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ABSTRACT

Five sets of activities for students are included in this document. Each is designed for use in secondary school mathematics instruction. Topics include: (1) "5-Con Triangles"; (2) "Striking Sequences"; (3) "Invariants"; (4) "On the Ball," (a geometry lesson); and (5) "Coloring Maps." These topics focus on geometric and statistical concepts. Each set contains a variety of activities, an extension section, and an answer key. Pages may be reproduced for classroom use without permission. (CW)

student math notes

5-con Triangles

In figure 1 four triangles have been placed in a spiral pattern. Your challenge is to extend this spiral by adding appropriate triangles to its ends. Carefully cut out the four triangles and label the interior of each angle with the letter of its corresponding vertex.

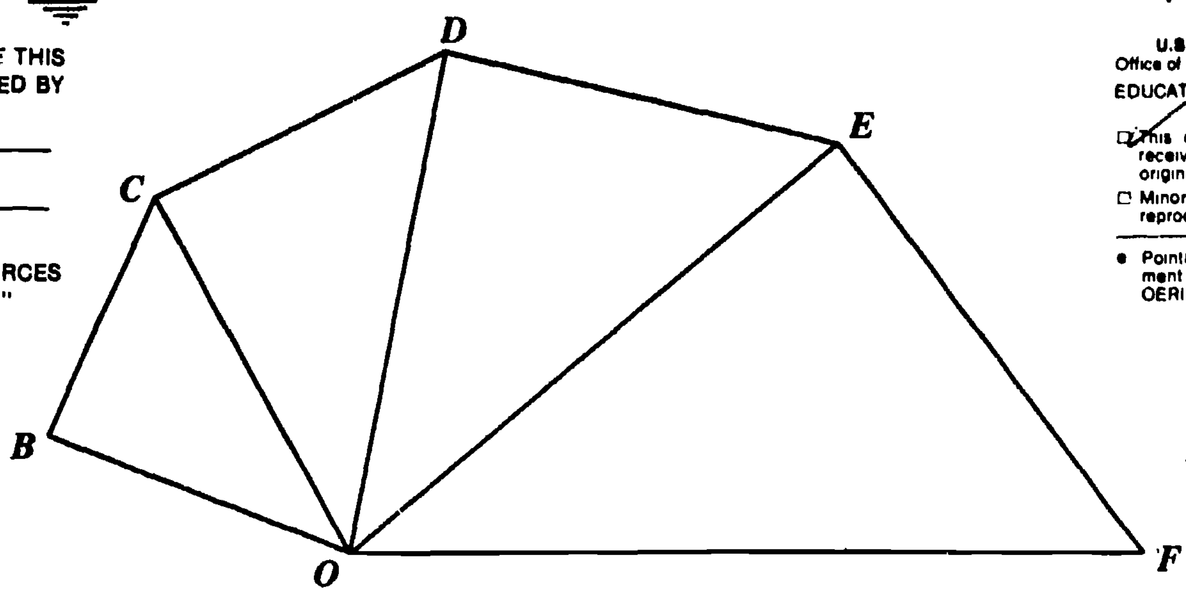

Fig. 1

1. In each column, list the congruent pairs of segments and the congruent pairs of angles in the two triangles by comparing the cut-out triangles one pair at a time. Note the example.

$\triangle OBC$ and $\triangle OCD$	$\triangle OCD$ and $\triangle ODE$	$\triangle ODE$ and $\triangle OEF$
$\overline{OB} \cong \overline{CD}$		
$\overline{OC} \cong \overline{OC}$		
$\angle CBO \cong \angle DCO$		
$\angle BCO \cong \angle CDO$		
$\angle BOC \cong \angle COD$		

2. What patterns can you find in the relationships between the sides and the angles of the consecutive triangles in the spiral? ______________________

____

3. If a $\triangle OFG$ is to be connected to $\triangle OEF$ to extend the spiral, what relationships should exist between $\triangle OFG$ and $\triangle OEF$? ______________________

____

4. If a $\triangle OAB$ is to be connected to $\triangle OBC$ to extend the spiral inward, what relationships should exist between $\triangle OAB$ and $\triangle OBC$? ______________________

____

The editors wish to thank Maurice Burke, Department of Mathematical Sciences, Montana State University, Bozeman, MT 59715, for writing this issue of *NCTM Student Math Notes*.

5-con Triangles—Continued

5. Using your cut-out triangles to help you with any necessary measurements, construct $\triangle OFG$ and $\triangle OAB$.

6. $\triangle OBC$ and $\triangle OCD$ are not congruent. However, $\triangle OBC$ and $\triangle OCD$ are called a *5-con pair of triangles,* since exactly five of their six parts (two sides and three angles) are congruent. Name the other 5-con pairs of triangles in figure 1. ____________________

____

Let's find out how to construct other spirals consisting of 5-con triangle pairs.

7. In each of the following sequences, find a pattern and use it to determine the next three numbers in the sequence.

   a) 6, 18, 54, 162, ____, ____, ____, . . .

   What pattern did you find? ____________________

   b) 16, 24, 36, 54, ____, ____, ____, . . .

   What pattern did you find? ____________________

8. Attempt to construct triangles whose three sides have the indicated lengths in millimeters. (A compass would be helpful.)

   a) 6, 18, 54   *b*) 18, 54, 162   *c*) 16, 24, 36   *d*) 24, 36, 54

5-con Triangles—Continued

9. Why is it impossible to construct triangles in cases 8*a* and 8*b*? ______

10. Cut out and compare the angles of triangles formed in cases 8*c* and 8*d*. What can you conclude about the triangles?

______

11. 16, 24, 36, and 54 are the first four terms of the sequence in 7*b*. Use additional terms from this sequence to guide you in constructing two more 5-con triangles on a separate sheet of paper. What are the lengths of their sides?

______

12. The sequences in 7*a* and 7*b* are both *geometric sequences,* since their numbers follow the pattern $a, ar, ar^2, ar^3, \ldots$, with $a > 0$. They are increasing sequences, since $r > 1$. Fill in the blanks:

    In the sequence in 7*a*, a = ____, r = ____.

    In the sequence in 7*b*, a = ____, r = ____.

    Some increasing geometric sequences like that in 7*b* generate spirals of 5-con triangles, and some like that in 7*a* do not. The generation of a spiral depends on whether three consecutive numbers in the sequence satisfy the triangle inequality, that is, that the sum of the lengths of any two triangles is greater than the third. The following theorem reveals the conditions under which a 5-con-triangle spiral can be formed.

> 5-CON THEOREM: *Two triangles will form a 5-con pair if and only if the measures of their sides can be expressed as (a, ar, ar^2) and (ar, ar^2, ar^3) where $a > 0$ and $1 < r < (1 + \sqrt{5})/2$. Furthermore, an increasing geometric sequence $a, ar, ar^2, ar^3, \ldots$, will generate a 5-con spiral of triangles if and only if $a > 0$ and $1 < r (1 + \sqrt{5})/2$.*

13. Measure the sides of $\triangle OBC$ and $\triangle OCD$ in millimeters to see if the measurements satisfy the 5-con theorem. (Remember, your measurements will be only approximate.)

    $\triangle OBC$: ____, ____, ____ $\triangle OCD$: ____, ____, ____

14. Suppose $\triangle KLM$ has sides that measure 27, 36, and 48. Suppose $\triangle NPQ$ has sides of lengths 36, 48, and 64. Do these triangles form a 5-con pair? Justify your answer. ______

15. Is it possible to construct two triangles that have exactly three sides and two angles congruent? Why or why not?

______

16. Is it possible to construct two isosceles triangles that form a 5-con pair of triangles? Why or why not? ______

5-con Triangles—Continued

Can you . . .

