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ABSTRACT

Research on computational estimation and mental computation has received a considerable amount of attention from mathematics educators during the past decade. These proceedings resulted from a meeting to explore dimensions of number sense and its related fields. The participants came from three groups: mathematics educators actively pursuing research on the topic; cognitive psychologists; and other mathematics educators who have adapted methods and models from psychology. The conference was unusual in that no formal papers were presented; instead, each participant submitted suggestions for preconference reading which was to serve as background for 2 days of discussion. The discussions were focused by the following questions: (1) What is number sense? How do we assess and teach it? How is it linked to mental computation and computational estimation?; (2) What research questions regarding these issues need to be addressed? What are the theoretical foundations for this research?; and (3) What do you want to see accomplished at the conference? Part I of the proceedings consists mainly of transcripts of the responses to three questions. After the conference had concluded, each participant was asked to write a short paper to include reflections on the themes of the conference. These papers are contained in Part II of the proceedings. Appendices list references and participants. (YP)

ESTABLISHING FOUNDATIONS FOR RESEARCH ON NUMBER SENSE AND RELATED TOPICS: REPORT OF A CONFERENCE

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and
Bonnie P. Schappelle,
Editors

August 1989

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Contents

Introduction

Judith Sowder

Research on computational estimation and mental computation has received a considerable amount of attention from mathematics educators during the past decade. For the most part these research efforts have been exploratory and self-contained, emerging from questions about classroom experiences, about testing results, or about curriculum change. While these efforts have furnished us with insights into how people perform computational estimation and mental computation, they often have lacked a theoretical framework for interpreting results, clarifying directions for subsequent study, and lending an overall coherence to the research.

During this same time span, cognitive and developmental psychologists have been writing on such topics as mathematical intuition, higher order thinking, mental schemas for understanding mathematics, and the development of quantitative thinking. It is not difficult to see that the work of the two groups is related. Furthermore, cognitive psychology has developed new methods for analyzing learning and task performance, and has formulated and tested new models for human learning. These methods and models have been used by both psychologists and mathematics educators to study the mathematical content of some topics in the mathematics curriculum and the psychological processes involved in performing them (Sowder et al., 1989). This body of work has not, however, so far included number sense and estimation.

These conference proceedings result from a meeting of some of the people interested in exploring the many dimensions of number sense and its related fields of study. The conference was held in San Diego on February 16 and 17, 1989, with participants from three groups: mathematics educators actively pursuing research on number sense, estimation, or mental computation; cognitive psychologists whose work has in some way been related to these topics; and other mathematics educators who have successfully adapted methods and models from psychology to establish a theoretical rationale for their own research efforts. The belief was that the collaboration of participants with these different sorts of expertise would clarify the research domain and help delineate appropriate theoretical foundations on which to build future research efforts within the domain.

The conference was a bit unusual in that there were no formal papers presented. Instead, each participant submitted suggestions for preconference reading (see Appendix B). The papers were either to be directly related to quantitative thinking, estimation, mental computation, or number sense, or they were to be included because the methodology or questions investigated were thought to be pertinent to the purpose of the conference. These readings provided a common background for the two days of discussion. The discussions were focused by the following questions:

(1) What is number sense? How do we assess it? How do we teach it? How is it linked to mental computation and computational estimation?

(2) What research questions regarding these issues need to be addressed? What are the theoretical foundations for this research? How does research in other areas of mathematics learning (exemplified in some of the preconference readings) relate to this agenda, and to the foundations for this research?

(3) What do you want to see accomplished during the next two days? Are there particular issues you would like to see addressed?

Each participant was asked to address the last of these points at the opening session of the conference. Part I of the proceedings includes an edited transcript of the responses. The last discussion of the conference, in which many of the ideas previously discussed were summarized and extended, is also included in edited transcript form in Part I of these proceedings. Between these discussions is a section that can best be described as a distillation of parts of the remaining conference discussion pertinent to the first two questions.

After the conference had concluded, each participant was asked to write a short paper to include reflections on the themes and subthemes of the conference, subsequent ideas related to the conference topics, and suggestions for next steps. These papers are contained in Part II of these proceedings, and they fall quite naturally into three categories, consistent with the backgrounds and interests of the three groups represented among the participants and their different purposes for attending the conference.

The first four papers are by psychologists who, through their research and interest in mathematics learning, have come to recognize the critical role that number sense plays in the acquisition of number concepts and skills. Since the primary purpose of the conference was to consider theoretical foundations for continued research on number sense and related topics, each of the psychologists addresses this issue. Resnick discusses the assumptions that underlie much of our thinking about knowledge and competence, assumptions that derive from associationist psychology. Because number sense is not a collection of "pieces of knowledge" and corresponding skills, it cannot be studied under these assumptions in the same manner as other mathematical knowledge has been studied. Rather, number sense can be characterized in much the same manner as higher order thinking, as Resnick demonstrates here. It is not surprising then that number sense is as difficult to define and assess as higher order thinking. Assessment, while not impossible, requires a different approach from the currently used methods of testing.

Marshall's analogy between number sense and common sense is not dissimilar to Resnick's characterization of number sense. Marshall portrays number sense as connected and integrated knowledge, and suggests that number sense can be examined from the multiple-dimension perspective that is used to investigate intelligence. She further asserts that it is possible to develop a researchable definition of number sense if we describe number sense as the richness of connections of mathematical knowledge. Such a definition would imply that we need to be more concerned with the development of links between individual pieces of knowledge than with the acquisition of the pieces of knowledge themselves.

Greeno characterizes number sense in terms of how it is exhibited--in flexibility of thinking, in the ability to estimate in computations, in the ability to make judgments and inferences about numerical quantities. But rather than advocating instructional activities to promote these skills, he suggests a more global view of number sense as a general feature of mathematical knowing and skill. His discussion of the acquisition of these skills, and of the social aspects involved in the learning of skills, goes considerably beyond what most mathematics educators see as being entailed in skill attainment.

Case directly addresses the issue of establishing appropriate models for studying number sense but gives two reasons why such a model is difficult to specify. He first notes that current cognitive models useful for studying mathematics are relatively "contained" and deal with explicit procedures rather than with intuitive knowledge such as number sense. His second point is that one's view of knowledge is, to a large extent, a function of the epistemological tradition which has most influenced that individual. Those whose backgrounds are from the empiricist tradition generally believe that a rationally designed curriculum should provide the optimal environment for children's acquisition of a necessary knowledge base. While recent advances in research following this tradition offer help for dealing with intuitive notions such as number sense, much work is yet to be done before clearly appropriate models will exist. In the rationalist tradition, knowledge is viewed as originating in the mind. Neo-Piagetian research (such as Case's), and neo-innatist research (stemming from work of Chomsky) offer different perspectives on children's learning from that which follows from the empiricist tradition. Mathematics educators who consider themselves "constructivists" operate in this second tradition. In a third tradition, sociocultural in nature, knowledge is viewed as a social construction, situated in the context of the group's activity. This tradition has so far had little influence on the thinking of mathematics educators, but its attraction is growing. (Greeno's paper is this volume explicates some of the ideas influencing current thinking in this tradition.) Case argues that there is bound to be difficulty in reaching consensus about issues pertaining to knowledge acquisition if the discussants come from different epistemological backgrounds.

The second group represented at the conference--the mathematics educators whose primary research over the past decade has been on topics related to number sense, primarily estimation and mental computation--had, in a sense, a larger stake in the conference. They were seeking theoretical models that would help prioritize research questions and guide them in designing studies, and they hoped that this conference would provide such models. In the sense that this did not happen, they were disappointed, and this disappointment is expressed in some of their papers. Another disappointment they expressed was that although the conference was to focus on number sense and related topics, discussion on the related topics of estimation and mental computation continued to drift back to discussions on number sense, perhaps because this was in the eyes of most participants the most interesting and/or elusive topic. In his paper, R. Reys warns of the dangers of making number sense the "umbrella" under which everything else falls, and argues that the distinct and unique characteristics of computational estimation and mental computation justify their treatment as distinct topics. Schoen, while sympathizing with this frustration, notes the progress made by the participants toward characterizing number sense and understanding the extensive assessment problems. He further notes that the goals implied by "teaching" number sense are certainly not new in mathematics education, but rather what is new is the emphasis on topics such as estimation as a means to "get at" number sense.

B. Reys and Trafton share the opinion that the "definition" of number sense given in the *Standards* (NCTM, 1989) and repeated in the Thompson and Rathmell introduction to the number sense issue of the *Arithmetic Teacher* (February, 1989) is adequate for most purposes: a sound understanding of the meaning of a number and of the relationships between numbers, a good understanding of the relative magnitudes of numbers, knowledge of the effect of using a number as an operator on other numbers, and awareness of appropriate referents for numbers used in everyday life. From a different perspective, Markovits attempts a working definition of number sense in terms of what must be undone. While these authors disagree with the belief stated by others that number sense will not be a product of schooling unless we do away with the current curriculum and design a new one, they do agree that styles of teaching must change. Trafton speaks of number sense as something that "unfolds" rather than something that is taught "directly." Markovits reminds us of the implications this style of teaching has for teacher education.

The insights provided from the final set of papers indicate the pervasiveness of number sense in other areas of mathematics education research. Hiebert's paper focuses on that part of number sense that involves sense of numerals. How do children develop facility with standard written symbols of numbers, which function both as records of something already known and as tools for thinking? He argues that good number sense requires both. Behr discusses notions of variance and invariance, flexibility of thought, and the process-name phenomenon of number representations. Is the student aware of the kinds of transformations that can take place on the operands without changing the outcome? He claims that the abilities to manipulate flexibly mental representations of physical quantities and to translate between the domains of representation are hallmarks of number sense.

Carpenter also claims that a critical aspect of number sense is the ability to operate flexibly with numbers. He views number sense and computational estimation as by-products of teaching for understanding and suggests that rather than studying estimation and number sense themselves, we need to situate this research in a broader context and draw upon the theory that is driving that research. This view is echoed in the Silver paper, where number sense is considered from the perspective of making sense of numbers. This perspective requires that students view mathematics as somehow connected with sense making. Such is not the case now, as Silver demonstrates from his own research on "making sense" of division quotients and remainders. For progress to be made in this area, Silver reminds us that instruction must address the matter of students' dispositions toward mathematical activity.

Much of what transpired during this conference served to reinforce my own thinking about number sense and helped clarify for me why investigating its development poses so many problems. A few weeks before the conference, I had given a presentation to teachers on the topic of number sense. On a transparency prepared for that presentation, I had written:

Number sense:

- Is a well-organized conceptual network that enables a person to relate number and operation properties.

- Can be recognized in the ability to use number magnitude, relative and absolute, to make qualitative and quantitative judgments necessary for (but not restricted to)
  - -- number comparison
  - -- recognition of unreasonable results for calculations
  - -- use of non-standard algorithmic forms for mental computation and estimation.

- Can be demonstrated by flexible and creative ways to solve problems involving numbers.

- Is neither easily taught nor easily measured.

This characterization of number sense is not too different from the characterization I would make now, after the conference. If I were planning the presentation now, however, I would rework and highlight the last point and clarify its relationship to the first point. For I would want my message to be that number sense is not a body of knowledge; rather it is a way of thinking, and so teaching and assessing number sense cannot be thought of in the same manner as we think about such other topics as operations on whole numbers. I would add that an individual must be disposed toward making sense of mathematics for number sense to develop. And if I were giving the presentation to researchers rather than to teachers, I would note the difficulty in locating theoretical models appropriate for investigating number sense. Estimation and mental computation are less elusive, and we have a body of research literature existing on these topics from which to build. But we need to continue to seek a better grasp of number sense, just as we are trying to obtain a better understanding of higher order thinking skills. The first step appears to be to locate good examples of number sense--cases for which we could achieve a fairly broad consensus that number sense is indeed being exemplified. We need to clarify when number sense is not being demonstrated, and why not. We should examine past and ongoing research, some of it probably in other guises, such as cognitively guided instruction, and decide what such studies teach us about number sense, its existence and its development. I suspect that in many cases where we would agree that an individual has exhibited number sense, the situations would involve computational estimation or mental computation, or at least some of the same concepts and skills that underlie these two computational processes. Formulating and testing theories of how these concepts and skills are acquired could serve the double purpose of consolidating our information base for the learning and development of computational estimation and mental computation, while also clarifying some of the instructional and assessment issues regarding number sense that so plagued the participants at this conference.

Part 1: Conference Notes

Expectations

Editors' note: The conference opened with statements in which each participant described his or her current thinking about number sense, estimation, and mental computation. The participants had been asked to reflect upon the following questions: What is your own personal agenda for this conference? What would you like to see accomplished? What particular issues would you like to see discussed? These statements, while reflecting the individual interests and backgrounds of the participants, provided a common ground for the ensuing two days of discussion. In turn, they provide the reader with some insights into the wide range of issues associated with number sense.

Merlyn Behr: Because of my work in rational numbers, I've been interested for a long time in why it is that children seem to deal with the symbols for rational numbers but don't indicate any sense that they perceive a symbol as an entity--that is, three fourths as being a single entity as opposed to two entities, which is what it looks like. From our NSF project work, there seems to be a lot of evidence that that's the way students *do* perceive a rational number, as a *pair* of entities. So I've been interested for a long time in how they develop that sense of number in the context of rational number notation. There are some things that seem, from my perspective, to relate to estimation, and that I'd like to see discussed here--the notion of flexibility of thought, for example. My sense is that people who are able to think flexibly, who seem to think on the basis of mental models or physical models or something like that, are able to come up with estimates or approximations or exact answers in ways that don't fit the usual symbol format. What the stages are of getting from the kind of flexible thinking that is based on mental models to a similar kind of flexibility when the situation is given in mathematical symbols, I think is an important question. The symbol system doesn't really have the flexibility that the mental models have because when we present symbols, we also give rules for manipulating those symbols, and those rules are extremely less flexible than the non-rules for manipulating quantities are.

Jack Hope: My own feeling is that there are probably dimensions of number sense. I think it's like problem solving; it's not easy to define. I'm interested in knowing how prevalent number sense is. I'm also curious about the context in which number sense occurs. I think sometimes that when we do experimental studies, the situations are very contrived, and I'm wondering if number sense only appears when it's put in a context like measurement. Many times when I've worked with students; they don't see any figures making sense, but if you put the figures in a familiar context, they do make some sort of sense of them. Obviously we can't have a thousand-meter rug, and carpenters know we don't have a standard board that is a hundred meters.

I am interested in number sense from the viewpoint of curriculum development. If we do define number sense in some way, how do we use the definition to evaluate number sense? I know from working with teachers that they're under a lot of pressure, and they say that if you can't evaluate something, then it really doesn't exist. So I think if we're developing curriculum materials for number sense, then we've got to have some sense of how to evaluate it. We also need to know how it develops. What are the conditions that foster or even inhibit the development of number sense? Right now the preoccupation with symbolic arithmetic is one condition that inhibits its development. Some of the students I've worked with who are very good at mental math didn't like working with a lot of the prototype, standardized procedures. They liked doing a lot of things in their heads, and they were very flexible in their use of numbers. I've had instances where teachers made them write down their work, and after several years, they just lost interest in mental computation. I've seen my own daughter change over the years from someone who had very good number sense to one who has literally no number sense. I remember that when I talked to her when she was in about Grade 1, she would say things like "Five and six is eleven, because five and five is ten, and one more is eleven. Seven and nine is ten, seventeen, and take away one"--the typical things that we've seen in the literature. I remember that later we were on a trip, at a time when Saskatchewan had a campaign, "Lights on for Life" where they were trying to encourage people to drive with their lights on during the daytime. She was about 14 at the time, and we decided to take a little survey. I said I would count cars with lights on while I was driving, and she would count the total number of cars. We stopped after fifty cars, and there were nineteen out of fifty cars that had their lights on. So I said to her, "Now, work that out as a percentage." I could hear rustling of paper while she divided fifty into nineteen. I said, "Can't you work it out in your head?" "Dad, don't confuse me; this is the way we do it in school." She got the exact answer, thirty-eight percent, but she didn't really see how it related to thirty-eight out of one hundred. I saw over the years what I felt was a diminishing of her number sense. I think that happens to a lot of students.

Barbara Reys: Bob and I started doing some research in the area of estimation back in 1978, basically because we were interested in the topic. We began because we wanted to help develop some curriculum materials, and we didn't see anything out there that we thought was very good. Over the years most of our work has been done with older children, junior high students and senior high students. Lately we've become interested in working with younger children. I'm convinced that if seventh graders don't have number sense, it's probably too late to turn them around. Something happens earlier, as Jack said, and I'm not sure what that is. So we want to begin some explorations with young children and trace the development of number sense, or the lack of development. We realize that we need to know more about young children and how they think and how they learn.

We spent October and November in Japan and interviewed some of Japanese students to find out how they estimate. That was very enlightening to me, given the fact that we know these children are very good on an international comparative basis. Yet they seem to lack good estimation skills. In some ways their schooling seems to have inhibited their development of estimation. They are very good at getting exact answers, even at mentally getting exact answers, but they are very uncomfortable with the notion of getting anything other than an exact answer. We're doing more analysis, but there are a lot of questions I have about where this all begins and how it ties into basic concept development. One of the questions you had on your list, Judy, was, "What is number sense?" And I'm still trying to decide how that is different from just good concept development. I think there are some differences. I think number sense includes some new dimensions, but I haven't been able to sort all those out yet, so I'm anxious to see what the rest of you think.

Jim Greeno: I think this problem of looking at what number sense is, is central and timely. I mainly want to get to a better understanding of number sense too. One small conjecture that I make is this: We tend to think of a metaphor where numbers kind of get into peoples' minds, where the mind contains these number concepts and has representations of them. I'm intrigued with what might happen if we turned that metaphor around. Suppose we think of number as an environment that peoples' minds can get into, so that it's much more a kind of territory with objects and resources and things that happen there. You can learn how to find your way around in that environment and learn how to do things with the things that are in that environment. It sort of puts the mind into mathematics, rather than putting mathematics into the mind. It might put an interesting perspective on our discussions, I don't know. It's something that I intend to be exploring, off on my notepad, as we go along.

The notion that what we're thinking about has to do with somehow objectifying these things, "numbers," and learning how they work as entities seems to be a helpful way to get started. The reason it's all so mysterious and fascinating is that we don't know how it happens...I don't know anybody who has a cogent story to tell about how that happens. So we have here, I think, a really good case in which to study something that has been discussed for a long time; Piaget called it reflective abstraction, and Vygotsky called it internalization. All these things that everybody has recognized as fundamental issues are things to ponder here. It is a nice case, because it's interesting for theory and yet maddeningly complex for theory as well, because there is a kind of fundamental interaction between what you intend to refer to in a formal symbol system on the one hand and the connection it has to how things in the world really work on the other hand. We know that part of what we haven't figured out well enough yet comes from the fact that getting to be good at the symbols by no means insures that you are good at kind of figuring out what it is that's going on in the system where things really work. There is the nice aspect of trying to get those two very important and interesting things understood within a single framework.

Lauren Resnick: Everything I wanted to say has already been said in one sense. Merlyn and Jim both spoke of a concern about the relationship of symbols to "something." I think there are several "somethings." There are physical materials, I mean really the stuff of the world, not Dienes blocks. Then there are also physical models, and also cultural models like money. And then there are the things we call mental models, which are some kind of representations that are in the head (although Jim just suggested we might want to throw out the head), and mathematizable situations that you could associate numbers with. Are those "somethings" the numbers? Well, not really, and yet they stand in correspondence, in some sense, to number itself. It seems to me to be a lot more complicated than I thought when, some years ago, I said, "Well, when we can get maps between the written symbols and the Dienes blocks, we've got it." And there's also language. There is a whole complex of ways in which we talk with each other. All of those things probably go into what we mean by number sense.

That leads me to think that we want to work on a definition of number sense, because it's almost become a talisman. It's being advocated in the *NCTM Standards* and all kinds of frameworks, but we don't know what it is! I think part of the problem is that we've been trying to define it in the way educators have usually made definitions: make it specific, look for key features, etc. Maybe what we have to do is look at a new kind of definition, one which describes the prototype cases that we can all recognize as being something we mean as number sense. This is a different kind of definitional work, more like Eleanor Rosch's idea of central prototypes that we carry around and that we can recognize as being closer to or further away from a particular case. This may also help on the evaluation question. We can find some prototype cases and set up situations where certain kinds of things might be expected to happen. We'll see if they do, and find some ways of scoring them. So a slightly different way of thinking of the definitional problem might get us somewhere. It's pretty clear that number sense is going to mean something sort of non-deterministic. It's going to be a set of actions or engagements for which there aren't fixed-in-advance, known responses; rather, there will be a fuzzy class of things that might be appropriate. That raises a question: What implications does adopting a definition, a description like that, have for the kinds of classrooms children need to live in?

I was struck with Jack's description of his daughter as going from having a lot of number sense to having none. The paradox I see in it is that if one takes the old view of number sense as a mental competence, if it's sort of there and stable and we know what it is, then it really couldn't disappear. The most that could happen is that, for some reason or another, it couldn't be used or accessed. That led me to wonder what we might mean by a decline in number sense with age in a given person? For one thing, we have changing standards. What you mean by number sense for a five-year-old is quite different from what you mean by it for a fourteen-year-old or for an adult, and so the ante keeps going up. Jack's daughter probably has all the number sense that she ever had for positive integers under 20, let's say, but she doesn't have some new things that we also mean by number sense--like the relationship between proportions and percents. There are changing contexts, changing sets of circumstances.

Meanwhile I'm trying to map all this into what I view myself as having been working on for six or seven years, the intuitions about mathematics that quite young children have. Is that what other people mean by number sense?

Bob Reys: I've been looking over the articles we all suggested for preconference reading; I think now that if I were going to suggest another one, I would suggest one by Brownell. Although he didn't explicitly use the term *number sense*, it's certainly heavily reflected in many of the examples of children's thinking he wrote about for many years. So there has been an effort, I think, to do what you're describing, Lauren, only maybe our labels are simply different now from the ones he was using.

Another thing that we might want to put in the hopper for discussion is the concept of what is reasonable or unreasonable. I know that as a math teacher I have often asked my students: "Is that answer reasonable?" As a regular classroom teacher, I asked this question, thinking I guess that the word *reasonable* was understood. But now, as I reflect back and have talked with more children, it's clear to me that a very high percent of children didn't even understand what I was asking. My motivation for that question was to have them look for more efficient ways of analyzing the problem, but reflecting back, I don't think that they understood the task.

Evaluation has been mentioned, and I think that is an important topic that we need to address. I particularly appreciate Hal's paper on that topic, but I think there is another part of evaluation that sometimes doesn't come through, that needs to be conveyed to teachers. Teachers may say to us: "They don't test estimation on the standardized test," or "They don't test mental computation." Yet I think that if we go through and analyze even the currently-available tests, although the tests may not directly assess estimation and mental computation, children who are strong in those skills are certainly going to be rewarded with higher test scores. I think that this point is not clearly understood by many teachers.

Robbie Case: I think that one of the possible areas in which breakthroughs might be made in my own particular field of cognitive development is in the area of mathematics, because the field has received so much careful analysis in past years from cognitive psychologists. The topics being discussed here are at the cutting edge of the work in mathematics, and I want to learn about them and take away what I can.

My colleagues and I are involved at the moment in a training program with children who are in kindergarten this year. We work in the schools with the children in small groups, about twenty minutes a day, for about forty days in the course of their kindergarten. This is preceded and followed by about fifteen days of evaluation. What we're teaching them I suspect, if one were willing to be really loose in one's definitions, is something related to number sense. These are children of immigrants to Canada, from rural, primarily agricultural, villages in Portugal, where numbers don't play a really major role in the cultural life of the communities. The children are darlings to work with. They're from excellent families, extended families, and they come socially and totally together. But they don't know much about numbers. Some come to school at the age of four-and-a-half not being able to count to five in Portuguese or in English. When taught the number string, they make the mistakes that normal two-and-a half-year-olds would make of double counting, missing numbers, and so on. Even those who can count wouldn't know, say, that five is bigger than three, that kind of thing. We have a program where we play a lot of games with them, and analyze what it is we want to teach them, and you could say that that has something to do with number sense. In the past, we've tested what happens on standard cognitive developmental measures like the balance beam. We've also used telling time and playing the piano. What happens on a lot of these measures is that these children go from behaving like normal middle-class two-and-a-half year olds to normal middle-class six-year-olds, although they've never seen a balance beam, never seen a clock, never seen a piano in the course of the instruction. Our project now is focused on a different question: Do we get them any more ready for Grade 1 math, where addition and subtraction are taught? We don't know; that's what the project is about. We'll know in a month or so, because we've been in there for sixty days now since October, developing what we could call number sense, to see if it has any effect on what they do.

Tom Carpenter: For the past four years I've been working on trying to understand how teachers come to understand children's thinking and use that knowledge in their classrooms, and how we can help them to understand better, specifically by sharing with them some of the research in that particular area. We've been studying a series of teachers now for three years. We certainly don't have all the answers, but I think we have some first-grade classrooms that provide a pretty good approximation of some of the things that we might like to see. We have teachers whom we consider to really be very good at attending to and listening to their students, who really do an incredible job of helping children to build upon their intuitive knowledge and of helping them extend that knowledge by providing regular opportunities to do so. If we take the idea of number sense as the ability to decompose problems in a variety of ways and reassemble them in a variety of ways, as a working definition, we see this in first grade. We see kids really able to do incredible kinds of things, at least in our star classrooms.

One of the star teachers followed her students into second grade, because she wasn't willing to give them up yet. In some ways it seems as though they have better number sense than she does, but she's not threatened by that. She is very, very good at listening to them and figuring out how they solve problems. So when they come up with very clever solutions to problems, she is good at understanding what they do. If they make mistakes, she may not catch them right away, or some student may catch them before she does. The interesting thing is that in some ways her real skill is her ability to really handle her students and to *really understand* them. She doesn't have a tremendous knowledge of mathematics. When we started out this program, we thought that one of the critical things would be to get teachers to fully understand how all of these things fit together, to have good cognitive maps. I'm not so sure about that anymore. Her cognitive maps may not be as good as those of the second graders. She herself sometimes won't see as many different ways to construct some of these relationships as some of her students will. She doesn't pick up errors as quickly as some of her students do.

Lauren Resnick: I'm wondering if there is a chance that what you're doing is concatenating fifteen or twenty kids and pitting that against one teacher? Because that together with the fact that the teacher has a few other things to attend to besides thinking of the clever ways to do a problem...

Tom Carpenter: I think the point is that she is not particularly quick and adept at doing *some* of those things. In one case she had her students halving, starting with five hundred, which is a pretty impressive problem for a group of second graders to solve toward the beginning of the year. So they halved five hundred and said that was two hundred fifty. And then they halved two hundred fifty, decided that was a hundred twenty-five; then they halved a hundred twenty-five and decided it was seventy-five. And she didn't notice at first that this was wrong. This particular class went on for awhile, until one of the students discovered, "No, it shouldn't be that, because seventy-five is half of one hundred fifty."

Lauren Resnick: You've pointed out one of the dimensions that we might want to get at. Because one meaning of number sense is knowing how numbers behave, for example that you can solve problems by using patterns. Another meaning of number sense is having very quick access to specific facts about numbers. Those two things are not necessarily the same. Again, maybe we mean several things by number sense. And maybe those two together would produce the performances we would like to see. But they might be fairly separate dimensions that could be looked at separately.

Sandra Marshall: I've been working on schema knowledge and problem solving for a couple of years, basically from a cognitive point of view. I'm interested in how instruction feeds into the way knowledge gets structured in a child's mind or an adult's mind and, related to that, how we are going to assess it. We sort of know what number sense is, but if we really wanted to try and break it apart and be able to say, in a diagnostic sense, who has some part of it and who doesn't have some part of it, then we don't yet have a handle on how to estimate students' levels of acquisition. I'm very interested in how we start to assess different parts of this schema. My own work's been done primarily on traditional story problems. I've been talking with Judy from time to time about her estimation project, starting to get a sense of what problem-solving schemas there might be in estimation.

Ed Silver: It seems to me that there are a couple of things that haven't been discussed yet. One is whether there's a kind of number sense, or a kind of phenomenon that we associate with number sense, that has to do with operating within the world of mathematical symbols all by itself without needing to think about references outside of the world of mathematics, with the possible exception of some kinds of special representations that we associate with a number. Running along side of that there might be a kind of number sense associated with things that go on in the real world that don't require you to actually get into mathematics in order to answer the question, but for which there are situational references in the task which allow you to make sense of whether the size of the answer or the result of the problem that you're considering is sensible and reasonable under the circumstances. In a lot of what we talk about, it is as if those two pieces are always connected. We're always talking about relationships between the math world and the real world. But maybe some of what we're interested in is going on within those worlds separately, and it's not simply a question of relationship. It might be useful to sort all that out. Similarly, of course, there's the school math world and then the world outside the school that kids live in. There might be a kind of reasonableness and number sense that goes on within the school math world and another kind that goes on within the real world, and maybe they don't always *have* to be connected. There's no reason to expect them to always be connected.

