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ABSTRACT

These conference proceedings include two invited lectures, four working group reports, five topic group reports, a list of participants, and a list of previous proceedings. The invited lectures were: "Teaching Mathematical Proof: Relevance and Complexity of a Social Approach" (Nicolas Balacheff) and "Geometry Is Alive and Well!" (Doris Schattschneider). The topics of the working group reports were: (1) using LOGO-based software (Benoit Cote, Sandy Dawson); (2) using the Computer Algebra Systems (Eric Muller, Stan Devitt); (3) aspects of communication and other functions of language used in mathmematics (David Pimm); and (4) using research methods of mathematical conceptions based on constructivism (Rafaella Borasi, Claude Janvier). The topic groups dealt with: computer uses in the elementary classroom (W. George Cathcart); constructions of fundamental mathematical concepts and the kindergartner's construction of natural numbers (Jacques Bergeron, Nicolas Herscovics); multicultural influences (Linda Davenport); meaning of grades assigned by teachers (Rose Traub et al.); and a discussion (not presented in this document) of the results of the 1988 International Assessment of Educational Progress Study (Dennis Raphael). Appended are lists of proceedings participants and of previous proceedings available through ERIC. (YP)

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIAN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

PROCEEDINGS

1989 ANNUAL MEETING

BROCK UNIVERSITY

ST. CATHARINES, ONTARIO

May 27-31, 1989

Edited by

Lionel Pereira-Mendoza

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CMESG/GCEDM Annual Meeting
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EDITORS' FORWARD

We would like to thank all the contributors for submitting their manuscripts for inclusion in these proceedings. Without their co-operation it would not have been possible to produce the proceedings so quickly.

Special thanks go from all of us who attended the conference to the organizers, and particularly to Eric Muller, who worked tirelessly before and during the meeting to ensure smooth sailing.

I hope these proceedings will help generate continued discussion on the many major issues raised during the conference.

Lionel Pereira-Mendoza
Martyn Quigley

April 10, 1990

ACKNOWLEDGEMENTS

The Canadian Mathematics Study Group wishes to acknowledge the continued assistance of the Social Sciences and Humanities Research Council for their support. Without this support the 1989 meeting and proceedings would not have been possible.

We would also like to thank Brock University for hosting the meeting and providing excellent facilities to conduct the meeting.

Finally, we would like to thank the contributors and participants, who helped make the meeting a valuable educational experience.

Lecture One

Teaching Mathematical Proof

Relevance and Complexity of a Social Approach

N. Balacheff

IRPEACS-CNRS

Introduction

What a mathematical proof consists of seems clear to all mathematics teachers and mathematics educators. That is: "A careful sequence of steps with each step following logically from an assumed or previously proved statement and from previous steps" (NCTM, 1989, p.144). This description is almost the same all over the world, and it is very close to what a logician would formulate, perhaps more formally. Comments made on mathematical proof as a content to be taught emphasize two points: first they stress that it has nothing to do with empirical or experimental verification, second they call attention to the move from concrete to abstract. Here is an example of such comments:

> "It is a completely new way of thinking for high school students. Their previous experience both in and out of school has taught them to accept informal and empirical arguments as sufficient. Students should come to understand that although such arguments are useful, they do not constitute a proof." (NCTM, 1989, p.145).

We can say that the definition of mathematical proof, as an outcome of these official texts is mathematically acceptable, but there is a long way from this definition to the image built in practice along the teaching interaction. More or less, teaching mathematical proof is understood as teaching how to formulate deductive reasoning: "Pour les professeurs, une démonstration, c'est très nettement l'exposé formel déductif d'un raisonnement logique" (Braconne, 1987, p.187).

The construction of this reasoning, and its possible relationships with other kind of reasoning, is hidden by that over emphasis on its "clear" formulation. That conception is so strong that some teachers can come to an evaluation of a mathematical proof just considering the surface level of the discourse. For example, in her requirement for teachers comments on a sample of students formulations, Braconne reports[1] that:

> "Les professeurs ont réagi aux longueur inutiles du texte de Bertrand, au désordre dans la solution de Karine, au fait que le texte d'Elodie ne suive pas le raisonnement déductif, etc. Toutefois, sept professeurs n'ont pas remarqué que, dans le texte de Bertrand, c'est la réciproque du théorème nécessaire à la démonstration qui était cité au premier paragraphe, et huit n'ont pas signalé que le texte de Laurent contenait la même erreur [...] Donc pour l'élève, et pour nous, les notes ne reflètent pas le fait que le professeur se soit apperçu de l'erreur ou non." (Braconne, 1987, p.99)

A report on proof frames of elementary preservice teachers shows a similar behaviour:

> "Many students who correctly accept a general-proof verification did not reject a false proof verification; they were influenced by the appearance of the argument - the ritualistic aspects of the proof - rather than the correctness of the argument. [...] Such students

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[1] The interviewees were 13 French Mathematics teachers.

appear to rely on a syntactic-level deductive frame in which a verification of a statement is evaluated according to ritualistic, surface features."

(Martin & Harel, 1989)

Thus, mathematical proof appears ultimately as a kind of rhetoric specific to the mathematical classroom, it is not surprising then that it appears as such to the eyes of students. The nature of mathematical proof as a tool to establish a mathematical statement is to some extent hidden by the emphasis on the linguistic dimension. What does not appear in the school context is that the mathematical proof is a tool for mathematicians for both establishing the validity of some statement, as well as a tool for communication with other mathematicians. Also, it is often forgotten that what constitutes the present consensus about rigor has not been created ex nihilo, but that it is the product of an historical and a social process within the community of mathematicians[2]. As Manin[3] recalls, ultimately "a proof becomes a proof after the social act of 'accepting it as a proof'."

There is another reason for considering so strongly the social dimension of mathematics teaching and learning. For as we recognized that learning is a personal process, we should also consider that its outcome is likely to be firstly a private knowledge: The students' conceptions. But that conflicts with two constraints specific to the teaching, which has to guarantee the socialization of students' conceptions for the following reasons:

- Mathematics is a social knowledge. Students should make their own the knowledge that exists outside the classroom. It has a social status in society, or in smaller social groups under whose control it is used. For example, the community of mathematicians or that of engineers can be taken as a social reference.

- The mathematics class exists as a community. The teacher has to obtain a certain homogeneity in the meaning of the knowledge constructed by students, and she or he has to ensure its coherence. Otherwise, the functioning of the class will hardly be possible. Because of the constructivist hypothesis we consider, the use of authority is not desirable. Thus, the homogenization can only be the result of a negotiation or of other specific social interactions such as the one Brousseau (1986) has described in the frame of his *théorie des situations didactiques*.

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2 The essay of I. Lakatos (1976) on the dialectic of proofs and refutations gives a good insight of this historical process.

3 Manin quoted by Hanna (1983).

Social Interaction and Situations for Validation

What is now clear is that as long as students rely on the teacher to decide on the validity of a mathematical outcome of their activity, the word 'proof' will not make sense for them as we expect it to do. In such a context they are likely to behave mainly to please their teacher, just as one of the British students interviewed by Galbraith (1979) told his interviewer: "To prove something in maths means that you have worked it out and it proves how good you are at working questions out and understanding them."

But it is not sufficient to propose a problem to the mathematics classroom and to tell the students that they have the responsibility of solving it. There is no reason for them, a priori, to consider that the problem is their problem and to feel committed to solving it; they can still think that they have to do so in order to please the teacher and thus their behaviour will not be significant.

Before going ahead, let us consider a short story told by Sir Karl Popper, which will throw a relevant light on what we want to suggest:

> "If somebody asked me, 'are you sure that the piece in your hand is a tenpenny piece?' I should *perhaps glance at it again* and say 'yes'. But should a lot depend on the truth of my judgement, I think I should take the trouble to go into the next bank and ask the teller to look closely at the piece; and if the life of a man depended on it, I should even try to get to the Chief Cashier of the Bank of England and ask him to certify the genuineness of the piece."
>
> (Popper, 1979, p.78).

And then Popper adds that "the 'certainty' of a belief is not so much a matter of its intensity, but of the *situation:* Of our expectation of its possible consequences." (ibid.)

Along the same line, I would like to suggest that if students do not engage in any proving processes, it is not so much because they are not able to do so, but rather that they do not see any reason. Even if they engage such a process, its level depends heavily on the way students understand the situation. Following a principle of *economy of logic* they are likely to bring into play no more logic than what is necessary for practical needs (Bourdieu, 1980, p.145).

Then the true meaning of the outcomes of students proving processes is to be traced in the characteristics of the situation in which they are involved.

In situations in which they have to decide a common[4] solution to a given problem, students have to construct a common language and to agree on a common system to

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4 By 'common', we mean here a solution supported by the whole classroom, or smaller groups of students as is usually the case.

decide of the validity of the solution they propose. The essential role of the social dimension, mainly in situations for communication, provoking a move from "doing" to "telling how to do", and their importance in the construction of meaning have been put in evidence by Brousseau in his *théorie des situations didactiques* (Brousseau, 1986). Here we would like to recall what this author wrote about the situations for validation:

> The situations for validation "will bring together two players who confront each other regarding a subject of study composed on the one side of messages and descriptions produced by the pupils and on the other side of the a-didactic milieu used as referent for these messages. The two players are alternately a 'proposer' and an 'opposer'; they exchange assertions, proofs and demonstrations concerning this pair 'milieu/message'. This pair is a new apparatus, the 'milieu' of the situation for validation. It can appear as a problem accompanied by the attempt at solving it, like a situation and its model, or like a reality and its description...
>
> While informer and informed have dissymmetric relations with the game (one knows something that the other does not know), the proposer and the opposer must be in symmetrical positions, both regards the information and means of action about the game and the messages which are at their disposal, and as regards their reciprocal relations, the means of sanctioning each other and the objectives vis-à-vis the pair milieu/message."
>
> (Brousseau, 1986, p.158).

We should realize that in such situations, behaviours that are more social than mathematical, would probably appear. For example, because of self-esteem, some students might refuse to recognize that they are wrong, or others might refuse to accept that their opponents are right.

Thus, to sum up, to provoke students proving behaviours we should design situations in such a way that students come to realize that there is a risk attached to uncertainty, and thus that there is an interest in finding a good solution. In order to obtain a significant scientific debate among students, we should provide them with a situation promoting contradiction, but also promoting acceptance. Otherwise systematic rejection could become an efficient defensive strategy. In other words, the situation should allow the recognition of a risk linked to the rejection of a true assertion, or to the acceptance of a false one.

Following these principles we have designed teaching situations as experimental settings in order to study students behaviours in such contexts, and the nature of these behaviours in relation to the characteristics of these situations. A priori, we thought that genuine mathematical proving processes will be observed, a deep analysis of our experiments shown that things are a bit more complex than what is usually acknowledged by innovative practice resting on social interaction.

In the following section we will report, in some detail, on one of these experiments.

A Case Study[5]: The Perimeter of a Triangle

A first principle we wanted to satisfy in designing the experiment was to obtain the devolution[6] of the responsibility for the validity of the problem's solution from the teacher to the students. For that purpose we have chosen a context of communication: We told the students that they will have to write a message for other students, of the same grade, in order to allow them to solve a given problem. In such a situation the criteria for success are left to be decided by students according to their own means for the evaluation of the efficiency and the reliability of the message they have produced. We thought that this setting would be sufficient to ensure that students will consider that they have the responsibility for the truth of their solution, and that they will not refer to the teacher expectation.

In such a situation there is usually some tensions because of the different individual motivation and commitment. For this reason, we think that it is not desirable to ask the students to work individually, but on the other hand it is not desirable to ask for a collective production form the whole class insofar as some students might feel that they are not concerned, leaving the job to the others. So, we decided to constitute small teams of three to four students working together, telling them that the final solution will be one of the ones proposed by the teams, or a modification of it. To promote collective work, each team must propose only one solution, and during the debate for the choice of the class solution the team will be asked to express its position through the voice of a chosen representative. That constraint obliges students to be explicit and to discuss a priori the correctness and appropriateness of what they want to be said. We think that the quality of the debate will rest on the motivation of each team, its willingness to have its message chosen, but also its commitment to the success of the class as a whole.

The mathematical problem we chosen was the following:

> *Write for other students, a message allowing them to come to know the perimeter of any triangle a piece of which is missing. To do it, your colleagues will have at their disposal only the paper on which is drawn a triangle and the same instruments as you (rules, etc.).*

Together with this text a triangle such as the following (fig. 1) was given to the students. All the teams in the classroom had the same materials.

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5 The case study reported here has been made possible because of the close relationships established by academics and teachers within a research group of the *IREM de Lyon*. It is a small part of a four year project which had allowed us to collect a large amount of data. The complete report is available from *IREM de Lyon, Université Clause Bernard, Lyon*.

6 Devolution: "A delegating of authority or duties to a subroutine or substitute" (The American Heritage Dictionary of the English Language, 1979).

The study we made before this experiment (Balacheff, 1988, pp 321-360), allowed us to think that all the students will be able to enter the problem-solving process, with quite different solutions. This diversity was expected to be the source of interesting debates. We know that some students, and thus some teams, will miss the fact that the solution must work for a general case and not only for the triangle given as an example. But we were sure that this will be pointed out during the debate, and then that it will be taken in consideration, even with more strength than if the teacher had warned about it a priori.

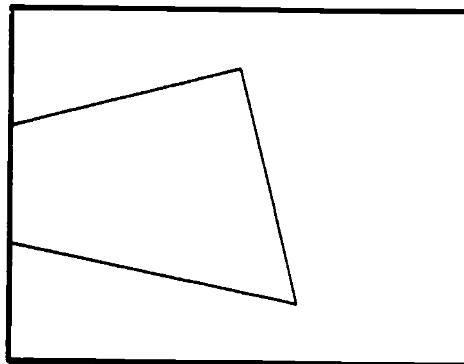

Figure 1

The role of the teacher was to present the situation, then not to intervene in any case up to the time when all the teams have proposed a solution; then the teacher's position will be to regulate the debate and to give the floor to the teams' representatives. The end of the sequence was to come from a general agreement on the fact that one of the solutions, or a new one obtained as a result of the interactions, is accepted. The debate was organised in the following way: The messages were written on a large sheet of paper and then they were displayed on a wall of the classroom. Each team had to analyze the messages and their representatives had to tell the class their criticisms and suggestions. These criticisms had to be accepted by the team which was the author of the message discussed. In case of an agreement of the class on a false solution, the teacher was allowed to propose to the teams a new triangle invoking that such a triangle might be considered by the receptors. (Such material had been prepared taking into account what we knew from the first study). On the other hand if more than one message was acceptable with no clear decision from the class then the teacher was supposed to organise a vote to make the choice, asking the students to tell the reasons for their choice.

I will not report here in detail on the analysis of this experiment. A complete report is available in Balacheff (1988, pp 465-562). I will here focus here on the outcomes relevant to my present purpose, as they are related to the observations which have been made in two different classrooms.

The First Experiment

The first experiment was carried out with students of the eighth grade (13 to 14 years old). The teacher was a member of the research team, which meant that we were in a good position to assert that the project was well known to her. The observations lasted for two sessions of 1:30 hours. After the first one we felt really happy with what had happened. The second phase raised a feeling of some difficulties ... beyond these feelings only the close analysis of the data gathered, led us to discover the existence of the parameters which have played a critical role in the teacher decisions, and thus in the students behaviours:

- First, a constraint of time, which made the teacher intervene in order to ensure that the whole process would keep within the limits imposed by the general school timetable in which the experiment took place.