- construct a pair of 5-con right triangles?
- construct a pair of 5-con acute triangles?
- use the Geometric Supposer to study 5-con triangles?
- explain how $\triangle OBC$ and $\triangle OCD$ are not congruent and yet $\triangle OBC$ can have two sides and their included angle congruent to two sides and an angle of $\triangle OCD$ and still not contradict the SAS property?
- explain how $\triangle OCD$ and $\triangle OBC$ are not congruent and yet $\triangle OCD$ can have two angles and their included side congruent to two angles and a side of $\triangle OBC$ and still not contradict the ASA property?
- prove that if $0 < a < ar < ar^2$ and if the lengths $a, ar,$ and ar^2 satisfy the triangle inequality, then $1 < r < (1 + \sqrt{5})/2$?
- prove the 5-con theorem?

Did you know that . . .

- the ideas of 5-con triangles can be extended to 7-con quadrilaterals and 9-con pentagons, and so on?
- $(1 + \sqrt{5})/2$ is the golden ratio?
- when r equals the square root of the golden ratio the 5-con triangles formed by the sequence $a, ar, ar^2, ar^3, \ldots$, are right triangles?
- you can read all about 5-con triangles in the *Mathematics Teacher*? See the References.

References

- Pawley, Richard. "5-con Triangles." *Mathematics Teacher* 60 (May 1967):438–42.
- Pagni, David, and Gerald Gammon. "The Golden Mean and an Intriguing Congruence Problem." *Mathematics Teacher* 74 (December 1981):725–28.

Answers

1. $\overline{OC} \cong \overline{DE}, \overline{OD} \cong \overline{OD}, \angle OCD \cong \angle ODE, \angle CDO \cong \angle DEO, \angle COD \cong \angle DOE;$ $\overline{OD} \cong \overline{EF}, \overline{OE} \cong \overline{OE}, \angle ODE \cong \angle OEF, \angle DEO \cong \angle EFO, \angle DOE \cong \angle EOF.$
2. One pair of sides in common, three pairs of angles congruent, a second pair of sides congruent
3. $\angle OFG \cong \angle OEF, \overline{FG} \cong \overline{OE}, \overline{OF} \cong \overline{OF}, \angle FGO \cong \angle EFO, \angle FOG \cong \angle EOF.$
4. $\angle OAB \cong \angle OBC, \overline{OA} \cong \overline{BC}, \overline{OB} \cong \overline{OB}, \angle ABO \cong \angle BCO, \angle AOB \cong \angle BOC.$
5. In $\triangle OAB$, $AO = BC$, $BO = BO$, and $AB \approx$ 24mm. In $\triangle OFG$, $OF = OF$, FG OE, and $OG \approx$ 104.5mm
6. $\triangle OCD$ and $\triangle ODE$; $\triangle ODE$ and $\triangle OEF$
7. a) 486, 1458, 4374 Each term is three times the previous term
   b) 81, 121.5, 182.25. Each term is 1.5 times the previous term.

9 The sum of the lengths of two sides of the triangle is less than the third side

10. They are 5-con triangles, three pairs of angles are congruent and two pairs of sides are congruent.
11. 36, 54, 81 and 54, 81, 121.5
12. 6, 3; 16, 3 2
13. $\triangle OBC$: 30, 37.5, 46.9; $\triangle OCD$: 37 5, 46.9, 58.6
14. 27, 36, 48, 64, a 27, and r · 4.3. Yes, they form a 5-con pair—the 5-con theorem is satisfied
15. No, because the sum of the measures of the three angles of a triangle is 180 degrees, and if two pairs of the angles were congruent, the third pair would also have to be congruent
16. No, because if the triangle was isosceles then in the triple a, ar, ar^2, the r equals 1, thus, the two triangles would be congruent and equilateral.

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student math notes

Striking Sequences

Throughout the ages, people have been fascinated with numbers in sequence. The *striking sequences* are formed from other sequences of numbers by striking out some elements and performing operations on the remaining sequences. The striking sequences given here were first discussed by the German number theorist Alfred Moessner in 1951 and have been extended and explained by the American number theorist Calvin Long, Washington State University. Try your hand at the following striking sequences.

I. Use the natural numbers and strike out every second number starting with 2.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 . . .

Use the remaining numbers to make a sequence of partial sums from the striking sequence. The first four terms are given here:

$1 = 1$
$1 + 3 = 4$
$1 + 3 + 5 = 9$
$1 + 3 + 5 + 7 = 16$

The terms found by determining the partial sums as illustrated form a new sequence.

1. Use the terms you found to complete writing the first nine terms of this new sequence.

1, 4, 9, 16, ____ , ____ , ____ , ____ , ____ , . . .

2. Describe the apparent pattern in the sequence that consists of the partial sums.

____

____

II. Construct another striking sequence from the natural numbers by striking out every third number starting with 3.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 . . .

As in part I, form partial sums from this striking sequence. The first three terms are $1 = 1$, $1 + 2 = 3$, and $1 + 2 + 4 = 7$.

3. List the first ten terms in the sequence of partial sums.

1, 3, 7, ____ , ____ , ____ , ____ , ____ , ____ , ____ , . . .

The editors wish to thank Carole B. Lacampagne, Department of Mathematical Science, Northern Illinois University, DeKalb, IL 60115, for writing this issue of *NCTM Student Math Notes*.

Striking Sequences—Continued

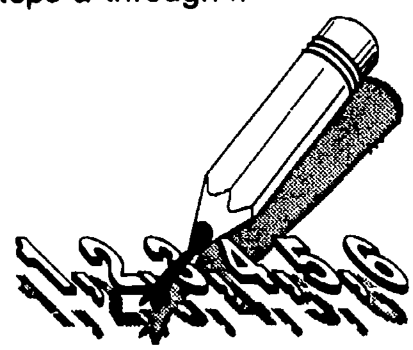
4. Next construct a new striking sequence from the partial-sums sequence in step 3 by striking out every other number starting with the second term.

5. Form a new sequence of partial sums from the sequence obtained in step 4.

6. Describe the apparent pattern in the sequence obtained in step 5.

III. Consider a striking sequence formed from the natural numbers 1–24 using all steps *a* through *f*:

(*a*) Strike out every fourth number.
(*b*) Make the sequence of partial sums.
(*c*) Strike out every third number from the sequence formed in step *b*.
(*d*) Make the sequence of partial sums from the sequence formed in step *c*.
(*e*) Strike out every other number from the sequence formed in step *d*.
(*f*) Make the sequence of partial sums from the sequence formed in step *e*.



7. Predict what the final sequence will be.

8. Construct the striking sequence using all the steps *a* through *f* in the order given.

9. What would be the result of repeating the process in part III by striking out every fifth number beginning with 5. (You may wish to start with a longer sequence of natural numbers.) Construct the sequence.

IV. In parts I–III striking sequences were obtained from the sequence of natural numbers. Next use the sequence of odd counting numbers and strike out every other number starting with 3.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 . . .

10. Form the sequence of partial sums from the striking sequence of odd numbers obtained above.

The sequence of partial sums obtained in step 10 may not look familiar. However, consider factoring each term as follows:

$$1, \quad 2 \cdot 3, \quad 3 \cdot 5, \quad 4 \cdot 7, \ldots$$

11. Predict the factorization of the next six terms in the sequence.

12. Verify your answer by extending the sequence in step 10.

V. Next construct a striking sequence using the odd numbers, striking every third number, getting partial sums, striking every other number of the partial-sums sequence, and getting a new sequence of partial sums.

13. What is the final sequence you obtained?

____

14. What is the pattern found by factoring the sequence?

____

VI. Next let's construct a striking sequence using the natural numbers. Strike out the first number, then the second number after that, then the third number after that, the fourth number after that, and so on.

15. List the sequence obtained.

____

16. Make a sequence of partial sums from the sequence obtained in step 15.

____

17. Construct a new striking sequence from the one in step 16 in the same fashion as before: Strike out the first number, then the second number after that, then the third number after that, and so on; then take the partial sums to form another sequence and write it.

____

18. Repeat the process on the sequence in step 17. Show the sequence of partial sums.

____

19. Repeat the process on the sequence in step 18. Show the sequence of partial sums.

____

20. Form a new sequence starting with 1, made up of the *first* stricken entries of the sequences in steps 16, 17, 18, and 19. Continue the pattern.

