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ABSTRACT

Mathematics and the use of mathematical thinking should be much more than what has been traditional school arithmetic. Much of the mathematical reasoning can be developed and experienced out of school, particularly in the home. This material is a teacher's guide designed to help parents support what is done with their children in class. Background material for parents is provided. End-of-the-year assessment material is presented. A total of 29 activities on the following concepts and skills are included: (1) equality and inequality; (2) use of operations; (3) place value; (4) computation; (5) geometry; (6) logic; (7) ratio; (8) use of calculators; and (9) use of LOGO. (YP)

MEANINGFUL MATHEMATICS

LEVEL FOUR

TEACHER'S GUIDE TO LESSON PLANS

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TABLE OF CONTENTS

TO: THE TEACHER

FROM: A. DEAN HENDRICKSON

Attached is something you can reproduce and send home to parents to encourage them to support what you are doing in the classroom.

MATHEMATICS IN THE HOME

A. Dean Hendricksen, University of Minnesota-Duluth

Introduction:

Mathematics and the use of mathematical thinking is much more than what has been traditional school arithmetic. The arithmetic of whole numbers, fractions and decimals constitutes no more than 10-15% of the mathematics we use throughout our lives. Much of the mathematical reasoning we use can be developed and experienced out of school, particularly in the home. Some of these suggestions may seem remote from the arithmetic you remember, but they will involve children in the THINKING essential to both the learning and use of mathematics in everyday life.

Pre-Mathematical Thinking:

Before a child can understand school mathematics, certain ways of thinking and skills must be available for use. These are continuously used throughout learning of mathematics, but particularly elementary school mathematics. These include: counting, comparing, ordering, using patterns, using grouped material, using language and establishing relations and relationships. Needed experience with these can be obtained around the home. Before describing things to do with children at home to help them with their school mathematics, here are some "golden rules" based upon research and experience with learning children.

1. You must not force children since this has negative effects, such as turning them away from doing things or from you. A child learns when ready, curious, and needing to make sense of something. This goes in spades for drill on memorizing so-called "basic facts."

2. Give children positive things to do when time is available, especially those things they can do and enjoy doing. Don't ask for things beyond the child's capacity to do.

3. Give lots of praise and encouragement. If what the child does or says doesn't seem to make sense to you, don't criticize or correct. Ask questions that might lead the child to consider it in a different way.

4. Don't look for day-to-day progress or change or for immediate results. Just as with many other things, such as walking or talking, a child may seem to be making no headway and then suddenly, it's all there. Children develop in spurts and unevenly, and have long plateaus where nothing seems to be happening. That's normal and accept it. There is probably a lot going on below the surface.

5. Don't compare yours with other children. Everyone is different - thank goodness!

6. Don't worry if a particular skill, such as using language, is

coming along more slowly than you'd like or than brother John's did. Somehow most of them seem quite a lot alike by the time they are 12 or so.

Words:

A number of words commonly used in mathematics and related to teaching mathematics should be used often outside of school as well. Some examples are *same-more, a lot, more than, less than, large, small, many, few, same as, different, alike, all, some, not, left, right, ahead of, behind, above, below, front, back, long, short.*

In addition to words associated with comparing, grouping and space, the number words are important. Children must know the counting words, but even more than that, they must see the pattern in the use of counting words. The cordial words like *first, second, third,* et. are also important. Use of these words around home helps children to count objects correctly and to identify position of things in ordered arrangements.

Comparing:

Have children compare things as to size, length, are and volume whenever possible. "Which glass has more?" "Which box holds more?" "which of these is heavier? heaviest?" "Put these sticks in order of length." "Arrange the silverware so the longest is farthest from the plate and the shortest is nearest the plate." Questions like these should be frequent. They should involve different kinds of things both indoors and outdoors. Combine these with questions that make the children estimate measurements of distance and height such as "Which do you think is as high as the shed, A or B?"

Comparing of quantity leads to better understanding of number and number relationships. "Are there more chairs or lamps in this room?" "Are there more cups or teaspoons on the table?" "Have we go more red roofs or green roofs on our street?" "Put enough table knives on the table so that there are as many knives as forks." "Do you have more boys or girls in your class?" These can be asked when out walking, riding in the car, watching TV or sitting in the boat. Ask children to do things that will make one group as large as another frequently All such activity helps children build number relations into their deeper understandings, instead of as memorized associations that have no meaning - like names and dates you once memorized to pass a history test!

Ordering things that can be counted is important. Bead stringing activities are good for young children. "String some beads so the third bead is red and the fourth bead is blue." "Make a string so every other one is green," etc.

Ordering things that have lengths, areas and volumes extends comparing beyond two things. Have children place three sticks of different lengths in order from shortest to longest; place three pieces of paper of different areas into orders; place three different sized cans of jars into order. Gradually extend the number of things to more than three for these activities.

Ask frequent questions about the ordering of events as to which happens first, second...last, etc. Connect these with time estimations, "How many minutes ago do you think this happened? How many days?" etc.

Counting:

Children should keep extending their memorized sequence of counting words. This is important. But being able to say the words in right order does not mean they can count things. They need much practice at this. Have them count everything around the house that is countable - the chairs, tables, legs on chairs; the tiles on the floor, in the ceiling; the number of windows in a room; the silverware in the drawer; the cans on the shelf; the pieces of wood in the woodpile; the telephone poles going by, etc. The more they count, the better able they are to count. When they are pretty good at counting forward, have them do some counting back. For example, start with 20 clothespins. One at at time put one into a can and count aloud those that are left as each one is removed from the pile.

Patterns:

Have children look for patterns - in the carpet, in the ceiling, in wallpaper, in the drapes, on the bedspreads. Patterns of shape, or color, or sound are all important. Beads can be strung in patterns. Collections of bottle caps, old keys, buttons, screws, nuts and bolts, and similar "junk" can be put into patterns. Ask children what would come next in a pattern, or what would go where something is missing in a pattern.

Number:

Help your child learn number size by having him see the same number, such as five, in many different arrangements and materials. Playing cards can be sorted into those all having the same number. Mixed groups of say, five marbles, three buttons, three keys, six spoons, can be used. "Find me the material there are five of," etc. Put some number, seven for example, of beads or marbles into three or four different shaped glass jars, "Find a jar with seven in it." "Find another." Put the same number of one kind of thing in one jar and another kind in a second jar, etc., and do the same kind of thing. Involve the child with numbers in as many different ways, with as many different kinds of material, and as many different sizes as possible. Gradually increase the number size as the child seems able to easily handle smaller numbers.

Using Numbers:

Comparing groups with number property; combining such groups; separating larger groups into smaller groups of a given size or into equal size groups - all of these activities help children to understand when each of the four arithmetic operations are used.

Some examples of things to do in the home of this kind are:

1. Compare two different sized groups in several ways. "How many more are there in this group than in that group?" "This group has how many fewer than that group?" "How many times as many are there here as there?" These kinds of questions used with groups of all kinds

of things - knives, forks, chairs, chair legs and table legs, buttons, marbles, pieces of candy, etc., help the child with what the school is doing.

2. Join together several groups of the same size into a larger group. Rows of pennies can be arranged into an array like this and can then be looked at a different way to see 5 groups of 6 pennies:

```
oooooo
oooooo
oooooo
oooooo
oooooo
```

Both lead to a total of 30 in the array. Do this in a row at a time, having the child tell you how many are there all together each time. Separate and take apart such arrays row by row and see what is left each time. Do this with different kinds of things, different size rows and different total numbers of things. Clothes pins, ceramic tiles, beans, corn are all good for this.

3. Join together groups of different size, such as seven things with five things. Have the child describe what is happening in words. Have the child add to one group of things enough to make it the same size as another larger group. Have the child make equal two unequal size groups without adding anything more to the collection. "Here are a group of 15 clothes pins and one of 7 clothes pins. Do something so you have two equal groups."

4. Give the child large amounts - in the 20's or 30's of things to:

a) make several groups of a given size from. Some numbers should make these smaller groups an even number of times and some should have some left that is not enough to make another of the smaller group.

b) make a certain number of groups that will all have just as many in them.

Examples:

"Put these 30 beans into 6 cups, so each cup has just as many. How many are in each cu?"

"Put these 43 beans, six at a time into cups. How many cups did you use?" "What should e done with what is left over?" "When do you have some left over?" "When don't you have anything left over?"

When you do for walks, have the children compare, add together, etc., things along the way. Do the same in the car, the supermarket, in the drugstore. "How many are there on the top shelf?" "How many re on the bottom shelf?" "How many are there on the top and bottom shelves together?"

Have the child do as much adding, subtracting, multiplying and dividing of this kind - always as related to things - as you can. <u>DON'T</u> try to drill your child on "addition" facts or "multiplication" facts. Let the child learn these in due time through the school activities and those you do at home as described here. <u>DON'T</u> have the child write number things - the school will do this. Accept verbal answers and descriptions. Get in the habit of asking your child why certain answers are given and <u>LISTEN.</u>

SOME FINAL HINTS:

1. Have your children count things as much as possible.

2. Ask children simple addition, subtraction questions about REAL things in the surroundings to give practice in mental arithmetic.

3. Play card games that require mathematics or related things like WAR, OLD MAID, CRIBBAGE, RUMMY (regular or gin).

4. Give thinking games for holiday gifts - CONCENTRATION, HUSKER DU, etc.

5. Get a Little Professor or some similar calculator-based program to give mental arithmetic practice.

6. Cheap mathematics games can be bought at Target, Woolworths, etc. Some examples are COVER UP, HEADS UP, SCORE FOUR, TUF, APOLLO, etc.

7. Give your child a simple four function calculator and let him or her fool around with it.

8. Encourage block play and building, sand play, making birdhouses, etc.

9. Key words are COMPARING, COUNTING, PATTERNS, COMBINING (groups), SEPARATING (large groups into smaller groups)

10. Point out mathematics wherever it is in the surroundings. Children must realize mathematics is:

    a. easy to learn
    b. useful
    c. fun

LEVEL FOUR

INTRODUCTION

The first three levels of this program are designed to develop basic ideas in mathematics. Many children will have developed sound ideas about addition, subtraction, number, numeration and place value, fraction, multiplication, division, equality, computation in base ten, and shapes in two and three dimensions by this time. A good number of children become concrete operational and hence, capable of reversibility in their thinking, and in the use of relational logic by age 8 or 9. Others will become concrete operational during this year. Hence, maintenance activities for these basic ideas will review for some and help crystallize ideas for others.

New topics to be introduced at this level include:

............the symbolic representation of fraction operations
............extended use of Guess My Rule to two step operations
............analysis of tables of numbers
............use of attribute blocks to introduce "if-then" reasoning
............multistep problems involving the four arithmetic operations
............working problems with extraneous data
............problems that incorporate ratio
............describing line relationships in geometric shapes
............quantitative descriptions of volume

Begin the year with a review of "basic facts." Children should have immediate recall of all 2 part and 3 part combinations for wholes up to 20, i.e., 12 + 8 = 20, 5 + 4 = 9, etc. Use games, relays, mental drills, etc. to have children practice recall of these.

Children should also know the 2 factor combinations factors up to ten, i.e., 8 x 7 = 56; 3 x 4 = 12, etc. Use the materials provided here and create others of your own. Oral drill on recall should be put into an interesting context of some kind. Take time to show how these are related to each other:

............x 4 is doubled and doubled again
............x 6 is doubled and tripled
............x 8 is doubled three times
............so 3 x 3 is 6 doubled = 12 doubled = 24

The lessons are arranged by topics. Stay with particular topics until mastery is achieved by most students before moving on. However, do review and make every attempt to relate the new topics to previously mastered topics.

Work with calculators and related counting activities should be done for 5 - 10 minutes of a period 2 or 3 times each week. Integrate these with lessons being worked on. Some lessons suggest related calculator or counting activities.

Repeat lessons as needed. Repeat or alter examples provided (by using different numbers, for example) as desired. Most lessons have ONE explanatory example. More similar examples may be needed before assigning seat work.

The use of number sentences as mathematical models of verbally described or concretely represented situations is important. Children must learn to USE mathematics and see how mathematics is used to represent real world phenomena.

Several commercial products are available to supplement the lessons provided. These include, by topic:

<u>Calculator</u>
KEYSTROKES SERIES

<u>Fractions</u>
FRACTION BARS WORKBOOKS I & II
EVERYTHING'S COMING UP FRACTIONS

<u>Tangrams</u>
TANGRAMATH

<u>Geometry</u>
GEOBLOCKS AND GEOJACKETS
GEOBOARD ACTIVITY SHEETS
DOT PAPER GEOMETRY

<u>Logic</u>
ATTRIBUTE GAMES AND PROBLEMS
ATTRIBUTE ACROBATICS

LEVEL FOUR

END OF THE YEAR OUTCOMES EXPECTED

1. Knowledge of all Addition and Multiplication combinations for numbers 1-12.

2. Ability to obtain the relationship connecting to variables from a table of values.

3. Recognition of equality as a relationship between two quantities.

4. Recognition of inequality as a relationship between two quantities.

5. Ability to estimate and measure lengths, areas and volumes of common shapes.

6. Ability to use appropriate arithmetic operations in all 14 situations leading to addition and subtraction.

7. Ability to use all four operations with fractions.

8. Ability to compute in the base ten system, using two digit multipliers and two digit divisors in division.

9. Ability to solve simple open sentences for the missing values.

10. Ability to recognize and calculate with tenths and hundredths in decimal form.

11. Ability to use the calculator to perform computations needed for problems.

12. Ability to recognize signed numbers and to add and subtract signed numbers.

13. Ability to use LOGO as appropriate for this age group.

14. Knowledge of logical, AND, OR, NOT and use of similarities and differences in properties.

15. Use of distributive property of multiplication over addition, subtraction and ratio.

16. Recognition of ratio relationships and equivalence of ratios.

LFVEL FOUR ASSESSMENT RECORD

Mathematician: ______________________________

	LFVEL OF MASTERY					
	Date:		Date:		Date:	
TOPIC	Progress Made	Mastery Attained	Progress Made	Mastery Attained	Progress Made	Mastery Attained
Equality						
Inequality						
Use of Operations						
addition						
subtraction						
multiplication						
division						
mixed						
Place Value						
Computation in Base Ten						
addition						
suotraction						
multiplication						
division						
Computation with Fractions						
addition						
subtraction						
multiplication						
division						

LEVEL OF MASTERY

TOPIC	Date:		Date:		Date:	
	Progress Made	Mastery Attained	Progress Made	Mastery Attained	Progress Made	Mastery Attained
Geometry						
recognition of shapes						
finding area						
finding perimeters						
finding volumes						
Using Logic						
Use of AND						
Use of OR						
Use of IF-THEN						
Ratio						
Signed Numbers						
Identification						
Comparing Value						
Addition						
Subtraction						
Calculator USE						
Correct inputting						
Correct readout						
LOGO Command Use						
Forward						
Backward						
Turning R & L						
Integraded use in						
Producing Shapes						

GRADE FOUR ASSESSMENT

1. Draw a ring around the largest number in each row. Put an X through the smallest.

Example:	16	~~14~~ (X)	(19)

A) 347 294 299

B) 2300 2003 2030

C) 10100 11000 10010

2. Write counting numbers that come before and after each of these numbers.

Example:	41,	42,	43

A) ______, 99, ______

B) ______, 1000, ______

C) ______, 420, ______

3. The first row, A, shows counting by threes. The other rows show counting by other numbers. Fill in the missing numbers in each row.

Example:	2,	4,	6,	8,	10,	12

A) 14, 17, 20, 23, 26, 29, 32, _____, _____, _____

B) 234, 334, 434, _____, _____, _____, 834

C) 31, 27, 23, _____, _____, 11, _____

4. How many HUNDREDS are indicated in the HUNDREDS place in this numeral? 1,382

A) ☐ 2 B) ☐ 5 C) ☐ 8 D) ☐ 1

5. Write the correct NUMERAL in each blank.

Example:	Six Tens, Five Ones = 65

A) Four Tens, Six Ones = __________
B) 3 Hundreds + 5 Tens + 2 Ones = __________
C) 432 = _____ Hundreds + _____ Tens + _____ Ones
D) 23 Tens + 4 Ones = __________
E) 14 Hundreds, 14 Ones = __________

6. How many HUNDREDS <u>in all</u> are indicated in the following number? 4836

A) ☐ 8 B) ☐ 480 C) ☐ 48 D) ☐ 3

7. Mark <u>all</u> the right ways of thinking about the numeral 2000.

A) ☐ 20 TENS C) ☐ 20 HUNDREDS

B) ☐ 2 THOUSANDS D) ☐ 200 TENS

8. Which of these will be 200 GREATER if <u>each</u> 5 is changed to a 7?

A) ☐ 5013 B) ☐ 2053 C) ☐ 582 D) ☐ 691

9. If a digit is moved THREE places to the LEFT, its value is:

A) ☐ 10 Times as great C) ☐ The same

B) ☐ Divided by 1000 D) ☐ 1000 times as great

10. If a digit is moved TWO places to the RIGHT, its value is:

A) ☐ One hundredth as great C) ☐ The same

B) ☐ One tenth as great D) ☐ 100 times as great

11. Write correct numerals for these.

A) (3 x 100) + (5 x 10) + (2 x 1) = ____________

B) (6 x 10) + (4 x 100) + (1 x 1) = ____________

12. Complete each number sentence.

A) 5 + 6 = 11, so 11 - _____ = 5

B) 9 + 7 = 16, so _____ = _____ - _____

C) 18 - 6 = 12, so _____ + _____ = _____

13. Draw a ring around the BEST way to find the missing number in each sentence.

Example: $3 + \square = 8$

A) $\begin{array}{r} 8 \\ - \square \\ \hline 3 \end{array}$ B) $\begin{array}{r} 8 \\ + 3 \\ \hline \square \end{array}$ C) $\begin{array}{r} 3 \\ + \square \\ \hline 8 \end{array}$ D) $\begin{array}{r} 8 \\ - 3 \\ \hline \square \end{array}$

I. $7 + \square = 15$

A) $\begin{array}{r} 15 \\ - \square \\ \hline 7 \end{array}$ B) $\begin{array}{r} 7 \\ + \square \\ \hline 15 \end{array}$ C) $\begin{array}{r} 15 \\ + 7 \\ \hline \square \end{array}$ D) $\begin{array}{r} 15 \\ - 7 \\ \hline \square \end{array}$

II. $35 + 15 = \square$

A) $\begin{array}{r} 35 \\ + 15 \\ \hline \square \end{array}$ B) $\begin{array}{r} 35 \\ + \square \\ \hline 15 \end{array}$ C) $\begin{array}{r} \square \\ - 15 \\ \hline 35 \end{array}$ D) $\begin{array}{r} 35 \\ - 15 \\ \hline \square \end{array}$

III. $25 - 15 = \square$

A) $\begin{array}{r} 25 \\ - \square \\ \hline 15 \end{array}$ B) $\begin{array}{r} 25 \\ - 15 \\ \hline \square \end{array}$ C) $\begin{array}{r} 25 \\ + \square \\ \hline 15 \end{array}$ D) $\begin{array}{r} \square \\ + 15 \\ \hline 25 \end{array}$

IV. $\square - 22 = 45$

A) $\begin{array}{r} 22 \\ + 45 \\ \hline \square \end{array}$ B) $\begin{array}{r} 45 \\ - 22 \\ \hline \square \end{array}$ C) $\begin{array}{r} \square \\ - 22 \\ \hline 45 \end{array}$ D) $\begin{array}{r} 22 \\ + \square \\ \hline 45 \end{array}$

V. □ + 34 = 51

A) □ + 34 51	B) 34 - 51 □	C) 51 - □ 34	D) 51 - 34 □

VI. 35 + □ = 50

A) 35 ÷ □ 50	B) 50 + 35 □	C) 50 - 35 □	D) 50 - □ 35

VII. 76 - □ = 29

A) 76 + 29 □	B) 76 - □ 29	C) 29 - 76 □	D) 76 - 29 □

VIII. □ ÷ 16 = 12

A) □ 12/ 16	B) 12 16/ □	C) 16 X 12 □	D) 16 - 12 □

IX. 18 X □)

A) 18 □ / 90	B) □ 18/ 90	C) 90 X 18 □	D) 18 □ 90

X. 16
□ / 48

A) □ 16/ 48	B) 48 X 16 □	C) 16 X □ = 48	D) 16 - 12 □

XI. 85 ÷ ☐ = 17

A) 17/ 85 with ☐ as quotient   B) 85 X 17 = ☐   C) ☐ / 85 = 17   D) 17 X ☐ = 85

14. Complete the following sentences.

Example: If ☐ X 4 = 20, then 4 = 20 ÷ ☐

I. If ☐ X 9 = 63, then 9 = _________

II. If ☐ ÷ 8 = 12, then 8 = _________

15. In working this example, which is regrouped?

```
  157
+ 262
```

A) ☐ 10 ONES into one TEN

B) ☐ 10 TENTHS into ONE

C) ☐ 10 TENS into one HUNDRED

D) ☐ 11 TENS into one HUNDRED

16. In working this example, which is the best way to think of the 343?

```
  343
- 196
```

A) ☐ 3 HUNDREDS + 4 TENS + 3 ONES

B) ☐ 3 HUNDREDS + 3 TENS + 13 ONES

C) ☐ 2 HUNDREDS + 14 TENS + 3 ONES

D) ☐ 2 HUNDREDS + 13 TENS + 13 ONES

17. In working this example, what is the product of the two circled places?

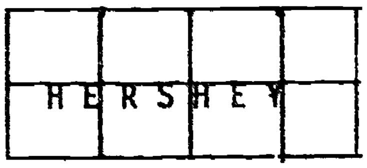
$$\begin{array}{r} 43 \\ \times\ 56 \\ \hline \end{array}$$

A) ☐ 1500 B) ☐ 15 C) ☐ 150 D) ☐ 20

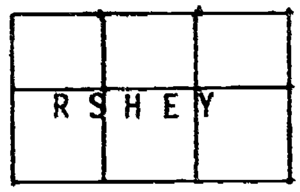
18. In a division by 9, the greatest remainder permitted is:

A) ☐ 1 B) ☐ 6 C) ☐ 9 D) ☐ can't tell

19. Here is a candy bar



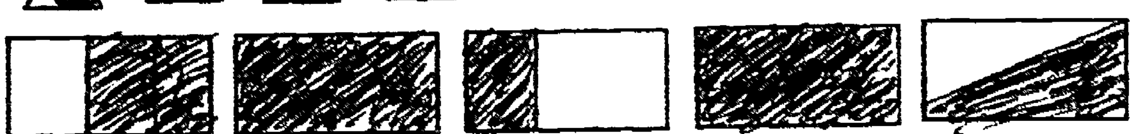
What fractional part of the bar is shown here?

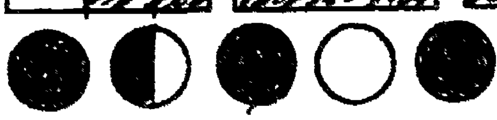


A) ☐ $\frac{6}{6}$ B) ☐ $\frac{6}{8}$ C) ☐ $\frac{2}{3}$ D) ☐ $\frac{3}{3}$

20. Which rows of pictures show $3\frac{1}{2}$ shaded shapes?

A) ☐ 

B) ☐ 

C) ☐ 

D) ☐ 

21. If you had FOUR pies and wished to share them equally among FIVE persons, what fraction represents how much pie each person would receive.

A) ☐ $\frac{4}{5}$ B) ☐ $\frac{5}{4}$

22. Draw a ring around each way that is another way to write what is given.

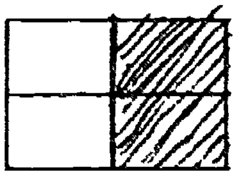
A) 20 divided by 4

$20\overline{)4}$ $20 \div 4$ $4\overline{)20}$

$\frac{20}{4}$ $\frac{4}{20}$

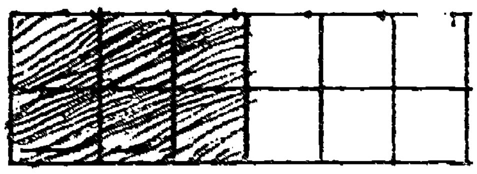
B) 6 multiplied by 5 6×5 $6 + 5$ $\begin{array}{r} 5 \\ \times\ 6 \\ \hline \end{array}$

23. Choose a pair of fractions shown by each figure.

I. 

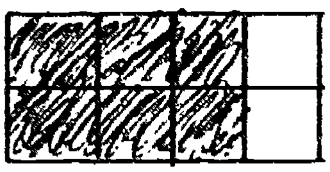
A) ☐ $\frac{1}{2}, \frac{2}{4}$ C) ☐ $\frac{2}{3}, \frac{4}{6}$

B) ☐ $\frac{1}{4}, \frac{2}{8}$ D) ☐ $\frac{1}{3}, \frac{2}{6}$

II. 

A) ☐ $\frac{3}{8}, \frac{1}{2}$ C) ☐ $\frac{3}{4}, \frac{6}{8}$

B) ☐ $\frac{3}{6}, \frac{1}{2}$ D) ☐ $\frac{3}{5}, \frac{6}{10}$

III. 

A) ☐ $\frac{2}{3}, \frac{4}{6}$ C) ☐ $\frac{6}{8}, \frac{3}{4}$

B) ☐ $\frac{6}{4}, \frac{3}{2}$ D) ☐ $\frac{3}{6}, \frac{1}{2}$

24. Write missing parts for these fractions.

A) $\frac{1}{2} = \frac{\quad}{4}$ B) $\frac{5}{10} = \frac{1}{\quad}$

C) $\frac{2}{3} = \frac{4}{\quad}$ D) $\frac{3}{4} = \frac{\quad}{8}$

25. Which group of fractions is arranged from largest to smallest?

A) ☐ $\frac{2}{5}, \frac{2}{3}, \frac{2}{7}$ B) ☐ $\frac{2}{9}, \frac{2}{7}, \frac{2}{5}$

C) ☐ $\frac{2}{5}, \frac{3}{4}, \frac{3}{5}$ D) ☐ $\frac{2}{3}, \frac{2}{5}, \frac{2}{8}$

26. Complete each number sentence.

Example: 4 + 5 = <u>3</u> + <u>6</u>

A) 7 + 8 = _____ + _____

B) 17 = _____ - _____

C) 19 + 6 = _____

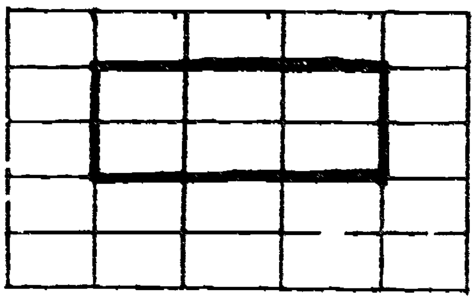
D) 14 - 3 = _____ _____

E) 4 + 11 = _____ - _____

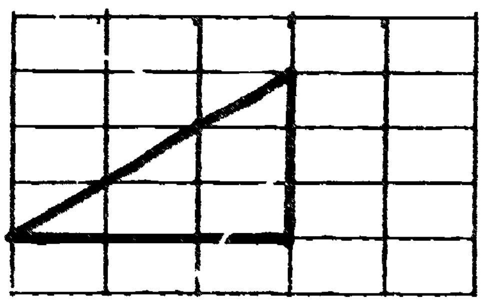
F) $3 = \frac{3}{3} +$ _____

G) $2 = \frac{\_\_\_}{3}$

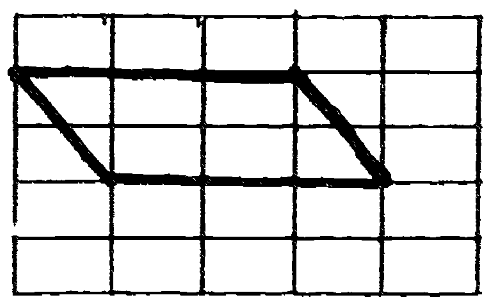
27. Circle each shape with the same area as this shape.



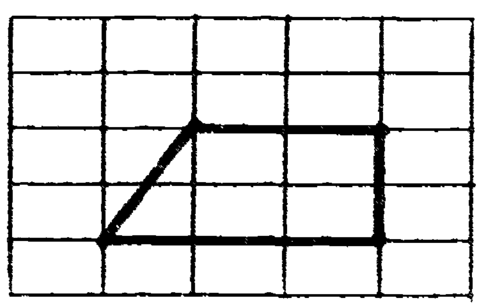
A)



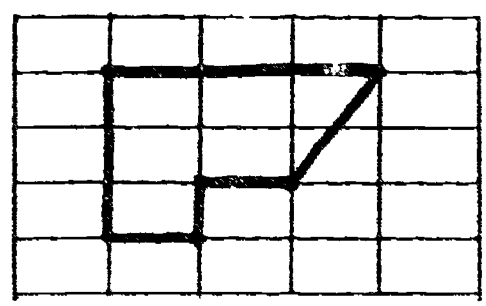
B)



C)



D)



28. Compute.

A) $\frac{1}{2} \times 4 =$

B) $\frac{2}{3} + \frac{1}{3} =$

C) $\frac{1}{4} + \frac{3}{8} =$

D) $\frac{1}{2} - \frac{1}{4} =$

E) $1 - \frac{1}{4} =$

F) $= \frac{2}{3} - \frac{1}{3}$

29. Compute.

A) $4.39 + $3.39

B) $5.00 - $2.98

C) 12
X 43

D) 43
X 24

E) 7/ 28

F) 5/ 36

G) 11/ 35

H) 9/ 90

30. Compute.

A) 153
+ 204

B) 404
- 183

C) 50
+ 148

D) 109 + 285

E) 308
+ 199

F) 308-199

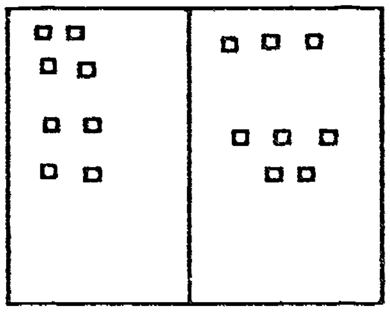
ASSESSMENT PROCEDURES

Use the tasks suggested to assess the student performance at the beginning of the year, midway through the year and at the end of the year. Check the topics as mastery is attained.

ASSESSMENT TASKS

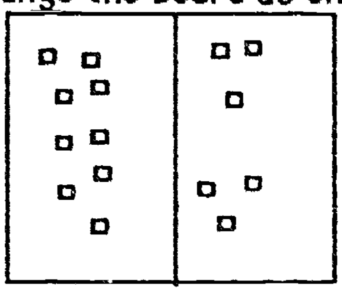
EQUALITY AND INEQUALITY

Show the students an arranged split board.



"Write a number sentence to show what is on this board." (4 + 4 = 3 + 5)

Change the board as shown:



"Write a number sentence to show what is on the board now." (9 > 6)

USING OPERATIONS

Give students a 34 problem test that has one of each of the fourteen situations that give rise to addition and subtraction and the sixteen situations that give rise to multiplication and division. Mix these and use the recording form provided.

COMPUTATION IN BASE TEN

Give students the attached nine item test.

PLACE VALUE

Show students the numeral 4,327

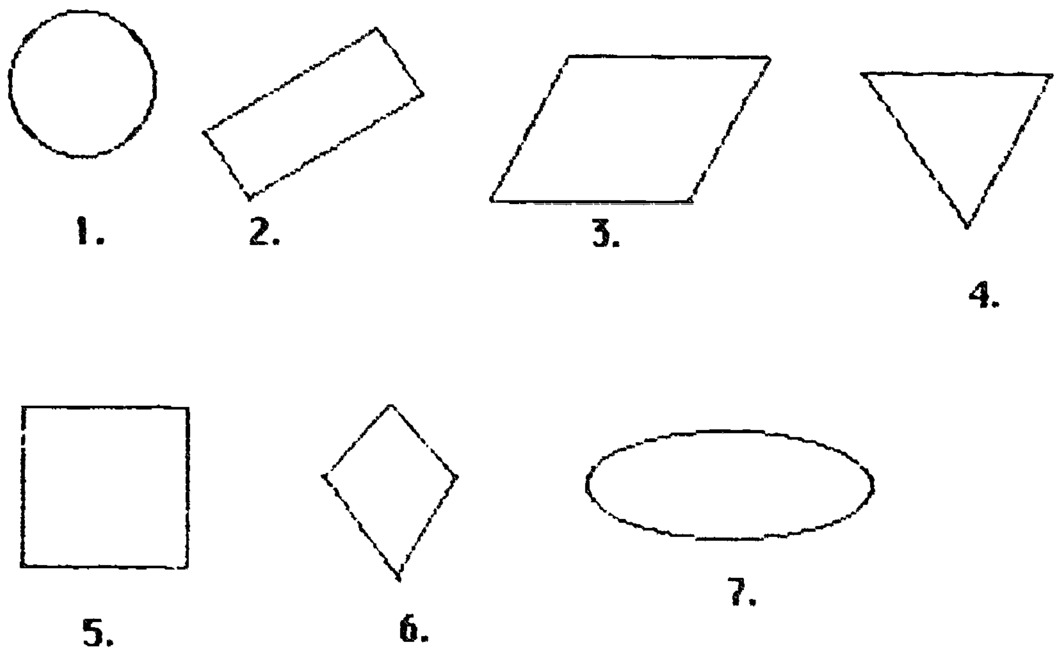
1. "Is this closer to four thousand or five thousand?"
2. "What numeral is in the hundreds place?"
3. "How many hundreds are in this number?"
4. "How many tens are in this number?"
5. "What numeral is in the thousands place?"

