ABSTRACT
         Reasons for poor performance in mathematics by
students in the United States are discussed. According to the
authors, too many students never experience arithmetic at a physical,
concrete level. Students are drilled in arithmetic facts without any
meaningful context and are given few opportunities to use numerical
concepts in real-life applications. Described in this fastback in the
form of a Decalogue, or ten commandments, are methods that can be
used to teach mathematics successfully. These commandments hold for
every grade level including postsecondary mathematics. Topics
include: (1) the use of manipulatives and visuals; (2) cooperative
learning models; (3) diagnosis of student development; (4) unit plan
development; (5) problem solving; (6) algebra and geometry; (7) the
use of computers and calculators; (8) mental computation, estimation,
and measurement; (9) probability and statistics; (10) integration of
skills and techniques from different branches of mathematics. (KR)

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A Decalogue
for Teaching Mathematics

by
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and
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Introduction

On 8 June 1988, the front page of the *Tacoma Morning News Tribune* carried the headline: "2 + 2 = 5: American Students Get An 'F' in Math, Study Shows." The study described is *The Mathematics Report Card, Ar* We Measuring Up? (Dorsey et al. 1988), one of the reports from the National Assessment of Educational Progress, conducted by the Educational Testing Service and funded by the federal government.

The study examined the mathematics performance of 9-, 13-, and 17-year-olds in 1972-73, 1977-78, 1981-82, and 1985-86. The achievement results reported were disheartening. Nearly half of 17-year-olds do not have skills beyond basic computation with whole numbers. Only about 6% can solve multi-step or algebra problems. Among black and Hispanic students the percentages for multi-step and algebra problems were just below and just above 1% respectively. Among 13-year-olds, 27% could not perform arithmetic problems normally taught in the elementary school. All of the students tested had studied most of the content of the test, but they no longer had command of the skills. The skills either had never been mastered or had been forgotten.

Why are American students doing so poorly in mathematics? Simply stated, too many never experience arithmetic at a physical, concrete level They are drilled in arithmetic facts without any meaningful context. They are given few opportunities to use numerical concepts

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in real-life applications. And when they encounter higher-level mathematics, they soon discover that it requires more than simple memorization of number facts. As a result, many drop out of mathematics in high school as soon as they can.

So much for the bad news. The good news is that there are methods f teaching mathematics successfully, which have been verified by research ranging from Piaget's stages of cognitive development (Copeland 1984) to Gagne's work on learning and transfer (Gagne 1970, 1974). These methods will be described in this fastback in the form of a Decalogue or ten commandments for successful mathematics teaching. These commandments hold for every grade level including postsecondary mathematics. The reader who wants to learn more about these methods than can be covered in the brief space of this fastback should consult the references in the bibliography.

By using these ten commandments of successful mathematics teaching, along with a good dose of emotional support, teachers (with the aid of parents) can help almost all students to succeed, even at college level mathematics. Can we afford to do less?

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I. Thou Shalt Use Manipulatives and Visuals

One of the best ways for young children to learn basic number concepts (and other concepts as well) is by manipulating objects — not on paper, not on a computer screen, but physically — by touching, picking up, throwing, hitting, moving, dropping, combining, and stacking The best objects are simple materials. blocks of different shapes and sizes, tongue depressors rods of different lengths, simple plastic squares, rectangles, circles, and triangles. For young children, simply playing with these objects is valuable for concept development. According to Parham (1983), elementary youngsters with significant experience in using manipulatives scored at the 85th percentile on standardized mathematics tests. As mathematics becomes more complex and abstract, teachers must continue to use manipulatives and visuals to help students master concepts. By this stage (concrete operational), manipulatives take the form of physical models, supplemented by diagrams, graphs, pictures, slides, and films.

Examples of manipulatives for elementary mathematics abound. For younger children, the mathematical sentence $3 + 2 = N$ can be represented by letting the child put a pile of three objects together with a pile of two objects and then count the resulting pile.

For children who are learning to add fractions, apples or chocolate chip cookies can be cut into equal sections, which they separate and

then put back together before eating them. The children can then move on to a slightly more abstract activity by drawing fractions of apples or cookies on paper, thus seeing how the parts add up to a whole. Then a worksheet can be assigned requiring the students to put combinations of fractions of apples or cookies together to make different fractional values. Of course, less messy materials, such as cardboard circles or paper plates, can be used to teach the same concepts, but the chance to "eat" one's lesson is an attention getting device that contributes to more concrete and better learning.

