Topic Group A

Implementation of an Apple Centre for Innovation and Year 1 Mathematics Results

W. George Cathcart

University of Alberta

Recent surveys (Petruk, 1985; Hubert, 1988) have shown an exponential increase in the number of computers in schools during the past decade. Hubert (1988) estimated nearly 27 000 computers in Alberta schools at the end of 1987. This translates into about a 1:15 computer to student ratio or an average of about 100 minutes per week of computer access for each child in Alberta. The actual time a child spends at a computer, of course, varies significantly from this theoretical average. Questions remain. If children had continuous access to a computer all day, every day, what could they do? What would they learn? Would their thinking patterns change? How would the school program change?

The Proposal

In an attempt to at least partially answer the broad and open questions stated above, a proposal was submitted to the Apple Canada Education Foundation (ACEF) for the establishment of an Apple Centre for Innovation (ACI) in a third grade classroom. The proposal called for the installation of 1 complete Apple II GS microcomputer workstation for each child in the classroom. The plan was to network the computers and printers and ultimately to incorporate a file server. With respect to the curriculum, the plan was to develop materials that would uniquely integrate the computer into the language arts and mathematics programs.

Implementation

Hardware

The proposal was approved by the ACEF early in the summer of 1987. Thirteen[1] complete workstations were set up on temporary furniture ready for the 26 grade 2/3's first day of classes in September.

Plans for new functional furniture were completed and the furniture ordered. The design consisted of an octagonal desk-like cabinet with rectangular wings emanating from every second side of the octagon to form a 4-student workstation as illustrated in Figure 1. The wings housed the keyboard on a pull out shelf below the table top. Two small shelf-like compartments and a longer one along one side of the rectangular wing provided storage for disk drives and the CPU respectively. Only the monitor sat on top of the wing. The octagonal area in the centre could be used for individual and group work.

Electricians rewired the classroom so that there were no floor or post outlets. AppleTalk cables were also strung through the walls to completely remove all wiring from places where it could be accidentally pulled or tripped over. AppleTalk was also extended to the school office and library at this time.

[1] In year 1 the computer to student ratio was 1:2. In year 2 (1988-89) the ratio was increased to 1:1.

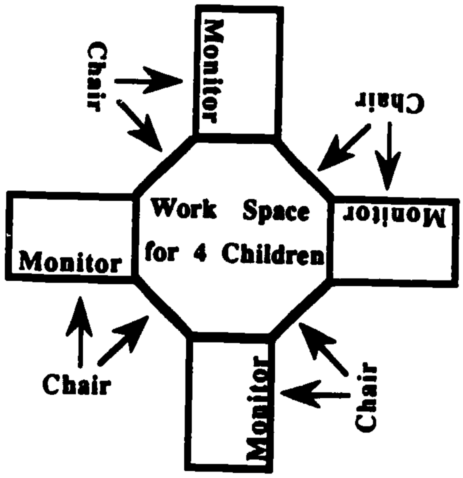

Figure 1. Octagonal 4-Student Workstation

Program

During the summer, the project director (grade 3 teacher) spent many hours preparing language arts materials which would incorporate the use of the computer so that there would be experimental materials in place for the beginning of the school term.

It seemed apparent that if grade 3 children were going to make productive use of the materials, efficient keyboarding skills would need to be developed. A professor in business education at the University of Alberta agreed to teach keyboarding to the 4 grade 2 and 22 grade 3 students in the ACI classroom. She taught a 30 minute keyboarding lesson 4 days each week for 3 months. Afterwards periodic keyboarding review lessons (approximately once a week) were conducted for the remainder of the year.

In language arts the computer was used as a tool in creative writing, responding to reading comprehension exercises, theme and book studies, research reporting, and for a variety of data base activities.

In mathematics the computer was used primarily as a means of providing practice during the first year of the project. Courseware included MAC 3 (Houghton Mifflin), MECC (Minnesota Educational Computing Consortium), graphing activities from National Geographic's Project Zoo, and some practice and problem solving software written by the project director.

Year 1 Mathematics Results

Data is available on the keyboarding and language arts component of the program as well as student and parent attitudes towards the project. The focus of this paper, however, is the performance of the ACI students in mathematics during year 1 of the project.

Instruments

In order to monitor achievement in mathematics a test based on mid to late grade 3 material was developed by the researcher. Part A of the test consisted of 30 open-ended questions. Part B contained 20 multiple-choice questions adapted from released items used by Alberta Education. There were 48 basic fact items in multiple choice format in Part C; 12 facts for each of the 4 operations. Students were given 1 minute to do each section. Parts A and B were not timed.

In addition to the 4 basic facts scores, the test yielded scores on the 5 strands in the Alberta curriculum, number, operations, and properties (25 items), numeration (12 items), graphing (2 items), measurement (6 items), and geometry (5 items). These 50 items formed what will be referred to as the concepts portion of the test. In addition 7 items from these strands were considered to be problem solving and were scored as a sixth strand.

The mathematics test was administered in September 1987 and again in late May 1988 in 2 sittings, usually before and after recess.

The Kuder-Richardson reliability for Parts A and B was 0.78 using pre-test scores and 0.81 using post-test scores.

Control Classes

For comparison purposes, 2 control classes were also given the mathematics test during the same week as the experimental class. One control class was in a neighbouring school, the other was in a very different part of the city. Table 1 shows the age and IQ scores for the 3 classes. There were no statistically significant differences among the 3 groups on the first 3 variables in table 1. There was, however, a significant difference among the classes on non-verbal IQ.

Table 1
Mean Age and IQ Scores

	Control 1	Control 2	Experimental
Age (months)	105.85	109.55	107.71
IQ			
Verbal	114.95	99.91	106.67
Quantitative	113.65	104.23	112.62
Non-verbal	105.45	97.95	112.48

Pre-Test to Post-Test Gains

It was expected that a significant positive gain in mathematics would be made over the course of one school year. A one-way analysis of variance with repeated measures for each class confirmed, in general, this hypothesis but there were some interesting exceptions. The actual (raw) gains made by each group on the mathematics measures are included in Table 2.

The gains made by the experimental class were all statistically significant. The 2 control classes, however, had a total of 7 non-significant gains. All but one of these were in the concepts portion of the test. Both control classes failed to register significant gains in graphing and geometry, the 2 strands with the least number of test items. Control group 2 did not reach the level of statistical significance ($p \leq 0.05$) on measurement and control class 1 did not reach that level on problem solving. Control 1 also failed to reach significance on the addition section of the basic facts test. Table 2 also contains a summary of the one-way analysis of variance with repeated measures.

Relative to the 2 control classes, the experimental group improved its rank from pre-test to post-test on 9 of the 13 scales, maintained its rank (highest) on 3 of the measures and declined in rank (highest to middle) on the numeration subscale.

Comparison of Classes

There were no significant pre-test differences (one-way ANOVA) among the groups on the major scales (concepts, facts, total score). There were, however, significant differences on 2 of the subscales of the concepts test (numeration and geometry) and on the subtraction section of the basic facts test.

The major analysis involved a two-way ANOVA (groups (3) by repeated measures (2)). A summary of this analysis is included in Table 3.

Table 2
Means, standard deviations, gains, and anova summary for each group

CLASS/VARIABLE	MEANS PRE	MEANS POST	GAIN	ST.DEV. PRE	ST.DEV. POST	F
CONTROL 1						
CONCEPTS						
Number, Operations, and Properties	12.40	17.70	5.30	3.66	4.01	55.54***
Numeration	6.15	8.95	2.80	1.60	1.47	39.62***
Graphing	0.75	1.25	0.50	0.79	0.64	4.13
Measurement	2.25	3.50	1.25	1.12	1.19	23.06***
Geometry	3.70	3.65	-0.05	0.92	0.93	0.03
Problem Solving	4.00	4.55	0.55	1.52	1.50	2.81
TOTAL CONCEPTS	25.25	35.05	9.80	6.00	6.36	59.71***
BASIC FACTS						
Addition	10.45	10.90	0.45	2.50	1.74	1.98
Subtraction	7.95	9.05	1.10	3.15	2.59	5.62*
Multiplication	3.60	5.85	2.25	2.78	3.10	8.83**
Division	1.20	5.55	4.35	1.58	3.47	20.75***
TOTAL FACTS	23.20	31.35	8.15	6.60	8.57	22.61***
total score	48.45	66.40	17.95	10.48	12.98	84.86***
CONTROL 2						
CONCEPTS						
Number, Operations, and Properties	11.09	15.73	4.64	4.24	4.63	37.46***
Numeration	5.91	6.96	1.05	2.62	2.15	5.00*
Graphing	0.64	0.77	0.13	0.73	0.69	0.59
Measurement	2.36	2.64	0.28	1.22	1.40	0.68
Geometry	2.36	2.59	0.23	1.40	1.40	0.63
Problem Solving	3.05	3.59	0.54	1.68	1.65	5.01*
TOTAL CONCEPTS	22.36	28.68	6.32	8.42	7.83	42.03***
BASIC FACTS						
Addition	10.00	11.59	1.59	2.29	1.01	12.53**
Subtraction	5.91	10.14	4.23	2.33	2.23	65.59***
Multiplication	2.91	5.14	2.23	1.54	2.34	16.87***
Division	1.77	5.41	3.64	1.51	3.08	27.63***
TOTAL FACTS	20.59	32.09	11.50	5.10	6.74	103.65***
total score	42.96	60.77	17.81	11.29	12.31	143.05***
EXPERIMENTAL						
CONCEPTS						
Number, Operations, and Properties	9.76	18.67	8.91	4.77	3.43	144.92***
Numeration	7.43	8.62	1.19	1.81	1.83	16.89***
Graphing	0.33	0.95	0.62	0.48	0.67	14.70**
Measurement	2.05	3.14	1.09	0.97	0.91	36.48***
Geometry	2.76	3.67	0.91	1.09	1.11	21.75***
Problem Solving	3.38	4.95	1.57	1.75	1.32	20.28***
TOTAL CONCEPTS	22.81	35.05	12.24	8.56	5.97	120.09***
BASIC FACTS						
Addition	10.62	11.86	1.24	2.54	0.48	6.32*
Subtraction	8.38	10.71	2.33	3.46	2.05	12.66**
Multiplication	3.10	7.10	4.00	2.59	3.83	39.53***
Division	1.62	6.38	4.76	2.04	4.56	31.77***
TOTAL FACTS	23.71	36.05	12.34	8.84	9.17	119.49***
TOTAL SCORE	46.52	71.10	24.58	15.96	12.66	223.39***

* $p \le 0.05$ ** $p \le 0.01$ *** $p \le 0.001$

Table 3
Anova summary (Group by Repeated Measures)

Scale	Source	df	F	P
CONCEPTS				
Number, Operations, and Properties	Group (G)	2,62	0.98	0.38
	Measures (M)	1,63	216.66	0.00
	G X M	2,63	9.66	0.00
Numeration	G	2,62	4.71	0.01
	M	1,63	50.31	0.00
	G X M	2,63	5.65	0.01
Graphing	G	2,62	3.08	0.05
	M	1,63	13.62	0.00
	G X M	2,63	1.64	0.20
Measurement	G	2,62	0.84	0.44
	M	1,63	31.96	0.00
	G X M	2,63	3.86	0.03
Geometry	G	2,62	7.57	0.00
	M	1.63	5.80	0.02
	G X M	2,63	3.51	0.03
Problem Solving	G	2,62	2.89	0.06
	M	1,63	24.95	0.00
	G X M	2,63	3.68	0.03
TOTAL CONCEPTS	G	2,62	2.57	0.09
	M	1,63	214.23	0.00
	G X M	2,63	7.08	0.00
BASIC FACTS				
Addition	G	2,62	0.68	0.51
	M	1,63	19.27	0.00
	G X M	2,63	1.84	0.17
Subtraction	G	2,62	2.31	0.11
	M	1,63	63.38	0.00
	G X M	2,63	8.04	0.00
Multiplication	G	2,62	1.13	0.33
	M	1,63	57.56	0.00
	G X M	2,63	2.49	0.09
Division	G	2,62	0.43	0.65
	M	1,63	78.57	0.00
	G X M	2,63	0.47	0.63
TOTAL FACTS	G	2,62	1.45	0.24
	M	1,63	191.45	0.00
	G X M	2,63	2.76	0.07
TOTAL SCORE	G	2,62	1.93	0.15
	M	1,63	423.71	0.00
	G X M	2,63	5.20	0.01

Testing Effects

The two-way ANOVA produced significant effects due to the testing on all 13 measures. In all cases the post-test composite mean was significantly higher than the pre-test composite mean.

Group (treatment) Effects

The study was primarily interested in differences among the groups. There were significant main effects due to treatment on 3 of the 13 scales; numeration, graphing, and geometry. All of these were components of the concepts test. There were no significant group effects on the basic facts test.

A Scheffe post-hoc pairwise comparison of unweighted main effects on the numeration subscale found a significant difference between control class 2 and the experimental class ($p \leq 0.02$). The experimental class had a significantly higher composite mean than control group 2.

The Scheffe comparison of unweighted main effects due to treatment on the graphing subtest found no significant differences among the groups. The difference between the experimental group and control 1 came the closest to reaching significance ($p = 0.08$). On the geometry subtest the significant main effects were primarily due to a significant difference between the 2 control groups although the difference between the experimental group and control class 2 was close to significance ($p = 0.06$).

Interaction Effects

There were significant interaction effects on 8 of the 13 scales used in the study. The graphs in Figure 2 picture the interaction for the 3 major scales (concepts, facts, total score). Figure 3 contains a graph of the interaction on the 3 subscales of the concepts test which had significant main effects due to treatment.

The interaction on the facts test was not significant. The interaction on the concepts test and the total mathematics score seems to be due to the steeper slope (greater gain) of the experimental group. On 2 of the 3 concepts subscales where there were significant treatment effects there were also significant interaction effects. On the numeration subtest the interaction seems to be due to the greater gain of control group 1 and on the geometry subscale it seems to be a greater gain by the experimental class that caused the interaction.

Summary and Discussion

On mathematics concepts, basic facts, and on the total score, the experimental class (extensive use of the microcomputer) made greater gains than 2 control classes (incidental computer use). In fact the experimental class made greater gains on 9 of the 13 measures used in the study.

There were statistically significant differences among the groups (based on a two-way ANOVA with repeated measures) on 3 of the 13 scales used in the study. These were the numeration, graphing, and geometry subtests of the mathematics concepts test. On the first 2 subtests the experimental group significantly outperformed one but not both control groups. On the geometry subtest, the difference was due to the difference between the 2 control groups.