FRACTIONS

Give students the attached test on fractions.

GEOMETRY

A. Show the students posterboard cut-outs of the following shapes:



"Write the numerals for all quadrilaterals." (2,3,5,6)

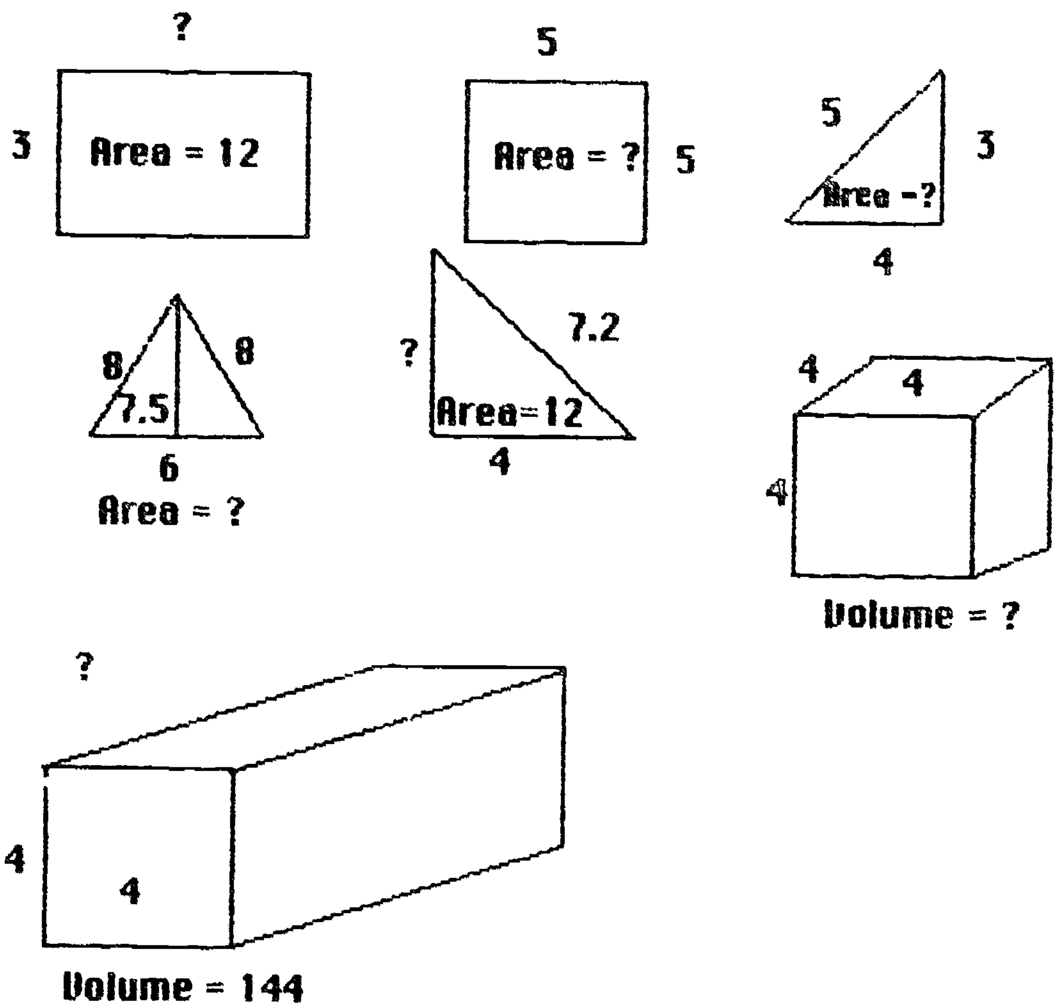
"Write the numerals for all parallelograms." (2, 3 5)

"Write the numerals for all closed curves. (1, 7)

"Write the numerals for all squares." (5)

"Write the numerals for all circles." (1)

B. Show the following figures to the students They are to find the number to replace "?" in each case.



RATIO

Show the students a ratio 3 : 5

"Write two ratios that are equivalent to this ratio."

SIGNED NUMBERS

Show the students the following: +5 -3

"Which of these is greater in value?" (+5)

"What is the sum of these numbers?" (+2)

"How many numbers apart are these numbers?" (8)

"Place these on a number line."

CALCULATOR USE

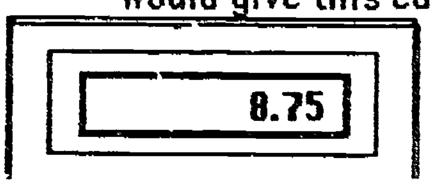
A. Have the students label the keys in sequence to show the given operations.

Blank Key Sequence

Problems:

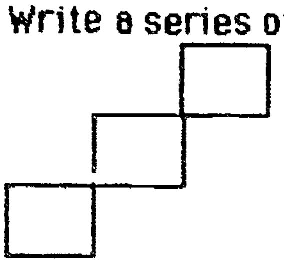
41.7 x .04 =
.52 + 1.8 =
53.4 ÷ 1.8 =
17 - 1.43 =

B. "Write a computation for each of the four arithmetic operations that would give this calculator readout."



LOGO

Write a series of LOGO commands that would result in this figure:



ASSESSMENT (NUMBER SENTENCES)

Mathematician:____

"I wrote the answer when the problem was read. Then I wrote the number sentence for the problem and circled the answer in the number sentence."

PROBLEM	ANSWER	NUMBER SENTENCE
1		
2		
3		
4		
5		
6		
7		
8		
9		

ASSESSMENT (NUMBER SENTENCES)

Mathematician:________________________________

"I wrote the answer when the problem was read. Then I wrote the number sentence for the problem and circled the answer in the number sentence."

PROBLEM	ANSWER	NUMBER SENTENCE
10		
11		
12		
13		
14		
15		
16		
17		
18		

ASSESSMENT (NUMBER SENTENCES)

Mathematician:________________________________

"I wrote the answer when the problem was read. Then I wrote the number sentence for the problem and circled the answer in the number sentence."

PROBLEM	ANSWER	NUMBER SENTENCE
19		
20		
21		
22		
23		
24		
25		
26		
27		

ASSESSMENT (NUMBER SENTENCES)

Mathematician:________________________________

"I wrote the answer when the problem was read. Then I wrote the number sentence for the problem and circled the answer in the number sentence."

PROBLEM	ANSWER	NUMBER SENTENCE
28		
29		
30		
31		
32		
33		
34		

Combine 1: Tony has 10 red marbles and 12 blue marbles. How many marbles does Tony have?

Combine 2: Jack has 26 pets. 11 are dogs and the rest are cats. How many cats does Jack have?

Change 1: Joni has 9 cassettes of her favorite groups. On her birthday she received 8 more cassettes. How many does she have now?

Change 2: Joyce has 23 poppies. She sold 18. How many poppies does she have left to sell?

Change 3: Before Willie gave him some more nails, Fred had 26. Now he has 40. How many nails did Willie give him?

Change 4: Connie had 13 extra valentine stickers. She gave some to Ruth. Now Connie has only 4. How many stickers did Ruth get?

Change 5: Tom had some hazelnuts in a basket. Jerry put 19 hazelnuts into the basket. Then Tom had 34 hazelnuts. How many hazelnuts were in Tom's basket at the start?

Change 6: Gloria had some pennies in her purse. She used 8 of these to pay for some buttons. She then had 9 pennies in her purse. How many pennies were in Gloria's purse to start with?

Compare 1: Petra has 11 baseball cards. Gerta has 18 baseball cards. How many more cards does Gerta have?

Compare 2: Tomas has 9 scout badges. Willie has 19 scout badges. Tomas has how many fewer badges than Willie?

Compare 3: Walter has 8 pencils. Jeannette has 4 more pencils than Walter. How many pencils does Jeannette have?

Compare 4: Sharon has 12 campaign buttons. June has 3 fewer buttons. How many buttons does June have?

Compare 5: Bobbie got 23 correct on his spelling test. She got 6 more correct than Barbara. How many did Barbara have correct?

Compare 6: Tess did 13 push-ups in physical education class. She did 8 fewer than Vera. How many push-ups did Vera do?

CHANGE 1: Freda has 4 boxes with 5 packets of seeds in each box. How many packets of seeds does Freda have?

CHANGE 2: Johanna had 30 cookies. She gave 6 cookies to each person in her troop. How many of her friends received cookies?

CHANGE 3: Paul had 24 marbles that he put into 4 bags. He put the same number in each bag. How many marbles were in each bag?

COMPARE 1: Joellen has 4 pairs of sandals. She has 5 times as many pairs of stockings. How many pairs of stockings does she have?

COMPARE 2: Irene has 30 pennies. She has 5 times as many pennies as Pat. Pat has how many pennies?

COMPARE 3: Donald has 6 marbles. Francis has 18 marbles. Francis has how many times as many marbles as Donald?

COMPARE 4: Bonnie has 16 white blouses and 4 colored blouses. She has how many white blouses for each colored blouse?

COMPARE 5: Her colored blouses were what fractional part of her blouses?

COMPARE 6: Our class has 16 boys and 12 girls. There are how many times as many boys as girls?

COMPARE 7: Our class has 15 boys and 12 girls. There are how many girls for a group of how many boys?

COMPARE 8: The girls were what fractional part of the class?

COMPARE 9: Fred has 25 baseball cards. He has 5/4 as many cards as Bill. Bill has how many baseball cards?

COMPARE 10: Tom as 25 baseball cards. Tim has 4/5 as many baseball cards as Tom. Tim has how many baseball cards?

COMPARE 11: Jack had some marbles. Dennis had 12 marbles or 2/3 as many as Jack. Jack had how many marbles?

SELECTION 1: Paula has 3 kinds of cheese and 2 kinds of sausage. How many different cheese and sausage pizzas can she make?

SELECTION 2: Frank's makes 18 different cheese and sausage pizzas. He uses 6 kinds of cheese. How many kinds of sausage does he have?

SELECTION 3: Rita is going to make a soapbox derby car. She has 3 sets of different size wheels, 4 different boxes for bodies, and 3 different windshields. How many different cars with a set of wheels, a body and windshield can she make?

SELECTION 4: Bonnie can wear 30 different outfits consisting of a skirt, blouse and shoes. She has 3 skirts, and 5 blouses. How many pairs of shoes does she have?

RATE 1: Lisa bought 6 cans of potato chips at "2 cans for 59¢". How much did potato chips cost her?

RATE 2: Corrine ran 100 meters in 15 seconds. What was her average speed in meters per second?

ASSESSMENT (Computation)

Mathematician: ______________________________

"I computed the answer for each computation given."

304 +127 ———	21⟌143	503 - 196 ———
53 + 49 =	54 x 9 =	46 x 28 ———
340 ÷ 21 =	104 - 87 =	92 + (4 x 16) =

ASSESSMENT (FRACTION SENTENCES)

Mathematician:________________________________

"I completed the following fraction sentences."

3/4 + 2/3 = ☐

4/5 − 2/3 = ☐

7/8 ☐ 3/4

6/9 ☐ 2/3

5/8 x 1/2 = ☐

3/4 ÷ 2/3 = ☐

☐ = 1 1/2 x 2 1/3

☐ = 2 1/2 ÷ 1 1/2

8/9 ☐ 15/16

5 1/2 ☐ 5 3/6

LEVEL FOUR

BASIC FACTS REVIEW

Background: These activities should be used for daily work for 6 or 7 weeks and then occasionally for maintenance the rest of the year.

ADDITION FACTS

1. Add on to the larger of the two by counting mentally:

   Example: 7 + 5 = ☐

   Seven, eight, nine, ten, eleven, TWELVE

2. Make a TEN:

   Example: 8 + 7 = ☐

   Two from the SEVEN with EIGHT, makes TEN
   FIVE is left so 8 + 7 = 10 + 5 = 15

3. Use doubles (these are quickly learned by most children)

   Example: 9 + 6 = ☐

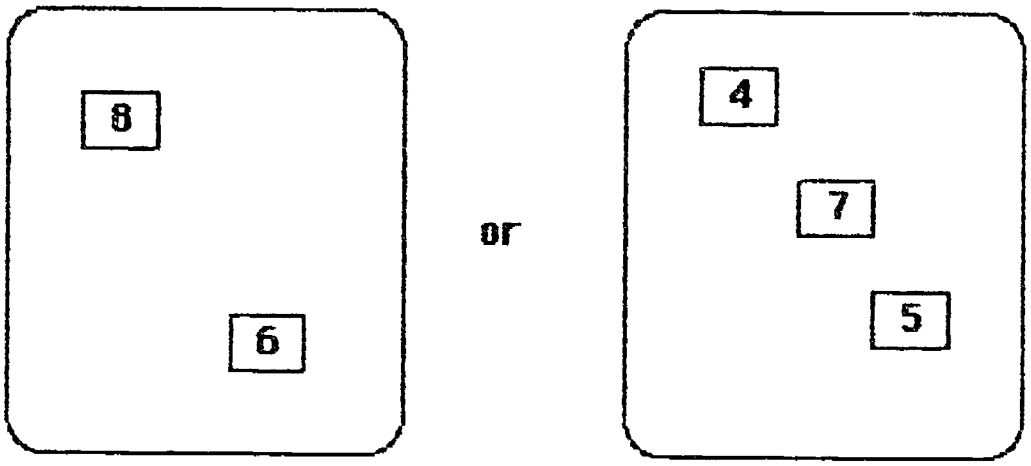
   double 6 = 12 + 13 = 15

4. Random addition table completion:
   Example:

+	3	0	9	4	1	7	5	2	etc.	
5										
1										
8										
2										
3										
7										
4										
e										
t										
c.										

5. Overhead "flash card" pairs. Make overhead numeral cards. With the overhead lamp off arrange 2 or 3 of these on the overhead. Turn it on and ask for the result.

Examples:



6. Roll dice with numerals, not dots, on them. Use dice with more than four faces - 6, 8, 10, 12, so all numerals 0-9 can be on the faces. These can be obtained from commercial sources.

7. Card games. Examples: KRYPTO, CRIBBAGE, RUMMY

8. Oral Drills. Examples:
"five plus what is nine?"
"eight minus what is seven?"
"sixteen minus what is seven?"
"sixteen minus nine is what?"

9. Other games. Examples: TUF, CROSS NUMBER ROLLS.

MULTIPLICATION FACTS

1. Master the 2's facts first by:
--doubling activities
--skip counting by two's
--using 100 tables

2. Master the 5/s family
--all end in 0 (evens) or 5 (odds)
--skip counting by fives
--using 100 tables

3. Master the 10's family
   --all end in zero
   --all start with counting numbers in sequence
   --relate these to place value activity

4. Master the 3's family
   --end in a pattern

3	6	9
2	5	8
1	4	7

   --skip count by 3's

5. Master the 4's family
   --double the doubles

   Example: 2 x 3 = 6 so 4 x 3 = 12

   --skip count by 4's

6. Master the 6's family

   --use 5's and add on more of the other

   Example: 6 x 6 = 5 x 6 = 30 + 6 = 36

General activities to develop the multiplication facts include:

--skip counting
--randomly arranged digit tables

Example:

x	4	9	0	2	7	1		etc.			
5											
2											
8											
2											
7											
4											
e											
t											
c.											

--card games
--dice rolls - 1st six sided dice with 0-5 on faces, then TEN sides dice with 0-9 on the faces
--flash pairs numerals on the overhead to find products
--oral drills

Stress the relationship between addition and subtraction "facts" and between multiplication and division "facts."

Examples: 3 + 4 = 7 so 7 -3 = 4 and 7 - 4 = 3
3 x 4 = 12 so 12 3 = 4 and 12 4 = 3

Have the children write fact families for a "number for the day."
Example:

8 = 2 x 4   8 2 = 4   8 4 = 2
8 = 5 + 3   8 - 5 = 3   8 3 = 5
8 = 4 + 4   8 - 4 = 4   8 - 4 = 4
8 = 7 + 1   8 - 7 = 1   8 - 1 = 7

Keep in mind that:

0 added or subracted leaves a number unchanged
0 multiplied gives a product of 0
DIVISION BY 0 IS NOT POSSIBLE! WHY?
Consider:
0 x 4 = 0
0 x 5 = 0
0 x 6 = 0   so 0 0 4, 5 or 6

REVIEW OF "BASIC FACTS"

Some worksheets are provided for this. Blank Masters are also provided so you can prepare other, similar worksheets.

Continue to use skip counting and other counting activities several times each week, especially at the beginning of the year.

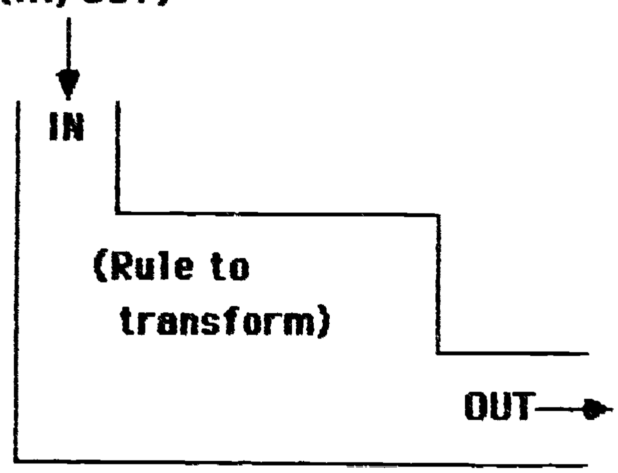
LEVEL FOUR

GUESS MY RULE

Background: Children should have been introduced to this in LEVEL THREE. However, this cannot be assumed so it is best to review the rules of the game and to give some experience with simple rules.

The basic idea is that of a function machine that transforms numbers put in according to a fixed rule and generates numbers that come out. The result is a set of ordered pairs of numbers of the form:

(IN, OUT)



An example:

IN	**RULE**	**OUT**
2	+ 3	5
4	+ 3	7

LESSON ONE: How to Play the Game:

CHILDREN ARE TO RAISE HAND AND SAY, "I KNOW THE RULE," WHEN THEY THINK THEY DO.

1. Children are to input numbers. Call on them one at a time.
2. For each number, you use a predetermined rule to orally give a number back to the class.
3. Repeat until a child raises his hand to test whether or not he/she knows the rule.

CHILDREN TEST BY SAYING, "IF I GIVE YOU (NUMBER), WILL YOU GIVE ME (NUMBER)?" THEY ARE NOT TO BLURT OUT THE RULE!

4. If you reply, "YES", the child is to keep silent and think of a better way to organize the information.

5. If "NO", remind all that they must always get further information if they cannot see the rule with what they have.
6. Periodically, ask those who think they have the rule to raise their hands.
7. Continue until:
   a. most have the rule; or
   b. they seem to be at a stalemate
8. Ask for tables of data and analyze. EXAMPLE:

DATA AS GENERATED

IN	OUT
2	6
5	9
1	5
3	7
10	14

Ask for suggestions as to what to do to make seeing patterns of numbers easier. Discuss: Encourage students to look for patterns in the OUT numbers after they have put the IN numbers in numerical order.

REORGANIZED DATA

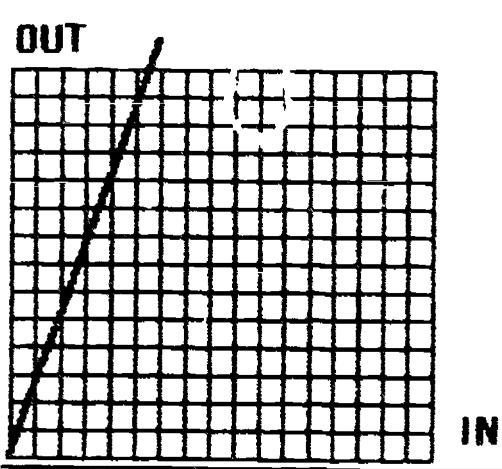
IN	OUT	
1	5	**"How are the IN numbers changing?"**
2	6	
3	7	**"How are the OUT numbers changing?"**
5	9	**"How do you get an OUT number for a given IN number?"**
10	14	RULE: OUT = IN + 4

Have the students supply numbers to fill in any missing number pairs in an organized table.

A sequence of rules to use:

1. Add a given number
2. Subtract a given number (there is a chance of negative numbers coming about here)
3. Multiply by a given number
4. Square the IN number

Graph all rules found with the IN numbers on the horizontal axis and the OUT numbers on the vertical axis. Example:



RULE: OUT = 3 X IN

LESSON TWO: (Guess My Rule with Other Than Numbers)

Use Guess My Rule with other than numbers.

1. Attribute Blocks:

Example: IN	OUT
"Large Red Squares"	"Large Red Triangle"
"Large Green Circle"	"Large Green Square"

RULE: CHANGE THE SHAPE

Other rules:

Change Color
Change Size
Change Color and Size
Change Shape and Size
Change Color and Shape, etc.

2. Geoblocks

Choose 2 ge blocks that are alike in some way and hold these up.

"What was my rule for choosing these two?"

This has the added outcome of focusing children's attention upon the properties - corners, edges, shape of the faces, volume, etc.

LESSON THREE: Combining Operations

Introduction: After students are comfortable finding rules and seem to be able to recognize patterns in the OUT numbers, introduce rules that combine operations. Example:

IN	OUT
0	2
1	5
2	8
3	11
4	14

The patterns in OUT numbers is that each differs by 3 from the previous one. At this point introduce a "d" (for difference) column:

IN	OUT	d
0	2	
1	5	3
2	8	3
3	11	3
4	14	3

The difference is constant. In other words, the OUT number increases by 3 each time the IN number increases by 1.

This rule is 3 x IN + 2 = OUT.

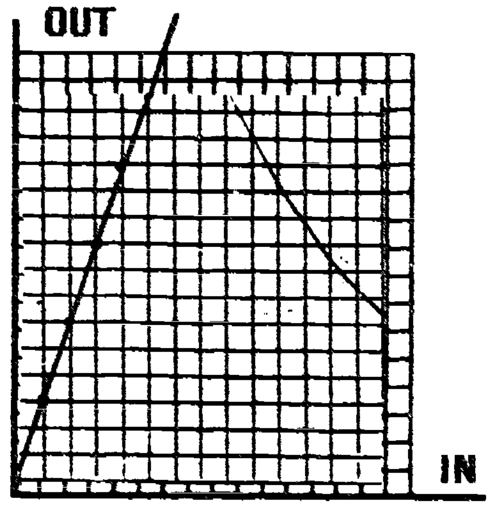
Two things to observe are that the number added is found when IN = 0, and the constant number "d" is the multiplier of IN.

Graph these ordered pairs as before:

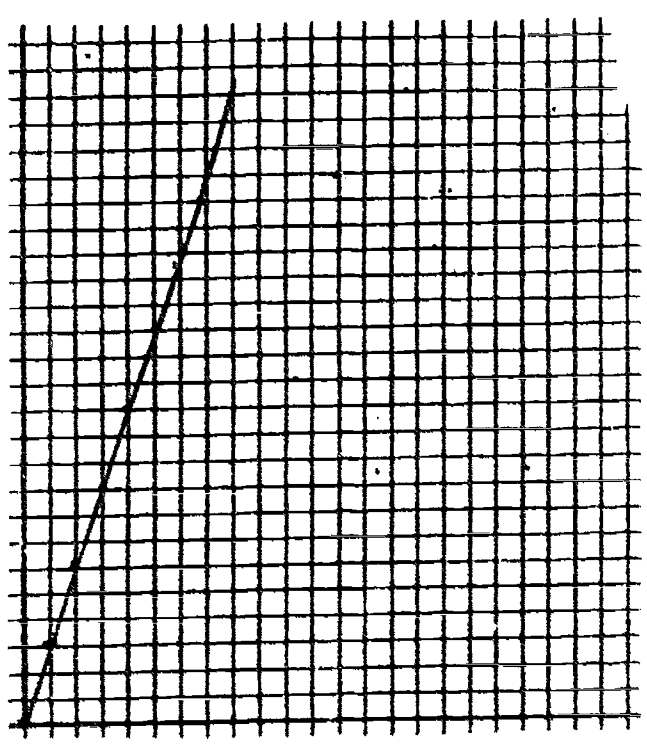
A sequence of rules to use:

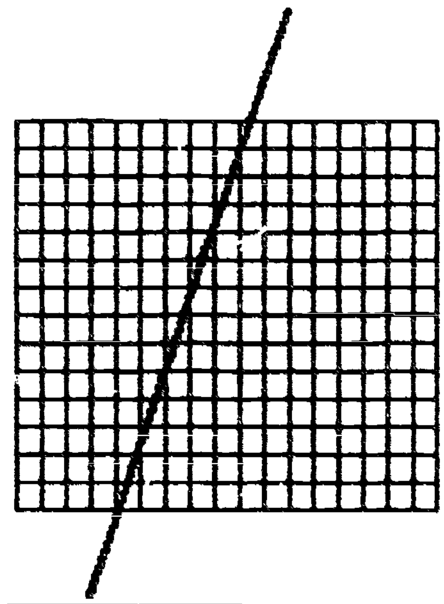
1. Add a given number
2. Subtract a given number (there is a chance of negative numbers coming about here
3. Multiply by a given number
4. Square the IN number

Graph all rules found with the IN numbers on the horizontal axis and the OUT numbers on the verticle axix. Exemple:

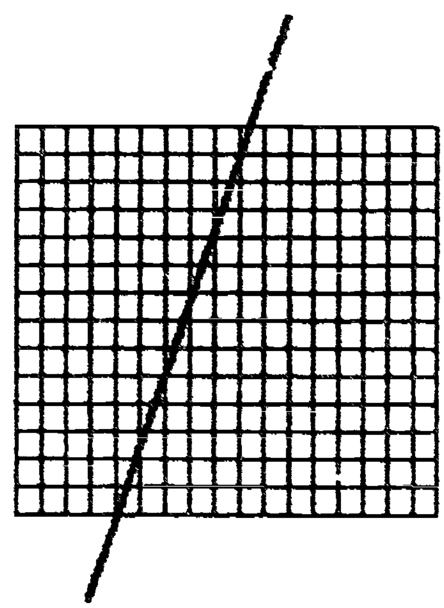


RULE: OUT = 3 X IN





Point out where the line crosses the vertical (OUT) axis, and that the change in OUT of 3 for each change of In of 1 is clearly shown on the graph. Draw these in:



Ask the children what else they see about the graph. Play the game with several rules of the form a x IN + b = OUT and have the children graph these.

When they are comfortable doing these introduce rules of the form a x IN - b = OUT.

LEVEL FOUR

THINKING

Background: Whenever educators are asked to list the goals of education for virtually any level of instruction, they invariably list "Teaching pupils to think for themselves", in the leading three, if not first. But what is thinking? Several definitions are possible. Edward de Bono, a British philosopher and developer of thinking curriculum materials, defines it as "the deliberate exploration of experience for a purpose." (Teaching Thinking, Temple Smith, London, 1976.)

Whatever the stated objectives, the PRACTICE in teaching school subjects is the acquisition of info..nation. Information is easy to teach and can be tested in exams using "objective" items. But obtaining information is no substitute for thinking and conversely thinking uses the raw material of information.

However, emphasis on content only assuming thinking skills will develop as a by product of knowledge acquisition is not effective. Content has its own momentum that makes the development of thinking skills difficult to attend to.

Pleas to "think about it" fall on deaf ears when the end-of-the line is a content based objective examination that measures the acquisition of isolated facts and unrelated bits of information. The operations of thinking must be developed as tools that can be used in any content area.

LESSON ONE: Treating Ideas (PAN)

Introduction: Before making an evaluative judgment of an idea, children must learn to look at the positives, negatives, and appeal of the idea (PAN the idea.)

This lesson gives practice in using the PAN thinking skills - listing of the good things (P), listing of the bad things (N) and listing of the appealing features of the idea (A).

Use this idea to teach the children the process.

"Cars should be banned from downtown business areas so people can walk wherever they want to."

. Ask the children to help you develop a list of what is good about this idea. Do not discuss the value of these.

Ask the children what is bad about the idea. List these but do not give value to them.

Ask the children to list everything that is interesting about the idea. Make no judgments and permit no judgments about the ultimate value of the suggested practice.

Other ideas to PAN:

"By law all cars should be painted blaze orange like hunting clothes."

"Every pupil should be paid to go to school."

"Every 4th grader should adopt a 3rd grader to help through school."

<u>LESSON TWO: Consider All Factors (CAF)</u>

<u>Introduction:</u> Before making a decision, children should consider all factors related to that decision (CAF).

This lesson gives children experience in listing all factors to consider relevant to a decision before it is made (CAF).

"A large midwestern city passed an ordinance requiring all new buildings to be built downtown to have a large garage in the basement for employees' cars. Three years later they repealed the ordinance. Why?"

Ask the children to list the factors that may have been overlooked in making the first decision.

Here are additional decisions to have children do CAF on:

"What must be considered in buying a used car?"

"What should be considered in designing school desks?"

"What should be considered before choosing to go along with what your three best friends want to do?"

LESSON THREE: Making Rules (MR)

Rules have many purposes:

............to prevent confusion - Example: STOP signs
............to allow enjoyment - Example: The rules for playing a game
............to maintain order - Example: Rules of conduct for military personnel
............to prevent a few from taking advantage of everyone else
Example: THOU SHALT NOT STEAL

In general, rules are to make life more pleasant and comfortable for the majority.

Good rules:

............are well known and possible to obey
............are not bad just because they are unpopular
............should work for the benefit of the majority
............have a purpose identified by those expected to obey it
............may need to change with changing conditions

Ask the class to use PAN to make a list of 5 rules needed for the school.

Some additional rules problems:

"A group of people go to Mars to start a colony. They abolish money, property rights and all the old rules. They soon find all do not wish to work. Invent some rules for this colony. (USE PAN AND CAF).

"Make a set of rules for basketball to be played by 4 persons with a basket 3 feet higher."

LESSON FOUR: Consequences of Decisions (COD)

Introduction: Children must learn to consider the consequences of actions taken and decisions made (COD). They need to realize that:

............other people may be able to see the consequences of your decisions and actions better than you can
............one must know if consequences are short term or long term
............one must know if consequences are reversible

.... consequences of one's actions and decisions affect other people

Ask the ch''dren to list the consequences of the following as short term or long term:

"A law is passed allowing children to obtain drivers licenses at age 12."

Other situations to do COD on are:

"A flashlight is invented that makes a person's face green if that person is lying."

"Foreign banks acquire ten U.S. companies each year for five years."

"All exams are abolished in school."

"The world runs out of oil."

"The average temperature on the surface of the earth rises 5 degrees C."

<u>LESSON FIVE: Identifying Goals & Objectives (IGO)</u>

<u>Introduction:</u> It is important for children to have practice in setting objectives and goals and seeing how these interact with each other. The issue of short term and long term goals is important. Some general characteristics of goals and objectives include:

.... they are easier to achieve if clearly known
.... in the same situation, different people may have different objectives
.... successively achieved objectives may help in reaching a goal
.... goals and objectives must be achievable
.... all goals and objectives are not created equal - some are more important than others

Ask the children to try to find what purpose the following might have:

"You captain a spaceship approaching Mars, having started out from Earth.:

Other situations to do an IGO on are:

"What objectives might the following have with regard to food:"

............homemaker
............restaurant chef
............grocer
............farmer
............U.S. Department of Agriculture

"The coach of a youth athletic team is holding a practice right after the team lost a game."

"The teacher who asks a student to correct each incorrect answer on a test."

LESSON SIX: Planning Ahead (PA)

Introduction: Have the children organized into 4 groups. Give each group the objective of making money by sale of objects or a product made from the objects provided. They must develop a plan for achieving that objective.

Remind the children that in planning:

............one must know what you want to achieve
............an alternative, or back up, plan should be ready
............the value of a plan depends on the consequences related to carrying out
............plans should be simple and direct
............all factors should be considered BEFORE making a plan

GROUP ONE:	5 bicycles, 2,000 old books, ten gallons of red paint
GROUP TWO:	5 bicycles, a horse, a printing machine
GROUP THREE:	a horse, a recipe for sausage, ten gallons of red paint
GROUP FOUR:	2,000 old books, a horse, a printing machine

Some other things to have plans for might include:

............a class carnival
............an expedition to find Noah's Ark
............a screening system to prevent the hijacking of airplanes

............a school track meet

LESSON SEVEN: Setting of Priorities (SOP)

Introduction: Some things are more important than others - objectives, factors, consequences. After doing PANs,
SOPs are done to determine: which things should be dealt with first. Children need to consider the importance of things. Sometimes it is easier to do this by looking first at the least important.