At the secondary level, teachers can devise creative uses of manipulatives. For example, in algebra, a second order expression like $X^2 + 5X + 6$ is more easily understood when students can manipulate paper, wood, or plastic squares and rectangles of the dimensions X by X, X by one, and one by one, as illustrated below.

| one (X by X) | five (X by one) | six (one by one) |
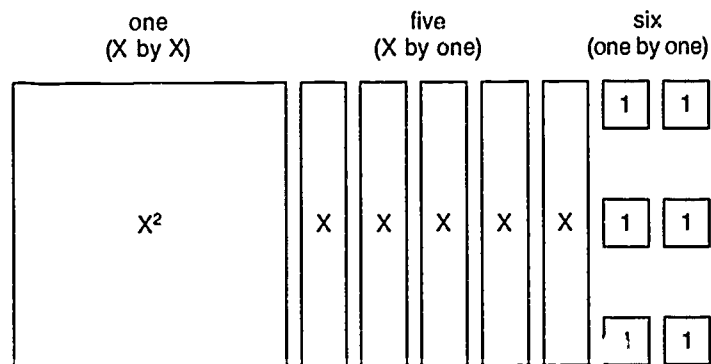
Similarly, these shapes can be manipulated to show multiplication of polynomials. For example, $(X + 3)(X + 2)$ can be represented as.
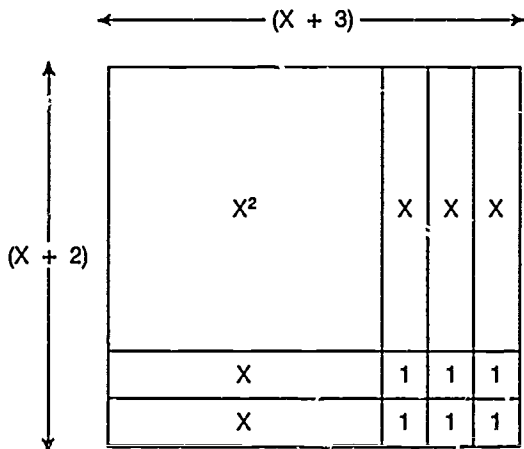
The area is composed of one square of area X^2, five rectangles of area X, and six squares of area one. Thus $(X + 3)(X + 2) = X^2 + 5X + 6$. These manipulatives are also useful for students learning to factor quadratic expressions.

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II. Thou Shalt Use
Cooperative-Learning Models

When children are using manipulatives, they need to verbalize what they are doing in order to connect appropriate words and symbols with what they are manipulating. Most classes are too large for students to verbalize one on one with the teacher. The solution is for students to work on their assignments in cooperative learning groups of two to six.

In classrooms using cooperative learning groups, students do not sit quietly and listen to the teacher's instructions and then do their work independently (see fastback 299 *Cooperative Learning* by Eileen Veronica Hilke). Rather, they work in small groups on a common task that draws on each group member's strengths. Teachers must provide instruction and practice in the social skills needed for effective group cooperation and provide time for students to evaluate how well they are using these skills. Further, teachers must develop assignments that require each member of the group to work cooperatively with others.

Teachers in cooperative learning classrooms spend less time lecturing and in direct instruction. Instead, they act as facilitators, observing group processes, and sometimes becoming a temporary member of a group that is experiencing problems. As a temporary group member, the teacher can model appropriate small-group behavior, ask probing questions to get the group back on task, or give direction to group members who are not participating. When groups

are functioning smoothly and independently, time is available for individual student-teacher conferences.

One way of using cooperative-learning groups in the mathematics class is to assign half of a problem set from the text for groups to work on and receive a group grade and assign the other half as homework for individual grades. In cooperative learning, the group assumes the responsibility of ensuring that each member understands the math problem solutions before turning in the group assignment. To reinforce comprehension, the teacher might select individuals to work some of the group problems on the chalkboard near the end of the period.

Another way of using cooperative-learning groups is with discovery lessons, where students work together with manipulatives to discover number relationships or to solve story problems. For example, a second-grade teacher might write a story problem on the chalkboard such as, "In the two second-grade classes in this school, there are 27 boys and 29 girls. How many boys and girls are there in the second grade?" The teacher organizes the children into groups of three to five, gives each group tongue depressors and rubber bands (to make groupings of 10), and directs each group to find the answer to the problem. The groups would be expected to find a solution with the manipulatives, draw a picture of their solution, and try to write it in symbols. Then the teacher would ask the groups to share the ways they pictured and wrote their answers using symbols. After the groups share their solutions, the teacher could summarize by showing them the standard way of writing the algorithm for two-digit addition with regrouping. Then several similar story problems would be worked through and shared before working on individual assignments.