For a long time I've been interested in what this world of number sense is all about. You know, we have mental computation, we have estimation, we have real world reasonableness of results; we have all these things sort of running around here. Are they all really part of the same package? Or are we talking about different packages that we want to conveniently assemble and disassemble for different kinds of discussion? There are a lot of things about mental computation that were interesting to me which came out of a study I once did with Margariete Wheeler and Merlyn Behr. One was why the effect of context and the presentation of the task seem to make such a great difference. I think there's a parallel question in estimation too. I look at the kind of estimation tasks that students get in school most of the time. They're told, "Today's estimation day, so I'm going to give you the same problem that I gave you yesterday when I wanted an exact answer, only today I want you to give me an estimate." There's a real problem with that. Especially if most of the time the best way to go is to place your bets on an exact answer; then it doesn't make a lot of sense to do a radical shift in your behavior now just to get through today when tomorrow we're going to be back to exact answers again. Why would a student want to invest the kind of energy necessary to learn that?

Similarly, in the mental computation world, it was clear that the elementary school teachers we studied had very clever strategies that they could use to do mental computation. And of course there's evidence from a lot of work that we read that children have those too, but there seems to be very little transfer, if you want to call it that, between what they know how to do and the flexibility that they can exhibit mentally. What do they do if they're given a paper-and-pencil calculation like 1000 - 784? It's difficult when you do it with paper and pencil because you have to borrow from all those zeros and you're prone to make mistakes, but mentally it is not a hard problem to do. Why is it that people don't do a mental transformation to make it easier when they're handed that problem on paper? This task has a lot to do with flexibility of moving around in these various spaces. It has to do with knowing that you might make mistakes in this situation, being sensitive to error, and trying to circumvent it. There are many things that might be involved. But part of the instructional story, I think, has to be building up a better understanding of what it would take to make somebody really flexible.

The main reason I was really glad I could be here is that I've just taken on a new project that has to do with middle school mathematics for educationally disadvantaged students in this country. It's clear that what's involved in this discussion is really important to that project, because if at the end we could somehow come to some agreement that what happened inside of that project was that we came to a better understanding of how to build a curriculum and adapt a curriculum that would get kids to develop number sense, whatever that is, by the time they left middle school, and that we had come to grips with some questions of how to assess it so that we knew what happened, that there was some sense in which we had been successful, then we would have come a long way toward accomplishing what we were setting out to do.

The last thing I want to throw on the agenda here is whether or not it's possible that there is sort of a dispositional issue here. Maybe some people are more inclined to want to make sense out of numbers. Maybe there is a sort of inherent reasonableness to this enterprise in the relationships among the various pieces that I've mentioned. And part of what the educational issue is is to instill more widely in the population of students the fact that it *should* all make sense and that there should be some reasonableness to all this.

Jim Hiebert: My interests in the past years have been primarily in how children make sense out of standard written notational systems in mathematics. For a reason I'm not sure about, I had often assumed that the work in estimation and number sense had sort of a different focus from the things I was interested in. But it struck me, while reading a number of papers in the package, that in fact there is similarity of interest.

One of the questions that arose for me while reading the papers was whether at least in my own mind I haven't partitioned things out a little too much, whether a classroom situation, let's say, that would promote number sense in whatever way we would like to define it, wouldn't also do exactly the kind of things that I would be interested in seeing happen in students' written symbol notation. And if not, what would the distinctions be? I couldn't come up with anything that would fit one situation and not the other.

As I said, my interest has been primarily in trying to get some description of how students make sense of written symbols, especially by looking at how they connect them with other things that already make sense to them. What sort of experience does it take to enable them to do that? So one of the specific questions I'm interested in has to do with Ed's question about whether there is a number sense that's contained entirely within the symbol system and maybe does not require links to anything outside of that. If there is such a thing, and it seems plausible that there might be, my interest would be in its origins, how it got established. I can ask the question in a little broader way, relating it to current writings about situated learning, learning in particular contexts in mathematics such as in the work on street math. What happens if we think about school learning as just another situation of learning? The issue is the boundaries that students place on the learning experiences that they have, in whatever situation they find themselves, in school or on the street or in the store. In some sense I think the street learning is somehow more satisfying for us because it seems to be more within the real life of the person, but I think the real question is how well that kind of learning extends into other situations in which it wasn't initially acquired. It is the same question we always ask about school learning: How do we get it to extend into other situations? Are there possibly some common mechanisms operating that restrict initial learning to within the context in which it was acquired, whatever the context may be, *and* help to extend it beyond the boundaries? Since my primary interest has been in the standard written symbol systems, one of the things that I would like to learn about is how number sense for those written symbols gets established. My suspicion is that initially they need to be connected to something outside of themselves. What is the nature of those connections?

Zvia Markovits: My dissertation at the Weizmann Institute was on estimation, and I'm continuing work on estimation and number sense. I have a lot of questions, some of them already asked, and what I'd like to do here is get some answers. On the research project I'm working on now here in San Diego, we've done some research in which we touched upon some things that are related to number sense, but I am missing the whole picture of what number sense is. I want to know where, exactly, my research fits in, and I'm just missing something. On one hand we have number sense, but then we also have common sense. Maybe common sense is what we use in real life, and number sense only with numbers, I don't know. And then we have number size, mental computation, computational estimation, and probably some other topics. On the other hand we talk about links, schemas, frames, conceptual knowledge, intuitive knowledge, high order thinking, super schemas, metacognition, and so on. I am hoping to somehow find some links between these two columns that I have on this paper.

I have found that kids think that they don't need number sense to do mathematics in school. So if we really want to teach them some number sense or some estimation, we need first to convince them that school mathematics is related to the real world, that they are not two separate things, and I do not know how to do this. Another question is concerned with when we should start instruction on estimation or number sense. Should it start in first grade, or should we wait until the students are older? By starting estimation when they are older, I think it's difficult for them to understand this concept because by then they are used to school mathematics where there is one answer, one algorithm, everything is exact, and then we come to them and say, "Now we have to switch; there is not one answer, not one algorithm." It's probably difficult for them to see it at once, so maybe it would be better to start in first grade telling them we can have exact answers, but we can also have estimates. I think one of the difficulties with including estimation in school mathematics is that estimation requires making decisions. Somehow with those multiple choice tests that you have in this country, you don't expect students to make many decisions. Just read the problem and do something, that's it. With estimation they need to make a lot of decisions before they start to do the problem. How can we educate them to understand that they should be making decisions?

Another question that I have is about teaching estimation and number sense. Somehow number sense cannot be a unit that we can teach for one month and that's it. Somehow it's a *way* of teaching. Are all teachers able to teach estimation and number sense? Can all of the teachers make number sense their way of teaching? I don't know. Maybe some teachers can do it; maybe some need a lot of help in order to be able to do it; and maybe some teachers cannot do it at all.

I have my own example at home. I have a daughter who is ten. She's in the fifth grade. They have been teaching estimation in her school, and you probably saw an example of one of her work pages in the package of papers sent to you. When she tried to solve estimation problems by first rounding and then compensating, she was told that it was a mistake; "You are not supposed to compensate, just round." So now she knows that at home when we talk about estimation, we mean estimation using number sense, but at school, she has just to do it by rounding and following rules. She's very confused.

Hal Schoen: I'm primarily interested in the curriculum and evaluation aspects of estimation. I'm pleased to see that there's quite a bit of interest in estimation. I originally got into this as the editor of the estimation yearbook that NCTM put out in '86. At that time I came to appreciate the potential power of estimation and some of the related number sense ideas that we're talking about. But I also came to appreciate the fact that we're not sure what we're talking about, and that we're often talking about different things. We don't know how various aspects of number sense, computation, and estimation are related.

As I'm sure many of you know, the *Iowa Test of Basic Skills* is produced on our campus. Over the years that I've been in Iowa, I've often complained about things the authors do or don't do on the tests. And one of the things I've complained about is not testing estimation. So when one of the authors asked for help on testing estimation, I couldn't say no. In the process of wrestling with how a standardized test might include estimation, I've tried lots of things. I've used open-ended tests where students were timed on items so they didn't have time to do just a purely symbolic kind of estimation. I was never really satisfied with that. For one thing, I'm not sure that the issue of time is a major one. Another thing we found is that students tended to make these estimations, if that's what we want to call them, almost entirely by a procedural, rote rounding method. I don't think that's estimation, if that's all they're doing. So we have tried a number of other things, including various kinds of formats on multiple choice items where we've tried to write the items--and the foils--in such a way that we are making sure that students can't do just rounding, or at least in some cases can't. One of the papers in the preconference reading pack is a report of some of the things that we found.

With *ITBS* we have access to large numbers of students and can try various items and formats. We've gotten some really strange results. We were working with fifth through eighth grades. On some of the items, we attempted to funnel students into choosing something that would involve compensation, where rounding would not give the closest estimate. We found that the older students and the students who tended to do best on the test as a whole did poorest on those items. They focused on a rote rounding kind of procedure. When we asked some of them to estimate, they reacted by simply saying "I don't know what that means; we haven't done that." Others who thought they knew what it meant almost always did the rounding. We've also more recently tried out some items with and without a context. We collected some data last fall where we did our best to place problems in a little context with just a picture, maybe some grocery items or something where we had a price tag on them, where the question would be to give us an estimate, and the very same item with the same numbers would also be given in a purely symbolic setting. Definitely, when you put a problem in a context, students could operate much better. We found that to be consistently true. It's especially true the more familiar the context is Children are much much better in a money setting than they are in other decimal situations. So there is definitely something different going on. We've just gotten that data back, and we haven't done very much yet to try to probe into that. At any rate, I'm hoping I'll get a lot of good ideas at this conference. The *ITBS* people have some interest in estimation now, and if we can get something acceptable that can fit their constraints, then perhaps we can get them to move in this direction.

Paul Trafton: I've been thinking about an article I wrote on establishing an estimation mindset. I didn't realize until I was sitting here right now that what I was really struggling with in that article was how number sense needs to be part of teaching about estimation. Then later, working on the K to 4 section of the *Standards* for NCTM, we came up with this notion of "developmentally appropriate." Putting young children and mathematics together draws a lot on the work of learning in context and the kind of thinking they can do. Well, out of these situations and the discussion so far today, there are four, five, six points that are going through my head. The first is that when situations and questions that seem natural to them are posed to children, they use very powerful kinds of reasoning, apparently without a whole lot of formal instruction. Ninety percent of the time, Carraher, Carraher and Schliemann found that posing a problem like 242 - 150 in a street context produced methods of thinking that would really represent a high level of understanding of mathematical relationships.

The second thing I think comes out in the discussion this morning is the problem of activating number sense in the school setting. Students do get locked into a school math thing. How do you activate number sense in the context of math? Jim (Hiebert), you touched on something important in an article you wrote. Even when you have spent a great deal of time and effort building references for symbols so that the symbols make sense to students, it seems that once you start working within that symbol system, although you've done some things earlier to insure that these symbols represent something beyond themselves, it seems there is something inherent in the manipulating of symbols that, in a sense, gets in the way of understanding. It's hard to get back to the meaning of the symbols.

I think that the third thing that comes through, particularly from successful practitioners, is the powerful effect of releasing children's intuitive thoughts. This has come out of some work from masters' students dealing with sixth-, seventh-, and eighth-grade teachers. They remark that they seem to be releasing something that was bottled up, and these older students just love it. They saw the amount of learning that took place as students listened to other students. Student teachers in remedial classrooms were saying, "My kids are feeling powerful". A very structured seventh-grade teacher in Chicago public schools was saying, "My class has turned from lectures into conversations." And with estimation/mental computation, when teachers have tried to go in and do something, with a little bit of guidance, something fairly powerful seems to happen, even for students who have been well-indoctrinated into the context of schooling in mathematics education. Maybe one thing we need to do is just to study what happens when teachers start doing this, and try to document in some way some of the changes that seem to occur, cognitively as well as in the climate and in students' orientation toward learning mathematics.

There's a fourth issue. I've read the Carraher, Carraher, and Schliemann article three or four times, and I guess I'd read past it every other time, until this last time, but they made the distinction between the manipulation of quantities and the manipulation of symbols. What strikes me is that much mental computation and estimation has a very definite physical quality. For example, I was asking a fourth-grade boy an interesting question, and the solution I got was quite different from the one I thought would be produced, but he ended up doing $6.00 - $2.85 mentally. He said, "Six minus two is four. If you take away the two dollars, that only leaves four dollars, and you knock off eighty-five cents, and that leaves $3.15." You get a lot of this "knocking off" language, and "glomming together" and "tack a little on here." You don't get just pluses and minuses. I think it comes through in Bob's and Barb's interview studies. It's very physical.

Just a couple more points. The fifth one is that I'm shifting from thinking of mental computation and estimation as being a specific body of knowledge. Much of the research has said: Let's choose examples of the type twenty plus fifteen to see how they do it. I'm finding myself pulling away from that kind of thinking toward: Let's pose some settings of certain types, and not worry about controlling example types and see what kinds of thinking emerge. Sometimes when we pick example types, we're forcing thinking into a certain kind of mold, and I don't think we're learning as much as we could otherwise. I'm also wondering about the effects of instructional programs that begin with a heavy dose of posing situations, allow several possible solutions to bubble out, and select a few of those for further highlighting so students are aware that these solutions could be drawn upon in other situations. There's a delicate balance between saying, "This is the way we want you to do it," versus, "Here's an interesting way to think; let's look at this a little more closely, so we all can make use of it if we want to."

Another thing is that I had trouble for a long time sorting out this "reasonableness of answer," and how it's tied to estimation. I'm using the word estimation now as it's commonly used, producing an estimate. Thinking about this conference and rereading articles, I'm more convinced that reasonableness of answers has very little to do with producing estimates. It's much more intuitive; it's much more global, more of an overall quantitative judgment about what can occur in a situation, which I think ties back more into number sense.

My last point deals with what would happen if we took a different point of view about computation and estimation. Much of our research now on how people handle estimation and mental computation with whole numbers is done after computation has been learned, as an add-on. One of the things that has come through so many of the documents like the *California Framework* and the *Standards* is that we begin with problem situations where either we need to estimate or we need to do the exact computation. We can produce something that's kind of in the ball park, or we can produce something that is exact. There is decision making here. Once we're done we go back and see if what we did fits with the purpose. What would happen if we worked very hard on instructional treatments that would help young children implement a point of view like this? It would be a total approach to computation.

Judy Sowder: In my first research study on computational estimation, I asked students to do problems such as telling me about how much 789 x 0.52 was. Many of the students rounded 0.52 to 1, 789 to 800, and said the answer was about 800. Others said 0.52 was close to 0, so the answer was 0. I began to understand then that number sense had a great deal to do with estimation. In later research I've looked at number size and mental computation as important components of estimation, and they too are closely related to number sense. I believe that instruction on number size, mental computation, and estimation will all lead to better number sense, but they are interrelated in such a way that some number sense is needed *before* proficiency can develop in the trio of related topics. In the research that Zvia and I are involved in now, we have prepared units for teachers on number size, mental computation, and computational estimation, but we are not completely comfortable with the approach of interspersing discrete instructional units on these topics between regular textbook lessons. One teacher, for example, used base ten blocks to teach our unit on number size of decimal numbers, but when I visited her classroom the following week, she was teaching addition of decimal numbers exactly as in the text, with no use of the base ten blocks, and no reference to the size of the numbers and what that might say about the size of the sums. The teacher was enthusiastic about the number size lessons and using manipulatives, but it didn't occur to her to carry that over into the work on operations. My feeling is that estimation and mental computation must be integral parts of all instruction on computation, that number sense will develop only when instruction focuses on quantitative thinking. But I'm not sure how to formulate viable research questions that address this, other than in very general terms, i.e., what types of experiences need to be provided in order for number sense to develop? I am hoping that we can establish a better framework for conducting research in this area, and that such a framework will guide us in developing researchable questions so that we can make some progress in this very important area.

Another question that I have...I don't know whether or not we'll have time to explore it here--is the whole questions of teacher preparation, and what we can do to help practicing teachers. What *do* they need to know, and do, to teach in a manner that promotes number sense in their students? It seems that instruction would have to be drastically different than it is in most classrooms at the present time.

Discussion Notes

Editor's note (Judy Sowder): These are a subset of my notes from the conference, reorganized and edited. I cannot take credit for the many good thoughts expressed here--in fact, an astute reader will be able to identify ideas and even phrases with particular participants. There may seem to be conflicting ideas side-by-side in some instances, but this is to be expected if these notes represent the thinking of individuals who did not always agree. These notes do *not* represent a group consensus. They are, and should be interpreted, as the collective thoughts and ideas of the participants, as I understood and interpreted them. There were many topics discussed that are not included here, not because they were less interesting or important, but because they either had less to do with the central focus of this conference or because they have been expanded upon in other parts of these proceedings.

Number Sense

There are several possible meanings for number sense. These might include general schemas for how numbers behave, judgments of reasonableness of numbers as they are used in particular situations, flexibility in the use of strategies for mental calculation and for approximating answers, the ability to use appropriate benchmarks, the tendency to want to "make sense" of situations involving number and quantity, the understanding and correct use of decimal and fractional notations, the recognition of the relative effect of operating with numbers, the possession of a sense of number size, the realization that number and mathematics are inherently approximate. The more we consider and discuss the topic of number sense, the greater is our realization that it is very complex, and perhaps impossible to define to anyone's satisfaction. No one aspect of number sense sufficiently describes it, so all aspects would have to appear, collectively. Sometimes we give examples of number sense, and say the example is evidence that the child "has number sense." Perhaps, perhaps not. All of the aspects need to be considered as a whole before we can say an individual has number sense. And certainly part of this whole is the self-monitoring that guides the work of the individual who operates with number sense.

Example of Number Sense

It can sometimes be difficult to interpret whether the manner in which a particular example is undertaken displays number sense or not. When children are asked to estimate a sum such as 152.621 and 49.23, they often round to 150 and 50, then round off the decimal parts to obtain an estimate such as 200.9. Is this solution an example of a lack of number sense as it has been interpreted in research literature, or is it an example of behavior based on the belief that something *has* to be done with all parts of the number, and thus a fault of instruction? How about students who reduce 4/9 to 2/3? Do these students lack understanding of equivalence of fractions, or is this an example of a common error they have not yet learned to focus on *not* making? In other words, is this a competence error or a performance error? There are natural sources of interference that crop up in funny little ways. We must be careful about how we interpret them.

Yet we must take care not to put everything in this basket we call number sense. If we do, we could make it impossible for anyone to ever have it. It is perhaps better to think about number sense as a way of thinking rather than as a body of knowledge that can be transmitted to others. Certainly number sense is not a finite thing that is either present or absent. It does not exist in a linear scale, but has different levels and contexts. What we need to do is to identify the conditions that nourish the growth of number sense. Some would say a different theoretical approach is needed. The view that knowledge is something that people construct together leads in a different direction than the view that each individual constructs his/her own knowledge. If we take the first view, we might want to compare the ways learners do problems as a social process, where the ability to understand another's thinking is an important aspect. If we take the second view, we might want to trace the development of the different aspects of number sense for individual students.

But either theoretical path would lead to an approach to the design of curriculum different from the one we currently have. Rather than thinking of designing a set of lessons that will develop number sense, we may need to reconceptualize the development of arithmetic skills and emphasize different ways rather than unique ways of finding answers. We see examples of this happening in Carpenter, Fennema, and Peterson's work in Wisconsin, in Cobb's work at Purdue, in Shuard's work in England. (There is the tendency to think of algorithms as bad, which is of course not true. But we need to enshrine the notion of *an* algorithm, not of *the* algorithm.)

Number Size

Much of what happens in non-quantitative areas of children's general cognitive development, such as in the areas of understanding stories or drawing pictures, can be thought of as changes that occur when things that begin as procedures for children ultimately generate mental objects that can then be treated as primitives. In early number learning, children first understand counting as a procedure; understanding of numbers as objects comes later. In instruction on fractions and decimal numbers, we expect this understanding of numbers as objects to develop, but we rarely make it a focus of instruction. We teach procedures, for example, for dividing one number into another, but we don't design lessons that show that 0.66 bears a relationship to 0.75; the numbers are not taught as *objects*. Number size is not a sense about numbers that once acquired for whole numbers spontaneously generalizes to all kinds of numbers.

Suppose a child says that 1/3 is less than 1/4. Is this an example of a lack of number sense, or is it instead a lack of real understanding of basic definitions? Many would say this answer, which demonstrates a lack of understanding of rational number size, displays whole number sense but not rational number sense. But suppose the problem is one of comparing 1/3 meter to 1/4 meter in one instance, and in another instance comparing 1/3 and 1/4 as proportions. In other words, is comparing intensive quantities equivalent to comparing extensive quantities?

The understanding that there are numbers *between* other numbers is an important aspect of number size understanding. Children realize that there is a space on the ruler between 1 cm and 2 cm, but they don't associate that space with numbers. There is ample evidence that students at various stages don't think there are any numbers between 3 and 4, between 1/3 and 1/4, or between 0.3 and 0.4. In instruction we need to build both a system of numbers as objects and a set of interrelations among the numbers. To do this requires more instructional time on ordering symbolic representations for rational numbers after students are able to associate meaning with the symbols.

Instruction on Estimation

Estimation is not a new topic in the curriculum. John Clark (principal author of World Book elementary mathematics texts from the 1950s) had many lessons focusing on quantitative thinking, on number sense kinds of questions, and on a variety of strategies for estimation and for mental computation. But with the text changes of the new math era, all of that was, unfortunately, put aside. At the present time many students, and many teachers, think that estimation means rounding numbers according to standard rules. For example, Barbara Reys related that at a recent workshop, teachers were asked to estimate the product of 26 and 37. When one teacher rounded the numbers to 25 and 40, this caused another teacher great consternation: according to the rules, 26 *had* to be rounded *up*, not down to 25.

Researchers and curriculum developers do not always agree on whether explicit strategies for estimation should be taught. One side of the argument is that such explicit teaching is counter-productive and fosters learning that bypasses conceptual development, with the result that estimation becomes another topic relegated to a set of rules that are not fully understood by the user. Others believe that a good teacher will generate situations where particular strategies are appropriately introduced, and that the students' exposure to such situations should not be left to chance. Yet most argue that there is a difference between having frequent open discussion on a variety of possible and appropriate ways to solve problems and making sure that every student has mastered every strategy.

Why *do* we want estimation in the mathematics curriculum? There are basically two reasons, which often become confused. The first reason is that estimation helps us develop conceptual structures for number. Learning to estimate is a way of learning something about number size, about the decimal system, about compensation, and so on. It is closely related to number sense. The other reason is that estimation is useful in everyday life. The distinction is important. Work on pure number problems without real-world contexts is appropriate if we are attempting to develop number understanding. But context is very important if we expect transfer to real world settings, since it is the context, the use of the estimate, that determines the degree of accuracy necessary.

When planning any type of instruction on estimation, we must consider the role of intention. If children don't know *why* they are given a problem, they don't know what to attend to. Classroom mathematics usually demands attention to all the numbers in a problem. So when children are asked to estimate 3475 + 5872 + 1983 + 6, it is very difficult for them to ignore or drop the 6. Intention also determines the required accuracy of the estimate. Usually, in school tasks, somebody else (the teacher) sets the parameters, controls the benchmarks. This need not be so; forming benchmarks should be a topic open to discussion in the classroom.

Another aspect of estimation that needs both instructional and research attention is relative versus absolute error. For example, which is bigger, the difference between 3 and 5, or the difference between 123 and 125? They are absolutely the same, but relatively very different. Or, if I want to estimate 48 x 1002, will I get as good an estimate by rounding 48 to 50 and finding 50 x 1002, as if I round 1002 to 1000, and take 48 x 1000?

Both teachers and students feel less comfortable with computational estimation than with mental computation, because in estimation problems there are many possible answers, whereas a mental computation problem has only one right answer. Perhaps we should devote time to having children make sensible quantitative judgments *before* they attempt formal estimation. For example, knowing that 3472 + 5481 is more than 8000 is adequate for third graders. Why do we try to get to estimation so early? It is a complex topic. The timing of instruction on estimation is related to the teaching of strategies. If students are not developmentally ready to estimate, then strategies become just rules to follow, and estimation is far removed from number sense.

Not getting the exact answer should not be a focus of estimation. Estimation is not in opposition to obtaining an exact answer. Rather, it deals with how close we need to be to an exact answer. There has to be some reason for stopping short of an exact answer (when obtaining one is even possible). The question of intentionality is again relevant. Sometimes students are asked to estimate then find the exact answer, to see if the estimate was close. This betrays the usefulness of an estimate, which should tell whether an *answer* is reasonable, not vice versa.

Views on Mental Computation

There are two very different views of instruction on mental computation. Some educators believe that instruction should focus on providing students with an opportunity to generate intuitive solutions to problems rather than having them follow the usual algorithms. This kind of mental computation both demands and extends number understanding. It fosters number sense by requiring students to look closely at the numbers and to think about what they mean, how they can be changed in appearance without change in value. For example, inventing a left-to-right procedure for a problem demands thinking about place value; other procedures require that numbers be decomposed and recomposed. A problem such as 999 x 27 is interesting since much can be learned from a discussion of methods for doing it mentally. According to this view, reflection needs to be a part of mental computation. In contrast, if one's view of mental computation is doing problems very quickly in one's head, instruction can be reduced to drill. In classrooms of teachers who hold this view, mental computation becomes operationalized as skill performance, "learning tricks" like multiplying by 11, or doing chain arithmetic using basic facts despite the questionable value of such problems. Unfortunately, the second view is what many people believe the *Standards* (NCTM, 1989) is advocating.

Theoretical ideas about skill acquisition that have bearing on this discussion deal with the growth of expertise in situations where the use of deliberate rules, that is, rules which call for reflection in their execution, tend to come in the early stages of the acquisition of the skill. For example, a student driver must reflect on the use of the clutch and its relation to shifting gears. But after the reflective stage, when reflection on the individual parts of the process is no longer necessary, there comes a stage marked by flexibility. Chess provides an example. An expert simply recognizes what is important about a position and responds to that. The power lies in what is not being articulated or formulated as a procedure. If we take as a conjecture, then, that much of what we are calling flexibility belongs in the more advanced stages rather than in the earlier stages of deliberation, how does this apply to mental computation? It raises the question of whether we want all of the "doing of" mathematics to be reflective, or whether there should be some areas where flexibility is no longer dependent on reflection.

This brings us to the question about the role of automaticity. Will this flexibility just happen? Will students automate their own procedures? Some people view automaticity as a good thing to have. All of us learned mathematics partly through drill and practice--partly through other things, too. We don't know yet what the right mix is, but, theoretically, we need to think about the role of automaticity in learning. Psychologically there are really two views on automaticity. The first one gets more play in textbooks and may be misleading. This view of automaticity focuses on a very narrow set of things--as in the training of an air traffic controller. This type of automaticity might involve thousands of trials, all doing the *same thing*. Although the repetitiousness of the trials might seem to be mind-deadening, it does lead to automatic responses. The other version of automaticity is less mechanical, much more flexible. The individual reaches a point where she/he has seen all possible variations, in so many versions and so many contexts, that she/he can immediately sense what to do. In mental computation, the ability to *sense* numbers may be part of what we're talking about--the ability to immediately recognize relevant features and relationships without engaging in deliberate cognition to get at them.

We need to think again about what *needs* to be automatic. With respect to mental computation, does anything besides knowledge of basic facts need to be automatic? Perhaps multiplying by powers of ten is also essential, but this is quite prescriptive. Knowing doubles is useful for problems like 24 + 26, but not for problems like 86 + 87. Yet *knowing* that using doubles is useful for 24 + 26 but not for 86 + 87 is really part of the second notion of automaticity--more flexible, less mechanical. The paradox is that people who are good at doing mathematics exhibit considerable mechanical automaticity. Can good number sense exist in the absence of this mechanical type of automaticity?

Another aspect of this discussion involves the many people, including students, who are very good at pencil-and-paper procedures, can do them quickly and proficiently. Such individuals might find $1.49 + $0.32 by the computational algorithm. Do we say these individuals lack number sense? Perhaps, like reading teachers who make students functional in reading but then are disturbed when they read comic books, we don't want our children using the algorithms we've spent weeks and months teaching them. Some current research by Sowder and Markovits has shown that students proficient in the use of algorithms are extremely resistant to changing to "invented" algorithms. They don't see the point of inventing procedures they think are more difficult to use.

Teachers need to know how mental computation can best be incorporated into mathematics classes. There is disagreement on the efficacy of devoting ten minutes a day to mental computation. This insures, on the one hand, that mental computation at least gets some attention. But on the other hand, there is a danger of focusing on mental computation drill rather than on a discussion of how problems are done. How *do* teachers use the 10-minutes-a-day method, on explanation and discussion, or on drill and practice? Is there a place for both, or not?

A subtext to this discussion is, of course, the ways we make *any* of these ideas accessible in schools. There is a deep difference of opinion on whether we should attempt to change the whole curriculum, or whether we must accept the basic structure as a given, and try to revise chunks of the curriculum. But this question reaches far beyond the topic of mental computation.

Assessment

We are dealing here with a new definition of knowledge. We are not talking about bits of things people know, disembodied from action. Yet the old theory of knowledge, knowledge as stable, discrete bits, is at the very heart of the standardized testing scene. We cannot fix the tests by simply changing the items. One solution might be to find a way of attaching single numbers to situations that need to be evaluated. It is possible to rate a complex performance on several dimensions, such as is done in gymnastics or diving competition, and combine these ratings in some fashion.