Ask the children to do a SOP on the father's actions in the following:

"Tom's father learns that Tom has taken a baseball home from the community playground. In dealing with Tom, a fourth grader, what should his father's priorities be?"

Some additional course of action, etc. to SOP include:

"Making a TV show interesting."
"Making a TV show educational."
"Buying materials to teach science in your grade."
"Spending your earnings or allowance."

LESSON EIGHT: Selecting Alternatives (SA)

Introduction: Children must learn not to jump to conclusions or select the first or most visible choice available. Consideration of alternatives is important. This lesson gives experience in that thinking skill.

"The county sheriffs found a smashed bicycle in the ditch and a dead boy five feet away. What could have happened?"

When the alternative explanations are all considered, have the children to a PAN to find the most plausible one.

Some additional situations to SA on:

"You find your best friend has stolen something from a store."

"June has been getting A's on all of her tests and suddenly starts getting D's and F's. What explanations are there for this change?"

"Fewer U.S. college students are choosing to become career scientists. Why is this so and what can be done?"

"Tom's older brother wants to go to college, but his father gets sick and has to leave his job. What can Tom's brother do?"

LESSON NINE: Making Decisions (MD)

Some decisions are easy and some are difficult. Some decisions are routine – like deciding which clothes to wear, and some are a choice of alternatives – like whether to "do drugs" or not. Some decisions are voluntary and some are forced upon you.

Ask the children to make a decision about the following:

"A policeman working alone sees a moving light in a downtown store. What should he do?"

In discussing possible courses of action, review the processes for looking at pluses and minuses, considering all factors, including consequences, making a plan, etc.

Here are a few more situations requiring decisions to be made:

"Fred is asked to play in the band, act in the school play, and to play on the school hockey team. They all have practice at 3:30 p.m."

"Joanne's sister has a choice of 2 of these subjects to fill out her schedule – typing, algebra, biology, art and creative writing."

LESSON TEN: Looking at Another Side (LAAS)

Introduction: Children who are concrete operational or becoming concrete operational are beginning to recognize points of view other than their own. As egocentric pre-operational thinkers, they were incapable of this. Considering another point of view could involve CAF, CA, PAN or COD.

Ask the children to consider the following and ask them first to describe what constitutes her point of view, then his point of view, then have them PAN each point of view:

"Sally's father forbids her 13 year old sister to smoke. What is his point of view and what is hers?"

Use previously developed thinking skills to explore this situation in depth.

Here are some other situations involving differing points of view:

"A family in an all white neighborhood sells their house to a black family. Some neighbors circulate a petition against this sale. Some people on the block sign it and others don't. What is the point of view of the family selling, the family buying, those signing the petition, and those not signing the petition?"

"Jack misbehaves and interrupts the class. He is sent to the principal's office. The principal calls the parents and asks them to come to bring him home. They come but object to having Jack miss school. What is the point of view of the teacher, of Jack, of the principal, of Jack's parents, of Jack's classmates?"

"There is an empty lot across from a school that the children use as a playground. The city government decides to build a tool shed on the lot. The children object. What is the point of view of the children, of the city officials?"

LEVEL FOUR

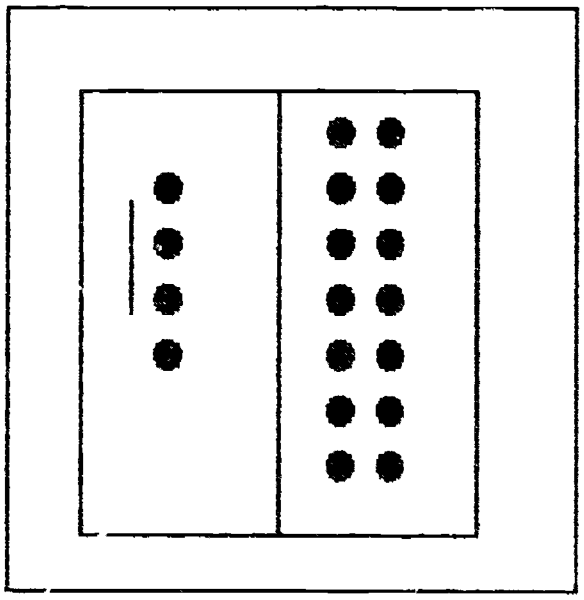
EQUALITY

Background: Children using this program will have had three years of emphasis on equality so should realize that this relation is reflexive and number sentences can be written many ways, i.e. $5 = 2 + 3$, $2 + 3 = 5$, $2 + 3 = 3 + 2$, $2 + 3 = 4 + 1$, etc.

Periodically, both equality and inequality should be reviewed.

LESSON ONE

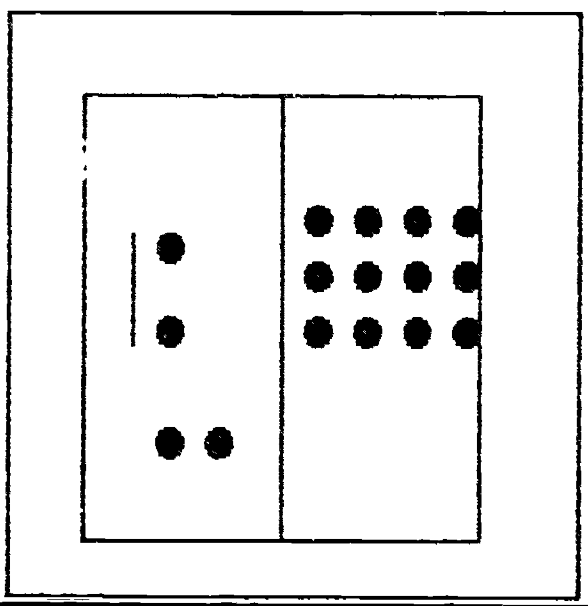
Give the children split boards and base ten TENS and ONES. Using the overhead projector, arrange the following:



Ask the children to write the number sentence

($10 + 4 = 7 + 7$)

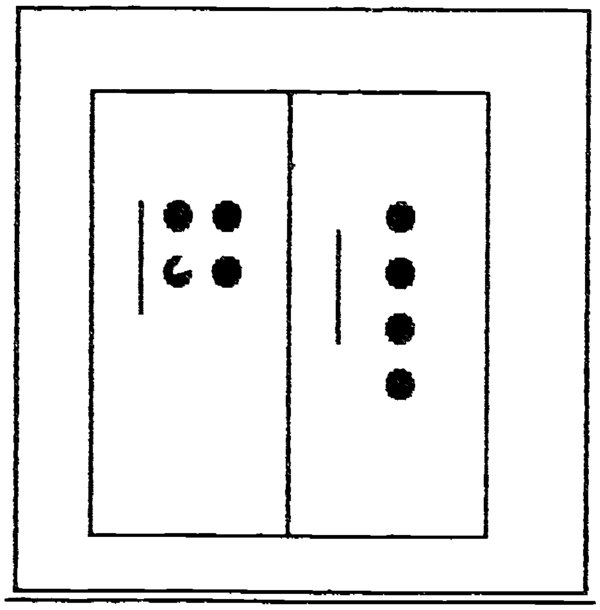
Rearrange as shown:



$10 + 2 + 2 = 4 \times 3 + 2$
$10 + 2 + 2 = 12 + 2$
$12 + 2 = 4 \times 3 + 2$
$12 + 2 = 12 + 2$

are some of the number sentences that could be generated

Trade ten ONES for a TEN



Some possible number sentences:

$10 + (2 \times 2) = 10 + 4$
$10 + 2 + 2 = 14$

Use different numbers for similar kinds of equality activites using combinations of TENS and ONES on the left side, right side or both sides.

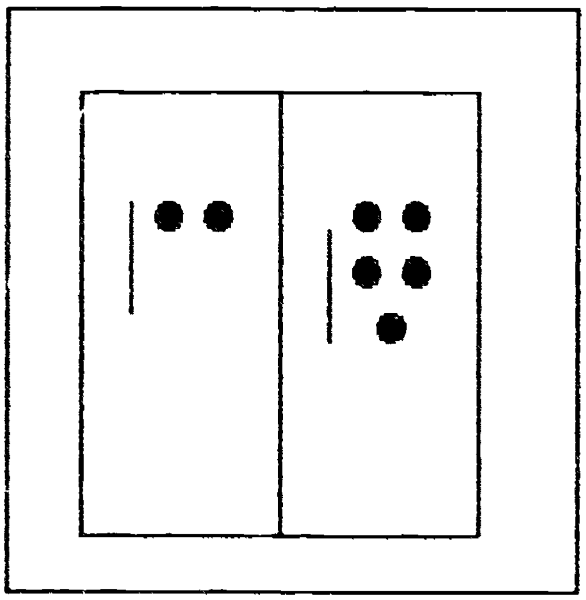
LEVEL FOUR

INEQUALITY

Background: Children will have had a brief introduction to this symbolism in the Level Three work. It should be emphasized at this level.

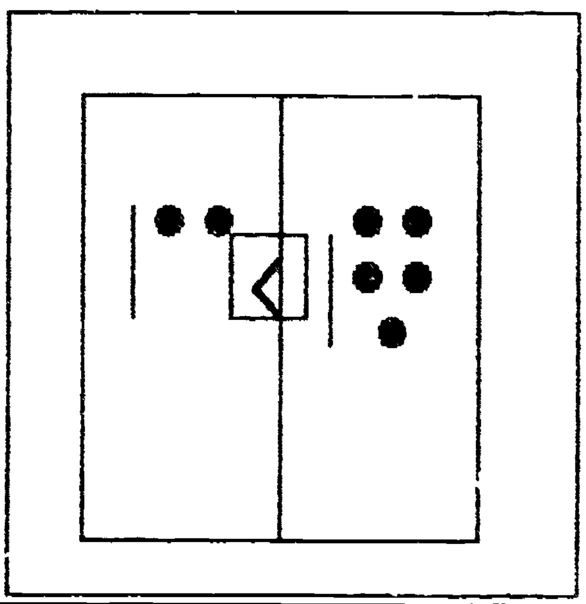
Introduction: Prepare 2 x 2 cards with the inequality sign on them. Children are to use split boards and orient the symbol to correctly show the relationship.

On the overhead projector, place an array of base ten TENS and ONES as shown:



Show the children a transparency inequality sign and ask how it should be pointed.

Then place the symbol.



Do another. Discuss how the wide part points to the LARGER quantity and the "point" to the smaller.

Children are to work in pairs. One sets base ten pieces on the split board, the other should place the sign correctly. Have the roles exchanged after each 5 "problems."

<u>LESSON TWO</u>

Have the children write the signs into the worksheet provided.

Arithmetic Problem Solving at Level Four

The goals for arithmetic problem solving at this level should be for children to:

1. Be able to apply all fourteen kinds of addition and subtraction problems
2. Be able to apply the simplest combine and change multiplication and division problems
3. Be able to do two or three step problems involving these kinds of problems.

Consider these factors when working with children with arithmetic problem solving:

Some students may still need some concrete materials to represent the things given orally or in written situations......

Students should write number sentences that model the conditions of the problem given.......

Students should write "story problems" that given number sentences would model......

It is not the size of the numbers that presents difficulty to students in problem solving. Thinking through the situations to determine WHICH operations are needed presents the difficulty. Number size only affects ease of computations........

Students should be given opportunities to see a variety of problem solving being used
........guessing and checking the guess
........drawing pictures
........making diagrams
........making tables or graphs
........estimating
........writing number sentences to model

Non-numeric problem solving situations should also be presented using Pattern Blocks, Tangrams and other right hemisphere related materials.

PROBLEM SOLVING: CHECKING UP

Background: In previous lessons, children will have been introduced to all fourteen situations that lead to addition and subtraction (see material in the Appendix).

Introduction: Place a transparency of the problems furnished on the overhead projector covered by a cardboard. Children should have answer sheets supplied.

Reveal and read slowly the problems one at a time. Give children plenty of time to think through the problems. Have children circle the answer number in the number sentence they have written.

Analyze the results to see which kinds of problems are still giving children difficulty so these can be reviewed and re-emphasized.

LEVEL FOUR

PROBLEM SOLVING

LESSON ONE: Making Sure

Background: From the analysis of your administration of the fourteen kinds of addition and subtraction problems, select those that proved to be most difficult - probably Compare 5 and 6 and Change 3, 5 and 6.

Write further problems of this kind, using a seasonal theme. Use these for instruction. For the COMPARE problems, emphasize "less than", "fewer than", and "more than" language. Give many examples, move materials on the overhead projector to illustrate these comparisons. Encourage children to ask questions like: "Which is larger?"

"Which is smaller?"

"What is the difference between them?"

For the CHANGE problems, encourage the children to think of questions like:

"What is being added to?"

"What is being added to something else?"

"What is being subtracted from?"

"What is being subtracted from something else?"

"What is not known?"

"What does the question ask for?"

Encourage children to make diagrams to represent the quantities. Number sentences to directly model the problem kinds are:

Problem	Number Sentence	Converted Sentence
Combine 1:	A + B = □	
Combine 2:	A + □ = C	□ = C − A
Change 1:	A + B = □	
Change 2:	C − A = □	
Change 3:	A + □ = C	□ = C − A
Change 4:	C − □ = B	C + B = □
Change 5:	□ + B = C	□ = C − B
Change 6:	□ − B = A	□ = A + B

Many children will mentally convert to the model they are most comfortable with. If you encourage them to make pictures to represent the objects in the problems, they are more likely to write correct number sentences - either direct models or conversions. Most of the COMPARE problems will yield proper number sentences this way.

<u>LESSON TWO: Emphasis on Modelling</u>

<u>Introduction:</u> In this lesson go through the problem types one at a time. Use materials - base ten blocks, counters or whatever is most appropriate. Make a diagram of each type. Write the number sentence, explaining how each numeral and symbol is derived from the verbal description.

A CHANGE 3 problem is used as an example.

"Betty has 12 star coupons. Here are her coupons."

Problem	Number Sentence	Converted Sentence
Combine 1:	A + B = □	
Combine 2:	A + □ = C	□ = C − A
Change 1:	A + B = □	
Change 2:	C − A = □	
Change 3:	A + □ = C	□ = C − A
Change 4:	C − □ = B	C + B = □
Change 5:	□ + B = C	□ = C − B
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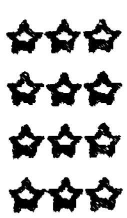


Write

12

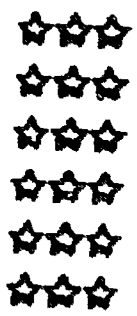
"Fred gives Betty some more coupons." Write: 12 + ☐

"Why did we write the plus sign?"

"Why did we write the ☐ sign?"

Turn overhead off. Add six more counters to the collection. Turn overhead on.

"Betty now has 18 of these."



Write

12 + ☐ = 18

"How many did Fred give Betty?" Write: 12 + [6] = 18

Circle the answer: 12 + ([6]) = 18

"SIX is the missing amount that we didn't know – the number Fred gave Betty."

<u>LESSON THREE: Writing Problems</u>

<u>Introduction:</u> Write a number sentence on the overhead or chalkboard:

23 + ☐ = 44

"Write a story problem so that this number sentence shows what is in the problem."

Ask for these from individuals and analyze them as a group. Remind children "+" can show parts or the action of joining. It may also show "how much more than." The ☐ always shows the number to be found – the answer to the question in the problem.

Activity: Pass out the worksheet and have children write a story problem for each. Allow the use of materials as needed.

LESSON THREE: Checking Up 2

Background: Children have had enough experience with some of the problems requiring multiplication and division to warrant finding out which of these need instructional emphasis. Read the material in the APPENDIX first.

Introduction: Each child should have 30 cubes of some kind and a calculator. Remind them to use these to represent objects in the problems. "Listen as I read the problem and show it to you. First, find the answer. I will read it a second time. This time concentrate on the number sentence to show the problem. Then CIRCLE the number in the sentence that answers the question in the problem."

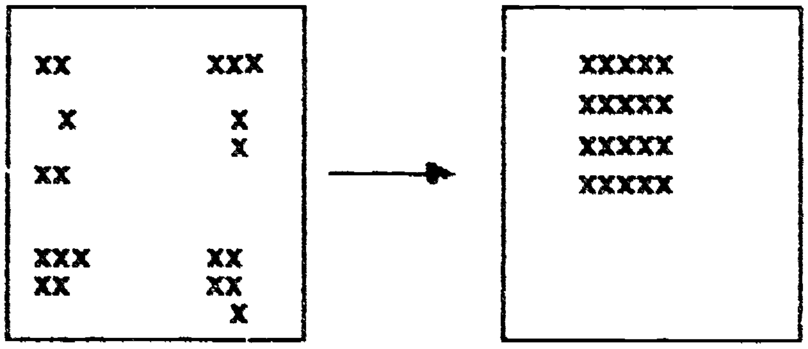
Reveal the problems one at a time. Read each one twice slowly. Give children time to THINK between each step. Analyze the results to determine which kinds of problems to emphasize.

LESSON FOUR: Multiplication and Division

Introduction: Go through the problem kinds for multiplication and division one at a time. Follow the same procedure as with addition and subtraction.

Allow the use of materials
Emphasize writing number sentences
Emphasize making diagrams to show the problem situations

CHANGE 1: "I put 5 pictures on each of 4 bulletin boards. How many picture did I use?" Materials to show this on the overhead:

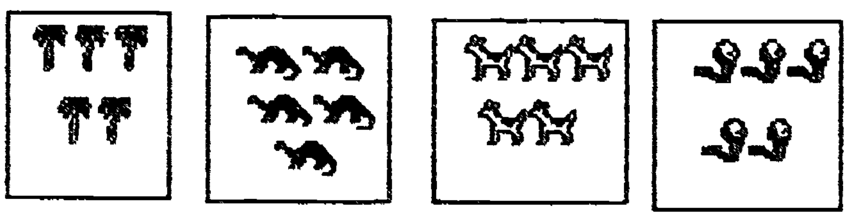


"There are 20 pictures altogether."

Number Sentence:

☐ = 4 x 5   "20 goes in the box."

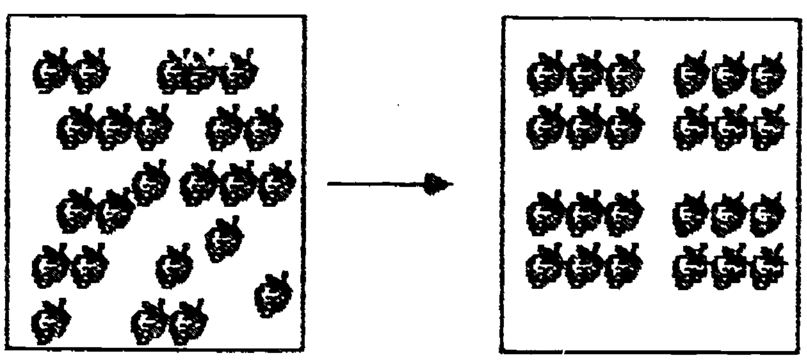
Picture:



"There are 20 pictures altogether."

CHANGE 2: Joyce had 24 strawberries.. She gave 6 to each of her friends. How many friends received strawberries? Materials to show this on the overhead:

Sort out by 6's:



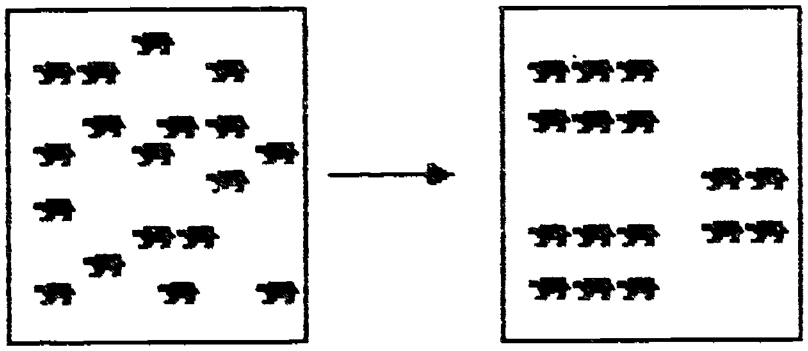
"There are 4 collections of 6, so 4 friends received strawberries."

Number Sentence: 24 ÷ 6 = ☐

"4 goes in the box."

CHANGE 3: George had 18 turtles. He gave an equal number to each of 3 friends. How many turtles did each friend get? Materials to show this on the overhead.

Sort into 3 groups one at a time:

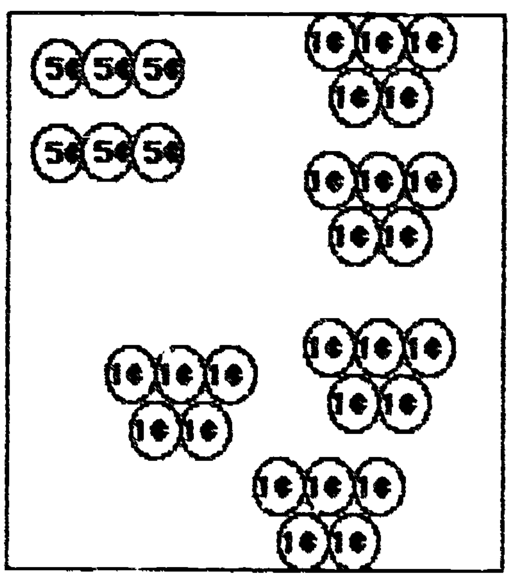


"Each of the 3 friends received 6 turtles."

Number Sentence:

☐ **= 18 ÷ 3 "6 goes in the box."**

COMPARE 1: Francine has 6 nickels. She has 5 times as many pennies as nickels.. How many pennies does Francine have? Materials to show this on the overhead:

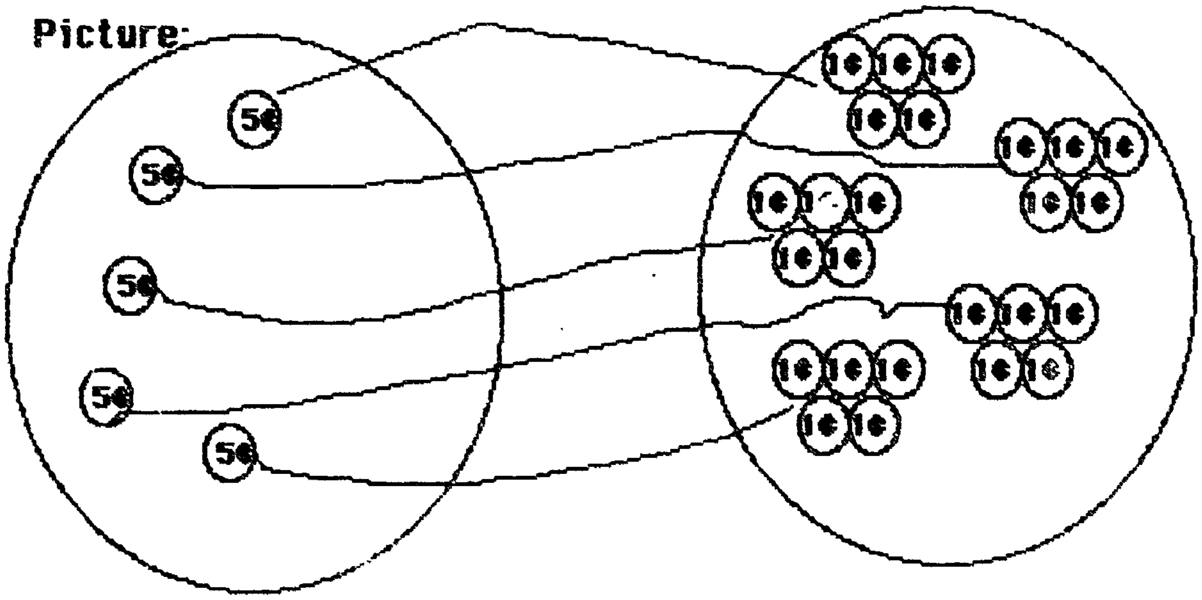


"For each nickel, there are 5 pennies. For 2 nickels there are 10 pennies. For 6 nickels, there are 30 pennies."

Number Sentence: 5 x 6 = ☐

"30 goes in the box."

Picture:

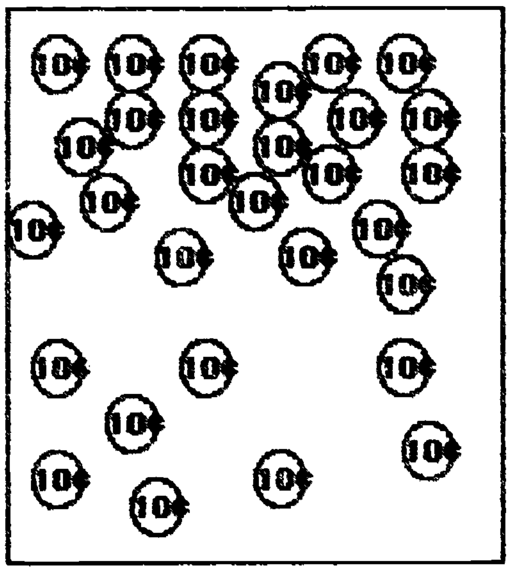


"For each nickel there are 5 pennies."

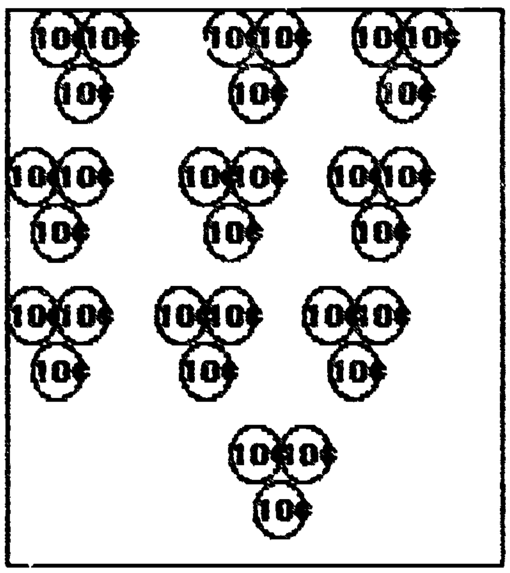
TABLE:

Nickels	Pennies
1	5
2	10
3	15
4	20
5	25
6	30

COMPARE 2: Jane had 30 dimes. She has 3 times as many dimes as Pat. How many dimes does Pat have? Materials to show on the overhead:



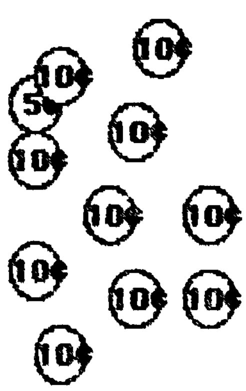
"Jane has 3 dimes for every one of Pat's."



"There are 10 groups of 3 so Pat has 30 dimes."

PICTURE: PAT

JANE



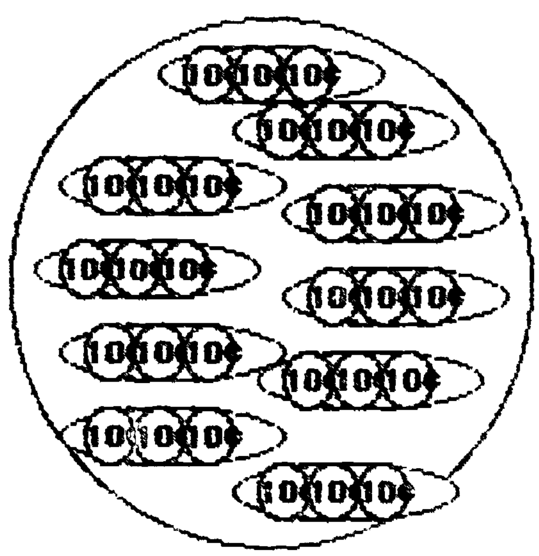
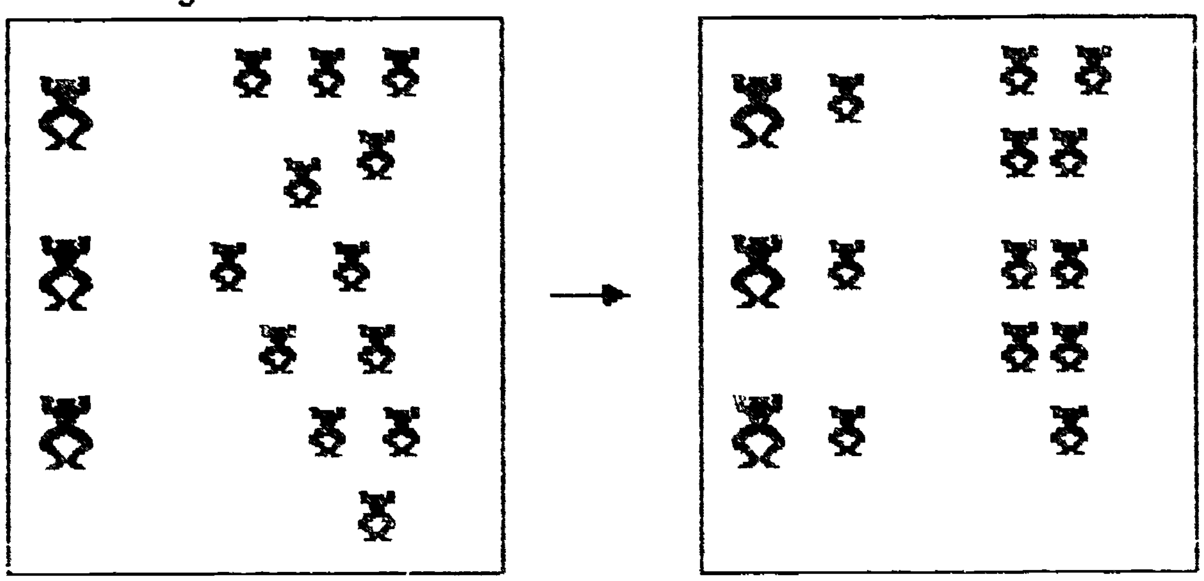


TABLE:

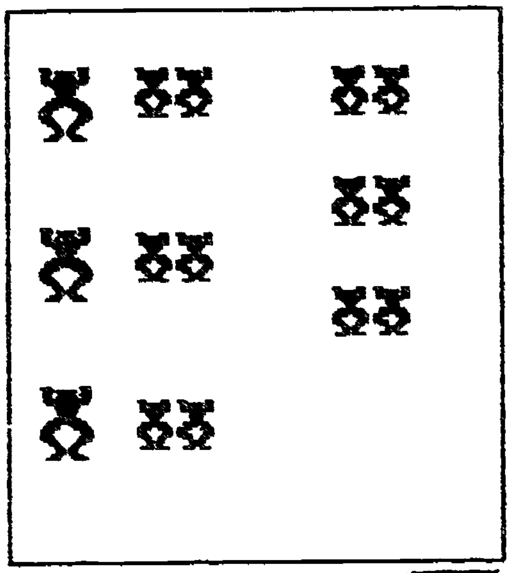
Pat	Jane
1	3
2	6
3	9
4	12
5	15
6	18
7	21
8	24
9	27
10	30

COMPARE 3: Davey has 3 frogs. Paul has 12. Paul has how many times as many frogs as Davey? Materials on the overhead.

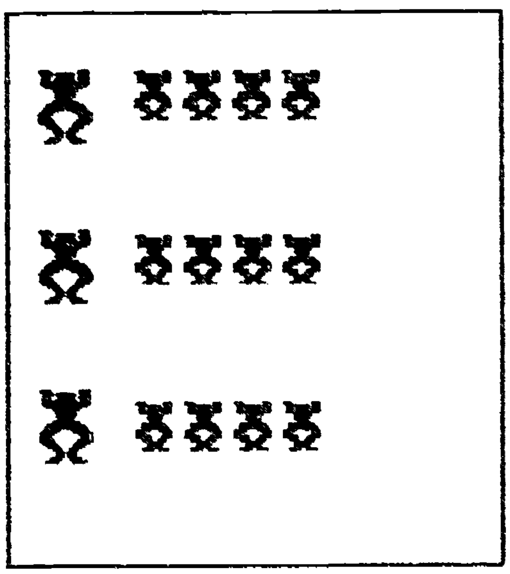
Let's arrange these first as one of each:



"We can use another so there are 2 for each."