- Second, the teacher's willingness to guarantee an acceptable end in her own eyes. There was a huge tension between this willingness and the willingness of not breaking the contract of "non intervention". This tension is the indication of what we would like to call in the future: *the teacher epistemological responsibility.*

Because of these two constraints, the decisions the teacher made tended to oppose the devolution of the problem. In particular, to guarantee that the problem solving phase would not be too long, the teacher invited students to propose a solution as soon as she thought that it was mature enough, but with no information about the real feeling of these students. Also, some teacher's interventions aimed at calling the attention of students to the word "any" (in the sentence "any triangle"), and doing so she did not think that it was a mathematical intervention, insofar as she thought that it was only due to the students' lack of carefulness. But all these interventions led students to a feeling of dependence and the idea of a possible responsibility of the teacher for the validity of their answer.

A significant phenomenon, is that the teacher (as well as the observers) did not realize what a continuous contact she kept with the students, making about one intervention every minute over an 80 minutes period. The content of these intervention could have been light, as: "Are you O.K.?", or more important as: "Are you sure you have carefully read the statement of the task? ". All together we have counted, within these interventions, 129 different items. We see this phenomenon as an indicator of the intensity of the relationship between the teacher and the students in a situation that we thought to be quasi-isolated from the teacher before we did a close analysis of the records.

The same constraints were an obstacle to the functioning of the second phase. After a first exchange of critiques by the teams' representatives, the teacher intervened because she thought that nothing positive will come out of the engaged process – at least within the time available. The teacher then tried to facilitate the progress in the discovery of a solution, calling explicitly for ideas and suggestions to start from them and go further. Actually, it was quite clear from her attitudes that not all the ideas were of the same value. The students' behaviours were deeply transformed by these interventions. They got confused and they were no longer committed to any real discovery of a solution.

The teacher thought that she had kept the spirit of the sequence, the basic frame being: search for a solution, critics, new ideas and suggestions to go ahead. But only the superficial aspects of the intended sequence were still there; its meaning for the students were fundamentally changed. They did not enter a true mathematical activity, as expected, but just a new school game not so different, beyond the new and exiting social setting, from the ordinary one.

The Second Experiment

We learned a lot from this first experiment, and we thought that it would be worthwhile to make a second one. We decided to keep the same general framework, but to overcome the obstacles we came to be faced with, we chose the three following modifications:

(i) To observe a tenth grade classroom in order to be sure that no mathematical difficulty will disturb the phenomenon we wanted to observe. Also, at this level students have already been introduced to mathematical proof. The situation could be an opportunity to evidence its power as a means for proving...

(ii) To open the time, that means that we decided to leave open when the end of the experiment will end up. We thought that three or four sequences of about one hour each would be sufficient.

(iii) To ask the teacher not to intervene, as strictly as possible, during the first phase (the initial problem-solving phase), and then to act just as a chairperson and as the collective memory[7] of the class during the second phase (the debate).

The first phase[8] did not present any special peculiarity. The teacher did not intervene at all, leaving students free to decide that they had a solution to propose. Four teams among the five reached a solution, the fifth one which was clearly close to surrender, finally proposed a "contribution" to the collective effort, as a response to the teacher demand.

During the second phase, she also followed the specifications we decided together. Then ...

More than a scientific debate, that is, proposing proofs or counterexamples, the data show that students entered a discussion with some mathematical content in it, but which mainly consisted of an exchange of arguments *pro et contra* not necessarily connected the one to the others. They argued about the different proposed solutions, but they did not prove mathematically.

The situation for communication has really been taken into account as such by students, as their remarks on the proposed messages show. The main critics are related to the fact

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7 To be the "memory" of the class means to take a record of what is said, in particular by writing students' decisions on the blackboard.

8 This phase took about one hour.

that this message must be understandable and usable by its receptors. But the problem of the validity of the proposed solution is not really considered. In that sense we can say that the situation does not realize a situation for validation. For a clear distinction between "arguing" and "proving" in mathematics[9], we refer to the distinction as formulated by Moeschler:

> "Un discours argumentatif n'est pas un discours apportant à proprement parler des preuves, ni un discours fonctionnant sur les principes de la déduction logique. En d'autres termes, argumenter ne revient pas à démontrer la vérité d'une assertion, ni à indiquer le caractère logiquement valide d'un raisonnement [...] Un discours argumentatif, et c'est là une hypothèse de départ importante, se place toujours par rapport à un contre-discours effectif ou virtuel. L'argumentation est à ce titre indissociable de la polémique." (Moeschler, 1985, p.46-47).

In that sense, what we have observed is first of all an exchange of arguments about the simplicity of the solution ... or of its complexity. The context of a communication with other students has favoured the feeling of the relevance of critics in that register. But what leads us to suggest that this debate is more an argumentation than a scientific debate, in the Moeschler sense, is the frequent lack of logical relationships between arguments. Even more, some students can pass in the same argumentation from one position to another completely contradictory. These arguments can have nothing to do with mathematics, or even with what is required by the situation ... and it could be the same for the objections opposed to an argument. Finally, the involvement of some of the teams in the game, I mean the fact that they are eager to win, had favoured the appearance of polemics: The strongest opponents to the "too complex" message are the authors of the "too simple", and conversely.

After a first period of debate the messages had been accepted, provided that some modifications were made, but their validity has not been really discussed. So, the teacher proposed a new triangle, in order to challenge the messages. This triangle was such that the wrong solutions will obviously fail. The debate following this checking phase, shows how strongly students are more involved in an argumentation than in a scientific debate. Finally, one solution being accepted as the solution of the class, the teacher asked students whether they were sure of that solution. They answer: "Yes, because we have done it in a lot of cases." So, it is even not sufficient to directly address the question of the validity. Note, that when later on the teacher asked the students about a possible mathematical proof of their solution, they gave one showing that technically it was possible to them.

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[9] We do not refer necessarily to formal proof, or mathematical proof in the classical sense.

Discussion

Efficiency Versus Rigour

Even if we are able to set up a situation whose characteristics promote content specific students' interaction, we cannot take for granted that they will engage a "mathematical debate", and finally that they will produce a mathematical proof.

A peculiarity of mathematics is the kind of knowledge it aims at producing. Its main concern is with concepts specific to its internal development. There is evidence that Egyptians used intellectual tools in practical situations for which we have now mathematical descriptions, but the birth of mathematical proof is essentially the result of the willingness of some philosophers to reject mere observation and pragmatism, to break off perception (the *monde sensible*), to base knowledge and truth on Reason. That actually is an evolution, or a revolution, of mathematics as a tool towards mathematics as an object by itself, and as a consequence a change of focus from "efficiency" towards "rigor".

It is a rupture of the same kind which happens between "practical geometry" (where students draw and observe) and "deductive geometry" (where students have to establish theorems deductively). Also in numerical activities, like the one reported by Lampert (1988), the same rupture happens when students no longer have to find some pattern out of the observation of numbers, but that they have to establish numerical properties in their "full" generality (using letters and elementary algebra).

Here we have to realize that most of the time students do not act as a *theoretician* but as a *practical man*. Their job is to give a solution to the problem the teacher has given to them, a solution that will be acceptable with respect to the classroom situation. In such a context the most important thing is to be effective. The problem of the practical man is *to be efficient* not *to be rigorous*. It is to produce a solution, not to produce knowledge. Thus the problem solver does not feel the need to call for more logic than is necessary for practice.

That means that beyond the social characteristics of the teaching situation, we must analyse the nature of the target it aims at. If students see the target as "doing", more than "knowing", then their debate will focus more on efficiency and reliability, than on rigor and certainty. Thus again argumentative behaviours could be viewed as being more "economic" than proving mathematically, while providing students with a feeling good enough about the fact that they have completed the task.

Social Interaction Revisited

Social interaction, while solving a problem, can favour the appearance of students' proving processes. Insofar as students are committed in finding a common solution to a given problem, they have to come to an agreement on the acceptable ways to justify and to explain their choices. But what we have shown is that proving processes are not the only processes likely to appear in such social situations, and that in some circumstances they could even be almost completely replaced by other types of interactional behaviours. Our point is that in some circumstances social interaction might become an obstacle, when students are eager to succeed, or when they are not able to coordinate their different points of view, or when they are not able to overcome their conflict on a scientific basis[10]. In particular these situations can favour naive empiricism, or they can justify the use of crucial experiment in order to obtain an agreement instead of proofs at a higher level (Balacheff, 1988).

Perhaps some people might suggest that a better didactical engineering could allow us to overcome these difficulties; indeed much progress can be made in this direction and more research is needed. But we would like to suggest that "argumentative behaviours" (i) are always potentially present in human interaction, (ii) that they are genuine epistemological obstacles[11] to the learning of mathematical proof. By "argumentative behaviours" we mean behaviours by which somebody tries to obtain from somebody else the agreement on the validity of a given assertion, by means of various arguments or representations (Oléron, 1984). In that sense, argumentation is likely to appear in any social interaction aiming at establishing the truth or falsehood of something. But we do consider that argumentation and mathematical proof are not of the same nature: The aim of argumentation is to obtain the agreement of the partner in the interaction, but not in the first place to establish the truth of some statement. As a social behaviour it is an open process, in other words it allows the use of any kind of means; whereas, for mathematical proofs, we have to fit the requirement for the use of a knowledge taken in a common body of knowledge on which people (mathematicians) agree. As outcomes of argumentation, problems' solutions are proposed but nothing is ever definitive (Perelman, 1970, p.41).

Insofar as students are concerned, we have observed that argumentative behaviours play a major role, pushing to the backside other behaviours like the one we were aiming at. Clearly enough, that could be explained by the fact that such behaviours pertain to the genesis of the child development in logic: Very early, children experience the efficiency

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10 I mean, content specific.

11 The notion of "epistemological obstacle" has been coined by Bachelard (1938), and then pushed on the forefront of the didactical scene by Brousseau (1983). It refers to a genuine piece of knowledge which resists to the construction of the new one, but such that the overcoming of this resistance is part of a full understanding of the new knowledge.

of argumentation in social interactions with other children, or with adults (in particular with parents). Then, it is quite natural that these behaviours appear first when what is in debate is the validity of some production, even a mathematical one.

So, what might be questioned is perhaps not so much the students' rationality as a whole, but the relationships between the rationale of their behaviours and the characteristics of the situation in which they are involved. Not surprisingly, students refer first to the kind of interaction they are already familiar with. Argumentation has its own domain of validity and of operationality, as all of us know.

So, in order to successfully teach mathematical proof, the major problem appears to be that of *negotiating* the acceptance by the students of new rules, but not necessarily to obtain that they reject argumentation insofar as it is perhaps well adapted to other contexts. Mathematical proof should be learned "against" argumentation, bringing students to the awareness of the specificity of mathematical proof and of its efficiency to solve the kind of problem we have to solve in mathematics.

Here negotiation is the key process, for the following reasons:

- First, because the teaching situation cannot be delivered "open" to the students, otherwise many of them will not understand the point and they will get lost. The following quotation from Cooney makes it clear:

> "Maybe not all of them but at least some of them felt 'I am not going to participate in this class because you [referring to the teacher] are just wasting my time'. It is so ironic because if I was doing the type of thing they wanted to do, they would be turning around in their seats and talking. So it's a no-win situation."
>
> (Cooney, 1985, p.332).

- Second, because of the rules to be followed, the true aim of the teacher cannot be stated explicitly. If the rules for the interaction are explicitly stated, then some students will try to escape them or to discuss them just as many people do with law. Also because interacting mathematically might then become "mastering a few clever techniques" which may turn into objects to be taught, just as teaching "problem solving" has often become teaching quasi-algorithmic procedures (Schoenfeld, 1985).

The solution is somewhere else, in the study and the better understanding of the phenomena related to the didactical contract, the condition of its negotiation, which is almost essentially implicit, and the nature of its outcomes: the devolution of the learning responsibility to the students. We cannot expect ready-to-wear teaching situations, but it is reasonable to think that the development of research will make available some knowledge which will enable teachers to face the difficult didactical problem of the management of the life of this original society: The mathematics classroom.

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Lecture 2

Geometry is Alive and Well!

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Geometry is Alive and Well[1]

The reports of my death are greatly exaggerated.
Mark Twain, 1897 (cable to Associated Press
from London, upon reading of his death.)

It is a widely held opinion that geometry is dead. At the Fourth International Congress on Mathematical Education held in Berkeley in 1980, a lively debate on the topic featured J. Dieudonné, B. Grünbaum, and R. Osserman — all well recognized research mathematicians with deep interests in and strong opinions about geometric questions and the teaching of geometry. In his address, Osserman noted "... to speak of the 'death of geometry' at the post-secondary or any other level is clearly an exaggeration, [though] it nevertheless reflects a reality."[2]

The evidence of the death of geometry as a vital part of the body of mathematics seemed convincing:

— The small role of geometry in the high school curriculum: rarely required, and typically a one year (or shorter) course.
— The insignificant role of geometry in College and University curricula: if offered at all, limited to a course for prospective teachers, or specialized courses (projective geometry, differential geometry).
— The dearth of research papers, conferences, and symposia devoted to geometry.
— The small number of geometry texts at the college level, and absence of any new texts.

Historically, a knowledge of geometry was considered the mark of an educated person. However, in recent times, a reverse kind of snobbery has occurred: a lack of knowledge about, and disinterest in geometric questions is a common profile of the mathematical research community. The view towards geometry is generally a mixture of one or more of the following beliefs:

1. Euclidean geometry, like Latin, is GOOD FOR YOU. It should be studied (in high school) for historical appreciation and to build character. The geometric content is not expected to lead (mathematically) anywhere.

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[1] An earlier version of this address was given at the Conference on Learning and Teaching Geometry, June 1987, at Syracuse University, New York.

[2] All quotations in this paper from ICME IV may be found in The Two-Year College Mathematics Journal, 12 (September 1981) 226-246, which contains the addresses given by Dieudonné, Grünbaum, and Osserman.

2. Euclidean geometry is where students learn logic, the axiomatic method, and deductive proof. The geometric content of the course is secondary to these aims. (This course could be titled GEOMETRY AS A MILITARY DISCIPLINE.)

3. Geometry provides some interesting low-level recreational problems to solve, but there aren't any important unanswered questions. Mathematicians who claim to do research in geometry are not considered as serious in their interests.

> Most mathematicians are totally unaware of the fact that the elementary, intuitive approach to geometry continues (and will continue) to generate mathematically profound and interesting problems and results. (B. Grünbaum)

4. The content of geometry has been integrated into (absorbed by) almost all of higher mathematics — linear algebra, analysis, algebraic geometry, topology, group theory, etc. — so there is no need to teach it apart from these.

> ...mathematicians have been extremely appreciative of the benefits of the geometric language, to such an extent that very soon they proceeded to generalize it to parts of mathematics which looked very far removed from Geometry.
> (J. Dieudonné)

This last view was met with a memorable rejoinder by Osserman, who summarized Dieudonné's position as follows:

> Geometry is alive and well and living in Paris under an assumed name.

Even in recent years, there has always been a small core of mathematicians who have done considerable research in geometry despite the prevailing mathematical fashion. H.S.M. Coxeter might be considered the "dean" of such researchers. In an interview in 1979 for The Two -Year College Mathematics Journal with David Logothetti, he gave testimony to his enduring interest in and excitement about geometry, and his belief in its vitality. The interview closes with a question by the interviewer, and Coxeter's reply:

> L. If I or my colleague Jean Pedersen start rhapsodizing about geometry, the reaction that we frequently get is, "Oh well, that's a dead subject; everything is known." What is your reaction to that reaction?
>
> C. Oh, I think geometry is developing as fast as any other kind of mathematics; it's just that people [research mathematicians] are not looking at it.

In his closing remarks at the 1980 ICME, Osserman echoed similar sentiments:

> ...geometry...has gone through a period of neglect, while the arbiters of mathematical taste and values were generally of the Bourbaki persuasion. On the other hand, ... that period is already drawing to a close. ... I would predict that with no effort on any of our

> parts, we will witness a rebirth of geometry in the coming years, as the pendulum swings back from the extreme devotion to structure, abstraction, and generality.