On the other hand, standardized tests have a strong hold on the educational system in this country. Is there anything we can do that might make these tests more amenable to the kinds of thinking we are talking about here? Are there ways we can exploit the link between what is on a test and what is in the curriculum? Suppose, for example, some squares are displayed, together with the size of a unit, and the student is required to choose the square with the area closest to 4 x 6. For the child who had learned only to multiply numbers, this would be a difficult item. What happens when teachers realize that it is important for their students to be good on these types of items? Will that push them more toward including activities in the curriculum that satisfy our intuitions about what number sense is?

One of the problems with assessment, of course, is that we don't want to reduce estimation and number sense items to things that can be trained for, and somehow bypass conceptual development. There are still a lot of people convinced that they can overcome the testing problem students have by very explicitly trying to teach all the strategies a student might encounter on a test. Such thinking runs contrary to what we believe is the essence of number sense--a way of thinking rather than a body of knowledge and skills.

Theoretical Framework

One purpose of this conference was to try to establish a better theoretical foundation for continuing research on quantitative thinking, number sense, mental computation, and estimation. How are these topics related to one another? What are the most important research issues? How can they be best addressed? Cognitive psychology has provided models for learning that have been very useful to investigators interested in how individuals learn certain kinds of mathematics. However, information processing models are less suitable to studying topics like number sense, which is less a body of knowledge to be transmitted that it is a way of thinking. Rather than wanting to know how someone goes about arriving at a particular answer, we prefer to understand how the individual relates different answers to one another. This research domain is perhaps a good example of where current cognition models are inadequate to address the essential issues that need to be investigated.

Maybe we should not be expecting or even seeking consensus on a framework. We come from different epistemological traditions. If we see people as existing within traditions, as often interacting with each other but at other times talking at cross purposes, then we must give up on the notion of theoretical consensus or even the notion of consensus on direction, because "we aren't such a we."

Yet we do have consensus that quantitative thinking, number sense, estimation, and mental computation are very important topics to investigate and that the kinds of investigations found in the preconference readings need to continue. We have some unanimity, some largely shared judgments about when we are observing a display of number sense or estimation. We can talk about the same event, the same cases, but our interpretations will vary. One of the benefits of this type of discussion is the opportunity to enlarge the space of interpretation. For example, if one thinks of knowledge as situated in environments, as created between people, then what predictions would that person make about developing quantitative thinking? That person's interpretations will be different from those of someone working within a Piagetian framework, for example.

Wrap-Up

Editors' note: Toward the end of the conference, we returned to concerns about a theoretical framework for research, about the nature of number sense, and about how to assess number sense. The following is an edited version of our final discussion.

Judy Sowder: I know that some of us feel the need to narrow our discussion somewhat and talk more about what some of the researchable questions are. What *can* we do? It might be nice if we could throw out the present curriculum and start all over, but we can't very well do that. What *are* some things that can be addressed in mathematics classrooms as they now exist? Bob, would you like to begin?

Bob Reys: I came here hoping to leave with a theoretical framework on which to base some research that we want to carry out. I must confess that, at this point, I don't feel that a lot of what has gone on helps me in terms of synthesizing and putting it all together into a framework that would guide the research I want to do. Part of that is probably due to my own limitations.

Judy Sowder: From what I understand of what's happened here, maybe that's not possible within the context of what cognitive psychology has to offer right now. According to what Jim was saying, maybe we need to look at research in a different way. Am I right, Jim, or can you rephrase that for me?

Jim Greeno: Can you say more about what sort of thing you were hoping for? What kinds of problems or research activities would you like to see addressed here?

Bob Reys: I guess what Barbara and I want to do is to build on to the framework that we proposed ten years ago and have continued to refine and develop. Robbie and Judy have provided a sort of structuring which I think is very important in terms of guiding both teaching and curriculum development. I see those models as very appropriate to guide further research. Yet my impression is that others think that what we're doing is not adequately theoretically based, that our research needs to tie into cognitive psychology to a greater degree than it has.

Jim Greeno: You have here at least one representative from cognitive psychology who's a little concerned about the current resources that we have to offer.

Bob Reys: I realize that now, and that has helped me. I came here thinking we'd hear more of "You ought to think about this; you ought to think about that." But what I hear instead are some nice ideas that haven't reached the state of closure that I had anticipated, and I'm glad to know this.

Lauren Resnick: I think that the framework that's in Robbie's and Judy's paper is very helpful and....

Judy Sowder: Very limited though.

Lauren Resnick: Well, it's limited to the particular domain of estimation that it addresses. If I remember correctly, it's all on the additive kinds of structures. Attempts to do a similar job for other parts of the mathematics curriculum would be extremely helpful. It's a harder job to do this for rational numbers, for multiplicative structures, but that needs to be done. It is absolutely crucial that the kind of work being done continues. We need a structured analysis of the domain of knowledge that we have in mind. But I'm struck by the fact that there is also something else, a new part of the space. It comes up when we ask, "Well, is knowledge really in the head?" That part of the space is another agenda in its own right. Maybe we could get the questions more focused, but I actually think that there is a big agenda out there, of trying to figure out what the environments are that would promote or permit or demand the kinds of behavior that we mean. I don't mean the *kinds* of knowledge per se, although knowledge will have to be considered, but rather the kinds of activities, and then we have to look at the relationships among those environments. If you get good in one kind of activity, does it make you good in another? If so, in what way? One of the reasons we can't get specific is that this is dramatically different for all of us, for anybody who's been in the American intellectual tradition studying learning. It's just radically new to think of activities and situations as being central to the question rather than simply being the dressing within which concepts are manifested. And so we have to figure out what it means to work that way.

Jim Greeno: There are a lot of things that we've been leading up to, and then we've stopped because we don't know of any research that has been done that supports those things. One of those things is this: Suppose you look at people who have gotten pretty good at quantitative reasoning within some domain that they work in, such as carpenters, or bakers. No one has yet identified the kinds of capabilities that they have in that environment, and then given them a set of tasks in some different environment and looked at what they carry into the new environment. That's a research question that I hope somebody will go after. Related to that is the question of interaction between success in school math and the ability to learn how to reason quantitatively in another environment. There are plenty of studies that show whether or not a person can pass a school math test, but they don't tell us a lot about whether the person will be good in some practical situations. But it would certainly be reasonable to ask that question about number sense. How does it carry over from school mathematics to other situations?

Lauren Resnick: We have zero as far as studies of learning in places other than school. We do have a few studies now of performance in work settings, like Sylvia Scribner's studies and the Lave ones. But I have been searching for three years now, and I can't find a single study that looks in any detail at how people move into a work setting and at what the intellectual resources needed are. There are plenty of them that look at the socialization into jobs and so on. But I can't find a single study of the kind we're talking about now. I shouldn't assert there are zero, but I've been searching, and I haven't found anything.

Jim Greeno: There isn't anything where the results are very interesting. And it founders on the fact that nobody has found an interesting way of assessing job performance.

Lauren Resnick: Job performance turns out to be a set of ratings by the person who is supervising the work and just doesn't get at the level of detail that we would need. What does get studied a little bit is something called on-the-job training, but that's really just learning in another kind of classroom. Some of the best studies are in the military. They're called on-the-job training, but when you look at what they're doing, you see a version of a standard curriculum, often little related to what is going to be needed on the job. The training is clearly functioning as a kind of gateway. So research of the kind Jim was suggesting would be very useful.

Judy Sowder: It just seems as though, with such things right now as the *Standards* document coming out with its emphasis on estimation and number sense, that we have to set some priorities. The fact is, those of us sitting around this table admit we don't really know what number sense is, and yet the *Standards* document is saying that we need to start teaching number sense, and this bothers me. What *are* the most valuable things that we could do first in terms of outlining what this domain is and how we think children learn it? For example, Robbie and I have looked at some developmental issues related to estimation, and I certainly think more work needs to be done in terms of when students are ready to learn particular things. I'd like to hear what other people have to say about priorities.

Robbie Case: Since there isn't complete agreement, or anything approaching it in the area of cognitive psychology, as to how to look at knowledge and its acquisition, short and long term, one thing we could say is that there is more than one possible agenda. One important research question, I believe I'm looking at a part of it, is what happens in populations that don't come to school with what we would call number sense? We've looked at a lot of examples--street kids, and so on--where they seem to have it, but the real problem is that they don't access it when they're in school. It's as though we say: Leave your number sense behind when you come to school; check it when you come in the front door. Then there are the examples of children who don't have anything to check, like the ones I work with. Number sense for a young kid can be as primitive as knowing that five is bigger than three, and if you don't have that number sense coming in, well, what happens? What if I define number sense as knowing a little structure, such as for the numbers 1, 2, 3, 4, 5 and a set of objects on the table, knowing some relations like "next to" for the number string and "plus one" for moving up in set size; what happens if you treat that as number sense, produce it in children, and then turn them loose in a regular school?

At the upper end we've got kids with concrete number sense. They have some rough sense of the decimal number system and of how you can break things apart, and so on. But they don't seem to have higher order number sense of the kind Merlyn has mentioned several times, in areas such as multiplication, proportional reasoning, probability. There are various higher order, quite abstract mathematical domains into which students are initiated in much the same fashion as they previously were into arithmetic, kind of algorithmically. They learn to pass the test. What would we have to do to get *real* number sense there? So that would be the question at the upper end. What happens long term when we do what almost everybody has agreed is useful, which is to make things a little less algorithmic? We have consensus that we want kids thinking mathematically. It seems to me that a possible research question is: What happens cumulatively if we've got classrooms operating as we think they should, and we try to build a sensible curriculum? Would the effects be really radical or would the effects be simply that now for school tasks, they wouldn't check their common sense at the door any more? What would be the long term effects? Those are some possible agenda items.

Paul Trafton: We've shifted from Bob's concern about a theoretical framework. Maybe it's part of our western tradition to feel that we have to have a correct logical foundation before we can proceed. But certainly within mathematics that's not true. Calculus was created and used long before the logical foundations were there to explain why it had to be that way. Maybe we need to shift from a question of theoretical framework to other questions. Do we understand the phenomenon under investigation slightly better than we did before? That's not a bad place to begin research, to try to get a better handle on the phenomenon, and I think we've done that. Do we have a set of guiding principles? From what Robbie just said, I could pull out four or five guiding principles for our research or for our question asking. One is that children apparently, we don't know how, spontaneously or intuitively or whatever, do have something called number sense. They have some intuitions about number; they have some things that seem sensible for them to say about numbers, like "five is more than three, not only that, it's pretty close to three." These are intuitions; we don't know quite how they develop, but somehow these intuitions don't get capitalized on or connected to school learning. That seemed to be one thing that could guide some work. A second would be that in certain well-bounded contexts people display a high level of number sense. It shows up in mental computation, in their strategies. What would be the possibility of trying to find some contexts that seemed plausible for school settings and then trying to look for ways to build connections from those?

Jim Greeno: I think one of the things we might want to do in a kind of systematic way is to explore situations in which number sense that youngsters do have can be displayed. A lot of the research we do sets up situations where maybe a few kids get it right, but most don't. We need to create situations where many will do well. Some of the research that I'm doing now has that as its goal and it seems to be working. It is another version of a research approach to the problem that says: Let's find out as much as we can about what number sense is in place for children who have reached a certain point. What has happened to different youngsters by the time they get to sixth or seventh grade that would enable them to do less well or better on a set of tasks that we design to make it as easy as we can to get at the process? Then we need to look at other kinds of situations where the relations between the different quantities are not so easy to figure out, and see what kind of number sense is there. There's a whole set of interesting questions in the spirit of trying to understand the relation between quantitative reasoning or number sense and how somebody can reason in particular situations that we devise.

Hal Schoen: Something that has been mentioned, but perhaps not emphasized enough is that just about every activity we've identified as falling under the broad category of number sense also has some characteristics of reflectiveness about mathematics. It struck me in our work in testing, for example, that we've been able to write test items to see whether students can estimate or whether they can judge reasonableness in certain very constrained situations, but we have not tested whether in fact they would do so in a broader situation. So the disposition to actually call upon these things that we're talking about in terms of number sense is an important part of what we want to look at. And another is, as we talk about teaching or having students learn in groups and all of these various ways that we want them to be reflective about what they do, a lot of issues are raised about teacher education. There certainly are highly researchable questions there for math educators who have an interest in that. I think that's a rich area for research.

There is another thing too, that Ed brought up at dinner last night. It could be that number sense is too broad a term to be very useful for us in research. We may want to use the term number sense in discourse in math education, and we may have a fair idea among ourselves what we mean, but, in fact, to try to define number sense in some way that would guide research may not be appropriate. I'm not sure that is exactly what you said, Ed. Maybe you want to correct me.

Ed Silver: Well, I was thinking about a parallel that I saw between this discussion about number sense and previous discussions about metacognition. There are several levels on which these parallels exist. At least some people in the field of metacognition have argued, I think persuasively, that maybe it is better to use language like "regulation," "monitoring," and so on, to describe those things that we were talking about when we were engaged in trying to think through a particular piece of the metacognition story. Maybe we don't need the word metacognition. It doesn't give us a coherent sort of concept to think about. We might be better off separating out the pieces and thinking about the pieces separately, understanding that perhaps they will all fit together in some way, eventually. What we need is to understand the separate pieces better in order to see how they might all tie together. In some way number sense is similar. It seems to be a paralyzingly large phenomenon that we don't quite know how to get a handle on, how to get it all inside the same loop, but we do have a better sense of knowing what it means, for example, to judge the relative size of numbers. I'm not arguing that you want to get really narrow about what that means, but that it might be helpful to think about those pieces sometimes, rather than trying to think about the whole area of number sense. It might be harmful too, and that's why keeping both levels in mind seems critical, to realize that they might be reassembled, and that their reassembly might give something greater than the sum of the pieces.

Judy Sowder: Would you make that same statement about estimation, that rather than studying estimation, maybe we should just be looking at reasonableness of results of a certain kind?

Ed Silver: I don't know, I haven't thought about that. I'm not sure. I feel more comfortable with estimation than I do with number sense. It seems more coherent.

Tom Carpenter: Do you think that estimation is coherent? I think that there are really qualitatively very different processes one brings to different kinds of estimation. I think estimating operations with fractions, for example, seems to depend on very different kinds of knowledge than estimating operations on whole numbers. I mean, you may wind up doing very different kinds of things with them. A lot of times we talk about estimates involving fractions when in fact we mean to ignore the fraction part and deal with the whole number.

Robbie Case: On another topic, one of the things in our language that we are socialized to is that the narrative form in our literary culture is different from what is used in a normal culture. Although we get students to generate answers to problems that we give them, we don't teach them much about how to write mathematics. What are the forms in which you cast such intuitions as you have so you can pass them along to someone else?

Lauren Resnick: This is an example of a kind of researchable question, but one that requires research of a different form from written tests or the well-structured interview studies that we have gotten good at. Even though we're saying that we don't have the definition of number sense here, there is a germ of consensus about the nature of this space, that it is open-ended, that it requires quick grasping of bench marks, that it requires being reasonable, all of these things. These push beyond skills to culture, to a creation of culture, to socialization. And we can't study that process except by either trying to create the culture and seeing what we've got, or by getting in there and trying to find structured ways to describe what is going on and what it looks like, what is in this messy environment. Maybe that's why it seems as though we weren't talking about a research agenda, because this doesn't appear in the kinds of papers most of us have been publishing for many years. I'm not making a distinction between math educators and cognitive or any other kind of psychologists. We've all been working within the same basic framework of what constitutes a research study, and what is partly uncomfortable here is that we're going to have to craft new forms of research to look at these questions. It's truly new because in the kind of stuff that has gone on before when people have looked at the messier environments, they have almost never cared about the details of the intellectual activity, and we care. And it's going to be new.

Judy Sowder: Is there anything in the literature on higher order thinking that would help?

Lauren Resnick: Well, I wrote a book on that (*Education and Learning to Think*). I have it here. I keep opening it up and saying, "Is this what we're talking about?" I wrote a so-called definition of higher order thinking in here. We could substitute number sense for every piece of that definition; let's just try it. Number sense "is *nonalgorithmic*. That is, the path of action is not fully specified in advance." Number sense "tends to be *complex*.

The total path is not "visible" (mentally speaking) from any single vantage point...." and so on. The question then was how much research on higher order thinking was around. It had to be pulled, tugged, screaming, out of domains, out of kinds of work that didn't call themselves being about higher order thinking, but called themselves being about motivation, called themselves being about making inferences in basic reading. The crazy thing is that you could say none of these are relevant, or all are. And I don't know where you want to land because there wasn't *a body* of research on higher order thinking in my judgment. There are lots of studies that are called higher order thinking studies that didn't find a role here. It isn't that I didn't see them. I made a definition that I think is consistent with what we've been talking about that excluded the things that turned out to define higher order thinking as "you can make an inference, you can separate fact from opinion," and all of those sorts of things. They don't show up in this book. It won't be the same to think about this specifically for mathematics as it was to think about it more generally. But we need to do it in mathematics. We can't substitute general work on thinking skills for what we're talking about. What is new is that we have to do it in every subject and in each of the topics in the subject that we think can survive this scrutiny.

Paul Trafton: I'm trying to tie Lauren's comments back to the idea that number sense is more a phrase that's used to evoke a certain approach to teaching, rather than a highly defined construct. Now there are some dimensions of number sense that have some particular usefulness to those of us who are interested in mental computation and estimation. One would be how children describe numbers in a variety of ways. Clearly the additive composition notion, that 24 is 2 dozens, or four 5s and 4 left over, or two bunches of ten and four more, or six less than thirty, those kinds of things, you might even call number fluency. It's an ability to describe a quantity in terms of other quantities in a variety of ways. That seems to be an aspect of numbers sense that, as I've looked at children and their doing of estimation and mental computation, would have some usefulness to me. The relative magnitude part, whether a number is small compared to a certain number, but big compared to another, which may be part of the same continuum, also has some possible usefulness. The ability to make reasonable quantitative judgments about a situation, two things, each costing 7 cents, have to be at least this much, but not as much as that, is another. One of the things I've been trying to clarify for myself is the ability of the student to sort of sense when a result just "can't be." This is an aspect of number sense that I think would be useful for those of us who want to look more at estimation and mental computation. Maybe what's come out of the last fifteen minutes of discussion is that pursuing number sense as number sense isn't going to be as useful for us as pursuing those aspects of number sense that seem to have some direct relation to our interest in how children process numbers in computational situations.

Judy Sowder: Does that mean we don't have to assess number sense now? Teachers, curriculum developers, and testers are going to ask about that, and we've talked about it now for two days. What are we going to tell these people?

Lauren Resnick: I would turn that question around. How would one change the whole testing system so that number sense was valued throughout it? Because what we've been saying for two days is that if you keep switching back and forth from calculation algorithms to all these other things, estimation, number sense, reasonableness, and so on, and if the tests value the exact answers and the fixed routines, then we just won't win. Everybody seems to be sharing that intuition. Well, if that's true, we've got to change the tests. What would it take to make the tests reflect this kind of thoughtful work on numbers instead of the usual routinized work on numbers?

Tom Carpenter: But I'm interested in the question of what we *can* assess. We really haven't spent much time on that. I think it's a really difficult problem. I think in our own research we tried to be as sensitive to that as possible, use as good measures as we could. But I'm convinced there was a lot more going on than we were able to measure.

Jim Greeno: Earlier, Tom, you said that teachers could tell that their students were learning something significant. Would that be a clue for us, in terms of how to develop ways of determining that significant learning outcomes are occurring? How is it that the teachers realize that these good things are going on?

Tom Carpenter: Well, I'm not sure that the teachers do anything a whole lot differently from what anybody does who's talking to kids on a regular basis. I think our point is that you don't have to prescribe ways for teachers to assess. You can give them the same kinds of techniques that we use and the same kind of understanding, and they're able to make very much the same kinds of judgments. The question I guess I was raising about assessment is that I think we can provide lots of rich examples of performance that are very impressive, and we can at least say 60% of the kids can do this and that. It's harder to come up with numbers to use to *prove* we've been successful.

Lauren Resnick: Tom, I think we can come up with numbers, if that is what we want. Olympic judges take a kind of messy performance, and put a series of judgments on it. It's dimensionalized in ways that the theory of diving, for example, says matter. There are maybe five or six dimensions that are judged for diving, or for figure skating, and then each judge also does a summary. Whatever the formulas are, they come out with numbers. So if people want a number, we can always get numbers if we're making good judgments. I think that a major research agenda is to figure out how to do this for individuals or pairs or small groups, conceivably even whole classrooms. I don't know where we'd get the right trade-offs of reliability in judgment versus ecological validity (I guess would be the word), how we figure out how to put some templates over student activities that would allow us to get adequate inter-judge reliability. That's the kind of problem we deal with whenever we do anything where there aren't fixed answers.

There are two basic patterns of assessing open-ended kinds of performance, and there are only two, I think. One is to set up a special performance and either to take videotape or to actually sit there and rate it. The other is the use of portfolios where work is collected over a period of time. The teacher and children together select what they consider to be the best work, and that is submitted to a panel of judges. Those are the two assessment techniques that we use in all performance domains, art juries, international violin competitions, olympics, and now writing.

Sandy Marshall: We're starting to do some of this in the CAP (California Assessment Program) tests. Reliability is a problem. Another problem is that we end up with many hundreds of assessments because testing is still locked into the concept of a broad sample. We've got to get rid of that notion.

Lauren Resnick: We have to get rid of the idea that we have to test every child every year or even every three years. If we're looking for a school profile, there's no need to do that. And we don't even need a score on individual students. If we decide that what matters is the ability of children to hold a mathematical conversation, then we take the conversation as the unit size and assess it. As soon as we get away from the idea that knowledge is the accumulation of individual bits carried by individual people, we can solve the assessment problem. Not easily, I don't mean the answers are sitting there, but the conceptual framework is available. The experts in the twenties gave us one method of getting reliable judgments on human competence, but only one. And it doesn't happen to be the right one for what we want to do now, so let's make another one.

Judy Sowder: Are there any last issues, questions, points, summaries, that people around the table would like to make before we finish up?

Ed Silver: I have one. Most of our discussion over the two days focused on whole number understanding at the K-6, maybe the K-4 grades. I think that it would be useful to keep in mind that a lot of the really important stuff that we're interested in, and a lot of things that would constitute good examples, might happen later in the curriculum. One interesting question is: What happens if you establish a certain sensible foundation at the primary grade-level for this kind of orientation to mathematics? How would that play itself out as students moved on in mathematics? I mean, there's a sense in which, when you move up to the high school level, it's a much tougher nut to crack. There's a kind of number sense, maybe not number sense, it's a kind of math sense, about algebra, for example, that's also something that we want kids to have, sensible answers to equations and sensible ways of manipulating them and so on. If students were oriented to make sense out of math, if they developed that kind of disposition early, what would the consequences of that be as they move up in school?

Paul Trafton: I want to go back to one of Lauren's general comments and try to place it in context. It seems to me that when we have a phenomenon like estimation that is being treated at best as an add-on to the curriculum, and when it's done in such ways that it doesn't seem to make sense, and then we try to insert a few items on tests that basically are measuring very small items of knowledge, procedural knowledge really, well, we're just never going to find that that's a productive way to go. Suppose we have whole tests devoted to estimation, that begin with quantitative judgments and things like that. There may be some ways of creating such a test, but this whole notion of inserting these items into a vehicle which is built around other structures is simply not going to trigger in kids' heads the kind of thinking that will produce the kind of results we're looking for.

Sandy Marshall: One of the things that we've been trying to do as we're building a CAP test is to get at this. As Lauren said, there's some sort of performance we want the kids to demonstrate, and it's not being done with current tests. We're basing some of our work on the Shell Centre material which has a long module that is part of the classroom activity. We will interrupt it at various points and get a report from the students, in small groups or individually, to get at some of these issues. This should get students to anticipate and think about where they are, where they are going. We can get a record of that. If we continue with the whole module, we have a progression of measurement, if you will, over time. Typically the modules would be about two weeks long. Many of the things we'll see will simply be the students jotting down where they are, or small groups will give a brief report to the class. This amounts to a kind of portfolio. These things get accumulated and also are a history of that process over a small period of time.

Ed Silver: One thing about a portfolio is that students can pick them up and carry them for the eight years of elementary school. We can begin to think about things like growth and change and measuring how thinking has developed over time. But we wouldn't get this by simply selecting out the best things. We could ask a different kind of question such as: Show me some evidence of how you're thinking about fractions, or show me some problems that you've worked on that show me that you really understand something about percents. Those are different kinds of questions that a portfolio kind of approach could be used to get at.

Paul Trafton: I have another idea about testing mental computation. The numbers themselves often trigger the extent to which kids are going to be successful or not successful. Sometimes you try to control that by teaching a skill with certain kinds of numbers, then testing it. What would be wrong with posing five situations from which the student or the group was to select two and try to explain how to work them out. We would choose five that could well be in the range of most kids at that level, children who had been having certain kinds of experiences, and they would select two to tell us about. That would solve the problem of the numbers dictating the response in mental computation.

Lauren Resnick: As these ideas come up, we can recognize that each of them has been used somewhere in the past, but usually not in a mass assessment approach. Your kind of assessment, Paul, is characteristic of Ivy League examinations: "Here are five questions; answer any two of them you want." The portfolio is used in certain private schools that don't do regular grading. So, the models exist. Actually, the trick is to figure out how to get them used more widely. Part of the problem is economics. Part is how to build a community of people in the public education system who know how to interpret them. Somebody in the CAP system told me of a district that was merging money for assessment with money for staff development because learning how to grade the essay tests in the CAP was one of the best ways they could think of to train writing teachers.

Judy Sowder: I think this does really get at what we were saying. Yesterday we talked a little bit about assessment, and agreed that perhaps we cannot assess number sense in the same way that we've assessed other mathematical topics in the past. We have to do something different. I think this conversation has addressed that issue. It's obvious that this is going to need a lot of attention. But there are models out there, starting to be used.

Part II: Reflections

Defining, Assessing, and Teaching Number Sense

Lauren B. Resnick

As I have reflected on our discussions at the San Diego meeting, what stands out most sharply is the difficulty we had in defining *number sense* (perhaps less so *estimation*, but that was not easy, either), and our concern about finding ways to assess it. I think that the two issues are related and that our difficulties reflect the extent to which educational discourse and practice are rooted in a set of assumptions about the nature of knowledge and competence that do not fit well with today's educational aspirations or, for that matter, with current cognitive/epistemological theory.

Our traditional ways of defining educational objectives and assessing achievement derive from assumptions built into associationist psychology and subsequently maintained in behaviorist intervention theories. These have entered into the mainstream of educational thinking to such an extent that we find it hard to imagine proceeding without "clearly defined objectives" and "objective measurements" of those objectives. What counts both as an objective and as an acceptable test of achieving that objective is influenced by two hidden assumptions that I call the *decomposition* and the *decontextualization assumptions*.

The decomposition assumption refers to the notion that competence can be completely defined by a collection of independent elements of knowledge or skill. According to this assumption, when all components of a complex capability have been identified, the capability is completely defined. E. L. Thorndike, a primary theorist of associationism and a proponent of its application to both teaching and testing, illustrated how the decomposition assumption works in educational thinking in his 1922 book, *The Psychology of Arithmetic*. In this book, he attempted to identify all the components (stimulus-response associations, which he called *bonds*) that comprise arithmetic. The book contains lists of hundreds of these bonds, such as number facts and steps of procedures. Thorndike claimed that these bonds, taken together, *are* arithmetic knowledge and skill; there is nothing more. The whole of arithmetic competence is the sum of its parts. From this definition flowed the instructional prescription that each of the bonds be practiced--*exercised* in Thorndike's terms--until it could be performed reliably and automatically. From it also flowed the proposal that arithmetic competence can be tested by observing performance on a sample of the individual components. This is what our standardized tests are designed to do.

The decontextualization assumption refers to the idea that competence exists independently of the performances that it enables; that there is some pure or abstract form of knowing that remains intact no matter what the conditions of use; that knowledge is fully defined as something inside an individual's head, independent of the situation in which the individual acts. In this view, if one *knows* a number fact, that knowledge always takes the same form. There is no room for the idea that one might know that 4 quarters make a dollar but not know that 4 times 25 is 100 or that 4 x .25 = 1.00--except that a child fails to "apply" his or her knowledge of quarters to the different symbolic form. In this conception of knowing, teaching, and assessment, everything is determined in advance. Since competence is fully defined by the components, there is no attention to the different ways in which the components might be used, how they might go together to produce personal and interesting new constructions in specific situations.

How can number sense be defined within the confines of the decomposition and decontextualization assumptions? It turns out to be nearly impossible, because number sense, upon reflection, is *not* a collection of things that one knows about numbers or of skills that one can exercise upon numbers. Rather, it is a set of not fully predictable things that one tends to do with numbers under certain circumstances on the basis of a body of interrelated concepts of number and knowledge of specific numbers.

The following is a list of possible indicators (not components) of number sense:

1. Using well-known number facts to figure out facts of which one is not so sure. Note that this will be observed only when the particular individuals involved feel they know the "benchmark" number fact much better than the fact to be figured out, when they value accuracy more highly than speed, and when there is no more efficient procedure (such as counting) available to them.

2. Judging whether a particular number constitutes a reasonable answer to a particular problem. Note that someone might be able to judge what is reasonable in some situations, but not in others. Note also that we would normally see this kind of number sense only when a person has generated a wrong answer.

3. Approximating numerical answers (rather than calculating exact answers). Note that this will be used only when the individual judges that an approximation is adequate and when it is easier (quicker, more reliable) for that person to approximate than to calculate exactly. The latter may depend, in part, on whether pencil and paper or a calculator is available.