"We can give each large frog 2 more, so:

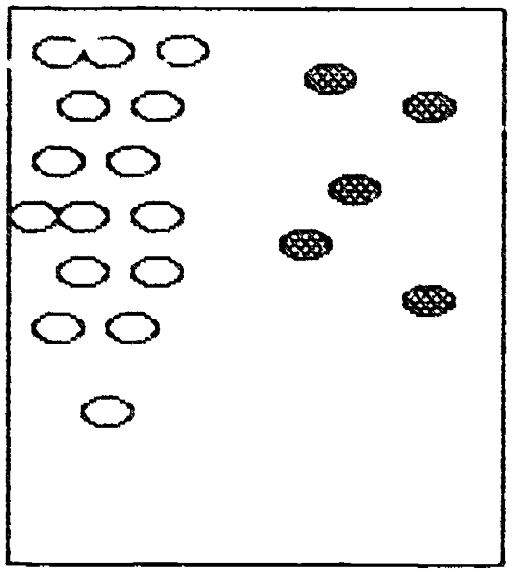


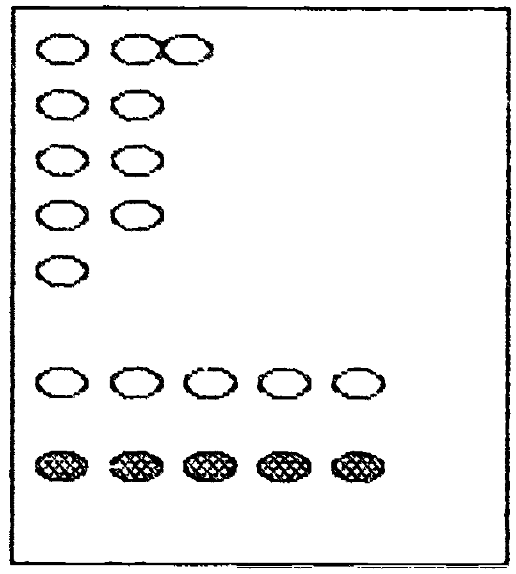
There are 4 for each one."

Number Sentence: ☐ = 12 ÷ 3 "4 goes in the box."

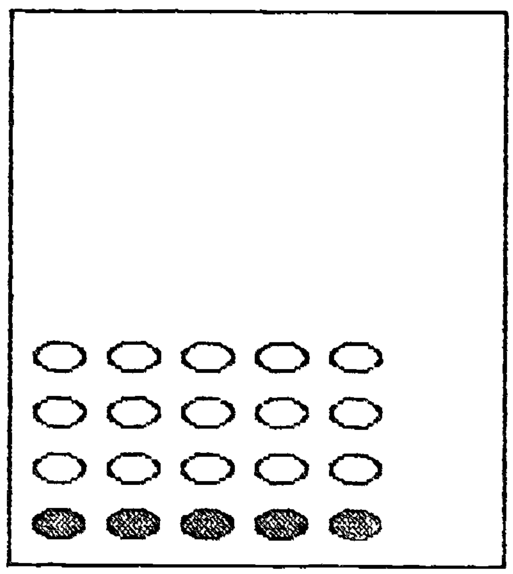
COMPARE 4: John has 15 pairs of white sox and 5 pairs of colored sox. He has how many pairs of white sox for each pair of colored sox? Materials to show on the overhead projector:

Let's arrange these first using 1 for 1:





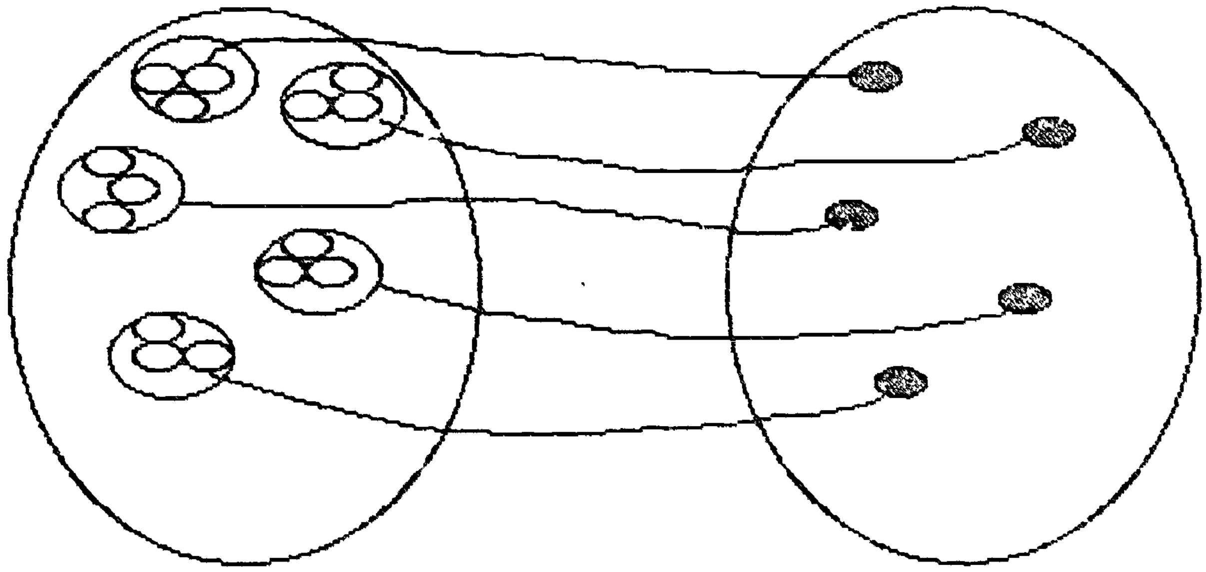
"How many more whites can we put with each colored one?"



"There are 3 whites for each colored one.."

Number Sentence. 15 ÷ 5 = _____   "3 goes in the box."

Picture:



COMPARE 5: His colored sox are what fractional part of all of his sox?

"How many sox does he have?" 20
"How many colored sox does he have?" 5
"5 is what part of 20?" 1/4

It is unlikely your assessment will yield many students who can handle many-to-many comparisons as in COMPARE 6-11. The ratio work that you do with that set of lessons will give the background needed so those kinds of problems can be worked on at the next level.

LESSON FIVE: More Multiplication & Division

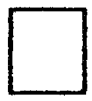
Introduction: This is to introduce children to the selections problems that lead to multiplication and division.

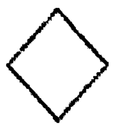
Selection 1: Materials to use on the overhead.

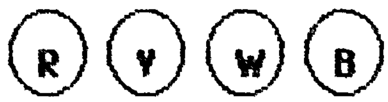
"Gloria has 3 dogs and 4 collars for these. In how many different ways can she match a dog with a collar?"

"Let's represent these. We'll let colored chips represent the collars and different shapes represent the dogs:

 is Fido

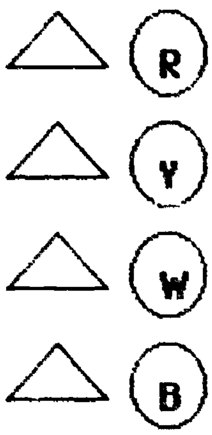
 is Spot

 is Rover

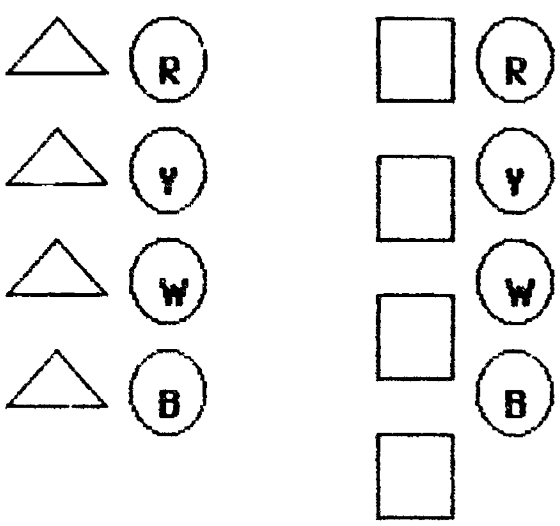


are the collars."

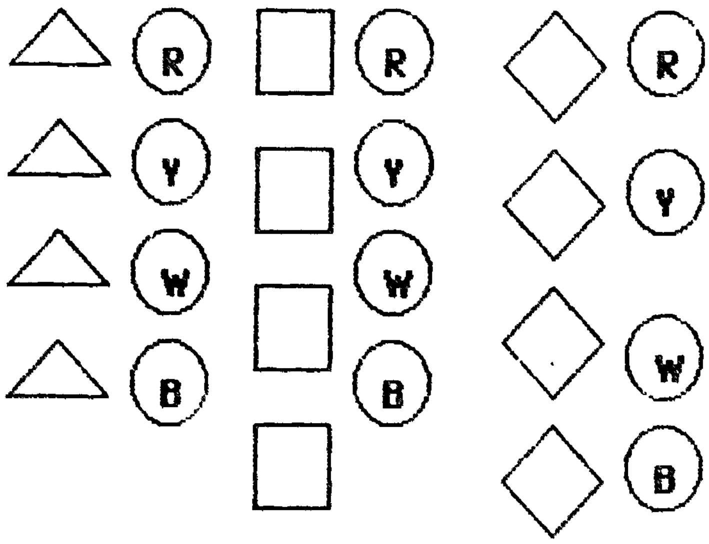
"How can we match the collars of different colors with the dogs.:



"Fido can wear a collar 4 different ways."



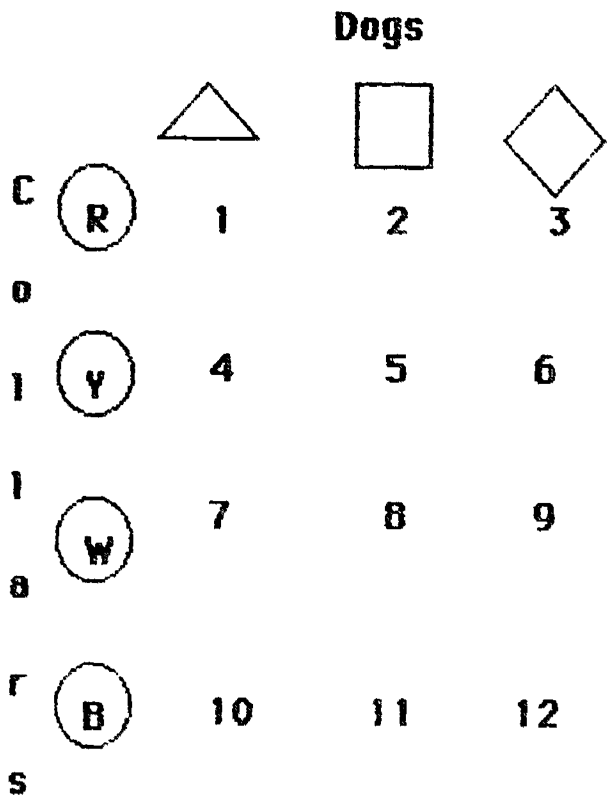
"Spot can wear one of these collars 4 different ways."

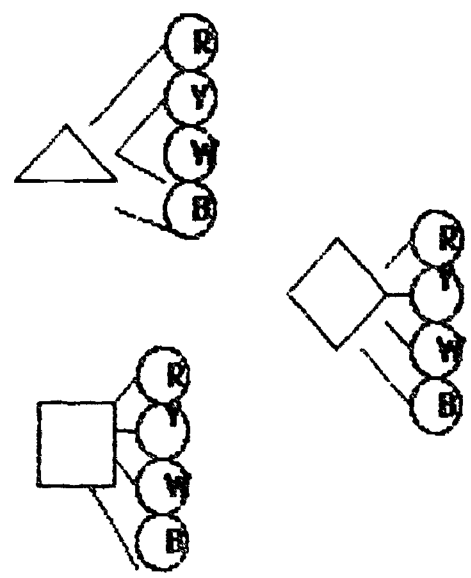


"Rover can wear one of the collars in 4 different ways, also."

"How many matchings do we have?"

"What are some ways that we could show this in a table or a diagram?"





Number Sentence: _____ = 3 x 4

Selection 2: In these the total number of pairs is known and the number of things possible for one part of the pair is known, so division is neded.

"Frank can make 15 outfits including a sweater and a pair of slacks. He has 3 sweaters to choose from. How many slacks does he have?"

Let's look at a collection of pairs of UNIFIX cubes and try to put them into an array by classifying them."

Have fifteen UNIFIX pairs:

5 with red followed by white, green, blue, brown or black;
5 with yellow followed by white, green, blue, brown or black;
5 with orange followed by white, green, blue, brown or black

These should be in random arrangement.

"How should we classify these in an arrangement of rows and columns?"

You should eventually get the following:

R W	R G	R Bl.	R Br.	R Bk.
Y W	Y G	Y Bl.	Y Br.	Y Bk.
O W	O G	O Bl.	O Br.	O Bk.

"The first UNIFIX in each pair is the same for every row and the second is the same for every column."

If we knew the 3 colors used first or the 4 colors used second, we could find the other one from the number of pairs. In this case: $12 \div 3 = 4$ or $3 = 12 \div 4$.

"You tell me what to do to work this problem."

"Tim has 15 hat and scarf combinations. He has 3 hats. How many scarves does he have?"

Activity: Give students some of the provided problems to work on. Allow the use of materials to represent objects in the problems. Have them write number sentences and circle the number that answers the question.

LESSON SIX: Two-step Problems

Background: These problems involve the application of one arithmetic operation followed by another. To do these, children MUST very thoroughly understand the situations that give rise to the four arithmetic operations.

Introduction: "Tom has 8 bags with 8 marbles in each bag. He has another bag with 26 marbles in it. How many marbles does he have in all?"

"Let's look at this one step at a time." "How many marbles are in all of the 8 bags?" 64.

"He has another bag with 26 marbles. What do we do with this 26?"

"How many marbles do we have in all?"

"Notice we multiplied first, then added something to that result."

Write: ☐ = 8 x 8 + 26

Do a second:

"Sally put 9 coins on each of 7 pages in her coin collection book. She had 77 other coins in a box. How many coins did she have in all?"

"What do we do first?"

"How many coins is that?"

"How do we get the number of coins IN ALL?"

"What is the number sentence?"

☐ = 9 x7 + 77

Activity: Pass out the activity sheets and monitor the work closely.

LESSON SEVEN: Non-Traditional Problems

Background: Problem solving does not all fall neatly into cases of adding, subtracting, multiplying and dividing, or combinations of these. Some problems require use of other strategies. These include:

Guessing and Checking
Making a picture
Making a table
Making a list
Finding a pattern(s)
But mostly, THINKING

Introduction: Work two or three problems that illustrate these. Some examples are:

"Sam has 7 coins, all nickels and dimes. Their total value is 50¢. What are the 7 coins?"

"Could all 7 be dimes? Why not?"

"Could all 7 be nickels? Why not?"

"What amounts could be in dimes?"

List these as given:

Dimes:
10¢
20¢
30¢
40$

"What amounts could be in nickels?"

Discuss why they must all end in "0".

List as given:

10¢

20¢
30¢
40¢

Make a table:

No. of Dimes	Amount	No. of Nickels	Amount
1	10¢	2	10¢
2	20¢	4	20¢
3	30¢	6	30¢
4	40¢	8	40¢

"How can we get 7 coins?" Circle with colored chalk.

"Which gives 50¢?"

So, we have 3 dimes and 4 nickels.

"The time is 7:35. What time will it be in one hour and twenty-five minuts?"

"What is the time are one hour?" 8:25

"What is 25 minutes added to this time?"

"So, it will be 8:50."

"Janis is thinking of two numbers. If she adds the two numbers, it is 17. The larger is 9 more than the smaller. What are the numbers?"

"What are some ways to do this?" Possibilities include looking at all 2 number combinations making 17. Another is looking at numbers 9 apart.

Possibility 1:
1. 16
2. 15
3. 14
4. 13
5. 12

Possibility 2:
1. 10
2. 11
3. 12
4. 13
5. 14

6. 11
7. 10
8. 9

6. 15
7. 16

"How many of these pairs have larger 9 more than smaller?

"How many of these add to 17?"

The result is clearly 4 and 13.

"How much do 30 eggs cost if eggs are 66¢ per dozen?"

"What must we do first?" Pick up on suggestion to see how many dozen in 30:

30 ÷ 12 = ☐

☐ = 2 1/2

"Now what must we do?"

"Let's summarize all we did in a number sentence."

30 ÷ 12 x 66¢ = ☐

"Could we have done this another way?" Make a table:

	Eggs	Cost
1	6	33
	12	66 What we know
2	18	99
3	24	132
4	30	156 What we need

Activity: The attached set of problems should be used as a source for giving problem solving work at least once a week.

Sometimes have the children work on these in pairs. Sometimes work with the whole group in analyzing and thinking through a problem. Sometimes have children list all of the different ways to work a given problem after having worked it.

LESSON EIGHT: Creating Problems

Introduction: Explain to the children that they will be given some information, but no question about it. They are to write as many questions about this that require a number as they can. Give one example:

> "The flag of the United States has 7 red stripes and 6 white stripes."

Try to get as many related questions as you can - for example:

> "There are how many more red stripes?"
> "How many stripes are on the flag?"
> "The number of white stripes is what fraction of all of the stripes?"

Activity: Use the problem sheets. Have children write questions about the information given.

ASSORTED PROBLEMS TO USE

1. As a problem of the day
2. For assignments
3. For pairs of children to work on, etc.

1. Bridgeman's is having an ice cream cone special. They have cake cones and sugar cones. They will let you choose from chocolate, vanilla or strawberry ice cream. You can add crushed nuts or marshmallow topping. How many different choices of an ice cream cone do you have?

2. In preparing for a marathon, Frank ran every day of the week for a total of 100 miles. He ran 13 miles on Saturday and 12 miles on Sunday. On the rest of the days he ran the same number of miles each day. How many miles did he run on the weekdays?

3. Several children in the class have dogs as pets. There are three times as many Collies as Spaniels. There are 6 more Spaniels than poodles. Three children have poodles. How many children have dogs?

4. June sold 28 paperbacks for 10¢ each at her mother's garage sale. She has 27 paperbacks left. She wants to get a total of $10.00. How much should she charge for each remaining paperback?

5. Dean used pattern block triangles to make large triangles. The largest triangle required 36 triangles. The side of this triangle is how many times as long as the side of a pattern block triangle.

6. Vicki, Shelley, Tom and Chris each have a dog. The dogs are Spaniel, Spitz, Terrier and Labrador. Vicki has the Spaniel; Shelley does not have the Spitz or the Labrador. Chris does not have the Labrador. Who has which dog?

7. Al's 3 stage rocket model is 120 centimeters long. With the first stage removed, it is 74 cm. With the second stage removed as well, it is 42 cm. long. Which stage is the longest? How much longer is it than each of the other two?

8. Linda paid for her records with $2.25 in quarters and dimes. She used 2 more quarters than dimes. What coins did she give the clerk?

9. Terri's mother made a cake in a pan 9 in. x 12 in. When it cooled, she frosted it and decided to put chocolate soldiers around the top edge of the cake. If she put the soldiers 3 inches apart, how many did she use?

10. Of the 30 players on the football team, all but 2 are going to the awards banquet. They will be seated 4 to a table. How many tables are needed?

11. Tabitha numbered the pages of her diary. It has 150 pages. How many times did she use the digit "4"?

12. Lynn, Iris and Sue found a boxful of marbles. They set aside half of them to use later. Each girl took 1/3 of the rest of the marbles? Lynn received 12 marbles. How many marbles were in the box?

13. Larry went smelting. Each time he dipped the net, he had 2 more smelt than the last time. How many smelt were in the net when he dipped it the tenth time?

14. The 4th grade class has 5 gerbils and 2 cages. In how many different ways could they be put in the cages, without any cage being empty?

15. Phyllis's mother went on a diet for 30 days. To make it more challenging, Phyllis's father said he would pay her $2.00 for every day she lost weight, but would charge her $1.50 for every day she gained weight. At the end of 30 days, Phyllis's father paid her $25. On how many days did Phyllis's mother lose weight?

16. Denny made a deal with his neighbor to mow his lawn. He said he would charge $4.00 for each of the first 5 times and $5.00 for each time more than 5. He mowed the lawn 12 times. How much did his neighbor pay Denny?

17. Shelley and Gregg went to the movie. It started at 6:30. Previews of coming attractions took 12 minutes. Commercial ads took 7 minutes. The film lasted 1 hour and 30 minutes. Their bus left at 8:30. How long did they have to catch the bus?

18. Tickets for the football game were numbered 500-1000. Each person having a ticket with only one 6 and no other digit smaller than 8 received a free banner. How many banners were given away?

19. Fritz tried to make the longest UNIFIX link in the class. He used 900 UNIFIX cubes. Each is 3/4 inch long. How long was his UNIFIX cube?

20. Red pencils are 3 for 89¢ and yellow pencils are 4 for 89¢. If Tom's bill was $5.34 and he bought more yellow pencils than red pencils, how many pencils did he buy?

21. 5 pound bags of potatoes cost 89¢ and 10 pound bags cost 1.59¢. How much cheaper is it to buy all 10 pound bags if you need 50 pounds of potatoes?

22. 132 people attended Rocky VIII. Adult tickets cost $4.00 and children's tickets are $2.50. How many children attended if the total receipts for tickets were $480?

23. Two-thirds of Mrs. Runions class are boys. To even things, 5 boys go to Mrs. Davern's class and 5 girls come to Mrs. Runions' class. Now only one-half of Mrs. Runions' class are boys. How many students are in Mrs. Runions' class?

24. Write 5 division problems that have an answer 7 R 3.

25. Gerte is now 7 inches taller than her brother Dean. She grew 3 inches last year while Dean grew 4 inches. A year ago Dean was 4 feet 3 inches. How tall is Gerte?

PROBLEMS TO USE ON ASSIGNMENTS FOR STUDENTS

These are also models of kinds of multiplication and division problems for you to use in writing more of these.

"Bridgeman's has 10 flavors of ice cream and 5 different toppings. How many different ice cream cones can be made?"

"I put 5 pictures on each of 4 bulletin boards. How many pictures did I use?"

"I have 6 bags of apples with 9 apples in each bag. How many apples do I have?"

If 7 boxes of crayons have 8 crayons in each box, how many crayons are there?"

"Bill has 9 pennies. His brother has 4 times as many pennies as Bill. How many pennies does his brother have?"

"Fred's marble bag has only 4 marbles in it. John's bag has 7 times as many marbles. How many marbles are in John's bag?"

"Jane has 18 crackers. She put them into piles of 6. How many piles did she have?"

"If 24 apples are put into 3 bags so that each bag has just as many apples, how many will be in each bag?"

"John caught 3 times as many perch as sunfish. He caught 15 perch. How many sunfish did he catch?"

"Bill's box has 18 washers and 6 nails. He has how many times as many washers as nails?"

"Paula has 24 different sweater and skirt outfits. She has 6 sweaters. How many skirts does she have?"

"How many 6-packs of pop do you have to have to buy so each one in a class of 54 can have one bottle?

"If 5 tires are sold with each car, how many tires in all have to be supplied with 8 cars?

"Jet planes have 4 engines on each plane How many engines are there on 8 of these planes?"

"A wren weighs 4 ounces. A crow weighs 9 times as much. How much does a crow weigh?"

"John has 6 shirts and 4 pairs of trousers. How many different outfits of a shirt and a pair of trousers can he wear?"

"Jean has 4 packages of jacks. There are 16 jacks in each package. How many jacks does she have?"

"If the hockey team plays 2 games a week for 3 months, how many games will it play?"

"A chess set has 4 castles. How many castles are in 9 sets?"

"John bought 5 bags of marbles. Each bag had 12 marbles. How many did he buy?"

"Jean has 6 baseball cards. Tom has 8 times as many. How many baseball cards does Tom have?"

"72 eggs are put into cartons with 12 spaces. How many cartons are used?"

"64 pencils are put into boxes with an equal number of pencils in each box. 8 boxes are used. How many pencils are in each box?"

"If 72 straws are put into bundles of 9 straws, how many bundles are used?"

"63 marbles are put into 7 bags so each bag has just as many marbles. How many marbles are in each bag?"

"If you have 54 peanuts and give 5 peanuts to each squirrel, how many squirrels will be fed?"

"A classroom has 56 hamsters in 8 cages. How many are in each cage if they are equally divided?"

"32 Boy Scouts were divided into 4 patrols Each patrol had just as many Scouts. How many boys were in each patrol?"

"Jer works at the deli after school. She put 48 cans of peaches in boxes with spaces for 8 cans. How many boxes did she fill?"

"If 72 apples are put into 8 bags with the same number in each bag, how many apples are in each bag?"

"How many 20 cent candy bars can you buy for one dollar?"

"How many weeks are there in 49 days?"

"What is the cost of six 8 cent pencils?"

"How many hours are there in six days?"

"How many apples costing 20¢ can you buy for 80¢?"

"In cleaning up the playground, 34 students worked on Thursday and 48 students worked on Friday. How many students worked on the two days?"

"The students picked up 432 cans and 172 bottles. How many more cans than bottles were picked up?"

"In starting a school garden, the sixth grade bought 7 packets of seed, 15 tomato plants and 12 pepper plants. How many things did they buy?"

"Janet picked 14 tomatoes after school. Her sister picked 9 more than Janet did. How many tomatoes did her sister pick?

"Terri had some beads for a necklace. When Elly gave her 15 more, she had enough to make a 34-bead necklace. How many beads did Terri have to start with?"

"Peter has 12 cassette tapes that play for 60 minutes each. How many minutes would it take to listen to all of the tapes?"

"A factory has 260 chairs. One store ordered 144 chairs. Another ordered 152. How many more chairs does the factory need to fill both orders?"

"A carton holds 15 screwdriver sets. Each set has 6 screwdrivers. How many screwdrivers will be sent out in 8 cartons?"

"If the school store has 120 pencils at the beginning of the week and 63 at the end of the week, how many pencils were sold during the week?"

"If the school store sold 29 green binders and 34 blue binders, how many binders were sold in all?"

"If the store sold 93 notebooks in April and 134 notebooks in May, how many fewer were sold in April?"

"The football team agreed to share the cost of the hamburgers and cakes equally. If these totalled $17.92 and 11 players shared the cost, how much did each pay?"

"If 45 seats are in 5 equal rows, how many seats are in each row?"

"How many cookies would each person get if a bag of 144 cookies was shared by 24 students?":

"How many oranges are in 6 crates that hold 86 each?"

LEVEL FOUR

NON-NUMERIC PROBLEM SOLVING

At this level children chould continue to work on problems that involve the use of Pattern Blocks and Tangrams.

Select materials from the following sources:

- Pattern Block Activities A
- Pattern Block Activities B
- Let's Pattern Block It
- Moving On with Pattern Blocks
- Pattern Blocks Games - Geometry
- Pattern Blocks Games - Logic
- Tangram Patterns
- Moving On with Tangrams
- TANGRAMATH

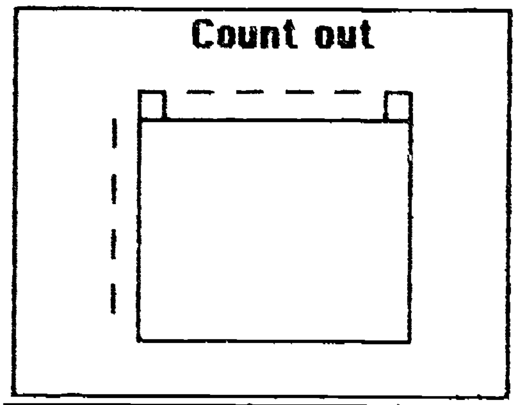
A few pages from these have been reproduced to illustrate activities appropriate for this level.

LEVEL FOUR

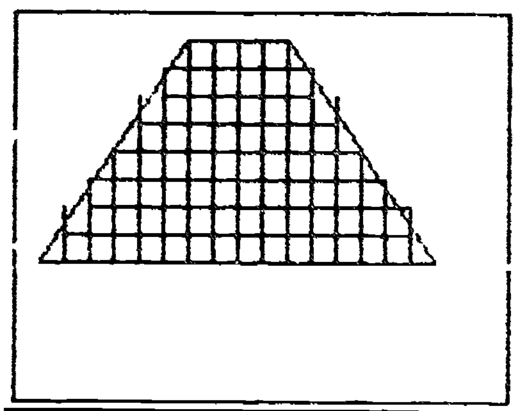
MEASUREMENT

LESSON ONE: Length

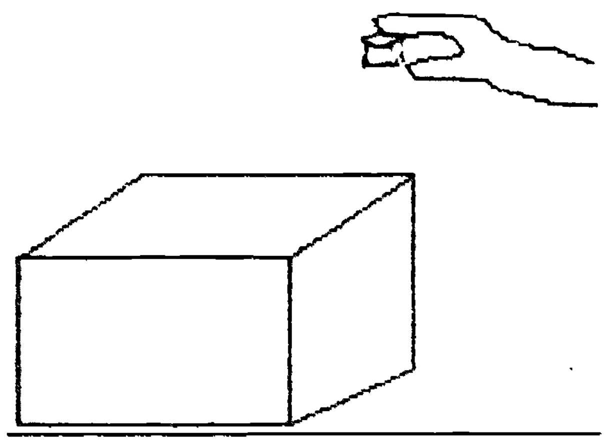
Introduction: Place a piece of cardboard on the overhead. Measure it with cuisenaire white rods. Review the idea of a standard unit to measure length.



Place a transparent geometric shape on the overhead. Measure its area as best you can with small squares of the same size. Review the use of an area unit to measure area.



Take a small box. Count out the cubes you use as you fill this box with cubes of the same size.



Review the use of volume units to measure volumes.

Ask the children what an appropriate unit would be to measure the length of the desk instead of the C white

Discuss the responses - the C orange, a longer string of paper clips, etc.

Ask the children what would be a better unit to measure the area of the top of the desk.

Discuss the responses - base ten hundreds, etc.

Emphasize why using larger units that are TEN times smaller units are easier to use

LESSON TWO: Standard Units

Introduction: Cuisenaire white rods are 1 cm. on a side. A cuisensaire orange rod is a decimeter (q0 cm.) in length. The end face of any cuisenaire is a square centimeter (sq. cm.) Cuisenaire rods can be used to yield metric measures of short lengths and distances, and small areas sq. cm. squares made from graph paper and cut out of light poster board to measure small areas. The cuisenaire white is a cubic centimeter (c.c. or cm 3) in volume.

Hold up a small open box:

"What is there about this box that can be measured?"

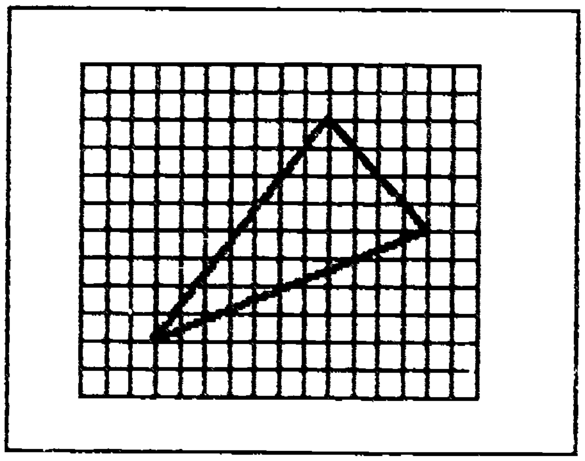
Discuss these responses. Then consider LENGTH, AREA and VOLUME separately as to how cuisenaire rods and graph paper squares could be used to measure each of these.

On the overhead projector, put a transparency with a line on it. Use a metric transparent ruler to measure the line in centimeters. Then use a foot-inch transparent ruler to measure the same line. Compare the two measurements (nearest centimeter and nearest inch.)

Assign a set of lengths for groups of three children to measure in centimeters. Have them record these and discuss the results as a large group.

<u>LESSON THREE</u>

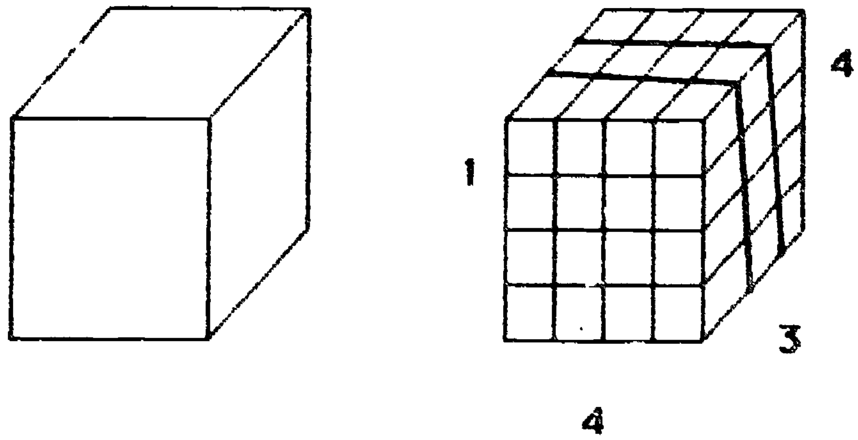
<u>Introduction:</u> Place a transparency of a geometric shape on a transparency of a cm2 graph paper. Have the children count the squares and estimate additional squares made up of pieces of the region inside the triangle:



Give each group of three children graph paper and several cut out geometric shapes. Have them (1) estimate the area, then (2) find the area in cm2 of each shape.

<u>LESSON FOUR:</u>

Give each group of children a block (such as in Geoblocks) or a box and MULTILINK cubes. Demonstrate making a Multilink "block" the same size as a given block:



Point out the THREE lengths in the block that determine the number of cubes. The block can be thought of as 4 layers with 12 blocks in each layer.

Then show an open box and ask the children if the same number of multilink cubes would FILL it as would be needed to build another one like it. Discuss.