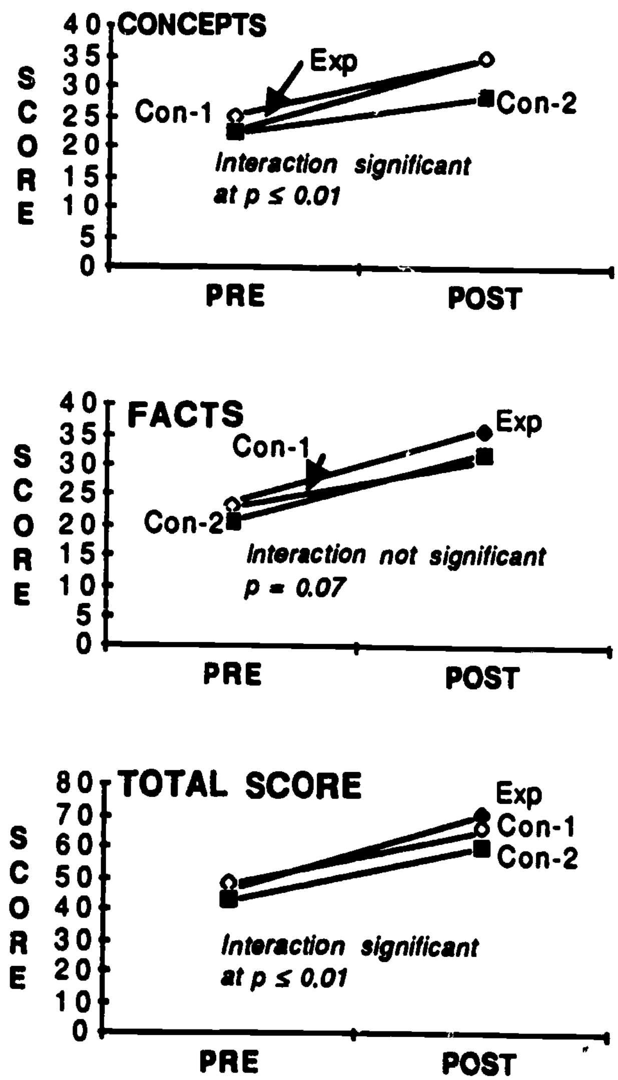

Figure 2. Interaction Effects on Major Scales

The results of this study lend some, although not strong, support to the thesis that supplementary computer experiences enhances mathematical skills. Given the nature of the treatment (major emphasis on the language arts and a lesser emphasis on mathematics), the results are not surprising. If the same time and energy could have been given to mathematics as to language arts, the results may have been more definitive.

Computer use was carefully controlled, the teacher factor was minimally controlled, but there are many variables such as teaching style, school philosophy, use of manipulatives, and others which were not controlled in this study. These certainly could have a bearing on the results.

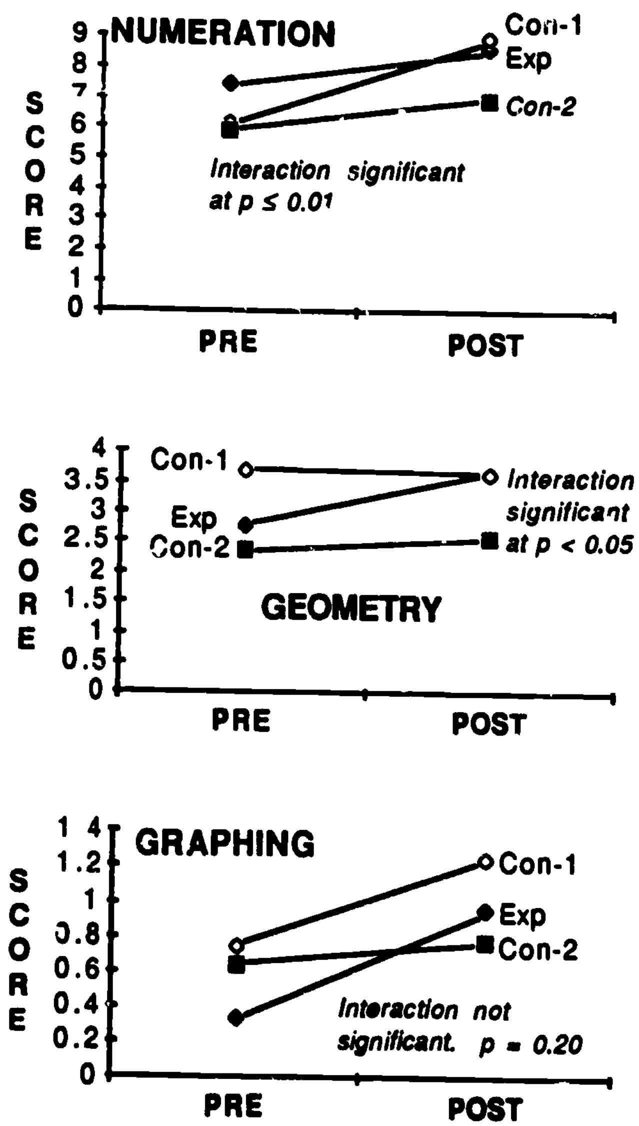

Figure 3. **Interaction Effects on Subscales which had Significant Treatment Effects**

References

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Topic Group B

A Model to Describe the Construction of Mathematical Concepts from an Epistemological Perspective

J. Bergeron

Université de Montréal

N. Herscovics

Concordia University

Introduction[1]

For the last fifteen years, we have witnessed extensive discussions on the need to define what is meant by "understanding". Ephraim Fischbein (1978) stressed the importance of **intuition** for the understanding of mathematics and Richard Skemp (1976) provided an early model which distinguished between **instrumental understanding** ("rules without reason") and **relational understanding** ("knowing what to do and why"). Using Skemp's model and combining it with Bruner's distinction between analytic thinking and intuitive thinking (Bruner, 1960), Byers and Herscovics (1977) suggested the tetrahedral model of understanding which identified four complementary modes of understanding: **instrumental, relational, intuitive, and formal.** Later on Skemp (1979) extended his model to three modes of understanding (instrumental, relational, and logical) each one subject to two levels of thinking (intuitive and reflective) and, three years later, he added a fourth mode, that of symbolic understanding (Skemp, 1982). A more extensive survey can be found in **Models of Understanding** (Herscovics & Bergeron, 1983).

The reasons for finding better answers to the question "What does it mean to understand mathematics?" are not purely aesthetic and academic, they are also very practical. Without some answer to this question, one can hardly expect to train teachers to "teach for understanding". The training of teachers in the analysis of mathematical concepts through the use of models of understanding was attempted with a class of practising primary school teachers (Bergeron, Herscovics, & Dionne, 1981). Results proved to be most promising since these teachers ended up de-emphasizing the value of the written answer and instead assigned equal importance to the thinking processes underlying these answers (Herscovics, Bergeron & Nantais-Martin, 1981).

The early models of understanding were heavily oriented towards **problem solving** and proved inadequate to describe the comprehension involved in **concept formation** (Bergeron & Herscovics, 1981). Thus, a new model identifying **four levels** of understanding in the construction of mathematical concepts (**intuitive understanding, initial conceptualization, abstraction, and formalization**) was suggested (Herscovics & Bergeron, 1981). In the early eighties, this initial model was constantly improved in the sense of providing clearer criteria for the different levels of understanding (Herscovics & Bergeron, 1982, 1983, 1984). By 1982 we had characterized our second level of understanding as "**procedural understanding**" instead of "**initial conceptualization**", and by 1983 we were distinguishing between "abstraction" in the psychological sense (detachment from the concrete) and "mathematical abstraction" (the construction of mathematical invariants). In 1984 we adjusted our definition of "procedural

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Based on the analysis of number developed in this paper, we undertook an international study to assess the kindergartners' knowledge of natural number. Some highlights of this study appear in a companion paper "The kindergartners' construction of natural numbers: an international study".

understanding" to include both the acquisition of mathematical procedures as well as the ability to use these appropriately.

The continued attempt to provide an epistemological analysis of the various conceptual schemata taught at the elementary level has proved to be the whetstone on which we have refined our evolving model. Of course, we are using the term "epistemological" very broadly in the sense of "growth of knowledge" but also within a pedagogical context which acknowledges the impact of instruction. On one hand, our model of understanding provides us with a new perspective raising new questions such as "What kind of knowledge could be considered as evidence of intuitive understanding?". On the other hand, the research results force us to refine our initial model.

The objective of this paper is to present our two-tiered model of understanding and to illustrate how it can be used to describe the understanding of a fundamental mathematical concept such as natural number.

The Understanding of Preliminary Physical Concepts

Back in 1983, we described intuitive understanding by pointing out that

> For most of the (arithmetical) notions taught, one can find some pre-concepts which can be viewed as embryonic to the conceptual schema whose construction is intended...There is not yet any (numerical) quantification, maybe at most some simple (visual) estimation. These are situations which lead to what Ginsburg (1977) describes as 'informal knowledge'.
>
> (Herscovics & Bergeron, 1983, p. 77)

The above characterization of intuitive understanding served us well, since it forced us to search for appropriate situations in the child's experience that could be used as starting points for each intended concept. The acquisition of new knowledge would thereby be endowed with meaning and relevance. This last year, we achieved some kind of breakthrough when, in our analysis of the number scheme we decided to apply our existing model to the two notions we consider as pre-concepts of the number concept.

We have identified the notion of **plurality**, that is, the distinction between one and several, and the notion of **position** of an element in an ordered set, as two physical concept preliminary to the concept of number. We can then define 'number' teleologically, that is in terms of its initial uses and functions, as a **measure of plurality** and as a **measure of position.**

Applying our existing model to the notion of plurality and to the notion of position meant we had to find non-numerical criteria which might be interpreted as representing intuitive understanding, procedural understanding, and logico-physical abstraction of these two concepts. We would not attempt to find a fourth level of understanding, that of formalization, since, in effect, the construction of the number concept could be viewed as the mathematization of plurality and position. We have been successful in identifying the needed criteria and in converting these into tasks which have been used to assess the kindergartners' understanding of plurality and position. We provide here a brief summary of the criteria and tasks used to evaluate each level of understanding.

Regarding the **intuitive understanding** of plurality, quite early in our work we had designed tasks involving discrete sets that children could compare on the basis of visual estimation in order to decide which one had **more**, which one had **less**, where there were **many**, where there were **few**, or if one set had **as many** objects as another one. More recently, we developed tasks in which children used visual estimation to decide if an object was **before** (or in front), **after** (or behind) another one, if two objects were **together** (or at the same time), whether an object was **between** two other ones. The ability to estimate these notions visually could be considered as evidence of intuitive understanding of plurality and position since neither needed to be determined with any precision, rough approximations proving to be sufficient.

To identify a level of **procedural understanding** of plurality and position one had to find logico-physical procedures that were non-numerical, in which no counting was involved, but which provided precision to the notions introduced at the intuitive level. Procedures based on one-to-one correspondences answered this requirement since they provided accuracy and reliability to questions regarding plurality and order. Our investigations have shown that by the time children complete kindergarten, most of them can use one-to-one correspondences to generate sets that are larger, or smaller, or equal, or that have one more element, than a given set. They can also generate ordered sets subject to positional constraints such as before, after, at the same time.

Abstraction in the logico-physical sense was also easy to identify. We used as criterion the children's ability to perceive the invariance of plurality or position under various surface or figural transformations. The logico-physical processes which enable them to overcome the misleading information they obtain from their visual perception provides them with more stable conceptions of plurality and position. The **abstraction of plurality** was assessed through tasks in which sets of objects laid out randomly were rotated and displaced within the same space, dispersed, and contracted. Two tasks dealt with the visual impact of the elongation of a row, the first task involved a single row, and in the second task, one row was stretched while another one was kept fixed (Piaget's conservation of plurality). The invariance of plurality with respect to the visual perception of the elements was tested by hiding some of the objects. The **abstraction of position** was evaluated by assessing the invariance of position with respect to the elongation of a row, with respect to the visibility of all the objects in a row, and with respect to conservation of position when one of two parallel rows was translated. The abstraction of position was also assessed by verifying if the child was aware that the position of an element changed when one of the preceding objects was removed.

As can be seen from the above outline, it is quite possible to identify criteria that will clearly describe three levels of understanding of preliminary physical concepts. These three levels replace advantageously the level of understanding which in our previous model we described as "intuitive understanding of a mathematical concept" since they enable us to provide a full blown epistemological analysis of the preliminary concepts rather than view them as merely the initial embryonic stage in the construction of the intended mathematical concept. Of course, a model of understanding applied to physical notions needs to be distinguished from a model applied to mathematical ones. For instance, the procedural understanding evidenced by the use of a 1:1 correspondence between two sets of objects can be considered as a logico-physical procedure whereas the 1:1 correspondence between objects and the number-word sequence (counting) is of a logico-mathematical nature. A similar distinction applies to the construction of invariants. These comments provide us with the following description of the levels of understanding of physical concepts:

Intuitive understanding refers to a global perception of the notion at hand; it results from a type of thinking based essentially on visual perception; it provides rough non-numerical approximations.

Procedural understanding refers to the acquisition of logico-physical procedures which the learners can relate to their intuitive knowledge and use appropriately.

Logico-physical abstraction refers to the construction of logico-physical invariants (as in the case of the various conservations of plurality and position), or the reversibility and composition of logico-physical transformations (e.g. taking away is viewed as the inverse of adding to; a sequence of increments can be reduced to fewer steps through composition), or as generalization (e.g. perceiving the commutativity of the physical union of any two sets).

The Understanding of the Emerging Mathematical Concepts

We distinguish mathematical concepts from physical concepts when explicit mathematical procedures and invariants are involved. We then can identify three distinct constituent parts of understanding: procedural understanding, logico-mathematical abstraction, and formalization. Once again, we illustrate this with the number concept. In our opinion, the number concept is present only when enumeration (counting) is involved. Of course, knowledge of the number-word sequence by itself does not imply numerical knowledge. However, it is an essential pre-requisite to counting. Fuson, Richards & Briars (1982) has described different skills in the child's handling of the number-word sequence (reciting from one, reciting on from a given number, reciting backwards, etc.).

The **procedural understanding** of number involves explicit counting procedures. Since we defined number as a measure of plurality and of position, we had to design various tasks in which all the counting procedures could be used. For instance, asking children to count up a pile of chips "as far as they could go" would assess their mastery of the counting-from-one procedure and their numerical range. Asking them to generate a set of a given cardinality or to identify an object of a given position would assess their ability to count and stop at a given number. A task which might favour the counting-on procedure was developed (cf. Steffe et al, 1983) a row of thirteen chips was glued to a cardboard and the first six were hidden in front of the children. They were reminded how many were hidden and then asked: How many there were altogether? Could they find the ninth chip? Could they find the position of an indicated chip? Another task which might favour counting backwards also involved a row of twelve chips, some of which were hidden: with six chips hidden and the tenth chip pointed out. The children would then be asked: How many are hidden? With three chips hidden and the tenth chip identified, they were asked to find the seventh chip and afterwards, to find the position of an indicated chip. Finally, even more sophisticated tasks were selected, tasks that would involve double counting forwards or backwards. For instance, children might be asked to count out loud five number words from a given number, or to find how many number words are between two given ones. As can be seen, many tasks can be designed to evaluate procedural understanding.

In view of our definition of number as a measure of plurality and of position, the logico-mathematical abstraction of number must reflect both the invariance of plurality and the invariance of its measure, leading to the **abstraction of cardinal number**. It must also reflect both the invariance of position and the invariance of the measure of position, leading to the **abstraction of ordinal number**.

Over twenty years ago, Piaget's collaborator Pierre Gréco (1962) felt the need to distinguish between plurality and the measure of plurality. He modified the original conservation task involving two equal rows of chips by asking the children to count one of the rows before stretching the other one; he then asked how many chips were in the elongated row while screening it from view. Those who could answer the question were said to **conserve quotity**. Gréco found that many five-year-olds claimed that there were seven chips in each row but that the elongated row had more. Thus, these children **conserved quotity without conserving plurality**. For these children, to conserve quotity simply meant that they could maintain the numerical label associated with the elongated row, but their count was not yet a measure of plurality, since they thought that the plurality had changed. It is only when both plurality and quotity are conserved, when both invariances are perceived, that number becomes a measure of plurality. At that stage, one can claim that the child has achieved a logico-mathematical abstraction of cardinal number. Of course, the Piaget and the Gréco tasks are not the only ones by which abstraction of cardinal number can be assessed. These involve a specific type of transformation. All the other tasks previously used to assess the invariance of plurality can also be used here by modifying them to include enumeration.

An entirely analogous approach can be used to describe the logico-mathematical abstraction of ordinal number. Similar to the notion of quotity, one can introduce its parallel in the context of position. We define **ordity** as the ability to maintain the numerical label associated with the position of an element in an ordered set subject to various transformations such as elongation, translation, hiding part of a row. And of course, there are children who perceive the invariance of ordity without perceiving the invariance of position. Only when both are present can one claim to have achieved a logico-mathematical abstraction of ordinal number.