4. Using the decimal structure of the numerical system to decompose and recompose numbers to simplify calculations (especially mental calculations). But note that such methods can only be taken as evidence of number sense if they are flexible and "invented" by the child (something we can ascertain only by knowing a considerable amount about the child's past arithmetic performance and experience). Otherwise, they no more reflect number sense than does the correct performance of a standard written algorithm. Note also that such mental arithmetic "simplifies" only if one does not already have well-learned standardized procedures.

5. Tending to want to "make sense" of situations involving number and quantity. Talking about numbers and their relationships.

6. Having a sense of the relative size of numbers and the quantities to which numbers refer. Note that the values of numbers depend on the situational context; the same number can refer to "a lot" or "a little" in different situations.

7. Substituting flexibly among different possible representations of a quantity (e.g., 24 for 2 dozen, a little less than 1/2 for 0.4). Note that a substitution's usefulness depends on the particular problem to be solved and on details of the individual's knowledge of numbers.

The above is only a partial list, but it shows the difficulty of capturing number sense in a definition or test based on the assumptions of decomposition and decontextualization. None of the performances that we might take to indicate number sense will *always* be performed, even by someone whose number sense is very good. Each will occur only when the problem to be solved and social expectations make it more efficient or appropriate than other ways at the individual's disposal for solving the problem. What is more, no single performance reliably demonstrates an individual's number sense. Children might be taught a particular decomposition procedure, for example, and apply it with no more sense of why it works than they have of written algorithms. What seems a "flexible" substitution of less than 1/2 for 0.4 may be a rote rule that leads to very inappropriate reasoning in some other situation. Using some number facts to derive others may rarely be an efficient procedure and, therefore, rarely be observed. People may appear to approximate but, in fact, do detailed calculations and then report a rounded answer.

There is no way to build a standardized test--one in which the answers are prescribed in advance and in which only the results of reasoning, not its process or justification, are examined--that can give us a reasonable judgment of people's number sense. This is because number sense is a form of reasoning and thinking in the number domain. Like other forms of reasoning and thinking, it is nondeterministic, open-ended, and dependent on a complex interaction among an individual's knowledge, skill, the details of the problems to be solved, and the expectations of social performance that are inherent in the situation. In fact, one can characterize number sense rather well by substituting the term *number sense* for the original *higher order thinking* in the following characterization of thinking skill that I offered in *Education and Learning to Think* (Resnick, 1987).

> Number sense resists the precise forms of definition we have come to associate with the setting of specified objectives for schooling. Nevertheless, it is relatively easy to list some key features of number sense. When we do this, we become aware that, although we cannot define it exactly, we can recognize number sense when it occurs.
> Consider the following:
>
> Number sense is nonalgorithmic.
>
> Number sense tends to be complex.
>
> Number sense often yields multiple solutions, each with costs and benefits, rather than unique solutions.
>
> Number sense involves nuanced judgment and interpretation.
>
> Number sense involves the application of multiple criteria.
>
> Number sense often involves uncertainty.
>
> Number sense involves self-regulation of the thinking process.
>
> Number sense involves imposing meaning.
>
> Number sense is effortful.

None of this means that number sense cannot be assessed. But it does suggest that there is little likelihood that we will be able to assess it in anything like current standardized test forms. Instead, we are going to have to observe children solving various kinds of problems involving number, and then make judgments about the extent to which the children seem to be reasoning effectively about number. A current movement to develop *performance assessments* as a supplement to (and, eventually, a substitute for) standardized assessments (see Resnick & Resnick, in press) offers a more promising environment than we have seen in some time for new forms of testing that will permit genuine assessment of higher order abilities such as number sense.

The characterization of number sense as a nondeterministic *tendency* to display certain kinds of performances based on a body of interrelated concepts also has implications for the kinds of learning environments in which number sense can be developed. It helps to think of *socializing* children into a way of thinking about number and a view of themselves in relation to number, rather than of *instructing* children in specific skills or pieces of knowledge. Several members of the San Diego seminar expressed frustration at children's tendency to want to calculate and use exact methods, noting that children identified calculation and exactness as the essence of math. Such responses are not really surprising. Most children have been socialized in a mathematics environment that leads them to think of numbers as elements in calculations and of themselves as primarily responsible for generating the "right" numbers on demand.

During the past year, I have collaborated with a teacher (Victoria Bill) who has been redesigning her primary-grade mathematics teaching. This redesign effort is based on research on the development of mathematical intuition and problem-solving abilities. The following set of broad principles has guided the program development:

1. **Draw children's informal knowledge, developed outside of school, into the classroom.** For first graders this knowledge consists primarily of a set of *protoquantitative* schemas (see Resnick, 1989) that allows them to reason about relative sizes and amounts (the protoquantitative *compare* schema), changes in amounts (the protoquantitative *increase/decrease* schema), and the relations between parts and whole (the protoquantitative *part/whole* schema) without numerical quantification. It also consists of counting procedures that children can use to establish "how many" are in the set (a first level understanding of cardinality). An important task of the program is to stimulate joint use of counting and of the protoquantitative schemas in order to help children develop quantified, more powerful versions of these intuitive schemas. This is done through extensive problem-solving practice, using both story problems and actual situations. Use of manipulatives and of finger-counting is encouraged as part of this process.

2. **Develop children's trust in their own knowledge.** By focusing on specific procedures and on special mathematical notations and vocabulary, traditional instruction tends to teach children that what they already know is not legitimate mathematics. To develop children's trust in their own knowledge *qua* mathematics, our program stresses the possibility of multiple procedures for solving any problem, invites children's invention of these multiple procedures, and asks that children explain and justify their procedures using everyday language. In addition, the use of manipulatives and finger counting insures that children have a way of establishing for themselves the truth or falsity of their proposed solutions.

3. **Use formal notations (identity sentences and equations) as a public record of discussions and conclusions.** The goal here is to begin to link what children know to the formal language of mathematics. By using a standard mathematical notation to record conversations that are carried out in ordinary language and that are rooted in well-understood problem situations, the formalisms take on a meaning directly linked to children's mathematical intuitions. Most of the children began to write equations themselves only a few weeks into the school year.

4. **Introduce the whole additive structure as quickly as possible.** Children's protoquantitative schemas already allow them to think quite powerfully about amounts of material, how they compare, increase and decrease, come apart and go together. In other words, they already know, in non-numerically-quantified form, something about properties such as commutativity, associativity, the complementarity of addition and subtraction, and the constraints on decomposing and recomposing. A major goal of the first year or two of

school mathematics is to "mathematize" this knowledge--that is, quantify it and link it to formal expressions and operations. It is our conjecture that this is best done by laying out the additive structures (addition and subtraction problem situations, the composition of large numbers, regrouping as a special application of the part/whole schema) as quickly as possible and then allowing full mastery (speed, flexibility of procedures, articulate explanations) of the whole system to develop over an extended period of time. Guided by this principle, Ms. Bill found it possible to introduce addition and subtraction with regrouping in February of first grade. However, no specific procedures were taught; rather children were encouraged to invent (and explain) ways of solving multidigit addition and subtraction problems, using appropriate manipulatives and/or expanded notation formats that they developed.

5. **Encourage everyday problem finding.** This principle deliberately uses the term *everyday* in two senses. First, it means literally *doing arithmetic every day,* not only in school but also at home and in other informal settings. Children need massive practice in applying arithmetic ideas, far more than the classroom itself can provide. For this reason it is important to encourage children to find problems for themselves that will keep them practicing number facts and mathematical reasoning. Second, *everyday* means non-formal, situated in the activities of everyday life. It is important that children come to view mathematics as something that can be found everywhere, not just in school, not just in formal notations, not just in problems posed by a teacher. We want to get them in the habit of noticing quantitative and other pattern relationships wherever they are and of posing questions for themselves about these relationships. Two aspects of the program represent efforts to instantiate this principle. First, as already mentioned, the problems posed in class are drawn from things children know about and are involved in. Second, homework projects are designed so that they use the events and objects of children's home lives: for example, finding in the home as many sets of four things as possible; counting fingers and toes of family members; recording numbers and types of items removed from a grocery bag after a shopping trip. From child and parent reports, there is informal evidence that this strategy is working: that children are noticing numbers and relationships and setting problems for themselves in the course of everyday activities.

It is too early to offer a full evaluation of this program, which Ms. Bill has used with first through third graders. However, we can point to some promising early indicators. First observations make it clear that there are multiple occasions, every day, in which children are exhibiting performances like the ones listed here as indicators of number sense. Thus it appears that we have created a socializing environment in which number sense, according to my open-ended characterization, can develop. Second, number sense is not developing at the expense of traditional arithmetic competence. In March, this year's first graders (the only ones for whom standardized test scores are available at this time) had a median score on the CAT math test at the sixth stanine, compared with a median score at only the third stanine for the previous year's first graders (of the same entering ability) who had had traditional instruction with the same teacher. We are now planning more systematic evaluation with special attention to the development of number sense over the entire primary school period.

Retrospective Paper: Number Sense Conference

Sandra P. Marshall

My reactions to the Conference on Number Sense center on two thoughts. The first is that a definition of number sense in some way reflects back to the definition of common sense (and has the same problematic features). Under this perspective number sense also ties closely to intelligence and the difficulties in defining and measuring it. The second thought is that number sense is an informal way of describing the "connectedness" of a person's mathematical knowledge. The more connected and integrated that knowledge, the more number sense we attribute to that person.

Listening to the participants of the conference, I heard much frustration because we could not provide an operational definition to that which we all call number sense. Each of us had a slightly different idea of what constitutes number sense, and yet at the same time there was little outright disagreement or challenge about including each new aspect into the group's description of this phenomenon. At one point, this expansion resulted in a list of the different meanings of number sense, and this list consisted of more than a dozen things.

It is not surprising that we have difficulty in coining a narrow definition for number sense. It is equally difficult to define "common sense"--and yet we have no trouble recognizing either one when displayed. The term common sense comes from the notion that something is common to all five senses. Thus, to use common sense is to use incoming information from all sensory channels. It may be profitable to look at number sense as also having multi-dimensional input. What are the input channels? Perhaps they are the different arenas in which numerical information arises.

It is also difficult to define intelligence, and yet it too has similarities to number sense. Without pursuing the analogy too far, it is useful to look at these similarities, particularly with respect to observation and measurement. Psychologists have studied intelligence for quite a long time with little agreement about what it is. On the one hand, we have the opinion that intelligence is one broad characteristic. An individual is either intelligent or is not. Measurement of this characteristic usually comes from a test having a variety of different test items. One aggregates the student's performance on these items and calls it an estimate of intelligence. An interesting aside here is that these tests were originally designed to identify deficient individuals, not outstanding ones.

A more widely accepted view is that there are different types of intelligence, and that individuals may exhibit one and not another. Many early testing theories reflect this perspective, such as Thurstone's Primary Mental Abilities, Cattell's Fluid and Crystallized Intelligence, Guilford's Structure of Intellect, and more recent theories, such as Sternberg's component processes, also involve multiple aspects of the phenomenon we call intelligence.

Perhaps number sense can be profitably examined from the multiple-dimension perspective reflected in characterizations of both intelligence and common sense. Again, what would those dimensions be? What is number sense with respect to fractions, story problems, geometry, calculus? Will it simply boil down to using all available information to the senses? or retrieving all relevant information from memory?

At this point I come to the second thought expressed above: perhaps we can make progress by describing number sense as a function of the "connectedness" of mathematical knowledge. Suppose that an individual's knowledge is stored in memory as a vast network of nodes (individual pieces of knowledge). These nodes are useful only if they can be easily retrieved from memory when they are needed. Psychological models of memory have recently focused on some of the features related to this type of memory storage.

Three particular features of networks deserve mention: saliency, strength of association, and activation. Saliency is a characteristic of the individual node. It refers to the prominence of an individual node. Some information is more striking or bold than other information, and this is manifested in the encoding of it. Such nodes are highly salient in the initial encoding of the knowledge. Other nodes are less noticeable at first but are accessed frequently, are related to many other nodes, and consequently develop high saliency through use. Still others are not initially prominent, have few connections, and maintain low saliency.

Strength of association is a characteristic of a pair of nodes. It refers to the strength of connection between one node and another. If the link between two nodes is used often (i.e., if two pieces of information are retrieved and used together frequently), the strength of association will be high. On the other hand, two pieces of information might be connected (because they originally occurred together) but might have low strength of association (because they do not reoccur together very often).

Finally, activation refers to the way in which the retrieval of one piece of information leads to the retrieval of other pieces that are strongly connected to it and also to retrieval of additional information related to them but not to the originally retrieved piece. Activation depends on both saliency of nodes and strength of links between nodes. It is characterized as a fan, so that activating one node results in the activation of a large part of the network if there is a high degree of connectivity. Clearly no activation can occur if the starting node has no connections to other nodes.

How can this network model help to explain number sense? It may be easier to think about what isn't number sense and how the network reflects this. Suppose a student learned something about properties of numbers and something about the addition of whole numbers. It is perfectly possible that the student encodes these topics in memory as separate domains with no connections among them. That is, the student can retrieve information about specific properties of numbers when asked to do so, or the student can demonstrate ability to add two three-digit numbers. However, without connections between these two networks of knowledge, the student cannot, for example, explain how one might solve the problem of 153 + 148 by adding 150 + 150 and then adjusting the answer according to the difference between the original numbers and 150. Activation is limited to the nodes within the particular sub-network accessed and does not spread to other necessary sub-networks because there are no links connecting them.

By describing number sense as the richness of connectedness of mathematical knowledge, we may begin to develop a researchable definition. There are certain implications that follow from this conceptualization, and these have bearing on research issues. It becomes clear that number sense is not a body of information that can be taught because that would imply that it consisted of a set of nodes. Thus, we cannot develop materials to "teach" number sense as we would teach addition. Under the network model, we should be more concerned with the development of links among groups of nodes than with the acquisition of additional nodes. The issue is how to get students to create more and stronger links in their existing knowledge.

One particular tie that is expected is the link between formal symbol systems and applications drawn from an individual's experience. This link serves to connect abstract representation and episodic memory, allowing the individual to draw simultaneously upon different domains of knowledge.

Several research questions emerge from this view of number sense. First, what connections do students already have among the various sub-networks of mathematical information developed in elementary school? Can we observe these from students' performance? Can we develop interview procedures to elicit them? Second, which instructional materials lead to better and stronger connections? Materials that have self-contained sections for each central topic (e.g., probability, geometry) may hinder rather than help create rich networks. Third, how does the network change over time? Clearly, what we take as number sense in a first grader is different from what we look for in a sixth grader. What is the difference? Is it the inclusion of more sub-networks? Is it a fuller network?

The question of how to measure number sense is still open. One way networks have been studied is to examine the length of time individuals take to access different parts of a network. This is related to automaticity and reaction time. With the appropriate model, one might study children's ability to respond to mental arithmetic tasks. Under the model one can develop tasks that call for the access of different types of knowledge. Highly connected networks with strong associations among nodes will be more rapidly accessed and will result in lower reaction times.

The context in which individuals display number sense can also be explored through the network model. By varying the situations we can determine whether individuals have easier access to their knowledge and whether there is a higher degree of activation for some situations than for others.

One important implication of the network model is that it is the degree of connectivity that is important, not the presence or absence of specific links. Thus, some students will have some pieces of information in their network and others will not. Yet all of them may have the same demonstrable level of number sense. This can be explained in part by reference to models of connectionism, especially the parallel distributed processing models (PDP). An essential feature of these models is that they depend upon general patterns of nodes and links, not upon the presence of specific ones. All contribute to an overall level of activation, so that a network may be highly active even though one specified element is missing. Similarly, a sparse network might contain the specified element and nevertheless have a low level of activation because there are few links within it.

A second implication is that there is no single path through the network. Thus, individuals may use different solution paths, and indeed, one individual might use different paths on different occasions. Again, it is the total pattern that matters, and the degree of activation that occurs. One might wish to look for multiple solutions as an indication of a highly connected network. One might vary the initial stimulus (e.g., task situation) in an attempt to direct the student's retrieval. If some solutions are dependent upon task characteristics, one might want to examine what features of that task are responsible for the particular pattern of activation that emerges with the student's performance. At the heart of this conception of number sense is the notion of multiple paths among nodes in the network and ease of access to every node. Thus, both the amount of knowledge and the degree to which it is connected to other knowledge becomes important. It seems probable that there will be no core body of knowledge (i.e., set of nodes) that characterizes number sense. It is unlikely that we could ever develop a curriculum to "teach" number sense. Rather, it is hypothesized to be a general pattern of knowledge in which no single element is required. Different students will have slightly different networks and paths of access through them. The specific linkages developed by students will depend in part upon their own unique experiences in the world and in part upon the mathematics education they receive.

Some Conjectures About Number Sense

James G. Greeno

The term "number sense" refers to several important but elusive capabilities. Part of the idea involves flexibility in operating on numbers. For example, in a study of mental multiplication by high school students by Hope and Sherrill (1987), some students solved the problem 25 x 48 by converting the problem to 100/4 x 48, then to 100 x 48/4, then to 100 x 12, to obtain the answer, 1200. In contrast, one of the students solved 25 x 480 saying, "Let's see. 480 on the top and 25 on the bottom. 5 times 0, 5 times 8 is 40, carry 4, and 4 is 24. I have to realize that the second number is one over. 2 times 0, 2 times 8 is 16, carry 1; 2 times 4 is 8 and 1 is 9, so 960, 9600. So 9600 and 2400 is 0, 0,...,19 thousand and ... 860." (p. 101) The second answer reflects a lower level of number sense than the first because it operates on a mental analogue of paper-and-pencil symbols, rather than transforming the problem on the basis of useful equivalences.

Examples of flexible computation involving quantities have been provided in studies of everyday reasoning. In one example, Carraher, Carraher, and Schliemann (1985) observed young business persons who sold produce in street markets in Recife, Brazil. One 12-year-old saleswoman was asked the price of 10 coconuts that she had said cost 35 cents each. The reply was, "Three will be 105; with three more, that will be 210. I need four more. That is--315--I think it is 350."

In another example, Scribner (1984) observed young men whose job was to assemble orders of dairy products for delivery. Orders were written in a special notation. For each product and container size (e.g., quarts of chocolate milk) there is a positive integer and a signed integer (e.g., 2, -5). The first integer indicates a number of cases, and the signed integer indicates an additional number of containers if it is positive or a number of containers less than the number of full cases if it is negative. There are different numbers of containers per case, depending on the size of the container (cases of different size containers contain 4 gallons, 9 half-gallons, 16 quarts, 32 pints, or 48 half-pints). Thus, "2, -5" for quarts of some product would require one full case of 16 quarts and another case with 11 quarts. Scribner's observations focused on situations where use of an available partial case would allow an order to be filled more efficiently than use of only full cases. For example, if the order is "2, -5" for quarts, and there is a case with nine quart containers, then it is easier to add two quarts to that case than to take a full case and remove five from it. Scribner used the term "literal" to describe a solution in which a completely full or a completely empty case is used, with the designated number of cases either taken away or added. Sometimes, of course, a literal solution is optimal. Of 53 situations that were observed in the work setting, literal solutions were optimal in 25 situations, and the workers used literal solutions in all of them. In 28 situations where an available nonliteral solution was optimal, workers used that solution in 23 of the situations.

Estimation is a second capability that we associate with number sense. One form is computational estimation. For example, given the problem

$$\frac{347 \times 6}{43}$$

one ninth-grade student said, "It would be easiest to divide the 6 and 43 first, which is about 7; so 347/7 is about 50" (Reys, Rybolt, Bestgen, & Wyatt, 1982, p. 188). An egregious example of students' using computational techniques (incorrectly) instead of attending to numerical quantities, noted by Reys et al. (1982) is in the results of an exercise in the National Assessment of Educational Progress (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980), where students were asked to estimate the answer to

$$\frac{12}{13} + \frac{7}{8}$$

with the instruction, "You will not have time to solve the problem using paper and pencil." Fifty-five percent of 13-year-old students and 36% of 17-year-old students answered either 19 or 21, rather than 2, which was chosen by 24% of the 13-year-olds and 37% of the 17-year-olds.

Another form of estimation involves judging and making inferences about quantities with numerical values. As an example, Paulos (1989) suggested that a numerate person should be able to think correctly about the time it would take to level Mount Fuji with trucks that arrive, are loaded, and leave every 15 seconds. A notorious example from NAEP, noted by Schoenfeld (1988), involves a problem with 1128 soldiers to be transported in buses that hold 36 soldiers per bus. When asked how many buses would be required, the answer chosen by the largest number of students was "31 remainder 12," indicating that they had applied an arithmetic operation without regard for the meaning of the answer.

Educators and mathematicians concerned with education have recognized that number sense, in its various forms, should be an outcome of mathematics education. It plays a significant role in the *Curriculum and Evaluation Standards for School Mathematics* compiled by the National Council of Teachers of Mathematics (1989), the recommendations of the Mathematical Sciences Education Board (National Research Council, 1989), and the recommendations for mathematics education by the 2061 Project of the American Association for the Advancement of Science (Blackwell & Henkin, 1989).

A natural response is to treat flexible mental computation, computational estimation, and quantitative judgment and inferences as skills and try to design instructional activities in which students can acquire the skills. There might be considerable benefit in this approach, but there also are some potential limits to treating mental computation and estimation as objectives to be achieved directly through use of specific instructional tasks.

Another approach, which I will discuss in this essay, would treat flexible mental computation, computational estimation, and quantitative judgment and inference as symptoms of a more basic and general condition of knowing in the conceptual domain of numbers and quantities. This alternative approach suggests a global treatment of the question of number sense. In this view, all of the activities of mathematics instruction should contribute to the growth of students' number sense, rather than treating number sense as another objective of mathematical instruction with its own allocation of time and activity. Rather than designing additional instructional activities for the teaching of number sense, we should, in this view, redesign all the instructional activities in the curriculum so that they all contribute to students' growth of number sense.

In this essay, I examine the global view of number sense. First, I present a metaphor for knowing in a conceptual domain as knowing how to live in an environment. This metaphor provides an alternative to viewing domain knowledge as a set of cognitive structures and procedures and emphasizes knowing and reasoning as being cognitively connected with features of an environment at multiple levels. Second, I discuss characteristics of number sense, viewed as knowing in the domain of numbers and quantities, that seem to be captured naturally by the environmental metaphor. Third, I consider ways in which this view of knowing in domains is related to some current trends in research about cognition and learning. Finally, I discuss suggestions of this view about learning and teaching of number sense in mathematics education.

Knowing in Conceptual Environments

We usually think of a subject-matter domain as a structure of facts, concepts, and principles, and of knowledge in the domain as a subset of the structure that a person has learned. This fits well with the current information-processing framework of cognitive science, in which knowledge is a set of representations that are stored in the mind, including procedures for manipulating symbolic expressions. Learning in a domain, in the information-processing framework prevalent in cognitive science, is construction of the knowledge structures and procedures that correspond to the concepts, principles, and rules of inference of the domain.

An alternative characterization views the domain as a collection of resources for the cognitive activities of knowing, understanding, and reasoning. Using a spatial metaphor, the domain may be thought of as an environment, with resources at various places in the domain. In this metaphor, knowing in the domain includes knowing what resources are available in the environment as well as ability to find and use the resources that can be helpful in understanding and reasoning. Knowing includes interaction with the environment in its own terms--exploring the territory, appreciating its scenery, understanding how its various components interact. Knowing the domain also includes ability to recognize how resources in the domain can be used in activities that are not directly "environmental," recognizing features of situations that afford use of the domain resources, as well as ability to find and use the resources productively. Learning in the domain, in this view, is analogous to learning to get around in an environment and to use the resources there in conducting one's activities productively and enjoyably.

The environmental metaphor highlights the importance of social interactions in the process of using and learning to use the resources of the domain. We construct and use concepts in our collaborative activities of understanding and communicating about the events that are important to us, and the concepts that we construct and use help determine what it is that is important. An important aspect of learning in a conceptual domain is the acquisition of a social capability--the capability of communicating and constructing meanings using terms in the domain with other persons who are similarly engaged. To acquire that capability, we engage in the activities of communicating and constructing shared understanding in the domain, as we engage in communication about the features of the environment in which we interact with other people.

People who live in an environment engage in their various collaborative activities, and the environment's resources are used in ways that require coordination of the individuals' actions. Individuals work together in ways that make use of differential proximity to things in the environment. Different individuals contribute differently in activities depending on their resources (e.g., who has access to a car; whose schedule allows for stopping at a store; one individual can provide a map, and another a pair of binoculars on a scenic hike). The environment provides a setting in which mutually enjoyable and productive interactive activities can occur (e.g., planning and enjoying a trip in scenic country; conducting the activities of farming, building a deck, or planting and tending a garden).

Social aspects of learning in a domain are also highlighted by the environmental metaphor. When we are unfamiliar with an environment--either because we are visiting or because we have recently moved there--we depend on others to introduce us to the features and resources of the environment. The guidance we receive from people who know the environment takes many forms: expositions about features of the environment, simple directions for getting from one place to another, and accompanying us on tours and in other activities.

Knowing an environment has two special features: it is multileveled, and it is multiconnected. When we reason about an environment that we know, we can think about features at different levels of detail. A plan for a trip uses knowledge about general locations and requirements for getting between large divisions of the environment, as well as knowledge of specific routes that can be taken (Chase, 1983). As we move about in an environment that we know well, we maintain a continuously updated sense of where we are in the environment and where other things are in relation to where we are as well as in relation to each other (Rieser, Ashmead, & Pick, 1988). This sense of location is generated from locally available information in a flow of optical information (Lee, 1980), but that information provides much broader information locating us in the environment in a general way (cf. Hutchins, 1983).

The environmental metaphor is useful in another way if we think of the students' and teachers' roles in constructing students' capabilities of knowing, understanding, and reasoning. Like a physical environment, a conceptual environment contains many things that the student does not create. Examples in mathematics include numbers and the distinction between addition and multiplication. But, also like a physical environment, there are many routes from one place to another, and each person constructs a rich set of interconnections among the locations in the space that provides his or her capability of moving about in the environment.

The view of a subject-matter domain as a conceptual environment has much in common with ideas discussed by Toulmin (1972), who characterized the development of ideas in an intellectual discipline in evolutionary terms. Toulmin developed an idea of conceptual growth as a set of concepts with explanatory functions in the domain, and with concepts introduced into the system and selected through a social process depending strongly on forums of debate. An important implication of Toulmin's idea, recognized by Anderson (1988), is that concepts have to fit into an environmental niche in the conceptual structure that an individual develops as he or she learns in the domain. The metaphor that I have presented here focuses on the domain as a set of concepts and treats that as an environment that individuals learn to live in, cognitively. That knowing of an environment involves an adaptation that could be understood in evolutionary terms, with many features similar to those of Toulmin's and Anderson's analyses.

The Conceptual Environment of Numbers and Quantities

The metaphor of a subject-matter domain as a conceptual environment, and knowing in the domain as ability to find and use the concepts and principles in the environment as resources, seems to apply well to the domain of numbers and quantities, particularly in its suggestions regarding number sense. Numbers and quantities are important objects in the domain, with a structure of relations and operations. Persons with number sense know where they are in the environment, which things are nearby, which things are easy to reach from where they are, and how routes can be combined flexibly to reach other places efficiently.

The environmental metaphor highlights the multileveled and multiconnected nature of knowing in the domain of numbers and quantities. With number sense, persons understand where they are either in a general sense or in detail, as is appropriate in the context. Relations and possible operations of different kinds are known at multiple levels. Persons are sensitive both to the relations between quantities regarding their general magnitude (e.g., one quantity is about 50 times another) as well as their specific values (e.g., one number is 2 less than a multiple of 10, while another number is 2 greater than a different multiple of 10). They also are sensitive to multiple ways in which quantities and numbers can be transformed, recognizing either the approximate or specific results of operations.

Flexible mental computation involves recognition of equivalence among objects that are decomposed and recombined in different ways. Finding objects that are the same as other objects or combinations of objects is one of the abilities one would expect of a person who knows the objects in the environment and knows where they are. This results in flexible mental computation. For example, recognizing that 25 x 48 is the same as 100/4 x 48, is the same as 100 x 48/4, so that 25 x 48 is the same as 100 x 12 = 1200 involves quite a wonderful set of cognitive achievements, including knowing that an equivalence involving 100 will be more useful than many other equivalences (e.g., 25 is also equivalent to 125/5, which might be quite helpful in another circumstance) and that the divisor 4, which is introduced when 25 is replaced by 100/4, is related to the other term of the problem, 48, in a way that will be useful.

Computational estimation involves cognizance of objects in the domain at different levels of resolution. Regions of a spatial environment can be treated as objects; a person can know that he or she is in one region rather than another, such as being in one's office rather than in the room where one teaches a class, and a person in one region can think about a different region and decide to go there. At the same time, regions have internal structure. Within a region, objects have specific locations, and one finds one's way around, arriving at specific locations in order to perform specific actions, such as sitting in a chair and typing at a keyboard. Similarly, the activities of numerical reasoning can be carried out at different levels of resolution. If 12/13 and 7/8 are cognized as objects that are both in the neighborhood of 1, then their sum is an object in the neighborhood of 2. Of course, if the issue is how much their sum is less than 2, the appropriate resolution has to be more fine-grained, but the required detail might still be less than complete. It might be quite useful to realize that 1/13 is smaller than 1/8, and that the difference between 12/13 + 7/8 and 2 is greater than 1/8 but less than 1/4. This is a conclusion that we should expect if a student knows his or her way around in the space of numbers, with an ability to focus in the domain at different levels of resolution. It involves a capability that might not be taught effectively with exercises on computational estimation with correct answers.