1, 2, 6, ______________________________

21. Use the factorizations of the first three terms of the sequence in step 19 to try to discover a pattern. Then factor the next two terms similarly.

1, $1 \cdot 2$, $1 \cdot 2 \cdot 3$, ______________________________

22. Describe the sequence obtained in step 20.

____

Did you know that . . .

- a sequence is said to be *arithmetic* if the difference between consecutive terms is the same? The natural numbers form an arithmetic sequence because the common difference between terms is 1. The odd numbers form an arithmetic sequence because the common difference between terms is 2. Any sequence of the form a, $a + d$, $a + 2d$, $a + 3d$, . . . , $a + nd$ is arithmetic. The first term is a, and the common difference between terms is d.

Did you know that . . .—Continued

- the sum of the first n terms of an arithmetic sequence is

$$S_n = (1/2)n[2a + (n - 1)d]?$$

- as a school boy, Karl Freidrich Gauss discovered the sum of the first n natural numbers?

Can you . . .

- generalize the results in parts IV and V? What will the final sequence be if you start with the odd counting numbers, strike out every kth number, get partial sums, strike out every $(k - 1)$st number, get partial sums, and so on?
- find Gauss's formula and use it to prove the formula for the sum of the first n terms of an arithmetic sequence?
- write an arithmetic sequence of your choice, construct a striking sequence from it by striking out every kth number, get partial sums, and so on? Can you generalize about your striking sequence?
- write an arithmetic sequence and construct a striking sequence, as in part IV?
- write a computer program to complete the process in part VI? (Since the numbers in your sequence will grow rapidly, you may need to use double-precision variables.)

Have you read . . . ?

- Long, Calvin. "Mathematical Excitement—the Most Effective Motivation." *Mathematics Teacher* 75 (May 1982):413–15.
- ———. "On the Moessner Theorem on Integral Powers." *American Mathematical Monthly* 74 (October 1966):846–51.
- ———. "Strike It Out—Add It Up." *Mathematical Gazette* 66 (December 1982):273–77.
- Moessner, Alfred. "Eine Bemerkung uber die Potenzen der naturlichen Zahlen" [One Observation on the Power of Natural Numbers]. Munich: Bavarian Academy of Sciences and Mathematics. Nat. K1 S. -B., 29.

Answers

1. 1, 4, 9, 16, 25, 36, 49, 64, 81
2. The sequence consists of the squares of the counting numbers:

   $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, \ldots$
3. 1, 3, 7, 12, 19, 27, 37, 48, 61, 75, . . .
4. 1, 7, 19, 37, 61, . . .
5. 1, 8, 27, 64, 125, . . .
6. The sequence consists of the cubes of the counting numbers:

   $1^3, 2^3, 3^3, 4^3, 5^3, 6^3, 7^3, 8^3, \ldots$

7, 8. The sequence consists of the fourth powers of the counting numbers:

   1, 16, 81, 256, 625, 1296, . . .

9. The sequence consists of the fifth powers of the counting numbers:

   1, 32, 243, 1024, 3125, . . .
10. 1, 6, 15, 28, 45, 66, 91, . . .
11. $5 \cdot 9, 6 \cdot 11, 7 \cdot 13, 8 \cdot 15, 9 \cdot 17, 10 \cdot 19$
12. 120, 153, 190, 231, . . .
13. 1, 12, 45, 112, 225
14. $1, 2^2 \cdot 3, 3^2 \cdot 5, 4^2 \cdot 7, 5^2 \cdot 9, \ldots$
15. 2, 4, 5, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 20, 22, . . .
16. 2, 6, 11, 18, 26, 35, 46, 58, 71, 85, 101, 118, 136, 155, 175, . . .
17. 6, 24, 50, 96, 154, 225, 326, 444, 580, 735, . . .
18. 24, 120, 274, 600, 1044, 1624, . . .
19. 120, 720, 1764, . . .
20. 1, 2, 6, 24, 120, . . .
21. $1 \cdot 2 \cdot 3 \cdot 4, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5, \ldots$
22. The sequence consists of the factorials of the counting numbers:

   1!, 2!, 3!, 4!, 5!, 6!, 7!, 8!, . . .

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student math notes

Invariants

The knobs of appliances, the heads of screws, and a road sign are seen in figure 1.

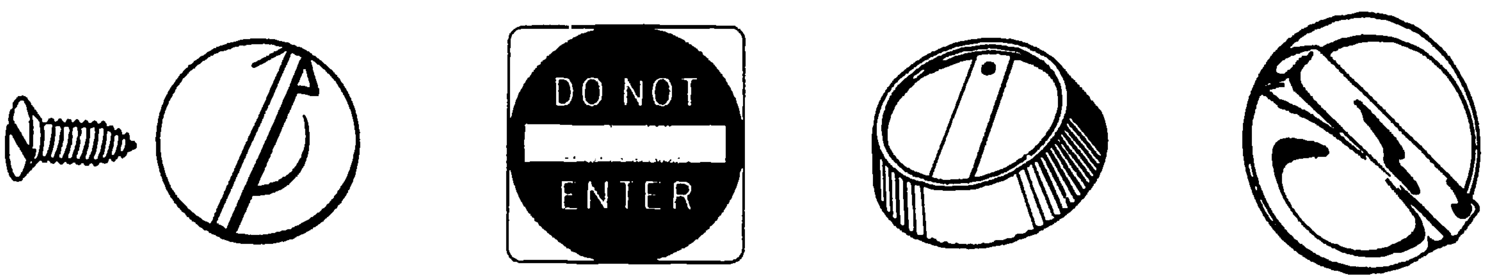
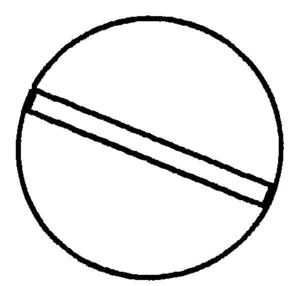


Fig. 1

A two-dimensional representation of the tops of these common items can be depicted as a rectangle inscribed in a circle (that is, the vertices are on the circle), as in figure 2.

Fig. 2



Carefully inscribe five rectangles in the circle in figure 3 and draw one diagonal in each rectangle.

1. What common properties do all your rectangles in figure 3 share?

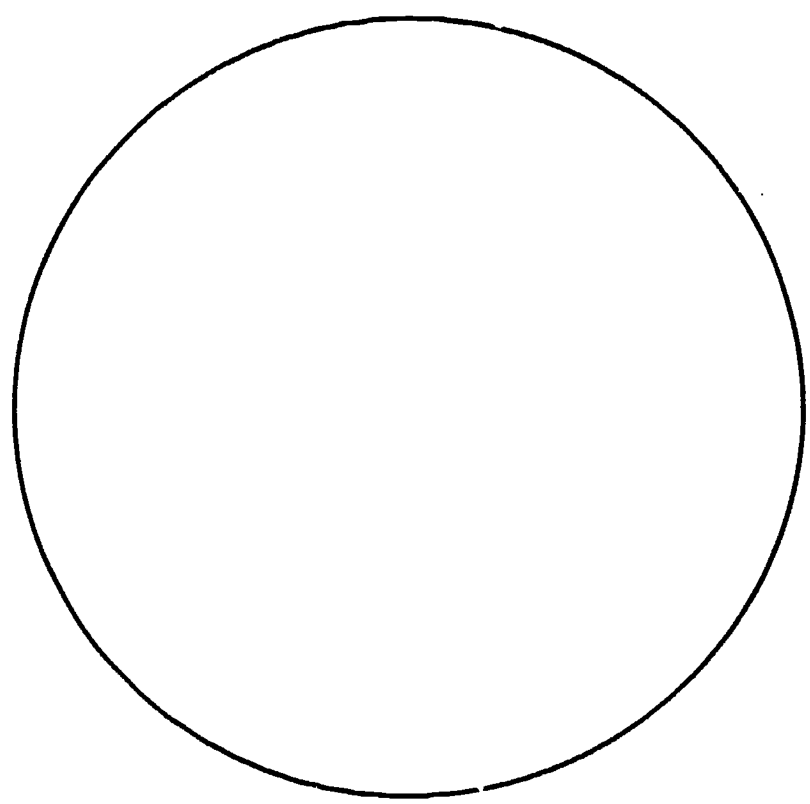


Fig. 3

The editors wish to thank Melfried Olson, Western Illinois University, Macomb, IL 61455, and Glenn Bruckhart, Araphoe High School, Littleton, CO 80120, for writing this issue of *NCTM Student Math Notes*.