By the **formalization** of number, we mean the gradual development of various mathematical notations. When asked to send a message indicating how many objects are in front of them, children will represent each one by a drawing and later on by a tally mark. Once they learn to write their numerals, they may write the sequence 1, 2, 3, 4, 5, 6, 7 to represent the cardinality of a set of seven objects, thereby indicating their need to rely on a 1:1 correspondence between the objects and the numerals; by the end of kindergarten, most of them can use the numeral '7' with its intended cardinal meaning.

In fact, many of them can write down numbers exceeding nine. Of course, this does not imply any awareness of place value notation. Nevertheless, it indicates that they perceive the concatenation of two digits globally (e.g. '12' no longer means 'one and two' but 'twelve'). However, even the understanding of positional notation grows gradually: from mere juxtaposition (numerals are written next to each other without regard to relative position), through a chronological stage (the order of production prevails over the relative position), to a final conventional level.

The above discussion of number suggests the following description of the understanding of mathematical concepts:

Procedural understanding refers to the acquisition of explicit logico-mathematical procedures which the learner can relate to the underlying preliminary physical concepts and use appropriately.

Logico-mathematical abstraction refers to the construction of logico-mathematical invariants together with the relevant logico-physical invariants (as in the abstraction of cardinal number and ordinal number), or the reversibility and composition of logico-mathematical transformations and operations (e.g. subtraction viewed as the inverse of addition; strings of additions reduced to fewer operations through composition), or as generalization (e.g. commutativity of addition perceived as a property applying to all pairs of natural numbers).

Formalization refers to its usual interpretations, that of axiomatization and formal mathematical proof which, at the elementary level, could be viewed as discovering axioms and finding logical mathematical justifications respectively. But two additional meanings are assigned to formalization, that of enclosing a mathematical notion into a formal definition, and that of using mathematical symbolization for notions for which prior procedural understanding or abstraction already exist to some degree.

As can be seen from the first two definitions above, the understanding of a mathematical concept must rest on the understanding of the preliminary physical concepts. We thus end up with a **two-tiered model of understanding**. However, this does not imply that the understanding of a mathematical concept needs to await the prior three levels of understanding of the preliminary physical concepts. For instance, our research shows that kindergartners master counting procedures and the formalization of number well before they perceive all the invariances of plurality and position. Nevertheless, due to the very definition of logico-mathematical abstraction, this component part of understanding cannot occur without the prior logico-physical abstraction of the preliminary physical concepts. The non-linearity of our model is expressed by the various arrows in the following diagram:

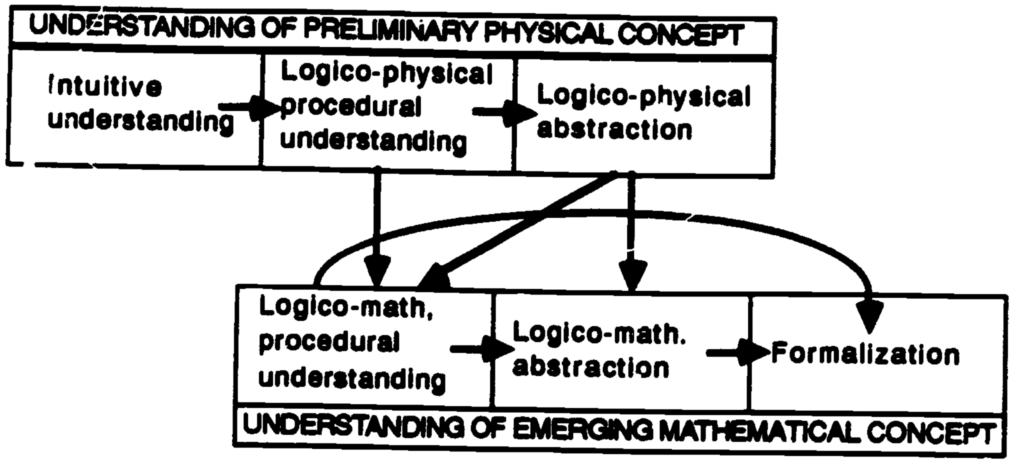

Figure 1. The two-tiered model of understanding

Two further important changes in the model need to be brought out. The first one pertains to our definition of 'formalization'. Whereas in our earlier models we required prior abstraction in order to recognize formalization as comprehension, we have now loosened this restriction to include procedural understanding. For instance, when sending a numerical message, the child may write out the whole sequence of digits and this can be considered as a formalization of the counting procedure. The other change is more general. We have avoided using the word 'level' to describe the understanding of mathematical concepts and replaced it with the expression 'constituent part' in order to prevent an overly hierarchical interpretation.

The following tables summarize the criteria used to assess the child's understanding of natural number:

Table 1. Understanding of preliminary physical concepts of number

INTUITIVE UNDERSTANDING	PROCEDURAL UNDERSTANDING (LOGICO-PHYSICAL)	ABSTRACTION (LOGICO-PHYSICAL)
Plurality: Visual determination of more, less, many, few, as many	1-1 correspondence used to generate a set that has more, less, as many as, one more than a given set	Invariance of a single set wrt dispersion, displacement within a given space, rotation, elongation, the non-visibility of some of its elements. Invariance of plurality in Piagetian test
Position: Visual determination of before, after, between, at the same time, first, last	1-1 correspondence used to generate an ordered set subject to positional constraints (before, after, at the same time)	Invariance of position of an object in a single row when the row is elongated, when some of its elements are visible. Invariance of position of two corresponding objects when one row is moved forward. Variability of the position of an object in a row when the first element of the row is removed.

Table 2. The understanding of number

PROCEDURAL UNDERSTANDING (LOGICO-MATHEMATICAL)	ABSTRACTION (LOGICO-MATHEMATICAL)	FORMALIZATION
Counting from 1; from 1 and stopping at a given number; from a given number M; from a given number M and stopping at a given number N>M; backward recitation from a given number; from a given number N and stopping at a given number M<N **Double-counting** Recitation of the N number-words following a given number-word; Recitation from A to B>A, keeping track of how many number-words are pronounced; Backwards recitation of the N number-words preceding a given one; Backwards recitation from B to A, keeping track of how many number-words are pronounced.	**Cardinal number:** uniqueness of cardinality; invariance of card of a row wrt the direction of the count; perception of the invariance of **plurality and quotity** of a single set wrt dispersion, wrt elongation, wrt the non-visibility of some elements; perception of the invariance of **plurality and quotity** of two equal rows when one of them is elongated; synthesis of counting-on and cardinality. **Ordinal number:** perception of the variability of **position and ordity** of an object when the first object of the row is removed; perception of the invariance of **position and ordity** of a single set wrt elongation, or the non-visibility of some of its elements; perception of the invariance of **position and ordity** of two corresponding objects when one row is moved forward.	Ability to recognise a numeral and generate a corresponding set of objects or identify an object of corresponding rank; ability to represent the cardinality of a set: - by drawing an equivalents set of pictures of the objects - by putting down an equivalent set of tally marks - by writing out the equivalent sequence of numerals - by writing a numeral as the cardinal of the set; ability to write the rank of an object in a given row; **Positional notation** (for those who can recognize or write two-digit numbers) - as juxtaposition - chronologically - conventionally

By way of conclusion

The construction of a fundamental concept in mathematics involves many different ideas that need to be related to each other into some kind of cognitive grid. As opposed to the acquisition of isolated parcels of knowledge, the development of fundamental mathematical concepts involves linking together several different notions into some organic whole forming some kind of cognitive matrix. But all this knowledge needs to be significant and relevant. This can be achieved only when it can be related to problem-situations, that is, situations in which this knowledge provides answers to some perceived problem. For instance, what would be the point in learning the sequence of the number words unless these were used to answer questions about cardinality and rank? We use the expression **'conceptual scheme'** to convey both the idea of a cognitive grid or cognitive matrix, as well as the relevant problem-situations.

As can be inferred from the theoretical part of our paper, our intention has been to study the learner's construction of a conceptual scheme and not just a part of it. It is with this objective in mind that we have developed our models of understanding. These models were to provide a frame of reference in which we could follow each learner's construction. In this sense, our models can be called 'epistemological'. Of course, this type of work actualizes what is meant by a constructivist approach to mathematics education. For instance, since the acquisition of fundamental concepts taught in primary school mathematics require two or three years, the conceptual analyses obtained by using our models provide the teachers with an overview of a given conceptual scheme. Without diminishing the importance of mathematical procedures, our models situate these in a broader context and emphasize the thinking processes involved. We thus realize a Lakatosian (Lakatos, 1976) perspective in the co' xt of concept formation.

Our latest model of understanding suggests a basic structure that distinguishes between a first tier dealing with preliminary physical concepts and a second tier involving the emerging mathematical concept. This distinction is somewhat analogous to the one Piaget makes between 'simple' abstraction (or 'physical' abstraction) based on the properties of objects, and 'reflective' abstraction (or 'logico-mathematical' abstraction) that is based on the coordination of actions or operations. This distinction can be justified as long as actions and operations are in the mental domain. However, one cannot justify it as readily when the actions and operations are carried out on concrete objects. In fact, Piaget has acknowledged this when he suggested two forms of reflective abstraction:

> We will speak in this case of "pseudo-empirical abstractions" since the information is based on the objects; however, the information regarding their properties results from the subject's actions on these objects. And this initial form of reflective abstraction plays a fundamental psycho-genetic role in all logico-mathematical learning, as long as the subject requires concrete manipulations in order to understand certain structures that might be considered too 'abstract'. (Piaget, 1974, p.84, our translation)

The existence of two tiers in our model takes into account the subject's action on his or her physical environment. The two forms of reflective abstraction are comparable to the two aspects of understanding in our model: Piaget's pseudo-empirical abstraction is equivalent to our logico-physical abstraction, his logico-mathematical abstraction is the same as the one we mention in our second tier.

Our new model has several pedagogical implications. It links up explicitly the children's mathematics to their physical world and thus strongly suggests using the latter as a starting point in the construction of their mathematical concepts. One cannot over-emphasize the importance of this approach, for Ginsburg's work (1977) has brought to light the gap that may exist in the children's mind between their school mathematics and their 'informal' mathematics, that is, those acquired outside of school. The informal knowledge that Ginsburg identifies as System 1 corresponds to what we call 'preliminary physical concepts'. Hence, with its two tiers, our model encompasses the two forms of knowledge.

Other implications of a more practical nature involve applications to instruction and evaluation. While our model of understanding is definitely not a model of instruction, nevertheless, its use for the analysis of a conceptual scheme brings out several aspects of understanding that are often neglected. For instance, few teachers or textbooks assign to the ordinal aspect of number the importance it deserves. Moreover, logico-mathematical procedures are often introduced prematurely, thus neglecting the prior development of logico-physical procedures. Activities that may provide the children with the possibility of achieving some degree of abstraction, at both tiers, are usually ignored. Following the analysis of a conceptual scheme, teachers can develop tasks related to every aspect of understanding of a given concept. They could thus present to the children a far broader range of activities whose complementarity adds up to a much richer cognitive environment.

Such analyses also provide a frame of reference for the evaluation of a child's knowledge. They enable the teacher to assess the shortcomings in his or her background. For instance, a child who cannot recite the number words backwards will not be able to deduce the new rank of an object in a row following the removal of one of the preceding objects. They also enable the teacher to verify if appropriate linkages have been made, if different aspects of a conceptual scheme have been integrated. For example, the child who can count-on from a given rank in a row of chips, but cannot tell the cardinality of the row, has not yet achieved a synthesis of the counting-on procedure and the notion of cardinality.

We do not claim that this model will be suitable to describe the understanding of all mathematical concepts. Up to now we have applied it successfully to the analysis of the addition of small numbers (Herscovics & Bergeron, 1989) and early multiplication (Nantais & Herscovics, 1989). Héraud (1987, 1989) has applied it to length and the measure of length, to surface and the measure of surface (area). Dionne and Boukhssimi (1989) have applied it to algebraic concepts: to physical point and algebraic point (coordinates); to the physical notion of steepness and to the measure of steepness (slope); to physical straight line and to linear equation.

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Topic Group B

The Kindergartners' Construction of Natural Numbers: An International Study

N. Herscovics

Concordia University

J. Bergeron

Université de Montréal

Introduction[1]

In a companion paper, "A model to describe the construction of mathematical concepts from an epistemological perspective", we presented a two-tiered model that could be used to follow the evolution of the child's understanding. The first tier dealt with the physical pre-concepts of the notion involved and consisted of three distinct levels: intuitive understanding, logico-physico procedural understanding, and logico-physical abstraction. The second tier described the comprehension of the emerging mathematical concept and involved three component parts: logico-mathematico procedural understanding, logico-mathematical abstraction, and formalization. This model was then applied to identify criteria that might be used to characterize each of the six aspects involved in the understanding of natural number.

Initially, each of the six components of understanding of natural number was the subject of an assessment study. Several of these studies have been reported. Two papers have dealt with numerical procedures (Bergeron, A., Herscovics, N., & Bergeron, J. C., 1986 Herscovics, N., Bergeron, J. C. & Bergeron, A., 1986a). Results on different tasks dealing with logico-physical abstraction of plurality and logico-mathematical abstraction of number have also been reported (Herscovics, N., Bergeron, J. C. & Bergeron, A, 1986b). The kindergartner's symbolization of numbers has been studied and discussed (Bergeron, J. C., Herscovics, N. & Bergeron, A., 1986). Following these assessment studies which were carried out with different groups of kindergartners, we experimented all the different tasks on the same children in four case studies (Herscovics, N., Bergeron, J. C. & Bergeron, A., 1987; Bergeron, A., Herscovics, N., & Bergeron, J. C., 1987).

However, to identify some general tendencies in the children's construction of number, a few case studies were not sufficient. This is why we extended our study to a larger group of kindergartners. We first experimented the tasks on the preliminary physical pre-concepts with a group of 30 Montréal kindergartners (Bergeron & Herscovics, in press). The following year we were ready to investigate both levels of our two-tier model with another group of French speaking Montréal children. And to determine if the cognitive structures observed here were comparable to those of other urban children from a different culture but with the same language or from the same culture but speaking another tongue, kindergartners from Paris, France, and Cambridge, Mass., were also assessed.

The samples used in our study involved 29 Parisian kindergartners of average age 5:8 whose school was situated in a lower socio-economic neighbourhood (lower middle class and working class); 30 kindergartners of average age 5:10 whose school was located in a lower socio-economic neighbourhood in Cambridge, Mass.; 14 of these children were

1 Research funded by the Quebec Ministry of Education (F.C.A.R. EQ-2923)

in regular classes whereas 16 of them were following an activity oriented program for early childhood based on Mary Baratta-Lorton's **Mathematics Their Way** (1976); 32 kindergartners of average age 6:2 from 4 different schools in the Montréal area, two being situated in higher socio-economic suburbs and two located in lower socio-economic neighbourhoods. For the overall project, which dealt with all the different aspects of understanding number, three to four individual interviews lasting on average 30 minutes were carried out with average children selected by the school authorities. Here are some highlights of this international study.

Enumeration skills

Pre-requisite to any mastery of the enumeration procedures is the child's memorization of the number word sequence. However, prior research has shown that a majority of kindergartners perform better on the enumeration of a large set of objects than on the mere recitation of the number-word sequence (Bergeron, A. et al. 1986). Thus in order to assess the extent of their knowledge of the number-word sequence, a set of 76 chips was provided with instructions to "Count as far as you can". The following table indicates the distribution of their enumeration skills.