The sense of quantities, and numerical values of quantities, includes effortless transitions between relations between quantities and operations on numbers. Operations on quantities have counterparts in operations on numbers, such as concatenations of motions through space, involving additions of distances and durations, and the corresponding additions of the numbers of miles and the numbers of hours. Relations among quantities such as distances, times, and speeds correspond to relations among and operations on numbers.

An individual can have a strong sense of quantities without sensing significant relations between the quantities and numbers. An example was included in observations made by de la Rocha (1986; see Lave, 1988) in a study of recent recruits to the Weight Watchers dieting program. One dieter was asked what would happen if he decided to use three fourths of his day's allotment of cottage cheese, which was two thirds of a cup. His solution was to measure 2/3 cup of cheese, drop it on the counter, pat it into a circle, divide it into quarters with a horizontal and a vertical line, and remove one of the quarters. This observation has provided cognitive science with a productive example because it illustrates, better than any other true story yet to occur, a process of reasoning that interacts with and depends on the features of the situation in which it happens.

It is interesting, too, to reflect on aspects of number sense that the dieter example does not show. If the relation between quantities and rational numbers was evident to the dieter, he might have recognized that 3/4 of 2/3 is the same as 2/3 of 3/4, and that 2 of the 3 fourths are 2/4, or 1/2 (Magdalene Lampert and Judith Sowder have noted this possibility in conversations). Another possibility would be to recognize that 2/3 is the same as 4/6, and that 3 of the 4 sixths are 3/6, or 1/2.

Knowing in the domain of numbers and quantities also includes fluency in the notations of arithmetic, and this is also informed by the environmental metaphor. Someone who knows an environment well can "see through" a map of that environment, understanding relations between places in the environment that are represented by the symbols on the map. There is similarity with mathematical symbols and the concepts and relations that they refer to. We often think of notations of place-value positions, decimal points, fraction bars, and ratio colons as impediments to students' understanding, and they often are. At the same time, they can be resources for reasoning. The positions of digits in an integer or the location of a decimal point, a fraction bar, or a ratio colon conveys important information about the relations among numbers or quantities that are denoted by the numerical symbols. They also are crucial cues in the performance of procedures on the symbolic representations that include them. Graphs, too, are a problem for students in their learning of mathematics. But they also can be a resource for reasoning, providing compact representations that present properties of functions in visible form.

Uses of symbolic notations in mathematics can be surprisingly complex. One example, noted by Carraher et al. (1987), involves the procedures for computation of multidigit numerals, where the difference in denotations of the digits in the different place positions is ignored in the procedures. Another, more elementary, example is involved in counting on with fingers, where a numeral, associated with a finger, simultaneously refers to that finger's ordinal position and the cardinality of the set of fingers that have that finger as the last member (Neuman, 1987). The simultaneous but formally incompatible meanings of a single term seem not to impede children's ability to use the notational system in their numerical reasoning. Indeed, the systematic ambiguities of symbols may play an important role in their usefulness. For example, the often-noted differences in meaning of plus, minus, and equals signs may play a critical role in allowing efficient manipulation of expressions by permitting differences in interpretation in different procedural contexts.

The metaphor of numbers and quantities as a conceptual environment also conforms to significant social aspects of mathematical reasoning and learning. Mathematics can provide a setting for groups of individuals to explore their ideas and solve problems together. Individuals can express their understanding of the domain to each other through the use of representations, thereby enabling each other to appreciate new features of the environment and find new routes in it, analogous to drawing maps to show others the locations and routes in an environment.

The metaphor is quite suggestive regarding the role of a teacher. As learning is analogous to acquiring abilities for finding one's way around in an environment, teaching is analogous to the help that a resident of the environment can give to newcomers. Someone who already lives in the environment is an important resource for a newcomer, helping by indicating what resources the environment has, where they can be found, what some of the easy routes are, and where interesting sites are that are worth visiting. Effectiveness in providing guidance to others is not equivalent to knowing the environment one's self; one can be fully effective in finding and using the resources of an environment, but be of little help to someone else. An effective guide for learners needs to be sensitive to the information they already have, to connect new information to it, to provide tasks and instructions that can be engaged in productively by beginners, to be aware of potential errors that can result from newcomers' partial knowledge, and to help beginners use errors as occasions for learning.

Teachers and students work together on the construction of mathematical knowing, but they are hardly ever engaged in the construction of new mathematical knowledge. As in the case of learning in a physical environment, most features of the environment are given, and persons learn how to move about and to conduct their own activities there. The main features of the mathematical landscape have been constructed by professional mathematicians, and students and teachers work on understanding the resources, routes, and paths that are there. This does not prevent occasional new constructions, in which a new path is discovered or a new bridge is built, and these occasions are particularly valuable for students and their teachers, when they occur

Related Trends in Cognitive Research

The view of number sense as knowing for living in a conceptual environment is congruent with several trends in research on cognition and learning.

Situated Cognition

Most generally, the idea fits with an effort that is being made in cognitive research to develop an alternative to the framework of information processing. The analytical resources that have been available in the theory of information processing (Newell & Simon, 1972) and the behavioristic theory that preceded it (Gagne, 1965) enable the dissection of knowledge and cognitive processes into components. An information-processing analysis assumes that a person constructs a representation of the situation and his or her goal and reasons by manipulating the symbols in the representation. The person's knowledge includes information structures corresponding to concepts and propositions. Knowledge also includes procedures that make inferences and set new goals by constructing additions to the representation of the situation. Analysis of a specific cognitive capability, in this framework, involves hypothesizing a set of processes for representing situations, a set of knowledge structures--most frequently in the form of schemata--that provide organization for the information and the reasoning processes, and a set of processes that make inferences of various kinds, including setting new goals that become appropriate as the reasoning process proceeds.

This framework has been used productively in analyses of several instructional tasks used in mathematics teaching, including arithmetic computation (Brown & Burton, 1978), elementary word problems (Riley & Greeno, 1988), and geometry proof exercises (Greeno, 1978; Anderson, 1982). Some interesting instructional methods and systems have been developed using the ideas (Anderson, Boyle, & Yost, 1985; Carpenter, Fennema, Peterson, Chiang, & Loef, 1988; Thompson, 1989).

The alternative view assumes that cognition is situated in contexts, and treats perception and reasoning as relations between cognitive agents and the social and physical situations they are in (see, e.g., Brown, Duguid & Collins, 1989; Clancey, in press; Greeno, 1989; in press; Lave, 1988; Suchman, 1987; Winograd & Flores, 1986). Following ideas of Heidegger (1926/1962), in this view cognition does not typically depend on representations, although the ability to construct and use representations plays an important role in some reasoning. The primary role of symbolic representations is in communication, when people use language or other symbolic systems to coordinate their activity and reach shared understanding of events and ideas. Symbolic representations are often constructed by individuals to provide physical objects that are useful in reasoning, such as road maps, musical scores, diagrams of physical or mathematical systems, shopping lists, and notes to use in giving a lecture. Cognitive representations, in the alternative view, are mental versions of physical representations, such as mental images or silent speech, and while they can play an important role in reasoning, they are by no means ubiquitous.

Several lines of evidence and theoretical work have motivated the development of the alternative view of cognition as a relation between agents and situations. The ideas are consistent with the theory of perception developed by Gibson (1986), and the evidence and argumentation that support that theory also support the effort to conceptualize cognition in relational terms. Some of the evidence regarding cognition is in studies of everyday cognition, including those by Carraher et al. (1985) and Scribner (1984), mentioned earlier, and also including work by Lave (1988) and Suchman (1987). Suchman presented a particularly direct case against the view that behavior is determined by cognitive representations in the form of plans.

Growth of Expertise

Another line of evidence is in a recent study of the nature and acquisition of expertise in cognitive domains by Dreyfus and Dreyfus (1986). According to an influential model of skill acquisition in psychology (Fitts, 1964), the process is viewed as having three main stages: an initial cognitive stage in which one learns general properties of actions by being given verbal descriptions and rules and observing performance by others, an associative stage in which the person learns to perform actions correctly, and then a stage of automatization in which the actions that have been learned come to be performed rapidly without conscious control. Dreyfus and Dreyfus's discussion is consistent with part of the standard psychological analysis, but also provides a new insight into the process. Dreyfus and Dreyfus agreed that initial stages of learning a skill are guided by verbal descriptions and rules and that translating cognitive representations into actions can get one to an intermediate stage of skill. They disagreed, however, about the way that performance improves toward expertise.

Rather than assuming that expertise consists of automatizing previously learned procedures, Dreyfus and Dreyfus hypothesized that expertise includes many capabilities that were never dependent on rules or descriptions. Rather than assuming that automatic skill is mainly a converted form of skill that once was controlled deliberately, they hypothesized that most of what matters in expert performance is learned without being deliberately controlled. These nondeliberate aspects of skill involve recognition of holistic patterns and interactive engagement with situations including flexible adjustment to nuances and configurations that the person does not consider consciously, and probably never considered consciously.

One example that Dreyfus and Dreyfus considered is the skill of driving a car. The earliest stages of learning include following directions for the sequence of things to do to start the car, rules for signaling before making a turn, and so on. A beginning driver practices the basic activities of driving until he or she performs them without making major violations of rules such as turning from the wrong lane or driving into the crosswalk when stopping at an intersection. When the driver is proficient enough, he or she can obtain a license to drive. If the person drives for several years, however, the skill he or she acquires includes abilities to adjust the speed of the car to allow another car to enter a lane, to identify a space between cars in another lane he or she wants to enter, to turn corners at different speeds and on different road surfaces so that the ride is smooth, and many other capabilities that were not taught deliberately (although there may have been remarks made about them, in the manner of coaching).

Dreyfus and Dreyfus studied expertise in piloting airplanes, playing chess, driving cars, communicating in a second language, and nursing. They characterized the features of expertise in these domains as global, holistic, and not derived from rule-based learning. As an example, they reported that one of them had played chess fairly seriously but had not progressed beyond a medium level of skill. In light of the analyses that they presented in *Mind Over Machine*, he came to believe that his progress was limited by treating the game too analytically, trying too much to identify rules and principles of successful play and blocking himself from acquiring the holistic and configural aspects of skill that are not amenable to descriptions in abstract terms.

An important distinction regarding expertise was made by Hatano (1988) between routine and adaptive expertise. Hatano's example was expertise in computing with the abacus, a skill that is highly developed by children who join clubs and become highly skilled. The expertise that these children achieve goes beyond rule-based knowledge, consistent with Dreyfus and Dreyfus's (1986) characterization of expert knowledge. The abacus experts have mental models of the abacus that they manipulate in flexible, problem-sensitive ways. Being situated in a conceptual model of the abacus, however, did not provide experts with concepts of numbers and quantities that enable them to generate ways of understanding and reasoning in a broad range of situations.

Conversation and Communication

Another set of results and analyses are provided in studies of conversational communication (e.g., Clark & Shaefer, 1989; Clark & Wilkes-Gibbs, 1986; Shegloff, 1981). Conversation has typically been considered as a series of turns, in which a speaker presents a message and the listener(s) either do or do not understand the message's meaning. The more recent analyses take another view, that meanings are achieved jointly by the speaker and listener(s) through a process of collaboration and negotiation. Clark and Shaefer drew an important distinction between presenting information and contributing information to the conversation. Presenting is done by an individual. Contributing is a collaborative act that is accomplished only when the speaker and listener(s) agree that something meaningful has been added to the collection of information that is shared in their common ground. In this view, meanings of words and phrases are not fixed, to be known or not by participants. Rather, meanings of terms are part of what emerges in the conversational context as the participants develop uses of terms that support their communicative activities. Of course, conversations do not start at ground zero with respect to word meanings, and members of a linguistic community share most of the terminological agreement that they need to communicate successfully in most situations. But a process of semantic negotiation occurs to some extent in all situations, and to a significant extent in many situations, especially when some participants in the conversation have less experience and familiarity with the topic than others.

Conceptual Competence and Conceptual Growth

In many domains, children's understanding grows naturally as a result of their ordinary interactions with adults and other children along with their natural curiosity and inclinations to learn to participate in social interactions. Young children become skilled in speaking and understanding their language, and in using language in ways that are practiced and valued in their social and cultural environments. For example, Heath (1983) described differences in the linguistic practices of two communities, and children's gaining different linguistic and communicative abilities in the two settings. In one of the communities children learn to tell stories that have morals in accord with community values, usually about some event that illustrates negative consequences of a wrongful act, with strict constraints of accuracy in statements of fact and chronology. In the other community, children learn to participate in elaborate expressive word play, including playsongs and ritual insults presented creatively, and stories with real events as the bases of the plots, but with details elaborated and exaggerated to emphasize truths about life rather than to report facts accurately.

Significant aspects of conceptual understanding also develop naturally in many conceptual domains. Properties of knowing, understanding, and reasoning in several domains have been investigated by developmental psychologists in studies of conceptual growth. Knowing in a domain includes distinguishing the entities and phenomena in the domain and knowing the principles of identity, invariance, composition, transformation, and causality that can be used to explain phenomena and events. It includes knowing the conceptual and theoretical entities of the domain, the principles of invariance, composition, transformation, and causality that apply to the theoretical entities. It also includes knowing how to use the terms in language that are used to designate properties, state principles, refer to theoretical entities, and construct explanations of events in the domain. All of these conceptual and linguistic capabilities develop through activities in which individuals communicate and negotiate the meanings of terms to coordinate their actions and to achieve mutual understanding of events.

As an example, in the domain of biological phenomena, a principle of invariance was studied by Keil (1986). Keil showed children of different ages pictures of different kinds of animals and different kinds of artifacts. The interviewer showed two animal pictures, for example a raccoon and a skunk, and said that some scientists had changed a raccoon's color, shape of tail, and other features to those shown in the picture of a skunk. The interviewer then asked whether it should be called a raccoon or a skunk. Similar questions were asked about a bird feeder and a coffee pot. Young children replied that the second animal should be called a skunk and the second artifact should be called a coffee pot, because their features had been changed. Older children said that if the animal was a skunk, it should still be called a skunk after its features were changed, although they said that the artifact had been changed into a coffee pot. The finding shows that older children come to understand that principles of identity for natural kinds are different from those of artifacts.

Another example involves principles of transformation, including constraints. Carey (1985) reported that young children answer the question "Why do people eat?" in terms of psychological concepts of desires and feelings--people eat because they feel hungry or they want food. Older children understand that eating is necessary for continuing to live and for growth. Hatano and Inagaki (1987) found that if 6-year-old children are asked whether a baby rabbit can be kept small by not feeding it, they understand that this will cause the rabbit to die. Apparently the younger children have some understanding of the functional principles of nutrition and growth, but do not spontaneously use these principles as explanations in some situations where older children and adults use those biological principles.

A third example illustrates understanding of theoretical entities and processes. Wellman and his associates (e.g., Johnson & Wellman, 1980; Wellman & Estes, 1986) have shown that children as young as 3 years understand the difference between mental states such as thinking about a dog and the physical objects that the mental states are about, and that they use concepts such as a person's knowledge in explaining behavioral events. Thoughts and states of knowledge are not observable, and children's use of these to explain the behavior of another person illustrates the role of unobservable conceptual entities and processes in a domain that they understand well early in their lives.

In the domain of numbers, Gelman and Gallistel (1978) showed that children have significant implicit understanding of counting principles that provide a beginning of conceptual knowing in the domain of numbers and quantities. Resnick and Greeno (in preparation) are working on an analysis of conceptual growth that starts with intuitive understanding of the sequence of numerals, ability to count sets of objects, and intuitive understanding of quantitative relations of comparison, change, and combination.

Intellectual Practices, Cognitive Apprenticeship, and Personal Epistemologies

The idea of learning in a subject-matter domain by being guided by someone who already is familiar with the conceptual environment has much in common with a view of learning through cognitive apprenticeship, which is being developed by Brown, Collins, and their associates (Brown, Duguid, & Collins, 1989; Collins, Brown, & Newman, in press). This idea emphasizes that knowing in a domain is an activity, and learning in the domain is acquiring the capabilities of understanding and reasoning that the domain affords, a kind of practice.

A characterization of the practice of mathematics was provided by Kitcher (1984) in a philosophical and historical analysis of mathematical knowledge. Knowing in the domain of mathematics, according to Kitcher's idea, includes (1) ability to understand and generate meaningful mathematical questions; (2) ability to appreciate and use methods of mathematical reasoning that are accepted as supporting conclusions; (3) understanding of metamathematical views that characterize goals and structures of mathematical knowledge; (4) ability to understand and use mathematical language; and (5) knowledge of the statements findings and conclusions that are accepted as established. The idea of number sense as an outcome of general knowing in mathematics implies that these capabilities all contribute to the understanding and reasoning power that constitute number sense.

Recent research and discussions by Belenky, Clinchy, Goldberger, and Tarule (1986) and by Dweck and Legett (1988) have identified important ways in which the beliefs that individuals have about the nature of knowing, and about themselves as knowers and learners, influence their activities in intellectual settings, including learning. The idea of learning in a domain as coming to know a conceptual environment emphasizes the active nature of learning, encouraging an epistemological view in which the learner explores ideas and understandings of his or her own, as well as those of other learners and teachers, rather than passively receiving information. We learn about environments through descriptions of their features in travelogues and other expositions. A travelogue about a place, however, does not teach us how to live there. For that, one has to be in the environment and be engaged in learning through activity in the situation.

Suggestions for Instruction

The view of number sense that I have tried to develop here suggests that number sense develops through a range of meaningful activities that students can engage in. It may be a good idea to have activities that are specifically designed to strengthen capabilities of flexible numerical computation, computational estimation, and quantitative judgment. It may be more important, however, to consider all the activities of mathematical instruction as potential contributors to students' development of number sense. In addition to designing specific activities for the growth of number sense, we should think about how the rest of the activities of the curriculum can be designed and organized so they contribute positively to the growth of number sense.

The idea that number sense is a general feature of knowing in the domain of numbers and quantities was expressed well by Howden (1989):

Number sense can be described as good intuition about numbers and their relationships. It develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms. Since textbooks are limited to paper-and-pencil orientation, they can only suggest ideas to be investigated, they cannot replace the "doing of mathematics" that is essential for the development of number sense. No substitute exists for a skillful teacher and an environment that fosters curiosity and exploration at all grade levels (p. 11).

The "environment that fosters curiosity and exploration" is a social construction in which students interact with the teacher and with each other about quantities and numbers. Examples of conversations in the domain of numbers and quantities are found in classrooms that are organized around the activities of collaborative learning and understanding. These include primary-school classes such as those based on problem solving (Cobb & Merkel, 1989) and construction and discussion of methods of solving word problems (Fennema, Carpenter, Keith, & Jenkins, 1989). Lampert's (1986; in press) teaching of mathematics in fifth grade is designed as a collaborative activity in which she and the students work together to understand mathematical concepts, notations, and procedures. At the Farm School in Irvine, California, students spend a significant amount of time in games involving numbers, including a continuing game where one student provides a few examples of numbers related by a function, and other students then construct further examples based on their hypotheses about the function (Lave, Smith, & Butler, 1988).

Conversations about quantities involve different levels of resolution that arise naturally. If one is talking about the amount of food that a whale consumes or the amount of money that is needed to purchase supplies for a trip, it is clear that exact answers are unnecessary for most purposes. Therefore, socially organized activities that include discussions of quantities may be more successful for students' learning to reason at appropriate levels of detail than exercises with directions to find approximate answers.

Most importantly, participation in discussions of ideas about numbers and quantities can result in benefits for students' beliefs and understandings of mathematical knowing and of themselves as knowers and learners of mathematics. If number sense is an aspect of students' general skill in the domain of numbers and quantities, it probably depends on students' beliefs that they are competent and potent knowers and learners in that domain, and on their willingness to act intellectually in the domain. Such beliefs and actions include believing that the domain has meaning that they can understand, that they can infer relations and construct methods that they have not been taught, and that they can achieve understanding through their own inquiry that goes beyond explanations that they have been given. Beliefs and attitudes of competence are fostered in situations where properties of numbers and operations are topics of discussion, with different opinions offered and evaluated by members of the group, rather than with correct propositions delivered authoritatively. Evidence of positive changes in students' epistemic motivation from participation in class discussions was presented by Hatano and Inagaki (1989).

The view of number sense as knowing an environment provides an added reason for emphasizing active exploration and problem formulation by students in their educational experience. Solving problems that other people present, using methods that other people have demonstrated, is analogous to following routes in an environment that have been laid out in detail by someone else. This can be valuable in an early stage of learning an environment, but if it is all one ever does, one's knowledge of the environment will be limited. To get to know one's way around it is necessary to explore, try alternative routes, attend to locations of places from different perspectives, and actively look for relations between routes and different places, including landmarks. It is just as important to attend to things that one passes or that one comes close to in moving about as it is to attend to one's exact destination.

In the domain of numbers and quantities, activities that explore relations include construction of patterns and identification of equivalences of many kinds. Viewing mathematics as the science of patterns (Steen, 1988), rather than a technology of procedures, would be consistent with this emphasis. At the same time, the patterns and equivalences have utility, and finding alternative ways to find answers can contribute to students' knowing the environment of numbers and quantities, as well as to their understanding of how mathematics is related to situations involving physical objects, quantities of money, and other concrete things.

The activities of learning to recognize patterns and equivalences may include some deliberate pattern-finding and descriptions, but the relations involved may be largely implicit, involving holistic and configural understanding rather than deliberate rule-based procedures. Indeed, it may be more fruitful to view number sense as a by-product of other learning than as a goal of direct instruction. In Cobb and Merkel's (1989) terms, "The activities were not designed to lead pupils to 'see' specific relationships, ... Instead, their function was to give the children opportunities to think about what they were doing as they solved arithmetic problems" (p. 72).

The skills needed to live in an environment of numbers and quantities undoubtedly include a considerable amount of automatic recognition and generation of low-level relations and patterns of a kind that requires significant amounts of practice. As with reading, recognition of patterns in numbers requires decoding written symbols into their meanings. In reading, it has been shown that automatic recognition of low-level patterns can be facilitated by use of rapid computational displays with students asked to respond to occurrences of target phonemic patterns (Weaver, Frederiksen, Warren, Gillotte, Freeman, & Goodman, 1982). It would be interesting to develop and test analogous programs for mathematics; for example, pairs of numbers could be displayed rapidly with a student requested to press a key each time the presented pair has a specified sum, difference, product, or ratio. A version of the exercise could ask students to indicate when an operation on a pair of numbers was near a target amount (say, plus or minus 10%), thus providing practice in computational estimation.

The view of number sense as knowing how to live in an environment has quite strong implications for the education of teachers. If someone is to serve as an effective guide for newcomers in an environment, it is essential that the guide himself or herself should be a comfortable resident of the environment. This would require developing a different relationship between teachers and the subject-matter of mathematics than many have now, since many teachers have little or no sense of participation in the community of mathematicians or of active intellectual construction of mathematical ideas and information. Teachers as guides might naturally interact with each other, sharing information that they had discovered in their own explorations in the environment, as well as in their activities of showing students some of the interesting sights.

The instructional suggestions of the environmental metaphor that seem most central involve the role of a social group in learning. Each individual needs to know his or her own way around in the environment. In learning that, however, social groups play very important roles. Groups of individuals can explore parts of the environment together and the understanding that they develop is both richer and more valuable because it is based on shared experience. Different individuals can explore different parts of the environment and share their experiences, providing valuable information for other members of the group.

Classrooms could be forums in which conjectures are discussed, inquiries are launched, and discoveries are both reported and made. Students' sense of numbers could hardly help from being developed.

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Fostering the Development of Children's Number Sense: Reflections on the Conference

Robbie Case

Definition of Children's Number Sense

For the purpose of the present paper, I shall define children's number sense as their intuitive knowledge about numbers and numerical transformations.

Hypothesized Properties of Number Sense

On the basis of the conference readings and discussions, the following hypotheses about children's number sense would appear to be warranted.

- Number sense is not static; it gradually grows, as children engage in mathematical activity.
- Mathematical activity is present in all cultures, whether or not any formal schooling is present.
- As a consequence, some growth of number sense takes place in all cultures as well. Number sense is thus both natural and universal.
- Ironically, one of the places that children are least likely to display their natural number sense is in the execution of formal scholastic tasks, even those that clearly require it.
- To say that number sense grows, and that it is natural, is not to say that all individuals, age groups, or cultures possess the same *degree* of number sense. In fact, there is strong evidence to the contrary.
- Some of the observed variability in number sense within a culture no doubt stems from endogenous differences in numerical ability. However, it seems likely that a good deal more variance stems from variability in the type and amount of mathematical activity in which young children engage.
- It follows that our schools are not doing as much as they could to foster the development of children's number sense, or its application to scholastic tasks.

Hypothesized Conditions that Foster or Inhibit the Development of Children's Number Sense.

The following additional hypotheses also seem congruent with many of the readings and discussions.

- For small numbers, children's number sense is facilitated by experience in executing standard arithmetic transformations (e.g, simple addition and subtraction).
- For larger numbers, children's number sense tends to be inhibited by carrying out these same tasks, as least by standard algorithms. Several possible reasons may be suggested for this:

  (1) Standard algorithms are often not well understood at the time they are learned. Procedures that are non-sensical from the child's point of view are likely to minimize the development of good number sense

  (2) Even when they *are* understood, standard algorithms are executed "mindlessly." Indeed, we would not want matters otherwise, since the power in these algorithms lies precisely in the fact that they *can* be executed in this fashion.

  (3) The standard algorithms all draw on the properties of the decimal system which permit large-number calculations to be reduced to small-number calculations. Thus, children do not get much chance to experience the properties of large numbers in the course of executing standard algorithms. By contrast, non-standard algorithms tend to reduce difficult large-number calculations to easy

large-number calculations, thus preserving (and even highlighting) some of the fundamental properties of large numbers.

- Number sense for large numbers is most likely to develop in programs that encourage children to explore the properties of different problems and alternate solutions.
- Number sense may also be facilitated by the opportunity to share and discuss these alternative solutions and procedures with other students.
- Some focus on mental computation may also be beneficial, since it will lead to the development of the non-standard algorithms mentioned above. Some of the properties of numbers that may be highlighted as a result include associativity and commutativity, as well as certain properties of the decimal system, "even numbers," "powers of ten," and "doubles."
- Attempts to teach "standard methods" for such tasks may well be self-defeating, for the reasons mentioned above (i.e. that they will convert a mindful problem-solving task to an algorithmic one).
- Number sense for negative numbers, fractions, proportions, probabilities, etc. is *not* universal or natural. Not only does it not appear to develop with any frequency in non-schooled populations, it often does not develop in schooled populations for more than a small percentage of children in any given class.
- The sort of altered experience that will improve children's number sense for these more complex kinds of numbers cannot be specified with any certainty at the moment. However, it seems likely that techniques which afford a rich variety of referents for the numerical entities that are involved would be a help, as would be simulations of the operations involved, either in the real world, or in computer "microworlds."
- Ideally, improved instructional situations should provide immediate and intuitively clear feedback on the accuracy of the results of numerical transformations, as well as referents for the transformations themselves. For example, in a computer micro-world, if one writes an equation involving a proportional multiplication of an area, one should be able to see the operation carried out visually and to see the result as well, thus building up an intuitive set of expectations regarding both the operation and the result.
- A greater amount of experience of the sort that is provided for more elementary operations (such as practice in magnitude comparisons, certain kinds of mental computation, discussion of alternative, "invented" methods, etc.) might also be helpful, as might mental estimation.

Psychological Basis of Children's Number Sense

One might expect that cognitive psychology, which has undergone a period of unprecedented growth over the last 30 years, would be able to offer a clear model of number sense. Producing such a model might constitute a good "research agenda" for cognitive psychologists who are interested in improving the theoretical basis of our understanding of mathematics. However, there are several factors that make it difficult for cognitive psychologists to specify such a model at the moment, or even to agree on the form such a model might take.

1. One factor is that, in the area of mathematics, at least, cognitive psychologists have tended to build models of knowledge that are relatively "contained" or "local," and that give rise to explicit procedures, rather than to an "intuitive sense."

2. A second factor is that, while most investigators now agree that knowledge is not simply a "collection of facts and skills," the way in which knowledge is viewed still varies a good deal as a function of the epistemological tradition by which the investigator has been most strongly influenced. Moreover, while there is some tendency towards convergence and "eclecticism," each one of the major traditions is in the midst of a period of potentially

productive renewal. This means that there are a variety of possible ways to think of children's knowledge within as well as across epistemological traditions.

Since there was considerable discussion of recent events in one epistemological tradition at the conference--both in the sessions themselves and at mealtimes--a brief digression on this general topic seems worthwhile.

Epistemological Roots of Cognitive Psychology

1. In the *empiricist* tradition, knowledge has customarily been viewed as having its origin in the external world, and the mind has traditionally been viewed as a device for registering this information, detecting the patterns that it contains, and developing responses for dealing with them. It was this tradition that gave "scientific" psychology its first learning theories (Watson, Hull etc.), and its first theories of intelligence (Galton, Burt, etc,). It was also this tradition that gave educational psychology its first formal curriculum planning techniques (Skinner, Gagne, etc.), as well as its first assessment devices for determining what children already know (see Thorndike's "Achievement Tests"), and how efficient their basic "machinery" is for acquiring new knowledge (see Binet's, Wechsler's IQ tests).