Students then would work on individual assignments that incorporate practice of their newly learned skill. For extended practice, students might make BINGO like boards in which the the squares contain the answers to two digit addition problems. The teacher presents two-digit addition problems (requiring regrouping), which the students have to solve in order to find the answers on their BINGO boards.

Cooperative-learning methods have considerable support from research. In a meta-analysis of 122 studies on cooperative learning, Johnson, Johnson, Holubec, and Roy conclude:

> Cooperative learning experiences tend to promote higher achievement than do competitive and individualistic learning experiences. These results hold for all age levels, for all subject areas, and for tasks involving concept attainment, verbal problem solving, categorization, spatial problem solving, retention and memory, motor performance, and guessing judging-predicting. For rote-decoding and correcting tasks, cooperation seems to be equally as effective as competitive and individualistic learning procedures. (1984, p. 15)

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III. Thou Shalt Become a Skilled Diagnostician

Mathematics is a hierarchical subject. Basic concepts must be mastered before moving on to the next level. Mathematics teachers must be able to diagnose students' intellectual development as they prepare lessons and other mathematics activities. For example, kindergarten children have little or no capacity for understanding place value. Therefore, having them learn to count beyond 10 is little more than a meaningless exercise in memorization. Until children have matured enough to understand place value, there is little sense in giving them addition or subtraction problems involving two digit numbers. Failure to diagnose developmental levels can result in learner frustration, with children coming to think of themselves as failures. In addition, children may begin to believe . . learning is only memorizing.

Another facet of diagnosis is determining whether students have the prerequisite knowledge for comprehending newly introduced material. For example, if students do not understand the physical model of multiplication as the area of a rectangle, they will not be able to solve story problems that involve multiplication of polynomials in algebra. Similarly, if students do not comprehend the concept of common factors, they will have great difficulty adding or subtracting fractions. At the beginning of a new unit, as well as during the unit, teachers need to determine whether each student has the prerequisite concepts for understanding the new material. Such diagnoses need not be pencil-and-paper tests. Much can be learned simply by observing what type of problems are causing frustration for a particular child.

With young children especially, a one-on-one conference can yield better diagnostic information, because they can talk through their problem lem even if they don't know how to show it on paper.

Effective diagnosis can help in setting up temporary groupings of students. This is illustrated by a fourth-grade teacher who discovered after administering a diagnostic test at the beginning of the year that about half of her students knew less than 80% of the multiplication facts for the multiples of 1 through 5 (small facts). The other half had mastered the small facts (80% or better) but still knew less than 80% of the multiplication facts for the multiples of 6 through 9 (large facts). This teacher developed 15 lessons using manipulatives to teach the small facts to the first group and 15 lessons also using manipula tives to teach the 6 through 9 multiples plus zero (large facts) to the second group. After about two months the teacher conducted another diagnosis and discovered that students who had studied the small facts were ready to study the large facts and the role of zero, while most of the other group were ready to study multiplication of two- or three-digit numbers by a one-digit multiplier. Several months later, diagnosis determined that all students were prepared to work on multiplying a two- or three-digit number by either a one-digit number or a two-or three-digit number. Systematic diagnosis by this teacher allowed her to group students in ways that made it possible for all students to succeed, albeit at different times.

Conducting diagnostic procedures and preparing lessons for different groups does take time. This can be a source of frustration to the teacher who feels she must cover the textbook in a year, no matter what. But there is little sense in covering the textbook if students are not learning the sequential concepts that success in mathematics requires.

Without diagnosis many students will be forced to sit through instruction that they do not or cannot understand. Many other students, who have mastered the material long before, are now bored and unchal lenged. For both of these groups of students, who often make up more than half the class, these hours of class time are hours of wasted time.

IV. Thou Shalt Develop Unit Plans that Provide for Conceptual Understanding and for Sufficient Practice to Establish Mastery

Teachers need to develop their own unit plans (or perhaps by collaborating with colleagues) that are consistent with students' developmental levels and with the hierarchical nature of mathematics. Each unit should be designed to teach a few closely related concepts and mathematical rules. Daily lesson plans should be designed to develop a thorough understanding of these concepts and rules with examples, non-examples, general cases, specific cases, and unusual cases. Then applications of these concepts to daily life should be the focus of classroom discussion.

Units for young children will focus on simple concepts, like matching two sets of objects in one to one correspondence to develop the concept of numbers. A more complex concept, like fractions, can be presented by manipulating slices of a cardboard pie or an apple or cookie. As students develop an understanding of mathematical concepts by using concrete examples, they are building a foundation for logical thought, which is necessary for comprehending many concepts in algebra and geometry.