Table 1. Enumeration skills

City	N	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70+	Ave.
Cambridge										
Regular classes	14	0	0	3	3	0	1	3	4	53.7
Lorton classes	16	0	0	0	0	2	1	1	12	70.8
Totals	30	0	0	3	3	2	2	4	16	62.8
Paris	29	1	10	6	5	1	3	0	3	32.4
Montréal										
Higher Soc. Ec.	16	0	4	1	4	1	0	4	2	45.1
Lower Soc. Ec.	16	1	3	4	4	0	1	1	2	37.3
Totels	32	1	7	5	8	1	1	5	4	41.2

What is striking at a first glance is the similarity between the Parisian and Montréal samples, this, in spite of the fact that the French children were six months younger than the Canadian ones. But even more striking is the shift in the distribution of the Cambridge children. Not a single child is in the 0—19 range, in contrast with the 37% and 25% in the other two cities. Moreover, half the Cambridge children can count beyond 70, compared to 10.3% and 12.5% in the other two cities.

The distributions provide another interesting fact. It seems that for the Cambridge *regular* classes, as well as for the Parisian children and the two Montréal groups, the number 39 constitutes a temporary limit point: 42.9%, 75.8%, 56,3% and 75.0% respectively are within the 0—39 range. Perhaps this might indicate that these children have not yet learned the sequence of multiples of ten. That two decades, from 20 to 29, and 30 to 39, are sufficient for the generalization of the decade structure, seems evident from the fact that when the children learn their multiples of ten, their range jumps up to the sixties and seventies. Few of them remain in the 40 to 59 range.

A greater frequency of the Parisian children in the 50—59 range might be explained by a lack of knowledge of the multiples of ten beyond 50. One might conjecture that the 5 Montréal children in the 60—69 range (16.7%) have difficulties with 70 since in French, the tens pattern changes (..., cinquante, soixante, soixante-dix, ...). However, the data does not bear this out, since in the regular Cambridge classes, 3 out of 14 children (21.4%) are in the same range.

Understanding counting—on

Fuson, Richards & Briars (1982) report that when the number word sequence becomes a breakable chain, children can start **counting-up** (reciting-up) from a given number and that this skill translates into a cardinal operation, that of **counting-on** in the context of addition (p.52). In our study, we have experimented numerical tasks requiring counting-on in non-additive situations involving both cardinal and ordinal contexts.

Our results indicate that 84 of the 91 kindergartners could recite up from a given number and that most of them did not even need a running start. Comparing the performance in the three cities shows that nearly all (90%) the Cambridge children can start at 12, that about two thirds of the Montréal children (68.8%), and about half of the Parisian sample (48.3%) can also do so. However, when asked to recite up starting from 6, 100% of the Cambridge children, 93.8% of the Montréal ones, and 75.9% of the Parisian ones succeeded. These differences can easily be explained by the emphasis on counting found in the Cambridge school and by the age difference of the Parisian children who were six months younger than the Montréal ones.

Having assessed the children's reciting-up skills, some special tasks were designed to determine their spontaneous use in the solution of cardinal and ordinal problems. Initially, these tasks were similar to the one used by Steffe, von Glasersfeld, Richards and Cobb (1983). Each child was presented with a row of 13 chips glued to a cardboard, the interviewer stating:

Here is a cardboard with some chips. Look, I'm putting it in this bag (while inserting it in a partially opaque plastic bag)

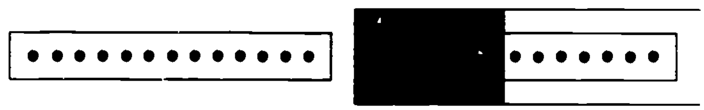

Figure 1: Inserting row in bag

Look, six chips are hidden here (indicating the opaque part)

Can you tell me how many chips are in the whole bag?

The results indicate that with the exception of the Lorton classes, the predominant procedure used was that of figural counting (counting first the hidden objects by pointing at each imagined unit and then continuing the count with the visible part): 50% of the children used it (50.0%, 48.3%, 43.8% and 56.3% respectively in the usual order of presentation).

Following this cardinal task, the same material was used for an ordinal task that required locating the chip corresponding to a given rank. The interviewer asked:

Remember, there are six chips that you can't see. Here is the first one (pointing out the one on the extreme left of the hidden part)
Can you put this little arrow next to the ninth chip?

The data show that once again, with the exception of the Lorton classes, figural counting is the most common procedure: 78.6%, 65.5%, 62.5% and 75.5% respectively in the usual order of presentation. Although most children can recite-up, the use of the counting-on procedure is relatively low, except for the Lorton classes. Less than a third of the children who possess the reciting-up skill think of using it in the above tasks, 21.4%, 4.5%, 31.3% and 28.6% respectively for the cardinal task and 21.4%, 27.3%, 31.3% and 28.6% respectively for the ordinal tasks.

These results bring into question the meaning of counting-on for most of these children. To investigate their interpretation, a simple task in which they were asked to count-on was proposed. The interviewer presented them with 11 chips glued to a cardboard. This cardboard was then inserted in a partially opaque plastic bag so that 4 chips would no longer be visible:

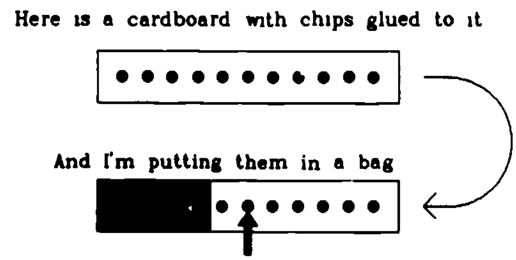

Figure 2: Counting-on a partially hidden row

Look, there are some hidden chips. When I counted them, I started from here (pointing to the first hidden chip on the left) **and when I got here** (putting a small arrow next to the sixth chip) **this was the sixth. Can you continue counting from here on, from the sixth one?**
When the counting was completed:
Can you tell me how many chips are in the whole bag?

The results show that out of 87 subjects who could count-on (compared with 82 who could recite up), only 33 of them (37.9%) could tell how many. Thus a full 60 % could not! Of course, this brings into question the children's interpretation of the counting-on procedure. It is evident that they have not yet been able to achieve a synthesis of the counting-on procedure and their cardinality scheme. The surprisingly poor performance on this task might be explained in terms of three conjectures: (1) Perhaps it is the non-visibility of some of the objects that affects the children's capacity to relate the counting-on procedure with the cardinality of the set; (2) Perhaps it is their need to still establish a one-to-one correspondence between the number-words and the objects; (3) There might be a gap in the children's integration of the cardinal and ordinal aspects of number. Further details on the procedural understanding of number appear in Bergeron & Herscovics (1989).

Logico-mathematical abstraction of cardinal number

Several different tasks were used to assess the children's logico-mathematical abstraction of cardinal number. We provide here details of the two most difficult tasks as well as an overview of the results obtained for the six criteria.

Piagetian tasks. One of the tasks used to assess the invariance of cardinality was the classical Piagetian test on the conservation of plurality and the Gréco modification mentioned earlier. The following table shows the results obtained:

Table 2. Success rates on Piagetian tasks

City	Invariance of plurality	Invariance of quotity	Invariance of both
Cambridge			
Reg. classes (n=14)	8 (57.1%)	12 (85.7%)	8 (57.1%)
Lorton cl. (n=16)	16 (100%)	16 (100%)	16 (100%)
Totals	24 (80.0%)	28 (93.3%)	24 (80.0%)
Paris (n=29)	7 (24.1%)	21 (72.4%)	7 (24.1%)
Montréal			
Hi soc.ec. (n=16)	13 (81.3%)	14 (87.5%)	12 (75.0%)
Low soc.ec. (n=16)	8 (50.0%)	12 (75.0%)	7 (43.8%)
Totals	21 (65.6%)	26 (81.3%)	19 (59.4%)

Results indicate a maximal rate of success among the children following the Barrata-Lorton program. On the invariance of plurality, the sample from the regular Cambridge classes compares with the sample from the two Montréal lower socio-economic neighbourhoods. The sample of Parisian children achieves a much lower rate (24.1%). This can be attributed in part to their younger age. However, this result is fairly consistent with their earlier performance on the elongation of a single row, for their success rate there was about 25% lower than the lowest results obtained in Montréal.

Invariance with respect to the visibility of the objects. In another set of tasks dealing with the invariance of cardinality, children were given in the first interview a row of 11 chips glued on a piece of cardboard. They were told: "Here is a large cardboard with little chips glued to it. Look, I'm putting the cardboard in a bag (the interviewer inserting the cardboard in a transparent bag). Good, are all the chips in the bag?". Following confirmation: "Look, I'm putting a plastic strip in the bag (the interviewer inserting a plastic strip with an opaque part large enough to cover three chips). And now, are there more chips in the bag, less chips, or the same number as before?". Usually in the second interview, this task was repeated but the children were asked to count up the number of chips before they were inserted in the bag. The following table shows the results obtained:

Table 3. Success rates on partially hidden row

City	Invariance of plurality	Invariance of quotity	Invariance of both
Cambridge			
Reg. classes (n=14)	1 (7.1%)	10 (71.4%)	1 (7.1%)
Lorton cl. (n=16)	5 (31.3%)	14 (87.5%)	5 (31.3%)
Totals	6 (20.0%)	24 (80.0%)	6 (20.0%)
Paris (n=29)	8 (27.6%)	6 (20.7%)	1 (3.4%)
Montréal			
Hi soc.ec. (n=16)	3 (18.8%)	13 (81.3%)	2 (12.5%)
Low soc.ec. (n=16)	0	12 (75.0%)	0
Totals	3 (9.4%)	26 (78.1%)	2 (6.3%)

Whereas the results on the invariance of quotity are similar in Cambridge and in Montréal, their discrepancy with those obtained in Paris is hard to explain. But it is the uniformly low results on the invariance of plurality that are most astonishing. They indicate that among most kindergartners, including those in the Lorton program, the visibility of the objects is still primordial. This is not a question of the permanence of the objects since it is acquired well before the age of five. Nor is it a question of the enumerability of the partially hidden set, as evidenced by the invariance of quotity. Visibility of the objects affects these children's conception of plurality.

In order to have an overview of the children's understanding of cardinal number, the results (in percents) obtained on the various tasks are summarized in the following table, invariance of cardinality signifying the invariance of both plurality and quotity:

Table 4. Hierarchy of criteria for cardinality

Invariance	Cambridge		Paris	Montréal	
	Lorton classes	Regular classes		Lower income	Higher income
Uniqueness of card.	93.8	100.	96.6	100.	93.8
Inv.wrt direction of count	100.	92.9	86.2	81.3	100.
Inv.wrt elongation of row	93.8	71.4	48.3	75.0	81.3
Inv.wrt dispersion of set	93.8	57.1	65.5	62.5	87.5
Inv.wrt Piagetian tests	100.	57.1	24.1	43.8	75.0
Inv.wrt visibility of objects	31.3	7.1	3.4	0.	12.5

What is most striking about this table is that apart from the Parisian results obtained on tasks involving the elongation of a set, the basic hierarchy is similar in the three samples. By and large, the uniqueness of the cardinality of a set and the invariance with respect to the direction of the count seem to be achieved in this age group. The Cambridge and Montréal results on the elongation of a row and on the dispersion of a set are similar in the two regular classes (71.4%) and the two lower income classes (75.0%). Compared with the dispersion of a set, the Piagetian tests are more difficult for both Parisian and Montréal children. The invariance with respect to the visibility of the objects has the lowest rate of success in all groups.

Also remarkable is a comparison of the success rates in the three middle columns. Again, if the odd results obtained in Paris on the elongations tasks are ignored, very similar rates are found among the Cambridge children from the regular classes, the Parisian children (who also come from a lower middle class and working class area), and the two Montréal classes situated in comparable neighbourhoods.

Logico-mathematical abstraction of ordinal number

Four different criteria were used to assess the logico-mathematical abstraction of ordinal number. We present here the details on the tasks dealing with the variability of ordinality and its invariance with respect to the visibility of the objects and with respect to translation. We also provide an overview of the results obtained for the four criteria.

Variability of the rank. In order to determine if children perceived the variability of the rank of an object with respect to the number of elements preceding it, we used the following task. A set of 8 little cars of different colours were aligned in a row.

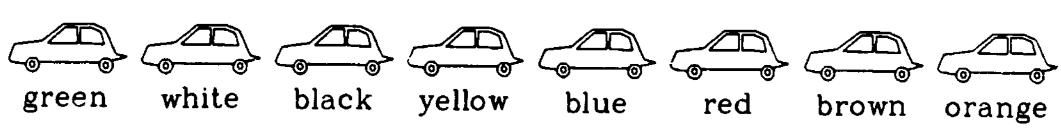

Figure 3: Variability of rank

Once a common vocabulary was established using the word "number" in an ordinal sense (Herscovics & Bergeron, 1988), we told the following little story: "The parade is now stopped because the green car broke down. The tow truck is coming to get it (the interviewer removing the green car), and it won't come back in the parade. Now look at the little blue car. Do you think that the blue car still has the same number as before in the parade or do you think that its number has changed?"

In the second interview, the variability of rank was investigated by repeating a similar question but with an important addition. As soon as the parade of cars was laid out in front of the children (in the same order as before), they were asked "Can you tell me the number of the brown car?". Once the children had found (by counting) that it was in seventh position, they were again told that the green car (the first one) had broken down, and the interviewer then removed it. At this point they were asked: "Now, without counting, can you tell me the number of the brown car?" while screening the parade with both hands or the forearm to prevent the possibility of counting. The following table shows the results obtained :

Table 5. Success rates on variability of rank

City	perceived change of blue car's position	were able to find new rank of brown car	succeeded both tasks
Cambridge			
Regular classes (n=14)	12 (85.7%)	5 (35.7%)	4 (28.6%)
Lorton classes (n=16)	14 (87.5%)	13 (81.3%)	12 (75.0%)
Totals (n=30)	26 (86.7%)	18 (60.0%)	16 (53.3%)
Paris (n=29)	25 (86.2%)	16 (55.2%)	15 (51.7%)
Montréal			
High soc-econ (n=16)	14 (87.5%)	14 (87.5%)	13 (81.3%)
Low soc-econ (n=16)	15 (93.8%)	10 (62.5%)	10 (62.5%)
Totals (n=32)	29 (90.6%)	24 (75.0%)	23 (71.9%)

Column 1 shows that without any counting, most children perceived that the position of the blue car had changed. However, the success rates shown in the second column vary widely. A low of 35.7% is obtained in the regular Cambridge classes. Results found in the Parisian group and the two Montréal classes in lower socio-economic neighbourhoods are comparable (55.2% and 62.5% respectively) as well as for the Lorton classes and the two other Montréal classes (81.3% vs 87.5%). These comments apply to the third column, when both tasks are considered.

These results are somewhat surprising. Except for the Parisian children, it is difficult to explain the poorer results obtained in finding the new rank of the brown car. Among the French kindergartners only 18 of them (62%) could recite the number-word sequence backwards from at least 6. Thus that only 55% succeeded in identifying the brown car's new rank (6) is reasonable. But for the Cambridge and Montréal samples, absolutely all children were able to recite backwards. Clearly, this indicates that the cognitive problem at hand is much deeper than that of mastering recitation skills. Indeed, the very integration of cardinality and ordinality is at stake here since by removing the head car, the number of cars preceding the brown car is reduced by one and this should induce a corresponding change in the perception of ordinality.

Invariance with respect to the visibility of the objects. A task introduced during the first interview dealt with the invariance of position when part of the set is hidden. A row of 9 little trucks was drawn on a cardboard, each truck coloured differently. The children were told: "Look, here is a parade of trucks. Can you show me the white truck?" (in sixth position). After it was duly pointed out, the interviewer announced "The parade must now go under a tunnel" and then proceeded to slide the cardboard under the tunnel in such a way that part of the first truck was still visible but three trucks were hidden. The children were then asked: "Do you think that the white truck has kept the same number in the parade or do you think that it now has a different number?"

