the answer demonstrate creative problem solving based on an understanding of fundamental number concepts. In the early elementary school years, children are capable of much more sophisticated thinking than adults have assumed. Children do not start school as blank slates but bring with them a rich store of mathematical knowledge that they have already acquired.

The research on children's solving of addition and subtraction problems demonstrates that children enter school with a rich store of informal knowledge that can serve as a basis for developing meaning for the formal symbolic procedures they learn in school. But the research does more than demonstrate that children know more and are capable of learning more than they have been given credit for. It also provides a principled framework for selecting problems and analyzing students' thinking that allows teachers to understand better their own students' thinking so that they can select appropriate instruction to build on the mathematical knowledge that their students have already acquired. Disadvantaged children, as well as advantaged children, have interacted with numbers in a variety of ways. They have counted many things, have some knowledge of money, and have had many natural interactions with numbers.

The analysis of the addition/subtraction problem space and the related research on children's solution strategies comprise a systematic body of knowledge that is useful in developing an approach to mathematics teaching and curriculum for all children in the primary grades. We consider now the Cognitively Guided Instruction (CGI) project in which we have been studying the use of this knowledge by teachers and children.

The CGI Approach

The CGI approach is based on two key assumptions: first, that knowledge of children's thinking about addition and subtraction problems can be useful to teachers;

and second, that just as children interpret and make sense of new knowledge in light of their existing knowledge and beliefs, so do teachers.

Sharing Research Knowledge with Teachers

Rather than attempt to use this research to specify a program of instruction, we decided to share the research-based knowledge about children's mathematical knowledge and thinking directly with teachers and to let teachers interpret for themselves what it meant to their instructional programs. Our approach is similar to how we believe children learn and is also compatible with site-based approaches to school improvement. Each child has to make sense of the world for himself or herself. Understanding comes only when a child is able to assimilate new knowledge in a way that is not in conflict with what she or he already knows and believes. Why should teachers be any different? In fact, research suggests strongly that teachers' understandings and beliefs profoundly influence their instruction (Clark & Peterson, 1986) and that teachers gain understanding in much the same way that children do (Duckworth, 1987; Lampert, 1984). Teachers in their classrooms are the ones who make the decisions that influence learning, and they make decisions that are congruent with what they understand and believe (Fennema, Carpenter, & Peterson, 1989b).

At the beginning of our National Science Foundation-supported CGI project, 40 experienced first-grade teachers from the Madison, Wisconsin area agreed to work with us. We assigned teachers randomly to one of two groups. The first group (CGI) participated in the training workshop during the summer of 1986. These teachers spent 20 hours per week for 4 weeks with us learning about children's thinking in addition and subtraction. The second group served as a comparison or control group during the first year and participated in a similar workshop in the summer of 1987.

During the workshop, we shared with the teachers the framework of problem types shown in Table 1 and the related children's solution strategies. The teachers viewed videotapes of children solving addition/subtraction word problems until the teachers could identify both problem types and strategies with relative ease. The teachers also interviewed five- and six-year old children to ascertain whether children actually used the solution strategies that had been discussed.

We did not tell the teachers what to do with the knowledge they had gained. We discussed the importance of a teacher's knowledge of how each child solves problems, the place of drill on number facts, and the necessity for children to think and talk about their own problem solutions with each other and with the teacher. We talked about adapting the problems (by type of problem or size of number in the problem) given to a child, depending on what the child understands and can do. We discussed writing problems around themes related to children's lives and classroom activities. (For a

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complete description of activities and readings used in the workshop, see Fennema & Carpenter, 1989.)

We gave the teachers time to plan how they would use their new knowledge in their classrooms during the following year. Teachers talked extensively with us and with other teachers about possible implications of the knowledge about addition and subtraction. Most teachers wrote examples of all the problem types to use in their classrooms, and tentatively planned one unit that they would teach sometime during the school year.

What Research Says About the Use and Effectiveness of CGI

We pre- and posttested children in the CGI and control teachers' classes, and we observed these teachers' mathematics teaching regularly during the 1986-87 school year. We also assessed the teachers' knowledge and beliefs about teaching mathematics both before the workshop and at the end of the school year (Carpenter, Fennema, Peterson, & Carey, 1988; Peterson, Carpenter, & Fennema, 1989; Peterson, Fennema, Carpenter, & Loef, 1989). We compared the instructional practices, beliefs, and knowledge of the CGI teachers and the learning of CGI students with those of the control group of teachers and their students.

When compared with control teachers, the CGI teachers spent significantly more time on word problem solving in addition and subtraction, and they spent significantly less time drilling on addition and subtraction number facts. CGI teachers also encouraged their students to solve problems in many more different ways, listened more to their students' verbalizations of ways they solved problems, and knew more about their individual students' problem-solving strategies. CGI students outperformed control students on written and interview measures of problem solving and number fact knowledge, including a measure of complex word problem solving on the Iowa Test of Basic Skills, and they reported greater understanding and confidence in their problem-solving abilities. Although CGI teachers spent only half as much time as control teachers did in teaching number fact skills explicitly, CGI students demonstrated greater recall of number facts than did control students.

Those teachers who believed more in the ideas of CGI and had more knowledge about their children listened more to their children's verbalizations of their thinking, and they implemented CGI more than did those teachers who had lesser knowledge and weaker beliefs. In sum, at the end of only one year, the research evidence demonstrated that teachers' use of the knowledge of children's mathematical thinking that they had gained from the workshop and developed in their classroom practice made a significant difference in their children's confidence and abilities to solve mathematics problems. (For complete descriptions of these results, see Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Peterson, Carpenter, & Fennema, 1989; Peterson, Fennema, & Carpenter, 1988/1989.)

But would a CGI approach be effective with disadvantaged children in inner-city schools? Indeed, some would argue that disadvantaged children still need drill on computation skills and number facts. However, we have recent evidence that a CGI approach can be quite successful in this setting.

Significant effects of CGI on students' problem solving were reported recently by Villasenor (1990), who worked with first-grade teachers in inner-city public and private schools in Milwaukee, Wisconsin. Villasenor participated in a CGI workshop taught in Madison, Wisconsin, by two members of our original project staff. He then used the Cognitively Guided Instruction program implementation guide, readings, and materials developed by Fennema and Carpenter (1989) to conduct a one-week, four-hour-per-day workshop (for a total of 20 hours) for 12 inner-city Milwaukee first-grade teachers in the summer of 1989. These teachers volunteered to participate, and they became the CGI "treatment" group. Villasenor also recruited another group of 12 first-grade teachers from schools in inner city Milwaukee who formed the "non-treatment" control group. During the workshop, teachers in the CGI group focused on understanding the different types of word problems in addition and subtraction and on understanding students' strategies for solving these word problems. Teachers explored ways to assess students' mathematical knowledge as well as ways to use this knowledge to design instruction, and they planned their instruction using CGI for the upcoming school year. During the school year, these CGI teachers met once a month on Saturday mornings to share their ideas about CGI and talk about their implementation of CGI ideas in their first-grade classrooms. Teachers in the control group participated in two 1-1/2-hour workshops on problem solving in October and in January.

To assess students' problem-solving achievement at the end of the year, Villasenor used the written test of problem solving that we developed (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Students in CGI teachers' classes achieved significantly higher scores than did students in control teachers' classes, achieving an average of 9.67 out of 14 items correct, a mean score that was nearly 4 standard deviations higher than the average score of 2.92 for students in the control group. CGI students also showed significantly greater knowledge of number facts, achieving an average of 4.75 out of 5 items correct, or about 5 standard deviations above the mean of 2.29 for the control group students. These significant results are shown in Figure 1.