Have the children make MULTILINK cube blocks as close to several given blocks as they can make.

LESSON FIVE

Children are to be organized into groups of three. Each group should have several blocks of different sizes and weights and several of the same kind of objects to use as weight units - Unifix cubes, washers, etc.

Introduction: Put a balance beam so all children can see it. An OHAUS bucket beam is good. Put a block in one bucket. Explain you will put (washers) in the other bucket until the beam balances. Do this, counting the blocks as you do it. "This block weighs __________ washers."

Activity: Have the children weigh several geoblocks, using whatever "units" of weight have been given to them - washers, marbles, Unifix cubes, Cuisenaire whites, etc. The recording forms should be completed and discussed later as to the variation in results when different units are used for the same block.

LESSON SIX

Introduction: Show a meter stick to the children. Point out the subdivisions on it. Emphasize that the units are all related by TEN - 10 millimeters in one centimeter, 10 centimeters in one decimeter, 10 decimeters in one meter.

Compare the meter stick with a yardstick so the children can see how close they are in length and that either can be used for measuring the same lengths with close to the same results.

Emphasize the use of "centi" by referring to money.

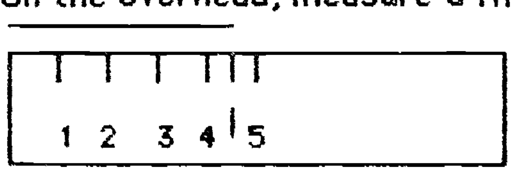
"How many CENTS are in a dollar?"

"How many CENTS are in a dime?"

"How many dimes are in a dollar?"

"How many CENTimeters are in a meter?"

On the overhead, measure a line using a transparent decimeter ruler:



closest to 4☐ MM. and to 5 MM.

Discuss measuring to the NEAREST unit on the ruler.

Activity: Have groups of 3 children measure several lengths in the room to the nearest CM. and to the nearest MM. If not enough meter sticks are available, have them use string, put labels on the string lengths, then use a meter stick to measure the string lengths.

LEVEL FOUR

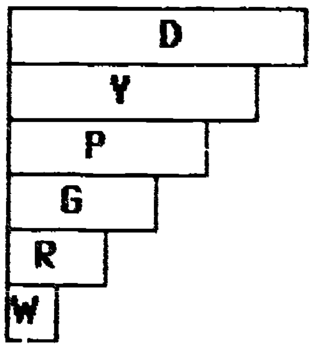
FRACTIONS: REVIEW

Background: Children will have had an introduction to fraction operations with Cuisenaire Rods in Level Three. LESSON ONE is to review that work.

LESSON ONE: Sixths Family

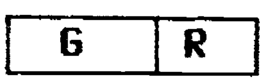
Introduction:

Place the SIXTHS family on the overhead projector:



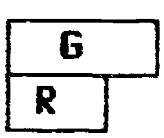
"The D rod is one."
"What fraction does W represent?"
"What fraction does R represent?"
"What fraction does G represent?"
"What fraction does P represent?"
"What fraction does Y represent?"
"How many 1/6ths in 1/2?"
"How many 1/6ths in 1/3?"
"How many 1/6ths in 2/3?"
"How many 1/6ths in ONE?"
"What are 2 names for the R rod?"
"What are 2 names for the G rod?"
"What are 2 names for the " rod?"

Join G and R as Shown:



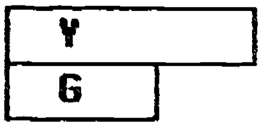
"What fraction is G rod?"
"What fraction is R rod?"
"What fraction is the JOINING of G & R?"

Compare G and R as shown:



"What is the difference between G and R?"
"What fraction is 1/2- 1/3?"

Compare Y and G:



"What is the difference between Y and G?"
"What fraction is 5/6 - 1/2?"

Write: 5/6 - 1/2 = 1/3
5/6 - 3/6 = 2/6 = 1/3

Why is it easier to use the SIXTHS name when using this family of rods?"

Activity: Give students D, Y, P, G, R and W rods and the worksheets. Have them put missing rods in first. Then have them work the fraction exercises.

LESSON TWO: Comparing Fractions

Introduction

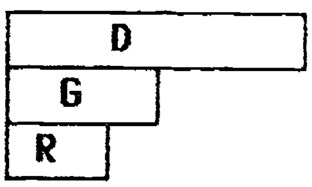
Place a D rod on the overhead projector.

"This is ONE."

Place a G rod under it.

"What fraction is this?"

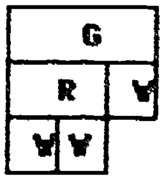
Place an R rod under it.



"What fraction is this?"

"Which fraction is larger?"
"How many times as big?"
"How many R rods can be made from a G rod?"

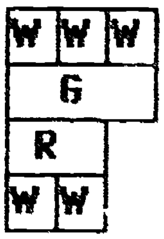
(Be sure here to put the W rod difference below the R rod to see the difference makes half of an R rod.)



Write: 1/2 1/3 = 1 1/2

"The R rod is what part of the G rod?"

You probably will have to put the following on the overhead.



"2 W is what part of 3 W?"

Write 1/3 1/2 = 2/3

"You expect to get only part of a 1/2 because IT IS BIGGER THAN 1/3!"

Remind them that if you try to get smaller things from a bigger one, you get one or more. But if you try to get something bigger from something smaller you get only a part of it!

Do a second example using Y and P to get 5/6 2/3 = 2 1/2
2/3 5/6 = 4/5

<u>Activity</u>: Pass out the worksheets for children to do using the rods. This work must be monitored since some children will need help in seeing how the W rod equivalents can be used to compare them.

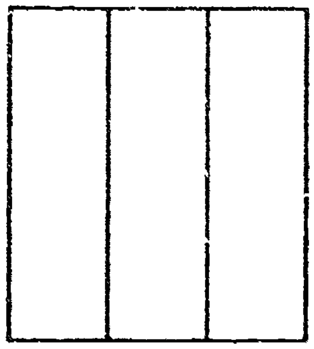
<u>LESSON THREE:</u>

<u>Introduction</u>

Once children have mastered joining fractions and comparing them as to which are larger, the difference between them and how many (or part.) of one can be made from another, they should take parts of fractions.

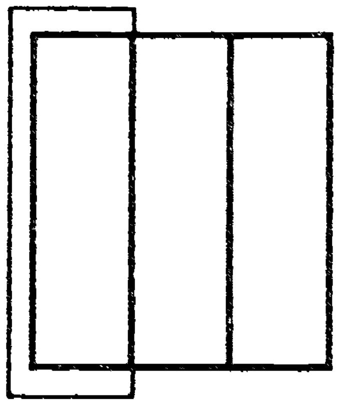
Make transparency squares of different colors from the masters provided. Use an opaque piece of cardboard to cover parts up.

Show a 1/3 square:



"Each part is what fraction?"

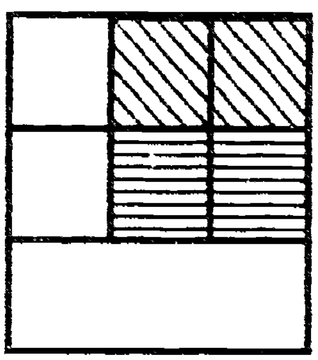
Cover one third:



"What part is now showing?"

"How would I take one-half of this?"

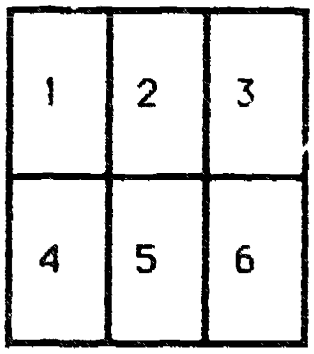
Use a second "half" square to do this by laying it over the 2/3.



"How many parts do you see?'

"What fraction of the whole square are each of these parts?"

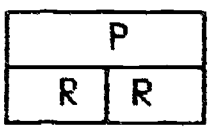
Draw a picture if you have to



"So we have two of the sixths in half of the two thirds."

Write this as: 1/2 x 2/3 = 2/6 = 1/3

Remind them of the Cuisenaire rods: P = 2/3:



"Each R is half of P or 1/3 is 1/2 of 2/3"

"We write: 1/2 x 2/3 = 1/3"

We get fractions by dividing ONE into parts in different ways. Thirds from 3 equal parts, fourths from 4 equal parts, etc."

We divide a fraction into equal parts to take a part of it in the same way."

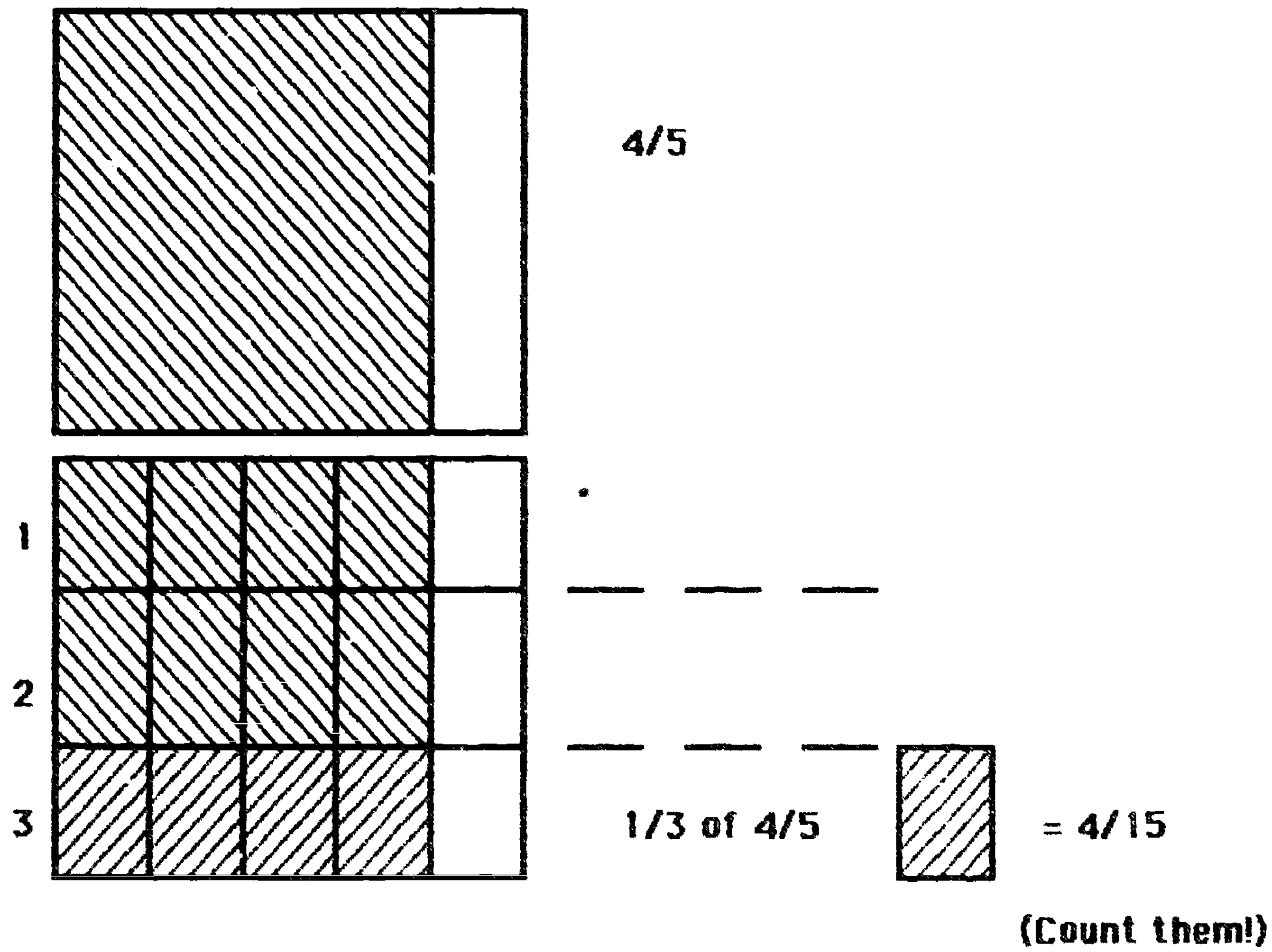
1/2 x 2/3

Multiplying these gives the number of new parts.

1/2 x 2/3

Multiplying these gives the number of these parts we have. So multiplying fractions is just taking parts (or multiples) of other fractions.

1/3 x 4/5 means taking one of the 3 equal parts 4/5 has been divided up into:

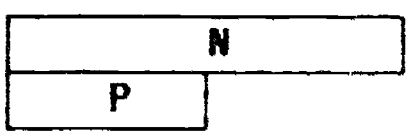


<u>LESSON FOUR:</u>

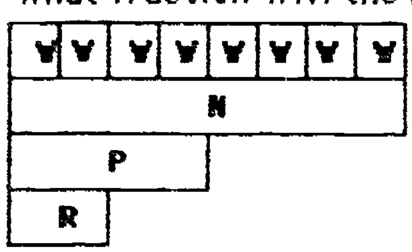
<u>Introduction</u>

Tell the children the N rod is now to be ONE. Put this rod on the overhead:

"What rod will be 1/2?" Put the P rod under N:



"What fraction will the R rod be?" Put the following rods on the overhead:



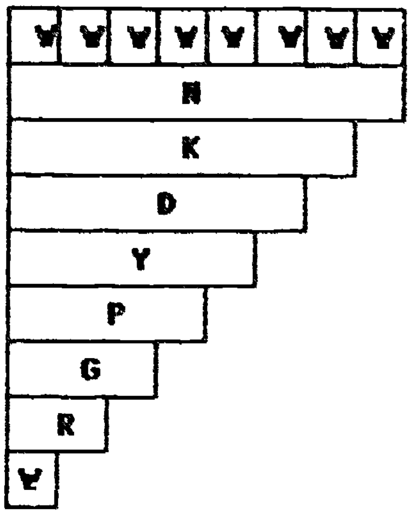
"What fraction is the W rod?"

"How many EIGHTHS in 1/2?"

"How many EIGHTHS in 1/4?"

Write: 1/2 = 4/8 1/4 = 2/8

Put the full array of rods on the overhead.



Have the children identify each rod as to its value in EIGHTS. Repeat the activities in the previous lessons. Join rods and write the number sentences. Compare rods by:

Finding the differences between 2 rods
Finding what part a shorter rod is of a longer rod
Finding how many shorter rods can be made from a longer rod
Write number sentences to show all of these

Activity: Have children use rods to do the worksheets.

LESSON FIVE: More Than Two Fractions

Introduction

In the family of eighths, join three rods together such as shown:

G	R	Y

3/8 + 1/4 + 5/8 = 10/8
(2/8)

Show this by:

N		R
G	R	Y

So: 10/8 = 1 + 2/8 or 1 1/4

Do a few of these and then pass out the worksheets. Have the children use rods to work the extended joining worksheets.

<u>LESSON FIVE: Twelfths</u>

<u>Background:</u> This family is the most versatile of the fraction families since it incorporates 1/2, 1/3, 1/4, 1/6 from the earlier families and adds 1/12. The same emphasis should be placed on combining, finding the difference, finding how many of the smaller in the larger, how many of the larger in the smaller, parts of fractions, multiples of fractions as before.

<u>Introduction:</u> Children should have Cuisenaire Rods to work with in pairs. Place the following on the overhead projector:

O	R

"This length is ONE."

"Make this train of rods."

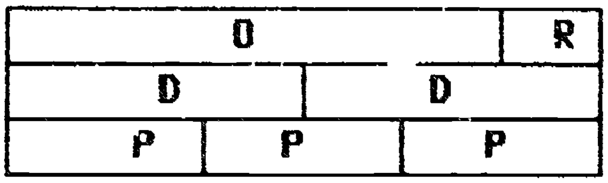
"Find the rod that is ONE HALF."

Place the following on the overhead:

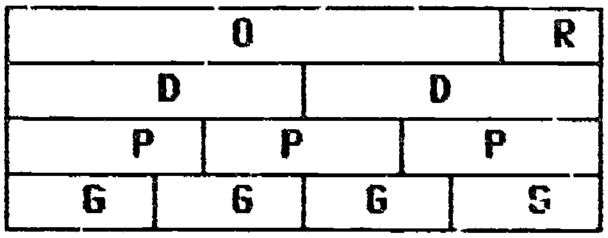
O		R
D	D	

"Find the rod that is ONE THIRD."

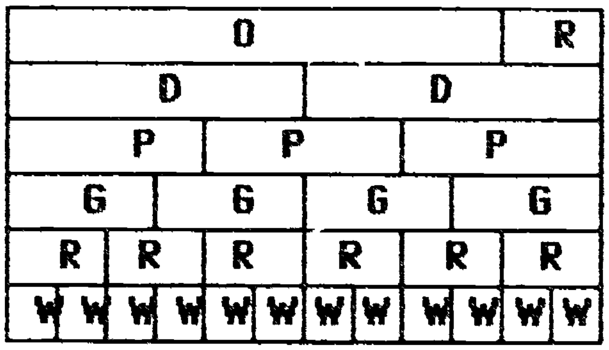
Place the following on the overhead projector:



"Find the rod that is ONE FOURTH." Place the following on the overhead projector:



"Find the rod that is ONE SIXTH." Place the following on the overhead.



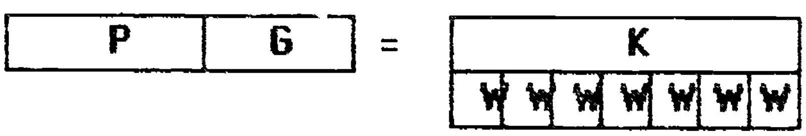
"What fraction is the W rod?"

Write: D = 1/2, P = 1/3, G = 1/4, R = 1/6, W = 1/12

"We are going to combine some of these rods."

Write: 1/3 + 1/4 = ☐

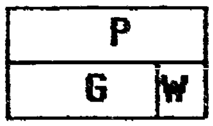
"Use rods to find the fraction in the box." Place these on the overhead projector:



= 7/12

"Now find the difference between 1/3 and 1/4."

Then place these rods on the overhead:



"The difference is 1 1/2 so 1/3 - 1/4 = 1/12."

"How many twelfths in 1/3?"

"How many twelfths in 1/4?"

"So we could write 4/12- 3/12 = 1/12 in place of 1/3 - 1/4 = 1/12."

"How many 1/4's can be made from 1/3?"

"We write: 1/3   1/4 = 1 1/3."

"Can we make a whole 1/3 from 1/4?"

"What part of a 1/3 can we get? Use the W rods to find this."

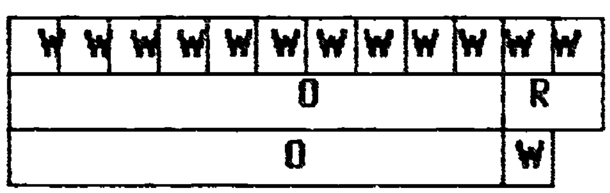
"We write: 1/4   1/3 = 3/4."

<u>Activity</u>: Have the children work in pairs on the activity sheets, using C rods. Carefully monitor this activity, asking frequent questions of the children about what they are doing.

<u>LESSON SIX: More Twelfths</u>

<u>Background</u>: Children should get extensive practice with fractions in the family of twelfths since it includes most of the everyday fractions we use. If they understand these operations, they will use them with EIGHTHS on the ruler and in other contexts.

<u>Introduction</u>: Place the twelfth ONE on the overhead projector along with O + W.



"What fraction is the length O + W? Use the W TWELFTHS to find this."

"What fraction is O?"

Write: O + W = 11/12
O = 10/12

"Find the fractions shown by E, N, K, and Y."

Then have the children help you complete:

O + W = 11/12
O = 10/12
E = 9/12
N = 8/12
K = 7/12
Y = 5/12

Point out how since R measures O and N, these can be written as 5/6 and 4/6. But P measures N also, so 8/12 = 4/6 = 2/3, 10/12 = 5/6 and 9/12 = 3/4. Remind the children they can use any name for a fraction that has more than one name.

<u>Combining:</u>

Put these rods on the overhead:

G	P	R

1/4 + 1/3 + 1/6

then put

<table>
<tr><td>G</td><td>P</td><td>R</td></tr>
<tr><td colspan="3">E</td></tr>
</table>

"The E rod is 9/12 or 3/4, so 1/4 + 1/3 + 1/6 = 3/4."

Comparing by difference. Put these rods on the overhead:

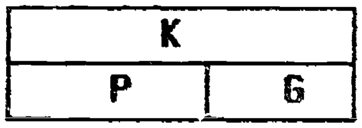
<table>
<tr><td colspan="2">K</td></tr>
<tr><td>P</td><td></td></tr>
</table>

"The K rod is what fraction?":

"The P rod is what fraction?"

Write: 7/12 1/3

"What is the difference between these?" Place:



and write: 7/12 - 1/3 = 1/4

"The G rod is 1/4 in the TWELFTHS family."

Finding Smaller in Larger:

"How many P rods could be made from the K rod?"

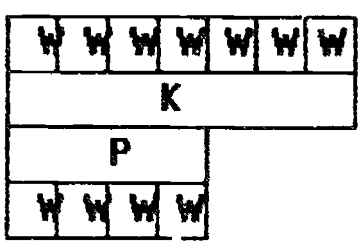
"One plus what part of another? Use W rods to see what part of a P rod a G rod is."

Write: 7/12   1/3 = 1 3/4

"How many K rods can you get from a P rod?"

"What part of a K rod, is a P rod?"

Use W rods to find this. Place on the overhead:



Write 1/3   7/12 = 4/7.

"Take the E rod."

"Find the rod that is 2/3 of this."

Write: 2/3 x 3/4 = 1/2

"3/4 of 2/3 is the same as 2/3 of 3/4."

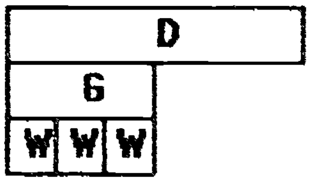
<u>Activity:</u> Give pairs of children rods to use to complete the activity sheets.

<u>LESSON SEVEN: Finding the ONE rod</u>

<u>Background:</u> Finding a common denominator for fractions is the clue to operating with them. A common denominator is a common measuring unit. A

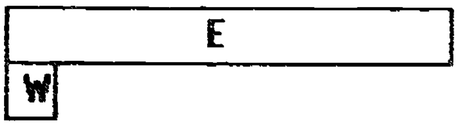
certain common measuring unit in Cuisenaire rods is the W rod. However, others are also; R measures P,D,N and O. G measures D and N.

<u>Introduction:</u> "Consider 1/2 and 1/6. Can we measure 1/2 with SIXTHS?"



"We can measure 1/3 three times with sixths so this is a common denominator. By renaming 1/2 and 3/6, we have a common name for the two. Now we can join them, compare them, see how many of one from the other, etc."

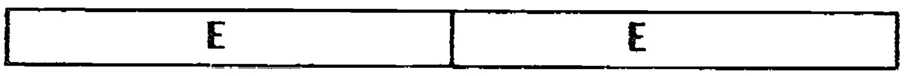
"Consider 1/2 and 1/9. If we use E as ONE, W is 1/9:



Try to find a rod that is one half of E."

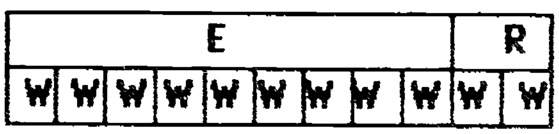
"There is no such rod so we cannot name 1/2 using NINTHS."

"What can we do to E so that we can always find 1/2 of it?"



"How many W rods to make this length?"

"How many red rods make this length?" R = 1/9 E = 1/2



11 W = 11/18 so 1/2 + 1/9 = 11/18

"We had to find a new ONE rod so both fractions would be in the same family."

"What is an easy way to see the units needed in the ONE rod?" 2 x 9 would give EIGHTEENTHS, the unit needed.

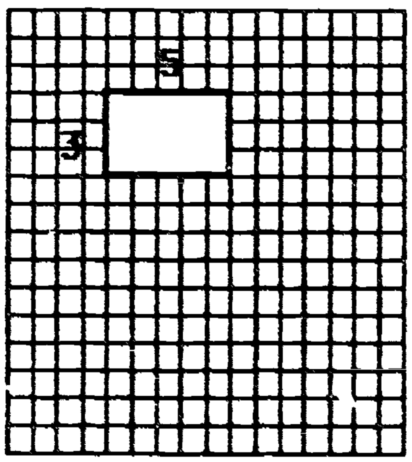
<u>Activity:</u> Give each pair of children a set of rods to work on the activity sheets. Keep reminding the children that W measures ALL of the rods and all fractions can be thought of in terms of W.

LESSON EIGHT: Multiplying the Easy Way

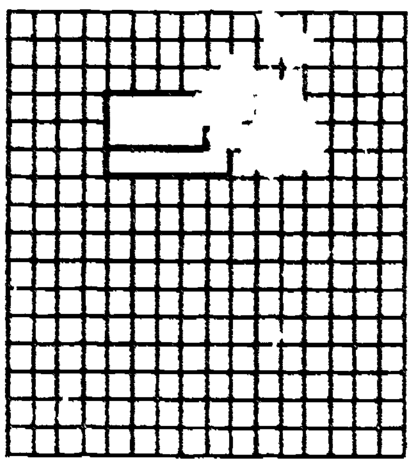
Background: After the initial exposure to multiplying fractions using colored transparencies, introduce using a diagram to show the multiplication. This reminds the children of the simply "multiply numerators - multiply denominators" process.

Introduction: Write 2/3 x 4/6 on the board.

Refer to the denominators of 3 and 5. On overhead transparency graph paper, outline a 3 x 5 rectangle:



"How many units in this rectangle?" Now outline the 2 x 3 rectangle inside this:



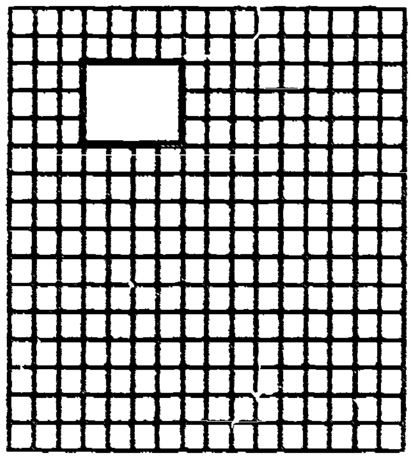
"The rectangle inside has how many units?"

"The product of 2/3 x 4/5 = 8/15.

Do a second problem.

1 1/3 x 3/4

Ask the children how many THIRDS are in the "1". Rewrite this as 3/3 + 1/3 = 4/3 so the problem is: 4/3 x 3/4:



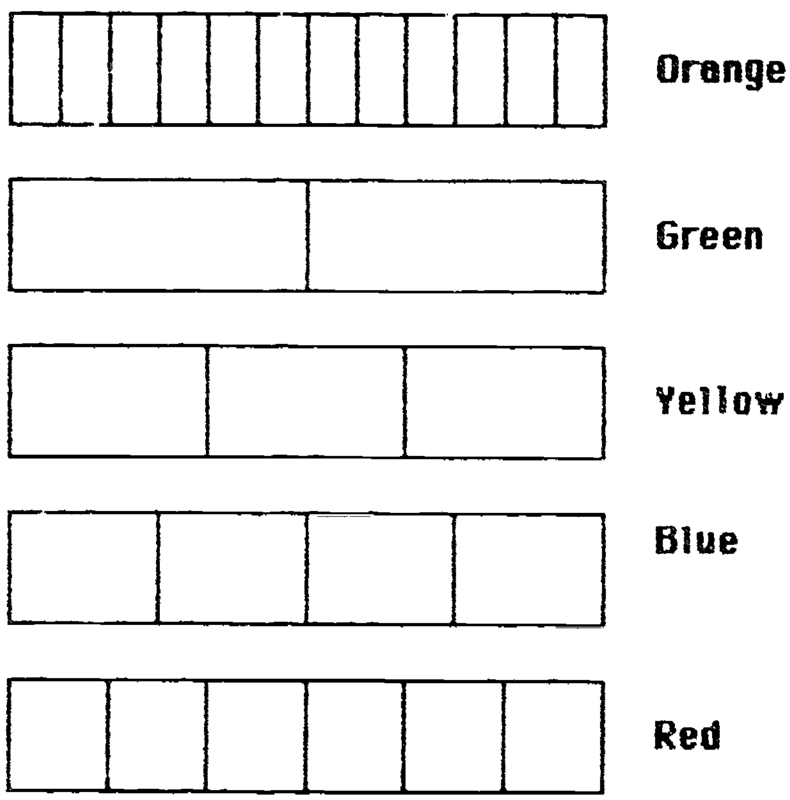
The two rectangles are the same so 12/12 = 1.

Activity: Have the children work in pairs to find the products for exercises on the activity sheet.

LESSON NINE: Introduction to Fraction Bars

Background: Fraction bars are a different model for fractions that make it easier to work with mixed numbers and to play some games involving fractions.

Introduction: Use overhead transparency models of fraction bars. Array all of these "zero" fraction bars on the overhead as shown.



Ask the children to identify the number of equal parts of each bar in turn.

Remind them that this is the TWELFTHS family, but that only the orange bar has the smallest common measuring unit.

Activity: Group children into fours. Each group should have half of one set of fraction bars, so there is ONE of each kind of bar. This is 13 orange bars, 7 red bars, 5 blue bars, 4 yellow bars and 3 green bars = 32 bars. Give each group the instruction sheet.

LESSON TEN: War

Background: This is based on the card game. The purpose is to get children to compare fractions.

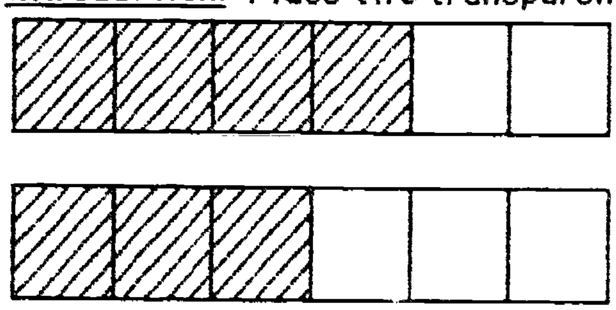
PLAYING THE GAME: Each pair of children should have SIXTEEN different bars, EIGHT per child. Each player puts the stack of bars face down. They each flip over the top bar of the stack and compare. The player with the largest area shaded (the largest fraction) wins both bars. If two bars are equally shaded, e.g, 6 of the orange and 2 of the blue, each player flips over a second bar. The winner takes all bars. When stacks are depleted the bars in the possession of each child are re-stacked and the game continues until one or the other had won all of the bars.

A related game: A pair of players is given SIXTEEN bars. These are placed in a stack face down. The top three are turned face up on the table. The first player turns over the top bar of the stack. If it matches in SHADED AREA one of the bars facing up, the player wins both bars and gets another turn. If it doesn't match any, it is placed face up with the others and the turn passes to the other player. When the stack is gone, the player with the most bars wins.

Similar games to find equivalent fractions and to compare fractions can be found in the FRACTION BARS INTRODUCTORY card set. This is available from almost any publisher distributing manipulative materials.

LESSON TEN: Dividing

Introduction: Place two transparent fraction bars as shown:



"How many of the 1/2 can be made from the 2/3?"

The children should see that ONE + ONE THIRD of another can be made. The extra shaded part is clearly one of the threes in the ONE HALF. Discuss this if necessary. Write: 2/3   1/2 = 1 1/3

"How many of the 2/3 can be made from the 1/2?"

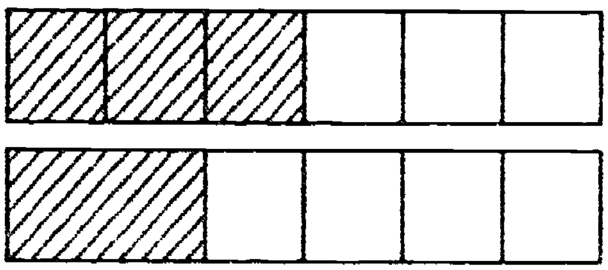
Children should be able to see that only 3/4 of one can be made since the parts are there to count. Discuss if necessary.

Activity: Give pairs of children all of the bars of a given color, different colors for each pair. Have them use these to do the worksheet. They are to find all of the ways smaller fractions can be made from larger and vice ..

LESSON ELEVEN: Adding and Subtracting Fractions

Introduction: This lesson will give practice in adding and subtracting fractions. Each pair of children should have all of the bars of a single color.

Put a pair of transparent fraction bars on the overhead projector:



"Which of these has the greater amount of shaded area?"

"What fraction is shown by the top bar?"

"What fraction is shown by the bottom bar?"

"What fraction is the DIFFERENCE between the shaded area?"