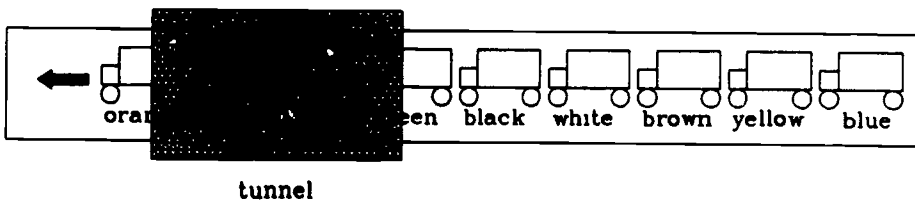

Figure 4: Trucks in tunnel

The above task was repeated in the second interview with an important variation. The children were now asked to find the rank of the white truck. After they had counted to determine its rank (sixth), the parade was moved forward into the tunnel and the interviewer asked again "Now, can you tell me the number of the white truck in the parade?". The following table provides the data obtained with this task assessing the invariance of position and of ordity:

Table 6. Success rates on partially hidden row of trucks

City	Invariance of position	Invariance of ordity	Invariance of both
Cambridge			
Regular classes (n=14)	4 (28.6%)	6 (42.9%)	3 (21.4%)
Lorton classes (n=16)	11 (68.8%)	12 (75.0%)	8 (50.0%)
Totals (n=30)	15 (50.0%)	18 (60.0%)	11 (36.7%)
Paris (n=29)	6 (20.7%)	10 (34.5%)	5 (17.2%)
Montréal			
High soc-econ (n=16)	10 (62.5%)	12 (75.0%)	9 (56.3%)
Low soc-econ (n=16)	5 (31.3%)	7 (43.8%)	3 (18.8%)
Totals (n=32)	15 (46.9%)	19 (59.4%)	12 (37.5%)

An examination of the results in the first column indicates that the number of children who think that the white truck has kept the same position in the row is comparable in the regular Cambridge classes, the Parisian children, and the Montréal kindergartners from the two schools situated in a lower socio-economic neighbourhood. The results of the other Montréal children compare with those of the Lorton classes. On the invariance of ordity alone as well as on the invariance of ordinality based on both position and ordity, again the results regroup themselves in two comparable sets, the Lorton classes and those of the Montréal children from the higher socio-èconomic areas in one set, and the other three samples in the other set.

Invariance with respect to translation. The last set of tasks we developed in our assessment of ordinal number were somewhat similar to the Piagetian conservation tests for plurality and quotity , for they involved the comparison of two parallel rows. The interviewer aligned a row of 9 identical cars, and asked the children "Would you make a parade just like mine and next to it?" while handing over another 9 cars. Then using a blue coloured sheet of paper (the river) and a small piece of cardboard to represent a ferry she explained: "The parades must cross the river on a little ferry boat. But the ferry can only carry two cars at a time, one car from each parade. When we are ready, we take one car in my parade (putting her lead car on the ferry), and one car from your parade" (asking the children to put their lead car on the ferry). The ferry then crossed the river with the two cars, unloaded them, and came back for two more:

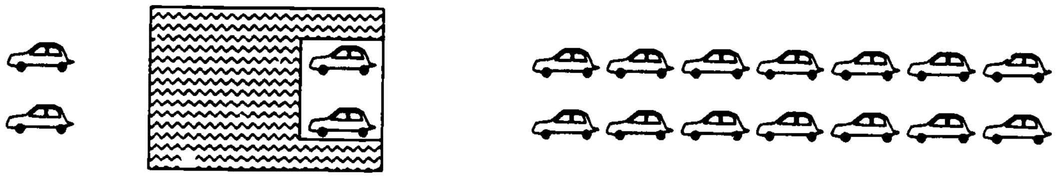

Figure 5: Two parades crossing a river

The cars were then put back in their initial position and the children were told: "Now I'm putting this little arrow on this car (the seventh car in the interviewer's row). Can you put this other arrow on the car in your parade which has the same number as mine?"

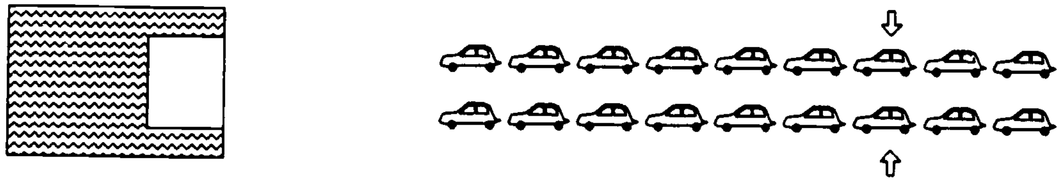

Figure 6: Two cars having the same rank

Once this was done, the interviewer announced "Now look, the parades move on" while moving the child's parade a small distance and moving her own parade somewhat further by the length of two cars:

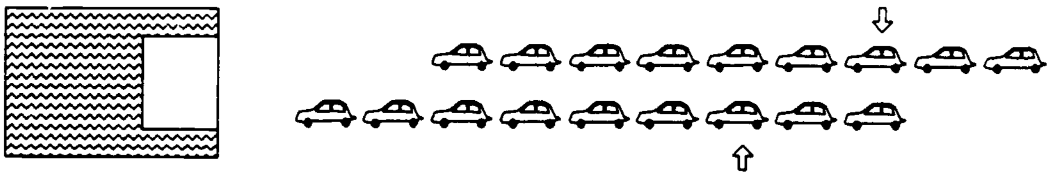

Figure 7: "Will the two cars cross at the same time?"

The children were then asked: "Do you think that the two cars with the arrows will cross the river at the same time?" Following their answer, they were asked to show the interviewer how the two parades were to cross the river in order to verify that they were aware that the cars had to be ferried in pairs. Following the above task, the invariance of ordity was immediately assessed. The next table provides data on the invariance of position and on the invariance of ordity:

Table 7. Success rates on translation task

City	Invariance of position	Invariance of ordity	Invariance of both
Cambridge			
Regular classes (n=14)	2 (14.3%)	9 (64.3%)	2 (14.3%)
Lorton classes (n=16)	4 (25.0%)	12 (75.0%)	3 (18.8%)
Totals (n=30)	6 (20.0%)	21 (70.0%)	5 (16.7%)
Paris (n=29)	2 (6.9%)	18 (62.1%)	2 (6.9%)
Montréal			
High soc-econ (n=16)	1 (6.3%)	11 (68.8%)	1 (6.3%)
Low soc-econ (n=16)	3 (18.8%)	10 (62.5%)	2 (12.5%)
Totals (n=32)	4 (12.5%)	21 (65.6%)	3 (9.4%)

The results on the invariance of ordity with respect to translation vary but little between the groups. The very low results on the invariance of position induce a very low rate of success on the invariance of ordinality as shown in the third column. Quite clearly, even the children in the Lorton classes and in the Montréal classes in higher socio-economic neighbourhoods do not manage to overcome the visual effect of the translation of one of the rows.

In order to have an overview of the children's understanding of ordinal number, the results (in percents) obtained on the various tasks are summarized in the following table, variability and invariance of ordinality signifying the variability and invariance of both position and ordity:

Table 8. Hierarchy of criteria for ordinality

	Cambridge		Paris	Montréal	
	Lorton classes	Regular classes		Lower soc-ec	Higher soc-ec
Variab. of ordin. no	75.0	28.6	51.7	62.5	81.3
Inv.wrt elongation	100.	64.3	27.6	56.3	68.8
Inv.wrt visibility	50.0	21.4	17.2	18.8	56.3
Inv.wrt translation	18.8	14.3	6.9	12.5	6.3

Again, as with cardinality, there are marked distinctions in the performance of the Lorton classes as compared with the regular Cambridge classes and important differences between the two Montréal samples.

As was the case with cardinal number, the similarities are quite striking. We find essentially the same hierarchy in the last three columns. For the two Cambridge groups, the success rates on variability and invariance with respect to elongation are inverted when compared with the other groups. Aside from this difference, the general hierarchy is the same. As mentioned earlier, the low performance of the regular Cambridge classes on the variability of ordinal number is somewhat surprising. The Parisian children's poor performance on the invariance of ordinal number with respect to elongation is similar to their poor performance on the other comparable elongation tasks dealing with the invariance of cardinal number. Regarding the comparison of the Lorton classes with the Montréal children in the schools located in higher socio-economic suburbs, the similarity is still quite strong.

By way of conclusion

A most important conclusion implied by the international study on the construction of natural number is that kindergartners in Western urban environments evolve similar cognitive structures, despite some cultural differences, despite linguistic differences. This is evidenced by the hierarchies we uncovered among the different criteria used to assess each component part of understanding. It is particularly true for the three groups that were comparable in terms of classroom activities and socio-economic background. But this remains true for the other two groups, those Cambridge classes using activities based on the Baratta-Lorton program and those Montréal children belonging to classes in schools situated in wealthier neighbourhoods. Although these last two groups had markedly higher success rates on the various tasks, the hierarchy of the success rates was essentially the same as for the other three groups.

In terms of logico-mathematico procedural understanding, Table 1 shows that, with the exception of the Lorton classes, the other four groups were still developing their counting skills and that in each group the number 39 constitutes a temporary limit point. Again, in each of these four groups the predominant procedure used to solve cardinal and ordinal problems in the context of a partially hidden row was figural counting, while less than a third of the children who could recite up from 6 used counting-on.

Regarding the children's abstraction of cardinality, Table 4 shows that for the regular Cambridge classes, the Parisian classes and the Montréal pupils in the lower income group, not only is the hierarchy of the criteria essentially the same, but the success rates are also comparable if we except the Parisian results on the tasks involving elongation. For the Montréal children in the higher income group, if we ignore a difference of 6% due to a difference of one child out of 16, the first two criteria are met by all, the next two criteria meet with comparable success rates (81.3% and 87.5%). And thus, the hierarchy obtained is essentially the same as for the three previous groups. For the Lorton group, since they all meet the first five criteria, one cannot order them. Nevertheless, even they have not achieved the invariance of cardinality with respect to the visibility of the objects. Regarding the abstraction of ordinality, Table 8 shows that, except for the inversion of the first two criteria for the Cambridge classes, all five groups indicate the same hierarchy.

Our critical analysis of the children's performance may have obscured the fact that these kindergartners possessed a surprisingly extensive knowledge of number. For instance, nearly all could recite up from either 6 or 12, as well as recite backwards. Fuson et al (1982) have shown that such children are dealing with the number word sequence as a bi-directional breakable chain. While they did not use counting-on in order to solve problems in which some of the chips in a row were hidden, nevertheless they showed great ingenuity in inventing a new procedure, that of figural counting. In terms of abstraction too, we did not anticipate that nearly all these pupils could perceive the uniqueness of the cardinality of a set and its invariance with respect to the direction of the count. But their knowledge was far more extensive than reported in this paper, for almost all could leave a numerical message indicating the number of objects in front of them. About half the Parisian children and almost all the North American ones could read and write numbers beyond 10.

This international study has also brought to light the existence of four unexpected cognitive obstacles. The first one relates to the task of counting-on from the sixth chip on a partially hidden row (see Fig. 2). A full 62% of the children who could count-on were unable to answer the question "How many?". Even the Lorton group did not succeed much better (50%). Three possible conjectures that might explain this problem were suggested in the analysis of the data. A second cognitive obstacle involves the children's perception of the variability of the rank of an object in a row (see Fig 3). Table 5 shows that in three of the five groups, many children still had problems in determining the new rank of a car following the removal of the lead car.

If the first two obstacles might show evidence of a lack of integration of specific counting procedures (counting-on and counting backwards) into the children's notions of cardinality and ordinality, the last two obstacles seem to be more of a developmental nature. For instance, in all five groups, the visibility of the objects affects the kindergartners' perception of the invariance of cardinality (see Table 3) and to a lesser extent the invariance of ordinality (see Table 6). A similar interference due to the visual apprehension of the objects seems to be present in the task involving the translation of a row of cars (see Figure 7).

Comparing the different results obtained from the two Montréal groups, it is clear that the overall success rate of the children from the classes in the higher socio-economic suburbs is greater. We are but mentioning the difference here without any pretence at a scientific investigation since the objective of our study was to assess the children's understanding. While we did not control for the quality of teachers, nevertheless it is well known that quite often, better schools manage to attract better teachers. Moreover, since officially there is no mathematics program for Quebec kindergartens, teachers are free to choose their classroom activities, with the result that these may differ both qualitatively and quantitatively from school to school. Finally, it is also well known that children in the wealthier neighbourhoods are more likely to experience at home a richer variety of educational activities. These are all variables that would have to be taken into account in any investigation of the effect of the pupils' socio-economic background. However, in our study, the reason for choosing schools in different socio-economic neighbourhoods was to provide us with a wider variety of subjects.

A comparison of the different results obtained from the two Cambridge groups indicates much higher success rates for the children following the Baratta-Lorton program. A closer look at this program shows that it makes extensive use of concrete material, games and rhythmic body movements, thus touching upon some aspects of the preliminary physical concepts. Regarding the logico-mathematico procedural understanding of number, it goes well beyond simple enumeration from 1, and teaches explicitly counting-on and counting backwards, procedures that are then used primarily in cardinal tasks. In some of the kindergarten classes, the early arithmetic may even include addition and subtraction of small numbers. Children are also taught numerals and the conventional symbolization of the operations. While some of the tasks involved the invariance of cardinal number with respect to the partition of a set, most of the activities related to procedural understanding and to formalization. But these activities seem to have had a marked impact on the children's logico-mathematical abstraction. Table 4 shows that on the last four tasks dealing with the invariance of cardinality, the Lorton group scored much higher than the regular one. This pattern holds for the first three tasks on ordinality listed in Table 8

The results obtained by the Lorton group have serious pedagogical implications. They bring into question the various government policies specifying that nc mathematical program ought to be assigned to the kindergarten level. These policies stem from a laudable desire to allow these children time to play and develop without any curriculum pressure. However, without confining them into a rigid mathematical program, one can envisage many numerical activities allowing them to play and develop their mathematical thinking. But for many kindergarten teachers, these activities are limited to simple counting tasks. As our conceptual analysis and our international study have shown, kindergartners possess intellectual abilities that far exceed those needed o master such simple procedural skills. In fact, our work suggests that many different numerical activities could be developed that would enable the child to progress along the different component parts of the understanding of number.

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Topic Group C

Multicultural Influences in Mathematics Education

Linda Ruiz Davenport

Portland State University

I know I've been asked to speak about multicultural influences in mathematics education, generally, but what I would like to do in this topic group is focus specifically upon cultural influences among Black, Hispanic, and Native American students. These are three populations of students which historically have been underachieving and underparticipating in the area of mathematics and often cultural influences are associated with their low levels of achievement and participation. Over the past year I have been working with a project designed to increase the levels of achievement and participation for Black, Hispanic, and Native American students and to look at cultural influences on their achievement and participation. Consequently, in this presentation I would like to discuss this project and our observations regarding the role of multicultural influences.

A review of the literature on the mathematics achievement and participation of Black, Hispanic, and Native American students contains a great deal of data documenting their low levels of achievement and participation. There are few studies which look empirically at possible causes of this low achievement and participation. Until recently, the only major studies addressing this issue were a clinical study of the mathematical understanding of Hispanic algebra students focusing on the relationships between language proficiency and mathematical understanding (Gerace & Mestre, 1982a, 1982b) and an examination of the role of Black English on the mathematical understanding of Black high school students (Orr, 1987). More recently, a series of chapters devoted to linguistic and cultural influences on mathematics achievement has been published (Cocking & Mestre, 1988), although these tend to address opportunities to learn in specific multicultural teaching contexts as well as the role of language in learning rather than multicultural influences on mathematical thinking itself. Incidentally, I did not feel that Orr made a convincing case for the effects of Black English on mathematics learning, although her book is rich with examples of student work demonstrating their difficulties with particular mathematics concepts.