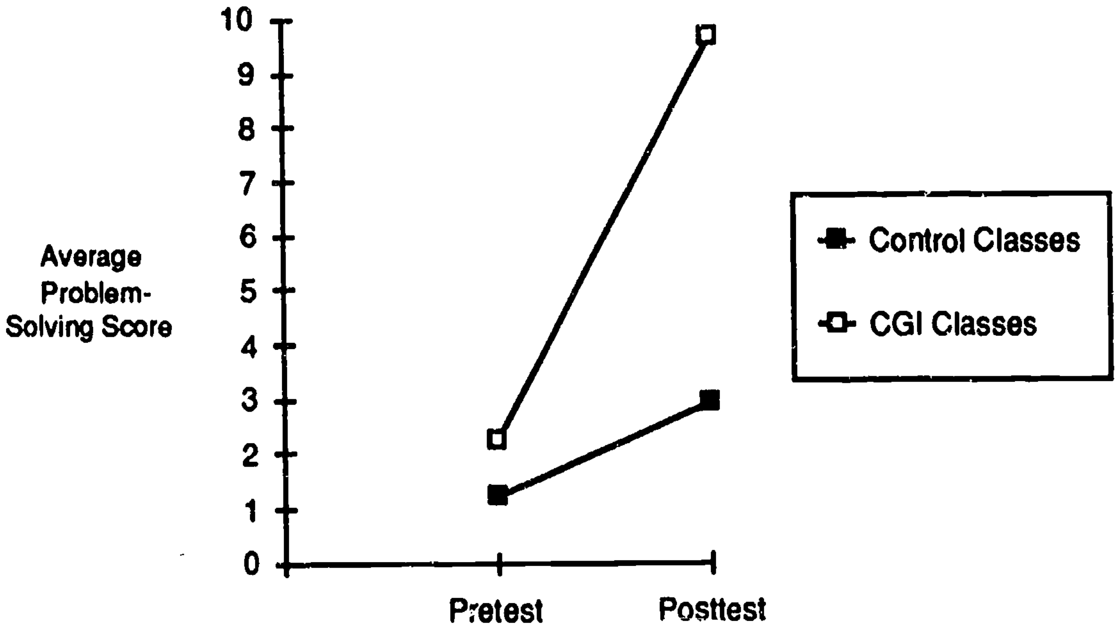

FIGURE 1 AVERAGE PRETEST AND POSTTEST PROBLEM-SOLVING SCORES OF STUDENTS IN CGI CLASSES AND CONTROL CLASSES IN VILLASENOR'S (1990) STUDY

One possible limitation of the study involves the comparability of the control group to the CGI group of teachers and students. The groups were similar in at least one respect. In both groups, teachers taught in schools with an enrollment that averaged 76% minority students, drawn from predominantly Hispanic and black populations. However, the groups were dissimilar in that teachers in the CGI group had students who performed 1 standard deviation higher on the problem-solving pretest than did students in the control group (although in both groups students' averages on the pretest were only 1 or 2 problems correct out of 14). Nonetheless, CGI students' average increase in problem-solving achievement was still significantly greater than that of control students, even when initial pretest scores were taken into account. In sum, Villasenor's results are important because they provide concrete evidence for the effectiveness of the CGI approach with a disadvantaged population of students—the same kinds of students who often participate in compensatory education programs.

A Look At CGI Classrooms

After the original year of studying CGI and its impact, we have continued to study CGI teachers. As a result, we know that CGI classrooms are different from traditional classrooms. Consider, for example, the following descriptions of a traditional second-grade classroom and a first grade CGI classroom.

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Traditional Classroom

The lesson begins with a three-minute timed test in which each student tries to beat a personal best in writing answers to number facts with speed and accuracy. Then Ms. K. reads aloud two word problems involving addition or subtraction of multidigit numbers. Students work on these problems at their tables. Ms. K. calls on a few students to explain their strategies. The students respond by stating the algorithms they used. Finally, the teacher passes out worksheets containing more word problems that are result-unknown problems. The students are to solve these with traditional multidigit algorithms. She asks the students to complete three of these problems before the end of the lesson. All students work the problems with the traditional algorithms while Ms. K. circulates to help them.

In the traditional primary mathematics classroom, children are on task and doing what the teacher has told them to do. Most activities focus on learning a computational procedure to solve each word problem. The teacher expects all children to do the same routine and to have the same knowledge in mathematics. The word problems seem to serve as a context for children to practice their algorithms rather than as a context for children to make visible and share their thinking and problem-solving strategies. The teacher expects all children to use the same strategy—the standard algorithm—to solve each word problem. The teacher bases all her decisions on what she thinks is important for children to learn—in this case, the addition and subtraction procedures.

CGI Classroom

While most of the class are solving word problems independently or in small groups, Ms. J. is sitting at a table with three students, Raja, Erik, and Ernestine (Ern). Each child has plastic cubes that can be connected together, a pencil, and a big sheet of paper on which are written the same word problems. As the students peruse the problems, they notice their names:

Raja:	My name is already there!
Ms. J.:	Your name is there? Yes!
Ern:	My name is on! Her name is on!
Ms. J.:	Yeah, and so is your name, eventually. Okay. Who wants to read the first one?
All:	Me!
Ms. J.:	Well, let's read them together.
All:	[reading] Raja made 18 clay dinosaurs. Ernestine has 9 clay dinosaurs. How many more clay dinosaurs does Raja have than Ernestine? [A compare problem]
Ms. J.:	Okay [reads problem again as students listen].

The students work on the problem in different ways. Raja puts together 18 cubes. She removes 9 of them and counts the rest. She gets 11. She writes the answer down, then looks up at the teacher for confirmation. Ms. J. looks at the answer, looks back at the problem, and then says, "You're real close." As Raja recounts the cubes, Ms. J. watches her closely. This time Raja counts 9.

Ernestine exclaims, "I've got it!" Ms. J. looks at Ernestine's answer and says, "No. You're real close."

Erik connects 9 cubes, and in a separate group he connects 18. He places them next to each other and matches them up, counting across each row to make sure there are 9 matches. Then Erik breaks off the unmatched cubes and counts them. "I've got it!" he announces. Erik writes down his answer. He says to Ms. J., "Got it. Want me to tell you?" Ms. J. nods "Yes." Erik goes to Ms. J. and whispers his answer in her ear. Ms. J. nods, "Yes" in reply. Turning to the group, she queries, "Okay now, how did you get your answers? Remember, that's what's always the important thing is: How did you get it? Let's see if we can come up with different ways this time. [Erik has his hand raised.] Erik, what did you do?"

Erik:	I had 9 cubes and then I had and then I put 18 cubes and then I put them together. And the 18 cubes, I took away some of the 18 cubes.
Ms. J.:	Okay, let's see if we can understand what Erik did. Okay, you got—show me 18 cubes.
Erik:	Okay. [He puts together two of the three sets of 9 he has lined up in front of him.]
Ms. J.:	Okay, so you have 18 cubes. Then you had nine.
Erik:	[He takes 9 cubes in his other hand and puts them side by side.] Yeah.
Ms. J.:	Then you compared.
Erik:	[simultaneously with Ms. J.] Then I put them together.
Ms. J.:	Then you put them together.
Erik:	Then I took...
Ms. J.:	Nine away.
Erik:	Nine away, and I counted them [the ones left], and there were 9.
Ms. J.:	Okay. So that's one way that you did it. Nice job, Erik. Which way did you do it, Raja?

Raja: [She has a set of 18 cubes connected in front of her.] Well, I counted 18 cubes.

Ms. J.: Erik, let's listen to her way.

Raja: Then I counted 9. [She counts 9 of the 18.]

Ms. J.: You counted 9. [Another student comes up to the table and puts a paper in front of Ms. J. Ms. J. puts her arm around this student while she continues listening to Raja.]

Raja: Then I [she breaks the 9 cubes away from the 18], then I got 9.

Ms. J.: Okay, great! [To Ernestine]: What did you do? [To Raja as she pats her on the hand]: Nice job.

Ern: I knew 9 plus 9 was, 9 plus 9 is 18. I took away one 9, and it was 9.

Ms. J.: Okay. Say that again. I'm sorry, I missed your problem.

Ern: I said I knew 9 plus 9 was 18.

Ms. J.: You knew that 9 plus 9 was 18. Okay.