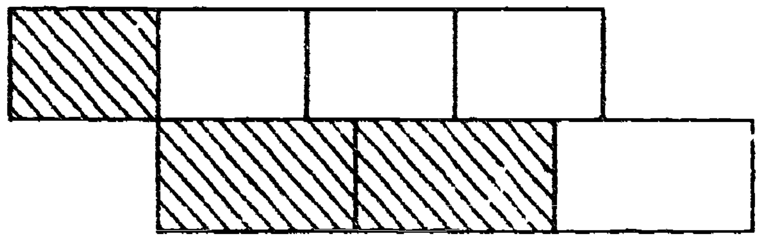
"What is the SUM of the shaded areas?"

Write: 1/2 - 1/3 = 1/6
1/2 + 1/3 = 5/6

Activity: Have the children working in pairs complete the worksheet using the bars they have. Repeat this so that each pair of children have a chance to use ALL colors of bars.

LESSON TWELVE: Common Denominators

<u>Introduction:</u> Children should have a half set of fraction bars of each pair. Place the transparency bars on the overhead:

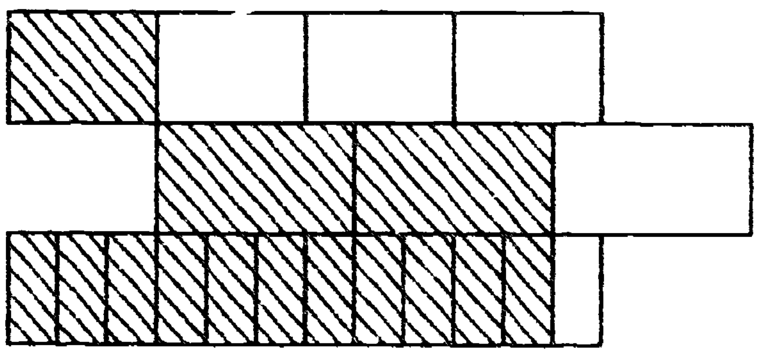


"Is there a bar that has parts that will measure both of these?"

"Match the shaded part of a bar of that color against the joined parts of these bars."

"What fraction results?"

Put these up:



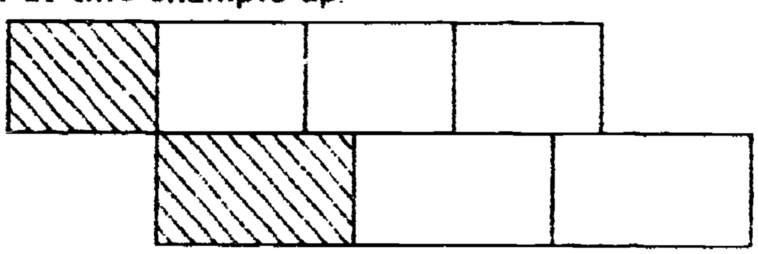
Write 1/4 + 2/3 = 11/12

Emphasize the fact that TWELFTHS measure FOURTHS and THIRDS; so both can be described in TWELFTHS.

1/4 = 3/12 AND 2/3 = 8/12

SO 1/4 + 2/3 = 3/12 + 8/12 = 11/12

Put this example up:



The SIXTHS measure the THIRDS so both can be shown as SIXTHS.

1/6 = 1/6   1/3 = 2/3

1/6 + 2/6 = 3/6 (1/2)

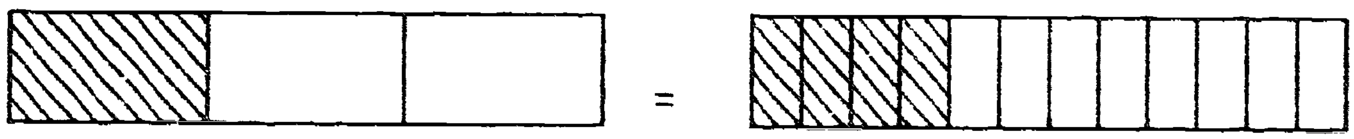
1. If the larger denominator can be EVENLY divided by the smaller denominator, name both using the larger:

1/6 + 1/3 - USE SIXTHS

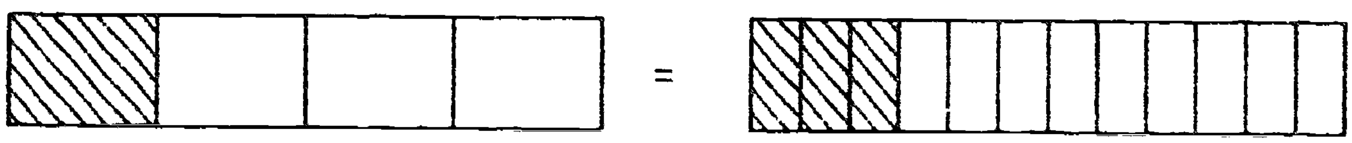
2. If not, multiply the denominators together to get A common measure:

1/3 + 1/4 = 4/12 + 3/12 = 7/12

Remind the children that they can find two bars of the SAME COLOR that have the same shaded area of the two given bars and add or subtract or divide using these:



1/3 = 4/12



1/4 = 3/12

So 1/3 + 1/4 = 4/12 + 3/17 = 7/12 like

4 apples + 3 apples = 7 apples
or
4 UNIFIX = 3 UNIFIX = 7 UNIFIX

The key is giving the fractions the same NAME using the same measuring unit.

<u>Activity</u>: Have pairs of children use fraction bars to complete the worksheet. Keep reminding them of the need to find the common denominator (NAMER!).

Background: Children must understand "equal parts of a whole" before understanding decimals. They may have been exposed to rotely "reading" decimals on calculators or in other places, but it is unlikely they have integrated this into their understanding of place value representation.

There are several models that can be used to introduce decimals:

1. Cuisenaire rods when O = ONE. The rest of the rods represent numbers of TENTHS.

2. Base Ten blocks: These have the advantage of "seeing" the ten equal parts on the TENS pieces or the hundred equal parts on the HUNDREDS piece.

3. Ten-part fraction bars: This model is used in this introductory lesson.

These will have to be made since they are not available commercially. Use the template provided to make these from a heavyweight cardboard or light posterboard.

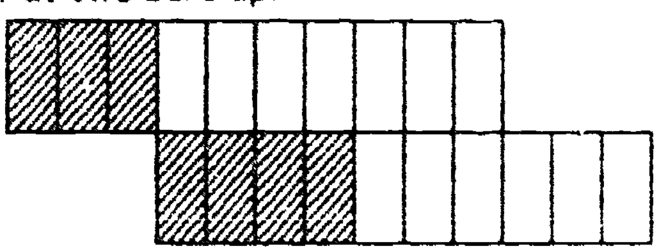
Introduction: Use transparency models of the bars. Children should be working in pairs with a set of bars. Put one bar on the overhead:

"What fraction does this represent?"

"We write TENTHS as . , so this three tenths is written as .3

Write: 3/10 = .3

Put two bars up:



"What is the result of joining these?"

Write .3 + .4 = .7

"If we have more than ten TENTHS or ONE, we write "the whole number to the left of the decimal point."

Write: 3.7

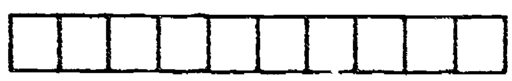
"This is three AND seven tenths."

<u>Activity</u>: Pairs of childre should have fraction bars and worksheets. Move around the groups so that you can observe proper placement of the decimal point.

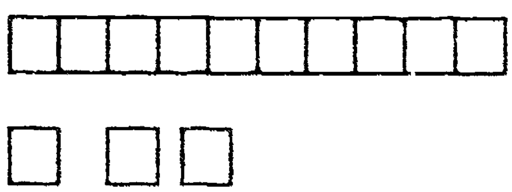
<u>LESSON TWO: Base Ten Pieces</u>

<u>Background:</u> Teaching decimals has two main goals: (1) relate to fractions as a particular kind of fraction, and (2) relate to place value in base ten. Both concepts must be well understood by children before decimals are understood, not merely manipulated with the use of rules.

<u>Introduction:</u> Put a transparency model of a base ten TEN on the overhead projector. Children should have base ten TENS and ONES. Tell the children the length divided into TEN parts is now going to be ONE and each part is to be ONE TENTH.



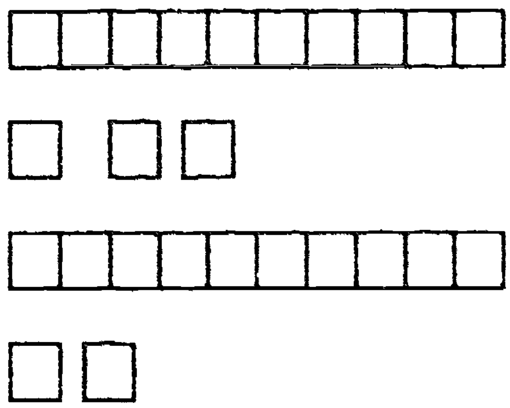
"Make ONE AND THREE TENTHS with the blocks."



Write: 1.3 alongside the blocks.

Remind the children that regrouping the small cubes that are now showing TENTHS gives ONES.

Add this to the overhead:



"What is the result of JOINING these?" (TWO and FIVE TENTHS)

Write:

$$\begin{array}{r} 1.3 \\ \underline{+1.2} \\ 2.5 \end{array}$$

"What is the DIFFERENCE between these?" (ONE TENTH)

Write:

$$\begin{array}{r} 1.3 \\ \underline{-1.2} \\ 0.1 \end{array}$$

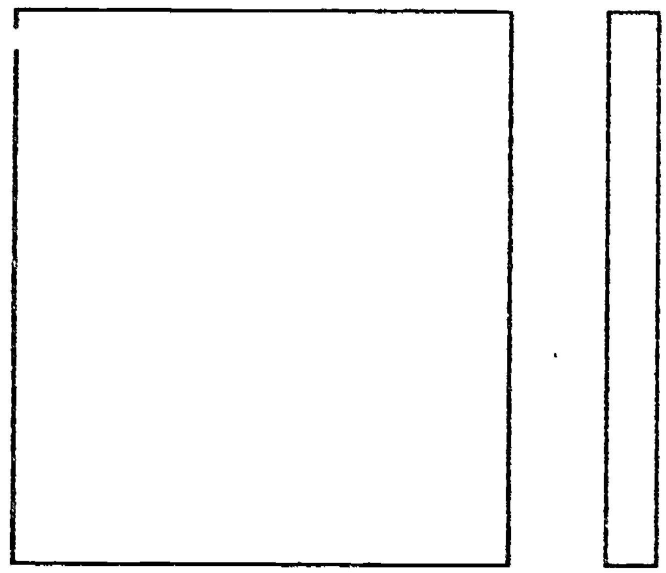
Do a couple more examples of joining and finding the difference.

<u>Activity:</u> Have the children work in pairs on the worksheets, using BASE TEN BLOCKS.

<u>LESSON THREE: Hundredths</u>

<u>Background:</u> When children are proficient at adding and subracting decimals and understand tenths, introduce hundredths.

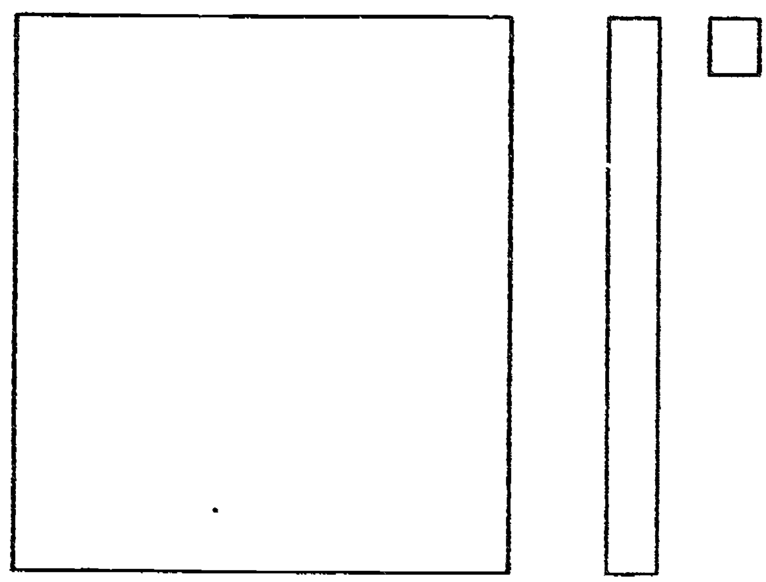
<u>Introduction:</u> Put a hundreds square (unmarked) and a Cuisenaire Orange rod beside it:



"How many of these orange longs will it take to build the square?" (TEN)

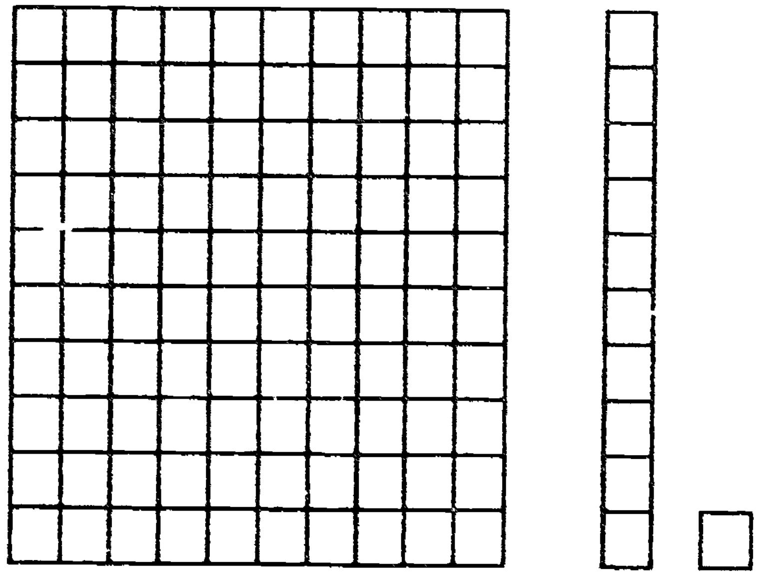
"If the square is ONE, what fraction will each long be?" (ONE TENTH)

Show a Cuisenaire white "unit.":



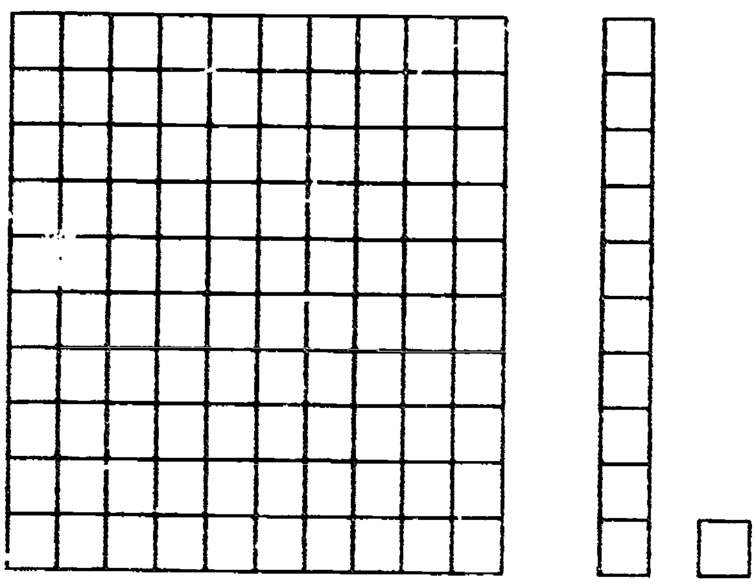
"How many of these will it take to build the ONE?" (ONE HUNDRED)

Replace the blank square by a hundred's square and a ten's long:



"This is ONE. It has one hundred of these (point to the small square), so the long is ONE TENTH and this is ONE HUNDREDTH."

Write below each piece as shown:



1.0 0.1 0.10

Write: Ten	TENTHS	=	ONE
One hundred	HUNDREDTHS	=	ONE
Ten	HUNDREDTHS	=	one TENTH

10 x .1 = 1.0 10 x .01 = 0.1
100 x .01 = 1.0

Activity:

Pass out base ten pieces and worksheets. Have the children work in pairs on the worksheets.

LESSON FOUR: Numeration

Background: Reading numerals accurately is necessary for recognizing number size, reading calculator output, reinforcing place value understanding and for learning to use scientific notation. Practice in reading and writing numerals should be given until the group as a whole develops this to a high level of proficiency.

Introduction: Write a numeral, e.g. 483 on the chalkboard:

"Read this in words." (four hundred eighty three - do not permit "and" after the hundred!!!!)

"What digit is in the tens place?" (8)

"How many tens are in the number?" (forty eight - elaborate on this so they see why the number has FORTY EIGHT TENS - Differentiate between this and the digit in the tens PLACE)

"How many tens are in the four hundreds?" (forty)

Write: $483 = 4 \times 100 + 8 \times 10 + 3$

Point out the 4 in the HUNDREDS PLACE, the 8 in the TENS PLACE, the 3 in the UNITS PLACE.

Write: 4.83 on the chalkboard

"What does the 4 count?" (ONES)

"What does the 8 count?" (TENTHS)

"What does the 3 count?" (HUNDREDTHS)

"How many HUNDREDTHS are in the eight TENTHS?" (EIGHTY)

"How many HUNDREDTHS are in the number?" (eighty-three)

Remind the children that TH means "part of"

Contrast HUNDRED and HUNDREDTHS

"One hundred has how many ones?"
"One hundred has how many TENTHS?"
"One hundred has how many HUNDREDTHS?"

<u>Activity:</u> Pairs of children should have base ten blocks and worksheets. Monitor closely for:

1. Correct placement of the decimal point
2. Recognition of TEN TENTHS = ONE
   TEN HUNDREDTHS = TENTHS, etc.

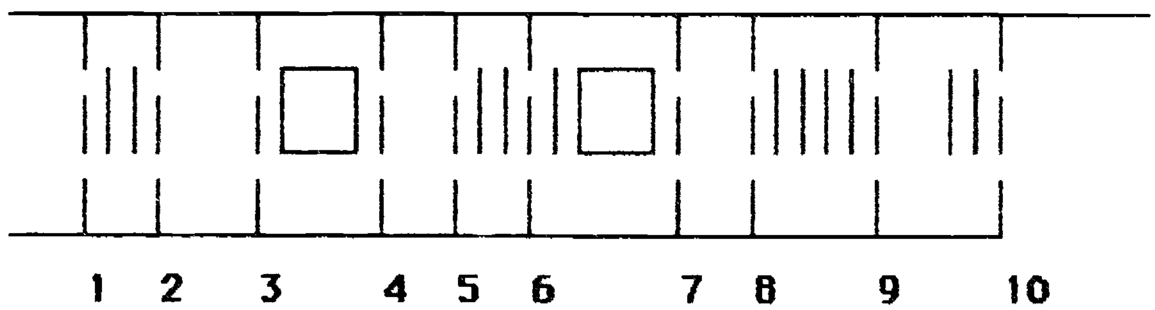
LEVEL FOUR

NUMERATION

LESSON ONE

Background: This lesson is a review of mentally counting place value materials.

Introduction: Children should have the recording form. Use overhead transparency base ten materials. Place an array of base ten pieces on the overhead projector. Cover with a cardboard. Gradually slide this cover left to right to Position 1, then Position 2, etc., to Position 10. Children are to record the amount visible on their recording forms:



Cover again and reverse the order of revealing the pieces - Position 10, then Position 9 to Position 1.

Repeat periodically with a different arrangement of pieces for a different experience in mentally seeing numeration.

LESSON TWO: Thousands

Background: The regroupings from ones to tens, from tens to hundreds and from hundreds to thousands must each be done carefully. They are not the same and, because children understand regrouping tens into hundreds does not mean they will automatically understand regrouping hundreds into thousands.

Introduction: Show children a THOUSANDS block and a HUNDREDS square:

"How many hundreds are needed to make the thousands piece?"

"How many tens are needed to make the thousands piece?"

"How many tens are needed to make the hundreds piece?"

"How many ones are needed to make the tens piece? the hundreds piece? the thousands piece?

Build any of these as needed from the smaller materials in order to confirm the answers.

Activity: Pairs of children should have base ten blocks - several ones, tens, hundreds nad one thousand piece.

They are to complete the worksheet provided using the blocks as needed.

COMPUTATION IN BASE TEN: BACKGROUND INFORMATION

Children should have had previous experience with oral rehearsal of multiplication facts for small numbers less than ten such as 7x8 = ?. They should also have had experience with responding to open sentence stimuli like 9x6 = _____ and _____ = 4x7 for these same numbers. The idea that multiplying by the tens place so as to change each place value digit to count the next larger group size should also have been well developed and the children should have had oral practice and written practice with stimuli like 13x 10, 24x10, etc. They also should have immediate response knowledge of the effect of multiplying by the hundreds lace on which group sizes the digits in the positional notation representation are counting, i.e., 100 x 7 = 700, 100 x 14 = 1400. Such knowledge is prerequisite to using multiplication in a place value system. The "basic facts" of place value multiplication must be done with understanding. These are before regrouping.

Counters x Counters = Counters
Ones x Base = Base
Base x Base - Base Squared
Base x $(Base)^2$ = Base Cubed, etc.

In base ten these translate into seven x three = twenty-one three = 21 (after regrouping).

10 x 7 = 70 (7 tens)
30 x 20 = 600 (6 hundreds where hundred = $(ten)^2$
20 x 400 = 8000 (8 thousands where thousand = $(ten)^3$

Give some oral review of ten times various one and two digit numbers and one hundred times these. Then try 20 times 30. Remind students that these products have 2 important features - the counter multiplication is like that with small numbers and the place value group "size increase" feature.

LEVEL FOUR

COMPUTATION IN BASE TEN

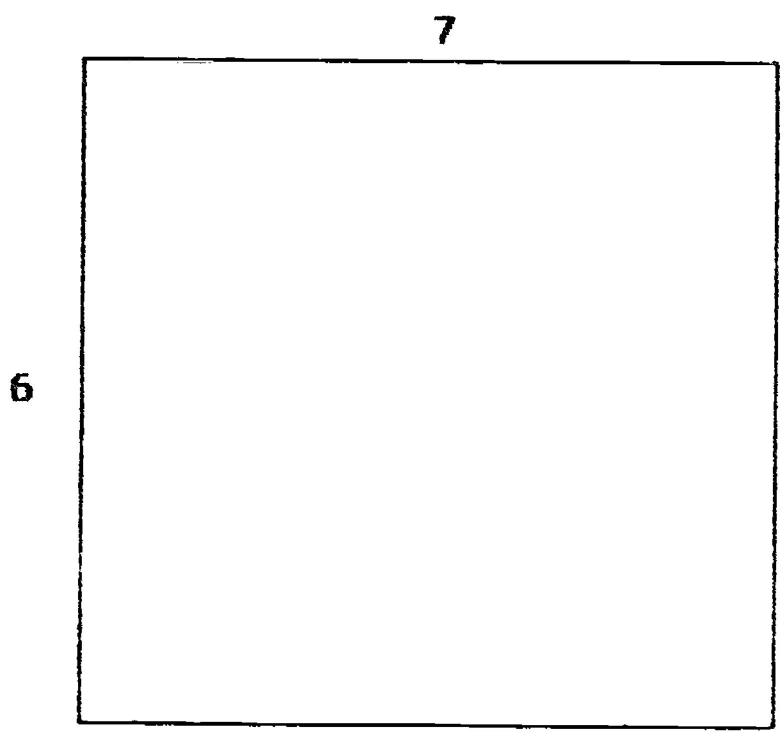
Background: Paper and pencil proficiency with the arithmetic algorithms for computation is not a high priority. Calculators and computers are used for computation. The only place students will see incomplete computation forms to fill in is on standardized tests or in workbooks. However, it is important for students to understand what algorithms are, how they work and how to design algorithms.

Students will have had an introduction to use of the rectangle model to perform base ten multiplication and division. This set of lessons will review that and expand on it.

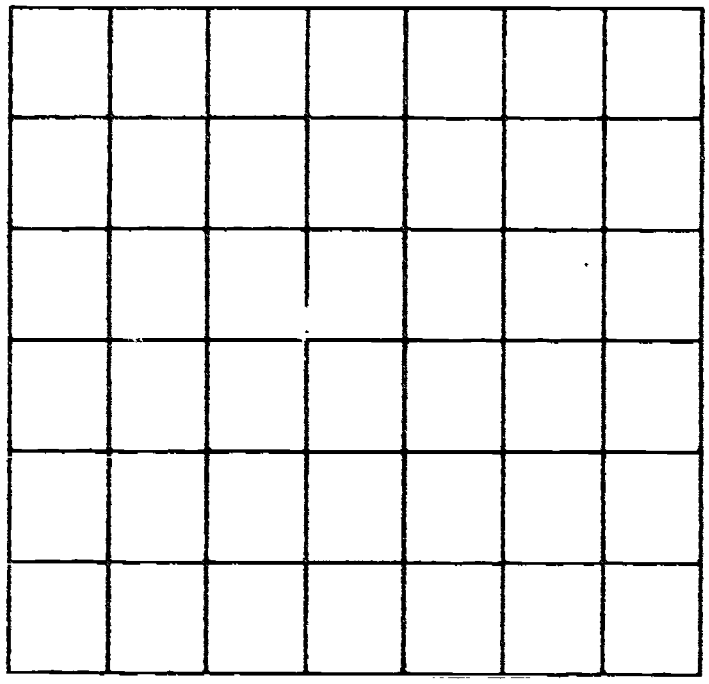
LESSON ONE: Multiplication

Introduction: This is to review the area model for multiplication.

Put an overhead transparency rectangle with dimensions so it can be filled with base ten ones:



Use a piece of centimeter graph paper over this rectangle to simulate the base ten ones. Have the children calculate the number of these 'ONES' in the rectangle:



Do a second rectangle in the same way. Answer all questions. Remind children the length and width of the rectangle represent the numbers being multiplied (FACTORS) and the area represents the result (PRODUCT).

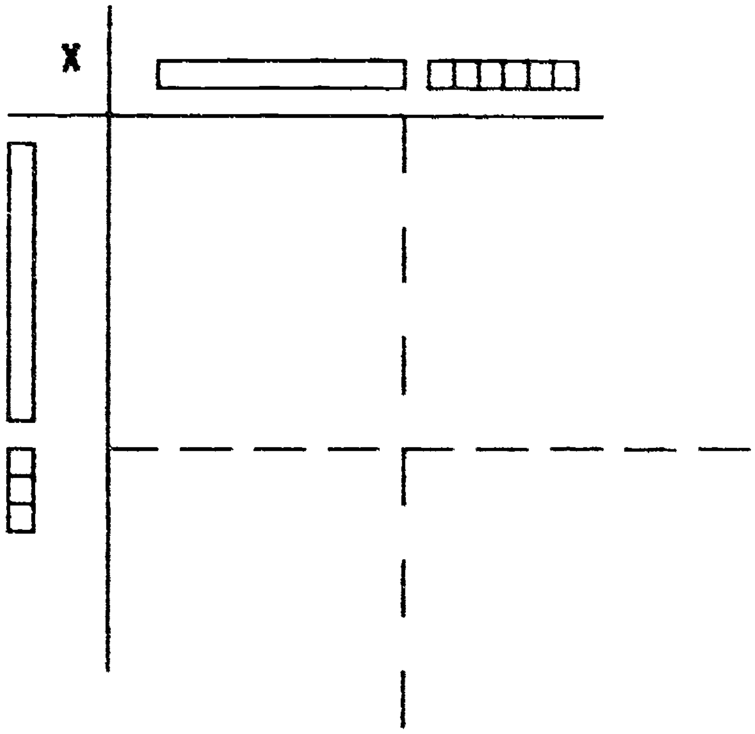
Activity: Pass out the worksheets for children to work on in pairs.

LESSON TWO: Multiplication

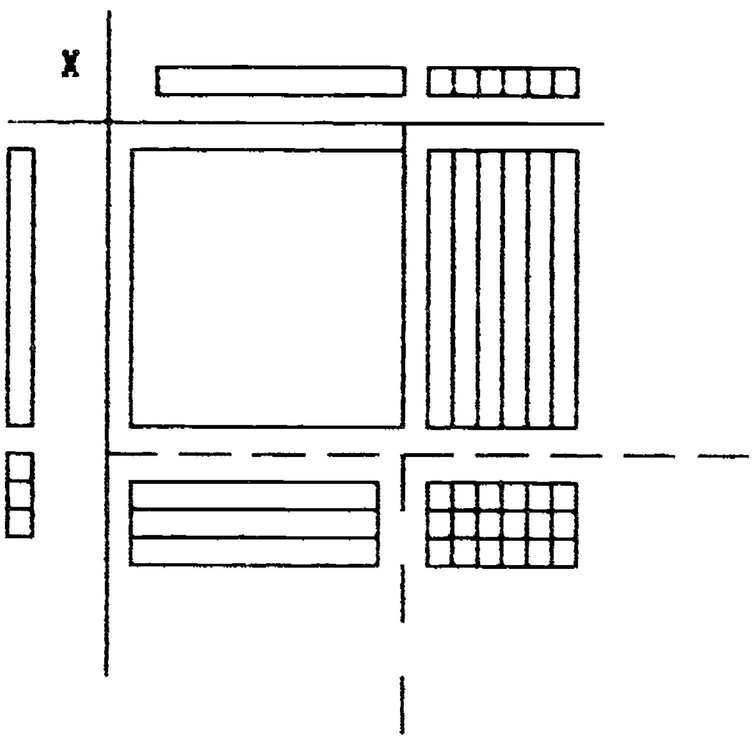
Introduction: Use overhead transparency base ten blocks. Children should have base ten hundreds, tens and ones to follow what you demonstrate.

Put multiplication on the chalkboard: 16 x 13

Remind the students that these can be thought of as the length and width of a rectangle. Array the blocks as:



Have a student come up to the overhead and place the blocks (with help, if needed). The result should be:



"How many hundreds pieces?" (1)

"How many tens pieces?" (9)
"Can we make another hundreds piece?"

"How many ones do we have?"

"What do you get when you multiply tens by another ten?" by 6 ones?"

"What are the four partial products?"

List the origins:

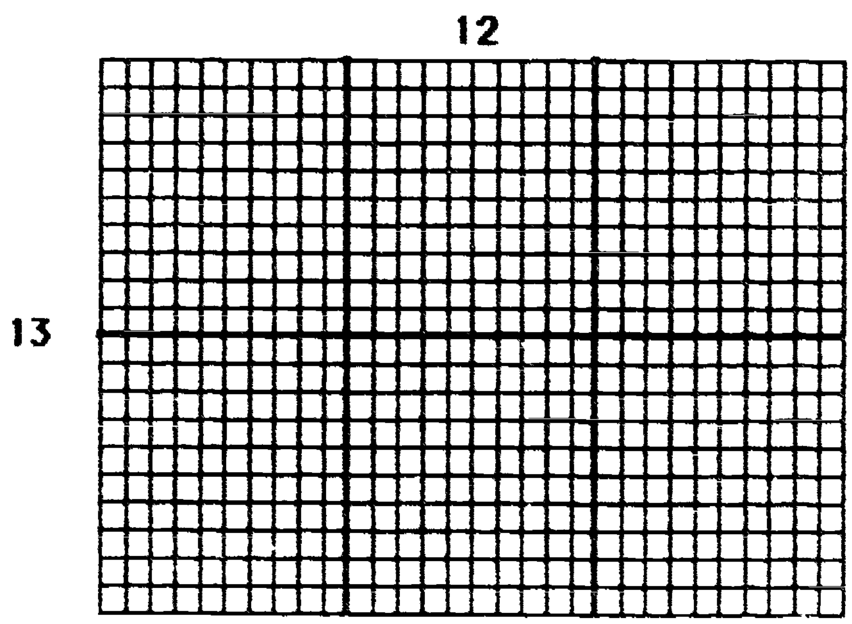
1. tens x tens
2. tens x ones
3. ones x tens
4. ones x ones

Keep asking related questions until it is clear the process has been reviewed with them.

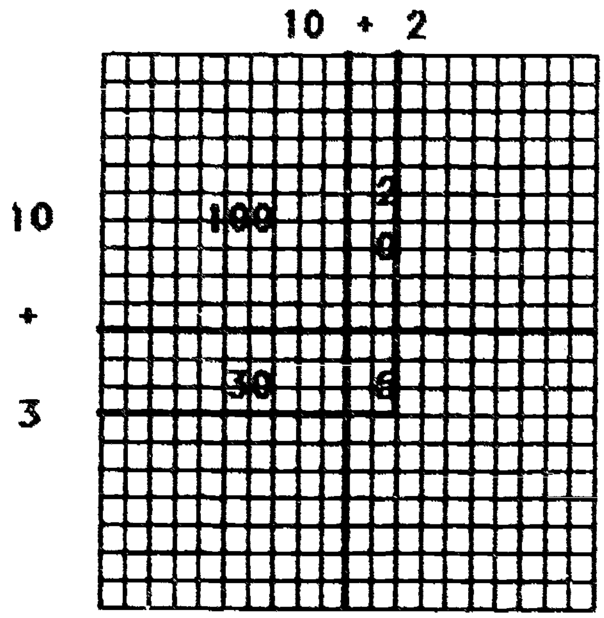
LESSON THREE

Introduction: This lesson is to simulate work with base ten pieces by using graph paper. Explain that the rectangles on graph paper are like those made with base ten pieces. The graph paper squares are ONES.