Textbooks often do not provide sufficient examples for conceptual understanding. Therefore, teachers must supplement texts with a variety of activities using manipulative objects, visual aids, and student discussion. Concept development occurs best in small group activity where students can interact with others and verbalize what they have

come to understand. Understanding must come before practice. Practice without understanding is meaningless activity.

Practice can be a class activity, a small-group activity, or an individual activity. In the optimal classroom, there will be all three types of practice. Textbook problem sets are one source of practice activities, but these need to be supplemented with mathematical games, computer practice problems, student-created story problems that use real-life applications of the concepts being studied, and unusual problems that push students to discover new mathematical relationships based on the concept under study (Kissane 1988). For mastery to occur, students must engage in the kind of practice that causes them to think about and use the concepts in ways that are interesting, lifelike, and challenging.

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V. Thou Shalt Teach Problem Solving

Students must relate the mathematics they learn in school to their experiences outside of school. Unfortunately, the mathematics curriculum in many schools provides few opportunities for students to apply their mathematics learning to real-life problems. Story problems found in textbooks typically are presented directly after the explanation of the very techniques used to solve them and contain only as much information as is needed to solve them (often the answer is in the back of the book).

Real problems are rarely so conveniently presented. Solving a real problem involves much more than simply arranging a group of numbers as an algorithm and making a computation. Real problem solving requires such skills as assessing the situation, defining the components of the problem, interpreting the problem in mathematical terms, isolating the useful and disregarding the irrelevant information, solving the mathematical portion of the problem (often requiring several different operations), and applying the mathematical solution to the problem. The problem-solving process may involve trial applications, a search for additional information, and use of several different mathematical operations and may have more than one solution. Or the problem may be unsolvable. Students must be given more opportunities to learn how to work with such problems.

Teaching students problem solving requires the use of real problem situations. The situations might be those encountered in adult life.

For example, "How many yards of fabric will it take to make three sundresses?" or "How much will it cost to build an 8' x 10' redwood deck two feet above level ground?" Also, the problems do not always have to be solvable by methods known by the students. Students can appreciate the value of mathematics when they are faced with a problem that requires more mathematics than they currently have. Knowing when to ask for help also is part of the problem-solving process.

"Wait!" many teachers are thinking, "My students hate word problems and don't have the patience even to solve the simple ones in the textbook. How can I expect them to take on complex problem-solving activities?" This response may be true for students working individually, but in cooperative-learning groups, students can brainstorm different approaches to solving the problem, share information, assign different individuals specific data collection tasks, correct each others' mistakes, and generally work together to develop a plan for solving the problem. By working together, students learn from each other. Less capable students learn how to solve problems by working with other students. They often can supply practical information that contributes as much or more to the problem-solving process as the brighter students contribute. As a result they gain confidence in their ability to do mathematics.

Another problem-solving activity that is neglected in classrooms is students writing their own story problems. The act of writing story problems provides a way for students to make real applications of their mathematical learning. In creating story problems, students are learning to use general solution methods, which they can then apply to similar categories of problems in real life situations. Students also become more adept at solving story problems when they challenge each other to solve problems they themselves have created.

Problem-solving can begin in the primary grades with the use of manipulatives and visuals. Using manipulatives and visuals helps children to make connections between objects and words and mathemat-

ical symbols. They help them to make sense of mathematical sentences that describe a problem. After using manipulatives to show a mathematical operation (like putting two sets of blocks together to form one set of blocks), then the teacher can write this manipulative experience as a story problem. (If John puts a set of four blocks together with a set of three blocks, how many blocks are in the single resulting set?) Then the teacher can use mathematical symbols to show the computation needed to solve the problem (4 blocks + 3 blocks = 7 blocks).

These processes can be altered by giving children a set of mathematical sentences, which they then develop into story problems with the manipulatives and visuals they have been using. Or children can be given story problems, which they then develop ir. manipulative and visual displays and the accompanying mathematical sentences

In exercises like these, children are learning how to approach problems in different ways. At the simplest level, this might mean drawing a picture or making a model with manipulatives to represent the problem. As problems become more complex, students can be introduced to graphing, constructing tables, estimating answers, identifying mathematical patterns, and generating hypotheses as ways of approaching problems. Students will have their preferences for different ways of approaching problems; some will come up with their own highly original approaches, which they should be encouraged to share. By practicing problem solving, students will become confident in using skills — skills that will serve them for the rest of their lives.