Ern: I took away one 9 and it was 9.

Ms. J.: Okay. Good. So we had—how many different ways did we do that problem? Erik, you did it one way, right? Raja, was your way different from Erik's? [Raja nods "Yes."] Was your way different from Ernestine's? [Raja nods "Yes."] So that was two ways. Ernestine, was your way different from Raja?

Ern: Yes.

Ms. J.: Was your way different from Erik?

Ern: Yes.

Ms. J.: So we did the problem in three different ways. Let's read the next problem.

The three children in Ms. J.'s class worked on one problem for about five minutes. These children, who could be characterized as disadvantaged, were solving a relatively difficult comparison problem. Ms. J. had written the problem just for these children and had even put two of the children's names in the problem. In deciding on the type of problem and the size of the numbers in the problem, Ms. J. drew on her knowledge of these children's mathematical knowledge and thinking. Each child figured out a way to solve the problem and described clearly what he or she had done. Ms. J. is experienced at listening to her children talking about their thinking. She is adept at understanding what they are trying to say and at gently probing when she doesn't understand how they are thinking. When the above dialogue occurred, Ms. J. had been working with these children for a number of months, so she knew the kinds of thinking they might do. She also had a well-organized body of knowledge that included how young children typically solve comparison problems. She recognized each strategy as one that many children use to solve comparison problems.

In the CGI classroom, the teacher poses problems that each child can solve at his or her level of mathematics knowledge and understanding. The teacher encourages each child to solve mathematical problems using ways that make sense to the child. Ms. J. encourages each child to tell her how he or she solved the problems and uses what the child tells her to make instructional decisions. Children are aware that their thinking is as important as the answer and are not only comfortable, but determined that Ms. J. understand how they have solved each problem.

In other CGI classrooms, teachers work with the whole class and take into account individual differences in students in two ways. First, teachers pose problems that can be solved in a variety of ways so that each child can solve the problem according to his or her level of mathematical knowledge. Second, teachers substitute smaller or larger numbers in the same word problem, depending on their judgments of the size of numbers that different children can work with and use. After posing a problem to the whole class, teachers typically call on several children, one at a time, to say and show how they solved the problem. Often the teacher expects the next child to provide a different solution strategy from the ones given previously.

Key Elements of CGI Classrooms

Because teachers were not given prescriptions, they have adapted CGI ideas according to their own teaching styles and according to what is comfortable and right for them. Thus, each CGI classroom is, in some sense, unique. However, we have observed three key elements that all CGI classrooms have in common. As we have discussed already, one important element is that *multiple solution strategies to problems are recognized, encouraged, and explored.* These solution strategies are brought out as the thinking of children becomes visible within the context of solving problems. This points to a second key element of CGI classrooms: *a focus on problem solving.* A third key element of CGI classrooms is that *teachers have an expansive view of children's mathematical knowledge and thinking.* CGI teachers believe that all children know something about mathematics and that, as teachers, they need to figure out continually what children know about mathematics and then use this knowledge to plan and adapt their mathematics instruction. We consider these last two elements further in the sections that follow.

Focus on Problem Solving

Problem solving is the focus of all CGI classrooms. Teachers pose many problems for children. They carefully write or select these problems to be appropriate for their children. Generally, teachers construct problems relevant to the children's real lives in school and out, such as a forthcoming class trip to the school forest or the real need to figure out at the beginning of each school day how many children will eat hot and cold lunches. Problems emerge during social studies lessons, or from a book the teacher happens to be reading to the class, or from a fantasy world constructed by the class, such as the Friendly Forest where raccoons can change the number of stripes on their tails depending on the problem. Some teachers, like Ms. J., usually write problems for each child, but not all teachers do. Because all problems can be solved in a variety of ways, teachers find that only a few problems can occupy the entire class for a day. Table 2 presents a set of problems that one teacher constructed for her children as she was reading her class the book *The Berenstein Bears and Too Much Junk Food.*

Problem solving is not limited exclusively to word problems. For example, children invent their own ways to solve multidigit number problems. The following solution shows the thinking of a child in a CGI classroom as she added two three-digit numbers, 248 and 176, to solve a word problem:

> *Well, 2 plus 1 is 3, so I know it's two hundred and one hundred, so now it's somewhere in the three hundreds. And then you have to add the tens on. And the tens are 4 and 7...well, um, is you started at 70, 80, 90, 100. Right? And that's four hundreds. So now you're already in the three hundreds because of the (40+70). But you've still got one more ten. So if you're doing it: 300 plus 40 plus 70, you'd have four hundred and ten. But you're not doing that. So what you need to do then is add 6 more onto 10, which is 16. And then 8 more: 17, 18, 19, 20, 21, 22, 23, 24. So that's 124. I mean 424.*

An Expansive View of Children's Mathematical Knowledge

CGI teachers believe that all children know something about mathematics and that, as teachers, they need to consider and use their children's mathematical knowledge in planning instruction and in making decisions during instruction. CGI teachers realize that they need to be learning continuously about their children's mathematical knowledge and thinking as it is developing. They continually assess what each child can do, formally through individual interviews and informally as part of ongoing classroom discourse when children solve problems. During classroom discourse, the teacher typically encourages the children to solve a problem any way they wish. Then the teacher asks individual children how they solved the problem and listens carefully to each child's explanation. Teachers' knowledge of the organized framework of problem types and related solution strategies helps them understand and keep track of individual students' thinking as well as the kinds of problems a student can solve.

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Table 2

WORD PROBLEMS WRITTEN BY A SECOND-GRADE TEACHER (Van Den Heuvel, 1990) to accompany *The Berenstein Bears and Too Much Junk Food* (Berenstein, S., and Berenstein, J.)

1. Sister Bear used to weigh 55 pounds. Then she ate too much junk food, and now she weighs 71 pounds. How many pounds did she gain?

2. Brother Bear weighed just the right amount for his height. Then he ate too much junk food and gained 24 pounds. Now he weighs 90 pounds. How much did he weigh to begin with?

3. If Brother Bear weighed 84 pounds, and he weighed 37 pounds more than Sister Bear, then how much would Sister Bear weigh?

4. Papa Bear weighed 21 pounds more than Brother Bear and Sister Bear combined. If Brother Bear weighed 45 pounds and Sister Bear weighed 31 pounds, then how much would Papa Bear weigh?

5. While the bears were getting back in shape, Mama kept track of how much weight they lost. Brother Bear lost 23 pounds. Papa Bear lost 47 pounds more than Brother Bear lost. How many pounds did Papa Bear lose?

6. Sister Bear was a little "chubby" as Mama Bear put it. Then she began to eat healthier food and exercise more. After one month she lost 11 pounds. Then she weighed 60 pounds. How much did she weigh when she was "chubby?"

7. Write your own problem about the Berenstein Bears and their exercise program.

A Limiting View of Children's Knowledge: The Case of Ms. W. and Adam—To clarify what we mean when we say that CGI teachers have an expansive view of children's knowledge and that they use their understanding of what children know to build on children's thinking, we first describe a teacher who takes a limiting view of her children's mathematics knowledge. In the following example, Ms. W.'s limited view of Adam's knowledge leads her to miss opportunities to use that knowledge to help other students learn (Lubinski, 1989):

Ms. W.: [writes on the board]:
 35
+35

Ms. W.: Now, who can add this for me? Adam.

Adam: 70

Ms. W.: How did you get that?

Adam: Well, I knew that 3 and 3 is 6, so 30 and 30 is 60. And 5 and 5 is 10, and 60 plus 10 is 70.

Ms. W.: OK. You have the right answer. However, if I did 3 plus 3 is 6, and then I went to 5 plus 5 is 10, and I put that down, Adam, I'd have 610.

Ms. W.: [writes on the board]:

$$\begin{array}{r} 35 \\ \underline{+35} \\ 610 \end{array}$$

Ms. W.: Is that the right answer?

Adam: No.

Ms. W.: You have the right answer, but how could I do that to show it?

Adam: You could do 5 and 3.