Other Invariants of Inscribed Quadrilaterals

All the properties found in question 1 are called *invariant properties* of this set of rectangles. One such property is that for any circle or circles having the same radii, the diagonals of all inscribed rectangles are the diameters of the circles and have the same length.

In figure 4 carefully inscribe one nonrectangular quadrilateral in each circle.

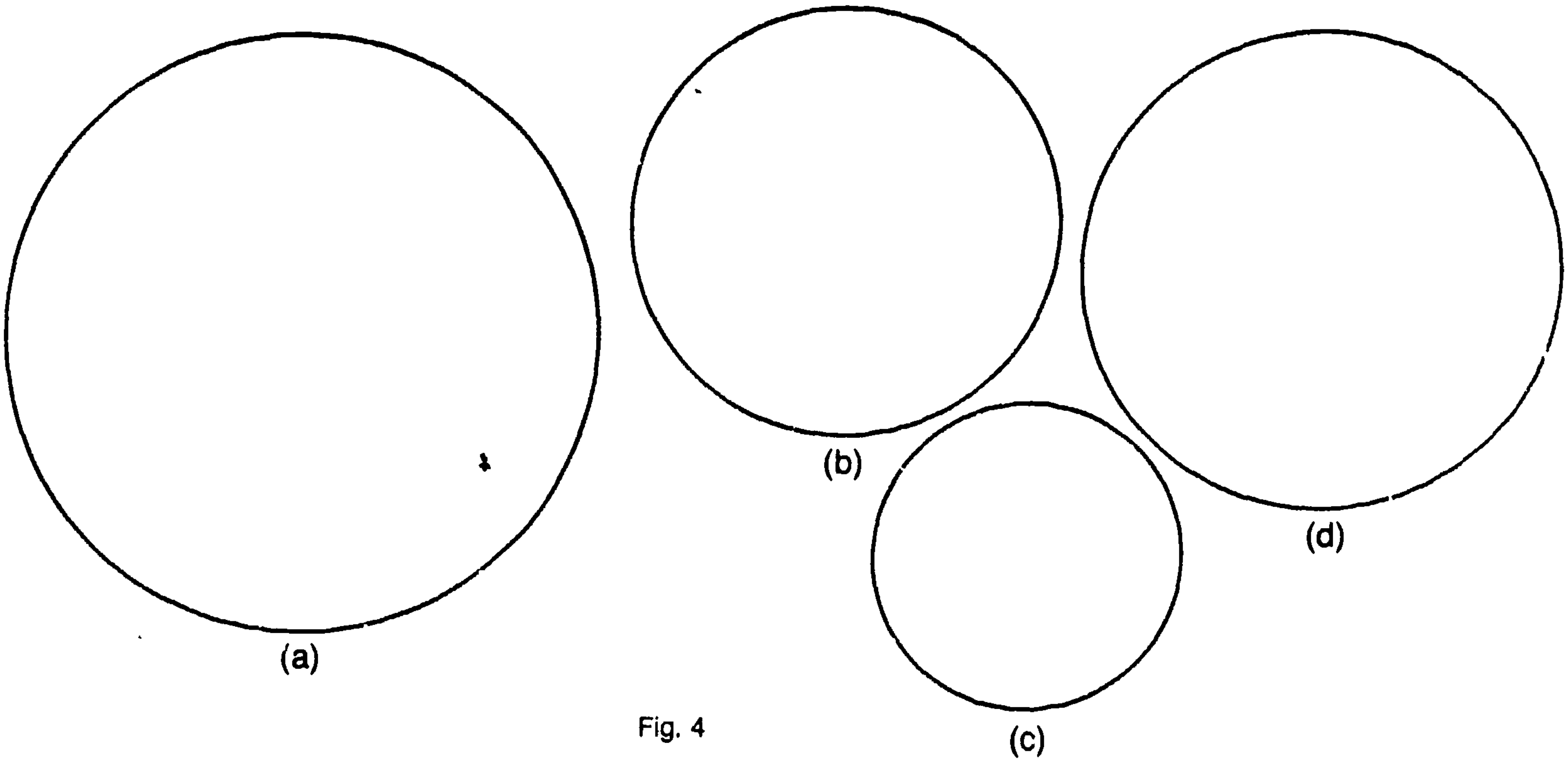


Fig. 4

2. Label each quadrilateral in figure 4 as $ABCD$. Draw diagonals $\overline{AC}$ and $\overline{BD}$ and let E be the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Record the measures of each of the following angles and segments in table 1 (measure segments in millimeters).

Table 1

	Figure 4a	Figure 4b	Figure 4c	Figure 4d
a) $m\angle ABC$				
b) $m\angle BCD$				
c) $m\angle CDA$				
d) $m\angle DAB$				
e) $m\angle DAB + m\angle BCD$				
f) $m\angle ABC + m\angle CDA$				
g) AE				
h) EC				
i) BE				
j) ED				
k) $AE \cdot EC$				
l) $BE \cdot ED$				

More Invariants

3. Use 2e and 2f to describe the sums of the measures of the opposite angles of each quadrilateral in table 1. What did you discover?

4. Use 2k and 2l to find the products of the lengths of the segments of the chords in table 1. What did you discover?

Table 2 shows several pairs of numbers whose product is 12.

Table 2

a	1	1.5	2	2.5	3
b	12	8	6	4.8	4

In table 2, we have an invariant property that $a \cdot b = 12$ for each pair in a column. Can we relate this invariant property to circumscribed circles? In figure 5, two segments intersecting at E have been drawn. One segment, $\overline{AC}$, is 7 cm long and broken into pieces of lengths 3 cm and 4 cm. The other segment, $\overline{BD}$, is 8 cm long and broken into pieces of lengths 2 cm and 6 cm. (Note that the two pairs, 3 and 4 and 2 and 6, are from table 2 and that $3 \times 4 = 2 \times 6$.) Construct a circle that contains points A, B, C, and D. (Hint: Find the intersection of the perpendicular bisectors of $\overline{AC}$ and $\overline{BD}$.)

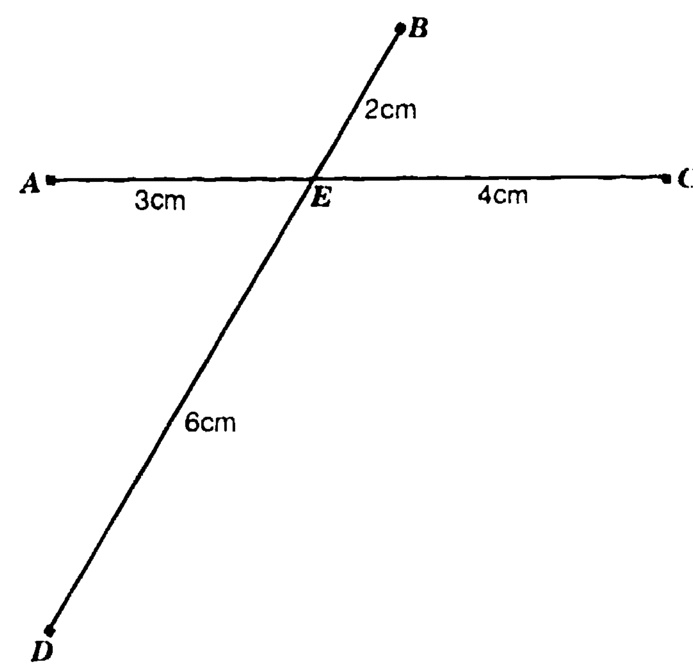


Fig. 5

Use another two pairs, 1.5 and 8 and 3 and 4, to draw segment $\overline{AC}$ of length 9.5 cm, or (1.5 + 8), and segment $\overline{BD}$ of length 7 cm, or (3 + 4), which intersect at point E as in figure 5. Construct a circle that contains points A, B, C, and D.