The first five commandments of our Decalogue cover fundamental components of effective teaching in mathematics (and in other subjects as well). The remaining five commandments will explore more fully the interrelationships among the different branches of mathematics and the real world. These are the links that make mathematics valuable for all people in their everyday lives.

VI. Thou Shalt Teach Algebra and Geometry to All Students

With the advent of computers and other forms of technology, busi ness and industry are demanding higher mathematics competency from employees. In order to meet this demand for higher mathematics achievement, schools must restructure their mathematics curricula so that by graduation almost all students will have mastered algebra and ometry, the gateway courses to higher-level mathematics.

The first step toward this goal is to introduce algebraic and geometric concepts in the elementary school by using manipulative and visual experiences similar to those used to teach basic arithmetic concepts. The linking of algebra and geometry with arithmetic helps to build a more complete model of mathematics that includes basic computation (addition, subtraction, multiplication, and division), abstract concepts like the axioms of addition and multiplication, which are the bases of algebra, and abstract geometric concepts like similarity, parallelism, and formal mathematical proofs.

Perhaps one of the most important changes needed is for elementary students to learn basic operations in terms of horizontal mathematical sentences, like $5 + N = 10$, instead of the traditional algorithms in the form of a column of numbers. Horizontal mathematical sentences are used in every mathematical and scientific discipline as well as on computer spreadsheets and in programming. In the same way that a succession of English sentences is used to develop an idea, mathematics sentences are placed together to build more

complex mathematical statements. As students come to understand mathematical sentence structures, they learn they can use them to communicate mathematical statements. In learning to read mathematical sentences horizontally, students are doing mathematics in a form that approaches the structure of word or story problems, with which students often have difficulty. And by using horizontal mathematical sentences, students are one step closer to the types of problems they will be working with in algebra and geometry.

For example, consider the problem $15 + N = 36$ and the following number sentence solution pattern:

1. What kind of number sentence is $15 + N = 36$?
   (It is an addition sentence.)
2. What is missing in the number sentence?
   (One of the addends is missing.)
3. How do you find the missing part?
   (A missing addend is found by subtracting.)
4. Find the missing part.
   $(36 - 15 = N, N = 21)$
5. Check to make sure.
   $(15 + 21 = 36)$

Note how the pattern above using an arithmetic solution is similar to the pattern for an algebraic solution, as follows:

$$
\begin{array}{ll}
\begin{array}{r} 15 + N = 36 \\ -15 \qquad -15 \\ \hline \qquad N = 21 \end{array} &
\begin{array}{l}
\text{Axiom: If you subtract equal amounts from} \\
\text{equal amounts, the remainders are equal.} \\
\text{Solution.}
\end{array}
\end{array}
$$

$15 + 21 = 36$ Check.

The use of algebraic axioms is too complex for most younger children to understand, but they can understand how to work number sentences to arrive at arithmetic solutions. And the similarity between the solution of arithmetic sentences and algebraic problems is close enough that students will feel confident when they begin the formal study of algebra.

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To incorporate geometry into the elementary mathematics curriculum, teachers need to use manipulatives – plastic, wood, or cardboard representations of two and three-dimensional geometric figures like squares, rectangles, triangles, parallelograms, trapezoids, circles, spheres, cubes, pyramids, and dice with different numbers of sides. Let students put these shapes together to form new shapes, flip them over, draw them from different angles, look for similarities, and place them in sets according to similar characteristics. By manipulating, observing, and comparing the shapes, students will begin to understand such concepts as planes, points, lines, angles, congruency, and parallelism. At the elementary level, students should not be learning formal geometry proofs, but they can become familiar with the concepts that serve as the basis for more abstract learning at a later time.

Even in secondary mathematics courses, teachers should continue to make extensive use of manipulatives to build conceptual understanding before moving on to abstract algebraic axioms and postulates and geometric proofs. This is a departure from traditional methods of teaching algebra and geometry, which present axioms and postulates at the beginning of the course and give students examples of methods of solution or proof, after which they are expected to apply them to problem sets that are nearly identical to the sample problems. Unfortunately, what typically happens is that students memorize the methods of solution or proof without really understanding the abstract axioms and postulates.

One example of using manipulatives in algebra is the use of an old fashioned balance scale to demonstrate the need to perform operations on both sides of an equation in order to solve the equation. Manipulative work in geometry with secondary students allows them to begin making hypotheses about geometric "truths" that they can practice "proving." Manipulatives give students concrete examples of how mathematics works. In this manner, students gain a conceptual base for understanding more complex and abstract mathematical processes.