Ms. W.: Well, I can't. They live in different houses.

Adam: You add the fives and then you add the threes.

Ms. W.: Well, I'm over here in the ones' house. What do I have to do? I'll bet, Linda, you remember what I did. . . .

In an interview following the class, Ms. W. described her thinking about the above situation as follows:

> *He [Adam] had a very good way to explain it, but he wasn't explaining that I wanted him to carry the 10. You have so many children that will write down the 10 and then go to the tens' column and put down a number there too and come up with a three digit answer when it should be two. I thought his process—his thinking—was excellent, but he would not have been able to record it. He would have known it was wrong, but he wouldn't have known how to change it.*

This episode illustrates that children are capable of sophisticated mathematical thinking. But, like many adults, Ms. W. missed the opportunity to capitalize on a child's thinking because she looks at the problem only from her point of view rather than the child's and attempts to teach (in this case, the carrying procedure) rather than to listen, understand, and facilitate the child's development of mathematical knowledge. However, we cannot be too critical of Ms. W. She was concerned with more than whether Adam had the right answer. She did ask Adam to explain how he got his answer, and she seemed to understand his explanation. Yet she was unwilling to let go of her role as the dispenser of knowledge to try to build on Adam's thinking. Although she expressed a concern that Adam's procedure would result in errors for him or for other students, nothing in Adam's response suggests that he could *not* have written his answer correctly. Indeed, many students who had difficulty understanding Ms. W.'s procedure of "carrying the 1" may have understood Adam's thinking about the problem, and might have been able to solve the problem after listening to him. Ironically, Ms. W. did not seem to recognize that Adam was modeling exactly the kinds of mental computations that are advocated by authors of mathematics education reform documents such as the NCTM *Standards* (1989, pp. 46-47).

Expanding Teachers' Views of Children's Knowledge—Although Ms. W. participated in a CGI workshop, she does not yet take the expansive view of children's mathematical knowledge illustrated by Ms. J. However, Ms. W. is starting to listen to how her students solve problems. This is an important step. Some CGI ideas conflict with central beliefs of many teachers—that teachers are the source of knowledge and that teachers have a responsibility to "cover" all the mathematics content specified in a mandated curriculum. Changing beliefs and attitudes takes time. The CGI workshop alone did not change teachers; teachers changed most when they began to listen and attend seriously to their own students' thinking as the children solved problems. We found that the impact of the CGI approach and the change in teachers' behavior were related significantly to how carefully and closely teachers listened to the ways their children solved mathematics problems.

The research-based knowledge of the problem framework and children's strategies gave teachers a context for thinking about children's knowledge and for helping them make sense of their children's thinking. As one teacher said to us, "I've always known that I should ask the children questions that would tell me what they were thinking, but I never knew the questions to ask or what to listen for."

Teachers also have to believe that children's thinking is important. Consider contrasting statements about the role of the teacher and the role of the learner made by the same teacher, before the CGI workshop and a year later, after the teacher had been using her knowledge of the problem types and solution strategies in her classroom. Before the workshop, the teacher asserted that "It is the job of the teacher to make sure that she starts out very basic regardless of the math ability of the children." She saw the student's role as "following directions, to be listening, to be looking at the teacher, to be quiet so that they can absorb" what the teacher is saying. After using CGI for a year, she said the teacher "should be a leader, yet an allower of the children to express their ideas and a listener, so she can find out also what they are able to give her on their own." She asserted that the student "should first be definitely thinking and be thinking about what they do know."

An Expansive View of Children's Knowledge: The Case of Ms. J. and Billy—To illustrate the thinking and teaching of a CGI teacher who takes an expansive view of children's knowledge, we return to the classroom of Ms. J., whom we visited earlier. Ms. J. is a first-grade teacher who teaches many disadvantaged children. She took the CGI workshop in the initial year, and since then she has used and built on her knowledge of children's thinking in her mathematics classroom. She has children of all ability levels in her classroom, and she has children who do quite remarkable mathematical thinking in

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her room. Rather than look at these high-achieving children, we focus here on Ms. J.'s work with Billy, one of the lowest-achieving children in her classroom. By so doing, we emphasize our belief that all children have knowledge on which teachers might build mathematics instruction. When teachers make instructional decisions that take into account a child's mathematical knowledge, they enable that child to learn more mathematics.

Billy wa a disadvantaged child who had arrived in Ms. J.'s classroom in the middle of Octobe six weeks after school had started. He had not been in school previously that year because of a teachers' strike in the community from which he came. When he entered Ms. J.'s first-grade class, Billy could neither count nor recognize numerals. Ms. J. and the other children helped Billy learn to count objects, first to five and then to 10. Billy learned to count to 10 verbally, and when he continued to have great difficulty recognizing numerals, Ms. J. gave him a number line with each number clearly identified. Billy carried the number line with him continuously, and if he needed to know what a numeral looked like, he would count the marks on the number line and know that the numeral written beside the appropriate mark was the numeral he needed. As soon as Billy could count, Ms. J. began giving him simple word problems to solve. She would write a word problem on a sheet of paper such as, "If Billy had two pennies, and Maria gave him three more, how many would he have then?" (a joining problem with the result unknown). Either Ms. J. or another child would then read the problem to Billy, who would get some counters and patiently model the problem. In this problem, Billy made a set of two cubes and a set of three cubes and then counted all the cubes. Ms. J. would then ask Billy to explain how he got his answer. He would tell her what each set meant, and how he had counted them all and gotten five and then counted up on his number line to know what five looked like.

During mathematics class, Billy might solve only two or three of these simple problems, but he knew what he was doing, and he was able to report his thinking so that Ms. J. could understand what he had done. When Ms. J. was sure he understood the simple problems such as the joining or separating result-unknown problems, she moved on to somewhat harder ones and to somewhat larger numbers. She encouraged Billy to make up his own problems to solve and to give to other children. Almost all of Billy's time in mathematics class during the year was spent in solving problems by direct modeling or in making up problems for other children to solve.

When we interviewed Billy near the end of the year, he was solving problems more difficult than those typically included in most first-grade textbooks. Billy had become less reliant on his number line, and he could solve result-unknown and change-unknown problems with numbers up to 20. Although at that point he was not yet able to recall basic arithmetic facts, he nonetheless, understood conceptually what addition and subtraction meant, and he could directly model problems to find the answer. Billy was no less proud of himself or excited about mathematics than any other child in the classroom. As he said to the school principal: "Do you know those kids in Ms. J.'s class who love math? Well, I'm one of them."

The case of Billy is true. By the end of first grade, this child, who would have qualified for any program for the disadvantaged, had made progress in learning mathematics; he understood the mathematics he was doing; and he felt good about himself and about mathematics. In his eyes, and in his teacher's eyes, Billy was a successful learner, and clinical interview data also confirmed his success.

What enabled this success to occur? Although Ms. J. was acknowledged as an expert teacher before she took the CGI workshop, she developed significant new knowledge of and beliefs about children's mathematics learning during the year following the workshop. The knowledge that she developed and used enabled her to work more effectively than ever before with all children, including children like Billy.

Implications for Compensatory Education

The story of CGI is a story of teachers working with young children in a way that enables them to learn mathematics with understanding, including children who are less advantaged or less advanced in their mathematical knowledge. It is a remarkable story because it demonstrates the professionalism of teachers who work with children in schools. It shows that when teachers are given access to research-based knowledge that is robust and is directly useful in helping them fulfill their perceived roles, they use that knowledge as they teach, and it directly benefits the children with whom they work. To be useful, knowledge needs to be well organized so that teachers can use it on a daily basis as part of their ongoing mathematics instruction. The research-based knowledge on children's problem solving in addition and subtraction proved to be an example of well-organized, robust, and useful knowledge. Once teachers were given access to this knowledge, and they perceived that it helped them to understand their own children's thinking, they used the knowledge in their teaching of addition and subtraction. This knowledge helped teachers think about the mathematical knowledge of each child in their mathematics classrooms and to design mathematics curriculum and instruction so that each child could learn. It helped teachers understand the thinking of students who were having trouble learning, as well as the thinking of students who were more advanced in their mathematics learning.