Today we witness a renewed interest in teaching and learning geometry. In 1987, the NCTM yearbook and an international conference in Syracuse, New York were devoted to the topic. The newly announced NCTM standards (1989) address the need to strengthen geometric content in the K-12 curriculum, and an article by Marjorie Senechal in a collection of position papers (to be published in 1990 by the National Research Council) on the mathematical content in the K-12 curriculum, identifies shape as a major content strand at all levels of learning. These are timely events, since there is convincing evidence that points to a renaissance in geometry. There is strong interest in geometric figures in the plane and 3-space — exploration of their properties, their interrelationships and enumeration of their types. In what follows, I want to convince you that reports of the death of geometry (in 1980, and even more so today) are greatly exaggerated. The remarks by Coxeter and Osserman in 1980 were prophetic — for whether or not the official teachers and researchers in the mathematical community choose to lead (or even join) in this renaissance, it is happening.

The Evidence

Activity outside mainstream mathematics

While mathematicians were neglecting (or ignoring) geometry, its importance grew in many other fields. Those areas in which geometry has always been central — art, architecture, design and engineering — make direct use of geometry to create and build forms which satisfy aesthetic desires and structural needs. The three-dimensional Euclidean world which we inhabit demands answers to complex geometric questions, and manufacturers, craftsmen, architects and engineers have not waited for the mathematical community to provide answers — they always have and still continue to solve geometry problems, sometimes in an ad hoc and ingenious manner. Renewed interest in geometry related to structure is evidenced in the recent publication of several books concerned with the geometry of spatial forms,and the topics of incidence and symmetry in design (see, for example, [Baglivo and Graver], [Blackwell], [Gasson]). One especially active site of research into structure and form is the University of Montreal, and its associated "Structural Topology" group, which seeks to have investigators from many disciplines contribute to the common search for a better understanding of and solutions to geometric problems.

Many other fields have found geometry a rich source of ideas for creating models to understand complex forms, relations, and processes which cannot be viewed directly. Historically, artists and artisans as well as mathematicians have been interested in polyhedra (Leonardo da Vinci and Albrecht Dürer, as well as Johannes Kepler and Leonhard Euler to name but a few), but today, it is not likely that students or their teachers even know why a soccer ball has hexagon and pentagon faces, or why it must have exactly 12 pentagon faces. Polyhedra, sometimes viewed by mathematicians merely as pretty ornaments, rather than a rich source for study, are indispensable as models in diverse fields. The idea of ball and stick polyhedra models to represent molecules gained wide acceptance by the late nineteenth century. This modelling of chemical structure (the balls representing atoms, the sticks the bonds between atoms) has been one of the most productive ideas of modern chemistry. Tetrahedron is the name of an international journal of organic chemistry, signifying the importance of the model which considers carbon atoms to be situated at the centers of tetrahedra. Inorganic chemistry as well has recently developed simple and successful polyhedral models; an international journal in that discipline is named Polyhedron.

Some of the most exquisite polyhedra can be found in nature as crystals. But the inner atomic structure of crystals is also highly geometric — it is modeled by a vast lattice of atoms which can be viewed as packed polyhedra, and has been the subject of intense investigation in recent times by crystallographers, chemists, mathematicians and physicists. In biology, polyhedra serve as useful models for the structure of viruses which often (surprisingly) have icosahedral symmetry. The investigation of how information is carried by viruses, and how viruses self-replicate has led to the study of repeating patterns on polyhedra, and to questions on polyhedral packing. Soap bubble froth has been used to study aggregates of polyhedra which model biological structures. Difficult questions concerning packing of spheres are of interest to those who model chemical (atomic) structures and biological processes; these same studies have important applications in algebraic coding theory.

Another active area of geometry research which has recently emerged involves dynamic polyhedral models — here investigators might attempt to model the growth of a rigid plant stem through the division of packed polyhedral cells, or model the functioning of a robot mechanism. An extremely readable and well illustrated overview of the rich topic of polyhedra — history, properties, occurrences in nature and man-made design, importance as a modelling device, activities, questions — can be found in the book *Shaping Space*.

Symmetry is a concept that encompasses very diverse fields; here geometry also plays a central role. Symmetry is not only a powerful tool for creating or analysing beautiful designs in the plane or space by means of Euclidean and affine transformations; it is also a profound idea that gives an approach to understanding many of nature's structures and processes. Recently there have been several conferences, articles, and books devoted to symmetry and its many manifestations and applications. A large and varied collection of articles on symmetry, by authors representing many disciplines and countries, is contained in the collection *Symmetry: Unifying Human Understanding*; a sequel volume has just been published. A newly recognized 'type' of symmetry, that of "self-similarity", has revealed not only beautiful graphic images of dynamic processes, but offers a new view of forms and dynamic systems that were previously viewed as random or unpredictable in shape or behaviour. (See, for example, [Gleick], [Mandelbrot], [Barnsley].)

Activity within the mathematical community

Two measures of the vitality of activity in a mathematical field are the output of research articles and the number lectures, seminars and conferences devoted to the topic.

In recent years, the number of pages in *Mathematical Reviews* devoted to reviews of articles on geometry has grown dramatically. Indeed, the category 51, simply titled "Geometry", now has 14 subtitles (51A - 51N), and category 52, "Convex sets and related geometric topics" has become a catchall for the large number of papers on geometric topics for which a separate category has not yet been designated. (Differential Geometry and Topology have their own category numbers.) This increase in publication reflects not only a proliferation of articles, but also the establishment of several new journals devoted primarily to research in geometry. In 1989 alone, two new journals, *Combinatorial Geometry* and *Symmetry* were launched.

Two new areas of research activity in which the publication of papers has been especially prolific are signalled by the titles of recently published books: *Tilings and Patterns*, and *Computational Geometry*. Artisans of all cultures have designed decorative patterns and geometric tilings, and many popular recreational problems concern tilings of geometric figures. Yet mathematicians B. Grünbaum and G.C. Shephard found when they set out to write a work on "visual geometry":

> Perhaps our biggest surprise when we started collecting material for the present work was that so little about tilings and patterns is known. We thought, naïvely as it turned out, that the two millennia of development of plane geometry would leave little room for new ideas. Not only were we unable to find anywhere a meaningful definition of pattern, but we also discovered that some of the most exciting developments in this area (such as the phenomenon of aperiodicity for tilings) are not more than twenty years old.
>
> (p.vii, Tilings and Patterns)

Their book brings together the work of many who have investigated tilings, sets out definitions and classification schemes, and, most importantly, indicates many avenues for further investigation.

The title "Computational Geometry" is simultaneously suggestive and ambiguous — I doubt that agreement could easily be reached on what it is and what it is not. The authors Preparata and Shamos indicate in their introduction that several contexts have been clothed with that title, but make clear that the essence of computational geometry is the design of efficient algorithms (for computers) to solve geometric problems. Classically, the restrictive tools of compass and straightedge and the algorithms of Euclidean constructions were used to solve geometry problems. With Descartes and later Gauss, algebraic and analytic tools could be employed to solve geometry problems, and in addition, the question of what constructions were feasible could be discussed. Today's researchers may use computers as restrictive tools and so the problems as well as the methods of solution must be recast:

> One fundamental feature of this discipline is the realization that classical characterizations of geometric objects are frequently not amenable to the design of efficient algorithms. To obviate this inadequacy, it is necessary to identify the useful concepts and to establish their properties which are conducive to efficient computations. In a nutshell, computational geometry must reshape — whenever necessary- -the classical discipline into its computational incarnation. (p.6, Computational Geometry)

A few of the concerns are the development of new coordinate systems to encode geometric information, the creation of very accurate data bases for geometric objects, and the visual (screen) representation of geometric objects in 2, 3, and higher dimensions. The emphasis on computation has even changed the way in which many geometry questions are asked. Instead of asking "How many different types of polyhedra are there with n vertices?", the researcher asks "How can the computer determine whether two given polyhedra are of the same type?" and "What is the complexity of the best algorithm to do so?"

Conference activity on geometry is decidedly on the upswing, with the participants representing many areas of mathematics and other disciplines. Here are just a few special conferences largely concerned with geometry held during 1984-87:

"Shaping Space", an interdisciplinary conference on polyhedra, Smith College, April 1984.
"International Congress on M.C. Escher", Rome, April 1985.
"Eugene Strens Memorial Conference on Recreational and Intuitive Mathematics", University of Calgary, July 1986.
Special semester devoted to the Geometry of Rigid Structures, CRM, University of Montreal, January-May 1987.
"Computer-aided geometric reasoning", INRIA, Sophia Antipolis, France, June 1987
"SIAM Conference on Applied Geometry", Albany, July 1987.

In the last two years, the number of such special conferences on geometric topics has risen dramatically, and in addition, at the National MAA and AMS meetings the number of lectures, minicourses, and special sessions reflects the growing interest and diversity of research in geometry. Here is a list of items on the program of just one such meeting, the AMS-MAA meeting held August 7-10 in Boulder, Colorado:

Colloquium Lectures:	"Geometry, Groups, and Self-Similar Tilings", William P. Thurston
Special Session:	"Mathematical Questions in Computational Geometry"
Minicourses:	"Chaotic Dynamical Systems", Robert L. Devaney
	"Group Theory Through Art", Thomas Brylawski
Invited Addresses:	"The dynamics of billiards in polygons", Howard A. Masur
Jean E. Taylor:	"Crystals, in equilibrium and otherwise"
Progress in Mathematics Lecture:	"Liquid Crystals", Haim Brezis

The impact of technology

Perhaps the greatest single impetus to renewed activity in geometry has been the availability and proliferation of technological tools. This has created a two-way interaction involving geometric activity and technology.

On the one hand, the design and implementation of computers and other high-powered research, design, and diagnostic tools require a high level of understanding of traditional geometry and the solution of many new geometric problems. For example, computer-aided design (CAD) and manufacturing (CAM) (imaging and robotics), communications (networks and coding), and diagnostic imaging (computer-aided scanning devices) are areas in which geometry plays a central role. On the other hand, technological tools can also be utilized to investigate and even prove geometric statements. The ability to make and test conjectures in geometry (or any subject) is greatly enhanced by looking at a large number of specific cases. Complicated geometric forms can be shown rapidly in many aspects on a computer screen, changed and modified effortlessly, and data recorded and compared. Plausible conjectures based on such experimental data can be subjected to traditional methods of proof, or in some cases, proved by computer programs. As high-powered "eyes", technological devices can reveal the inner geometry of crystals, plant cells, viruses, and even chemical molecules, making it possible to test the veracity of accepted models and provide challenging new geometry problems to solve.

Titles of several of the sessions at the meetings held in France and in Albany in the summer of 1987 (listed below) will indicate some of the areas in which there is strong interest and active research:

> **Image processing; Surfaces; Mathematical Methods and Design**
> **Packing and Tiling; Mesh Generation; Graphics; Computational Geometry; Robotics; Solids; Modelling for Manufacturing;**
> **Automatic Theorem-proving; Computer-aided design; Applications to Rigidity of Structures; Applications to Scene Analysis and**
> **Polytopial Realization; Algebraic, Topological and Combinatorial Aids to Geometric Computation.**

The availability and use of technology, especially microcomputers, has also begun to affect the teaching of geometry at all levels. Exploratory activities with LOGO ("turtle geometry"), computer-aided Euclidean constructions ("The Geometric Supposer", "The Geometric Constructor", "Cabri") , and transformations using computer graphics can enrich the teaching and learning of geometry in elementary and secondary school. To construct a computer program which produces an image on a computer screen — the first task of computer graphics — requires a good knowledge of geometry, and affords an excellent opportunity to teach some traditional college geometry in a new light. A recent text, Projective Geometry and its Applications to Computer Graphics, develops the geometric machinery necessary to understand the representation and transformation of geometric objects in order to produce a screen image. Along the way, the main theorems of projective geometry are proved analytically. The strong purpose of the book linking the subject to computer graphics makes a compelling case for learning the geometry. On page 1, the authors make clear that a knowledge of Euclidean geometry is assumed:

> The primary purpose of this [first] chapter is to introduce projective geometry and discuss it in relation to Euclidean geometry. The reasons for doing this are twofold. First, Euclidean geometry is well-known and is a good foundation for the discussion of a "new" geometry. Second, the geometry of real objects is Euclidean, while the geometry of imaging an object is projective; hence the study of computer graphics naturally involves both geometries.

Controversy

A subject can be declared moribund only when people cease to ask questions and never challenge assumptions or methodology. Controversy is a certain measure of health in research. We are accustomed to announcements of new theories, new interpretations, and public squabbles among scientists as they seek to explain nature's phenomena — revision of old tenets, and even simultaneous acceptance of competing but equally convenient theories is not unusual. But controversy in geometry? That has not happened since the reluctant acceptance in the nineteenth century of non-euclidean geometries as consistent systems apart from Euclidean geometry. In fact, perhaps more so than in any other branch of mathematics, the view of geometry has been one of orthodoxy, ruled by the views of F. Klein's Erlangen program, in which geometry is primarily the study of invariants of transformation groups, or by the influence of 20th century seekers of complete axiomatic systems, perfecting the original Euclid. The narrowness of these confines is being challenged by many.

Among those most vocal is Grünbaum, whose provocative piece "The Emperor's New Clothes: Full Regalia, G string, or Nothing?" decries the arrogance of those mathematicians who will only analyze geometric figures from the standpoint of symmetry groups, and who declare decorative art as "wrong", or a "mistake" if it doesn't fit that scheme. The plea is made to look for other ways to understand and analyse; to look to the motives and methods of the creators of the works. As if to underscore this very point, in the last couple of years scientists have seen nature mock the orthodox geometric model of internal crystal structure, which postulates a periodic repetition of cells, and hence forbids the occurrence of crystals with five-fold (pentagonal) symmetry. Yet imaging technology has revealed that such "crystals" do exist, and now mathematicians, physicists, and crystallographers are scrambling to try to explain how this can occur (see [Steinhart] and [Jaric]). Adding a bit of extra irony, these "quasicrystals" appear to have lattice patterns related to aperiodic tilings discovered by Roger Penrose — about which the symmetry group theory gives absolutely no information, since no symmetry leaves these patterns invariant. This incident also illustrates the fact that so-called "recreational" mathematics (as Penrose's tilings were viewed) is largely a matter of fashion — now researchers are making "serious" attempts at understanding aperiodic tilings. (See [Gardner] and [Grünbaum and Shephard, Chapter 10].)

Open questions

By now it should be apparent that there are more unanswered than answered questions in geometry — even geometry in the Euclidean plane and Euclidean 3-space. It may seem, from the applications and illustrations that I have given, that most are extremely technical in nature, and are difficult to formulate and understand, much less to solve. Of course, many are, but many are deceptively simple to state, and point to how little we really do know about the geometric structure of the space we inhabit. Many are amenable to experimental investigation by students and amateurs — they will yield (at least partially) to patient enumeration, or to ingenious insight rather than to what may be inappropriate and complex mathematical structure and theory.

The subject of packings and tilings is rich with such unanswered questions. Many can be found in *Tilings and Patterns*; I would like to point out just a few which are easy to state.

1. *Describe all of the convex pentagons which can tile the plane.*

Although congruent regular pentagons cannot tile the plane (fill it completely, without gaps or overlaps), there are many pentagons which can be used as paving blocks to tile the plane. But the list of such pentagons has not been proved to be complete. The problem was thought to have been solved by a mathematician in 1918, and again in 1968 by another mathematician, yet each was wrong. After Martin Gardner discussed the problem in his Mathematical Games column in *Scientific American* in July 1975, several new types of pentagon tiles were discovered by amateurs. In addition, in 1976, a high school summer class in Australia discovered all but one type of equilateral pentagon that tiles the plane. (See [Schattschneider: 1978, 1981, 1985].)

2. *If a tile can fill the plane by half-turns only, must there exist a periodic tiling of the plane by that tile?*

Tiles that can fill the plane in a periodic manner using only half-turns were characterized by J. H. Conway; analysing and creating tiles using his criterion is an enjoyable exercise. The question above, however, has not yet been answered. (See [Schattschneider, 1980].)