VII. Thou Shalt Us Computers and Calculators

Microcomputers have become a powerful mathematical tool for elementary and secondary students. They can be used to personalize instruction and to offer conceptually richer instruction. One of the successful applications of computers in the mathematics classroom is for drill and practice to supplement assignments from the textbook, which often do not provide sufficient practice exercises to establish mastery. Computers also have the capability of providing feedback in the form of hints or clues and of reteaching or backing up to an easier level when a student encounters difficulty. For students who are ahead of the rest of the class, software is available for a variety of enrichment activities. Additional features of computers include keeping records of each student's progress, program branching to accommodate differences in student ability, and teacher designed program inputs in the form of additional content or questions.

Because computers are designed for individual use, teachers can select software to accommodate each student's skill level. This is especially helpful with students who have skill deficiencies, since it allows them to finally receive instruction appropriate to their levels. They can proceed in private and at their own pace. If they make a mistake, the computer will tell them so they can correct it without their teacher or peers passing judgment on them. Moreover, when slower students are placed in front of the computer keyboard, they feel they have some control over the situation and thereby can concentrate on learning.

Computer database programs are available that students can use to create files on everything from personal record collections to statistics on their favorite sports stars. Students also can use these files in mathematics classes to learn about the various ways statistics can be used.

Perhaps the most creative use of computers in the mathematics class room is having students develop their own programs. Even elementary students can create simple programs to develop tables of the counting numbers and to perform various operations on them, such as addition by two, or multiplication by one, or nine, or ten. Students can use these programs to study the effects of odd and even numbers added together or multiplied together, or they can study examples of arith metic versus geometric sequences. Secondary students can study the characteristics of exponential functions, or discover the relationship between linear, second order, and other ordered functions, or discover the progression of values as a function approaches a limit.

Even with all that computers have to offer the mathematics class room, teachers should not forget the versatility of the hand held cal culator. Calculators are essentially mini microcomputers that offer many of the same benefits as computers do. The first of these is per forming time consuming calculations, thus freeing students to con centrate on applications, proofs, patterns, generalizations, and interpretations of mathematical results.

Calculators also provide student a tool other than pencil and paper to link mathematics learned in school to other situations. For example, students could be asked to accompany parents on a shop ping trip to the grocery store and use a calculator and a pad of paper to keep track of how much the groceries cost, how much is discounted due to coupons, and how much sales tax is involved. Students then bring the store receipt along with the calculator's tally to class and compare the results.

In most classrooms, there are always a few students who have never memorized the multiplication tables. Without such skills, students of

ten are destined to fail in mathematics for the rest of their lives. If they are permitted to use hand-held calculators, they can keep up with their peers in more complex mathematical concepts, even through algebra and geometry. This is not to say that these students shouldn't continue to practice solving arithmetic problems involving such skills with paper and pencil, but if they can experience success using a calculator, they are much more likely to continue to develop their mathematical skills and to overcome their arithmetic handicaps.

In short, the calculator and computer make mathematics accessible and understandable to students who might otherwise experience (or have previously experienced) failure and frustration in mathematics. These tools allow students to apply mathematics in problem-solving situations without getting bogged down in time-consuming computations. Further, they are powerful tools with many uses in advanced mathematics through college-level courses or on the job. Mathematics teachers can no longer afford to ignore them.

VIII. Thou Shalt Teach Mental Computation, Estimation, and Measurement

If students are expected to use mathematics to solve real problems in their everyday lives, they must learn how to estimate and compute mentally, without the use of paper and pencil or calculator. Many situations in life call for quick computations that are approximations and don't have to be exact. Examples of estimating and mental computation include tipping at restaurants, staying within a budget while shopping for clothes, dividing the price of a gas fill up among members of a car pool, estimating sales tax, estimating the number of gallons of gas required to travel from one city to another, determining the number of hours needed to complete a project, estimating the number of pages to complete a partially finished report, doubling or halving a recipe, calculating the approximate cost of a banquet for a group of people, and so on Research has shown that development of estimation and mental computation skills improve computation skills using standard paper-and-pencil algorithms and also help to develop higher-order thinking skills (Schoer and Zweng 1986).

Students learn to estimate through estimating. Research shows that successful estimators use a variety of different techniques, and that no single technique is superior to any other. Different situations require different strategies, and students need to experiment to discover which techniques are most effective for them. Students should be encouraged to use unconventional methods to estimate, as long as they come up with reasonable answers.