Since the "cognitive revolution," investigators in the empiricist tradition have focused more and more on the internal devices by which patterns, principals, etc. are represented in the mind (whether appropriate or misleading), and the processes and procedures that are available for accessing and deploying these devices, in various types of problem-situations (schema theory and analysis would be one example). The basic epistemology has remained the same, however, as has the general educational notion that the environment should teach the necessary elements of children's knowledge, via a rationally designed curriculum that takes children from where they are to where the curriculum designer feels they should be, in an optimally efficient fashion.

2. In the *rationalist* tradition, knowledge has traditionally been viewed as having its origin in the mind of the organism and in the structures with which the organism is endowed for creating order out of the rather chaotic environment around it. In Piaget's work, these structures were seen as going through a universal "life-cycle," as it were, as a result of interaction with the physical and social environments and of reflection on the internal regularities which this interaction revealed. In Chomsky's work, these structures were seen as innate, and as emerging at biologically determined points in time. In either case, however, a sharp distinction was made between learning and development: development was not simply seen as being the result of cumulative experience or learning, as in the empiricist tradition. Rather, development was seen as determining what sort of learning was possible at what points in the human life cycle.

The same events that brought about the cognitive revolution produced a great interest in both Piaget's and Chomsky's work, especially among mathematicians (for whom the properties of numbers have always had a rational "life of their own," which was dependent on an internal reflection). The general educational methods that were advocated by Piagetians were ones that were child-centered, and which encouraged this sort of "internal reflection," accompanied by attempts by teachers to point out any contradictions to which this reflection led. There was also a concern that certain types of material might be inappropriate for certain age groups, and th[illegible] there were attempts to match the form of learning opportunities with which children were presented to their stage of cognitive development.

Notwithstanding the popularity of Piaget's ideas, many of his notions (especially those related to stages) took quite a beating at the hands of empiricists and cultural relativists during the 1960s and 1970s. One result was the emergence of neo-Piagetian theory, in which these difficulties were acknowledged and through which an attempt was made to salvage the positive aspects. From an educational point of view, the change that these theories brought resulted from the fact that neo-Piagetian theorists often admitted the possibility of direct instruction, and indeed attempted to specify in some detail how it could best be conducted. However, they also still leaned towards child-centered approaches, in which different age groups received different sorts of treatment and children did a great deal of the cognitive "work' in the curriculum as a result of their own problem-solving activity.

3. The third epistemological tradition is the *sociohistoric* one. This tradition was the one that generated the most discussion at the conference, as a result of recent developments within it and the way in which current representatives of this tradition interpret the recent work on the arithmetic knowledge of unschooled children. According to the classic sociohistoric view, knowledge is not simply "recorded" by the senses. Nor is it simply "constructed" by the individual as a result of the application of age-typical cognitive structures and reflection on the invariant internal patterns which these reflections yield when applied to the content of a particular domain of knowledge.

Rather knowledge is essentially a *social* construction, with certain particular forms of knowledge constituting the intellectual tools for generating new knowledge and being passed on from one generation to the next. These tools (including forms of representation) reflect the culture's own unique dilemmas at the point in time at which they are invented, and continue to evolve with the culture. As a result of Vygotsky's influence, language has often been seen as one of the most important of these "cultural tools."

During the time of the cognitive revolution, Vygotsky's ideas were made popular by Bruner, who added his own unique twist to Vygotsky's theory and developed it considerably. A number of important educational notions also were developed at this time, including the notion that children should learn the "voice of their discipline" (which is itself a form of culturally inherited "intellectual tool") from those who were at its growing tip. The movement for "language across the curriculum," as well as the curriculum reform movement and New Math, drew much of its inspiration from this general epistemological tradition.

Recent Developments in the Three Epistemological Traditions

In the *empiricist* tradition, there is a good deal of exciting turmoil at the moment as to whether children's knowledge should be represented by symbols and symbol systems (of either the declarative/conceptual or procedural variety), or whether it should be represented as a large associative network, in which rules are not explicitly represented but simply emerge, as a result of solutions which the existing weighted "association strengths" in the network cause it to "settle to," when confronted with a particular sort of problem.

This sort of modeling is ideally suited for dealing with intuitive knowledge. However, much remains to be worked out, both as to how such models might deal with mathematical principles and with regard to how they might relate to models of the more conventional "schema" and "process" sort.

In the *rationalist* tradition, as well, there are a number of new developments. One of these concerns the role of innate "naive theories" in shaping children's views of particular disciplines (cf. Carey, Weiser, Spelke)--a development that owes its presuppositions more to Chomsky's brand of rationalism than to Piaget's. Another development, this time . the neo-Piagetian tradition, involves the assertion that Piaget's internal structures, appropriately re-construed, may in fact constitute psychological realities whose properties must be understood if instruction in any domain is to be made conceptually meaningful and age appropriate (Case & Griffin, in press).

Finally, in the *sociocultural* tradition, there has also been an interesting set of new developments, as investigators from artificial intelligence (traditionally guided by either rationalist or empiricist principles, or both) have become intrigued by alternative epistemologies and have taken particular interest in the notions (1) that knowledge results from social activity, (2) that knowledge cannot simply be understood as existing in the head of the individual, but rather should be understood as being distributed across the social group, and (3) that knowledge is thus always *situated* in the context of the group's physical and social activity and in the purposes for which this activity is designed.

Relevance of the Recent Work in the Various Traditions for Improving our Understanding of How to Foster the Development of Children's Number Sense

As I have tried to indicate in the above digression, cognitive psychology is alive and well, and has considerable potential to offer mathematics educators who are interested in understanding and fostering the development of children's number sense. However, the particular line of research that various conference participants are engaged in does vary widely, and is still influenced to a considerable degree by the underlying epistemological assumptions they have inherited. As a consequence, there is unlikely to be complete consensus with regard to what sets of topics should be researched, what sorts of psychological models developed, and what sorts of instructional programs introduced. It would thus appear that a "multi-front" attack on the problem would be the most likely to yield fruit, and that the sorts of contributions one might expect from each front may well depend on how the issues currently being debated unfold, as much as on the specific way in which investigators decide to tackle the problem of understanding and developing children's number sense per se.

For example, investigators in the sociohistoric tradition may well want to think of mathematics as a discipline which has its own "voice" and to work on defining the nature of this voice, and how it can best come to be "heard." Even more probably, they may want to re-situate children's mathematical activity in the social and physical contexts for which it was originally designed, and for which it may still be used by real mathematicians. Thus, proofs may be seen not as things that students are to learn, but rather as things they are meant to generate and to dispute with other students, etc.

By contrast, investigators in the empiricist tradition may wish to focus on new ways for fostering children's number sense via direct instruction--ways that take into account the deficiencies of conventional instruction that were noted in earlier sections. Whether one is talking about conventional classroom instruction, however, or more modern "microworld" techniques, the particular programs that are developed may well vary substantially as a function of whether the investigator in question is more predisposed toward a "symbol system" or a "neo-associationist" model of children's number sense (or some combination thereof).

A similar point can be made for recent developments in the rationalist tradition. Those who are working on neo-innatist views of naive theories may have one perspective to offer on children's learning while those who are working on neo-Piagetian notions of "central conceptual structures" may have another. And while the contributions of these two lines of work may end up overlapping each other, as well as the work being done in other traditions, they may also end up being relatively orthogonal.

Notwithstanding (and indeed because of) this diverse set of developments, then, the prognosis for the future interaction between mathematical education and cognitive psychology seems to be to a reasonably bright one, and indeed brighter than if agreement were present on all fundamental issues.

Potential Contribution of My Own Work, to Building an Improved Psychological Understanding of Children's Number Sense

As implied above, my own personal belief is that all three of the major epistemological traditions have a legitimate window to offer on human-kind's acquisition of knowledge in general, and on the acquisition of mathematical knowledge in particular. I also believe that all three traditions have at least something to offer mathematics educators in the way of hypotheses regarding children's number sense and the sorts of educational innovations that can be developed to improve this number sense. I also believe that cognitive psychology has a great deal to learn from mathematics educators regarding children's intuitions about mathematics, and the way in which these intuitions will be affected by various educational interventions.

This having been said, however, the fact remains that my own work has been primarily in the area of cognitive development and has been influenced more strongly by one of the three traditions (namely the rationalist tradition) than any other. I shall therefore conclude by giving an example, from my own work, of the sorts of theoretical problems that assume relevance regarding the underlying nature of children's number sense and the educational directions which emerge as being most promising, when seen from the vantage point that this tradition provides.

In the paper I wrote with Sowder (Case & Sowder, in press), I proposed a model of children's computational estimation that derived from the more general theory of children's intellectual representations on which I have been working for the past three or four years (Case & Griffin, in press). According to the general theory, 6-year-olds are capable of conceptualizing a single quantitative dimension. According to the model in the Case and Sowder paper, this ability should find its expression in the domain of mathematics in an ability to carry out one-column mental addition. Then, at the ages when children are capable of representing two dimensions, and then two dimensions with compensation, they should become capable of carrying out two-column mental addition (8 years) and then two-column mental addition with carrying (10 years). In a parallel set of developments, they should become capable of making one-column "nearness" judgments at 6 years, two-column nearness judgments at 8 years, and two-column nearness judgments with some compensation at 10 years.

A major shift in children's representational competence is hypothesized in my general theory between 10 and 12 years, as it is in Piaget's. This shift now allows children to coordinate two activities which they could execute only in isolation at the age of 10. As a result of this coordination in the domain of mathematics, we (Case & Sowder) proposed that children should become capable of coordinating the capability for making complex "nearness" judgments with the capability for carrying out two-column mental arithmetic. As a consequence, they should be able to solve a very lengthy two-column addition problem in a new way, namely by rounding each number to the nearest "ten," (or for that matter, to the nearest five or double) and then mentally adding these rounded values. In short, they should now be capable of "computational estimation," as long as they need only keep track of and compute one "running, rounded, total." Since the general theory is a "recursive" one, it follows further that by age 13 or 14, they should be able to keep track of two such running, rounded totals. Finally, at approximately 16 years, they should become capable of keeping track of three such running totals, or two totals with sophisticated compensation.

The relevance of this model to the topic of the conference was never made explicit. (Nor was the model itself very clearly presented.) One potentially relevant aspect, however, is to suggest a testable hypothesis, namely that a strong focus on computational estimation before the age of 11 is likely to be misplaced, unless the topic is introduced in a new way and adapted to children's existing representational capabilities.

Another potentially relevant aspect of this line of work is more general. The general theory of children's cognitive development, and the mathematical embodiments of it that were suggested in the paper, may suggest a basis for building a model of how children's number sense changes with age, as their underlying representational competence changes.

Consider the specific mathematics model that generated the prediction for 6-year-olds' capabilities regarding mental addition and judgments of "nearness," namely that both these capabilities should be restricted to "one-column." These predictions were based on the assumption that children's underlying representation at this age takes the general form indicated in Figure 1.

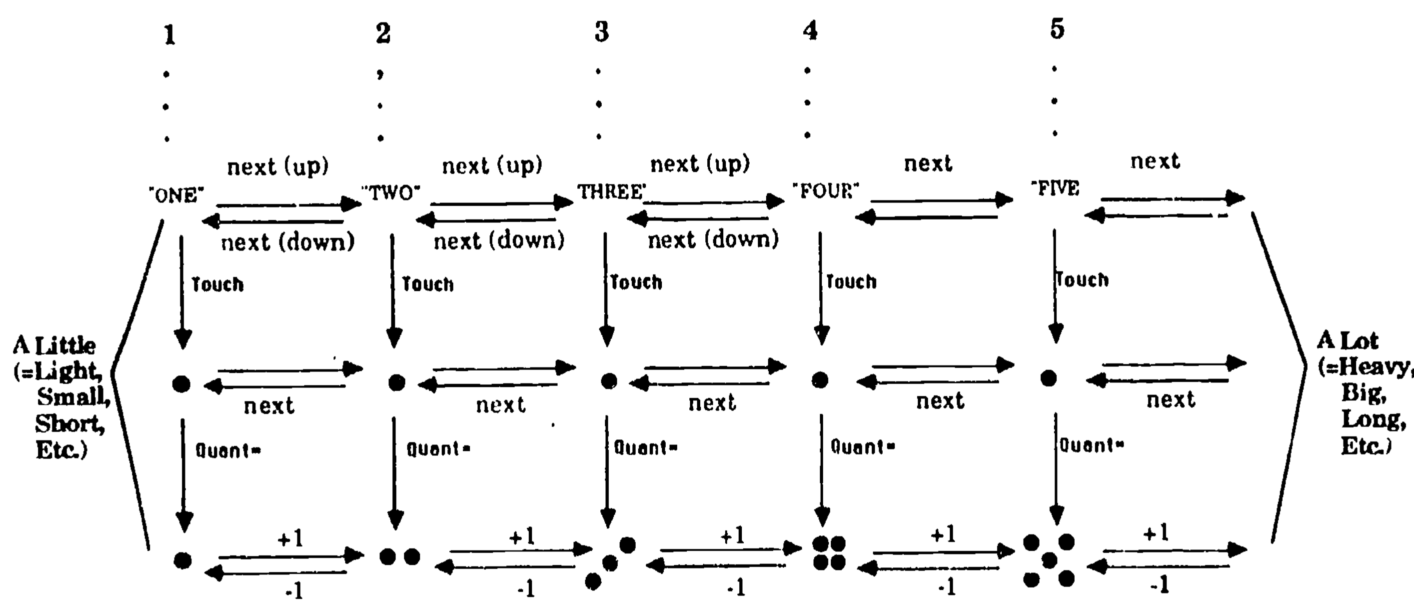


Figure 1. Cognitive structure underlying 6-year-olds' numerical understanding. (Dotted lines indicate "optional" (i.e. non-universal) notational knowledge)

As may be seen, the central notion is that children at the age of 6 are able to think in terms of the mental analog of a single number line. It is this underlying representation which is hypothesized to underlie the ability to perform one-column mental addition or to execute one-column "nearness judgments." More importantly, however, it is this underlying representation which is hypothesized to constitute *their sense of what a number is and of what the legitimate things that can be done with numbers are.*

Now this underlying conception can be seen as being composed of an intrinsically structured set of insights, which are spelled out in the Case and Griffin paper. Moreover, since similar work is going on for each of the other age groups from 8 to 18, we hope ultimately to specify the structured set of insights which each of these age groups possesses, as well.

Once specified, these insights can form the basis for several types of educational research of some importance. For example, in my present research, I have been collaborating with Bob Sandieson and Sharon Griffin, exploring the possibility that lower class immigrant children, and others who do poorly in Grade 1 arithmetic, may experience the difficulties they do because the curriculum presupposes a great deal of the number sense implied by the above structure, even in the "math readiness" exercises it presents (for evidence see Case & Sandieson (1988) or Case & Griffin (in press)).

This being the case, it would seem that three directions for further research on number sense can be suggested. First, we can see whether providing games and exercises that develop the standard o-year-old conception of number will enable disadvantaged students to profit from a standard mathematics curriculum, in the way that other students currently do (with the aid of the McDonnell Foundation, we are currently exploring this possibility). Second, as we further refine and develop the theory's structural models for older age groups, we can see if they can serve a similar function at later points in time, for the same general populations. Finally, as we reach the ages for which a great many number concepts are poorly understood by all children (see section on Hypothesized Conditions, above), we may find that the functic.n the models can serve is considerably broadened and that they can in fact serve as a basis for improving the mathematics curriculum for all children, so that our students emerge with a much stronger sense of such entities as proportions, irrational numbers, and so on. Given the increasing importance of such numbers in the technical and everyday worlds, all three of the above lines of research would appear to be ones that are worthy of further exploration.

Some Personal Reflections on the Conference on Number Sense, Mental Computation, and Estimation

Robert Reys

I came to the meeting as a researcher in mathematics education, with a special interest in learning about how students estimate. Over a decade of previous research related to computational estimation had suggested that both number sense and mental computation skills are important foundations for the development of estimation techniques. Critiques of a recent research proposal I had submitted to the National Science Foundation suggested there was much research in cognitive learning that I should not only be aware of, but that such research should be used to provide "psychological hooks" for my proposed research. Thus one specific motivation--and one that is self-serving--for participating in this conference was to become familiar with some of this relevant research and learn more about these "psychological hooks."

The papers provided for reading prior to the conference provided a wide range of ideas, many of which directly addressed some key research issues and others of which seemed to be loosely related to what I perceived to be the focus of the meeting. The two-day meeting left me with mixed emotions. I found some discussions, particularly about number sense, to be very interesting and stimulating. Number sense, perhaps because of its current visibility in the NCTM *Standards* and elsewhere, dominated the agenda. The fact that number sense is so many different things to different people produced much discussion. However, the amorphous nature of number sense made it impossible to get beyond identifying characteristics of people who exhibit a good number sense. It is clear that number sense is not a discrete quality that a person either has or does not have; rather it is a continuous quality that is possessed at different levels, and these levels have the potential for constantly expanding to reflect new experiences and higher levels of understanding.

The discussion of number sense has caused me to rethink my own schema involving the relationships among number sense, estimation, and mental computation. The conference discussions treated number sense as a structural umbrella which included estimation and mental computation. My earlier research focused on computational estimation, which is a subset of estimation. Rightly or wrongly, this research encouraged me to keep estimation as a focal point of my thinking, so I subsequently thought of number sense and some of its characteristics (such as flexibility in working with numbers) as building blocks for estimation. Similarly I thought of mental computation as another building block for estimation and not necessarily as a part of number sense. Although I can certainly accept number sense as providing a broad structural umbrella, I am also still concerned with the dangers of making number sense (already an amorphous topic) more expansive and thereby even more amorphous by collapsing mental computation and computational estimation under the guise of number sense. I think the distinct and unique characteristics of both mental computation and estimation justify their treatment as distinct topics.

Even though number sense may provide an overall structure, I think too much time was spent discussing number sense and too little was said about mental computation and estimation. Although many important ideas about how students learn mathematical concepts and how classroom teaching can be modified were discussed, I heard virtually no discussion devoted to exploring learning models which apply to estimation. It may very well be that existing general cognitive frameworks are sufficient to address the issue. While reflecting, I have looked back over the preconference papers. I found three of them (Reys, Rybolt, Bestgen and Wyatt, 1982; Siegel, Goldsmith and Madson, 1982; Sowder and Case, in press) that offer cognitive frameworks which specifically address estimation. Do these papers contain sufficient structure to provide "psychological hooks" to guide further research? I thought so going into the conference, but based on the previous feedback from the NSF proposal, I assumed there was an entire body of cognitive research relevant to estimation. Consequently, I anticipated that several alternative cognitive models based on that body of knowledge would be offered during the conference to guide further exploratory research in estimation. I did not hear anything of this kind. I did, however, hear the need for additional research mentioned numerous times, as well as the need for researchers in mathematics education to work cooperatively with researchers in cognitive psychology.

Perhaps some time should have been given to people who are currently doing research in the area to update participants on their latest thinking. I know that in our case the estimation research projects that we have been doing this year in Japan and Mexico have been very interesting. We have much more analysis and synthesis to do. Yet we have learned that even though the performance varies greatly, the estimation processes used by good estimators in each of those countries is highly consistent with our earlier research with students in the United States. Maybe some updates from active researchers would have stimulated more discussion about cognitive models and estimation processes.

I came to the conference fully expecting to have some new developments in cognitive psychology unveiled. Furthermore, I anticipated that their relevance to research in estimation would be clear and direct, and that perhaps some of the discussions would focus on speculating about different models to use in guiding future research. These things did not occur, and upon completion of the conference, I was disappointed because these specific expectations were not met. My reflecting time has convinced me that my entering expectations were not commensurate with the conference agenda.

During the past two weeks I have been assimilating and accommodating some of the ideas put forth during the conference. I learned much and am glad to have been involved. It is clear that much research needs to be done in number sense. It is equally clear that joint research efforts involving cognitive psychologists as well as mathematics educators are likely not only to lead to research projects that are funded but also to produce results that will make substantial contributions to the mathematics learning of students.

Reaction to the Conference on Number Sense

Harold L. Schoen

The conference provided a stimulating mix of people. There was the usual healthy tension between the viewpoints of the mathematics educators and those of the psychologists. This tension stimulated those holding each point of view to examine their assumptions and sharpen their thinking, but the differing backgrounds and views of the world also appeared to cause some frustration to those who had a definite agenda in mind. For example, I am not sure that we made much progress toward a theory of number sense that is useful for mathematics education or cognitive/developmental psychology researchers.

In this paper, I present my perspective on some of the progress that I see we made on two issues--defining number sense and assessing number sense. I also present some ideas not brought out at the conference that are intended to add further clarification of, at least, my perspective on these issues.

Defining Number Sense

"Number sense," as it is used in mathematics education, seems to be an umbrella term that refers to an insightful, reflective approach to doing arithmetic. Students are said to have good number sense if they routinely apply, in appropriate situations, (at least) the following knowledge and understanding:

a) the concept of number, especially with respect to relative sizes of numbers and the ways that they can be decomposed and combined;
b) the relationships among and between numbers, such as 1/2 = 0.5 = 50%;
c) the properties of numbers under the various operations and the effect on numbers of each operation; and
d) the role of numbers as measures of various quantities in real-world settings and especially the homomorphism between the numbers under operations in the world of mathematics and the quantities under appropriate transformations in the real-world setting.

Presumably, good number sense and an understanding of number and operation properties, such as distributivity and inverse operations, develop simultaneously. In addition, having good number sense also seems to include the ability to be completely comfortable with the use of numbers and operations to represent and manipulate quantities in any of an unspecified, but large set of real-world contexts, and to be able to move easily between each of these contexts and the world of pure number. Furthermore, a student with good number sense has a monitoring system for the reasonableness of her arithmetic results, which functions almost automatically for her in much the same way that a good player "sees" the implications of the positions of the pieces after each move in a game of chess.

One argument that emerged at the conference was that this characterization of a student with good number sense requires a new view of what it means to know mathematics, one that is profoundly different from the prevailing view in schools today. Most participants (including me) seemed ready to embrace this idea, but it now seems to me that I was glossing over some important issues. The goals implied by the term "number sense" are not new for mathematics educators; good teachers, curriculum developers, textbook writers, and even test developers have long held them. Our best students have even achieved these goals. What is new is the term "number sense" and the emphasis on specific components such as estimation and reasonableness of results in settings within and outside mathematics, and a desire to emphasize these in instruction and assessment.

A simplified form of the position advanced in the meeting seems to be: We hold this view of knowing mathematics; therefore, elementary school curriculum and teaching practices that will best help students attain our broad goal are those that mirror our epistemology. That seems to me to be a jump in logic of mammoth proportions that does not warrant uncritical acceptance (although I agree that a great deal of change is needed in most schools). The argument appears to arise from a juxtaposition of two extremes--the straw man of the worst case scenario often used to describe the present state of practice by teachers, textbooks, and tests as contrasted with the idealized best case scenario of a number sense, now rarely seen in educated adults, somehow becoming commonplace among the general population of elementary school students.

For my purposes as a mathematics educator, neither of these extremes is useful. My daily practice requires me to deal with a real-life classroom context, which is neither as bad as the worst case scenario nor likely to become as good as the best case (and surely not without a great deal of thoughtful, empirical work to move the school setting from its present state toward this ideal). I also am acutely aware that teachers construct their own views of teaching, basing it on their existing conceptual frameworks, in much the same way that students construct their own mathematics. Movements aimed at reforming curriculum and instruction must take that fact into account.

For me the possibility and desirability of the new emphasis on number sense is primarily a result of technological developments which make paper-and-pencil skills less important for their own sake than they may once have been, and only secondarily based on epistemological considerations. How these newly emphasized goals can best be attained (or approached) in real classrooms seems to me to be an empirical question, perhaps *the most important* question that mathematics education research has to address.

For selfish reasons, I was disappointed that so little of our discussion focused on assessment. This all too brief discussion was especially interesting to me as were one or two of the papers that dealt with assessment issues. Interaction with those who have been experimenting with innovative modes of assessment was stimulating, too. On the other hand, it seemed appropriate that we should first struggle to arrive at a definition before we discussed assessment issues.

Although testing is likely to change drastically in the near future, knowledge about some micro-level number sense issues will still remain important for mathematics educators and for teachers. For example, it may inform our holistic view of number sense to know what estimation processes are prevalent among good estimators, even on constrained tasks. Such knowledge could provide useful diagnostic information for dealing with students who fail to meet expectations in a holistic classroom setting. Such micro-level knowledge may also be an important part of a good teacher's repertoire of pedagogical knowledge (his "teaching sense"), even for one who is teaching ala Lampert. Good teachers need to have a detailed understanding of individual children's cognitive abilities and preferences, no matter what pedagogical methods they employ.

An assessment issue that has always fascinated me is the extent to which tests affect the practice of teachers and students. I understand that Tom Romberg has studied this effect. To what extent can and do assessment techniques drive curriculum and instruction? If the effect is as great as many people believe, can testing be used as an instrument to change curriculum and instruction for the better? The CAP (California Assessment Program) approach for example, seems to rest on the assumption that the answer to this last question is yes.

On the whole, I believe that good tests and testing modes can have a positive effect on the curriculum. However, I was disappointed (but not entirely surprised) to learn recently that high school students who participate in mathematical olympiads are taught highly specific techniques for solving the types of problems that are likely to be on the test. They also spend a great deal of time solving problems from old olympiad exams. The contestants are not all particularly good mathematics students, but they develop a well-honed set of skills for solving olympiad-type problems. There may be some value in helping students develop test-taking skills that work well on such non-routine problems as those on olympiad examinations, but the moral to me is that bright students will almost surely look for ways to maximize their probability of success on any test. Thus, developers of innovative test modes need to be fully aware of the sometimes unexpected test-taking skills that are encouraged by their assessment procedures.

Conclusion

This conference was a stimulating experience for me. It was a pleasure to have the all too brief opportunity to interact with new psychologist friends and to see old mathematics educator friends as well. We struggled with difficult issues, but I believe we made progress in clarifying some important ideas.

Conference on Number Sense: Reflections

Barbara Reys

My first plan in writing this reflective paper was to take one aspect of number sense and report the comments of the conference group as well as elaborate by providing my own thoughts. However, I found it difficult to choose one topic. Rather, I felt the most important outcome of the conference for me was what I describe as "the filling in" of many small pieces in my own thinking covering a variety of topics discussed. Therefore, this paper is a report of the new thoughts I've had as a result of the discussion of the group.

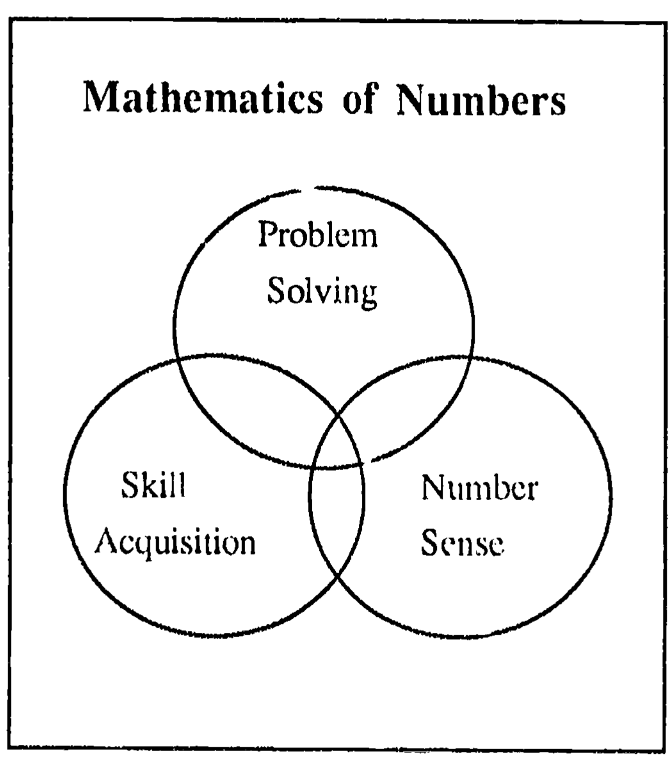
Number Sense

Although there was much discussion as well as a demonstrated concern over not having a comprehensive definition of number sense, I do not feel that such a definition is altogether necessary. To my knowledge, there is no definitive statement on what problem solving is, yet that makes it no less important nor has it deterred a concerted effort to promote the teaching of mathematics as developing problem solving.

The explanation of number sense provided by Thompson and Rathmell in "By Way of Introduction" in the recent *Arithmetic Teacher* number sense special issue is indeed adequate to guide discussion with teachers, mathematics educators, and researchers.

*

One of the comments made during the discussion of number sense which caused me to do further thinking was, "Since we seem to be having trouble deciding what number sense is, perhaps it would be helpful to describe what it is *not*." I propose this diagram for the reader's consideration:



As is illustrated, I believe it is possible for the learner to be working within any of the main areas of problem solving, number sense, and skill acquisition apart from the other areas, or there is also the possibility of overlapping work. For example, a fourth-grade student who correctly finds the sum of 45 and 59 may be demonstrating a working knowledge of the addition algorithm without any conceptual base for the numbers 45 or 59. In this instance, the student is working in the area of skill acquisition alone. Similarly, a student may be able to find the sum of two fractions such as 1/2 and 1/4 without having been introduced to or having used the standard algorithm for adding fractions with unlike denominators, relying instead on his/her understanding of the fractions and the operation of addition. In this case, the student is working within the realm of number sense alone (or perhaps the intersection of number sense and problem solving).