Put the graph paper transparency on the overhead:

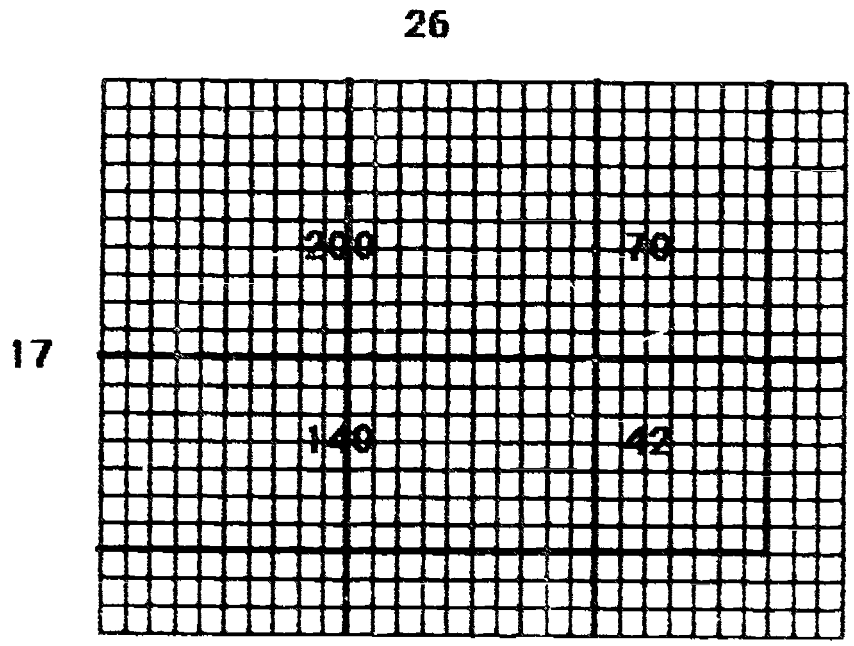


Write in the numbers to show the four partial products:



Remind the children that 10 x 10 gives the hundred. 10 x 2 gives 20. 3 x 10 gives 30 and 3 x 2 gives 6. Adding these gives 156 – the product.

Do a second as shown:

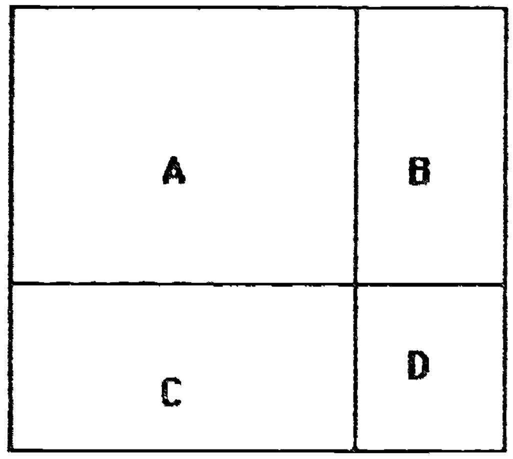


Point out the

20 x 10 = 200
20 x 7 = 140
6 x 10 = 60
6 x 7 = 42

The total is 442

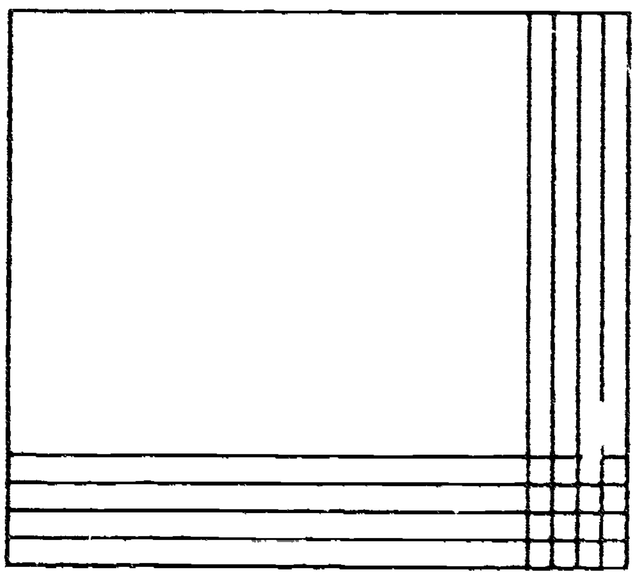
<u>Activity</u>: Give pairs of students graph paper, multiplication problems and recording sheets. Have the children record partial products as:



LESSON FOUR: Expanded forms in Multiplication

Introduction: Children must realize that multiplication can be expressed in many ways. Expanded notation emphasizes place value ideas.

Place the following on the overhead projector.



"What multiplication does this show?" (14 x 13)
"How can we write this multiplication? Write as suggested or else write to add to suggestions.

$$\begin{array}{r} 14 \\ \underline{\times 13} \end{array} \qquad 13 \times 14$$

"How is fourteen expressed in tens and ones?"

"How is thirteen expressed in tens and ones?" Write.

```
  14   =   10 + 4     13 x 14 = (10 + 3) (10 + 4)
 x13   =   10 + 3
```

"What is ten times ten?"
Write:

```
  14 = 10 + 4    13 x 14 = (10 + 3) (10 + 4) = 100
 x13 = 10 + 3
      100
```

"What is times times four?" Add to what is written:

```
  14 = 10 + 4    (13 x 14 = (10 + 3) (10 + 4) = 100 + 40
 x13 = 10 + 3
       100 + 40
         30
```

"What is three times four?" Add

```
  14 =  ◡ + 4    (13 x 14 = (10 + 3) (10 + 4) = 100 + 40 + 30 + 12
 x13 = 10 + 3
    100 + 40
         30 + 12
```

"Add up the totals"

```
     182                          182
```

Do a second similar problem.

Activity: Have pairs of children use the problem sheets that require the writing of an expanded form.

Keep emphasizing the partial products with 2 digit multipliers:

Tens x Tens	Tens x Ones
Ones x Tens	Ones x Ones

LESSON FIVE: Expanded Form Abstracted

Introduction: When children understand (1) expanded form and (2) the original of partial products, introduce using "T" to show TEN, "T^2" to show HUNDRED and "O" to show ONES. Write the following:

$$\begin{array}{rcl} 13 & = & 10 + 3 \\ \underline{\times 12} & \underline{=} & \underline{10 + 2} \end{array}$$

"Since TEN begins with T, we'll use the letter T to show TENS. How would we write this with T?"

$$\begin{array}{rcccl} 13 & = & 10 + 3 & = & T + 3 \\ \times 12 & = & 10 + 2 & = & T + 2 \end{array}$$

"When we multiply TEN times TEN, we get one HUNDRED. What shape is a base ten HUNDRED piece?"

"What are the sides of this square?"

T and T

"Since the HUNDRED is TENS x TENS, we can write T x T. A short way to show two equal multipliers is using T^2. This shows T is a multiplier twice.

"How would we show 3 used as a multiplier twice?"

Write 3^2

Complete the forms:

$$\begin{array}{lcl} 10 + 3 & & T + 3 \\ \underline{10 + 2} & & \underline{T + 3} \\ & & \\ 100 + 30 & & T^2 + 3T \\ (1) \quad (3) & & (2) \end{array}$$

"Now we'll multiply by the 2."

$$\begin{array}{lcl} 10 + 3 & & T + 3 \\ \underline{10 + 2} & & \underline{T + 2} \\ & & \\ 100 + 30 & & T^2 + 3T \\ 20 + 6 & & 2T + 6 \end{array}$$

"And add them"

$$\begin{array}{r} 10+3 \\ \underline{10+2} \\ 100+30 \\ 20+6 \\ \hline 100+50+6 \end{array} \qquad \begin{array}{r} T+3 \\ \underline{T+2} \\ T^2+3T \\ 2T+6 \\ \hline T^2+5T+6 \end{array}$$

We could write this horizontally as

$$(T+2)(T+3) = T^2+3T+2T+6$$
$$= T^2+5T+6$$

In doing multiplications, try writing them as many different ways as you can.

Activity: Have children go back over their multiplication worksheets and write each one in (1) expanded form and (2) abstracted by representing TEN by T. Keep reminding them of partial products and the relationships (before trading)

ONES X ONES → ONES

ONES x TENS → TENS (the ones count the tens)

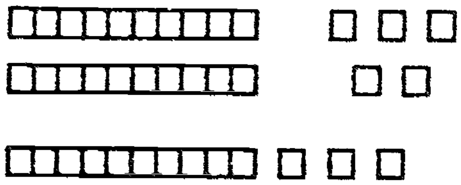
TENS X ONES → TENS (same here)

TENS X TENS → HUNDREDS

Multiplying by TENS always make the digits count the next larger group size is the basic idea to keep emphasizing.

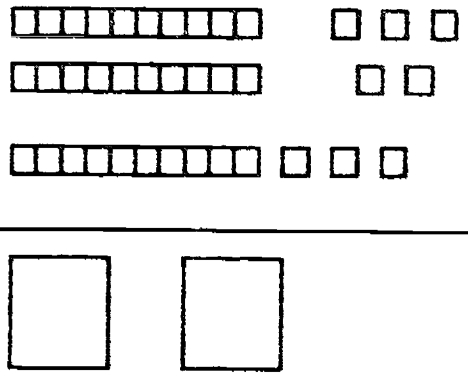
LESSON SIX: Multiplication Standard Form

Introduction: Use base ten blocks on the overhead projector. Arrange the following:



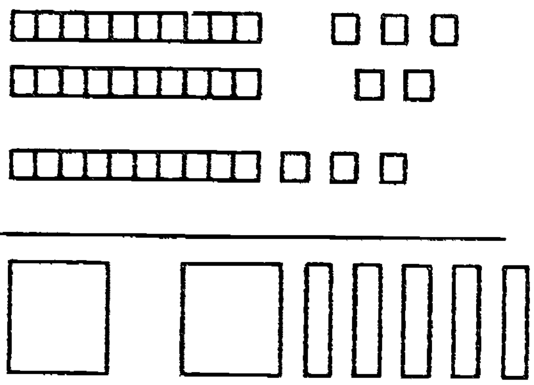
25
x 13

Point to the lower ten: "What is the result of multiplying this ten times the other two tens?" (2 hundreds) Rearrange:



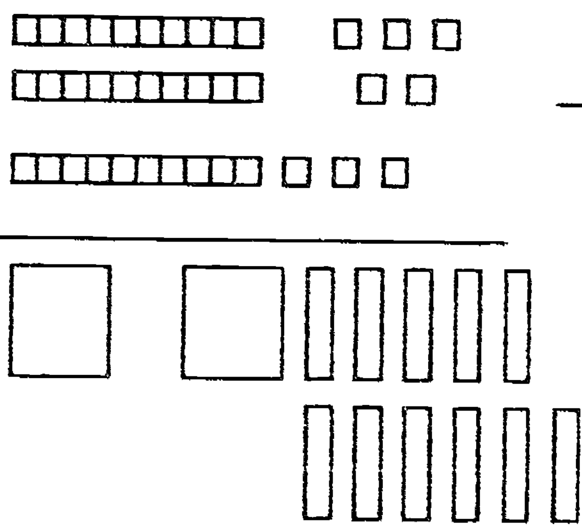
25
x 13
200

"What is the result of multiplying the ten times the 5 ones?" (5 tens) Rearrange:



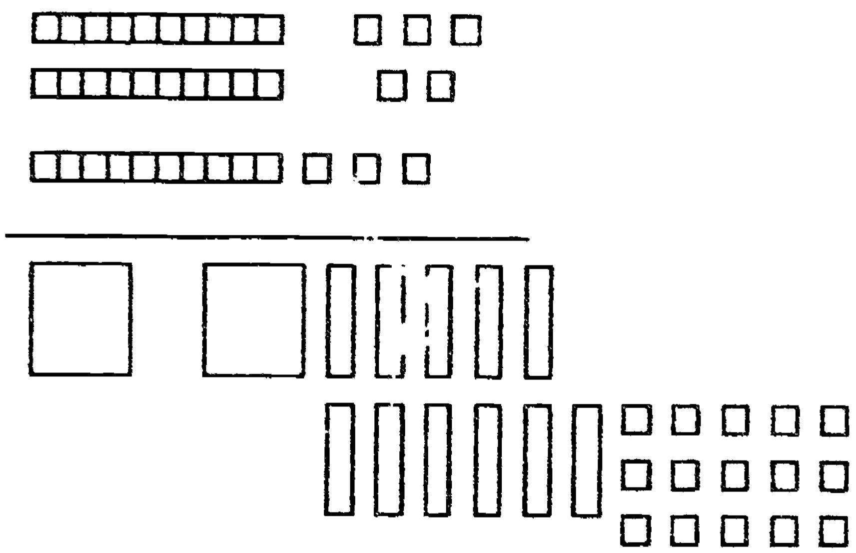
25
x 13
200
50

"What is the result of multiplying 3 ones times the 2 tens?" (6 tens) Rearrange:



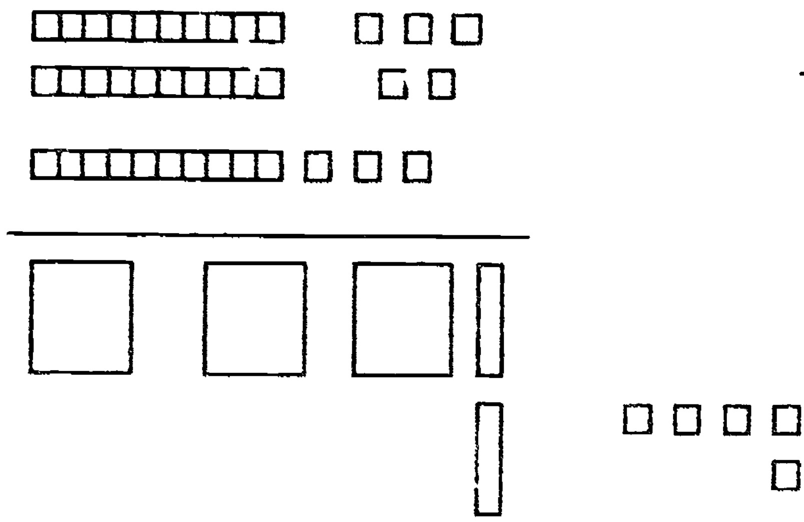
25
x 13
200
50
60

"What is the result of multiplying the 3 ones times the 5 ones?" (15 ones) Rearrange:



```
  25
x 13
----
 200
  50
  60
  15
----
```

"What r.:grouping must we do?" "(10 ones for a ten and ten tens for a hundred)" Rearrange:



```
  25
x 13
----
 200 + 100
  10 + 10
         5
```

325 325

Do a second problem. Emphasize the result of multiplying by tens and ones.

<u>Activity</u>: Have children use base ten blocks to do the appropriate worksheet.

<u>LESSON SEVEN: Hundreds</u>

<u>Introduction</u>: After develoning understanding of the hundreds place, the grouping of tens into hundreds, the decomposing of hundreds into tens in numeration activities, this place can be introduced into computation.

Using base ten blocks, build a stack of hundreds pieces, ten units high. Place the ten rod along the edge of the stack:

"The TENS place times the HUNDREDS place give the THOUSANDS place."

Remember that the TENS place as a muliplier makes each digit count the next larger group.

Write:
```
   231
x   14
```

Point to the 1 in 14.

"What place is this?" (TENS)

Point to the 2 in 231

"What place is this?" (Hundreds)

"What place results from multiplying the tens x hundreds?" (Thousands)

"What is 1 x 2?"

"2 WHAT?" (Thousands)

Point to the 3 in 231

"What place is this?" (Tens)

"What is the result of multiplying tens x tens?" (hundreds)

"What is 1 x 3?"

"3 WHAT?" (hundreds)

"Multiplying is using the number facts. You MUST know WHAT the result is. Write as expanded form:

```
    231  = 200 + 30 + 1
  x  14  ___________10 + 4
```

"Ten times 2 hundred is 2 WHAT?" (2 THOUSANDS)

"1 ten times 3 tens is 3 WHAT?" (3 hundreds)

"1 ten times 1 one is 1 WHAT?" (1 ten)

"4 ones times 2 hundred is 8 WHAT?" (8 hundreds)

"4 ones times 3 tens is 12 WHAT?" (12 tens)

"12 tens is 1 hundred plus 2 tens"

"4 ones times 1 one is 4 WHAT?" (4 ones)

Write all products:

```
    231  = 200 + 30 + 1
  x  14  ___________10 + 4
        2000 + 300 + 10
        ____________800 + 100 + 20 + 4
        2000 + 12(100) + 30 + 4
```

"12 hundred is WHAT?"
Write: 2000 + 1000 + 200 + 30 + 4 = 3234

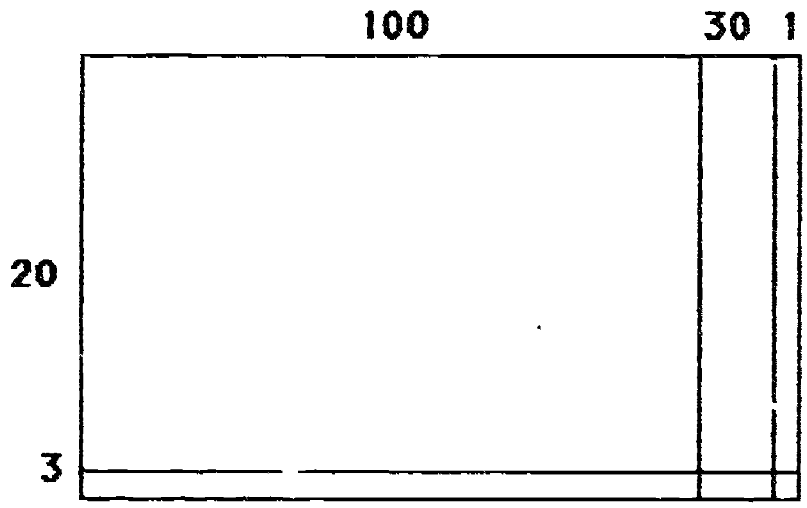
Do a second example in expanded form to emphasize the products coming from multiplying places.

LESSON TEN: Larger Rectangles

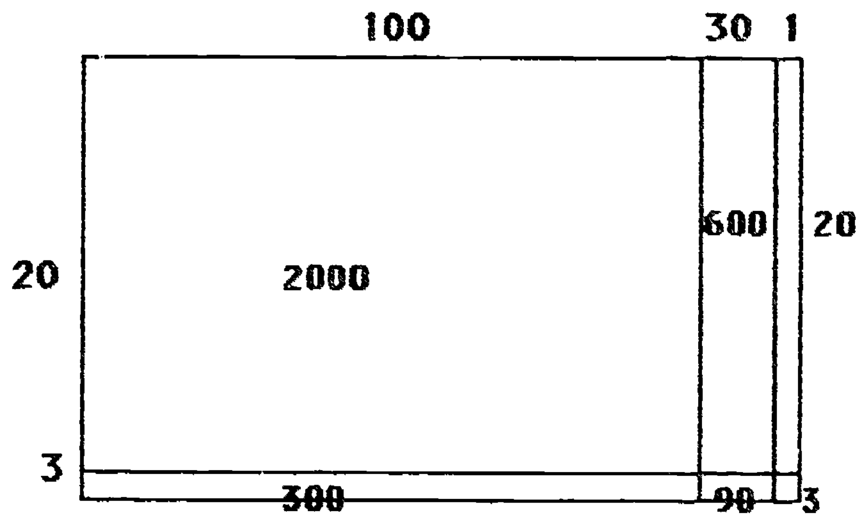
Introduction: On the overhead, write 131 x 23: "How long is the rectangle?"

"How wide is the rectangle?"

Put the following transparency on the overhead. Emphasize that this is NOT the same size as base ten blocks:



Write in products.



"There are SIX partial products:

TENS x HUNDREDS	ONES x HUNDREDS
TENS x TENS	ONES x TENS
TENS x ONES	ONEX x ONES"

Point to each of these on the diagram.

Add these together:

2000
600
20
300
90
3013 is the products

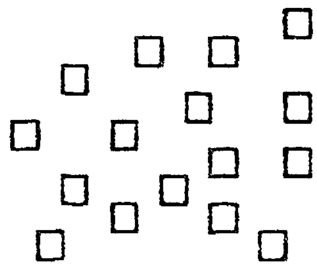
Do a second example that emphasizes the place value characteristics of this process.

<u>Activity</u>: Have pairs of children work together on the worksheets that give practice with this process.

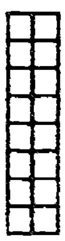
<u>LESSON ELEVEN: Division</u>

<u>Background:</u> This is a review lesson to remind children of the rectangle model for division as developed in Level Three.

<u>Introduction:</u> Use tiles on the overhead projector. Children should have some inch square tiles like ceramic floor tiles to follow what you do.

Randomly place sixteen tiles on the overhead: 

"How many rows of tiles can I make?" (8) Rearrange tiles.



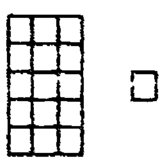
Write

$$2\overline{)16}^{\;8}$$

and **16 ÷ 2 = 8**

"This is how we show this division."

"How many rows of 3 can I make?" (5 with one left over.) Rearrange tiles.

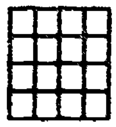


"We write this as " **5 R 1** over **3⟌16** **and 16 ÷ 3 = 5 R 1**

"The R stands for the remainder - what is left that is too small to make another 3."

"How many rows of fours can we make?" (4)

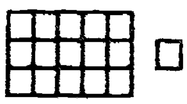
Rearrange tiles:



"Notice we get a square - 4 on each side."

Write: $4 \overline{)16}$ with 4 above, **and** $16 \div 4 = 4$

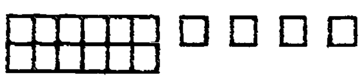
"How many rows of 5 can we get?" (3 with a remainder) Rearrange the tiles:



"How do we write this?"

Write $5 \overline{)16}$ with 3 R 1 above, $16 \div 5 = 3 \text{ R } 1$

"How many rows of six can we make?" (2 with a remainder) Rearrange tiles:



"How is this written?" Write: $6 \overline{)16}$ with 2 R 4 above, **and** $16 \div 6 = 2 \text{ R } 4$

"How long would rows have to be to get no remainder?" (2, 4 or 8)

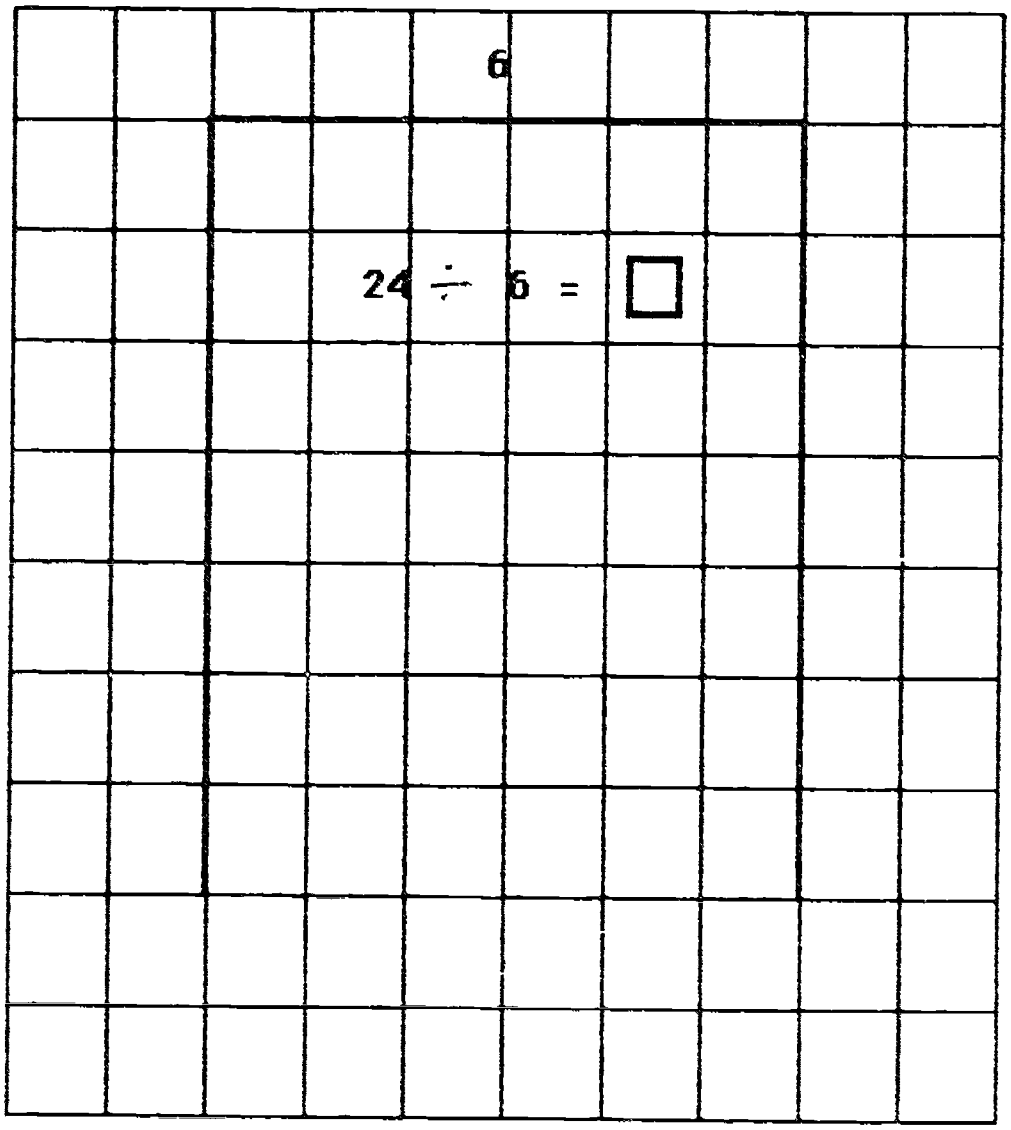
"16 is EVENLY divided by 2, 4 or 8. Other divisors give a remainder."

Place fourteen tiles on the overhead.

"Divide these by 2, 3, 4 5 on up to 7. Write the computation form and the number sentences on your form.

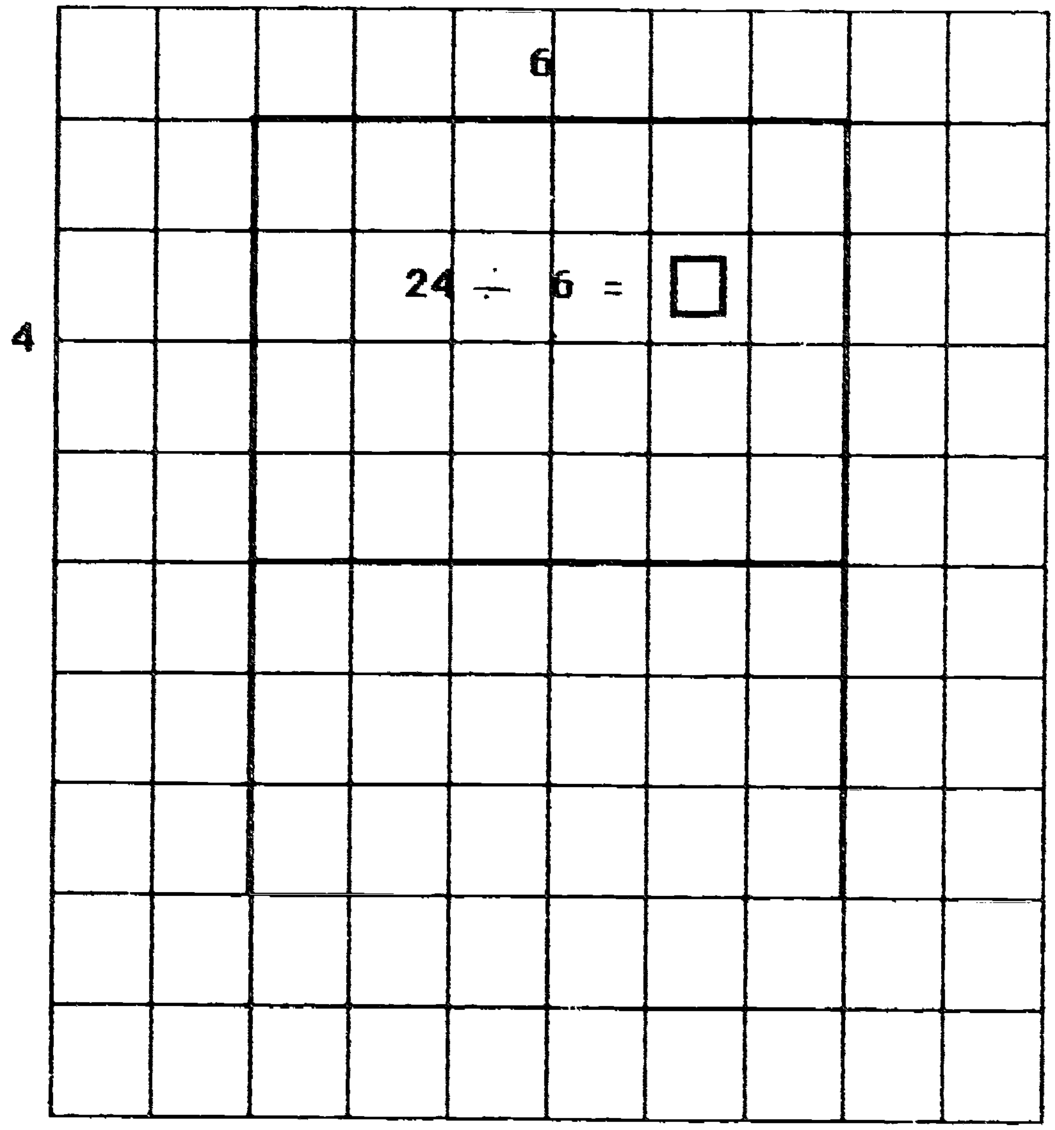
LESSON TWELVE:

Introduction: This is a review of the introduction to division in previous levels. Use the overhead projector with a graph paper transparency as shown:



"SIX is the divisor and one side of the rectangle. Twenty-four must be the area of the rectangle and we need to know the other side. How long should the other side be so the area is twenty-four?"

When the class has agreed it is to be FOUR, draw the line in and complete the diagram as shown:

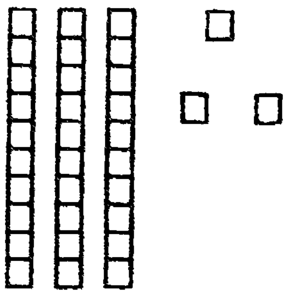


Do a second and as many additional examples as needed to get the children ready to practice on the worksheet.

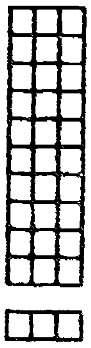
Activity: Have the pairs of children work together to complete the worksheet. Monitor the work to see the second sides are being determined correctly.

LESSON THIRTEEN: Division

Introduction: Use base ten blocks for the overhead projector. Children should have base ten blocks to follow with. Place the following on the overhead projector:



"I want to divide this by 3. How wide should the rectangle be?" (3)
Rearrange:



Write: 33 ÷ 3 = 11 and

$$3\overline{)33}^{\,11}$$

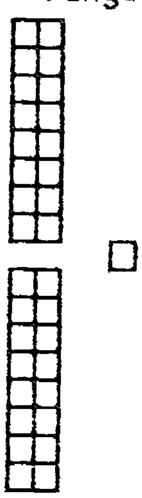
Point out (1) there is no remainder, and (2) no trading of TENS for ONES was necessary.

"Build the largest rectangle you can with one side of 2 units."

"Did you have to trade a TEN for ONES?"

"Did you get a remainder?"