Classes can play games in which students are given three seconds to estimate an answer to a problem. The closest estimation is rewarded with a point, or the first student to give an answer within 5% of the correct answer is rewarded with a point. When working problems at the chalkboard in front of the class, the teacher might ask for estimated answers before working through the problem. The teacher records the students' estimates on the board and gives credit to the students with the closest approximation. Giving students timed tests periodically provides them with practice in estimating and also allows them to evaluate their progress in estimating. These should probably be informal tests without grades.

Students learn mental computation mostly through practice. But they also can profit from discussion of mental-computation techniques. They can estimate an answer first and then modify the estimate to accommodate the difference between the actual numbers given in the problem and the numbers used to produce the estimate. If given the problem 398 + 255, the student can estimate the answer to be about 650, which is equal to 400 + 250. That also is equal to (398 + 2) + (255 − 5), which is equal to 398 + 255 − 3, so 398 + 255 = 650 + 3 = 653. Describing this process takes more time than the mental computation itself would take, but it shows how doing a mental computation is faster than using the standard addition algorithm. Teachers can go through this process of estimating and then modifying the estimate orally to show students how other people compute mentally. But again, students will develop their own, often unconventional, methods of mental computation, and this should be encouraged. Students need many opportunities to discover methods they feel comfortable with if they are to use them outside of the classroom.

Estimation and mental computation skills go beyond the ability to do arithmetic problems in one's head. Students also need to learn how to estimate in order to solve real problems they encounter. Children used to learn to estimate distance and area by working in the fields

or in the woods; they learned to estimate amounts or quantities by working in the stables or in the kitchen. Today, children's exposure to concrete examples of using numbers to describe their world is very limited, except perhaps for reading the price tags on things they want to purchase.

To develop children's understanding of how mathematics is used to describe the real world, teachers must provide them with multiple opportunities to measure real objects and use these measurements in meaningful computations. The simplest form of measurement is counting Preschool and early elementary students should practice counting everything they can get their hands on. Once they have mastered counting, they should start measuring larger objects in terms of smaller objects For example, students could measure desktop dimensions in terms of paper clips, or use an old-fashioned balance scale to measure the mass of an apple in terms of dominoes. Let the students estimate before actually measuring, and help them realize that most measurements are approximations.

As students develop some proficiency in measuring, they should be introduced to both our American standard and the international metric systems of measurement and should learn to measure and estimate in both systems. In the increasingly interdependent international marketplace, students will need to be thoroughly familiar with both systems in order to compare prices, read recipes, or even to order parts for their cars.

By providing manipulative activities that integrate measurement, arithmetic, and problem solving, students will learn mathematics in contexts of real-life applications. In fact, measurement can serve as a unifying theme throughout the elementary and secondary mathematics curriculum. After all, measurement is the very reason mathematics was invented. Students should know something about the ancient systems of measurement and the forms of mathematics that developed from them. With this historical perspective, students can understand how mathematics developed as a tool to solve real problems.

IX. Thou Shalt Teach Probability and Statistics

In the Information Age in which we live, measurement in the form of probability and statistics is a pervasive influence. People are barraged with statistical laden communications every day in the form of advertisements, reports on crime rates and health risks, and economic indices of inflation and interest rates, to name a few. International studies of quality of life or the status of education are all described in statistical terms. Government agencies use demographic statistics to allocate federal funds. Physicians use statistics on potential health risks to determine whether or not to advise surgery. Store managers use statistics to analyze buying patterns from year to year. Statistics are used to determine which intersections need traffic lights and how those traffic lights should be timed.

Probability is the foundation for establishing insurance rates. Probability is used in safety campaigns to document the risks of driving without using a seatbelt. The weather forecast every day is reported in terms of the probability of rain, snow, or other weather phenomena.

Yet, despite all the media exposure to examples of statistics and probability in daily life, many people still don't understand how to interpret them. As a result of this lack of understanding, the public is often deceived by advertisers, special interest groups, and politicians who use statistics in ways that are misleading or biased. Understanding statistics and probability is essential for making sense of what is happening in the world. Therefore, all students from an early

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age must learn basic statistical concepts and know how to use and interpret them in real-life contexts.

At the elementary school level, students can learn to use bar graphs and circle graphs. For example, they can make graphs to depict characteristics of members of their class (number of boys and girls, number of blue-eyed, brown-eyed, green-eyed students, number whose birthdays fall in different months, etc.). They can graph the amount of rainfall by day or by month, or they can graph the rate of growth of bean plants in a science classroom.

A popular way of understanding probability is repeated flipping of coins, recording the results, and then analyzing the results to discover just how likely it is that a flipped coin will come up heads. Students might investigate whether a coin is any more likely to come up heads if the past five flips have yielded only tails. More complex probability tables can be developed using dice or decks of cards. Students also can begin to study correlations, say between foot size and height or between outside temperature and the time of day.