More Invariants—Continued

5. Will the circle you obtain be congruent to the circle a friend makes? ________

   Repeat this experiment using other pairs from the table.

6. Write an invariant property for intersecting segments (the products of whose parts are the same).

____

____

____

Did You Know That . . .

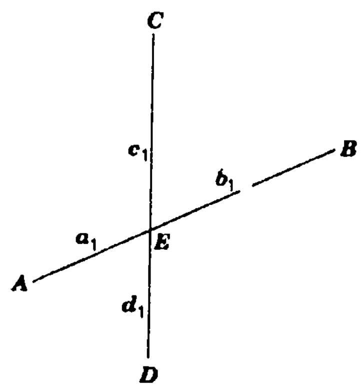
- defining properties are invariant properties, but not all invariant properties are defining properties of geometric objects?
- transformational geometry was begun as a study of invariant properties?
- the Pythagorean theorem is an invariant property of right triangles?
- in an inscribed quadilateral, the angle between a side and a diagonal is equal to the angle between the opposite side and the other diagonal?
- in an inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides?

Can You . . .

- find other invariant properties involving segments and circles?
- find other invariant properties involving measures of angles inscribed in circles?
- use the Geometric preSupposer software Points and Lines to do the explorations suggested in figure 5?
- describe the locus of the centers of the circles you constructed in question 5 if $\overline{BD}$ is rotated at E until it is collinear with $\overline{AC}$?

Answers

1. Answers may vary. Examples may include the following: All diagonals are the same length; all angles are right angles; two pairs of congruent chords and arcs are determined by each rectangle; the opposite sides are equal, the opposite sides are parallel.
2. Answers will vary.
3. Opposite angles are supplementary; that is, the sum of opposite angles is 180 degrees.
4. The product of the lengths of the segments of one diagonal equals the product of the lengths of the segments of the other diagonal. From the actual measurements used, more than likely the products will be close, not exactly equal.
5. Probably not
6. If two segments AB and CD intersect at a single point E that is not the endpoint of either segment, such that $a_1 \times b_1 = c_1 \times d_1$ (in figure), then a circle can be drawn that contains the endpoints of both segments. However, the radius of the circle is dependent on the angle "between" the segments.



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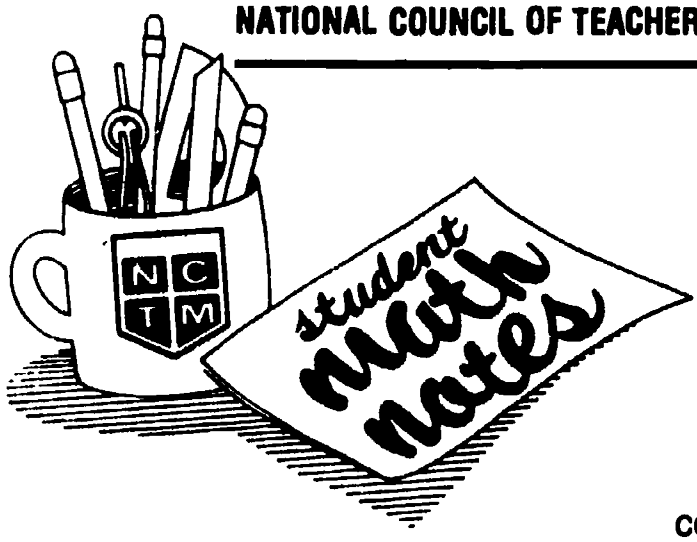


On the Ball

Sionilli Mathematics Academy is on the ball. Their basketball coach is a mathematics teacher who uses every opportunity to teach mathematics. To work on notions of greatest common divisors and least common multiples, the coach devises special warm-up drills and practice schedules.

Warming Up

For a pregame warm-up, the members gather into a circle and the captain has the only basketball. The coach asks the captain to call out a number, say, 3, and pass the ball to the third person to the right. After catching the ball, that person continues the drill by passing the ball to the third person on the right, and so on.

1. Figure 1 represents the beginning of a warm-up for a team of ten. The captain called the number 3, and the first three passes are recorded. Sketch in the remaining passes until the ball returns to the captain. Note that each player touches the ball once before it returns to the captain. When such a cycle occurs, we shall refer to it as a *complete cycle*.

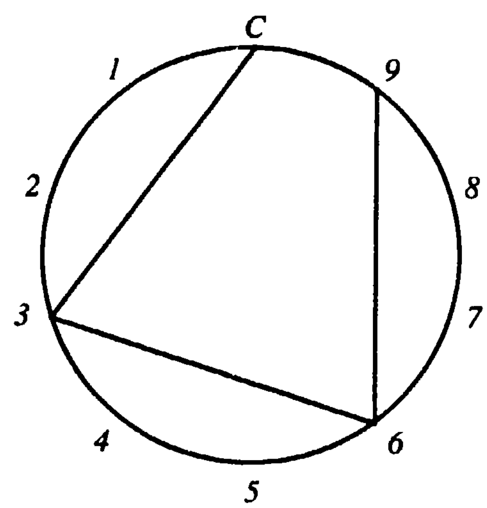


Fig. 1

2. If the captain calls the number 7, does this number lead to a complete cycle? (Record passes for this warm-up on fig. 2.) __________

3. What happens if the captain calls the number 6? (Record passes for this warm-up on fig. 3.)

____

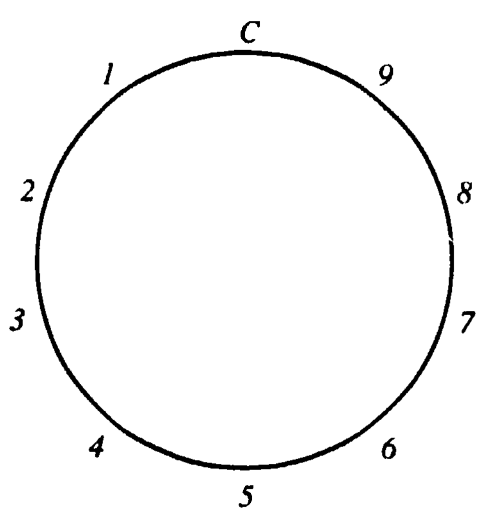


Fig. 2

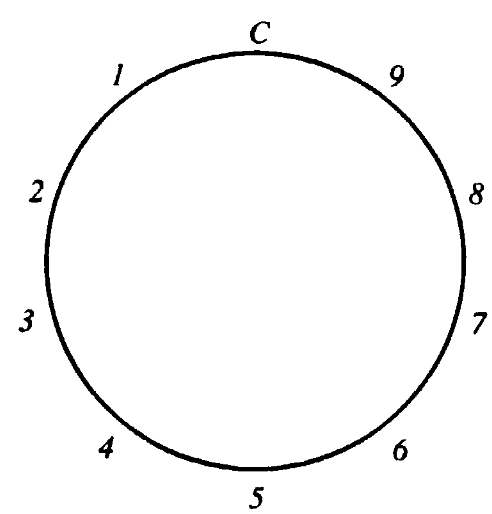


Fig. 3

The editors wish to thank Melfried Olson, Western Illinois University, Macomb, IL 61455, for writing this issue of *NCTM Student Math Notes*.

Warming Up—Continued

Complete table 1 to determine which of the whole numbers 1 through 9 the captain can call to form complete cycles.