*

At one point in our discussion, knowledge of the various operations of arithmetic was included as a part of number sense. It struck me at the time that perhaps there is such a thing as "number sense" and something else called "operation sense." More likely, the latter is a part of the former.

*

It was clear from our discussion that the consensus of the group was that number sense is as much a way of teaching as it is a set of "things" to be taught. In this regard, it is analogous to the teaching of problem solving.

*

I don't believe number sense can decline with age (as was suggested by several examples highlighting individuals who demonstrated number sense early in school, but not later in their school experience). Rather, the environment which would value number sense and cause it to come to the surface of intellectual thought is likely not available to the learner at the later point in time. I am convinced that over time, many students do not believe that mathematics is supposed to make sense or be logical; therefore they do not require it to make sense.

Estimation

Estimation seems to be a very good vehicle for studying conceptual knowledge and acquisition. While a body of research has established that a working knowledge and proficiency with algorithms can develop absent a conceptual base, I do not think a working knowledge and proficiency in estimation can develop without a good conceptual base.

*

Standardized tests, by asking test-takers to use arithmetic algorithms, seem to measure whether one can <u>do</u> mathematics rather than whether one *knows* mathematics.

*

I heard several participants counsel against the teaching of specific estimation strategies. I am in agreement that such teaching should not lead to a new set of algorithms (in this case, estimation algorithms) for the student to memorize. However, if the teaching is presented in a manner which promotes conversations among students as to the variety of approaches one might take to forming an estimate, then I believe these conversations could be extremely helpful to the learner. In this atmosphere, it would also seem appropriate for the teacher to propose strategies not highlighted by the students.

*

The notion of reasonableness was discussed only briefly at the conference. I would like to see more focused research attention given to this issue. Specifically, in what context is a user of mathematics likely to reflect on whether a calculation (whether produced by paper/pencil, estimation, or calculator) is reasonable? Is this attention to the calculated answer most likely when the result of such a check on reasonableness has physical consequences to the user (e.g. when they will be overcharged at a grocery checkout), or is it the problem itself (numbers, operation) which cause some learners to reflect on the calculated answer? In either case, can such a sensitivity to calculated answers be "taught?"

*

Paul Trafton proposed the idea of a research effort that would study the effect of developing number sense before algorithm work. Specifically, he suggested the area of percent be taught with early emphasis on the meaning of percent, attention to establishing key benchmarks (10%, 50%), mental computation with these benchmarks, estimation of percent problems using these benchmarks, and finally, calculation methods (paper/pencil or calculator) with percent problems.

I think such a concentrated effort on a specific area, namely percents, would be most useful. Other areas where parallel investigations could occur are angle measure, summing fractions, and decimal multiplication.

*

Several reasons for including estimation in a mathematics program were offered during the conference. The three with which I concur are:

- vehicle for developing number sense / concepts
- social utility
- understanding the nature of mathematics.

*

One way of teaching estimation is the "classroom conversations" approach. That is, start with an estimation problem which causes students to search for a solution method. Follow with a discussion of various estimates and the processes which led to these estimates.

Mental Computation

Paul Trafton made a very important distinction with regard to mental computation which I had never heard or read or even considered in my own thinking. He stated that perhaps there are at least two levels of mental computation. Problems which include operating with powers of ten or multiples of powers of ten are one level. This most basic level requires an "algorithm" for determining place value of the answer but does not generally require an invented strategy on the part of the learner. Likewise, problems from the area of fractions such as 1/2 + 1/4 or 2/5 + 4/5 require a conceptual foundation in fractions but no specific strategy. Such problems are actually "EXTENDED Basic Facts". On a second level, problems such as 7 x 99 or 15% of $14 generally require knowledge of properties of numbers and/or operations and a self-developed technique which is understood by the user.

*

Little attention was focused on the practical aspects of teaching number sense, estimation, or mental computation during the conference. It is, of course, easy to highlight existing poor examples of teaching these areas. It is also easy to make a blanket statement that none of these notions can be taught without a major overhaul in our curriculum and current teaching methods. However, there are many examples of students who have developed these notions within the existing curriculum and teaching methodology. Therefore, it seems reasonable to assume that minor (?) changes made to either curriculum or methodology can lead to more students developing more fully these notions. For example, if mental computation is given systematic attention using the "classroom conversations" model outlined earlier, such effort can lead not only to proficiency at mental computation tasks, but might lead also to a personal global view of mathematics as a way of thinking about and working with numbers.

Reflections on the Number Sense Conference

Paul Trafton

To say that participants struggled with the notion of number sense is an understatement. We believe we recognize number sense when we see it, but it is difficult to describe and discuss precisely. Nonetheless, I think we all profited from exploring this idea and its connections to mental computation and estimation. The lack of a precise description of number sense should not be a barrier to further work on it; after all, we all have a sense about "intuition" and "insight" although these ideas lack a precise definition--and number sense is in the same arena as they are. I found the conference stimulating, and several thoughts have been going through my mind as I work on clarifying my own thinking.

1. Most children possess number sense, at least with whole numbers. It is closely linked to primitive notions of how quantities can be composed and decomposed (additive composition, to use Resnick's term) in a variety of ways. Children seem able to grasp intuitively that 250 can be partitioned into 200 and 50, when sharing 250 M & Ms with a friend. Some think about taking half of 200 and half of 50 and combining the results. Other children might "build up" to the answer by trying 100 for each one, realize that there are 50 left, and "know" that 25 and 25 are 50. These responses occur often with both second- and third-grade students who have not learned the typical procedure for computing 250 divided by 2.

2. Number sense is more related to intuitions and insights associated with numbers as quantities, rather than numbers as abstract, formal entities. There is a "physical" aspect of responding to numbers; parts are "tacked on" and "knocked off" as needed. The discussion by Carraher, Carraher, and Schliemann (1987) about the *manipulation of quantities* associated with oral mathematics (mental computation), rather than the *manipulation of symbols* associated with written computation, is significant here.

3. As children encounter symbolic and rule-driven mathematics in typical school settings, they tend to ignore their informal insights and attempt to deal with number tasks within the more formal mathematical arena. This poses problems in attempting to measure number sense, because children interpret such tasks from a school learning perspective, rather than in terms of their natural understandings. Perhaps this is why a fourth grader responded in the following way to a number sense task involving the percent correct on a test where 23 of 25 items were correct (the choices were 46%, 60%, and 90%): "We haven't studied percent yet." The interviewer talked about what a 50% chance of rain and a 100% chance of rain meant and other common uses of percent. The student then said, "Oh, you mean *that kind of percent*," and then went on to reason that it must be 90% since the person got them almost all correct and 100% is the whole thing. The student's initial response may have meant something different, but it is suggestive of what many of us have noted.

Hilde Howden's discussion in the February 1989 focus issue of the *Arithmetic Teacher* on number sense illustrates these opening points as well. Children in a lower middle-class school were able to give several ways of thinking about 24, whereas upper middle-class children gave the standard response and stopped. The various examples of "marketplace" mathematics, the strategies used by children and adults in familiar settings, also seem to support this point.

4. While number sense is difficult to describe precisely, there are some components of it that we *are* aware of. I think the points made in the K-4 *Standards* in discussing number sense come close to getting hold of significant components of number sense. The first point deals with having a sound understanding of the meaning of a number. Children have many contacts with whole numbers at a young age and are able to connect models, oral language, and symbols relatively easily. Thus it is not surprising that it is in this area that we see "numbers sense" most clearly. On the other hand, meanings for fractions and decimals are much more difficult to construct, as shown by the abundance of literature in this area, and thus children have little awareness of what 2/5 and 0.46 represent. As a result, they are limited in thinking informally about them. Thus, we may say they lack number sense with fractions and decimals, whereas the real problem is that they lack any conceptual framework for these numbers, and have no insights about them.

The second and third points deal with relationships between numbers and with relative magnitude. The ability to describe how numbers are composed and decomposed is part of what I have in mind; 24 is 2 twelves, is almost 25, is about half of 50, and so on. Children are able to describe 8 as 100 minus 92 before they learn to compute this. Informal efforts at providing opportunities for children to engage in these kinds of experiences seem to suggest that they are productive in increasing children's abilities to see various relationships.

One clear arena in which number sense is displayed is operating with numbers. Tasks which have children focus on the quantities involved encourage this. For example, if one has the numbers 487, 169, and 257 and wants a sum between 400 and 500, or over 700, solutions can easily be found by thinking of the amounts involved. While fourth graders, for example, first approach this type of task fairly mechanically, they learn relatively quickly that such tasks can be solved in more intuitive ways. What is important to note is that their first response is in terms of the conventional procedures for operating on numbers in school.

5. One underlying question is how one goes about "teaching" number sense. When we use the word teaching, we often think of a series of lessons devoted specifically to this topic. While this may be possible, I think it is more a case of *developing* number sense, that is, a continuous emphasis on notions about numbers, informal relationships between them, and the effect of operating with numbers. Another way to say this is that number sense work needs to be an ongoing, informal emphasis in all work with numbers. In early grades, students need to realize that 18 is closer to 20 than to 10. This might be accomplished by showing reference chains of 10 and 20 connecting links, and then showing a chain of 18 links. Many children will "see" that 18 is 2 less than 20. They might be led to see that it is 3 more than 15 and 8 more than 10. Certainly part of concept development of fractions can include ongoing emphasis on suggesting fractions that might be used to describe a shaded portion of an unmarked bar. If 39% of a bar is shaded, children might note that it is a little less than one half, a little more than one third, about two fifths or three eighths, etc. Using marked bars, they can compare two fifths to one half, one third, three eighths, four tenths. This sort of ongoing treatment of number sense has a cumulative effect of building rich and deep relationships among numbers. Children become "aware" of the amount represented by a number. Thus, my best sense is that number sense is something that "unfolds" rather than something that is "taught" directly.

6. Evaluating number sense poses real problems. When tasks are posed within the more formal, symbolic dimension of school mathematics, students tend to try to apply various mechanical rules to these tasks. However, when these tasks are posed more informally, and often in real world contexts, they tend to respond from a different perspective. Children "sense" that when they have read 48 pages of a 100-page book, they have read about "half" of the book. Yet, when asked to respond to a formal task such as: 1/2 of 100 is about ___ (22, 49, 65), they will attempt to access their formal system of rules, become confused, and give meaningless responses. The evaluation is complicated by the fact that their instruction has been formal and abstract. It would be interesting to evaluate number sense in children who have had informal, exploratory instructional programs with problems and settings that encourage accessing this part of their thinking about number. It may be relevant to draw conclusions about children's number sense only when the approach to instruction has encouraged and fostered such insights. Certainly children who have had typical instruction have little awareness of what is being asked and few reference points for responding.

7. Mental computation and estimation are often viewed as specific topics within the mathematical sphere--that is, as quite specific and detailed procedures alternative to traditional procedures. When approached from this viewpoint, they tend to be treated as a series of objectives to be attained, and instruction focuses on teaching the specific procedures: adding 9 to a number, adding tens and then ones, front-end estimation, compatible-number estimation, et cetera.

   I feel it is far more productive to view these topics within the number sense sphere--as an extension of children's abilities to compose and decompose numbers in ways that make sense to them in given situations. Each situation may "trigger" a different way of thinking about numbers, and the sharing of these approaches makes other children aware of various ways of manipulating *quantities* rather than symbols. The studies of adults' and children's reasoning in naturalistic situations seem to suggest this. The approaches they use are quite inventive and insightful. Thus, perhaps these ideas should also be allowed to unfold, rather than be the direct objects of instruction. Posing problems set in realistic situations may help children use their informal network of ideas, rather than try to access the more formal network of rules for manipulating symbols. It might also prevent their viewing these topics as additions to this already overburdened system.

This discussion provides a different context for viewing instruction on mental computation and estimation. Rather than teaching lessons devoted to specific procedures, we might pose situations and let a variety of reasoning patterns emerge. Some of these reasoning patterns might receive additional focus so that other children could take advantage of them. Rather than beginning instruction of estimation by teaching a specific strategy, we might have students make informal quantitative judgments about a situation. For example, if one box contains 46 marbles and another contains 35 marbles, we might ask: "Are there more than 50 marbles? Are there as many as 100 marbles? Do you think there are as many as 70 marbles? About how many marbles do you think there are?" This line of reasoning encourages students to make reasonable quantitative judgments, rather than to attempt to "find an estimate." This introduction encourages a type of thinking different from that resulting from instruction on specific strategies. It is natural and reasonable, and it sets the stage for discussions of what would be reasonable to expect and of specific ways to determine this.

For me, this is a different way to view mental computation and estimation, and is a significant departure from conventional ways of thinking about these topics. It is number sense that should drive these topics rather than formal properties of the real number system under various operations.

Reactions to the Number Sense Conference

Zvia Markovits

At the time of the number sense conference, I was already involved in a research project on number sense, number size, mental computation, and computational estimation. So obviously before the meeting I had struggled with what number sense is and how it is related to the other topics. During the meeting, I think we all realized how extensive number sense is, and how difficult it is to define it, and perhaps this is because (as suggested at the conference) we need a different theoretical approach to number sense.

In this paper I will first suggest a working definition for number sense. Then I will raise some questions regarding number sense, and finally I will present results from some number sense tasks administered to preservice teachers.

Working Definition for Number Sense

What does it mean to have number sense? It seems that if one has number sense, one can solve problems that cannot be solved just by applying rules, or if the problems can be solved by using rules, one will solve them in a much easier and in a more efficient way by applying number sense instead of or in addition to the rules.

We can be more specific with a working definition in terms of "what we expect a student with number sense to be able to do when given a task". The definition (to follow) is meant to meet the needs of our project "Relating Mental Computation, Number Sense, and Computational Estimation" in late elementary and early junior high grades, and although this definition goes beyond the project topics, it of course does not answer all the questions regarding what number sense is.

The first question in seeking a working definition for number sense should not be, "What do we expect a student with number sense to be able to do when given a task?" but rather "What do we expect the student to *undo*?" Actually the students should undo most of what they have done in school mathematics for years, since in school mathematics there is almost always one correct response, one correct algorithm, and no relationship between mathematics and the real world. School mathematics is very rule oriented (and even when estimation is taught, in most of the cases it is only as another set of rules). Students are not supposed to make any decisions or judgments, so they do not need to use (and sometimes they are not allowed to use) their number sense and/or their common sense. So, only by keeping in mind that there is not always one answer and one algorithm, that new algorithms can be invented when necessary, that the methods they are using and the answers they are getting should reflect the real world, that mathematics is not something mechanical, but is an area in which they are supposed to make decisions and judgments, can students be prepared for using number sense.

Thus when given a task, a student with number sense is expected to have in mind that

- there is not always one answer
- there is not always one algorithm
- mathematics and real life are related
- decisions and judgments are expected

and

in a task without a context	*in a task with a context*
- to look at the whole problem first, rather than immediately applying an algorithm.	- to look at the whole problem first, rather than immediately applying an algorithm.
- to look for relationships among the , numbers,and between the numbers and the operations;	- to look for relationships among the numbers,and between the numbers and the operations,and between them and the context;
- to choose or to invent a method that takes advantage of those relationships.	- to choose or to invent a method that takes advantage of those relationships.
- to choose the most efficient representation for the given task, which sometimes means moving to a representation different from the one given in the problem.	- to choose the most efficient representation for the given task, which sometimes means moving to a representation different from the one given in the problem.
- to be aware at each step whether what is being done and the answer obtained are reasonable mathematically.	- to be aware at each step whether what is being done and the answer obtained are reasonable mathematically and in real life.

The following are examples of number sense tasks, and in order for them to be solved, all or some of the above points should be applied.

Do 47 + 26 + 18 - 26 in your head.

This task can be performed correctly the hard way without number sense, by adding the numbers in order from left to right, or much more easily and more accurately with number sense, by first looking at the whole problem and then taking advantage of the relationships between the numbers and the operations.

When you multiply 15.24 and 4.5, the answer is 6858, but the decimal point is missing. Place the decimal point where you think it should be.

15.24 x 4.5 = 6858

This task cannot be performed correctly without number sense. The rule for locating the decimal point cannot be applied here, and another method should be invented.

The height of a 10-year-old boy is 5 feet. What do you think his height will be when he is 20?

In this task one should not use a method that assumes a constant ratio between age and height because it contradicts the real world. But if this method *is* used, then the answer of 10 feet should signal that something is wrong.

Can you shade in 0.5 of

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?

If not, explain why. If so, shade in.

In order to be able to solve the problem one should first realize that the representation can be changed from a decimal to a fraction. (Just as, for example, when one works with functions, many times it is more convenient to work in a representation different from the one given.)

If we relate these and other numbers sense tasks to the definition, it seems that the only aspect of number sense that varies from one task to another is the kind of relationship between the numbers and the operations, which will automatically influence the choice of method and the appropriate representation. This means that the definition should be developed separately for each topic that requires number sense (mental computation, number size, computational estimation, etc), since in each topic, different relationships should be taken into account.

Questions Regarding Number Sense

If I had *some* questions regarding number sense before the conference, after the conference I had *lots*. Here are some of them.

• Can we say that every task can involve number sense? And then can the same task be done with or without number sense? (See the first example above.)

• What is the role of age in relation to the number sense involved in a task? Can we say that counting by fives, for example, involves number sense for kindergartners and first graders, but not for high school students? That is, can we say that a certain task involves number sense at a certain age, but does not involve number sense at a different age?

• What is the role of experience? For example, if we discuss with the students that it is more convenient to look at 578 + 49 as (578 + 22) + 27, and they solve ten problems using this method, can we still say that they are exhibiting number sense in an eleventh such problem?

• What is the relationship between number sense and common sense? Can we say that we need number sense in purely mathematical tasks, but that we need both number sense and common sense for tasks in context? And is one sometimes able to solve a problem in context but not without context, because the common sense "awakens" the number sense?

• What are the relationships between number sense and high order thinking, schemas, links, intuitive knowledge, metacognitive skills, etc.?

• What about teaching number sense? Number sense can not be another instructional unit; number sense is a *way* of teaching, and every topic should be taught with number sense. Are all teachers capable of teaching with number sense? What instruction should an inservice teacher get before he/she can teach for number sense, taking into account that this teacher probably was not taught with number sense and is not now prepared to teach with number sense? What about the preservice teachers? What about the textbooks? What about the tests?

Number Sense Questionnaire to Preservice Teachers

A few weeks ago I gave a number sense questionnaire to my students (n=49), who are preparing to become elementary school teachers. The questions were similar to those we use in our project with sixth and seventh graders. The questions were open ended, and the students were asked to explain how they did each problem. When we discussed the completed questionnaire in class, the students said that it was very difficult for them (and I think I agree after analyzing the results), not because of the content but because they had to explain their answers. They said that being required to explain made them think, that they had never thought before about how they do problems, and that actually that was the first time that they had been asked to explain !!! Until then, they had just had to circle the correct answer.

Some results:

- On the height problem (see above) 13% (n=49), said the answer is 10 feet.

- On placing the decimal point (for 15.24 x 4.5), 79% said it should be after the six, 6.858.

- Thirty-one percent were unable to order 0.53, 14/13, 5/12, and 0.993, from smallest to largest.

- When asked if the answer to 264 ÷ 0.79 would be more than, equal to, or less than 264, 49% said it would be less, because you divide. But if asked to do the computation, all of them could find the correct answer.

- Only 39% said that there would be an infinite number of decimals between 0.46 and 0.47. Others said, "There is no room between." Only 29% said that there would be an infinite number of fractions between those two decimals.

- When asked if the answer to 52 x 28 would be more than, exactly, or less than 1500, 38% said it would be exactly 1500.

Looking at these results makes one wonder, "What have these students been doing for so many years in school?" Probably nothing related to number sense.

Reflections After the Conference on Number Sense

James Hiebert

My comments are not intended as a summary of the conference. Rather, they are an attempt to articulate some thoughts that are gradually emerging and were boosted by discussions at the conference. The comments are related to the conference. I am sure of this because I have notes from the discussions, and the ideas expressed here are contained in the notes. However, I do not claim that the connections are obvious.

The comments pertain to that part of number sense that involves sense of numerals. I am especially interested in how children develop facility with our standard written symbols of numbers. This does not restrict the discussion to the written procedures or algorithms of arithmetic--strategies may involve mental manipulations--but it does restrict the discussion to processes that take the numerals as the arguments of thought.

Written symbols of arithmetic can function in at least two very different ways. They can function as records of something already known and as tools for thinking. I would like to argue that good number sense requires numerals (and operation and relation signs) to function in both ways. In fact, it seems to me that the two functions of symbols are necessary and sufficient for all of the specific skills indicative of number sense suggested by Lauren Resnick and compiled during the course of the conference.

As records of something already known, arithmetic symbols function as representations of quantities that have been experienced. Numerals are used to stand for quantities, and operation symbols are used to stand for actions on quantities. The reason symbols are used at all in this way is that they provide a convenient and permanent record of arithmetic activity. These records help one to remember the activity and to communicate it with others. In this sense, the symbols are put to public use. Written symbols support the sharing of an individual's activity with the group.

A primary contribution of the recording function of symbols for number sense is that symbols that act as records produce (at least initial) anchors of reasonableness. If symbols stand for familiar quantities, they can become sufficiently connected so that actions on quantities can be used to monitor actions on symbols. What is reasonable with quantities can be used to check what is reasonable with symbols.

If symbols function as records of what is already known about quantities, it appears that the knowledge of quantities can guide the development of initial procedures with symbols. For example, work with young children in beginning addition and subtraction suggests that children can use what they know about the quantities involved to deal with the written number sentences representing the situation. Also, our work with decimals indicates that if students see the symbols as records of quantities, they can work out their own procedures for simple problems, such as adding, subtracting, and ordering decimal numerals. I am not sure whether these initial procedures with written symbols impact on number sense directly, but they may be important indirectly by providing a link between the two functions of symbols. More will be said later about this linking possibility.

The second function of written symbols is as a tool for thinking. Symbols provide a means for organizing and manipulating ideas. Because many written symbols in mathematics stand for complex ideas, the symbols provide a very efficient way of dealing with complex ideas. As Alfred Whitehead pointed out: "By the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye, which otherwise would call upon higher faculties of the brain." This function of symbols is primarily a private rather than a public function. It supports individual thought.

Using the symbols as tools for thinking requires treating the symbols as objects of thought. What is needed here, I believe, is the recognition that symbols can be treated as objects of reality. Indeed, good estimators and mental calculators may do this as evidenced by Paul Trafton's remarks concerning the physically-oriented language used by students to describe their mental calculations. Numbers are "broken apart, built up, chopped in half," and so on. Symbols become things that can be acted on just as quantities can be acted on.

The primary contribution to number sense of using symbols as tools is the construction of mental calculation strategies that take advantage of properties of the system. Some numbers are easier to combine mentally than others, so rounded numbers can be used for approximations. Benchmark facts can be used to speed calculations. Numbers can be decomposed and recombined in various ways.

It is tempting to view the two functions of symbols as two separate roots that converge in some way to produce number sense. But that seems to be an oversimplified description. Perhaps a more accurate description requires the distinction between (1) the development of number sense and (2) its functioning at any point in time. An hypothesis about the developmental path of number sense will be presented later. With regard to its status at a particular point, it seems reasonable to assess number sense by assessing the two functions of written symbols and the potential links between them. Do students recognize symbols as records of quantity (and of actions on quantity)? What kinds of quantities (and what kinds of actions)? Do students use symbols as tools for thinking? (Because this is a private function, it is more difficult to assess.) Do students control the use of symbols (use them appropriately, as one would a tool), or are they controlled by them (follow a procedure in a slavish sort of way)? What links, if any, do students draw between manipulations of symbols and activity with quantities?

A final issue to be c nsidered is the nature of instruction that is likely to promote number sense. To a great degree, instruction that promotes number sense is no doubt the same kind of instruction that promotes meaningful mathematics learning across the board. But more specificity is needed. It is useful to consider again the two functions of symbols and to ask how instruction might support the development of one function or the other.

Designing instruction that encourages the use of symbols as records of something already known would seem to be relatively straightforward. Activity begins with problem situations in the context of quantity. Problems are resolved by working directly with the quantities involved. Quantities and actions on quantities or relationships between quantities are then represented on paper for some purpose. For example, the activity of one group in the class needs to be reported to another group. Perhaps the initial symbols are invented; the teacher then helps to guide students toward the eventual adoption of the standard notation. It seems that there is little question about the usefulness of these kinds of ac ivities. But they are rare in schools. Apparently, this is an area in which we know much more than we have translated into accepted classroom instruction.

Designing instruction that promotes the use of symbols as tools for thinking is not straightforward. An hypothesis is that the treatment of symbols as objects of thought is encouraged by the reflective analysis of procedures with symbols. That is, a thoughtful analysis of the way in which symbols behave as elements of a well-structured system may help students to see the symbols as things that can be acted on and as things that behave in a consistent and predictable way. A forum for this analysis is ordinary paper-and-pencil algorithms. A corollary of this hypothesis is that the value of algorithms is not only due to the products they generate. Algorithms also may have value in their potential to reveal the patterns and regularities of the symbol system. So, the notion that algorithms should be eliminated from the curriculum because they are overly restrictive mechanical procedures that have lost their efficiency value to technology and now serve only to suppress creative inventions of alternate strategies should be balanced, perhaps, with the possibility that reflective analyses of algorithms serve to inform and support the idea that symbols can be used as tools for thinking.

If the hypothesis is correct, one is faced with an immediate dilemma. How does the process begin? How do students get on the train? It is a chicken and egg problem. Analysis of algorithms is supposed to help students recognize the possibility and the value of treating symbols as objects of thought, but it would seem that to benefit from such analyses students would need to possess this recognition. A potential resolution of the dilemma lies in a link between the two functions of written symbols. Earlier, I suggested that if students see symbols as representations of quantities, they can use their knowledge of acting on quantities to develop simple procedures with symbols. The link comes in the fact that, once these are developed, they can serve as the first algorithms for analysis. Students can begin studying their own procedures from a syntactic rather than semantic perspective. Perhaps such analyses would provide an entrance to the world in which symbols function as tools for thinking.

The speculation that using symbols as records or representations of quantities can connect with using symbols as tools for thinking by crossing the bridge of analyzing "invented" symbol procedures is the key in suggesting the developmental path for number sense alluded to earlier. It may be that the development of number sense begins by using symbols as records of something already known. Then branches sprout from this root as symbols take on a life of their own. Different branches may be needed initially for different number systems. The growth of a branch may be stimulated by analyzing simple invented procedures, focusing on the way in which the written symbols behave as part of a symbol system.

A final comment raises the question of how far conventional instruction is from that implied by the discussion to this point. To get a glimpse of the distance, consider the way in which symbols are most often used in school mathematics and, in fact, the way in which symbols are first introduced to students: Symbols are used to present problems. Which function of symbol is being employed here? Or, as a better way to ask the question, how does this nearly exclusive use of written symbols constrain and support students' opportunities to develop facility with either function? I am not sure of the answer. It is probably not simple. But, it seems to me that using written symbols only to present problems fits neither function. For children first encountering them, the symbols do not stand for anything outside of themselves and they are not useful tools for thinking (they do not make an otherwise difficult problem easy). Using symbols to present problems early in school also violates what may be the more natural route to number sense. Many would guess that students who acquire number sense acquire it in spite of this kind of instruction.

Reflections on the Conference

Merlyn Behr

My reflections on the number sense and estimation conference center on the notions of variability (variance and invariance), flexibility of thought, and the process-name phenomenon of number representations. Variability has to do with the questions of whether the outcome of an operation changes as a result of a transformation on both or one of the operands and of how to compensate for the change when it occurs. Flexibility of thought has to do with the ability to form mental images of physical or iconic quantitative representations of the numbers and the operations, to flexibly manipulate these mental representations, and then to translate between these representations and number language, verbal or symbolic. The process-name phenomenon was initially raised by Davis, Jockusch, and McKnight (1978) and in the present context has to do with a child's ability to alternatively recognize an expression such as 8 + 7 both as an indicated operation and as a symbol for a number.

Variability

There seem to be two broad types of cognitive awareness involved in the notion of variability as it applies to number sense and estimation. One type of awareness is concerned with the questions of whether or not the computed outcome of an operation remains constant under a transformation on one or both of the operands, and of how to compensate for a change when it occurs. The second type of awareness is to recognize when a relationship between two numbers remains invariant under transformations on the two numbers in the relation. An operation involves a triple of numbers, two operands and the outcome. In the case of the variability of the result of an operation, the two "types" of strategies can be characterized according to which two of the three entities of the triple are involved in the transformation and to the timing of the compensating transformation with respect to the first transformation.

One strategy involves a transformation on one of the operands and a compensating transformation on the answer; the other strategy involves compensating transformations applied to each of the operands, respectively. When the objective is to find the result of an indicated operation, the one who uses a computation on a transformed expression needs to know in advance of doing the computation what the effect of a transformation will be and what the compensating transformation is. Invariance of operations under transformations is relevant when the question is one of getting an answer to an indicated computation and when the question is one of finding an expression which is equivalent to a given one, as in finding equivalent fractions or ratios. Some examples will help to clarify this point.