3. *Does there exist a single tile that can fill the plane only aperiodically?*

The first sets of aperiodic tiles (tiles that can fill the plane only with tilings having no translation symmetry) contained many differently shaped tiles; R. Penrose is credited with discovering the first such set containing only two different shapes. Other sets of two tiles which tile only aperiodically have since been discovered, but still a single tile that does so (or a proof that no such single tile can exist) has not been discovered. (See [Gardner], [Grünbaum and Shephard].).

4. *Which tetrahedra pack space?*

Dissecting simple forms that pack space, such as boxes and prisms, into congruent (non-regular) tetrahedra gives some answers to the question. But the list is far from complete. (See [Senechal].) Related to this question is the more general one: For a given n, describe all convex polyhedra having n faces which also pack space.

5. *Is there an upper bound on the number of faces of a convex polyhedron that packs space?*

It is known that no convex polygon having more than six sides can tile the plane. Although it seems plausible to believe that there cannot be a convex polyhedron which has a great number of faces and also packs space, no one has yet proved it. Amazingly, a convex polyhedron has been found that has 38 faces and packs space. (See [Danzer et al] (the answer to the question posed in the title of that article has been shown to be "no"); see also [Grünbaum and Shephard].)

Conclusion

I hope that the evidence has convinced you that, indeed, the reports of the death of geometry are greatly exaggerated — the reporters have not kept abreast of the many exciting developments which are contributing to its rebirth. The news needs to be spread — colleagues and students need to be made aware of the vitality of geometry.

What can teachers do to help bridge the gap between what is happening on the research frontier and what is learned in the classroom? Osserman ended his address by offering this advice:

> We can initiate and revitalize courses in which students become familiar and comfortable with geometric insights and methods. Perhaps most important and difficult of all is to develop courses where the fragile but vital ability to invoke geometric intuition will be fostered and nurtured. (R. Osserman, ICME IV, 1980)

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Working Group A

Using Computers for Investigative with Elementary Teachers

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All three sessions were devoted to discussions around a LOGO-based software called "Les deux tortues" presented by B. Côté[1] and a set of mathematical activities that it allows. Although the working group had been planned to focus on using computers in the context of elementary teacher training activities, we ended up spending most of our time looking at the mathematical activities presented, and discussing the role of computers in mathematical learning.

The system presented is the result of an effort to build a bridge between computer activities with the LOGO turtle and middle school mathematics curriculum. It is essentially based on two sets of ideas:

1. *Construction and exploration:*

Here the turtle is not used in the context of learning programming. A set of commands are provided that are used in direct mode to produce effects. So we have construction activities that deal with creating objects, usually geometric figures. The notion of procedure is used as a tool to create a bridge between the concrete world of actions and the symbolic level of descriptions of actions as sequences of instructions. The turtle belongs to both worlds. It is a "real" object that we can identify with, that we can simulate with our body or a paper clip. It is also a geometric object, a point that has an orientation. Construction has to do with going from one level to the other by simulating what needs to be done and describing our own actions in terms of instructions, or starting with instructions and simulating them in order to understand why they do what they do.

Questions arise naturally in the context of construction activities. Is it possible to do...? Are there other possibilities? What are all the possibilities? What will happen if...? Can we make a prediction of what will happen if we use such a number, or change that instruction...? The activity of formulating such questions and trying to generate an answer is what we mean by exploration. Construction has to do with "doing". Exploration has to do with "understanding". We have to explain or justify why something is like this or why it is impossible. It is a world of induction and deduction, where we try to establish what is true, what is false.

1 This project is funded through a research contract between UQAM and APO Québec research centre on uses of computers in education. Hélène Kayler, Lise Paquin and Tamara Lemerise have been involved in the first stage of the project.

2. *Working on paper:*

Working with the turtle at the computer will often result in an "interactive" way of functioning, where the student tries things until it works. This empirical mode of functioning is an important aspect of concrete intelligence and is certainly present in the mathematician's toolbox. But from the point of view of logico-mathematical development, it has to be put under the control of a more "reflexive" mode that works within explicit representations of problems with several kinds of reasoning. (These modes correspond roughly to what Hillel and Kieran (1987) call "visual" and "analytical".)

So we need to set up situations where the interactive mode does not work and the student has to switch to a reflexive mode. We also need to help students extract the mathematical knowledge that is interwoven in their interactive functioning (what Vergnaud (1982) calls "theorems in action"). The notion of turtle is actually a very powerful tool to build that bridge, provided 1) that these activities are overtly identified as mathematical, and 2) that the computer is used as an essential reference point but within a larger space that includes some work on paper. Working on paper means that you have to set up a representation of how the turtle works, that you can test afterwards with the computer. You can also use the computer to gather information that you write down in order to analyse a phenomena and try to understand it. It helps us keep in mind that the goal of all this is to learn mathematics and not particularly to get a computer to work.

Based on these ideas, we have redefined the basic turtle commands in order to facilitate work on square paper, with a metric ruler, protractor and compass. We have also added a set of commands that allow exploration of specific topics like fractions, polygons, integer operations, perimeter and area, motion geometry, variable ... Moreover, because of the central role of the notion of turtle, it was important to shape the basic commands in order to facilitate the understanding of its different aspects. So we ended up with *two turtles*, each with its basic commands and the possibility of working in a fraction or a decimal mode.

The *square turtle* is a simplification of the LOGO turtle that evolves on a square grid. It goes forward in terms of number of squares and turns a fourth of a turn, which allows only four possible orientations. It can also move along any diagonal. The *angular turtle* evolves on a blank plane. It goes forward in terms of centimetres and turns in a fraction of a turn that the user can set up. If we type TOURCOMPLET 360, the turtle turns in degrees. If we type TOURCOMPLET 12, it turns in twelfth of a turn. So we can simulate the square turtle on square paper and the angular turtle with a ruler and a protractor.

The *first session* of the group was spent looking at the basic commands of the two turtles and getting acquainted with the notions of construction and exploration in this context. Much time was spent around an exploration activity related to the command CYCLE, that asks for a sequence of instructions and repeats it until either the turtle comes back to its initial state (in which case it prints the number of repetitions) or it finds that it goes indefinitely away from it (in which case it prints VERS L'INFINI...).

Using this command with the square turtle, we can ask what are the possible answers of the machine and how we can predict them. It is an interesting problem in the sense that:

1) although an empirical approach can help find the possibilities (which are 1, 2, 4 and ∞), it does not help to find out why they are the only ones ; one has to identify what is the relevant factor and try to formulate a prediction rule;

2) one can develop a gradual understanding of the situation; that is, understand some cases before formulating the general rule;

3) the general solution comes from breaking all possibilities into a few categories and solving the problem for each of them. Although usually not obvious, the solution is quite accessible to 5th and 6th grade children (and their teachers) since there are only four possible orientations for the square turtle.

The *second session* was spent mainly discussing fractions and decimals. There are three representations of fractions in the system: turn, length and ratio. The notion of fraction is already involved in relationship with the command TOURCOMPLET. To understand it, we can fold a paper circle to separate it into 8 or 12 equal pieces and use it as a protractor. This makes the link with the traditional "pie or pizza" approach to fractions. We can also type instructions like D (right) 1/3 DE TOUR, that work directly in terms of fraction of a turn. In this context, no special distinction needs to be made between fractions that are smaller or larger than unity. The group discussed different ways to build the operations and some interesting situations like TOURCOMPLET 1/4.

We can also have fractions as lengths. One can make the square turtle move out of its grid by going forward fractions of squares. Using the ruler with the angular turtle, one comes naturally to want to express centimetres and millimetres, which is done with decimal numbers. The command POTEAU helps compare lengths and so create activities where one goes from fractions to decimals and vice versa. The command FUSÉE uses a fraction to specify the path of a rocket. It is basically the slope interpretation where the numerator is associated with the vertical component of a move and the denominator with the horizontal one. This creates activities on equivalence and order that promote the development of qualitative reasoning on fractions as ratios.

The discussion went around the notion of *microworld*. Is it a useful concept? Does it cover almost any software that is not based on direct teaching? In this case, we can talk about the system including several microworlds; that is, commands that create activities around a well defined topic. We can also think of domains of knowledge, for instance fractions, as microworlds. This is a way to see knowledge as a dynamic entity made out of a network of elements integrating formal and concrete aspects in a way that has to function. From this point of view, each learner has to build his own network.

The *last session* started with trying to characterize construction as *object formation* and exploration as *relating variables*. Although the activities with the two turtles start as construction of figures, the notion of construction gets eventually a larger sense. In order to build a figure, one has to choose the right command, with the appropriate number (build an instruction), formulate a sequence of instructions (build a procedure). Through the exploration activities, one has to manipulate objects like numerical and algebraic expressions, to build geometrical transformations like translation or rotation, to formulate rules ...

Exploration activities are generally based on some classification of objects. We have a set of commands or characteristics of commands, that can be put into categories, and a set of results, that can also be put into categories. What needs to be done is to formulate the relationship between variations in the command side and variations in the turtle side. For example, we might ask what will happen if we tell the square turtle to turn of a number larger than 4. All the possible turning instructions can be divided into four categories according to their end result and the question is which object (instruction) belongs to which category (orientation). We have the same thing with CYC[illegible] ere on one side we have procedures and on the other side number of repetitions ([illegible]); with FUSÂE, where on one side we have fractions, and on the other side the same or different paths, or above, below or equal to the middle path, or general ordering in terms of steepness of slope. We could have on one side possible items of addition and subtraction of integers (classified in terms of ⁺+⁺, ⁺+⁻, ... , ⁻-⁻) and on the other side the interpretation in terms of turtle move. Or we can have on one side the regular polygons divided into normal and stars, and on the other side the sequences of instructions that generate one or the other.

The session ended with a general discussion on a question raised throughout all workshop by S. Dawson: do mathematical activities defined around computers induce a reduced view of mathematics, in particular, and the real world in general?

Much debate throughout the three sessions focused on the supposed neutrality of the computer, a question centrally addressed by C. A. Bowers in his recent book *The cultural dimensions of educational computing*.

> "The question has to do with whether the technology is neutral: that is, neutral in terms of accurately representing, at the level of the software program, the domains of the real world in which people live. If the answer to this question is that it is not neutral, the critically important question of how the technology alters the learning process must be addressed." (Bowers, p. 24)

In particular, computers foster a digital, dichotomous, context-less, ultra rational form of world view, which though extremely productive in many ways, is also at the foundation of many misunderstandings about the world. To paraphrase Gregory Bateson, if we separate an object from its context we are likely to misunderstand it. Yet computer educators perpetuate the view that the computer is culturally neutral, that it is simply a 'dumb' machine.

But this overlooks the fact that "...the classroom strengthens certain cultural orientations by communicating them to the young and weakens others by not communicating them." (Bowers, p.6)

Bowers goes on to say:

> "By interpreting rationality, progress, and efficiency in terms of technological achievements, this mind-set has developed the hubris that leads to viewing the ecological crisis as requiring a further technological fix rather than the recognition that our most fundamental patterns of thinking may be faulty". (Bowers, p.8)

Much debate throughout the three sessions focused on the supposed neutrality of the computer and of Logo.

The conclusion which Bowers draws, noted below, was hotly debated:

> "Thus the machine that the student interacts with cuts out of the communication process (the reduction phenomenon) tacit-heuristic forms of knowledge that underlie commonsense experience. While the technology amplifies the sense of objectivity, it reduces the awareness that the data represent an interpretation influenced by the conceptual categories and perspective of the person who "collected" the data or information. The technology also reduces the recognition that language, and thus the foundations of thought itself, is metaphorical in nature. The binary logic that so strongly amplifies the sense of objective facts and data-based thinking serves, at the same time, to reduce the importance of meaning, ambiguity, and perspective. Finally, the sense of history, as well as the cultural relativism of both the student's and the software writer's interpretative frameworks, is also out of focus. As a symbol-processing technology, the computer selects and amplifies certain aspects of language... " (Bowers, pp 33-34)

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Working Group B

Computers in the Undergraduate Mathematics Curriculum

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Session I

At Brock University, the Department of Mathematics and Statistics has established an undergraduate computer laboratory as an instructional aid in teaching various undergraduate mathematics courses, particularlv introductory calculus courses. This laboratory contains thirty MacIntosh-SE desktop computers linked to an overhead video display unit. Working Group B was able to take advantage of this facility for some hands on experience.

During the first session, held in the laboratory, Eric Muller of Brock University presented a brief overview of the lab set-up and how it is utilized. The symbolic manipulation program MAPLE, developed at the University of Waterloo, is the computer environment in which sessions are conducted. Eric indicated that although MAPLE was not developed primarily for educational use, it is being used by a number of universities in the teaching of undergraduate mathematics. The availability of other software designed for specific educational use was mentioned.

The first session continued with a demonstration of some of the capabilities of MAPLE by Stan Devitt. Participants were given the opportunity to experience the considerable power of MAPLE as a calculator. The ability of the system to carry out routine as well as complex calculations was demonstrated. As a result, participants gained some appreciation of the capabilities of MAPLE as an instructional aid and this resulted in a discussion of some of the implications of this technology for teaching.

Stan Devitt indicated that the primary objective of current efforts to incorporate computer algebra systems (CAS) such as MAPLE in undergraduate mathematics instruction is to build an environment in which all so-called paper and pencil calculations can, with appropriate commands, be carried out on a computer screen. He suggested that in order to reach this objective, it will be necessary to design special routines so that students can easily utilize the full power of the system. For example, special routines, perhaps fairly advanced in nature, are necessary in order for students to realize the full potential of CAS as an aid in problem solving in areas like Linear Algebra and Number Theory.

Generally, during this session, participants had the opportunity to play around with the system and become familiar with some of its capabilities and potential problems. Even trivial problems such as how to change an expression after entering it were evident.

After coffee break, Stan Devitt gave a demonstration lesson using MAPLE. He indicated how the software evolved and introduced some of the commands, such as those for finding summations, evaluating definite integrals, and for performing numerical integration using Simpson's and the Trapezoidal Rule. Also, the use of computer graphics in estimating the area under a curve was demonstrated.

Towards the end of the session, several issues were raised by participants relative to possible implications of this technology on the mathematics curriculum. In particular, questions dealing with the evaluation of student learning and how to incorporate computer algebra systems such as MAPLE in the mathematics curriculum were discussed. The need to address such issues in a meaningful way was emphasized. The need to know what has worked well to date in the use of CAS and the need to identify some of the problems not just the advantages was emphasized.

Session II

At the beginning of the second session, Stan Devitt provided the group with some anecdotal experiences resulting from his own attempts to incorporate CAS in undergraduate mathematics courses. He pointed out that even though CAS have been around for some time, to date such programs have had very little evident impact on undergraduate teaching. One of the first available CAS programs was MACSYMA, developed at MIT and available on mainframes about 1980. MAPLE and other CAS programs were subsequently developed in an attempt to reduce the large amounts of computer memory that such programs require, and thus make the capabilities of CAS available to a much wider audience.

In 1986, the Sloan Foundation provided funding to eight institutions to establish computer laboratories using computer algebra systems. Included were the University of Waterloo and the University of Saskatchewan, both of which are using MAPLE. Other institutions are using different systems, such as Mu-Math at the University of Hawaii. These projects are now underway and workbooks have been produced. In fact, participants of this working group each received a copy of "Calculus Workbook; Problems and Solutions", compiled by Stan Devitt for the project now underway at the University of Saskatchewan.

The collective experience of the institutions funded by Sloan was reviewed at a conference held at Colby College in the summer of 1988. It was a disappointment to some that several of the projects were just getting underway after the initial eighteen month start-up period. Also, institutions reported varying experiences. For example, the reaction of students using CAS was not as positive as expected. Some students reported that they experienced more difficulty using CAS than with traditional instruction. On the other hand, most faculty members involved in these projects indicated that they would not consider teaching undergraduate mathematics without using CAS. In summary then, there appeared to be moderate disappointment with the extent to which progress had been made in implementing CAS into the undergraduate mathematics curriculum of the participating institutions, and some disappointment at the initial reaction of students exposed to CAS in their courses.