4. Which numbers can be called to form complete cycles? ________

5. Calling the number 3 allowed all ten people to touch the ball, but calling the number 2 did not. What relationship exists between 2 and 10 that does not exist between 3 and 10? ___________

6. Consider the number 4. Four is not a factor of 10, yet calling 4 did not allow all 10 people to touch the ball. What relationship exists between 4 and 10 that does not exist between 3 and 10? ________________

Table 1

Number of Players	Number Called	Complete Cycle?
10	1	
10	2	
10	3	Yes
10	4	
10	5	
10	6	No
10	7	
10	8	
10	9	

7. What is the greatest common factor (GCF) of 4 and 10? _________

8. What is the greatest common factor of 3 and 10? _________

When two whole numbers have 1 as their greatest common factor, the numbers are **relatively prime.**

9. One day a team member was absent. With nine players, what numbers can the captain call to assure complete cycles? ________________

10. Find the greatest common factor of 9 and each of the numbers less than 9. Which pairs are relatively prime?

11. For the regional tournament, the coach is allowed to add two players from the junior varsity team. With a team of twelve players, what numbers can the captain choose in the warm-up passing drill to form complete cycles?

Table 2

Number of Players	Numbers Called for Complete Cycles	How Many Numbers Called to Form Complete Cycles?
7		
8		
9	1, 2, 4, 5, 7, 8	6
10	1, 3, 7, 9	4
11		
12		
13		
14		
15		

Suppose a team had the following numbers of players (including the captain). Complete table 2 and record the numbers the captain could call to form complete cycles.

12. For *n* players, which numbers would form complete cycles? ________________

13. Which number of team members, 16 or 18, gives the greater number of warm-up options? ______

14. Why doesn't adding more players always give more options for warm-ups? ______________

Sionilli Practice Schedules

To maintain individual conditioning and skill level, the coach posted the following instructions and routines:

Tournament Practice

Rules

- Each player must practice **one hour** daily.
- Spend **five minutes** on each activity.
- Start day 1 at the top of the list.
- Do not change the order of the list.
- Return to the first activity after completing the seventh.
- Begin each day with the activity following the final activity of the previous day.

Routines

1. Field goals
2. Free throws
3. Lay-ups
4. Rebounds
5. Passing
6. Dribbling
7. Sprints

15. According to the rules, what activity will be the last one completed on day 1?________________

16. What activity will be first on day 3?________________

17. What is the fewest number of days after which each activity will have received equal practice time? ________________

18. How many five-minute intervals have been devoted to free throws after the end of fourteen days of practice?________________

After fourteen days, the coach decided to eliminate the last two activities, dribbling and sprints. Each individual continues to practice the remaining five activities, starting at the top of the routines list.

19. Does this change in scheduling affect the first day on which each activity will have received equal practice time?________________

20. If the coach also eliminates passing, on which days will an individual player start the day at the top of the list?________________

Use the information from questions 15–20 to complete table 3.

21. Make a conjecture about the relationship between the number of days between repeats and the least common multiples (LCMs) found in table 3.

____

____

Table 3
Twelve Five-Minute Practice Intervals

Number of Activities	Number of the Days on Which the First Routine Is Field Goals	Difference in the Arithmetic Sequence in Column 2	LCM of 12 and Number of Activities
3	1, 2, 3, 4, . . .	1	12
4			
5			
6			
7			
8			
9			
10			
11			

1C

Can you...

- determine a rule for finding the number of options the captain has for the warm-up drill with n players?
- determine the day of the week on which you were born?
- relate the warm-up exercise to procedures for creating stars in Logo?

Did you know that...

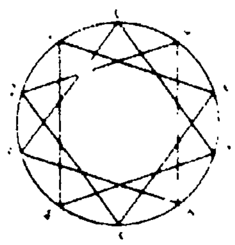
- Euler's phi function defines $\phi(N)$ as the number of numbers smaller than a given number N and relatively prime to that number. $\phi(N)$ can be determined from the prime factorization of the given number:

$$\text{If } N = P_1^{e_1} \cdot P_2^{e_2} \cdot P_3^{e_3} \cdot \cdots \cdot P_n^{e_n},$$
$$\text{then } \phi(N) = N(1 - 1/P_1)(1 - 1/P_2)(1 - 1/P_3)\cdots(1 - 1/P_n)$$

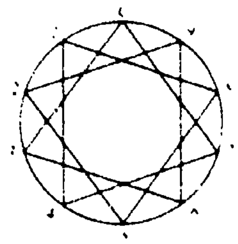
Answers

Page 1:

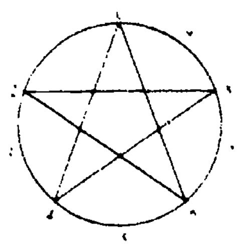
1



2 Yes



3 The cycle is not complete



Page 2:

Table 1

Number of Players	Number Called	Complete Cycle?
10	1	Yes
10	2	No
10	3	Yes
10	4	No
10	5	No
10	6	No
10	7	Yes
10	8	No
10	9	Yes

4 1, 3, 7, and 9

5 2 is a factor of 10
3 is not a factor of 10

6 4 and 10 have a common factor of 2
3 and 10 have no common factors except 1

7 GCF (4, 10) = 2

8 GCF (3, 10) = 1

9 1, 2, 4, 5, 7 and 8

10 GCF (1, 9) = 1 GCF (2, 9) = 1 GCF (3, 9) = 3 GCF (4, 9) = 1
GCF (5, 9) = 1 GCF (6, 9) = 3, GCF (7, 9) = 1, GCF (8, 9) = 1

11 1, 5, 7, and 11

Table 2

Number of Players	Numbers Called for Complete Cycles	How Many Numbers Called to Form Complete Cycles?
7	1, 2, 3, 4, 5, 6	6
8	1, 3, 5, 7	4
9	1, 2, 4, 5, 7, 8	6
10	1, 3, 7, 9	4
11	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	10
12	1, 5, 7, 11	4
13	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	12
14	1, 3, 5, 9, 11, 13	6
15	1, 2, 4, 7, 8, 11, 13, 14	8

12 All numbers less than n and relatively prime to n

13 The number 16 has eight options, whereas 18 has six options, thus 16 has more options.

14 The number of options depends not entirely on the number of players but on the number of numbers relatively prime to the number of players

Page 3:

15 Passing

16 Rebounds

17 Seven days

18 Twenty-four intervals

19 Yes, it will take only five days

20 Every day

21 The number of days between repeats is the quantity (LCM of 12 and the number of activities) divided by 12

Table 3
Twelve Five-Minute Practice Intervals

Number of Activities	Number of the Days on Which the First Routine Is Field Goals	Difference in the Arithmetic Sequence in Column 2	LCM of 12 and Number of Activities
3	1, 2, 3, 4, . . .	1	12
4	1, 2, 3, 4, . . .	1	12
5	1, 6, 11, 16, . .	5	60
6	1, 2, 3, 4, . . .	1	12
7	1, 8, 15, 22, . . .	7	84
8	1, 3, 5, 7, . . .	2	24
9	1, 4, 7, 10, . .	3	36
10	1, 6, 11, 16,	5	60
11	1, 12, 23, 34, . .	11	132

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Coloring Maps

Mapmakers follow two rules in coloring maps:

- Each country or state should be colored with a single color.
- Different colors should be used for countries or states that share a common border.