In our project, we were interested in looking at the relationship between culture and mathematics achievement. We did believe that Black, Hispanic, and Native American students were underachieving and underparticipating because they were not being provided with the kinds of opportunities that would help them construct meaningful mathematical knowledge. This seemed to fit with the research findings that these populations of students had little conceptual understanding of many topics in the mathematics curriculum. We also believed that the reason that these students had little conceptual understanding was that they were not being provided with opportunities to explore, discuss, and socially negotiate meaningful mathematical knowledge. Finally, we believed that since mathematical knowledge is constructed in a cultural context, there might be some cultural influences on the mathematical thinking of these students which we hoped would emerge during the course of the project.

I might add that I think the approach we are taking is a rather novel one considering the frameworks that are often used to examine the underachievement of Black, Hispanic, and Native American students generally. During the 1960's, efforts to explain the low levels of achievement of these students focused on issues of cultural disadvantage, followed by, in the 1970's, issues of cultural difference. Now, in the 1980's, issues of effective instruction for students at risk seems to be the framework for much of this research. I think it's interesting that the cultural disadvantage and cultural deficit frameworks were really not all that different. This can be seen if you look at some of the so-called cultural differences discussed in the literature of that period. For instance, in an article discussing the relationship between culture and school achievement, a chart entitled "Contrasting Values and their Effects on Mexican Americans" suggests that the chicano student "frequently lacks enthusiasm and self-confidence", "works more effectively in groups; usually noisy", and "apathetic in school; often embarrassed by deficiency in English and few successful experiences; may become a dropout" (Instructor, 1972). Oddly enough, the title of the article is "Building on Backgrounds". In the current framework of effective instruction, researchers are advocating greater academic learning times with learning broken down into smaller pieces. In this framework, cultural issues are ignored completely.

Our project involved examining changes in mathematical thinking of Black, Hispanic, and Native American middle school and high school students as they progressed through a *Visual Mathematics* curriculum (Bennett & Foreman, 1989) as opposed to a more traditional textbook-based curriculum. This *Visual Mathematics* curriculum, built around *Math and the Mind's Eye* activities developed through an NSF grant (Bennett, Maier, & Nelson; 1987), is highly student-centred, allows for student exploration and discussion, and encourages students to construct and share their personal visions of fundamental concepts in mathematics. We felt that this kind of mathematics instruction would provide opportunities for underachieving Black, Hispanic, and Native American students to construct and negotiate mathematical meaning as well as allow any culturally distinct mathematical views to emerge through their personal visions.

We planned to collect information on attitudes towards mathematics, using the Fennema-Sherman Mathematics Attitude Scales, and achievement in mathematics using both a standardized test and an open-ended mathematics test focusing on mathematical concepts and problem solving, for students using the *Visual Mathematics* curriculum and students using the traditional curriculum. In the open-ended test, for example, we asked students to explain their idea of multiplication and to draw a picture of multiplication. We also planned to develop case studies of the mathematical thinking of students in each of those instructional settings through the use of clinical interviews. We were especially hoping to be able to explore some cultural issues in these interview settings.

This is only the first year of our project and I have to say there have been some difficulties, the primary one being that the implementation of the *Visual Mathematics* curriculum has been quite challenging for many of the teachers. These teachers took a 3-credit course in *Math and the Mind's Eye* before the project began and, during the academic year, we have been meeting once a month for an all-day session designed to provide support as they implement the curriculum. I believe that the challenge lies in the fact that teachers are not accustomed to student-centred instruction. In the *Visual Mathematics* curriculum, the teacher is largely the problem-poser and the facilitator of discussion, and it is often impossible to know exactly where the lesson is going to go. This uncertainty seems to make teachers new to the approach a bit uncomfortable. Also, as students are encouraged to express their vision of mathematical concepts under discussion, teachers are often called upon to facilitate discussions about representations they may have not seen before. This also makes teachers a bit uncomfortable. Finally, we encourage teachers not to "show and tell", but let students try to figure things out for themselves (with the help of some probing questions from the teacher and lots of class discussion). Teachers seem a bit uncomfortable letting students go with their ideas and sometimes seem to want to show them "the right way". All in all, it is a big change for most teachers. I will say that when the *Visual Mathematics* curriculum is working well it can be very exciting for both teacher and students. We have seen it happen in some classrooms. Fortunately, our teachers have had enough of these exciting experiences with the curriculum to keep going and most claim that they could not go back to teaching from a textbook.

I would like to comment on what we have been seeing in our student interviews thus far, as this has been our primary vehicle for exploring cultural influences. The students we have been interviewing have shown very little conceptual understanding of mathematics topics usually included in the elementary school curriculum. Their facility with basic mathematical procedures has been quite limited and they seemed to have little in the way of visual models to help them solve the problems they were asked. We have seen some changes in students who have been using the *Visual Mathematics* curriculum. They seem more apt to say things like "This is how I see it" or "This is how I think about it". We are seeing a great deal of diversity in the approaches used by these students although I would have a hard time categorizing those approaches by cultural background. The only example I can cite that might even remotely suggest a cultural influence is when a young Native American girl drew several pictures of a measuring cup to help her think about adding fractions. I should say the questions we were asking were rather traditional and had a computational focus, for example, asking students to think about $1/2 + 1/3$. However, our intent was to explore how they were thinking about the problem and we did probe for any contexts in which the problem might be made meaningful.

Looking back on our beginning efforts to explore cultural influences on mathematical thinking, I think there are several issues which made our effort particularly difficult. One is that we were asking questions about school mathematics in the school context. Had we moved to a more culturally-relevant out-of-school context and asked questions about mathematical applications in that context, perhaps we would have found cultural influences. However, I think a larger problem lies in the area of defining "culture". Although these students were from several ethnic backgrounds, their affiliation with their ethnic culture varied tremendously. All of these students were born in the United States and were to some extent participating in its mainstream culture. Some students seemed uncomfortable with their ethnic identity and, for example, corrected us with an Anglified version of their name when we used its correct pronunciation. For all these students, I think there were numerous cultural influences such as those associated with television, contemporary films, rock music, and the many influences associated with the peer culture of middle school or high school.

As I have tried to think about perhaps better ways to uncover cultural influences on mathematical thinking, the work that I have found most helpful is a chapter in a recent *Review of Research in Education*. The chapter, entitled "Culture and Mathematics Learning" (Stigler & Baranes, 1988), provides a thoughtful overview of what is known about the role of culture in mathematics learning. The authors review cross-cultural research conducted both in and out of schools with both children and adults. In a discussion of the role of culture in mathematics learning, they suggest that:

> As children develop, they incorporate representations and procedures into their cognitive systems, a process that occurs in the context of socially constructed activities. Mathematical skills that the child learns in school are not logically constructed on the basis of abstract cognitive structures, but rather are forged out of a combination of previously acquired (or inherited) knowledge and skills, new cultural input. Thus, culture functions not as an independent variable that merely can promote or retard the development of mathematical abilities, but rather as a constitutive part of mathematical knowledge itself... In short, we are claiming that culture-specific representations of number do not merely influence the development of mathematical knowledge, but in fact remain part and parcel of that knowledge.

This view is similar, I believe, to that of D'Ambrosio in his several discussions of the concept of ethnomathematics (e.g. D'Ambrosio, 1985). If it is true that culture becomes and *remains* "part and parcel" of socially constructed mathematical knowledge, shouldn't it be possible to examine the influences of culture in any mathematical context? If it is true that we need to return to a culturally appropriate context in order to identify those cultural influences, what would that context be for many of the Black, Hispanic, and Native American students attending schools in some of the country's largest urban areas? These, I think, are some very intriguing questions which need to be answered if we are to more fully understand multicultural influences on mathematics education.

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Topic Group D

Teacher Assessment Practices in High School Calculus

R. Traub, P. Nagy, K. MacRury & R. Klaiman

The Ontario Institute for Studies in Education

Introduction[1]

The purpose of the research was to explore the meaning of grades assigned by different teachers of the same Grade 13 mathematics course, and to formulate possible explanations of any differences in meaning found to exist among the teachers. In Ontario, the high school curriculum is constrained by provincial guidelines, which specify minimum course content and length. Local school boards are responsible for implementing the guidelines in their schools, with differences consequently possible in topic emphasis, grading methods, and quality of achievement expected for a given mark.

The project began in the spring of 1986, with selection of a mathematics course. In making this choice, due regard was given to an ongoing transition from Grade 13 courses to Ontario Academic Courses (OACs). A comparison of Ministry Guidelines for Grade 13 and OAC mathematics courses (Ontario Ministry of Education, 1972, 1985) revealed that the overlap was substantial for Grade 13 and OAC Calculus. Choice of calculus also meant that the results of the survey of calculus examinations by Alexander (1987) would complement and inform the results of this study.

The choice of calculus was made in consultation with a five-member Advisory Committee for the project, consisting of:

- Dr. David Alexander, Faculty of Education, University of Toronto, and The Ontario Ministry of Education.
- Dr. Edward Barbeau, Department of Mathematics, University of Toronto.
- Dr. Gila Hanna, Department of Measurement, Evaluation and Computer Applications, The Ontario Institute for Studies in Education.
- Mr. George McNabb, Mathematics Teacher, Sudbury Board of Education, representing The Ontario Association of Mathematics Educators.
- Mr. John Scott, Mathematics Consultant, Toronto Board of Education.

The design of the study called for the recruitment of 20 teachers. Practical considerations limited choice to teachers working in the urban core of Southern Ontario. In the end, the participants were 17 teachers from 17 different schools in 13 different boards. (The teachers were guaranteed anonymity, so their names must remain confidential.) All participants held an undergraduate degree in mathematics, had at least five years experience teaching senior mathematics, and had taught the calculus course at least three times. Some descriptive information on the classes of the 17 teachers is provided in Table 1.

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[1] This is a short version of a long report under the title "Teacher Assessment Practices in a Senior High School Mathematics Course." A copy of the long report is available from the authors on request. This research was conducted with the support of the Ontario Ministry of Education, through Transfer Grant 52-1028 to the Ontario Institute for Studies in Education.

Several types of data were collected. The 17 teachers were asked to:

- complete a log throughout the Spring 1987 Semester for one Grade 13 Calculus class, recording the activities undertaken each class period and the time devoted to each activity;
- list on the log the homework and seat-work assigned each day;
- report the criteria used to arrive at student grades for the course, including tests, quizzes, examinations, and other factors (e.g., participation, attendance), along with the relative weights of each; and
- mark a common set of 20 final examination papers obtained from a class not involved in the study.

In the material that follows, the use of classroom time is considered first. Successively thereafter, attention is turned to the content of the teaching and testing, testing policy and practice, and, finally, the exam-marking study.

Use of Class Time

In their daily logs, most teachers provided descriptions that indicated the kinds, order, duration and focus of the teaching/learning activities undertaken during each class period. Four teachers, however, provided more information about topics covered in a class than about activities undertaken, and a fifth delegated the task of completing the log to different students, thus providing an uneven record. The logs of these five teachers were excluded from further consideration in this part of the study, leaving the data for 12 teachers.

There was considerable variation among the 12 teachers in number of class periods and length of courses. The number of class periods, ranged from 80 to 109 (Mean 86). Total class time for the calculus credit ranged from 96 to 114 hours (median 105). The scheduled length of class periods varied from 40 to 80 minutes, although on a given day the time actually spent in class might have been less than what was scheduled for any number of reasons.

Six categories of class activities were defined from terms used in the logs:

- Administration: taking attendance, making announcements.
- Direct teaching: presentations, demonstrations and discussions focusing on new material.
- Student practice: seat-work and board-work pertaining to new material (including handouts, assignments and orally presented problems considered in class), with opportunity for individualized instruction.
- Homework: tasks assigned for independent completion, either in class time or outside.

Table 1. Some Characteristics of the Schools and Classes

Teacher Number	School Size	Class Size	Teaching Hours	Number of Tests
1	1400	17	108	10
2	950	30	103	8
3	300	13	107	7
4	1200	15	97	6
5	1100	18	96	6
6	1200	28	105	7
7	900	22	106	6
8	1900	19	105	10
9	1900	16	103	7
10	1000	25	106	10
11	2050	24	105	13
12	1800	31	103	10
13	1500	27	110	9
14	1250	21	105	9
15	1300	13	104	8
16	1400	26	103	3
17	950	27	114	9

School size: Rounded to the nearest 50.
Teaching Hours: Including time for examinations and tests.
Number of Tests: Not including mid-course and final examinations, if either was administered.

- Review: class time used (i) to cover previously learned material, including prerequisite knowledge acquired in other courses (e.g., algebra), and content previously covered in the course, (ii) to prepare for tests and exams, and (iii) to mark or review tests, quizzes and exams.
- Assessment: quizzes, class tests and exams administered in class time.

Analysis of the time spent on each type of activity as a percentage of total time produced the following results: (a) Administration - 0 to 5 percent (median 1%); (b) Direct Teaching - 17 to 52 percent (median 26%); (c) Student Practice - 8 to 47 percent (median 29%); (d) Homework - 11 to 43 percent (median 18%); (e) Review - 4 to 14 percent (median 11%); and (f) Assessment - 8 to 16 (median 10%). Clearly, the teachers differed substantially in their use of class time.

In a search for patterns in these data, coefficients of correlation (over teachers) were computed. A negative correlation was found between total logged time (in hours) and the percentage of time spent on direct instruction. This suggests that teachers with greater amounts of class time tend to do less direct teaching. This interpretation is corroborated by the findings that percentage of time for direct instruction correlated negatively with percentage of time for homework and student practice, whereas the percentages of time for homework and student practice were each positively correlated with total time. The largest percentage of time, overall, was devoted to student practice. Negative correlations between the percentage of time for student practice and the percentages of time for review and for assessment suggest that teachers who place relatively high emphasis on practice in their teaching of calculus place a relatively low emphasis on review and assessment activities.

Content of Assignments and Tests

The information about content came from the daily logs of homework and seat-work assignments that were maintained by the teachers, and from the quizzes, term tests and exams (plus marking schemes) submitted by the teachers. For this and the remaining parts of the report, information has been included from all 17 teachers.

A comment is in order at the outset of this section lest our results be taken as implicitly critical of the teachers who participated in the study. The Guideline for the Ontario Grade 13 Calculus Course (Ontario Ministry of Education, 1972) mandates broad content areas, but not relative importance. Thus, we were not investigating whether some teachers exercise better or worse judgment as to what should be in the curriculum. Instead, we were investigating differences in the judgments made by qualified and experienced teachers.

A scheme was devised for categorizing the content of the course. The starting point was the 1972 Grade 13 Calculus Guideline (Ontario Ministry of Education, 1972) and the contents of two Ministry approved texts for the course. When a satisfactory version of the category system had been produced, two students, both about to graduate from a B.Sc./B.Ed. program in mathematics education and both experienced in practice teaching the calculus course, reviewed and revised the system, and then applied it to questions on teacher-produced handouts, quizzes, tests and exams.

The category system includes 126 topics. In applying this scheme, the two students achieved an inter-rater agreement of 88%. To simplify reporting, the 126-topic scheme was collapsed into 14 categories; at this level, inter-rater agreement was 97%. The 14 Content Categories were themselves classified into six Content Groups: I - the basic skills of calculus (limits, sequences and series, differentiation, and integration); II - proofs of basic theorems; III - applications of differentiation skills, also referred to as differentiation graphing (slope and equation of a tangent, curve sketching); IV - applications of integration skills, also referred to as integration graphing (area between curves, volume of revolution); V - situational problems (motion problems, related rates, maxima and minima); and VI - optional topics (complex numbers, polar coordinates), optional in that a school might decide to deal with these topics in one of the other senior mathematics courses - Algebra or Relations and Functions.