Example 1. A child (Guershon Harel, Personal Communication, Feb. 1989) was given the following division problem:

(Some number) ÷ 3.25 (In the given instance a specific number was used, but for purposes of illustration is irrelevant).

In response to this request the child responded something like the following: "Well I don't want to divide by 3.25; it's not a nice number to divide by." He next considered dividing by 6.50 (the computational representative is now (same number) ÷ 6.50), then 13, and went on to say, while pointing to where the problem format calls for placing the answer, "then I'd have to divide it by 4." In this example the transformation on the divisor, one of the operands, (equivalent to multiplying by 4) was to be compensated for by applying the inverse multiplicative operation (dividing by 4) to the result of the transformed computational representation ((same number) ÷ 13). While recognition of the need to make this transformation on the answer appears to be evident when the transformation is being made on the operand, it is made after a computation is completed. This example illustrates the first type of cognitive awareness; the second type is illustrated in the following example.

Example 2. The traditional algorithm for division of decimal numbers involves a transformation on both of the operands, assuming recognition that this transformation leaves the outcome of the computation unchanged. Consider the division 12.3 ÷ 4.25; the rule to "move the decimal point to the right in both numbers enough places so that the divisor is a whole number" involves a transformation on the divisor, "multiply by 100," which is compensated for by multiplying the dividend by the same number. Here the two compensating transformations are made before any "answer-getting" computations are performed. The emphasis on this type of transformation seems to be on getting an equivalent problem expression.

It would appear that at least implicit knowledge (i.e.as a theorem-in-action) of each of these two general transformation strategies is important to the ability to demonstrate number sense and to be able to carry out an estimation process. There are significant implications for instruction which follow from this observation, some of which I will try to call attention to later in this communication; moreover, the development of significantly different cognitive structures is involved in attaining an understanding of each of these two strategies; significant attention to this goes beyond the scope of this communication, but I will call attention to one perspective in this regard.

Children up through middle school are known to have an understanding of the equals symbol (=) as a pointer to the answer, rather than as a symbol which expresses a relationship between two mathematical entities. Recognition that the answer to the transformed division problem in Example 2 gives the same answer as the original problem ultimately rests on the realization that 12.3 ÷ 4.25 = 1230 ÷ 425; understanding this statement requires an understanding of the equals symbol as a relational symbol. The strategy used in Example 1 seems to rely more on equality as an "answer-pointer" than as a relational symbol because the concern is for finding an answer and then compensating in order to obtain the answer to the original problem.

Flexiblity of Thought

Flexibility of thought is nicely reflected in the division process shown in the following problem and solution. A young child was given the problem (Harel and Behr, in press) to find 42 ÷ 7. His process, which he explained orally, was as follows:

> I know 40 divided by 10 equals 4;
> 10 minus 7 equals 3;
> 4 times 3 equals 12;
> 12 plus 2 equals 14;
> 14 divided by 2 equals 7;
> 4 plus 2 equals 6: the answer is 6.

Further discussion with the child about how he attained the result indicated that he thought in terms of distributing objects to persons and was aware that division by 10, instead of 7, resulted in having distributed too many to each person. Subsequent steps in the process adjust for this over-distribution in ways that demonstrate the child's ability to mentally manipulate objects of his mental representation much more flexibly than is inherent in the constraints symbol manipulation must obey according to axioms of arithmetic operations. Moreover, the ability this child displayed in moving between the domain of mental representations of physical quantity and the domain of mathematical terminology was impressive. This ability to flexibly manipulate mental representations of physical quantity and to translate between the two domains of representation, physical quantity and mathematical symbols, seems to be a hallmark of good number sense.

Process-Name Phenomenon

The ability to recognize that the statement $12.3 \div 5.25 = 1230 \div 525$ or that $3/4 = 6/8$ requires that the individual cognize each of the expressions as a single conceptual entity (Greeno, 1983), i.e. see that each of the expressions is a name for a single entity, in addition to being an indicated operation on a pair of numbers. This recognition seems fundamental to being able to cognize either of the transformations below as resulting in *numbers* that are unchanged in value under a transformation from one representation to another.

1. $$12.3 \div 5.2 = (12.3 \times 100) \div (5.25 \times 100) = 1230 \div 525.$$
2. $$\frac{2}{3} \text{ --- } \times 3 \text{ ---> } \frac{6}{9}$$

To be able to cognize an arithmetic expression as a conceptual entity, as one number, seems to be critical to recognizing two arithmetic expressions as being equivalent. This recognition of equivalence is involved particularly in the invariance strategy which makes a transformation on each of the operands, but less in the strategy in which a transformation is made on one operand and the answer. In the strategy that involves a transformation on one operand and the obtained, or anticipated, answer, it would appear that cognition about the effect of the transformation is directed to the outcome number of an operation, and to a lesser extent to the arithmetic expression as a cognitive entity.

Implications for Research and Curriculum

It appears that both of the variability strategies that have been mentioned are essential knowledge for the development of number sense and estimation skills in children. The well-known strategy that many children learn on their own, called the near-doubles strategy, can be viewed from the perspective of both of these strategies. Research seems to suggest that in finding 7 + 8 by the near-doubles strategy, children's thinking is as follows: 7 plus 7 equals 14; 14 plus 1 equals 15; the answer is 15. This involves a transformation on one of the operands followed by compensation for this transformation with a transformation on a computed answer. This near-doubles strategy could also be represented by thinking which reflects the following:

$$\begin{aligned} 7 + 8 &= 7 + (7 + 1) \\ &= (7 + 7) + 1 \\ &= 14 + 1 \\ &= 15. \end{aligned}$$

Thought that would reflect this would seem to hold the several expressions as a single conceptual entity for which the representation is changed under a transformation that keeps the entity, and thus its value, intact. The first process seems to let go of the expression 7 + 8 as an entity when operating on 7 + 7 to create the cognitive entity 14, and then adds a process of putting the 1 back on to create the new entity, 15. Early experience for children that is based on manipulative aids might be developed to model the process. A chips model for the first process would be to display separate sets of 7 and 8 chips, remove from sight one chip from the set of 8, join the two sets of 7, and last bring the 1 chip back into sight and join it to the set of 14 during the joining process. The second process might be modeled by keeping the chip removed from the set of 8 in sight and in the proximity of the set from which it was removed. Similarly, children should be given experience with multiplicative situations of this type early in their learning of multiplication and division. For example, mental, or even paper-and-pencil computation, of 8 x 25 might be facilitated by considering a transformation so that 2 x 100, or (8 x 100) ÷ 4 is computed in its place. Different transformations, each of which helps to develop number sense and also foreshadows other mathematical principles, are the following:

1. $8 \times 25 = (8 \div 4) \times (25 \times 4)$
   $= 2 \times 100$
   $= 200.$

2. $8 \times 25 = (2 \times 4) \times 25$
   $= 2 \times (4 \times 25)$
   $= 2 \times 100$
   $= 200.$

3. 8 x 25 -----> 8 x 100 = 800 -----> 800 ÷ 4 = 200

Schemes 1 and 2 above seem to correspond to the second invariance strategy, while Scheme 3 corresponds to the first. Schemes 1 and 2 emphasize the conceptual entity represented by an arithmetic expression, and Scheme 3 exemplifies a principle of proportionality. Scheme 1 is easily modeled by starting with an 8-by-25 array or rectangle, partitioning the 8-dimension to make four 2-by-25 arrays or rectangles, and then rearranging the arrays or rectangles to form one 2-by-100 array or rectangle. Scheme 2 does not model as easily because the partitioning of the 8-dimension results in a representation which is interpreted as 4 x 2 rather than 2 x 4. Scheme 3 is modeled by starting in the same manner as Scheme 1, but by then replicating the array or rectangle 4 times along the 25-dimension, followed by partitioning this result along the 8-dimension into 4 parts and concentrating on one of the four parts. Whether children can interpret these models and connect them with verbal and symbolic mathematical representations which describe the physical manipulations and whether this facilitates children's development of number sense and ability to make estimates and do mental computation are questions for research. The questions of when and how such experiences can be incorporated into the curriculum are also in need of research.

Number Sense and Other Nonsense

Thomas P. Carpenter

The following remarks draw most directly on my work on the teaching and learning of number concepts at the primary school level. Although some of my comments may apply to older students and more complex number domains, I think we should be cautious in extrapolating too far. Number sense may mean very different things for young children, middle school students, and high school students; and instructional approaches that may work very well at one age may not be appropriate at another.

What is Number Sense?

I would say that a critical aspect of number sense is the ability to operate with numbers flexibly. One of the ways this is seen at the primary school level is in children's use of derived facts. I think the following examples for calculating the sum for 8 + 6 illustrate number sense:

> 8 + 8 is 16, and 8 + 6 is just 2 less. So it's 14.
> Take one from the 8 and give it to the six. That makes 7 + 7, and that's 14.

These examples involve the ability to decompose numbers (8 is the same as 6 + 2 or 7 + 1) and an understanding of how operations on sums can be changed without changing the answer. This is not as simple when the operation is subtraction. Consider the following solutions to 14 - 8:

> 16 - 8 is 8, so the answer is 2 less than 8. It's 6
> 14 - 7 is 7. But 8 is 1 more than 7, so the answer is 1 less, it's 6.

In this case it is necessary to recognize whether changing one of the terms increases or decreases the answer.

Most children exhibit some ability to use derived facts during their primary school years. In our longitudinal study of children in Grades 1 to 3, we found that for more than 40% of the children, derived facts were the primary strategy that they used to solve addition and subtraction problems at some time during the three years that we studied them (Carpenter & Moser, 1984). Furthermore, children infrequently made conceptual errors in inventing derived facts when they had not been a specific object of instruction. The facts involved a variety of relations among numbers, and they seemed to be based on conceptual knowledge about relations among numbers and operations rather than on procedures for figuring out sums and differences. In contrast, explicit attempts to teach derived facts often resulted in conceptual errors. For example, doubles might be used to calculate 8 + 6 as follows:

> 8 + 8 is 16, and 2 more is 18.

For multi-digit numbers, flexibility seems to depend upon a good understanding of place value. In many of the classes in our study, teachers have given children two-digit problems to solve without teaching them an explicit algorithm. Initially many of the children solve the problems using counters. Sometimes unifix cubes connected in groups of ten are used, and sometimes just loose counters. Many of the children, however, invent algorithms of their own that demonstrate what I would consider number sense.

The following is an example of how one first-grade child figured out the answer to 246 + 178. This protocol is relatively typical of the invented procedures that children construct themselves to solve multi-digit addition and subtraction problems.

> Well, 2 plus 1 is 3, so I know it's 200 and 100, so now it's somewhere in the three hundreds. And then you have to add the tens on. And the tens are 4 and 7 ... Well, um. If you started at 70, 80, 90, 100. Right? And that's four hundreds. So now you're already in the three hundreds because of the [100 + 200], but now you're in the four hundreds because of that [40 + 70]. But you've still got one more ten. So if you're doing it 300 plus 40 plus 70, you'd have 410 But you're not doing that. So what you need to do then is add 6 more onto 10, which is 16. And then 8 more: 17, 18, 19, 20, 21 22, 23, 24. So that's 124. I mean 424.

There are some interesting things about this example with respect to number sense and estimation. The child's solution involves successive estimates. Rather than starting at the right as we do with the standard algorithm, she starts with the largest digit, and each step provides a closer approximation to the exact answer. At each step she seems to have a sense of how large the answer should be. Estimation emerges naturally from this procedure. Although we do not have any hard data, the children in the classes we have observed seem to be able to adapt these procedures to estimate answers. In fact, when they get into discussions about whether answers are correct or not, they often use estimates to conclude that a particular answer is not possible.

At their best the classrooms we have been studying provide examples of primary school classrooms in which children have the opportunity to develop number sense. This is not attained by providing specific instruction designed to promote number sense, mental arithmetic skills, or estimation. What characterizes these classes is a focus on children's thinking. The teachers listen to their students and try to build instruction upon their informal knowledge. They provide the opportunity for children to invent their own solutions for problems and the opportunity to talk about their solutions with the other children. Because the children do not have the support of learned procedures, they are forced to be flexible and use relations among numbers to solve different problems.

Our observations suggest that number sense and estimation abilities may be a natural by-product of teaching for understanding. In fact, for number concepts and operations, it may be difficult to distinguish between number sense and understanding. However, the way that programs of instruction in mathematics typically operate, it is conceivable that someone might attempt to identify the components of number sense and teach them as distinct skills. The derived fact example about illustrates the limitations of this approach, and I certainly would not advocate directly teaching anyone to use the multi-digit algorithm in the above protocol. I think that we need to guard against characterizing number sense so that it can be subjected to this kind of analysis and fragmentation.

I would make the same argument for estimation. If students really understand the basic number concepts and the operations, they will be able to invent viable estimation procedures. I think that flexibility is important for estimation, but this does not imply that we should explicitly teach students a variety of different estimation procedures.

Related to the issue of instruction for number sense is the question of what levels of number sense are reasonable goals for instruction. I think that it might be useful to distinguish between number sense that characterizes highly expert performance and number sense that we think is necessary and sufficient for the majority of students. I think it is problematic whether the number sense observed in experts provides a reasonable model for instruction for all students.

Theoretical Links?

As noted above, I think that number sense and estimation skills are part of understanding basic number concepts and operations. It is not clear to me that number sense is a unitary concept and that students with good number sense for whole number operations will necessarily have good number sense for fractions. It seems to me that number sense is linked to understanding within number domains. It also is not clear that it is most productive to study number sense and/or estimation by themselves. I think that at this time it may be productive to situate the research in the broader context of research in particular content domains and draw upon the theory that is driving the research in those domains.

On Making Sense of Number Sense

Edward A. Silver

Given the work that I have been doing for the past decade or so, I approach the topic of *number sense* from the perspective of mathematical problem solving. I admit at the outset that I have not conducted research that is clearly directed at the topics of estimation or number sense. Therefore, the comments that I offer are made in an attempt to propose a few issues that I have investigated in the context of mathematical problem solving that might also be relevant for consideration in the area of number sense.

Whatever else the term number sense might refer to, it certainly is associated with judgments about the reasonableness of numbers in particular situations and flexible movement within the space of numbers and quantities. These defining characteristics of number sense appear to be connected to issues of representation and context that have been shown to be of central importance in cognitive analyses of mathematical problem solving. In what follows, I not only consider briefly these cognitive issues but also argue that number sense may be as much about dispositional considerations as it is about cognitive matters.

Number Sense and Making Sense of Numbers

As the term number sense implies, this complex phenomenon is about making sense of numbers and things mathematical. Behaviors like estimating before or after computing, judging the reasonableness of one's calculation, and using the relative size of numbers or numerical benchmarks (such as basic facts) to guide quantitative activity are all examples of sense-making actions associated with numbers and numerical activity. Such behaviors are notable when they are utilized by students because they are so rarely found in the general school population; hence, there is widespread interest in increasing students' number sense.

Given the high degree of interest in number sense, and given our tendency in mathematics education to identify discrete, teachable components of complex skills for the purposes of planning instruction, discussions of how to improve students' number sense often focus on overt cognitive behaviors whose manifestation will insure that number sense is present and in use. Instructional units may be developed to help students learn many different techniques for estimating the result of a numerical computation, learn information about the precision of measurement or calculation and techniques for approximation, or learn strategies for mental mathematics in order to increase students' flexibility with respect to calculation strategies. Although this instruction may increase the specific cognitive competence of the students who learn from it, it is unlikely to be completely successful in increasing students' number sense because it fails to address an important, more subtle aspect of number sense; namely, the disposition toward numerical activity, or more generally toward mathematics, that it should make sense.

On Not Being Disposed To Make Sense of Numbers

There are many anecdotal reports, and some empirical confirmation, that children do not generally develop in school a disposition toward making sense out of numbers or, more generally, of any mathematics they learn to use. Data from the Fourth National Assessment of Educational Progress (e.g., Kouba, Brown, Carpenter, Lindquist, Silver, & Swafford, 1988; Brown, Carpenter, Kouba, Lindquist, Silver, & Swafford, 1988) indicate that many students see mathematics as a subject that is not creative, that consists mainly of facts to be memorized, and that is about symbols rather than ideas. The prevalence among students of such views of school mathematics provides the underlying support for a belief that mathematics is not necessarily supposed to make sense.

Further confirmation for the assertion that students do not necessarily see mathematics as connected with sense-making comes from research I have been conducting over the past few years on difficulties students have in solving story problems involving division with remainders, such as the following: "The science teacher at Marie Curie School has been given 730 frogs. The frogs will be kept in tanks. Each tank holds 50 frogs. How many tanks are needed to hold all the frogs?"

The widespread failure of students to succeed in solving problems involving whole number division and remainders has been documented through the National Assessment of Educational Progress and several state assessments. For example, only 24% of a national sample of 13-year-olds was able to solve correctly the following problem which appeared on the Mathematics portion of the Third National Assessment of Educational Progress (NAEP, 1983): "An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?" Similarly, the following division problem, which appeared on the 1983 version of the California Assessment Program (CAP) Mathematics Test for Grade 6, was answered correctly by only about 35% of the sixth graders in California: "The 130 students and teachers from Marie Curie School are going on a picnic. Each school bus holds 50 passengers. How many buses will they need?" In both cases, students often erred by choosing answers that contained fractions or decimals (e.g., 2.6 for the school bus problem).

To understand better the basis for the observed difficulty that students have in solving division problems involving remainders, several investigations have been conducted with students in Grades 6, 7, and 8 (e.g., Silver 1986, 1988; Silver, Mukhopadhyay, & Gabriele, 1989). Overall, the findings of these investigations suggest that students appear to have little difficulty recognizing that division is the appropriate mathematical operation to be used in solving such problems and that they appear to have little difficulty carrying out the required computation. The widespread failure to solve these problems successfully appears to be directly related to students' failure to "make sense" of the result of their computation in the context of the problem situation.

Although students generally have little difficulty with different forms of expression for remainders in routine computation settings (e.g., 12 R2, 12 1/2, or 12.5), they often experience considerable difficulty when the computation is embedded in a problem situation. One source of the difficulty is that the same symbolic expression of a division problem can represent very different problem situations and have different answers that depend upon important aspects of the situational context and the quantities involved in the problem. For example, consider the following problems: "Mary has 100 brownies which she will put into containers that hold exactly 40 brownies each. (1) How many containers can she fill? (2) How many containers will she use for all the brownies? (3) After she fills as many containers as she can, how many brownies will be left over?" To solve each of these problems, one would perform the same calculation, 100 ÷ 40, but give a different answer to each problem. In the first problem, a quotient-only problem, the remainder is essentially ignored and only the quotient is given as the answer to the problem. For the second problem, an augmented-quotient problem, the existence of a remainder leads us to increment the quotient when answering the question. In the third problem, a remainder-only problem, the correct answer is the remainder itself.

Unlike most story problems that students solve in elementary school, these problems appear not to allow successful solution without semantic processing, without making sense of the situation, the quantities involved in the problem, and the contex. A successful solution appears to depend to a great extent on mappings between and among at least three referential systems: the story text, the story situation, and the mathematical model. The distinction between the story text and the story situation, which has been articulated by Kintsch (1986), appears to be of critical importance in the solution of division story problems involving remainders. Consider an hypothesized version of a successful solution of the brownie problems from this perspective. A successful solver would map from the story (natural language) text representation of the problems into a mathematical model representation of 100 ÷ 40, then perform the indicated computation within the referential system of mathematics, expressing the resulting answer with an appropriate mathematical representation. The solver would then map the computational result back either to the story text representation or to the implied story situation (in the "real world") representation in order to decide how to treat the quotient and remainder. Through such a process, the successful solver would finally obtain suitable mathematical and natural language representations of the solution that have accompanying interpretations and validity within the referential systems of real-world situations and the knowledge domain of mathematics. Figure 1 provides a schematic representation of the mappings involved in this idealized problem solution.

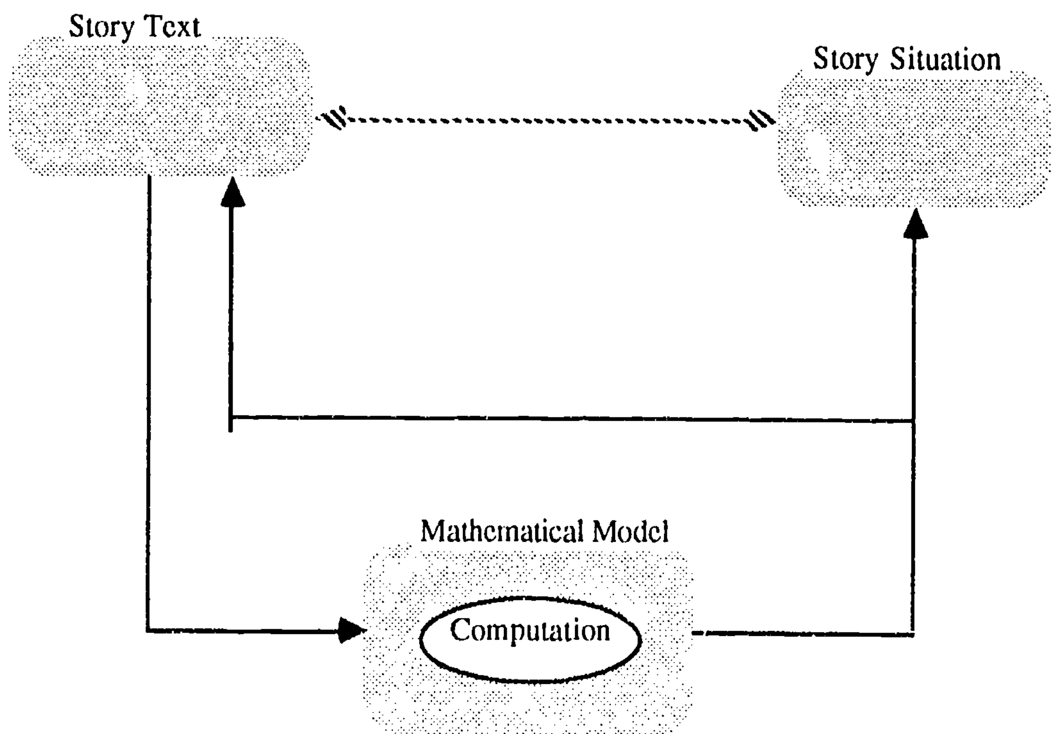

Figure 1. Schematic representation of an idealized successful solution.

An early study (Silver, 1986) examined students' performance on several alternate versions of the CAP school bus problem. It was hypothesized that students were failing to attend to relevant information implicitly represented in the problem situation but not explicitly stated in the story text (e.g., no one is to be left behind; on some bus there may be some empty seats). Several problem variants were created that made relevant structural information more salient and students' performance on these variants was examined. The results of that study suggested that variations in the presentation of the problem, designed to make explicit certain implicit information in the problem or the required solution, significantly enhanced students' performance. Unlike most considerations of "relevant information" for problem solving in elementary mathematics, the focus of attention in the study was not so much on information that would enhance the mapping between the story text and the mathematical model but rather enhance the mappings between and among these two reference spaces and the story situation.

In more recent research (Silver 1988; Silver, Mukhopadhyay, & Gabriele, 1989) students' performance on augmented-quotient division problems and other division problem types (e.g., remainder-only problems and quotient-only problems) was examined. Both studies examined the effects on students' performance of their solving other division problems that required the same computation and similar referential mappings. The results indicated that students' performance on each type of problem was enhanced by having students also solve related division problems. In general, the results were consistent with the explanation that enhanced performance was due to students' increased sensitivity and attention to the relevant semantic and referential mappings involved in the target problem solution. In particular, experience with the related problems may have drawn attention to the need for mapping into either the story text representation or the story situation representation after obtaining a solution to the target problem through use of a mathematical model.

Taken together, these results and the assessment findings suggest that poor student performance may be due, at least in part, to their incomplete mapping among the relevant referential systems. In particular, they appear to map from the problem text to a mathematical model (in this case, a division computation to be performed), compute an answer within the domain of the mathematics model, and fail to return to the problem story text or to the story situation referent in order to determine the best answer to the question. Figure 2 presents a schematic representation of an hypothesized canonical version of a student's unsuccessful solution attempt.

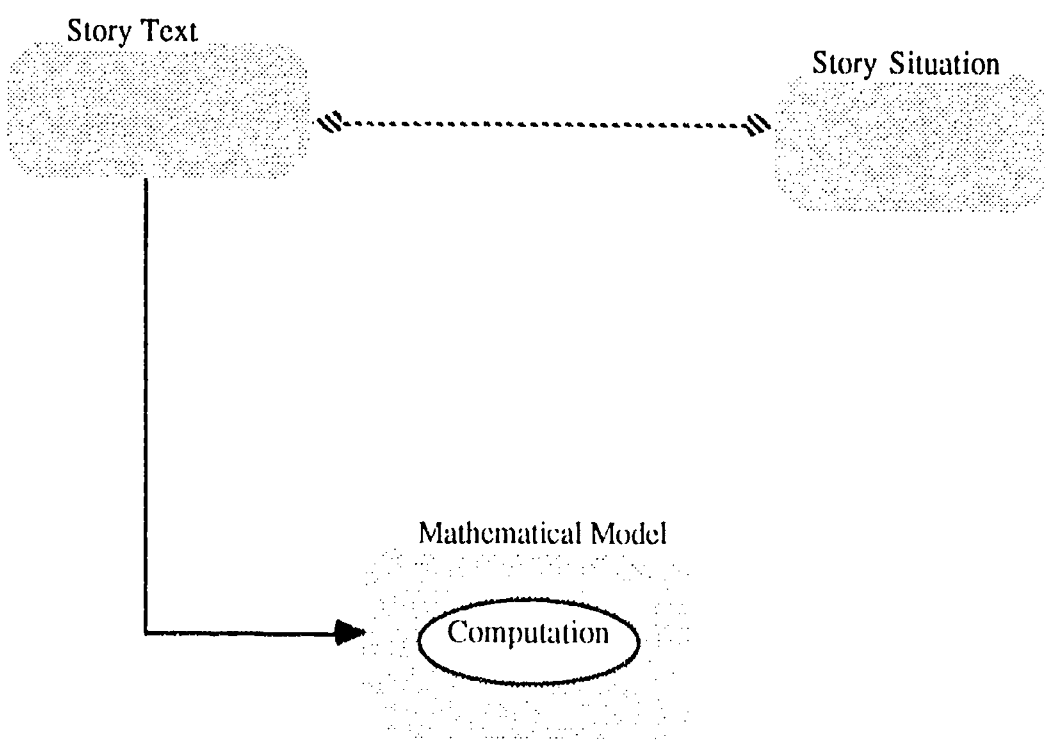

Figure 2. Schematic representation of hypothesized unsuccessful solution.

There are at least two important aspects of the relationship between these findings regarding incomplete semantic processing on division story problems and the general topic of number sense, which is the main focus of this paper. One aspect is fairly obvious: making sensible judgments about the reasonableness of answers (e.g., recognizing that one cannot have a fraction of a bus or a fish tank) is surely one component of what we refer to as number sense. The second aspect is somewhat more subtle and relates more to the issue of disposition toward sense-making that was discussed earlier. The findings regarding students' difficulties with division story problems provides further support for the notion that school mathematics education is not currently fostering the development in students of a general inclination toward sense-making in quantitative matters.

Increasing Number Sensibility

Isolated instruction on individual components of number sense, regardless of how well the instruction is designed, is unlikely to lead to the development of the kind of sense-making with respect to quantitative processes and products that we seek. Considering the division example discussed above, it is clear that teaching students simply to answer such questions correctly is not the entire story. To promote the development in students of a finely tuned number sense, what is needed is nothing short of cultural revolution in school mathematics education. Our curriculum and our instructional methods must be restructured to emphasize sense-making in all areas of mathematics instruction and at all grade levels from kindergarten to Grade 12. As curriculum reform is undertaken in response to the NCTM *Standards*, caution must be exercised in the deletion of topics from the curriculum, such as work with very large or very small numbers, so that we do not deprive students of important opportunities for the development of number sense. Moreover, we must not be deluded into believing that extensive use of calculators will remedy the situation and automatically develop good number sense. Although calculator activities can be used to promote number sense, they can also be used in ways that promote an over-reliance on the result found in the display and encourage failure to make sense of the situation, quantities, and numbers involved. It is worth recalling that performance on the NAEP bus problem in the Third Mathematics Assessment (NAEP, 1983) was even poorer when students used calculators than when they used only paper and pencil!

Research needs to be undertaken that examines the long-term consequences of early mathematics instruction which emphasizes sense-making on the part of students and teachers. Curricular and instructional experiments related to number sense also need to be conducted with older children as they learn more complex procedures and solve more difficult problems, and ways need to be found to embed sense-making into the fabric of mathematics courses for these older students as well. Although research must certainly also be done on specific components of number sense, we must be mindful that the sum of the cognitive components does not necessarily equal the complex whole of number sense; studies related to the general disposition toward quantitative sense-making must also be undertaken.

Appendices

Appendix A

Editors' note: The references deserve a few words of explanation. Besides the traditional formal references, we have attempted to include references made informally, particularly from Part I of the proceedings. In some cases this means inclusion of one or two references when a person's work has been mentioned. We apologize if these are not the selections the speaker would have chosen. We elected not to include references for authors whose work is extensive and well-known, such as that of Piaget.

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Appendix B

Preconference Reading List

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Appendix C

Participants -- NSF Conference: Establishing a Research Base for Number Sense and Related Topics

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