By way of elaboration on the above, Stan Devitt explained that at the University of Waterloo, where the MAPLE project has been underway for the past eight or nine years, it is not generally being used by faculty members in their teaching. Also, students at the University of Waterloo indicated that they were under a lot of pressure to get through their assigned work and the use of CAS meant additional work and material to cover.

At other universities, however, there was a more positive reaction. At Dennison, all students enrolled in undergraduate mathematics courses receive instruction in a computer laboratory environment. Also, at Brock University, all faculty members in the Department of Mathematics are involved in computer labs. However, at the University of Saskatchewan, with 30 members in the Department, only three members were seriously investigating the potential of CAS.

One explanation for the apparent lack of interest on the part of some faculty members is the fact that most are busy people and are not willing to invest large amounts of their limited time unless there is some evidence that the result will be worthwhile. Clearly, some faculty remain unconvinced that the result is worth the effort, and it is clear that much more thought and effort will be required before CAS can become widely accepted.

The above summarizes some of the comments of Stan Devitt at the beginning of the second session. Eric Muller then gave an overview of the Brock experience. He indicated that the original objective was to develop over a three year period, computer labs for all service courses offered in the Department. In the first year, VAX MAPLE was used by 100 out of 110 students enrolled in such courses, with students meeting in compulsory lab groups of 15. At the beginning of the second year, 30 Macintosh-SEs were purchased and used in the laboratory, with approximately 600 students now using CAS in the computer laboratory.

At the end of each year, a questionnaire was administered to participating students dealing with their attitudes toward the use of CAS. There were some obvious differences in the responses of the first group (1988) compared with those of the second group (1989). For example, 47 percent of the students in 1988 rated CAS as a good learning aid while 16 percent rated it poor. In 1989, the corresponding percentages are 11 and 67. Similar results were reported on such measures as confidence to do mathematics and enjoyment of mathematics. The course in which these students were enrolled was a traditional calculus course with applications.

In attempting to explain such results, it has been suggested that "better" students perform at a higher than normal level using CAS while weaker CAS students perform below normal.

Eric Muller then described how the computer lab at Brock was set up. He reviewed some of the practical considerations that received attention. For example, who is responsible for each lab session and who should be present in the lab with the students? At Brock, it was the practice to have one faculty member and one senior student (familiar with MAPLE) associated with each lab session. Each week, students would receive prior to the lab session a sheet of questions. A total of 29 lab sessions of one hour duration were scheduled over the 40 hour period per week available with about 28 students per session. There was a network server for each 10 machines in the lab (a total of 30 machines in the lab).

It was evident that using CAS resulted in changes to the style of teaching. There were more question and answer sessions than traditionally. However, in the lab setting, many of the questions were of a technical nature having to do with how to use the system to solve problems. There was open access to the terminal room during the semester and at the beginning of the year some introductory sessions outside of class time were scheduled to familiarize students with the system.

It was also evident that students at Brock preferred using the Macintosh to the VAX. However, one complaint, especially in multi-sectioned courses, was that some of the weekly assignments could be completed without the use of the computer and hence students did not see the need for the computer lab. This type of problem, however, seems to be one that could be solved if all faculty members teaching a course could agree on the nature of assigned work.

With respect to the attitudes of students using MAPLE relative to those of students in sections of a course not using MAPLE, it was reported that at Saskatchewan the drop-out rate in the MAPLE sections was higher. One explanation offered for this was that MAPLE students were left on their own more so than the others and the consequent lack of feedback when needed may have caused students to quit rather than persevere. In fact, the reaction of students left in the lab on their own was often very negative.

Some participants, as a result of the above discussion, questioned what possible good was resulting from this effort to incorporate CAS in the teaching of undergraduate mathematics. Did the costs justify the results? Is the use of computer/calculator technology being driven by a stick or a carrot? It was suggested that before many questions could be answered, there was the need for research on the impact of the technology in the classroom, and the only way to do this was via controlled experiments rather than anecdotal reporting of experiences.

Some of the drawbacks of the MAPLE system were mentioned. For example, the lack of a good graphing package and the fact that the user interface is not one that is very user-friendly. It was speculated that some of these problems would be addressed in future developments of the program. For example, a menu driven interface would improve matters considerably. One suggestion was that there could be developed an educational version of MAPLE to complement the scientific version. This led to a discussion of the pros and cons of MAPLE as opposed to a discussion of the pros and cons of symbolic algebra systems in general.

Session III

At the beginning of the third session, the group convened once again in the computer laboratory at Brock. Various reference materials were distributed. The session continued with a typical in-class CAS demonstration by Stan Devitt on limits and continuity.

The Group then reconvened for a group discussion. Eric Muller described the nature of an applied calculus course offered as a service course at Brock to non-math majors. A brief outline of the course was presented: functions, special functions, limits, continuity, differentiation, anti-differentiation, definite integrals, differential equations, probability distributions, and partial differentiation. In response to a question, Eric indicated that integration was not introduced as a limit of a sum, to which the question Why? was posed. This line of discussion raised the following questions: When is CAS a tool to help concept development? and When is it a tool just to compute? Where does one learn when to use an algorithm? This resulted in some discussion about the type of student being taught, that is math versus non-math students.

Perhaps the most interesting question posed was this: If the computer can draw pictures and compute derivatives, etc., why would a student have to learn any of this? How do we as mathematics educators deal with this question? Is there any attempt to try and show students that there are things in mathematics that the computer cannot do? The suggestion was that we need to give good examples to students that illustrate when it is (a) stupid, (b) hopeless and (c) inappropriate to use the computer. Perhaps good thoughtful examples to address the above questions would indicate to students why theory is so important in mathematics.

The end result of this question was: How do we teach intelligent uses of the computer? and Why is it important that we teach intelligent uses of the computer? The point was made that certainly the domain of computation in college courses is different than in the past or at least it should be. The discussion ended with some comments on potential dangers of using CAS in the teaching of undergraduate mathematics or a least a realization that if used inappropriately, certain undesirable outcomes may result. Again, the issue of the apparent negative attitudes of those students whom, we might assume, stand to benefit most from using CAS was raised. Also, the need for extra time perhaps to use CAS effectively.

In summary then, a synopsis of the activities of Working Group B is as follows:

1. A discussion of a philosophy of teaching mathematics using Computer Algebra Systems.

2. An overview of CAS in general, providing an awareness of the current state of the art and what efforts are underway to integrate CAS into the teaching of undergraduate mathematics.

3. An opportunity to experience in a laboratory setting how CAS can be used in a teaching situation with an actual demonstration of a lesson in introductory calculus.

4. An opportunity to become familiar with a Calculus Workbook incorporating CAS, produced at the University of Waterloo.

5. An overview of several projects at other universities that have been initiated since 1986 with the assistance of grants from the Sloan Foundation.

6. An overview of the Brock University experience of using CAS in the teaching of undergraduate calculus courses.

7. An indication of some of the problems associated with the implementation of CAS in undergraduate teaching, including the attitudes of students and faculty.

8. A look at what is likely to happen in the future. For example, the conclusion that the implementation of CAS requires a great deal of effort and planning for little evident initial payoff.

9. The opportunity to obtain a number of articles on CAS for retention and further use.

In conclusion, it is obvious that Working Group B accomplished much in a short time. However, it is also clear that as many questions were raised as were answered. It seems that before we can integrate CAS generally into the teaching of undergraduate mathematics, there is a need for much more thought, discussion, and investigation. There is no doubt that the availability of CAS has the potential to change dramatically how we teach and what we teach. It has the potential to remove much of what we might call the drudgery of elementary mathematics. However, care must be taken in the design of CAS based curricula that we do not replace one form of drudgery with another form that may be perceived by students to be equally distasteful.

There was a clear indication that CAS has a great deal of potential but at the same time that it can never be used to teach some of the fundamental understandings that are required of one whom we might classify as a mathematically literate person. Perhaps one of the important benefits of using CAS in undergraduate teaching is to makc available to instructors more time to concentrate on some of the essential ideas and concepts of mathematics than is available at present.

The need for the development of good research in this whole area was also evident. Controlled experiments on the effects of CAS on mathematics learning and retention seems to be called for before we jump on any bandwagon. The need for major curriculum reform efforts appear warranted and perhaps this should happen in any event. The past practice of permitting textbook writers to essentially determine the curriculum in calculus and other undergraduate mathematics courses, need not continue. It is possible with desktop publishing and sophisticated word and text processing capabilities for individual departments to produce their own curriculum materials and not depend on increasingly expensive and perhaps inadequate commercially produced textbooks.

In summary, this session proved to be interesting, informative and timely. Special thanks go to Stan Devitt for sharing his considerable experience with the group and to Eric Muller for superb local arrangements at Brock University, including of course, the use of the computer lab which made the session more than a speculative discussion group.

Working Group C

Natural Language and Mathematical Language

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> Science begins with the world we have to live in ... From there, it moves towards the imagination: it becomes a mental construct, a model of a possible way of interpreting experience. The further it goes in this direction, the more it tends to speak the language of mathematics, which is really one of the languages of the imagination, along with literature and music.
>
> (Northrop Frye, *The Educated Imagination*)

The descriptive advertisement for the three sessions went as follows.

> The group will examine aspects of communicative and other functions of language used in the service of mathematics and mathematicians. It will have a partly historical, partly linguistic and partly mathematical focus, exploring some of the means by which mathematical ideas are expressed and ways by which neophytes are encouraged to increase their command of the mathematics register.
>
> Further possible topics for discussion include the notions of metaphor and metonymy and their uses in mathematics as means for the creative extension of the expressive potential of language for the invention and control of mathematical notions.

I started the first session by attempting to share some of my current worries and concerns with the rest of the group. The first was the myth of learning by experience and the relation of language to that experience (see Pimm, 1986, in reply to Liebeck, 1986): in particular, the passive role often attributed to language in merely describing or representing experience, rather than being either a constituent component of the experience or the experience itself.

The second was an over-narrow conception of meaning in mathematics in terms of reference rather than connections in both form and content, and meaning in this restricted sense being claimed to be the most important, indeed only goal of mathematics teaching. In England, at least, an increasingly common dogma is if in doubt at any stage in anything mathematical, then told to go back to the 'meaning' (often the concrete) from which everything is presumed to stem. Valerie Walkerdine (1988) has recently drawn attention to the implausibility of such an account in the case of the teaching of place value. She offers a much more telling if complex account, one that intimately implicates the teacher's language and positioning within classroom activity. "Signifiers do not cover fixed 'meanings' any more than objects have only one set of physical properties or function" (Walkerdine, op cit., p. 30).

In an article entitled *On Notation*, Dick Tahta has claimed (1985, p. 49) that:

> We do not pay enough attention to the actual techniques involved in helping people gain facility in the handling of mathematical symbols. ... In some contexts, what is required - eventually - is a fluency with mathematical symbols that is independent of any awareness of current 'external' meaning. In linguistic jargon, 'signifiers' can sometimes gain more meaning from their connection with other signifiers than from what is being signified.
>
> Linguists have called the movement 'along the chain of signifiers' *metonymic* whereas 'the descent to the signified' is *metaphoric.*

The third concern I mentioned was one recently raised by Tahta (at the 1989 ATM Easter conference) of the current trend towards only stressing how we (or pupils) *differ* from one another, rather than what we have in common. How can we endeavour to develop ways of working together in relation to the learning of mathematics? One particular fear Tahta expressed was of the loss of consensus and commonality as a result of overemphasis on individual differences, with resulting isolation and lack of community. (I'm sure you will appreciate the political background of these concerns - in particular, following a decade of Thatcherism and the attempted wholesale destruction of collectivism at any level, whether inside education or outside it.)

Spoken language is one of the things that we share in common to a marked extent. It is socially acquired by considerable individual effort and little overt teaching. Language exists as a cultural repository, but also as a magnificent resource into which we can tap. A language both reflects and shapes the conceptual framework of its users. We can ask how thought is constituted in terms of and in relation to a system of signs, which by definition are social.

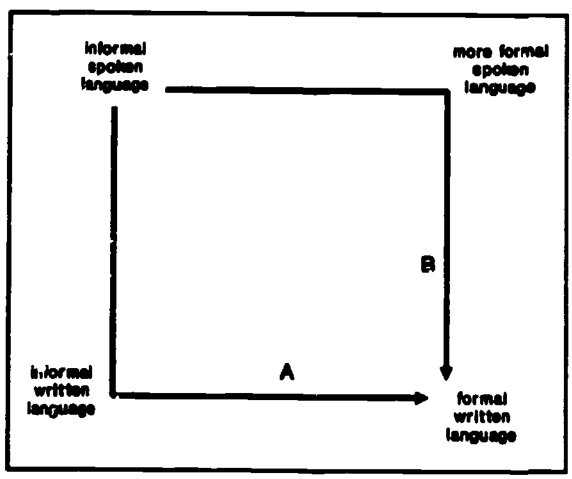
One way of describing the relation between mathematics and a natural language such as English is in terms of the linguistic notion of register. Linguist Michael Halliday (1975, p. 65, my emphasis) specifies this notion as 'a set of meanings that is appropriate to a particular *function* of language, together with the words and structures which express these meanings'. One function to which a language can be put is the expression of mathematical ideas and meanings, and to that end a mathematical register will develop.

Thus, while providing pupils with opportunities to gain access to the resources implicit in natural language can be seen as a common aim of all teachers (one interpretation of the 'language across the curriculum' idea), a particular aim of teachers of mathematics should be to provide their pupils with some means of making use of the mathematics register for their own purposes. To that end, a mathematics teacher needs knowledge about the language forms and structures that comprise aspects of that register. Part of learning mathematics is gaining control over the mathematics register so as to be able to talk like, and more subtly, to mean like a mathematician.

For these sessions, we were mostly in the realm of the signifier, and tried to explore to what extent signifiers can be used relatively autonomously from the signified they are taken to represent. In the second session, we worked on two classroom excerpts on videotape: Anne Tyson with a class on base five arithmetic and Irene Jones with a class working on a geometric poster (both from the Open University videotape PM644 *Secondary Mathematics: Classroom Practice*). In both cases, the pupils and adults were clearly engaged in a discussion - but about what? Where were the referents for what they were discussing - to what is the language pointing?

There are a number of different characteristics and functions of spoken and written language. One use of written language is to externalise thought in a relatively stable and permanent form, so it may be reflected upon by the writer, as well as providing access to it for others. One characteristic of written language is the need for it to be self-contained and able to stand on its own, with all the references internal to the formulation, unlike spoken language which can be employed to communicate successfully when full of 'thises', 'its' and 'over theres' due to other factors in the communicative situation.

One difficulty facing all teachers is how to encourage movement in their pupils from the predominantly informal spoken language with which they are all pretty fluent (see Brown, 1982), to the formal written language that is frequently perceived to be the hallmark of mathematical activity. There seem to me to be two ways that can be tried. The first (and I think far more common) is to encourage pupils to write down their informal utterances and then work on making the written language more self-sufficient (Route A in the diagram), for example by use of brackets and other written devices to convey similar information to that which is conveyed orally by stress or intonation.



A second route to greater control over the formal written mathematical language (shown as B in the diagram) might be to work on the formality and self-sufficiency of the spoken language prior to its being written down. In order for this to be feasible, constraints need to be made on the communicative situation in order to remove those features that allow spoken language to be merely one part of the communication.