Content of Assignments

Table 2 is a record of the percentage of homework and seat-work questions assigned during the course by each teacher, the questions having been classified according to the six content groups. In addition, the total number of question on which the percentages are based is given for each teacher. Differences among teachers in total number of questions assigned was great. The median was 1037 questions, but the range was from 460 to 1622 questions.

Several results in Table 2 stand out. First, the emphasis on basic skills (Content Group I) was high for all teachers, ranging from 41% of questions assigned to 76%. Second, the greater the emphasis on basic skills, the more almost everything else was de-emphasized. Third, little attention was paid to questions involving proofs and first principles, although increased emphasis on these issues is mandated in the new Ontario curriculum.

Table 2. Percentage of Assigned Questions by Content Group and Total Number of Assigned Questions

Teacher	Content Group						Total
	I	II	III	IV	V	VI	Number
1	49		12	5	13	20	1622
2	56		11	5	25	3	836
3	65		13	5	12	4	1498
4	53		21	9	17		456
5	74		8	8	6	5	1003
6	64		10	8	18		460
7	66	1	9	4	20		917
8	76	2	11	1	11		1388
9	49		14	11	14	12	1395
10	41		24	15	18	2	1341
11	57	2	17	5	19		967
12	70		10	5	14		1083
13	46		14	11	21	8	970
14	5[illegible]		19	5	23		1037
15	66		10	12	12		1343
16	50	2	13	5	18	12	850
17	54		18	8	19		1274

Note: The percentages for a row may not sum to 100 due to rounding error.

Note: See text for a description of the content groups.

Content of Tests

All questions used by teachers in quizzes, classroom tests, and exams were categorized. From teacher-supplied marking schemes and weighting systems, the relative (percentage) weight of every question in the calculation of final grades was determined. These relative weights were summed to yield the percentage of marks toward the final grade that were allocated by each teacher to questions in each of the six Content Groups (Table 3). The percentages in Table 3 indicate a relatively heavy emphasis in testing on basic skills (Content Group I). Emphasis on Content Group II (proofs of basic theorems) was relatively low. The teachers varied considerably in the degree to which they emphasized each content group, but this variation is especially noticeable for Group VI (optional topics).

Effect of exemptions. A study was made of the effects of exemptions from final exams on the content of the assessments of student achievement. Four teachers followed a policy, mandated by the board or the school, of exempting students with a high term mark (typically 65% or more) from the final exam. In one of these four classes, the final marks of the exempted students were based on assessments of substantially different content than the final marks of non-exempted students.

Table 3. Percentage of Test and Examination Marks by Content Group

Teacher	Content Group					
	I	II	III	IV	V	VI
1	35	2	14	12	20	17
2	50	3	13	3	20	11
3	51	7	16	4	15	6
4	33	7	18	12	30	
5	43	2	13	16	20	6
6	50	3	13	13	21	
7	46	2	18	8	26	
8	58	1	18		23	
9	37	3	14	11	22	14
10	33	2	23	12	27	2
11	35	6	27	8	24	
12	48	8	15	8	22	
13	26	4	19	15	28	9
14	33	5	24	7	31	
15	48	6	14	14	18	
16	38	4	15	6	25	13
17	34	10	15	14	25	

Note: The percentages for a row may not sum to 100 due to rounding error.

Note: See text for a description of the content groups.

Effect of discarding test results. One teacher divided the semester into five segments, referred to as terms. Each term contained up to six short quizzes and one test. The tests for Terms 3 and 5 were considered to be the mid-course and final exam respectively. Students were allowed to drop the test and quiz results for one of Terms 1, 2, or 4 from the calculation of their final grades. Dropping the test and quiz results for Term 4 produced final marks based on an assessment of somewhat different content than dropping the test and quiz results for either Term 1 or Term 2.

The foregoing results point to problems with the practices of exemptions and selectively discarding test results. The expectation of many consumers of high school grades is that they reflect achievement of the same curriculum. By exempting some students from final exams or discarding some term results from the calculation of final grades, with different results discarded for different students, marks within the same class will reflect achievement of different kinds. Unbeknownst to consumers, differences among such marks are uninterpretable.

Comparing the Contents of Tests and Assignments

A comparison of corresponding percentages in Tables 2 and 3 reveals a general tendency for teachers to do less testing than assigning of content in Group I (basic skills), slightly more testing than assigning of the content in Groups II and III (proofs and differentiation graphing), and considerably more testing than assigning of the content in Groups IV and V (integration graphing and situational problems). At least some of this pattern must be due to differences in the relative size of questions for basic skills compared to that for the other content groups. (Ten differentiation exercises may require less time to complete than one applications problem.) The greater emphasis on basic skills (Group I content) in assignments than in tests may also reflect the belief that practice makes perfect, not the belief that basic skills are especially important. Moreover, the greater emphasis on Group II content in testing than in assignment may mean that proofs are considered important, but are dealt with by class instruction and demonstration rather than by assigned exercises.

Grading Practices

The data collected about testing and grading practices were used to study the grading processes that were used and the actual grades that were assigned.

The Process

The 17 teachers were found to use 22 different grading systems. More than one system was in use by each of the four teachers who followed an exemptions policy - the grading system for a student of these teachers depended on whether the student had been exempted from the final examination. Also, the teacher who set aside some of a student's marks in calculating the final grade employed at least two different systems. (Refer to the description of this method given in the previous section of the report.)

Two grading criteria were used almost exclusively: tests, usually administered at the end of units of work, and examinations, administered near the mid-point of the course or the end or both. The number of tests ranged from three to 13, the number of examinations from one to two. For seven of the 17 teachers, tests and exams represented 100% of the students' grade. For the other 10 teachers, the weights for tests ranged from 30% to 80% of the final grade. The additional criteria used by these teachers included quizzes (six teachers, weight ranging from 2% to 20%), assignments (six teachers, weight ranging from 3% to 6%), and a subjective mark for participation (four teachers, weight ranging from 5% to 20%). Six of the teachers gave no mid-course exam, while the four who followed exemption policies had no final exam for the exempted students. The remaining grading systems included both mid-course and final exams. (We use the designations first-half and second-half of the course rather than first-term and second-term to avoid possible confusion over the meaning of term and semester. All our data were collected in semestered schools during the Spring Semester. The break between first-half and second-half of the course occurred about mid-April.)

The teachers combined the different test and examination marks into final grades in several different ways. The marks for a test or exam were either (a) weighted according to the number of marks in each (simple summation) or (b) re-weighted to make the weights of each test or exam equal or (c) re-weighted to reflect the teacher's perception of the relative importance of the topics covered by each test or exam. Similarly, the halves of the semester were either weighted equally or unequally. Five teachers weighted each half equally (at least for some students). The other eleven teachers weighted the first-half of the course less (about 30%) than the second-half (about 70%).

The time spent on testing activities (including exams) varied enormously, ranging from nine hours for one teacher to more than 17 hours for another. On average over the 17 teachers, 8.8 hours (range 3 to 15.2 hours) were spent in writing 8 tests (range 3 to 13 tests), not including mid-course and final examinations. The lengths of tests varied from 25 to 75 minutes. The total number of test questions administered during the semester averaged 104, and ranged from 53 to 180.

The cycles of teaching and testing throughout the course were examined. Nine of the teachers seemed to have more regular cycles of teaching and testing than the others. All teachers tested at more-or-less regular intervals throughout the first-half of the course, but the testing patterns for eight teachers became erratic in the second half. The teachers who followed more regular teach-test cycles also gave a greater number of tests on average (10 compared to 7).

Another factor that was considered is the number of different topics covered in a test. The taxonomy of 14 content categories was used to describe test coverage. The number of categories covered ranged from one to eight per test over all the tests given by the teachers. The average (for a teacher) of the number of categories per test ranged from

2 to 4.7. All but two teachers gave at least one test covering only one content category. For five teachers, the final exam included content categories that had not been covered in class tests. A possible reason is that the final examinations used by these teachers were set by other calculus teachers. When one teacher in a school sets a common final exam, that individual requires a certain amount of prescience to set questions that the students taught by other teachers have had an opportunity to learn. A class that proceeds more slowly than expected, for example, is likely to be disadvantaged by the exam.

The Grades

The average mid-course and final grades for each class and the difference between the two were calculated. The distinction between mid-course and final grades is important in Ontario for Grade 13 ccurses offered in the Spring Semester. Early in April, Ontario universities begin their admissions process. For the courses a student is taking in April, the school submits interim grades (normally the mid-course grades in semestered schools) to the Ontario University Applications Centre, and students receive a conditional acceptance or rejection based on these grades. A concern expressed by several teachers in the study is that students become less motivated to work once the interim grades have been submitted.

Averaged over all 17 teachers, the final grade was 67, six points less than the average mid-course grade of 73. For every teacher, the mean final grade was either lower than or at best equal to the mean mid-course grade. The range of mean final grades was 54 to 75, that of mean mid-course grades, 61 to 79. Although there is a high correlation between the two sets of grades for a teacher, the difference between a teacher's mean mid-course and mean final grades was as much as 15%.

A comparison of term marks with final exam marks turned up two results of interest: in the 13 classes in which all students wrote a final exam, the final exam marks were lower than the term marks by about 12 percentage points (58% compared to 70%). Also, final exams tended to discriminate more than term marks. For the 13 classes with no exemption policy, the standard deviation of final exam marks was about 50% larger than that of term marks.

Despite all the differences found in grading processes and in the grades themselves, no clear indication was found in the data provided by the 17 teacher-participants that the observed differences in grading process were related to the observed differences in grades.

Responses to the Common Marking Task

A study was made of the extent and nature of the variation among the 17 teachers in the standards they applied in marking a set of examination papers. A final calculus examination, administered in June 1987 to students in a school not otherwise involved in the study, yielded a set of 20 papers that spanned a range of quality. The 17 teachers were given the 20 papers, and each was asked to prepare a marking guide and then mark the papers against it. In addition, the teachers were asked to provide written comments, should they care to make any, about the examination and performance of the students.

The Examination

The exam contained 11 questions, several of which contained three or more sub-questions. In total, the examination consisted of 37 sub-questions. The content of each question can be described briefly as follows:

1. Find the point on a quadratic function where the tangent has a specified slope.
2. Obtain the derivative with respect to x for each of 13 different functions of x - six logarithmic or exponential functions, four polynomial functions, and three trigonometric functions.
3. Find integrals of nine functions - three trigonometric functions, four logarithmic or exponential functions, and two polynomial functions.
4. Integrate using the method of parts.
5. Find the limits of three polynomials.
6. Solve a problem involving a) acceleration, b) velocity, and c) the position of the particle in motion after specified amount of time has elapsed.
7. Find the area enclosed between two trigonometric functions of the same variable over a specified range of the variable.
8. For a cubic function, a) find the coordinates of all maximum and minimum points of the function, b) find the coordinates of all points of inflection, and (c) sketch the function.
9. Find the rate at which the distance between two moving objects is increasing or decreasing, given information about the direction and rate of motion of the two objects.
10. Prove that the formula (given) for the volume of a sphere can be obtained as a volume of revolution.
11. Find the radius and height of a cylinder, such that the cylinder will have a given volume and an unspecified but minimum surface area.

The exam was strongly weighted toward the testing of basic skills. According to the scheme for categorizing homework and test questions, the basic skills topics (Group I) contained most of the exam questions (26 sub-questions). (The total number of sub-questions for all other content groups combined was only 11.)

The Marking Guides

The marking guides prepared by the teachers indicate the maximum number of marks to be awarded for responses to each sub-question. There was considerable variation among teachers in the total number of marks allocated for perfect performance. The smallest of the maximum marks was 78, the largest 144, and the median 116. The teachers also differed in their allocations of marks to individual sub-questions. For example, wholly satisfactory performance of Question 7 was rewarded with as many as 12 marks by two teachers, and as few as 4 marks by one teacher.

What accounts for differences such as this? For the most part, they seem to stem from differences in the number of steps or stages to an answer that are awarded marks. Another difference was in the use of bonus marks and deductions. Several marking guides indicated bonus marks for good form and for stating the answer in a complete English sentence. Several others indicated deductions for failing to include the constant of integration in answers or for failing to specify units in answers to questions involving measured quantities. These bonuses and deductions, when used, were either one mark or one-half mark.

Despite the obvious disparities found among the teachers' marking schemes, the teachers were in general agreement as to the order of importance of the examination questions and sub-questions. A coefficient of correlation was computed for each pair of teachers between the maximum marks allocated to the questions and sub-questions of the examination. All the intercorrelations were substantial, ranging from 0.76 to 0.95, with a median of 0.89. Clearly, the teachers possessed very similar views of the relative importance of the questions and sub-questions of the examination.

This does not mean that the teachers thought the exam was particularly good, at least as judged by coverage of the calculus course described in the Guideline (Ontario Ministry of Education, 1972). Several teachers objected to the strong emphasis in the exam on integration. Three teachers noted the lack of coverage of polar coordinates and complex variables. Two teachers pointed to the coverage in the exam of trigonometric functions, with one feeling it was inadequate and another thinking it was overemphasized. It was observed by two teachers that volumes of revolution, trigonometric limits and differentials were given short shrift. And three teachers objected to the preponderance of skill-type questions, and the lack of questions involving problem-solving. Note that volume of revolution, polar coordinates, and complex numbers are optional topics. (We did not suggest that the exam was a model for all teachers to emulate; it was only a means to the end of studying differences in marking behaviour. In fact, for present purposes we eliminated the section of multiple-choice questions that appeared in the exam as originally administered.)

In a draft document entitled A Handbook for the Examination Component of Evaluation in the OAC - Calculus (Ontario Ministry of Education, 1987), attention is paid to the number of marks awarded for arithmetic and algebraic simplification in answers to OAC calculus examination questions. An analysis was made of the marking guides in an attempt to assess the extent of differences among them in the proportions of marks awarded for arithmetic, algebraic simplification, and other skills and knowledge (from earlier grades) compared to the calculus skills and knowledge to be acquired in the course. (This analysis was possible for 13 of the 17 guides; four guides indicated only total numbers of marks per sub-question.) The percentages of marks for calculus as opposed to other kinds of mathematical knowledge and skill ranged from 60 to 76, with a median percentage of 66. Thus, there was some variation, but not a lot, in the extent to which knowledge and skills peripheral or prerequisite to calculus were rewarded.

Total Student Marks

The comparability of the total marks assigned each paper by the 17 teachers was assessed by computing for each pair of teachers a coefficient of correlation between the total marks assigned the 20 papers. These coefficients were uniformly high, ranging from 0.81 to 0.97, with a median of 0.92. Obviously, there is close agreement among the teachers in the relative orders into which they placed the 20 papers.

Grading achievement in calculus and other subjects involves more than rank-ordering a group of students. Determinations of fail and pass and honours are usually required. How well, then, did the teachers agree as to which papers represented failing performance, which represented passing performance, and, of the passes, which represented honours? To address this question, the total mark a teacher assigned a paper was converted into a percentage of the total mark given in the marking guide. Here we find evidence of inconsistency in standards. Three teachers assigned no paper a mark in the honours range, and one teacher assigned failing marks to seven papers. On the other hand, seven teachers assigned no paper a failing mark, and one teacher assigned percentage marks of 80 or more to 10 papers. Variation in standards is apparent, despite the fact that the teachers ranked the papers for quality in very much the same way.