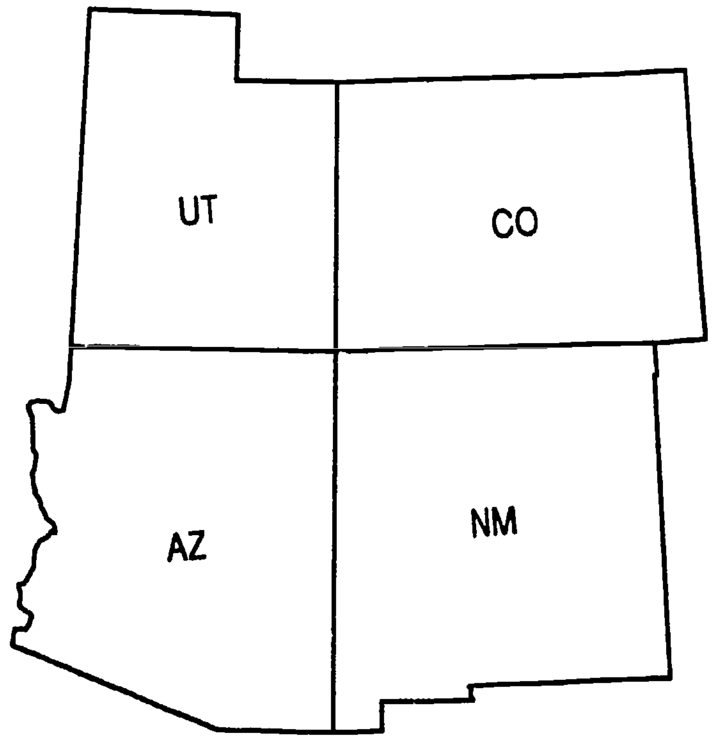


Fig. 1

Figure 1 is a map of Utah, Colorado, Arizona, and New Mexico. We could use a different color for each state, but following the mapmakers' rules, we can use fewer colors. Color the map with the fewest possible colors. If the states touch at only a single point, they can be of the same color.

1. How many colors are required?__________

Fig. 2

Figure 2 is a map of another portion of the western United States. How many colors do we need for this map if we follow the mapmakers' rules?

2. Are two colors enough?__________
3. If not, can the map be colored with three colors?

____

4. What is the minimum number of colors required to color the map that combines figures 1 and 2?

____

The editors wish to thank Thomas Dick, Oregon State University, Corvallis, OR 97331, for writing this issue of *NCTM Student Math Notes.*

Test your conjecture by coloring figure 3.

Fig. 3

For each of the regional maps in figure 4, determine the fewest colors necessary to color the states following the mapmakers' rules.

5. Which maps require only three colors?__________

6. Which maps require four colors?__________

7. Do any require five colors?__________

8. Is it possible for a map to require five colors?__________

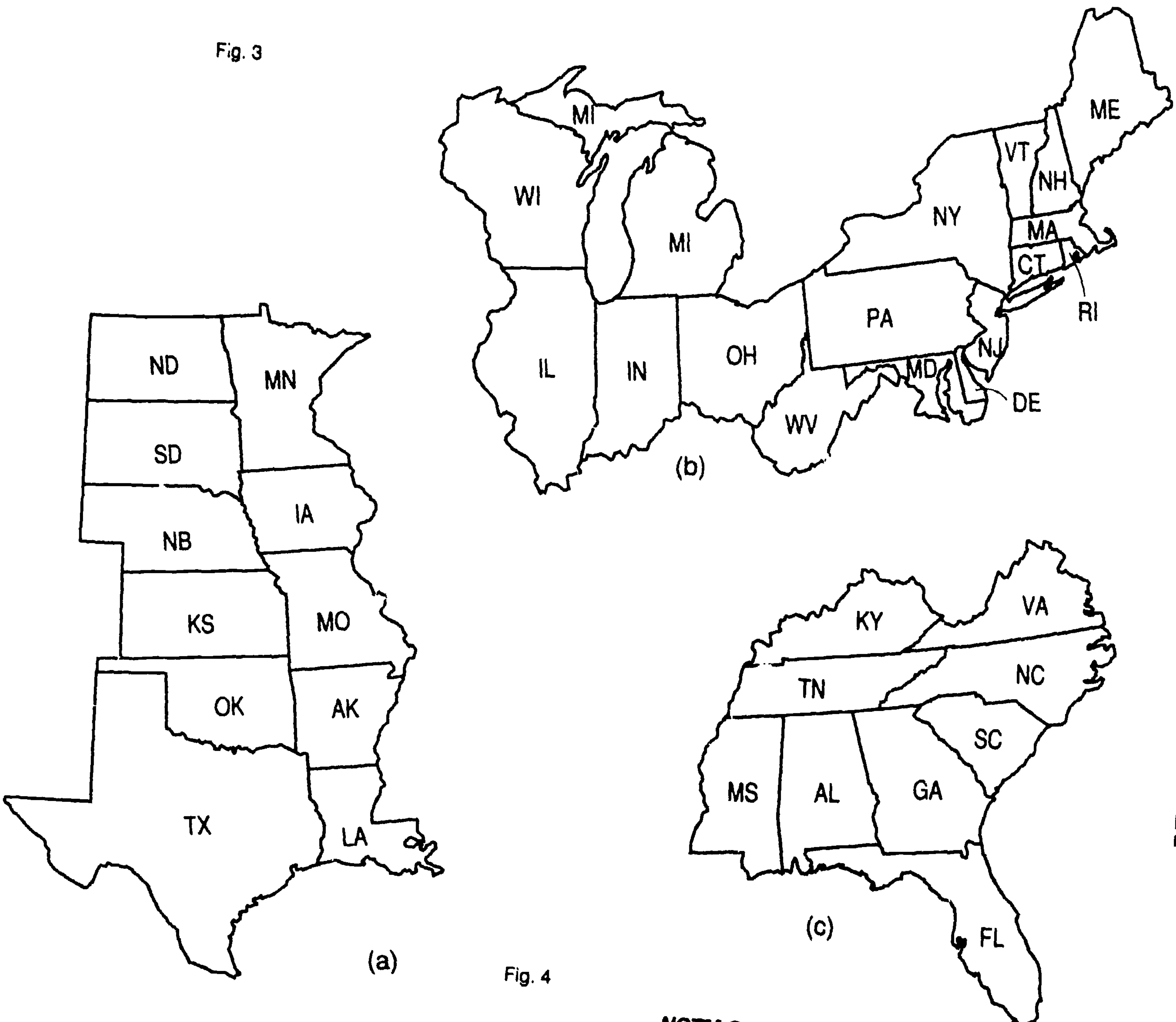


Fig. 4

Coloring Maps—Continued

9. Consider figure 5, in which country A is separated into two pieces. Are five colors necessary?________

10. Are any states in the United States broken into more than one piece?________Which ones?________________

11. Is it possible to create a map that would require six colors?________

12. Is it possible to create a map requiring five colors in which none of the countries are broken?________________

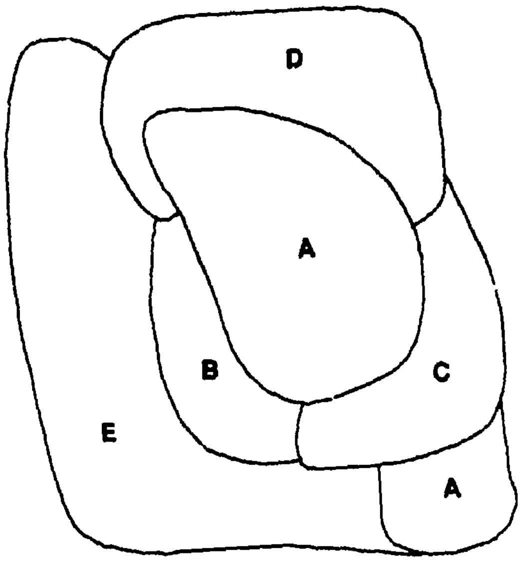


Fig. 5

Figure 6 shows a map of the continental United States. Four colors are enough to color this map. Color it using the mapmakers' rules.