Such situations often have some of the attributes of a game, and provided the pupils take on the proposed activity as worthy of engaging with, then those pupils have the possibility of rehearsing more formal spoken language skills. One such scenario is described by Jaworski (1985), where the focus of mathematical attention is a complex geometric poster. Pupils are invited to come and out and 'say what they have seen' to the rest of the class, under the constraints of 'no pointing and no touching'. These help to focus the challenge onto the language being used to 'point' at the picture. The situation is an artificial one: in 'real' life, one can often point and this is completely adequate for effective communication. However, if the artificiality is accepted by the pupils, natural learning can take place that would otherwise not have been so readily available. There is an interesting paradox here, one of how quite artificial teaching can give rise to natural learning under certain circumstances.

A second instance of such an approach comes from the contexts of 'investigations', when pupils are invited to report back to the class what they have done and found out. Because of the more formal nature of the language situation (particularly if rehearsal is encouraged), this can lead to more formal, 'public' speech and structured reflection on the language to be used. Thus, the demands of the situation alter the requirements of the language to be used. Reporting back can place some quite sophisticated linguistic demands on the pupils in terms of communicative competence - that is, knowing how to use language to communicate in certain circumstances: here, it includes how to choose what to say, taking into account what you know and what you believe your audience knows. A further example of these demands at work can be seen in the study by Balacheff (1988) on thirteen- year-old pupils' notions of proof, where he asked them in pairs to write down their claims about a mathematical situation to tell another pair what they had found out. By providing them with some plausible justification for them writing a message, he was able to gain access to their proficiency in this matter.

Educational linguist Michael Stubbs writes (1980, p. 115): "A general principle in teaching any kind of communicative competence, spoken or written, is that the speaking, listening, writing or reading should have some genuine communicative purpose". Pupils learning mathematics in school in part are attempting to acquire communicative competence in the mathematics register, and classroom activities can be usefully examined from this perspective in order to see what opportunities they are offering pupils for learning. Teachers cannot make pupils learn - at best, they can provide well-thought out situations which provide opportunities for pupils to engage with mathematical ideas and language.

For the third session, a couple of dynamic mental geometry activities were offered (see Beeney *et al.*, 1982, for further school examples), including the pole/polar construction between a point outside a circle and the two tangents to the circle passing through it. What happens when the point moves inside the circle?

In conclusion, the following quotation from the Second World Conference on Islamic Education (1980) was offered, which was their justification for the compulsory teaching of mathematics in school.

> The objective [of teaching mathematics] is to make the students implicitly able to formulate and understand abstractions and be steeped in the area of symbols. It is good training for the mind, so that they [students] may move from the concrete to the abstract, from sense experience to ideation, and from matter-of-factness to symbolisation. It makes them prepare for a much better understanding of how the Universe, which appears to be concrete and matter-of-fact, is actually *ayatullah*: signs of God - a symbol of reality.

Items which stood out for me during the discussion

A discussion of Helen Keller and her realisation by means of associating the running of water over one hand with a pattern being repeated tapped into her other of the *possibility* of symbolisation (the juxtaposition being essential in the creation of a sign - and the notion of sign itself) and her subsequent rapid 'linguistic' progress by demanding the symbols for many objects or phenomena. Valerie Walkerdine, in *The Mastery of Reason*, asks a fundamental question which has particular salience for mathematics teaching: "How do children come to read the myriad of arbitrary signifiers - the words, gestures, objects, etc. - with which they are surrounded, such that their arbitrariness is banished and they appear to have the meaning that is conventional?" This called to mind how we tend to project our understanding onto the symbols which can then trigger those meanings subsequently. We read the meanings into the symbols, and yet the projection can be so strong that we forget that the external manifestation is only the signifier and not the sign.

Being aware of structure is one part of being a mathematician. Algebraic manipulation can allow some new property to be apprehended that was not 'visible' before - the transformation was not made on the meaning, but only on the symbols - and that can be very powerful. "The sign $\sqrt{-1}$ represents an unthinkable non-thing. And yet it can be used very well in finding theorems." Johann Lambert, in a letter to Immanuel Kant.

Where are we to look for meaning? Self-reference *is* reference. Mathematics is at least as much in the relationships as in the objects, but we tend to see (and look for) the objects. Relationships are invisible objects to visualise. Caleb Gattegno, writing in his book *The Generation of Wealth* (p. 139), claimed:

> My studies indicate that "mathematization" is a special awareness, *an awareness of the dynamics of relationships*. To act as a mathematician, in other words, is always to be aware of certain dynamics present in the relationships being contemplated. (It is precisely because the essence of mathematics is relationships that mathematics is suitable to express many sciences.) Thus, it is the task of education in mathematics to help students reach

> the awareness that they can be aware of relationships and their dynamics. In geometry, the focus is on the relationships and dynamics of images; in algebra, on dynamics per se.

Mathematics has a problem with reference so it tends to reify its discourse in order to meet the naive desire for reference. "The questions 'What is length?', 'What is meaning?', 'What is the number one?', etc. produce in us a mental cramp. We feel that we can't point to anything in reply to them and yet ought to point to something. (We are up against one of the great sources of philosophical bewilderment: a substantive makes us look for a thing that corresponds to it.)" Ludwig Wittgenstein, *The Blue and Brown Books*.

'I can count faster than I can skip.'

There is an important difference between *wanting* to follow and *having* to follow the teacher. What is the teacher's role and responsibilities in attempting to create meaning for her students? Is it a pretence for the teacher not to be an authority? Who is the custodian of truth in a mathematics classroom?

Finally, two quotations about symbols:

> Civilisation advances by extending the number of important operations we can perform without thinking about them.
>
> (Alfred Whitehead, *Science in the Modern World*)

> Underlying the notations of mathematics there are verbal components; so the mastery of the spoken language means that it is possible to base mathematics on language.
>
> (Caleb Gattegno, *The Awareness of Mathematization*)

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Working Group D

Research Strategies for Pupils' Conceptions in Mathematics

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The report contains four parts and three appendices:

A) A summary of the discussions (C. Janvier and R. Borasi)
B) Crucial questions raised (R. Borasi)
C) Short descriptions of some presentations (R. Borasi)
D) A bibliography collected by Claude Janvier, annotations for 3 provided by R. B.
Appendix A, B and C

A) Summary of the Discussions

Our discussions on conceptions had taken place in a constructivist theory perspective: a theory in which individuals are actively involved when learning. Activity is two-fold: the individual decides to enter into the process of learning and he/she has to integrate into his/her past knowledge the new elements making the resulting knowledge a personal construct. Conceptions are important to consider in mathematics instruction because they influence such mental activities.

The discussions have shown that we could distinguish (for each individual) **cognitive conceptions** from **belief systems.** **Cognitive conceptions** could be considered as elements triggering the action in mathematics reasoning beyond or underneath a set of mathematical concepts. **Belief systems** can be regarded as a set of judgments that control the action of the individual in the sense that they determine his/her willingness to engage, to remain engaged and define ways of engaging (continuity, multiplication, circle...).

Note: Even though such a distinction was discussed and commented upon, **belief systems** and **cognitive conceptions** are not always distinguished in the summary. Firstly, the group has not analysed and described their difference. Secondly, it appeared all along the discussions that, most surprisingly, participants could argue their points and agree having in mind one concept or the other. We note then that the term conception is general enough so as it can convey the idea of beliefs.

Among the **belief systems**, it has seemed relevant to distinguish:

the one about the self
the one about mathematics
the one about school

Combinations of these such as those listed below are important:
self and mathematics,
mathematics in a school setting
school and self,
self and school mathematics.

In the group discussions, we have further enriched the following points: the beliefs about the self and the ones about mathematics (interpreted as of what mathematics is and what doing math is), the ones about school (its implicit role). We have also discussed about the self and mathematics (personal judgments and contrasting them in math and other disciplines), mathematics in a school setting, school and self (general history of success or failures in the regular school program and the expectations derived), self and school mathematics (specific success or failures in mathematics and the expectations derived).

It was proposed to consider conceptions as mental constructs induced by the observers (self-observation included) on the basis on specific behaviour (action and discourse) on the part of the subject. As a result, it is no easy matter to identify and describe a conception. It is important to distinguish between the **individual** conceptions and the **more general** categories that can link several individual conceptions. The latter are more abstract in nature. For instance, mathematics viewed as a set of rules to obey may be concretized differently in each individual. Equally of importance is the fact that conceptions are difficult to imagine without a theory that organizes the observations made with or on a specific individual.

Conceptions in the teacher-student relations

During the group discussions, it became clear that when considering conceptions relevant to mathematics education, teachers conceptions, students conceptions and the relations between these two categories should be considered.

The following three paragraphs are a personal version of the exchanges of ideas. (C.J.)

It has been suggested that when we envisage the teaching-learning relations between the teachers and the students, we must consider **STUDENT COGNITIVE CONCEPTION (SCC)** and **STUDENT BELIEF SYSTEMS (SBS)** not only per se but also as they are an integral part of the **TEACHER BELIEF SYSTEM (TBS)**.

It could be interesting to denote the teacher's version of SCC and SBS as SCC' and SBS'. This part of the **TEACHER BELIEF SYSTEM** also controls the action of the teacher as he/she interacts with students in classroom situations.

If the change of TBS becomes a concern, then one must minimally consider in addition a new variable the **TEACHERS TRAINERS BELIEF SYSTEMS**. Note the importance of this new category.

Are there right or wrong conceptions? Changing them!

The dichotomy right or wrong appears to be incorrect. In fact, the word functional describes more clearly what an appropriate conception is since one can evaluate a conception only in relation with the effect it has in achieving a set purpose. The essential factor is the fact that no value should be attached to a conception in absolute terms.

For example, if we take conceptions about what mathematics is, there should be some room for an informal kind of mathematics that would be distinguishable from "official mathematics". The idea becomes much more to focus our attentions on the mathematics activities such as reasoning, generalizing, formulating hypotheses... If one imagines that official mathematics results from an understanding between mathematicians, mathematics educators and mathematics teachers, one needs that informal mathematics by accepted as valuable by learners. **This is partly what has to be changed.**

If conceptions need to be changed, it must not be forgotten that teachers and students stay actors within the **school framework** constituting a system. And it is clear that taking into account the students' belief systems in the organisation of mathematics teaching would have to produce results within the actual school system. Perhaps, **assessment in schools** should be adjusted.

Changing the students' conceptions required that first of all they **become known** to the teachers or the researchers. **How can we determine conceptions? More, from the actions** then from the dialogue? But anyhow, how much do we need to know about students' specific and individual conceptions since similar past experiences will produce similar conceptions?

Should the students become aware of their own conceptions as a starting point for changing them? In the process of change in students' conceptions, should the teacher expect specific conceptions as goals? Should he/she consider replacement or adding something stronger? It would mean a **certain discontinuity** among conceptions: one being underivable from the others.

At any rate, the working group has agreed that conceptions cannot be directly taught, but rather developed or formed (implicitly or explicitly) in the individuals on the basis of experiences. Individuals are partially aware of their conceptions in the sense that they can only make a partial explicit account of them when solicited.

Acting on the conceptions cannot be achieved without taking into account the ways they develop. As a consequence, we cannot hope to change conceptions only by talking people into them or by teaching them directly. Individuals must be confronted with relevant or meaningful experiences.

It means that changing a belief system consists perhaps in introducing the seed for a new conception to emerge and that, as a result, the subject will be faced with a multiplicity of conceptions "available". This will imply on the part of the subject some abilities to **discriminate and choose how and when to resort to them**. Then the notion of **context awareness** appears to be of prime importance.

During the last sessions , we turn to the questions asked in the description of the working group work appearing in the announcement.

Difficulties involved in research and otherwise

Finding out a belief or a conception in children is time consuming and many teachers are not willing to envisage that it can be worthwhile. On the other hand, as we have said previously conceptions belong to a theory that is the mental framework enabling the researcher and the teacher to detect them. Many have claimed that the presence of a particular conception cannot be assessed if one has not been prepared mentally to notice it and, even then, the fact that a conception is effectively active remains a hypothesis.

Moreover, it is never sure whether a conception does belong to a more general conceptual system, a fact that would be more important for its pedagogical consequences. Also, conceptions are constantly changing and what can be really observed is not the presence of a conception but mainly the movement of conceptions, and the sudden action of one particular conception while the others are likely to be activated but not in action at a particular moment. In fact, we are back to the notion of an efficient model which requires the recourse to an appropriate conception among others.

The formulation or discovery of new conceptions by researchers does not seem to bring about unanimity in the group. On the one hand, some members of the group believe that the formulation of prior hypotheses and the relationships discovered between the previously analysed variables will lead **necessarily** to the conceptions that are involved in the more or less explicit a priori analysis. Others took more optimistic stands. Even though they agree that there is a discontinuity between the previously selected variables and the new variables, some people are able to reach the level of creativity needed for the discovery of a conception.

Are the conceptions personal or do they belong to a category of students?

Changing them

The conflict seems to be the "natural" technique. It involves that the teacher should introduce some facts or events that will clash with or contradict the conception held by the student(s). This method clearly depends on the capacity of the student(s) to be receptive to contradiction. Several examples were provided of students supporting contradictory positions. For instance, a few cases were reported of students believing that a specific fact could be false in arithmetic and true in algebra. In other words, mathematics for many is governed by a "special" logic (or by an absence of logic) which makes the contradiction that the teacher can see or appreciate strictly out of reach of the children.

As far as changing the conceptions is concerned, the "necessary but not sufficient reason" principle was very often mentioned. This was the case for having the students talk about the contradicting fact which is often either neglected or accepted with special sorts of reasoning. This was also the case for the reflection made possible via the use a daily journal. Even the list of key words that are slowly arrived at does not guarantee that the contradiction will be assumed. It is clear that the process requires two phases or stages: first the actions (and done meaningfully) and then the rejection often helped by the contradiction.

Reflection leads to awareness and then the chances that they will use their will to do it is magnified. One needs a motivation to deal with the contradiction. One often accepts things as they are and one doesn't mind since changing would be too costly for several reasons. In fact, there are always many things any individual doesn't understand. Consequently, there is nothing surprising in the fact that the contradiction is not the powerful tool to resolve issues as we would like it to be.

The interviews can be nice (a fruitful and efficient tool) because the students observe themselves. The actions during interviews are more meaningful and some participants think that the contradictions are thus more efficiently made explicit. However, it is not easy for the teachers to make the right moves and conduct interviews adroitly.

As far as the research goes, the word constraint is more appropriate in the circumstances than the vocable difficulty because it reflects the fact that there will always be a limit to the capacity of any research tool. Consequently, one should try to use a research approach that will maximize the outcomes in view to the objectives that are far from being unique.

Personal conclusions (C.J.)

The whole session was a real challenge and very fruitful. It is easily noticeable that the questions specific to research issues were less debated than the more fundamental problems. Thanks to the contributions of everyone, great steps were made in the understanding of the intricate network of conceptions of the many actors in the system.

B) Crucial Questions Raised in the Discussions (R. Borasi)

The questions/issues raised seem to cluster around three fundamental themes/topics:

(a) Determining and studying conceptions
(Whether they are teachers' conceptions or students' conceptions):

- How are conceptions determined:
  - through **verbal** reports of the subject?
  - through observation and interpretation of the subject's **action**?
  - what combination of the two?

- How can we take into account the researcher's frame in "interpreting" conceptions?

- How much do you need to know about **specific students'** conceptions? (Yet at the same time we may want to be aware of the *motivational* value that a teacher's research on his/her students' conceptions may have, **independent of results**, just because it shows the students that the teacher cares for them).

- Connection between "getting at" conceptions and "acting on them" (can we really do one and not the other?).

(b) Studying how conceptions are developed
(mainly for students)

- How does (past) teaching influence the development of certain conceptions?

- Are there crucial times/events/contents which can affect students' conceptions?

(c) "Changing" conceptions

- Can we talk of right/wrong conceptions? (or rather: dysfunctional? unrealistic? inappropriate?). Thus, can we really talk of "changing" conceptions?

- How can we "change" conceptions?

- How can we assess a change of conception?