Just as this specific research-based knowledge of children's learning of addition and subtraction was useful to primary teachers in Madison, Wisconsin, and in inner-city Milwaukee, so too should the knowledge be useful for primary teachers elsewhere who teach mathematics in compensatory education programs. This knowledge would provide a new framework for thinking about addition and subtraction problems as well as for thinking about young children's mathematical knowledge and abilities to solve addition and subtraction problems.

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In addition to this specific knowledge that could be used directly in compensatory education in mathematics, our work with CGI teachers suggests three important ideas that are important to consider in developing new approaches to teaching elementary mathematics in compensatory education. These ideas have to do with: (1) assessing students' mathematical knowledge and understanding, (2) building on students' informal and formal mathematical knowledge, and (3) constructing curriculum and teaching in ways that encourage mathematical thinking and problem solving by all children.

These "ABCs" may seem obvious to some teachers. Indeed, many elementary teachers, including compensatory education teachers, might agree with these ideas and even say they are implementing them in their mathematics teaching. However, we have found that these ideas mean very different things to different people, so it is important to discuss these ideas to gain an understanding of what they mean. Further, teachers need to consider what these ideas might mean for reforming their mathematics teaching in ways that will benefit students who have been served by compensatory education programs.

Assessing Students' Mathematical Knowledge and Understanding

Most elementary teachers believe that they are teaching for understanding in mathematics (see Cohen & Ball, 1990; Peterson, Fennema, Carpenter, & Loef, 1989), but their definitions of what it means to know and understand mathematics differ substantially from researchers' definitions derived from recent studies of children's mathematics learning. For example, as one rather traditional elementary teacher in California commented when discussing the state-level mathematics reform aimed at teaching mathematics for understanding, "What do they think we've been doing—teaching for *mis*understanding?" (Cohen & Ball, 1990). Of course, this teacher's concept of mathematical understanding differed substantially from that of researchers and curriculum reformers. Similarly, in a study of our first-grade teachers' goals and beliefs before the CGI workshop, we contrasted seven teachers who had initial beliefs that were more cognitively based and whose students did well on problem solving with seven teachers who had beliefs that were less cognitively based and whose students did less well on problem solving (Peterson, Fennema, Carpenter, & Loef, 1989). Cognitively based beliefs reflected strong agreement with the ideas that:

- Children construct mathematical knowledge.
- Math skills should be taught in relation to problem solving.
- Instruction should be sequenced to build on children's development of ideas.
- Instruction should facilitate children's construction of mathematical knowledge.

Although we found important differences between the less cognitively based teachers and the more cognitively based teachers in their goals, knowledge, beliefs, and reports of how they taught addition and subtraction, as well as in their students' problem-solving achievement, *all 14 teachers indicated that they placed the greatest emphasis on mathematical understanding*, compared with number fact knowledge and word problem solving. However, all seven cognitively based teachers rated fact knowledge as *least* important when compared with understanding and problem solving, while less cognitively based teachers placed number fact knowledge either *first* (tied with mathematical understanding) or *second*, after mathematical understanding. Thus, even though these teachers differed significantly in their beliefs and in their reports of how and what they taught in addition and subtraction, they all believed and reported that they were teaching for mathematical understanding.

How do teachers know whether a child knows and understands mathematics? To assess students' mathematical knowledge and understanding, most teachers rely on observed student engagement or on students' answers to mathematics problems on tests or worksheets (Ball, 1990; Peterson, Carpenter, Fennema, & Loef, 1989). In contrast, CGI teachers who have changed their ideas of what it means for children to understand mathematics adopt or invent new approaches and techniques for assessing students' knowledge and understanding. These include using whole-class and small-group discourse among children to learn about their mathematical knowledge and thinking, as well as interviewing individual children. CGI teachers are concerned with understanding the processes that children use to solve problems rather than focusing only on whether the answer is correct.

Building on Students' Informal and Formal Mathematical Knowledge

Before being able to use and build on students' mathematical knowledge in teaching, a teacher needs to realize that *all children know and understand some mathematics*. All too often, teachers focus not on what the student knows and how the student is understanding but on what the child doesn't know and on how the teacher herself or himself understands the mathematics problem. Like Ms. W., teachers often unwittingly encourage their own way of thinking about a mathematics problem and fail to listen to and try to understand a student's way of thinking about the problem. In contrast, teachers like Ms. J., who take an expansive view of children's mathematical knowledge and understanding, are continually astonished by what children know and understand. Ms. J. showed equal enthusiasm in extolling the virtues of her children's mathematical knowledge regardless of whether the child was one like Billy, who came to first grade with less knowledge and understanding than other children, or one like Chery' whose knowledge of division in December of first grade astounded Ms. J., who related to us the following story:

> *I was working with Cheryl the other day, and she had 12 cubes in her hand. The problem was Riva had 12 carrots, and she made 3 carrot cakes. She needed to divide them equally into each cake. And you know, Cheryl had these cubes, and go, go, go—she snapped it off real quick. I said, "How did you get that so quickly?" And she goes, "Oh, you know, the numbers, you know—first there were 3. If you put 3*

cakes, 3 carrots in each cake, and then I had 9. But if I add 1 more, that would be 4." So they [the children] are thinking. It's just so sophisticated. It just seems to come together for them.

Constructing Curriculum and Encouraging Math Thinking and Problem Solving

Our work suggests that knowledge about children's thinking can be an important influence on instruction and learning. Teachers' behavior can be changed by helping them gain knowledge about children's thinking, and this change in behavior results in better mathematics learning by their students. For CGI teachers, this means doing much more of the following in their teaching of addition and subtraction:

- Posing word problems.
- Listening to students' thinking.
- Encouraging the use of multiple strategies to solve word problems.
- Asking, "How did you get your answer?"

For CGI teachers, their use of the textbook and the way they think about the mathematics curriculum also changes.

Supplementing the Textbook–Although teachers had been told explicitly by the school district administrators that they did not have to use the textbook during the year of the experimental study, only two CGI teachers reported that they did not use the textbook at all that year. Eleven of the teachers reported using all or most of the textbook; the remaining four teachers used the textbook in some way either as a "backup" or "reinforcer" or for practice. When asked how she used the textbook, the following CGI teacher gave a response that was typical during the first year:

I used it as a resource for getting ideas on how to present material. I used it as practice pages for the kids. It is much less of a Bible than it has been in other years, which was really nice for me. I didn't feel as tied to it.

On the other hand, when asked, "Did you cover everything in your textbook?" the same teacher responded, "Well, yes, I've covered the objectives in the textbook."

Because CGI is premised on the idea of introducing addition and subtraction within the meaningful context of a wide variety of types of story problems, CGI teachers developed their own materials and supplemented the textbook with word problems. During the first year, CGI teachers admitted that, although they used their textbooks, they omitted whole pages of computation problems, and they took pages out of the textbook and rearranged them. In addition, they also used many word problems that they developed on their own or from supplemental enrichment materials. In describing how she supplemented the textbook, one teacher said she used all the kinds of the problems that she had constructed during the summer workshop. She added:

In fact, some of us said that we'd loved to spend more time on this and throw out the book and just sit and talk with kids about, "How would you work this one out? How would you work that one out?" You know, once they (the children) start catching on to those problems, I think that's where our emphasis should be.

Going Beyond the Textbook—As teachers developed their ideas about CGI over several years, many teachers ceased altogether to use a textbook. Fennema, Carpenter, and Loef (in press) describe the growth and development of one such teacher, Ms. G., over a four-year period as she learned to use children's thinking in her mathematics teaching. During the first year, Ms. G. reported, she used the textbook to guide her mathematics instruction explicitly. By the end of the second year, she began to supplement the textbook and to rely more and more on her knowledge of children. By January of the third year, Ms. G. used her textbook only as a guide; and as the year progressed, she spent more time on story problems, and completing textbook pages became a low priority. At the end of the third year, Ms. G. announced that she had received permission from her principal to teach mathematics the following year without a textbook. During the fourth year, Ms. G. reported that she "never even picks up her textbook." Rather, she constructs her curriculum from her knowledge of the variety of problem types and her children's understanding. She has decided that the textbook is too limiting and does not build on what children know and can do.