The teachers offered comments, several of which are relevant here. For example, the stiffest of the markers directed comments at student performance: the solutions were poorly developed, diagrams were missing, and the responses lacked clear, concise statements. These might be described as errors of form in the student responses. (Although the marking guide of this teacher indicated five marks for the first question, no student was awarded more than three. The apparent reason for this was the failure by all 20 students to include all the steps listed in the teacher's model answer. Thus, for example, no mark was awarded for finding the y-coordinate of the answer if the determination of this coordinate had not been made explicit, even when the student's answer did contain the correct coordinate.) Another of the hard marking teachers also noted the errors of form as a problem with student answers, but so did two teachers who were in the middle of the group as regards severity of marking. The fact that three other hard-marking teachers did not mention form of answer as a problem, suggests that this factor does not fully explain the source of severe marking standards.

In fact, no type of comment appears to distinguish the hard from the easy markers. The easiest marker described the marking exercise as boring. This teacher also described the exam questions as being all of the skill and recall type, and as not requiring higher level thinking skills. It is not apparent that adopting this point of view should cause one to be an easy marker, although another easy marker also commented on lack of problem solving questions on the exam. So too, however, did a teacher in the middle of the group for marking severity. Perhaps more in line with what might be expected, given the severity of his/her marking, was a teacher's registration of disappointment in the students' problem solving abilities. Other comments were made to the effect that the exam was too easy, was of uneven difficulty, with questions being either very easy or very difficult, was too long, was "too tricky by half", and was nicely balanced between straight-forward and challenging questions.

One factor, however, may distinguish hard from easy marking teachers. Ten of the teachers followed one textbook (published by Gage) and six others followed another textbook (published by Holt). (One teacher used a set of notes, and followed no published book.) The teachers who used the Gage text were, on average, relatively easy markers, whereas the teachers who used the Holt text were, on average, relatively hard markers.

In a final attempt to understand differences among teachers in marking standards, a study was made of the marks assigned by three teachers - the hardest and easiest markers, and a teacher at the centre - to three students - a high, middle and low scorer It was found that these teachers differed relatively little in the percentages of marks awarded for performance of the 26 basic skills sub-questions. Against this standard, however, the corresponding results for the other sub-questions are dramatically different. For example, one of the students was awarded about half the marks allocated by the easiest marking teacher for performance of the other-than-basic skills sub-questions, but the other two teachers assigned only one-fourth the marks they had allocated for performance of the same sub-questions. These results suggest that the main source of the difference among these teachers lies in their marking of the exam questions that test other-than-basic differentiation and integration skills.

Summary

The analysis of use of classroom time showed relatively substantial differences among the teachers in their allocation of class time to different categories of activities - administration, direct teaching, review, homework, practice, and assessment. Moreover, those with more class time available expended a smaller percentage of time on direct instruction, and allocated greater percentage to homework and practice.

Substantial differences were also found among teachers in content emphasis.

- Teachers varied widely in the number of questions assigned as homework - from under 500 to more than 1600. A large part of this variation was accounted for by differences in the number of questions on basic skills.

- Teachers varied considerably in the extent to which they emphasized different topics in their assignment of questions. For example, the number of questions on basics ranged from 40% to 75% of the total number of assigned questions.
- The emphasis on basic skills was less in the tests than the assignments, whereas the emphasis on other content groups was greater in the tests than the assignments.

Some of the within-teacher differences between assignment and testing emphases are probably intentional, as when a teacher decides to test only at the top of a small hierarchy of skills or knowledge, ignoring the prerequisite skills and knowledge that had been included in assignments. Conversely, a teacher may teach a difficult concept and choose not to test it because most students failed to grasp it. Whether or not discrepancies between teaching and testing constitute a problem to be corrected is a matter not addressed in the present study. All we have done here is provide evidence that such discrepancies as these exist.

From the analyses of the grading practices of the 17 teachers, it was learned that examinations and term tests were the two main determinants of student grades. All students in all classes wrote a minimum of one examination, as required by provincial policy. However, the nature of this examination varied. For the four classes following an exemption policy, the majority of students took their only exam on material learned in the first half of the semester. For six other classes, the only exam was a final exam based on the entire semester's work. In the remaining seven classes, both a mid-course exam and a final exam were required. The time that students from different classes spent in an examination situation ranged from 2 to 4.5 hours. The final examination mark was weighted from 15% to 40% of the student's final grade and, when a mid-course exam was administered, the resulting mark was weighted 9% to 30% of the final grade. In some cases, the calculus content topics that were tested during the semester were not emphasized to the same extent on the final exam. This might be attributed to the fact that, while the setting of term tests was usually the teacher's responsibility, the final examination was the mathematics department's, and not necessarily the participating teacher's, responsibility.

Term testing was found to vary in the following respects: (i) number of tests (ranging from 3 to 13), (ii) number of items comprising the tests (from 53 to 180), (iii) amount of classroom time used for test taking (from 3 to 15.2 hours), and (iv) schedule of tests (sporadic or regular). Term tests were weighted from 30% to 80% of the final grade.

It is evident from this study that students taking Grade 13 Calculus in the Spring 1987 Semester from the 17 teachers in this study did not demonstrate their achievement in calculus through a common process of assessment and grading. It is reasonable to question whether or not it would be beneficial for students to have experienced similar grading processes, and to have been judged according to similar standards on similar criteria of achievement.

The empirical study of the marking process revealed the following:

- The presence of substantial agreement among the 17 teachers as to the relative importance of the examination questions.
- Substantial agreement among the teachers as to the relative quality of the 20 student papers that were marked.
- Substantial disagreement among the teachers as to the absolute quality of the 20 student papers.
- The marking standards of teachers varied, to a limited extent at least, as a function of the textbook being used.

These results pose a challenge for the Ministry of Education in a jurisdiction where there is no external mechanism - no common, province-wide examination - for aligning standards of calculus achievement. This challenge has not been lost on critics of education in Ontario, and it has not been ignored by the Ontario Ministry. A Handbook for the Examination Component of Evaluation in the OAC - Calculus (Ontario Ministry of Education, 1987) was developed for the purpose of fostering a greater degree of consistency in calculus examinations across the province. The handbook addresses several problems found in the present study - (i) the practice of granting exemptions from final examinations and the variation in value of final examinations, (ii) the emphasis on basic skills to the virtual exclusion in teaching and testing of problem solving, and (iii) the wide differences in amount of testing and other assessment activities. But the results of the present study suggest that consistency in assessment will be increased only when other steps are taken as well.

These steps include the following: increase the consistency of what is taught; increase the consistency with which those examination questions that test other-than-basic calculus skills are marked; increase the consistency with which displays of other-than-calculus knowledge and skills are marked; have more than one teacher independently mark every student exam paper, and set the exam mark equal to the average of the several marks; and ensure that exams are sufficiently long and numerous so that all content is covered and so that the impact on a student's grade of performance on any one question or type of question is minimized.

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Topic Group E

ONTARIO IAEP RESULTS

Dennis Raphael

Ontario Ministry of Education

Dr. Raphael discussed the results of 13-year-old Ontario students on the 1988 International Assessment of Educational Progress. The results pertain to both Anglophone and Francophone student achievement in relation to achievement in the Canadian provinces and other countries.

The presentation was based on a paper read at the Annual Meeting of the American Educational Research Association in March 1989. The paper is available through ERIC. ED 306259

List Of Participants

LIST OF PARTICIPANTS A1 1989 MEETING

Nicolas Balacheff,
LSD-IMAG BP53X,
38041 Grenoble Cedex,
France.
76 564605
Email: balachef@imag.imag.fr

Jacques C. Bergeron,
Faculty of Education,
Université de Montréal,
Succ. "A", C.P. 6128,
Montréal, Quebec.
H3C 3J7
(514) 343-6061
Email: JCBERG@UMTLVR

Tasoula Berggren,
Mathematics and Statistics,
Simon Fraser University,
Burnaby, B.C.
V5A 1A6
(604) 291-3569
Email: TASO@SFU

Rafaella Borasi,
Graduate School of Education,
and Human Development,
Lattimore 425,
University of Rochester,
Rochester, N.Y. 14627

Linda Brandau,
Faculty of Education,
University of Calgary,
Calgary, Alberta
T2N 1N4
(403) 220-6288 (Bus.)
(403) 265-5395 (Res.)

Brian Calvert,
Department of Mathematics,
Brock University,
St. Catharine's, Ontario
L2S 3A1
(416) 688-5550 x 3299

W. George Cathcart,
Department of Elementary Education,
University of Alberta,
Edmonton, Alberta
T6G 2G5
(403) 492-4146
Email: USERWGC@UALTAMTS

François Conne,
Dep. Didactique,
Fac. Sc. Educ.,
Université de Montréal,
Montréal, Quebec.
H3C 3J7
(514) 343-2229 (Montréal)
01141 21 8087460 (Suisse)

Benoît Côté,
Section Didactique,
Dep. Math. Info.,
Université du Quebec à Montréal,
H3C 3P8
(514) 282-3326

Eleanor Cristall,
Brandon University,
18th Street and Lorne Avenue,
Brandon, Manitoba
R7A 6A9
727-9671 (Bus.)
727-4008 (res.)

Linda Ruiz Davenport,
Center for Urban Research in Education,
Portland State University,
P.O. Box 751
Portland, Oregon 97207
(503) 464-4692

A.J. Dawson,
Faculty of Education,
Simon Fraser University,
Burnaby, B.C.
V5A 1S6
Email: USERDAWS@SFU.BITNET

J. S. Devitt,
Department of Mathematics,
University of Saskatchewan,
Saskatoon, Saskatchewan
(306) 966-6114
Email: jsdevitt@watnum.uwaterloo.ca

Don Eastman,
Department of Math/Computer Science,
Brandon University,
Brandon, Manitoba
R7A 6A9
(204) 726-8680 (Res.)
(204) 727-9696 (Bus.)

Douglas Edge,
Faculty of Education,
University of Western Ontario,
London, Ontario
N6G 1N0
(519) 661-2086
Email: EDGE@UWOVAX.BITNET

George Gadanidis,
GR. 1, Box 28, R. R. 1,
Port Berry, Ontario
L0B 1N0
985-0390

Claude Gaulin,
F.S.E. 1082,
Universite Laval,
Quebec, Quebec
G1K 7P4
Email: CGAULIN@LAVALVM1

Gila Hanna,
OISE/MECA
252 Bloor Street West,
Toronto, Ontario
M5S 1V6
Email: G_Hanna@UTOROISE

Velmer B. Headley,
Department of Mathematics,
Brock University,
St. Catherine's, Ontario
L2S 3A1
(416) 688-5550 x 3803

Bill Higginson,
Faculty of Education,
Queen's University,
Kingston, Ontario
K7L 3N6
Email: higgins@qucdn

J. Hillel,
Math Dept.,
HB-234, Concordia University,
7141 Sherbrooke St. West,
Montreal, Quebec
H4B 1R6
(514) 848-3246
Email: JHILLEL@CONU1

Martin Hoffman,
33 Cambridge Place,
Brooklyn, N.Y. 11238
U.S.A.
(718) 789-7857
Email: HOFFMAN@QCVAX

Lars Jansson,
Faculty of Education,
University of Manitoba,
Winnipeg, Manitoba
R3T 2N2
(204) 474-9039 (Bus.)
(204) 269-7261
Email: jansson@ccm.manitoba.ca

Claude Janvier,
CIRADE-UQAM,
P. O. Box 8888, Station "A",
Montréal, Quebec
H3C 3P8
(514) 282-3225

Tom Kieran,
5208 - 142 Street,
Edmonton, Alberta
T6H 4B4
Email: USERKIER@UALTAMTS

David Kirshner,
Department of Curriculum and Instruction,
Louisiana State University,
Baton Rouge, Louisiana 70803
(504) 388-2298
Email: CIKIRS@LSUVM

Erika Kuendiger,
Faculty of Education,
University of Windsor,
Windsor, Ontario
N9B 3P4
(519) 969-0520
Email: SS@WINDSOR1

Lesley Lee,
1595 Rachel Est #606,
Montréal, Quebec
H2J 4A7
(514) 597-0610 (Res.)
(514) 598-2111 (Bus.)

Gary Lessard,
214-100 Blair Street,
Whitby, Ontario
L1N 6M5
(416) 430-0499

La Donna MacRae,
260 Woodfield Road S. W.,
Calgary, Alberta
T2W 3V9
(403) 251-3048

John P. Mayberry,
Department of Mathematics,
Brock University,
St. Catherine's, Ontario
L2S 3A1
(416) 688-5550 x 3294

Eric R. Muller,
Dept. of Mathematics,
Brock University,
St. Catherines, Ontario
L2S 3A1
Email: MFMULLER@BROCK.CDN

Lionel Pereira-Mendoza,
Faculty of Education,
Memorial University,
St. John's, Newfoundland
A1B 3X8
Email: lmendoza@kean.ucs.mun.ca

David Pimm,
Centre for Mathematics Education,
The Open University,
Walton Hall,
Milton Keynes, MK7 6AA
England
Email:
DJ_PIMM@VAX.ACS.OPEN.AC.UK

John Poland,
Mathematics & Statistics,
Carleton University,
Ottawa, Ontario
K1S 5B6
(613) 788-2162 (Bus.)
(613) 563-1697 (Res.)
Email: jpoland@carleton.bitnet

Yvonne Pothier,
Education Department,
Mount St. Vincent University,
Halifax, Nova Scotia
B3M 2J6
(902) 443-4450 x 182
Email: pothiery@msvu

Arthur Powell,
109 St. Marks Place,
Brooklyn, N.Y. 11217
(718) 783-1603 (Res.)
(201) 648-5434 (Bus.)

Ewa Puchalska,
Department de Didactique,
Université de Montréal,
C.P. 6128, Succ. "A",
Montréal, Quebec
H3C 3J7
(514) 343-7230

Martyn Quigley,
Faculty of Education,
Memorial University of Newfoundland,
St. John's, NFLD.,
A1B 3X8
(709) 737-8656
Email: mquigley@kean.ucs.mun.ca
...!uunet!garfie!d!martyn

Mary Rhodes,
Bishop's University,
Lennoxville, Quebec
J1M 1Z7
(819) 822-9600 x 651

Pat Rogers,
Mathematics Department,
York University,
4700 Keele Street,
North York, Ontario
M3J 1P3
(416) 736-5250 (Bus.)
(416) 924-4816 (Res.)
Email: PROGERS@YORKVM1

Joan Routledge,
Box 194,
Aurora, Ontario
L4G 3H3
(416) 727-1853

Robert Sangar,
26 Paris Road,
New Hartford, N. Y. 13413
(315) 724-1853

Constance Smith,
858 Drake Road,
Brockport, N. Y. 14420
(716) 964-8412

Doris Schattschneider,
Mathematics Department,
Moravian College,
Bethlehem, Penn., 18018
(215) 861-1373

Ralph Staal,
337 Whitmore Drive,
Waterloo, Ontario
(519) 885-4247

Ross Traub,
OISE/MECA,
252 Bloor Street West,
Toronto, Ontario
M5S 1V6
923-6641
Email: R_Traub@UTOROISE

Maria Do Carmo Vila,
1124 Ave. Landry #5,
Sainte-Foy, Quebec
G1V 3R7
(418) 651-4403

David Wheeler,
Dept. of Math.
Concordia University,
Montréal, Quebec
H4B 1R6
Email: DWHEELE@CONU1

Lorna Wiggan,
East Education Office,
Toronto Board of Education
885 Dundas Street East,
Toronto, Ontario
M4M 1R4
(416) 393-9682

Edgar R. Williams,
Dept. of Maths & Statistics,
Memorial Univ. of N' 'oundland,
St. John's, Newfoundland
A1C 5S7
(709) 737-8784
Email: ewilliam@kean.ucs.mun.ca

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