C) Short Descriptions of Some Presentations (R. Borasi)

About teachers and students' conceptions:

- **J. Bergeron, N. Herscovics and J. Dionne:**
  Description of a course for in-service teachers, consisting essentially of a re-examination of basic math concepts (such as NUMBER) and geared at changing the teachers' conceptions of maths and teaching mathematics.

  (*Research strategies* used to assess change in teachers' conceptions (JD): triangulation of:

(a) how the teacher graded (and justified) a set of students math tests
(b) questionnaire, asking teacher to rank and assign a weight, to the three views of mathematics: traditional (stress: algorithms); formalistic (stress: rigour); constructivist (stress: process)
(c) individual interview, also discussing previous tasks)

- **S. Brown and T. Cooney** (reported by R. Borasi):
  In-depth study of 4 math teachers' belief systems (of math, teaching, teaching math, etc).
  (*Research strategies*: classroom observations + ethnographic interviews, initiated through the teacher's discussion of several "episodes", transcribed; the teacher read the transcript and marked significant statements, and later categorized and labelled those).

- **Erika Kuendinger:**
  Study on teachers' conceptions of themselves as math teachers.
  (*Research strategies*: combination of:
  (a)learning history of the teacher (w.r.t. math)
  (b)questionnaire
  (c)classroom observations (to validate responses on questionnaire))

- **Linda Davenport:**
  An intervention study for students, but also addressing the necessity of dealing with the teachers' conceptions at the same time.
  (*Research strategies*:
  FOR STUDENTS: an open-ended math test and interviews addressing essentially their conception of specific math concepts — ex: asking to explain and draw what 1/2 means.
  FOR TEACHERS: questionnaire (by P. Ernest — see excerpt in Appendix A) addressing explicitly the teachers' conceptions of mathematics, learning math., teaching math and self w.r.t. math).

- **Arthur Powell:**
  Using writing (more specifically, dialogue journals) to help students' learning of mathematics (including a movement towards less dysfunctional conceptions of math.).
  (*Research strategies*:
  Analysis of what the students write (guided by questions, see Appendix B).
  NOTE: to help the students being more reflective and personal in their writing they had:
  •a peer and the teacher responding to their journal
  •a list of "processes involved in thinking mathematically" (see Appendix C) they were supposed to refer to).

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APPENDIX A

QUESTIONNAIRE ON THE TEACHING OF MATHEMATICS

Scale I: Attitude Towards Teaching Mathematics

a.	My knowledge of mathematical concepts is sound enough to teach basic math.	YES!	yes	??	no	NO!
b.	I am very enthusiastic about teaching math to students.	YES!	yes	??	no	NO!
c.	I am confident about my ability to teach math.	YES!	yes	??	no	NO!

Scale II: View of Mathematics

a.	Someone who is good at mathematics never makes a mistake.	YES!	yes	??	no	NO!
b.	Math consists of a set of fixed, everlasting truths.	YES!	yes	??	no	NO!
c.	Math is always changing and growing.	YES!	yes	??	no	NO!

Scale III: View of Teaching Mathematics

a.	If students learn the concepts of math then the basic skills will follow.	YES!	yes	??	no	NO!
b.	Students should be expected to use only those methods that their math books or teachers use.	YES!	yes	??	no	NO!
c.	Students should learn and discover many ideas in mathematics for themselves.	YES!	yes	??	no	NO!

Scale IV: View of Learning Mathematics

a.	In learning math, each student builds up knowledge in his or her own way.	YES!	yes	??	no	NO!
b.	Learning math is mainly remembering rules.	YES!	yes	??	no	NO!
c.	Most errors students make are due to carelessness.	YES!	yes	??	no	NO!

From the work of Paul Ernest

APPENDIX B

Professor Arthur Powell
Developmental Mathematics I

About Journals

You are asked to keep a journal on 8½" x 11" sheets of loose-leaf paper. Generally, one or two sheets will be sufficient for a week's worth of journal writing. Neither your syntax nor grammar will be a concern or checked; my only concern and interest is what you say, not how you say it. You are asked to make, at least, one journal entry for each meeting that we have, and, as a rule of thumb, you need not spend more than five to ten minutes writing each entry. Each week, the latest journal entries will be collected and returned with comments.

The focus of your journal entries should be on *your learning* of mathematics or on the *mathematics* of the course. That is, your reflections should be on what you do, feel, discover, or invent. Within this context, you may write on any topic or issue you choose. To stimulate your thoughts and reflections, here are some questions and suggestions.

1. What did you learn from the class activity and discussion or the assignment?
2. What questions do you have about the work you are doing or not able to do?
3. Describe any discoveries you make about mathematics (patterns, relationships, procedures, and so on) or yourself.
4. Describe the process you undertook to solve a problem.
5. What attributes, patterns, or relationships have you found?
6. How do you feel about your work, discoveries, the class or the assignment?
7. What confused you today? What did you especially like? What did you not especially like?
8. Describe any computational procedure you invent.

APPENDIX C

PROCESSES INVOLVED IN THINKING MATHEMATICALLY (OR HABITS OF THE MIND)

1. Posing problems and questions
2. Exploring a question systematically
3. Generating examples
4. Specializing
5. Generalizing
6. Devising symbols and notations
7. Making observations
8. Recording observations
9. Identifying patterns, relationships, and attributes
10. Formulating conjectures (inductively and deductively)
11. Testing conjectures
12. Justifying conjectures
13. Communicating with an audience
14. Writing to explore one's thoughts
15. Writing to inform an audience
16. Using appropriate techniques to solve a problem
17. Using technical language meaningfully
18. Devising methods, ways of solving problems
19. Struggling to be clear
20. Revising one's views
21. Making connections between equivalent statements or expressions, transformations
22. Making comparisons
23. Being skeptical, searching for counterexamples
24. Reflecting on experiences
25. Suspending judgement
26. Sleeping on a problem
27. Suspending temporarily work on a problem and returning to it later
28. Listening actively to peers

Submitted by Arthur Powell

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ABSTRACT

This paper describes the effects of science teacher subject-matter knowledge on classroom discourse at the level of individual utterances. It details one of three parallel analyses conducted in a year-long study of language in the classrooms of four new biology teachers. The conceptual framework of the study predicted that when teaching unfamiliar subject matter, teachers would use a variety of discourse strategies to constrain student talk to a narrowly circumscribed topic domain. The paper includes the results of an utterance-by-utterance analysis of teacher and student talk in a 30-lesson sample of science instruction. Data are broken down by classroom activity for a number of measures, including duration of utterances, domination of the speaking floor by the teacher, frequency of teacher questioning, cognitive level of teacher questions, and student verbal participation. When teaching unfamiliar topics, the four teachers in this study tended to talk more frequently and for longer periods of time, ask questions frequently, and rely heavily on low cognitive-level questions. The rate of student questions to the teacher was found to vary with the classroom activity. In common classroom communicative settings, student questions were less common when the teacher was teaching unfamiliar subject matter. The implications of these findings include a suggestion that teacher knowledge may be an important unconsidered variable in research on the cognitive level of questions and teacher wait-time. (Author/CW)

Teacher Knowledge and the Language of Science Teaching

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Running head: TEACHER KNOWLEDGE AND THE LANGUAGE OF SCIENCE

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ABSTRACT

This paper describes the effects of science teacher subject-matter knowledge on classroom discourse at the level of individual utterances. It details one of three parallel analyses conducted in a year-long study of language in the classrooms of four new biology teachers. The conceptual framework of the study predicts that when teaching unfamiliar subject matter, teachers utilize a variety of discourse strategies to constrain student talk to a narrowly circumscribed topic domain. This paper includes the results of an utterance-by-utterance analysis of teacher and student talk in a 30-lesson sample of science instruction. Data are broken down by classroom activity (e.g., lecture, laboratory, group work) for a number of measures, including mean duration of utterances, domination of the speaking floor by the teacher, frequency of teacher questioning, cognitive level of teacher questions, and student verbal participation. When teaching unfamiliar topics, the four teachers in this study tended to: talk more often and for longer periods of time, ask questions frequently, and rely heavily on low cognitive-level questions. The rate of student questions to the teacher was found to vary with classroom activity. In common classroom communicative settings, student questions were less common when the teacher was teaching unfamiliar subject matter. The implications of these findings include a suggestion that teacher knowledge may be an important unconsidered variable in research on the cognitive level of questions and teacher wait-time.

Teacher Knowledge and the Language of

Science Teaching

This paper describes the effects of science teacher subject-matter knowledge on classroom discourse at the level of individual utterances. This analysis complements broader analyses of the effects of teacher knowledge at the levels of conversations and multiple-lesson units of instruction, described elsewhere (Carlsen, 1988, 1989a, in press).

The conceptual framework of this study is a sociolinguistic model relating the scientific knowledge of new biology teachers to the structure of classroom discourse: it predicts that teachers will use discourse strategies that vary as a function of their subject-matter understanding. When teaching unfamiliar topics, the framework predicts that teachers will use instructional and discourse strategies that constrain students' opportunities to ask questions. Such strategies may serve to protect teachers from having classroom conversations move into unfamiliar areas. Teacher control of discourse may have pedagogical advantages, but it may also inadvertently undermine the teacher's intention to model the syntax of scientific inquiry.

The analyses in this paper utilize as data every audible recorded teacher utterance and every teacher-acknowledged student utterance in approximately 1500 minutes of recorded classroom conversation, a total of over 4000 utterances. The linguistic imprecision of the term utterance is acknowledged and has been addressed empirically. Using a computer and time-coded transcripts, a variety of algorithms were tested to computationally segment verbatim transcripts into utterances (Carlsen, 1989b). As a result of that work, an utterance is operationally defined here as a stretch of speech terminated by a pause of three seconds or a change in speaker. Although this definition is insensitive to the social functions of talk (in contrast to, for example, the treatment of utterances in speech act theory), it provides several advantages in this analysis.[1] For example, teacher domination of the speaking floor could be easily calculated in a variety of ways.

Pilot Study

A number of teacher strategies for constraining classroom conversations were identified during a pilot study, which contrasted lectures on familiar and unfamiliar topics by six new biology teachers (Carlsen, 1987). Compared to lectures on familiar topics, lectures on unfamiliar topics tended to: be dominated by the teacher (as measured by mean length of utterances, total number of utterances, and other measures), include more teacher questions, include a higher ratio of low cognitive level to high cognitive level questions, elicit fewer student questions and student utterances in general, and elicit shorter student remarks. Analysis was done using time-coded verbatim transcripts of pairs of lessons taught to the same groups of students in public school classrooms.

Although the pilot study findings were intriguing, they were not generalizable for two reasons. First, the pilot study was exploratory and emergent in design, rather than hypothesis-testing. Consequently, it was possible that the findings represent error variance, not knowledge effects. Second, the pilot study investigated a small number of lessons, all lectures. A broader study of a variety of types of instruction was therefore undertaken to provide a more general description of the effects of teacher knowledge on science instruction.

Design and Procedures

The effects of teacher subject-matter knowledge were documented in a year-long study of four new biology teachers. The teachers, all of whom had majored in biology, taught half days in public secondary schools as part of a master's-level teacher education program. As the teachers' university subject-matter supervisor, I regularly visited one of their biology classes and audiotape-recorded 12-15 lessons over the course of the school year. Visits were scheduled to sample a number of curricular topics over a range of teacher subject-matter expertise. The teachers in the study were aware of the discourse and curriculum orientations of the study, but were unaware that relative subject-matter knowledge was an independent variable.

Teacher subject-matter knowledge was assessed in several ways, including a card-sort task of 15 biological topics by self-reported subject-matter knowledge (administered as part of a curriculum workshop during the summer prior to the study), interviews on teachers' sources of knowledge (conducted at the end of the school year), and analyses of undergraduate and graduate transcripts. The principal contrasts in the study were within-teacher contrasts, comparing planning and teaching by individual teachers across a number of topics. Because of the wide variety of topics taught in high school biology, although all of the teachers were knowledgeable about many topics in biology, none had expertise on all of the topics that they were expected to teach.

Audiotape recordings of science lessons were made using a two-track portable tape deck and two microphones, one a wireless unit worn by the teacher. All transcription was done by the investigator. Transcripts of lessons were stored on a computer using a format that provided flexibility of analysis. The transcripts included data on the content of speech (e.g., the words being uttered, some paralinguistic features), the context (e.g., the type of classroom activity being conducted, the name of the speaker), and the time of utterance (data from which pause lengths, rate of speech, and other measures could be computed). Software was written to print transcripts using theoretically-sensible transcription rules, and models of classroom discourse could be played back on the computer screen in real time or at an accelerated rate. The software was also used to annotate transcripts. For example, teacher questions were coded by cognitive level while examining the questions in their conversational context.[2] Finally, statistics and graphs summarizing classroom talk could be easily generated.[3]

For each of the four teachers in the study, two relatively high-knowledge topics and two relatively low-knowledge topics were identified. For each of these four topics, two lessons were tape-recorded; consequently, data can be viewed as coming from a factorial design: 2 lessons x 2 topics x 2 knowledge levels x 4 teachers. For one teacher, Ms. Ross, two low-knowledge audiotapes (on different topics) were unusable, leaving a total of 30 lessons for this analysis. Because the teachers had similar educational backgrounds and because the focus of this paper is on within-subject variation, absolute levels of teacher subject-matter knowledge are not discussed here. As a rough metric, however, the teachers had all had several undergraduate or graduate-level courses (and, in some cases, research experience) on their high-knowledge topics and one or no college courses on their low-knowledge topics.

Among the contexts that are important in this analysis is the classroom activity, defined elsewhere in an analysis of these teachers' lesson plans and teaching (Carlsen, 1989a). Activities of 29 types were identified in this research, but communicatively-similar activities are combined in this paper to make the tables easier to interpret. Examples of combined activities are student Group work, teacher Instructions, Lectures, and Laboratories.

Results

Four general questions organize the findings in this paper. First, what is the relationship between teachers' subject-matter knowledge and the amount they talk during a lesson? The conceptual framework predicts that teachers will talk more when they have low subject-matter knowledge. Dominating conversation is one way of controlling discourse. Findings 1 and 2 address this point.

Second, what is the relationship between teachers' subject-matter knowledge and the frequency with which they question students? The conceptual framework predicts that teachers will ask more questions when they do not understand the subject well. Asking questions may be another way of controlling discourse. Findings 3 and 4 address this point.

Third, how does teacher subject-matter knowledge affect student verbal participation in lessons? The framework predicts that when teachers understand their subject well, their students will more actively participate in classroom conversation. Findings 5 and 6 discuss this prediction.

Fourth, what are the effects of teacher knowledge on the types of questions that are asked in class? The model predicts that when teachers understand their subject-matter well, they will ask questions which permit greater student flexibility in response. Finding 7 examines the effects of subject-matter knowledge on the cognitive level of teacher questions.

The analyses which follow do not report statistical significance, for several reasons. First, the data do not represent an empirically perfect and balanced sample of discourse. Data were collected in natural settings. The teachers and students knew that they were being audiotaped, but the class activities, the patterns of participation, and the topics of the lesson were under the control of the speakers, not the researcher. Hence, for example, although nine laboratory sessions were taped, no labs were taped in low-knowledge classes for Ms. Nims.[4] Second, there is a strong interaction between the teacher and most of the discourse measures used. For example, Ms. Town tended to talk more than Ms. Ross on most measures, including number of questions, number of utterances, and average utterance duration. Third, although data were collected for a large number of utterances (over 4000 utterances varying in duration from less than one second to several minutes), characteristics of the data make the application of parametric statistics problematic. Speech durations, for example, are highly skewed, with most utterances lasting six seconds or less. Finally, from a sociolinguistic perspective, conversation consists of topically-related utterances; any analysis of utterance data which relies upon an assumption of statistical independence is internally inconsistent.

Because the characteristics of classroom discourse depend on the type of activity that is taking place in the lesson, it is not enough to compare utterances in high-knowledge classes with utterances in low-knowledge classes. When that analytic strategy is used, differences in discourse attributable to classroom activity obscure differences attributable to teacher knowledge level. Many of the findings that follow are broken down by classroom activity.