Over the course of the four years, the mathematics that Ms. G. taught also changed dramatically. She increased the time she spent teaching mathematics and progressed to including mathematics in other subjects and at other times during the day. Most importantly, problem solving rather than drill became the focus of activity in Ms. G.'s mathematics teaching. For example, she taught place value in relation to ongoing problem solving. Although she consciously planned problems and activities in which children could explore place value ideas, and she had children focus on place value ideas where appropriate, she never taught a formal unit on place value. Rather, she completely integrated the teaching of place value with other mathematics. At the end of the year, she reported that she felt her children understood place value ideas better than any group of children she had taught previously.

During the fourth year, Ms. G. also reported doing multiplication and division work with her children. Previously, she would not have taught multiplication or division to her first-graders because she felt these topics were much too hard. In her teaching of multiplication and division, she described how she bought cookies for the students in her class, and the children had to find out how many cookies were in each package. She had selected the packages of cookies so that each package had a different number of rows of cookies and different numbers in a row. She reported that she posed the problem of "how many were in a row, how many rows, then they had to decide how many cookies there were altogether, and then how many cookies would each child get if I were to divide them up." She stated that "it took us the whole afternoon to do that problem...At the end of the day, then, we did divide the cookies up to see if they were right and to see how many were left over." This example of Ms. G.'s teaching illustrates both her attempt to give her children mathematics problems with real-life meaning and her recognition that her children could do many mathematics problems that she had viewed previously as too difficult for young children.

In sum, over the four years, Ms. G. learned from listening and observing her own students that her children have a lot of mathematical knowledge, and she learned continuously how to use and build on that knowledge in her teaching. Ms. G. continues to learn, as do her children.

Conclusion

In summary, the CGI approach is characterized by:

- Teachers who have a knowledge base for understanding their children's mathematical thinking.
- Teachers who listen to their students' mathematical thinking and who build on the knowledge they get by listening.
- Teachers who use their knowledge of students' mathematical thinking to think about and develop their mathematics instruction.
- Teachers who place increased emphasis on mathematics problem solving and decreased emphasis on drill and practice of routine mathematics skills.
- Teachers who provide their students with opportunities to talk about how they solve mathematics problems and to solve problems in a variety of ways.
- Classrooms in which students do a lot of mathematics problem solving and describe the processes they use to solve problems.
- Classrooms in which students demonstrate increased levels of mathematics problem-solving abilities while maintaining high levels of computational performance.

Frequently, the message of compensatory education in mathematics has been one of remediation and compensation for children's lack of mathematical knowledge. Our research with CGI teachers offers a strikingly different message. The message of CGI is that when teachers begin listening to and talking with their children, they come to realize how much more their children know than they had recognized previously. Teachers come to understand that children have a lot of mathematical knowledge on which they can build. When teachers know about children's mathematical knowledge and thinking, they can use it to facilitate children's development of mathematical problem-solving abilities. Teachers can achieve the goals of compensatory mathematics education by focusing on and using their children's mathematical knowledge and thinking in their classroom teaching.

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References

Ball, D. (1990). Reflections and deflections of policy: The case of Carol Turner. *Educational Evaluation and Policy Analysis. 12* (3).

Carpenter, T. P. (1985). Learning to add and subtract: An exercise in problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 17-40). Hillsdale, NJ: Erlbaum.

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. (1988). Teachers' pedagogical content knowledge of students' problem solving in mathematics. *Journal for Research in Mathematics Education, 19*, 385-401.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematical thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*, 499-532.

Clark, C. M., & Peterson, P. L. (1986). Teachers' thought processes. In M. Wittrock (Ed.), *Handbook of research on teaching* (Third edition) (pp. 255-296). New York: Macmillan.

Cohen, D. K., & Ball, D. L. (1990). Relations between policy and practice: A commentary. *Educational Evaluation and Policy Analysis, 12* (3).

Dossey, J. A., Mullis, I. V., Lindquist, M. M., & Chambers, D. L. (1988). *The mathematics report card: Are we measuring up*? Princeton, NJ: Educational Testing Service.

Duckworth, E. (1987). *"The having of wonderful ideas" and other essays on teaching and learning.* New York: Teachers College Press.

Fennema, E., & Carpenter, T. (1989). *Cognitively guided instruction: A program implementation guide.* Madison: Wisconsin Center for Education Research, University of Wisconsin-Madison.

Fennema, E., Carpenter, T., & Loef, M. (in press). Learning to use children's mathematics thinking: A case study. In C. Maher & R. Davis (Eds.), *Relating schools to reality.* Englewood Cliffs, NJ: Prentice Hall..

Fennema, E., Carpenter, T. P, & Peterson, P. L. (1989a). Learning mathematics with understanding. In J. Brophy (Ed.), *Advances in research on teaching, Volume 1* (pp. 195-221). Greenwich, CT: JAI Press.

Fennema, E., Carpenter, T. P., & Peterson, P. L. (1989b). Teachers' decision-making and cognitively guided instruction: A new paradigm for curriculum development. In N. Ellerton & M. Clements (Eds.), *School mathematics: The challenge to change* (pp. 174-187). Geelong, Victoria, Australia: Deakin University Press.

Fuson, K. (1988). *Children's counting and concepts of number.* New York: Springer Verlag.

Ginsburg, H. A. (1983). *The development of mathematical thinking.* New York: Academic Press.

Lampert, M. (1984). Teaching about thinking and thinking about teaching. *Curriculum Inquiry, 16,* 1-18.

Lave, J. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life.* New York: Cambridge University Press.

Lubinski, C. (1989). *Cognitively guided instruction and teachers' decision making.* Unpublished doctoral dissertation, University of Wisconsin, Madison.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education.* Washington, DC: National Academy Press.

Peterson, P. L., Carpenter, T.C., & Fennema, E. (1989). Teachers' knowledge of students' knowledge in mathematics problem solving: Correlational and case analyses. *Journal of Educational Psychology, 81,* 558-569.

Peterson, P. L., Fennema, E., & Carpenter, T. (1988/1989). Using knowledge of how students think about mathematics. *Educational Leadership, 46* (4), 42-46.

Peterson, P. L., Fennema, E., Carpenter, T., & Loef, M. (1989). Teachers' pedagogical content beliefs in mathematics. *Cognition and Instruction, 6,* 1-40.

Villasenor, A. (1990). *Teaching the first grade mathematics curriculum from a problem-solving perspective.* Unpublished doctoral dissertation, University of Wisconsin-Milwaukee.

Zucker, A. (1990). Review of research on effective curriculum and instruction in mathematics. In M. S. Knapp & P. M. Shields (Eds.), *Better schooling for the children of poverty: Alternatives to conventional wisdom. Volume 2: Commissioned papers and literature review.* Washington, DC: U.S. Department of Education.

DISCUSSION: APPRECIATING CHILDREN'S MATHEMATICAL KN⌃ VLEDGE AND THINKING IN ETHNICALLY, LINGUISTICALLY, AND ECONOMICALLY DIVERSE CLASSROOMS

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My reaction to the paper "Using Children's Mathematical Knowledge" by Penelope Peterson, Elizabeth Fennema, and Thomas Carpenter is based on 20 years of teaching in urban public school systems and on experience as an adjunct faculty member at Wheelock College and as a consultant on projects developing new curricula. For the past 17 years, I have taught in the Cambridge (Massachusetts) school system. I have taught in the Follow Through Program and for the last eight years have been teaching at the Saundra Graham and Rosa Parks Alternative Public School. The age-integrated classrooms at this K-8 magnet school serve a student body that is ethnically, linguistically, and economically diverse.