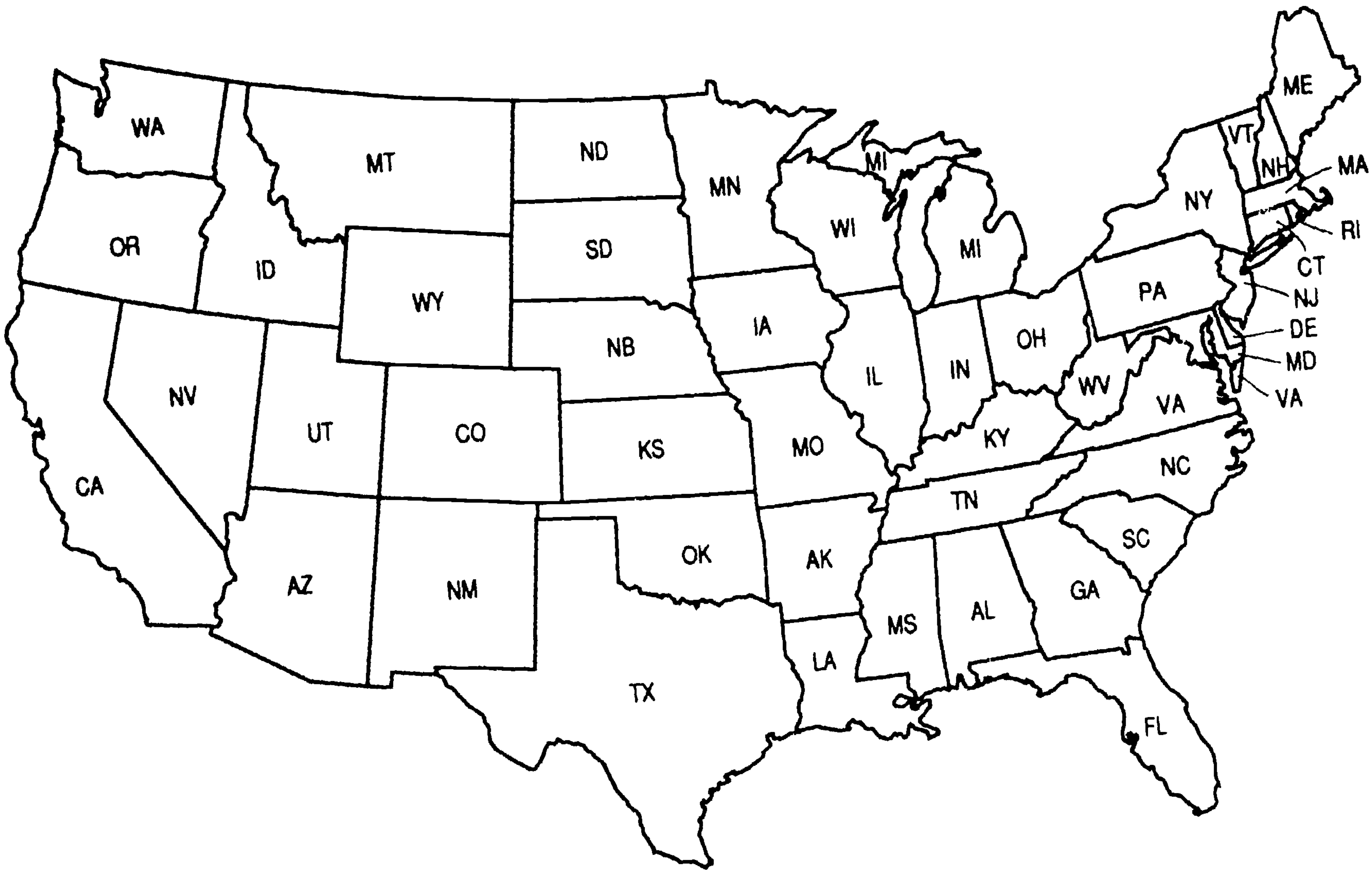
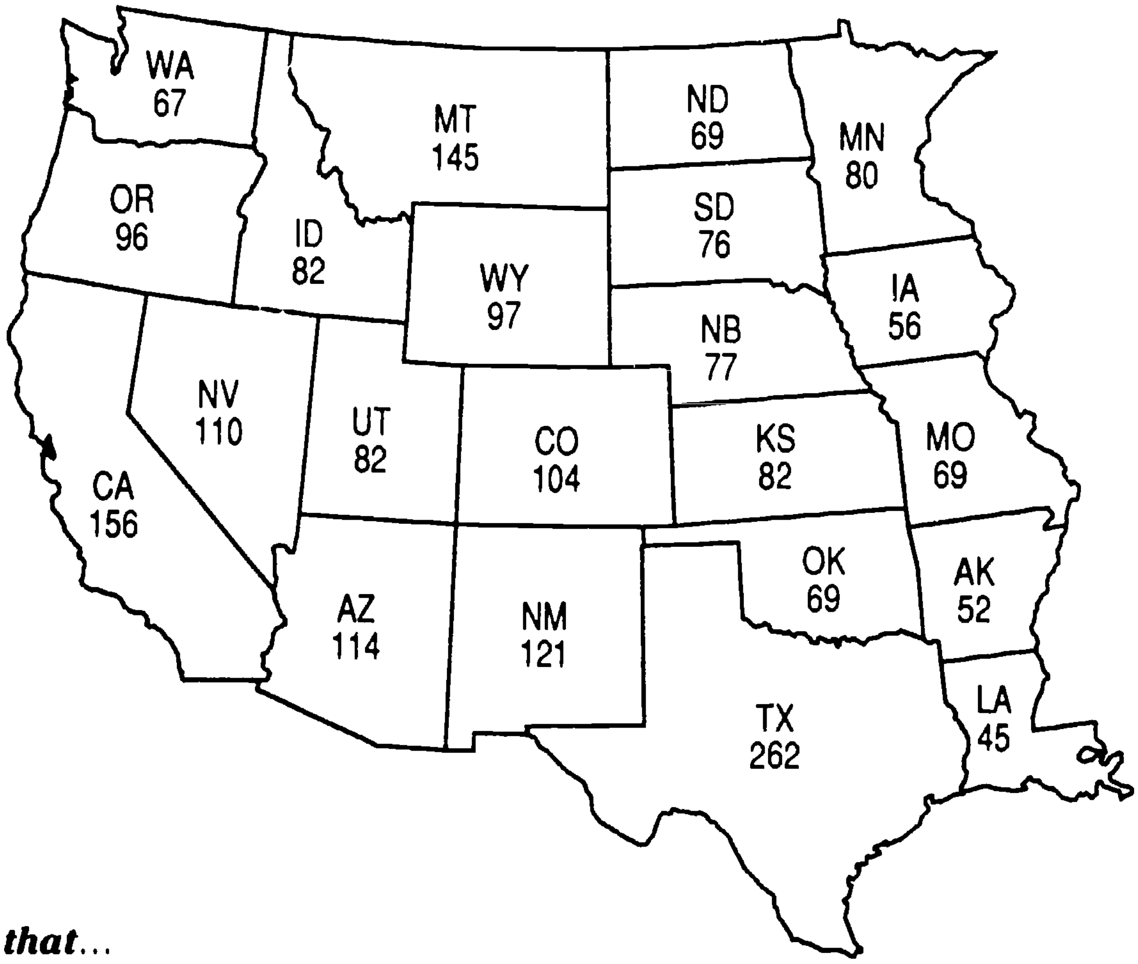


Fig. 6

Can you...

- color the map below at a minumum cost if the four chosen colors cost $1, $2, $3, and $4 per 1000 square miles, respectively? The number in each state represents thousands of square miles.



Did you know that...

- if a map is drawn on a flat plane or a globe and if all the countries or states are single, unbroken regions, then four colors always suffice to color the map (four-color theorem)?
- the four-color theorem was verified in 1976 at the University of Illinois with the help of computers?
- if we draw our map on a doughnut, the map may require as many as seven different colors?

Bibliography

- "Thinking with Ink." In Problem-solving Strategies. (Computer program) Available from Minnesota Educational Computing Corporation, 3490 Lexington Avenue North, St. Paul, MN 55126. Challenges students to color maps with inks of varying costs.

Answers:

1. Two	4. Four	7. No	10. Yes; Michigan and Virginia
2. No	5. a, b, c	8. Maybe	11. Yes
3. Yes	6. None	9. Yes	12. Not if the map is in a plane

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