In a similar fashion, I looked for teacher-specific effects on each of the Findings; where effects were found, they are noted. Where they were not found, pooled data are presented. My intention is not to suggest that pooled-teacher data represent a random sample of teachers, nor that the teachers' discourse strategies were identical. Breaking all statistics down by teacher and activity, however, would give the reader little sense of trends in the data.

Amount of Teacher Talk

Finding 1: Teacher talk as a percentage of class time. Table 1 displays teacher talk as a percentage of total class time, for all teachers combined. The data suggest that the teachers tended to talk more in lectures and recitations (shown here as the combined activity, "Lectures") when they were subject-matter knowledgeable than when they were not. This finding is consistent with the conceptual framework. The data suggest that the opposite effect occurred during Laboratories: teachers talked more when they were topic knowledgeable. This finding was not expected, and is explored elsewhere in more detail using discourse analysis (Carlsen, 1990).

Statistics for the other four categories in Table 1 are based on fewer minutes of observation. Group work, Routines and Seatwork activities had more teacher talk in low-knowledge lessons (although the size of the difference for routines was negligible); Instructions (for laboratory and other activities) had more teacher talk in high-knowledge classes. Overall, these statistics are consistent with the prediction of the conceptual framework that teacher knowledge is related to teacher talk, although the knowledge effect in Laboratories is opposite the predicted effect.

Finding 2: The duration of teacher utterances. Table 2 contains data on the average duration of teacher utterances, by class activity and teacher knowledge level. Although the differences were small, individual utterances were longer in low-knowledge classes for six of the seven activities listed in Table 2. Teachers tend to "hold on" to their speaking turns for slightly longer periods in low-knowledge classes. This finding is consistent with the

Table 1

Teacher Talk as a Percentage of Class Time (by Activity)

	Number of Minutes of Instruction		Percent of Time Teacher Talked		Teacher Knowledge Level of Classes with Most Teacher Talk
Activity	High-Knowledge Lessons	Low-Knowledge Lessons	High-Knowledge Lessons	Low Knowledge Lessons	
Group work	73 min.	169 min.	44.7%	53.3%	LOW
Instructions	82	102	70.0	61.3	HIGH
Laboratories	182	103	56.5	48.7	HIGH
Lectures	269	139	64.9	88.5	LOW
Routines	46	50	55.5	56.3	LOW
Seatwork	42	32	41.1	47.8	LOW
Total	694	595	59.1	62.0	LOW

Note: "Percent of time teacher talked" is the sum of the durations of all sequences of teacher talk, divided by the number of minutes of instruction. Sequences of teacher talk were measured in seconds from an initial teacher vocalization to the beginning of the next pause in teacher talk of three seconds or more. Note that, unlike the definition of utterances (as used in Table 2), sequences ignore overlapping student talk.

Table 2

Mean Duration of Teacher Utterances by Activity (in seconds)

Activity	Type of Lesson: High-Knowledge mean	High-Knowledge S.D. (N)[a]	Low-Knowledge mean	Low-Knowledge S.D. (N)	Teacher Knowledge of Classes with the Most Talk
Group work	9.78	10.3 (102)	10.25	22.5 (525)	LOW
Instructions	14.48	28.2 (237)	15.12	23.8 (248)	LOW
Laboratories	9.81	15.1 (630)	7.22	13.0 (415)	HIGH
Lectures	15.12	29.1 (693)	15.37	23.9 (479)	LOW
Routines	14.51	34.7 (106)	15.43	37.4 (110)	LOW
Seatwork	5.43	5.2 (190)	8.62	10.7 (107)	LOW

a Number of teacher utterances, all teachers combined.

conceptual framework of the study. Once again, the situation in Laboratories differed from that in other classes: in labs, the teachers spoke for longer periods of time on high-knowledge days.

The rate of speech in number of words per second and number of transcribed letters per second were also computed. Teacher knowledge level had no effect on either measure of the rate of teacher speech. Given that finding, a more labor-intensive calculation of speech rate in phonemes was not undertaken.

Rate of Teacher Questioning

Finding 3: Rate of teacher questioning. Table 3 presents teacher questioning rates across all activities for each of the four teachers.[5] For three of the four teachers, questioning rates were highest in classes on low teacher-knowledge topics. This finding is consistent with the conceptual framework, which predicts that teachers will ask questions more frequently in low-knowledge lessons than in high-knowledge lessons.

The trend for Ms. Kaye was clearly different from that of the other teachers: the average questioning rate in her high-knowledge lessons was much higher than the rate in her low-knowledge lessons. Inspection of questioning rates revealed that much of the effect was attributable to one lessons, which had a teacher questioning rate twice as high as any of Ms. Kaye's other seven lessons. The anomalous lesson occurred early in the school year, when Ms. Kaye was struggling to establish control of her class. Early in the lesson, the teacher stopped class and sent a disruptive student to the office, then began a 43 minute recitation with frequent low cognitive level teacher questions. In this case, problems of classroom management rather than teacher knowledge appear to have prompted a highly-inquisitorial teacher discourse strategy.

Table 3

Rate of Teacher Questioning by Teacher (questions/min)

	Type of Lesson				Teacher Knowledge of Classes with the Highest
	High-Knowledge		Low-Knowledge		
Teacher	mean	S.D.[a]	mean	S.D.	Ques. Rate
Kaye	0.99	0.52	0.38	0.23	HIGH
Nims	0.55	0.33	0.93	0.20	LOW
Ross	0.35	0.44	0.58	0.64	LOW
Town	0.73	0.40	0.98	0.55	LOW

a Between-lesson deviation. n=4 for all cells except Ross Low-Knowledge, where n=2.

Finding 4: Rate of teacher questioning during whole-class instruction. Although imbalances in sampling across teacher knowledge levels and activities prohibited activity-by-activity analysis of the effects of teacher knowledge on teacher questioning rates, combining communicatively-similar activities permitted some further analysis of teacher questioning rates. For example, combining all lectures, recitations, and teacher-conducted reviews of student homework and examination papers showed that these types of whole-class instruction were characterized by high rates of teacher questioning. The trend described in Finding 3 for all activities pooled also characterizes this subsample of all activities: three of the four teachers asked questions more frequently in low-knowledge classes. Again, Ms. Kaye was the one exception.

Student Questioning Rate

Finding 5: Rate of student questioning by activity. Table 4 presents student questioning rates and shows that the rate at which students ask questions to the teacher is highly dependent on which class activity is taking place. The highest average questioning rates occurred during laboratory activities, followed by seatwork, lectures, and teacher instructions. Students asked questions most frequently when whole-class instruction was not occurring. This finding underscores the necessity of contextualizing data about student talk within classroom activities.

For comparative purposes, Table 5 presents teacher questioning rates for the same activities. Across classrooms and activities, teacher questioning rates were almost always much higher than student questioning rates. The one exception to this generalization was laboratory exercises, where the rate of student questions to the teacher (0.702 questions/min.) exceeded the teacher rate to students (0.600 questions/ min.). Again, this suggests that lab

Table 4

Rate of Student Questioning by Activity (questions per minute)

Activity	Total Number of Classes	Total Number of Minutes	Total Number Questions	Mean Questioning Rate[a]
Group work	10	242	16	0.066
Instructions	24	184	19	0.103
Laboratories	9	285	200	0.702
Lectures	18	408	102	0.250
Routines	25	96	1	0.010
Seatwork	5	74	39	0.527

a This is the overall questioning rate, calculated over all classes and teachers. It is equal to the third column divided by the second column.

Table 5

Rate of Teacher Questioning by Activity (questions per minute)

ActType	Total Number of Classes	Total Number of Minutes	Total Number Questions	Mean Questioning Rate[a]
Group work	10	242	75	0.310
Instructions	24	184	73	0.397
Laboratories	9	285	171	0.600
Lectures	18	408	556	1.363
Routines	25	96	4	0.042
Seatwork	5	74	63	0.851

a This is the overall questioning rate, calculated over all classes and teachers. It is equal to the third column divided by the second column.

activities warrant attention as a special case in the study of communication in science classrooms. Although student group work was in some ways organizationally similar to laboratory exercises,[6] it was characterized by low rates of student questioning and moderate rates of teacher questioning.

Finding 6: Student questioning in student group activities and lectures. Imbalances in sampling prohibited an activity-by-activity analysis of the effects of teacher knowledge on student questioning. Nevertheless, by combining similar activities and pooling the data from all teachers, a rough test of the effects of teacher knowledge can be achieved. For example, when all activities involving student groups are pooled, the student questioning rate in high teacher knowledge lessons was 0.510 questions/minute, and in low-knowledge lessons was 0.391 questions/minute. Table 6 displays the relationship between teacher knowledge and student questioning rates in lectures and recitations. Again, student questioning rates were highest in high teacher knowledge lessons.

Cognitive Level of Teacher Questions

Finding 7: Ratio of high-level to low-level questions. Table 7 presents data on the cognitive level of teacher questions, by teacher and subject-matter knowledge. Each of the four teachers asked relatively more high cognitive level questions when they were teaching familiar topics. Differences between teachers were greater than subject-matter related differences, however.

Assessment of the cognitive level of teacher questions was done, as noted earlier, by the investigator through discourse analysis of questions in their instructional context. This method contrasts sharply with the relatively

Table 6

Student Questioning Rate by Teacher Subject-Matter Knowledge in Lectures (questions/minute)

Teacher Subject-Matter Knowledge	Minutes of Instruction	Number of Questions	Questioning Rate
High	269	77	0.286
Low	139	25	0.180

Table 7

Teacher Questions by Cognitive Level (all activities)

	Frequency of Teacher Questions					
	High-Knowledge Lessons			Low-Knowledge Lessons		
Teacher	High-Cog	Low-Cog	Ratio High:Low	High-Cog	Low-Cog	Ratio High:Low
Kaye	43	168	0.26	8	76	0.11
Nims	48	67	0.72	71	119	0.60
Ross	15	31	0.48	17	42	0.41
Town	72	106	0.68	33	66	0.50
Total	178	372	0.59	129	303	0.46

decontextualized semantic analysis of questions commonly used in process-product research on teaching. Although the findings on the effects of teacher knowledge on question cognitive level are intriguing, the possibility of investigator bias in coding cannot be dismissed.

Discussion

The findings presented in this paper are selected views of discourse from a microanalytic perspective. This analysis of the effects of teacher subject-matter knowledge on classroom language does not clearly indicate that new biology teachers deliver better (or worse) instruction on familiar topics than on unfamiliar topics. Little can be concluded about the quality of instruction by looking at individual utterances.

Nevertheless, this study suggests several ways in which teacher subject-matter knowledge is a critical contributor to the language of the science classroom. First, an heretofore undocumented relationship is identified between teacher subject-matter knowledge and the cognitive level of teacher questions. When these teachers were not topic-knowledgeable, they were more likely to rely upon low-level questions. Second, a relationship is noted between teacher knowledge and teacher domination of the speaking floor. When topic-knowledgeable, these four teachers gave their students more opportunities to speak. This raises the provocative possibility that naturally extended teacher wait times may be a function of teacher knowledge, and that wait-time training in the absence of subject-matter strength may be a wasted effort.[7] Third, this study points out that teachers ought not view questions as sociolinguistically inert. Teachers may ask more questions when they are unfamiliar with the content that they are teaching, and one effect of high levels of teacher questioning--particularly low-level teacher questioning--

appears to be a reduction in students' opportunities to speak.

While conducting these analyses, the data were examined in other ways, with less consistent results. For example, when student questions over all activities were pooled and displayed by teacher subject-matter knowledge. no consistent knowledge effect was seen. Activity-related differences in student questioning rate obscured differences related to teacher knowledge. Clearly, sensitivity to context is critical in discourse analysis even when relatively low-inference measures are used.

These results provide support for a sociolinguistic model of the effects of teacher knowledge on classroom discourse. Nevertheless, this microscopic view of discourse does not prove that teacher knowledge has a big impact on classroom discourse. Part of the difficulty in supporting such a claim is that the types of classroom instruction that teachers choose vary according to the teacher, the students, and the subject matter being studied. In order to make convincing assertions about the frequency of teacher questioning, the types of questions teachers ask, and student participation in lessons, one would need to observe many more lessons. Powerful multivariate techniques might be able to sort out the relative contributions to variance by teacher, lesson, academic level of the students, classroom activity and other factors. Several of the measures considered in this paper would need to be transformed or more carefully controlled in order to meet some of the assumptions of these methods, such as normality and homogeneity of variance.

An alternative approach to the problem of quantitative measurement of teacher and student verbal behaviors would be to more carefully delimit the teacher's task and the verbal behavior of the students. For example, one might ask teachers with different levels of subject-matter knowledge to answer scripted questions delivered by trained student actors (for an example of such a teaching experiment, see Carlsen & Wilson, 1988).

The problem with both of these approaches is that they present a very distorted view of what teaching in real classrooms is like. Numerical analysis of discourse obscures the give-and-take that occurs in actual conversation. Experimental manipulations ignore much of what we know about teaching. The language that teachers and students use is based on routines and shared meanings that are negotiated over time. When a researcher tells a teacher or her students to do certain things, the parameters of discourse are no longer defined by the natural inhabitants of the classroom. Some of those parameters are defined by an outsider.

Although this numerical analysis offers insights concerning the effects of teacher knowledge on classroom discourse, it should be viewed as only one part of a more comprehensive multimethodological analysis of classroom language, which also attended to the curricular decisions of teachers and the substance of discourse.

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Ochs, E. (1979). Transcription as theory. In E. Ochs & B. B. Schieffelen (Eds.), Developmental pragmatics (pp. 43-72). New York: Academic Press.

1. I do not mean to suggest that a computational algorithm can extract utterances from a transcript without reference to any linguistic theory. The transcript used as data for such an algorithm is the product of theory-based human interpretation (Ochs, 1979; Mishler, 1984). At best, an algorithm may be thought of as a tool for subsequent interpretation that is unusually theoretically explicit (Carlsen, 1989b). In this study, pauses and changes in speaker are more heavily weighted than they would be in a conventional analysis.

2. Questions were coded by the investigator as procedural or instructional. Procedural questions were excluded from the analysis in this paper. Instructional questions were further coded by cognitive level. Low cognitive level questions were questions that could be answered by recalling something stated earlier in the lesson or in an assigned reading. High cognitive level questions required evaluation, synthesis, computation, or other higher order thinking. Although this definition may seem fairly straightforward, what may appear on the surface to be a high level question may simply require recall from the previous day's lesson. Therefore, cognitive level coding was done only after: a) listening to the lesson at least three times, b) reviewing the teacher's lesson plans and related instructional materials, and c) interviewing the teacher about the content and the history of the lesson. The pilot study revealed that understanding the context of each question was

critical to cognitive level coding. This context includes the material in prior lessons, assigned readings, the communicative habits of speakers, and discourse from earlier in the lesson.

3. A more complete description of the software and its underpinnings in transcription theory and computer science is found in Carlsen (1989b).

4. To provide anonymity in such a small study, the four teachers were given pseudonyms and a coin toss determined the titles of all four teachers. In reality, two of the teachers were female and two were male.

5. The rates of teacher and student questioning were calculated using only instructional questions. Procedural questions (e.g., "When is this homework due?") were excluded.

6. As classified in this study, group work required two or more students to work together on reports, puzzles, art projects or similar activities, but did not include any data collection or the use of scientific equipment.

7. Space limitations preclude an extended development of this speculation, but it may be summarized as follows: 1) subject-matter knowledgeable teachers naturally wait longer after questions for student responses; 2) long teacher wait times serve as cues to students concerning the types of responses that the teacher expects; 3) wait time training in the absence of teacher subject-matter knowledge changes these cues, encouraging student responses that are inappropriate for the teacher's knowledge; and 4) the effects of wait time training decay with time.