This discussion responds to each of the key elements in the Cognitively Guided Instruction (CGI) approach described by Peterson et al., with specific attention to the role of the teacher in an urban classroom. I appreciate the opportunity to play an active role in the development of a new focus for preservice and inservice math education, in the empowerment of teachers in the design and implementation of curriculum, and in fostering an awareness of children's mathematical knowledge and thinking.

The CGI approach is one model for bringing current research in mathematics education into daily classroom practice. This approach is based on the following three principles:

- Teachers must have expansive views of children's mathematical knowledge and thinking.
- The mathematics curriculum must focus on problem solving.
- Teachers must encourage and recognize multiple strategies and solutions for problem solving.

I would suggest that this foundation is complete only when a fourth cornerstone is in place:

- The mathematics curriculum must be relevant to the events, daily lives, and rich cultural traditions of *all* our children.

Children's Mathematical Knowledge

The need for teachers to have an expansive view of children's mathematical knowledge and thinking is the basic premise underlying the construction of the new National Council of Teachers of Mathematics (NCTM) Standards. In fact, the first four standards are Mathematics as Problem Solving, Mathematics as Communication, Mathematical Reasoning, and Mathematical Connections. I believe that these standards are directly embodied in the three key principles of CGI and in the fourth cornerstone suggested above.

Teachers Need an Expansive View of Both Mathematics and Children's Mathematical Knowledge

Peterson, Fennema, and Carpenter begin their paper by acknowledging the skills and understanding that all children bring to a school setting, and note that this is not a widely held view among educators—particularly in schools with large populations of poor children. All too frequently, inner-city school math programs consist of rote-learning drill and practice (low-order skills) and either "neglect or de-emphasize the teaching of higher order skills" (Levine, Levine, & Eubanks, 1985).

School-aged children are certainly not empty vessels. As the authors of "Using Children's Mathematical Knowledge" stress, a large amount of recent research and classroom practices address the acceptance of the mathematical knowledge children bring to school. Resnick (1987) has studied the differences between children's out-of-school problem solving and their in-school thinking, and suggests that formal mathematics may in fact discourage students from bringing knowledge and intuition to school tasks by stressing memorization and written computation. Children can, and do, demonstrate problem-solving abilities and mathematical thinking independently of their mastery of school-based algorithms and number facts. Robert Moses' Algebra Project (Silva & Moses, 1990) with urban middle school students and Maggie Lampert's work with multiplication are examples of post-primary implementation of this practice of bringing children's mathematical thinking to bear in classroom mathematics.

In addition to the appreciation of children's mathematical understanding stressed by Peterson et al., teachers need a broader conception of the field of mathematics. My experience as an instructor in the teacher training program at Wheelock College leads me to believe that teachers must themselves have enhanced knowledge of a broad range of mathematical subject matter, including "the nature and discourse of mathematics, and the role of mathematics in culture and society" (Ball, 1989). They must have experience with a wide range of developmentally appropriate materials and must themselves experience problem solving as learners.

During the first session of my course, Teaching Mathematics to Young Children, I ask students to start a "working definitions" journal entry called "What is Math?" Over

95% of the students restrict their definition of mathematics to computation and numbers. This is in contrast to the NCTM Standards' decreased attention to early use of symbolic numbers and isolated computation.

We must help teachers in the United States develop a wider view of mathematics. Children throughout many other English-speaking countries call the subject *"maths."* I believe that this distinction is significant in the curriculum as well. In my view, some of the most innovative and progressive math educational curricula in the past two decades have come out of England (the Nuffield Series of the 1970s) and, more recently, Australia (Mathematics Curriculum Teaching Project). The need to bring an understanding of the broad field of mathematics and of children's thinking to teacher training is critical.

The National Science Foundation-supported project Cognitively Guided Instruction appears to embody this position. The teachers who took the CGI workshops seem empowered by the experience. Their testimonies are evidence of enormous growth. The teachers speak of textbooks as resources, but they have realized their own potential to make changes in the order of presentation. Textbooks necessarily have one page before another, a structure that dictates linear learning. Children actually learn naturally in a more geometric fashion, which can be supported only if instructional materials are sequenced flexibly.

How Do We *Know* What Children Are Thinking?

Children's *mathematizing* (Freudenthal, 1973) is often hidden and lost forever when they get the wrong answer for the "right reason" (sense making). For example, Xiamara, a sixth-grader, encountered the following problem on a test:

> Three walls of your room are covered with wallpaper. The fourth wall is 17' by 7'. Wallpaper is $2.99 a square foot. How much will it cost to finish the wallpapering job?

She initially came up with the "correct" answer ($355.81), but she found herself in a quandary. She believed that her calculations were correct but also believed that no one could spend that much money wallpapering a room. She doubted her own mathematical competence; and since multiplication with decimals was fairly new to her, she reasoned that she must have placed the decimal point incorrectly. She moved the decimal point one place to the left, rounded her answer to two decimal places, and arrived at the answer $35.58. In a subsequent conversation with her teacher, she was fortunate to have the opportunity to explain *how* she arrived at her answer. She came away from the conversation with a renewed sense of her own arithmetic skills, but also with a discouraging belief that school word problems do not have to be sensible, just arithmetically correct.

Xiamara's story also reminds us of the need to find ways to assess children's mathematical knowledge and thinking. Answers on test papers and textbook assignments do not bring us an understanding of children's thinking. The CGI approach, Robert Moses' Algebra Project (Silva & Moses, 1990), and the works of Magdalene Lampert (Lampert, 1990), Leah Richards (L. Richards, 1990), and Constance Kamii (Kamii, 1985) all suggest that teachers should offer and orchestrate lively classroom discourse. By observing and recording classroom discussions, teachers may assess children's problem-solving abilities. Peterson, Fennema, and Carpenter suggest that this practice allows teachers to continuously make curriculum changes during instruction. The teacher may also have an active role in the discourse.

Accepting the existence of children's mathematical knowledge and thinking brings the issue of teachers' expectations to the table. There is tremendous evidence to suggest that teachers' expectations drive the course of classroom curriculum and that teachers' attitudes influence student achievement. These effects are particularly evident when children are labeled "disadvantaged." The research of Jere Brophy, as cited by Eva Chun in 1988, describes this "down teaching" in detail. If teachers are to provide opportunities for excellence for *all* students, their expectations must be positive and equitable.

A Focus on Problem Solving in a Meaningful Curriculum

A focus on problem solving in the CGI classrooms allows children to be individually challenged and to use higher-level thinking skills and multiple strategies and learning styles. The problems presented are not simply a vehicle for children to practice the algorithms. I was pleased to see classroom descriptions that included the introduction of a small number of problems each day. The research literature suggests that this practice is atypical in the United States, yet is the norm in Japanese classrooms.

The approach used by the CGI teacher described in the paper incorporates two innovative instructional techniques. In the first, the teacher writes a story that uses classroom children's names and that reflects his or her perception of the students' math skills. The second technique involves having children share their strategies for solving equations in a group discussion.

I applaud the first of these techniques as a vast improvement over using traditional textbook word problems and a key-word approach. I would also propose extending this approach to give children an opportunity to author their own stories that describe problems for which arithmetic may be of service. This process also allows for story sharing and peer reactions (Kliman & Richards, 1990). The third-graders in our classroom took the California Achievement Test (CAT) after using this writing/response process for four months. Our children, without any textbook experiences with school-based word problems, did as well as the children in two control classrooms on the problem-solving subtest. Although the sample was too small for quantitative analysis, it might be noted that one difference did occur. All of our students, who speak English as a second language, scored in the "mastery range," while students with similar profiles (including an identical twin) in the control classrooms did not demonstrate mastery in this subtest on the CAT.

The authors' use of children's literature is an exciting idea that is equally successful with older children. Marlene Kliman and Glenn Kliman (1990) describe a wide range of mathematical activities and modeling through the use of Jonathan Swift's *Gulliver's Travels*. Children in our classroom spent a day trying to estimate Gulliver's actual height, armed only with Swift's description of a Lilliputian's size as being equal to the length of Gulliver's six-inch hand. Our children measured all their classmates' hand lengths and heights. They averaged these ratios to predict Gulliver's height—a mere five feet tall. They noted that this "made sense," since they had visited the Salem Witch House and knew that people were of shorter stature in the 17th and 18th centuries.