Although formal statistical operations are beyond the capabilities of most elementary students, they can begin to work with manipulatives to learn the concepts of average and median. By taking a string 53 centimeters long, placing it end-to-end with a string 41 centimeters long, and then taking the whole length and folding it in half, students can find the average. This activity can be extended by adding more lengths of string, folding them by the number of lengths, and figuring the average. Students can even plot how each of the original pieces of string is different from the average, thus developing a rudimentary understanding of standard deviation. Experiences like these can lead into discussions about the use of the term "average" in advertisements or news stories, or claims that "three out of four doctors recommend . . .," or what it means to say "one in every 3,000 children is susceptible to . . . ," or "33% of American children are living below the poverty level." These kinds of discussions are the beginning of statistical literacy.

In secondary school, students can be asked to bring in newspaper clippings or advertisements containing statistical information and discuss their interpretation of the statistics and their possible implications. At this level students can comprehend the concept of normal curve (plotting the grades on a class test is a good way to present this concept), and they will begin to get a sense of which kinds of data will fall into a normal distribution and which will not. The National Council of Teachers of Mathematics 1981 Yearbook, *Teaching Statistics and Probability*, provides many suggestions for teaching the use of statistics (also their misuse), as well as showing applications of statistics and probability in insurance rates, economics, inventory control, weather prediction, medicine, politics, and even work with subatomic particle physics.

As much as possible, students should learn statistical concepts through the firsthand experience of generating their own statistics. Through such experiences, students will have to deal with such questions as. What is the impact of the sample size? How do you detect sample bias? How do you adjust for sample bias? What are the limits for the curve of distribution? How much allowance should you make for chance error? How do you account for pieces of data that don't fit in with the rest? How will they affect your averages and distribution curves? How do you present your statistics in a way that provides an accurate and useful description of your data? By having students calculate their own probabilities, they will learn to see through common misconceptions about probability like. If there is a 50% chance that a baby will be born a girl, then in a family that has six children, three or 50% will be girls.

Teaching statistics and probability through firsthand experiences will prepare students to understand and interpret the vast array of statistical data they encounter in day-to-day life.

X. Thou Shalt Integrate the Different Branches of Mathematics

A persistent theme throughout this Decalogue has been the importance of mathematics as a problem-solving tool for real situations in students' lives. As students progress in using their mathematics in real situations (in contrast to the pre-digested problems from a textbook), they will have to integrate skills and techniques from several branches of mathematics. For example, designing and constructing a tool shed might require integration of problem solving methods involving addition, subtraction, division, multiplication, fractions, decimals, measurement, estimation, proportions, parallelism, perpendicularity, calculation of the angles in triangles, and the use of area formulas. Algebra will be used if there is need to make the shed fit within specific dimensions and still provide maximum volume for storage. Similarly, when students study geography, economics, physics, chemistry, astronomy, architecture, engineering, and market analysis, they will find it necessary to use statistics, probability, and other more complex integrations of mathematics.

Mathematics teachers should continuously point out the many ways the different branches of mathematics can be combined to solve problems, even including some of the less typical branches of mathematics like discrete mathematics, calculations in different bases, set theory, imaginary numbers, multiple dimensions, spherical and hyperbolic try, topology, fractals, sequences, and series, as well as mathe

matical systems designed for specific subject areas like acoustics, music theory, and computer design.

Working with the diversity of mathematics fields makes the subject more interesting for students and teachers alike. And more important, it gives students an awareness of the opportunities open to them if they continue to study mathematics.

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Conclusion

The authors of this Decalogue for teaching mathematics have cast it in the form of 10 commandments, which, if carried out, will overcome the obstacles that have hampered learning mathematics in the past and will result in almost all students being successful. Although the individual commandments have different emphases, we are convinced that they apply whether teaching mathematics to young children or adults. They are most applicable in grades one through six, where the foundation of mathematical understanding is established.

We invite teachers and administrators to take each commandment and recast it in the form of a question by substituting "Do we" for "Thou shalt" and then use these questions to evaluate the teaching of mathematics in their schools. Responses to the "Do we" questions could well serve as a way of determining a school's inservice needs for improving the mathematics curriculum. With help of a consultant and the use of such resources as the *Arithmetic Teacher* and the *Mathematics Teacher* along with yearbooks and other materials published by the National Council of Teachers of Mathematics, a school staff can identify strategies and develop materials that will permit them to answer "yes" to all the "Do we" questions.

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