I would further encourage teachers to "package" arithmetic situations in the cultural folktales of the children in their classrooms. This practice has allowed children of color to assume leadership roles in diverse (in terms of ethnicity, arithmetic skill, language, family economics, and gender) groups in our classroom. Over a third of the children at the Graham and Parks School are Haitian-American. Traditional Haitian folktales are an integral part of all areas of our classroom curricula. For example, I took a well-known problem concerning the sequence of fillings and pourings with a 3-liter and a 7-liter container to achieve exactly 5 liters, and "repackaged" it in the Haitian story of "Teyzen":

> *Do you remember the story of Teyzen? Well, one day Asefi and her brother Dyesel were going to the spring to get water. Their mother gave them each a calabash. Asefi's calabash held 7 liters when it was full. Dyesel's calabash held 3 liters when it was full. Their mother told them to bring home exactly 5 liters. Tell about the fillings and pourings that the timoun [children] must do in order to bring home 5 liters.*

If teachers demonstrate respect for children's knowledge and make room for it in their classrooms, they have an opportunity to develop a far richer curriculum. "To become meaningful, a curriculum has to be enacted by pupils as well as teachers, all of whom have their private lives outside school. . . . A curriculum, as soon as it becomes more than intentions, is embodied in the communicative life of an institution, the talk and gestures by which pupils and teachers exchange meanings even when they quarrel or cannot agree. In this sense, curriculum is a form of communication" (Barnes, 1976). Teachers from Eurocentric cultures must maintain high regard for, and an understanding of, other cultural traditions and styles. The misunderstanding of behavioral style can make it difficult to "establish rapport and to communicate" (Hilliard, 1989).

Multiple Solutions and Strategies

When problem solving is presented in meaningful contexts, all children are encouraged to bring their own learning styles to the process. Patricia Davidson's work from a neuropsychology perspective suggests that people learn math in one of two distinct styles (related to the functions of the brain's left and right hemispheres). She notes, for example, that style I learners (left cortical hemisphere preference) master formulas and have good recall of number facts (e.g., 6+8=14), while style II learners have stronger spatial and estimation skills. They usually know the "doubles" facts and might add 6+8 by thinking that 6+6=12 and since 8 is 2 more than 6, 2 is added on to make 14. What is particularly interesting about this research in light of Peterson, Fennema, and Carpenter's work with CGI is that some of the CGI teachers reacted to children that Davidson might call style II learners by remarking that their thinking was "sophisticated" or "abstract." Research indicates that teachers are more apt to be analytical (style I) in their own teaching and learning styles (Dunn & Dunn, 1988).

If these same teachers were to accept only single-answer, single-strategy problem-solving methods, they would lock out a large number of children from opportunities for excellence in mathematics. Peterson, Fennema, and Carpenter also describe a teacher with a limited view of children's knowledge. Mrs. W.'s resistance to Adam's mental computation strategy and insistence on a meaningless recipe is, unfortunately, typical of many teachers. In *Young Children Reinvent Arithmetic,* Constance Kamii details many classroom scenes where children like Adam are encouraged to share multiple strategies for mental arithmetic. Whereas children group numbers in many ways, the practice of adding first the tens and then the ones makes absolute sense. The recording is quite simple; the partial sums are listed and then combined:

$$\begin{array}{r} 35 \\ \underline{+35} \\ 60 \\ \underline{+10} \\ 70 \end{array}$$

During my own childhood, I was taught that the one (and only) way to compute mentally was to imagine a "chalkboard in my mind" and then to "see" the numbers and "carry" as I would with paper and pencil. Unfortunately, the numbers always disappeared before I could finish the calculation. I was convinced that I was not "good at math" and did not reach for advanced work in the field. It was not until much later that I learned Adam's strategy (to add larger units first). This offered me a renewed sense of confidence in my own abilities in mathematics.

While I applaud the new empowerment of teachers in the design of meaningful curricula, I am well aware of the position of power that teachers have always held in the classroom and the need to give status to all children's knowledge and thinking. When teachers and children have different "ways of doing math" (strategies), the teacher's approach is usually regarded as "right" or given greater status. Typically, if children do not understand a teacher's explanation, the teacher raises her or his volume and delivers the information in exactly the same way. The teacher's style becomes the normative reference, and many students who are sent to remedial classrooms may simply be "learning different."

In CGI classrooms, children are encouraged to share their strategies for solving equations. This is reminiscent of Constance Kamii's studies in Alabama classrooms and Dr. Kiyonobu Itakura's Hypothesis Experiment Instruction (HEI) method, a system developed in Japan for the construction of knowledge through discussion and demonstration. I have adapted the HEI method for use in my own classroom (J. Richards, 1990). While I applaud the approach for acknowledging and giving status to diverse strategies, I am concerned about the number of children who actually share their strategies in a large group setting. Teachers must be mindful to bring less-frequent speakers into the discourse.

Conclusion

We have a reform document in mathematics that is unparalleled in other areas of the curriculum. The NCTM Standards offer a new opportunity to bring about real change.

The question then becomes: "Where do we start?" If we were able to start (on a national scale) with the teachers of young children, we would still lose almost a generation of children and young adults. It would seem, therefore, that the universities and teachers' colleges need to embrace these changes early in the 1990s if we are to affect the greatest number of classrooms during the next decade. In addition, I would love to see the CGI approaches described in "Using Children's Mathematical Knowledge" become a genuine and integral part of the math classrooms of all children. The framework provided by Chapter 1, Follow Through, and other federal target projects might be a good conduit for beginning this transformation. As individual states review their teacher certification standards, the state boards of education might insist that the key elements of CGI be infused into teacher training for future teachers in undergraduate and graduate programs and for current practitioners in inservice programs. Through CGI, Peterson, Fennema, and Carpenter offer the scaffolding for teachers and children to develop a new curriculum.

ERIC

References

Ball, D. L. (1989). *Teaching mathematics for understanding. What do teachers need to know about the subject matter?* Washington, DC: National Education Seminar on Teacher Knowledge.

Barnes, D. (1976). *From communication to curriculum.* London: Penguin Books.

Dunn, R., & Dunn, K. (1988, October). *Presenting forward and backward. Teaching K-8.* Norwalk, CT: Early Years Inc.

Freudenthal, H. (1973). *The first person who used mathematizing: Mathematics as an educational task.* Dordrecht, Holland: D. Reichel Pub. Co.

Hilliard III, A. (1989, January). Teachers and cultural styles in a pluralistic society. *National Education Association Journal.*

Kamii, C. (1985). *Young children reinvent arithmetic.* New York: Teachers College Press.

Kliman, M., & Kliman, G. (1990). *Life among the giants: Mathematics writing and exploring Gulliver's world.* Newton, MA: Center for Learning Technology, Education Development Center.

Kliman, M., & Richards, J. (1990). *Now that we've done the calculation, how do we solve the problem? Writing, sharing, and discussing arithmetic stories.* Newton, MA: The Literacies Institute, Education Development Center, Inc.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27* (1) 29-63.

Levine, D., Levine, R., & Eubanks, E. (1985). Successful implementation of instruction at inner city schools. *Journal of Negro Education, 54* (3).

Resnick, L. B. (1987). Learning in school and out. *Educational Researcher, 16,* 13-20.

Richards, J. (1990). *Group discussion, influence and learning in science: A teacher's perspective on a Japanese curriculum in U.S. classrooms.* Paper presented at the annual meeting of the American Educational Research Association, Boston.

Richards, L. (1990). Measuring things in words: Language for learning mathematics. *Language Arts, 67* (1).

Silva, C. M., & Moses, R. P. (1990). The Algebra Project: Making middle school mathematics count. *The Journal of Negro Education, 59.*