Arrange as:

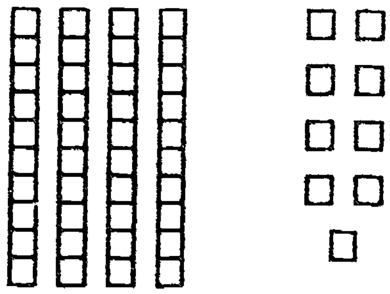


Write: 33 ÷ 2 = 16 R 1 and

$$2\overline{)33}^{\,16\text{ R }1}$$

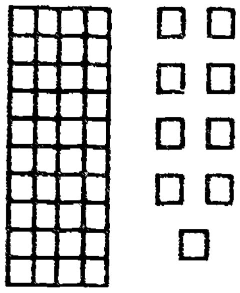
Point out the division in the computation form and the number sentence is the given side of the rectangle

Put the following on the overhead projector:



```
4 ⟌ 49
```

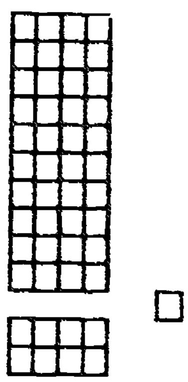
"What is the given side of the rectangle?" (4) Arrange the four tens as shown:



"We have used four tens or 40.

```
   10
4 ⟌ 49
   - 40
     9
```

"How many of the ones can fit into a rectangle?" (8) Rearrange (2 ones with each ten):



```
      2
     10
4 ⟌  49
   - 40   (tens used)
      9
   -  8   (ones used)
      1   (Remainder)
```

so: $49 \div 4 = 12$ R1

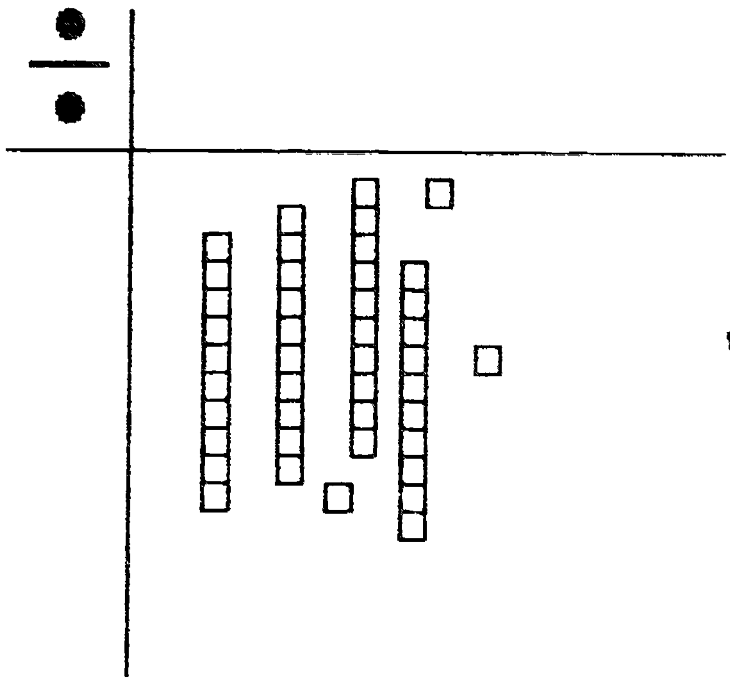
Activity: Pairs of children should use base ten blocks to make rectangles to solve the division examples.

LESSON FOURTEEN: Division

Introduction: Use overhead transparency base ten blocks. Children should have base ten blocks to follow. In this lesson the emphasis is on division by ten "shifting" digits to counting the next smaller group size. This is the inverse of multiplication by ten. Give the example:

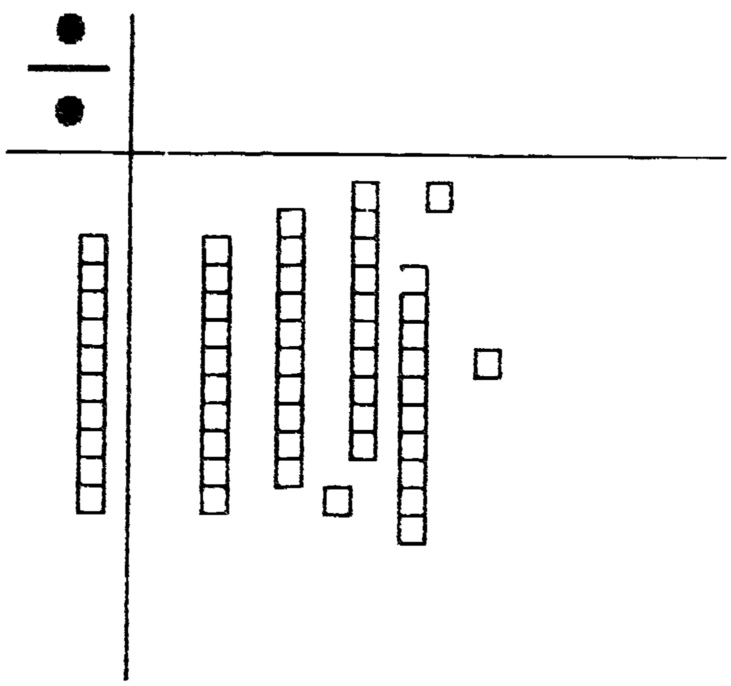
$$17 \times 10 = 170 \text{ and } 170 \div 10 = 17$$

Arrange the following on the overhead projector on a transparency of the workmat:



Write: 43

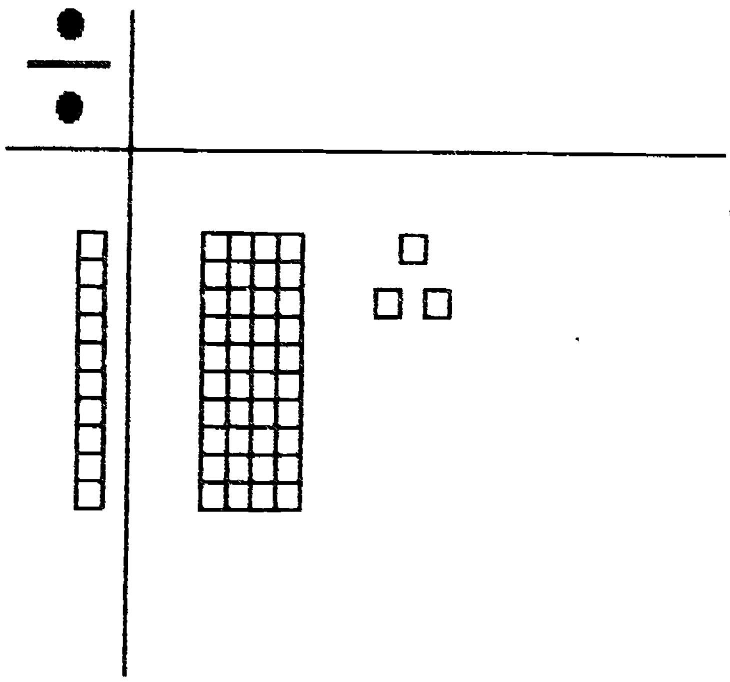
Put a ten to show one side:



43 ÷ 10

$$10\overline{)43}$$

"How should I arrange the pieces to make a rectangle with this side?" Rearrange:



43 ÷ 10 = 4 R 3

```
     4 R 3
10 ) 43
   - 40
     --
      3
```

"How long is this other side?" (4)

"Is there enough to make the rectangle one unit longer on that side?" (No)

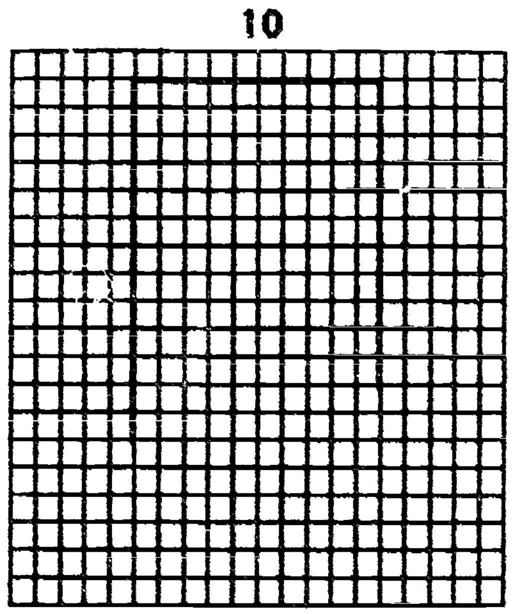
"How many more ones would be needed?:" (7)

Do a second example of division by ten.

<u>Activity</u>: Pairs of students should use the base ten pieces and workmats to complete the worksheets.

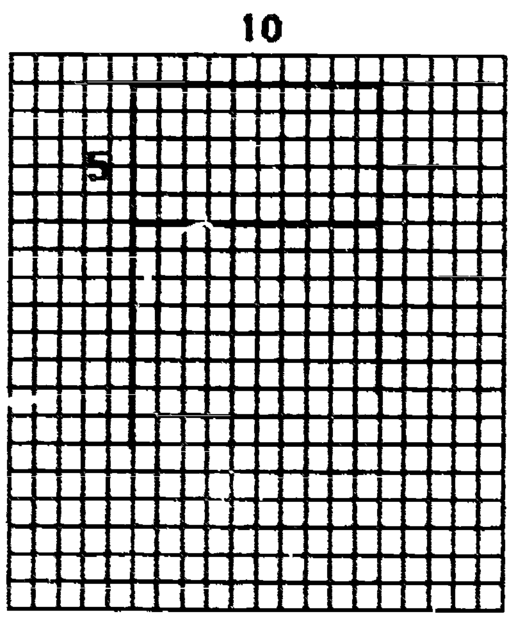
<u>LESSON FIFTEEN: Graph Paper</u>

<u>Introduction</u>: This lesson extends graph paper simulation of base ten blocks to the use of graph paper and division by the base-ten. On the overhead projector, use a graph paper transparency to illustrate this:



"Where should we put the line when we want the largest rectangle and have 54 of the units?" (5)

Draw the line.



"How many of the 54 did we use?" (50)

"Are there enough ones to make another row of ten?" No.

Write:

```
     5 R 4
    ______
10 | 54
    -50
    ----
      4
```

and 54 ÷ 10 = 5 R 4

"Notice the 5 now counts ones instead of tens. Dividing by ten shifts each digit one place to the right because the next smaller group size is being counted. TENS   TENS = ONES.

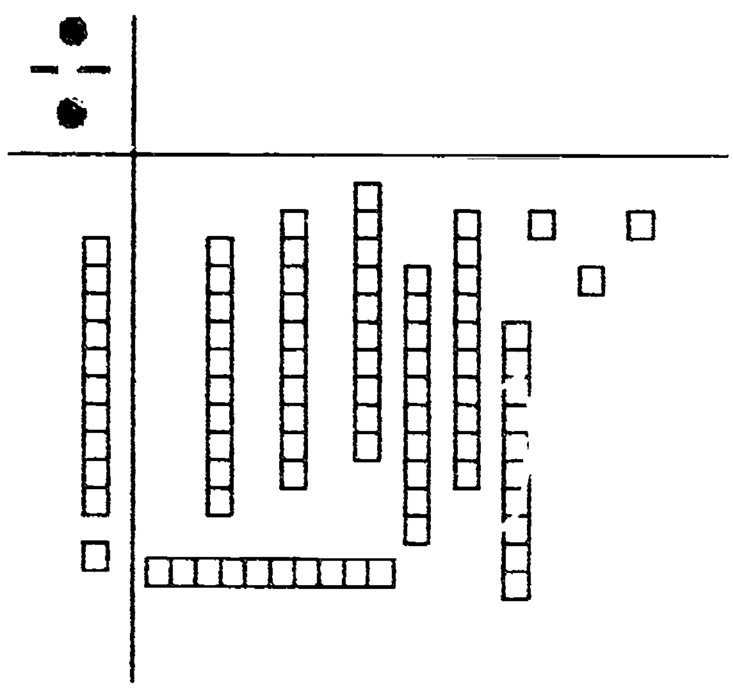
Do a second example of dividing by ten.

Activity: Give pairs of children the graph paper worksheets and recording forms to do. See that the computation forms and number sentences are completed correctly.

LESSON SIXTEEN: Two Digit Divisors

Background: With the availability of calculators, there is no good reason to develop division in base ten beyond 2 digit divisors. This lesson introduces this and is followed by the same topic using different models.

Introduction: Use base ten pieces on the overhead. Children should have base ten pieces, the division workmat and the recording forms. Put the following on the overhead projector:

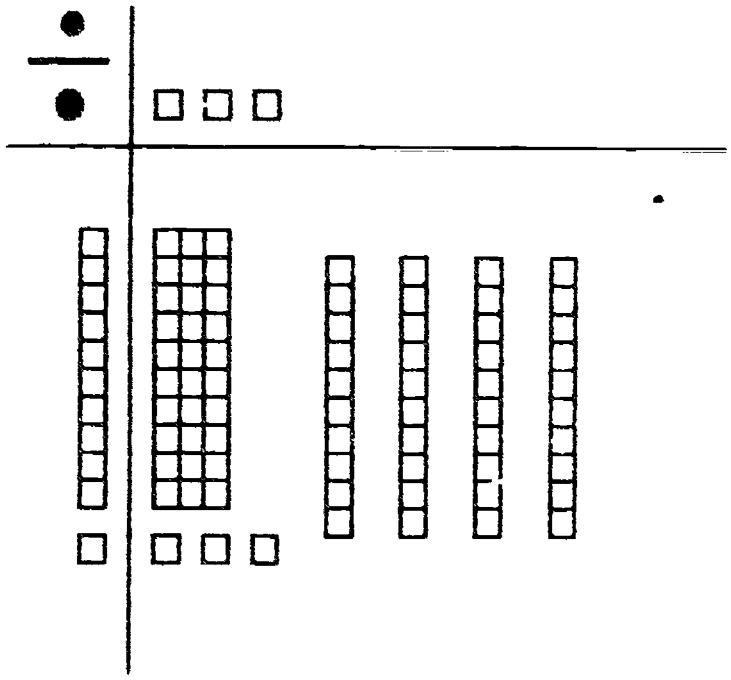


"What number is to be divided?" (73)

"What number is the divisor?" (11)

"Arrange the materials into as much of a rectangle as you can."

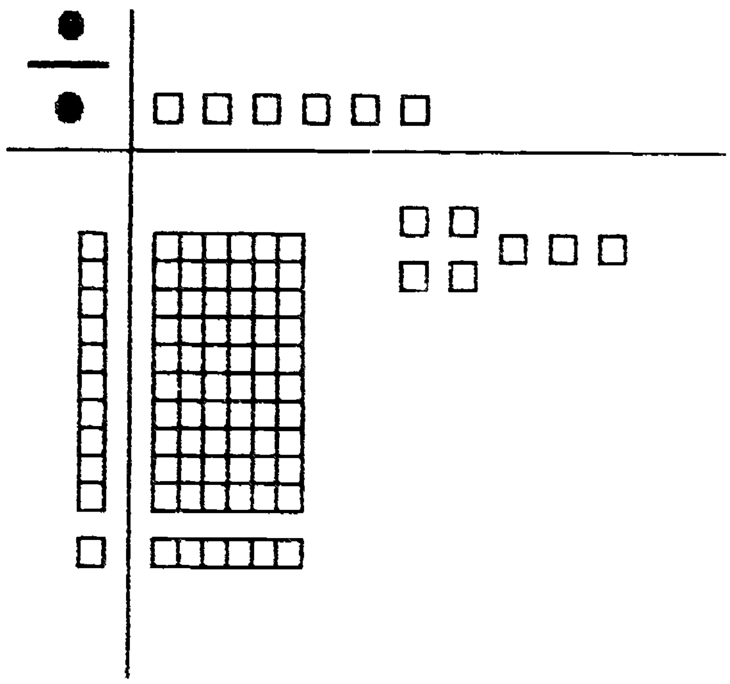
Wait and put this up.



"We could use 3 tens and 3 ones."

"What must we do now that all of the ONES are used?"

If no one suggests it, you suggest trading one TEN for some ONES. Rearrange as:



"Are there enough units to make the rectangle larger?"

"How many units did we use?" (66)

"How many are left over?" (7)

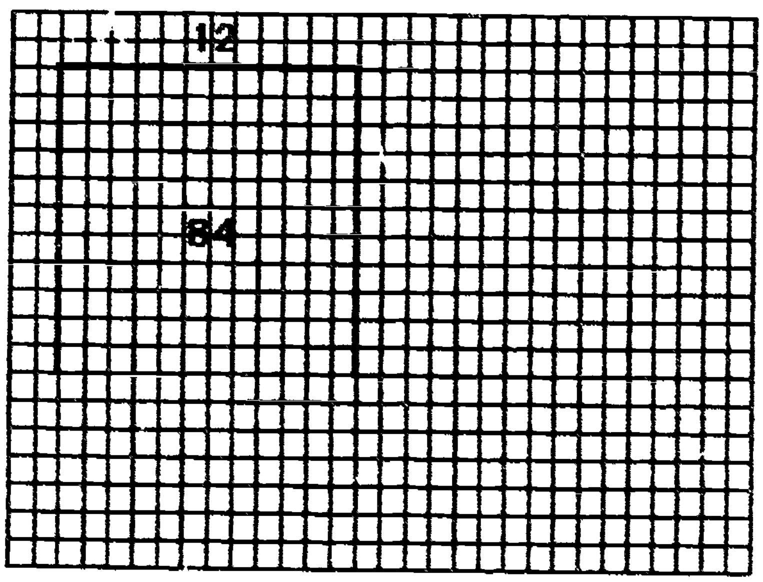
Write: $11\overline{)73}$ with quotient 6 R 7 and 73 ÷ 11 = 6 R 7

Do a second example with a different divisor, e.g. 73 ÷ 12.
Repeat until children see the need to trade to get enough ONES to make the rectangle as large as possible.

Activity: Give pairs of children base ten blocks, workmats and the worksheets. Monitor the work to watch for (1) trades as needed, and (2) use of the remainder and (3) writing of correct number sentences and computation forms.

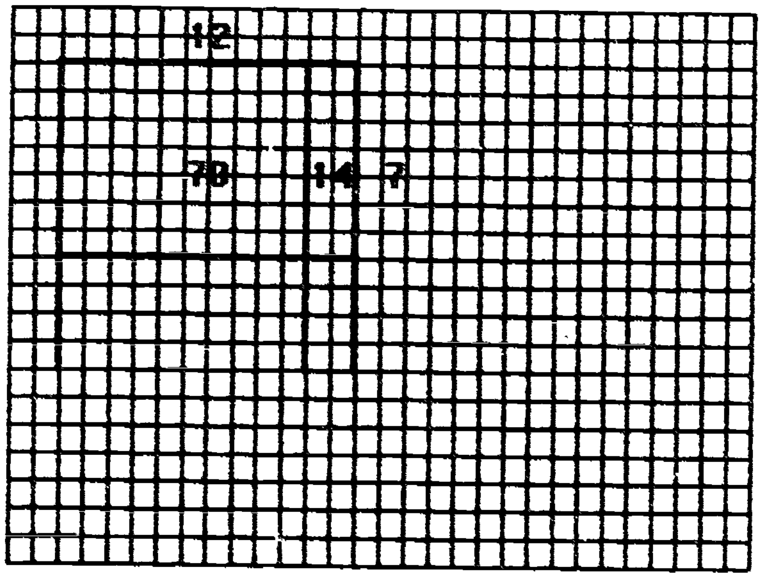
LESSON SEVENTEEN: Graph Paper

Introduction: This lesson involves use of graph paper to find the second side of rectangles from a given number of units and with one given side. It extends this to 2 digit ivisors. Use overhead transparency graph paper:



"We have eighty-four units to use and one side is twelve Where should I draw the line to make the largest possible rectangle?"

Discuss the responses. Some children may be willing to see what a side of 5 used first. Others may want 6 or 7. If someone suggests 8, point out that this would require more than is available. Eventually get to:



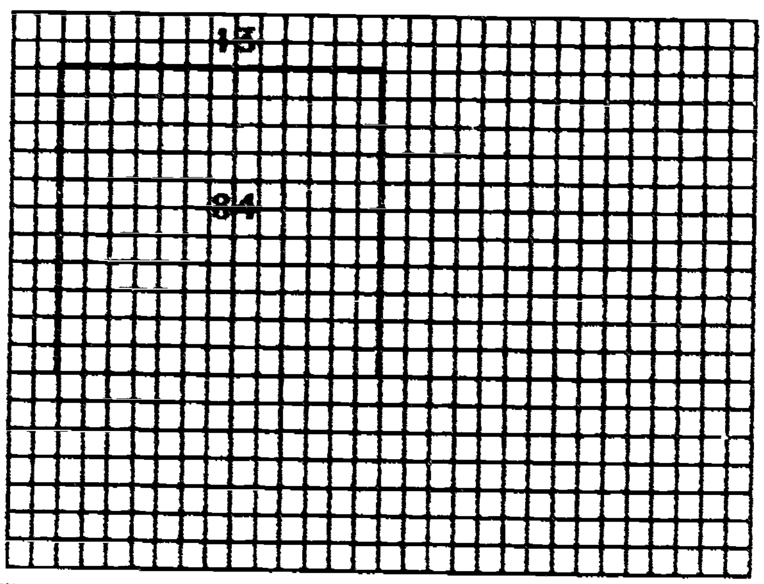
"How many squares are used with a side of 7?" (84)

"Are there any left?"

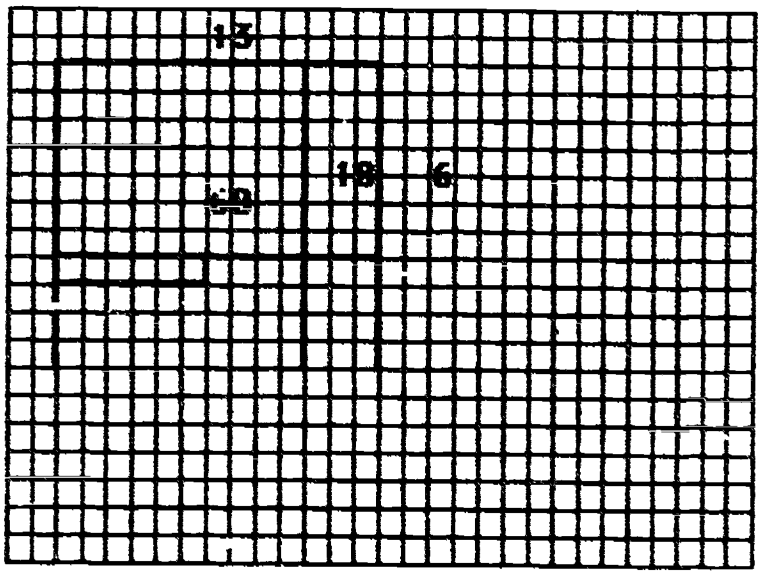
Write: 84 ÷ 17 = 7 and

$$\begin{array}{r} 7 \\ 12\overline{)\,84} \\ \underline{84} \\ 0 \end{array}$$

"We'll divide the eighty-four by thirteen."



"How long a side can we make this time? Why must it be shorter than before when we divided by twelve?" Put the line in.



"Are the six left enough to make the side longer?"

"We have a remainder." (Circle the remainder)

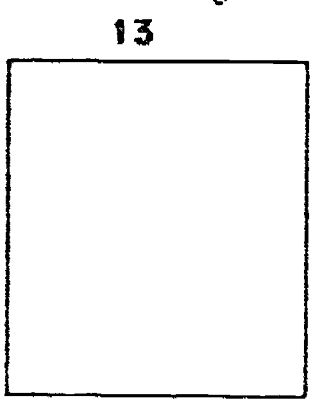
Write: 84 ÷ 13 = 6 R ^

```
     6 R 6
13 | 84
    -78
      6 remainder
```

Activity: Give pairs of students the graph paper problems and the recording forms.

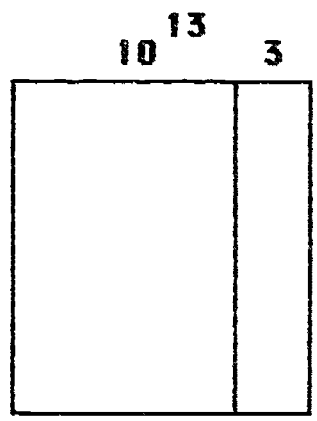
LESSON EIGHTEEN: Unmarked Rectangles

Introduction: Put a prepared rectangle on the overhead. Tell the children it is not as large as it would be made from base ten blocks or on graph paper:

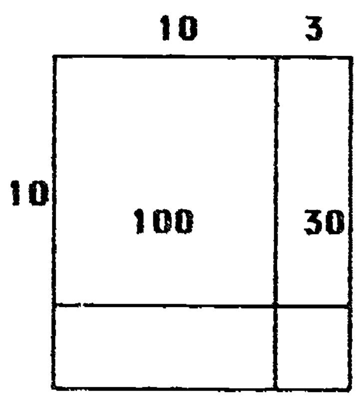


"We want the rectangle to have an area of 156. What must the other side be? Let's put lines in to show the tens and ones."

"We want the rectangle to have an area of 156. What must the other side be? Let's put lines in to show the tens and ones."



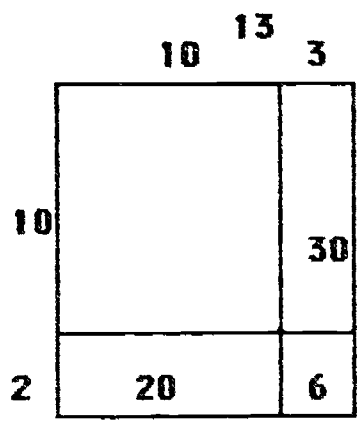
"How much would we use if we made a line at ten?"



"We used 130 of the 156. How many are left?" (26)

"How much longer can we make the side?"

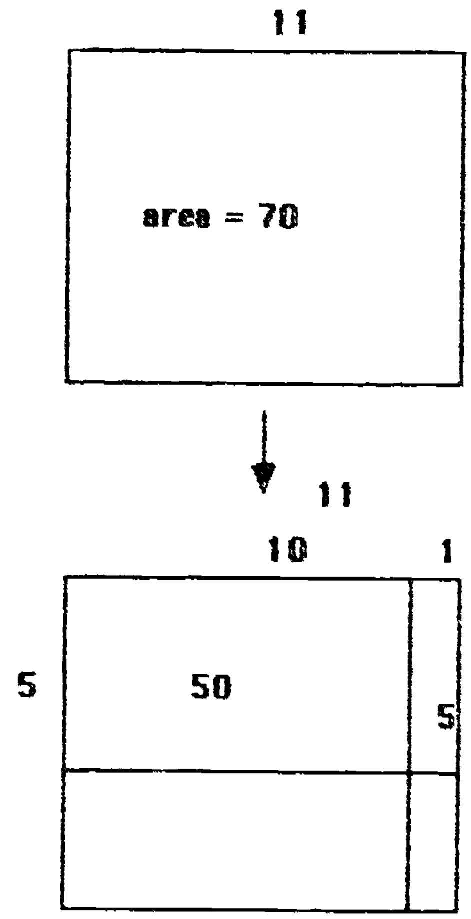
Complete the diagram and show all 156 are used:




```
Write:      2
           10
   13 | 156        156 ÷ 13 = 12
        - 130
           26
        -  26
            0
```

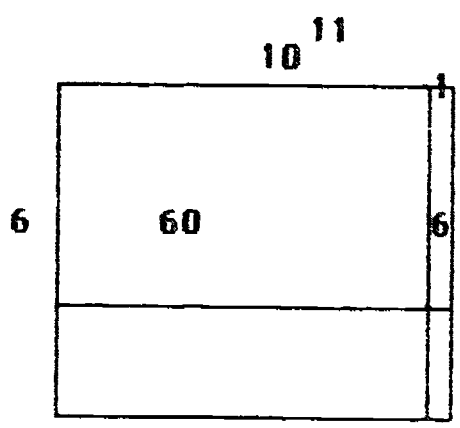
"There is no remainder."

Do a second example:



$$\begin{array}{r} 70 \\ -\ 55 \\ \hline 15 \end{array}$$

"We can extend one more to make the side 6":



$$\begin{array}{r} 70 \\ -\ 66 \\ \hline 4 \end{array}$$

Write:

$$\begin{array}{r} 6\ R\ 4 \\ 11\,\overline{)\,70} \\ \underline{-66} \\ 4 \text{ remainder} \end{array} \qquad 70 \div 11 = 6\ R\ 4$$

<u>Activity</u>: Have children work in pairs on the worksheets provided.

LESSON NINETEEN: Division by Places

Introduction: This is to emphasize the place value properties:

HUNDREDS ÷ TENS = TENS
HUNDREDS ÷ ONES = HUNDREDS
TENS ÷ TENS = ONES
TENS ÷ ONES = TENS
THOUSANDS ÷ HUNDREDS = TENS
THOUSANDS ÷ TENS = HUNDREDS
THOUSANDS ÷ ONES = THOUSANDS

Write on the board:

$100 \div 10 = 10$
$200 \div 10 = 20$
$200 \div 20 = 10$

Remind the children that this is:

1 HUNDRED ÷ 1 TEN = 1 ÷ 1 = 1 TEN
2 HUNDREDS ÷ 1 TEN = 2 ÷ 1 = 2 TEN
2 HUNDREDS ÷ 2 TENS = 2 ÷ 2 = 1 TEN

Similarly:

600 ÷ 30 = 6 ÷ 3 = 2 WHAT?
2 TENS = 20
800 ÷ 20 = 8 ÷ 2 = 4 WHAT?
4 TENS = 40

Give oral drill on:

x hundreds ÷ y tens
x thousands ÷ y tens or z hundreds, etc

Examples:

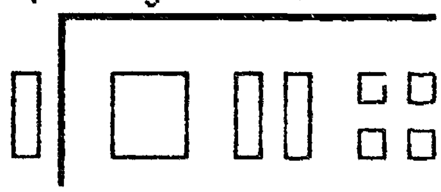
"200 divided by 20?"
"600 divided by 30?"
"500 divided by 30?"
"2000 divided by 200?"

Follow up Activity: Give children a speed drill on sight reading division by decades and hundreds.

LESSON TWENTY: Hundreds Division

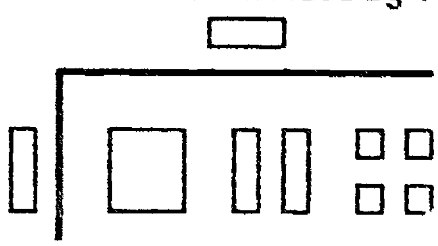
Introduction: This lesson will relate the standard division form to the rectangle model. Use an overhead transparency form and base ten materials.

Put these on the overhead projector:

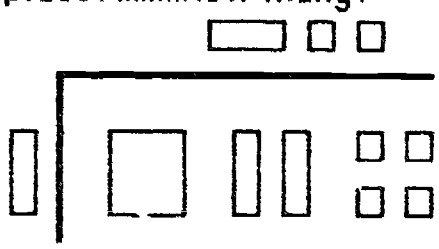


"One hundred twenty four is to be divided by ten in steps - the largest place in the DIVIDEND by the largest place in the DIVISOR first."

"One hundred divided by TEN is what place?.......How many?"



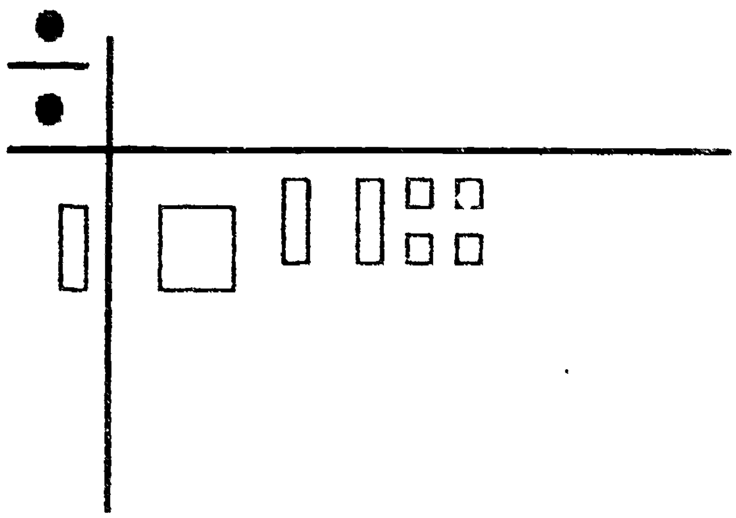
"We divide the next place by TEN. Two TENS divided by TEN gives what place?........How many?"



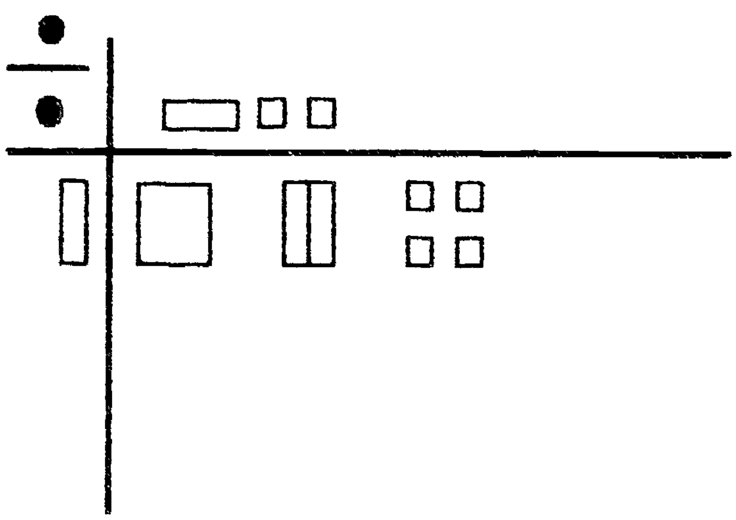
"Can we divide four ONES by TEN?" ...So four is a REMAINDER. We write:

```
      12  R 4
 10| 124
   - 100
      24
    - 20
       4 Remainder
```

Replace the |‾‾‾‾ by the division workmat transparency.



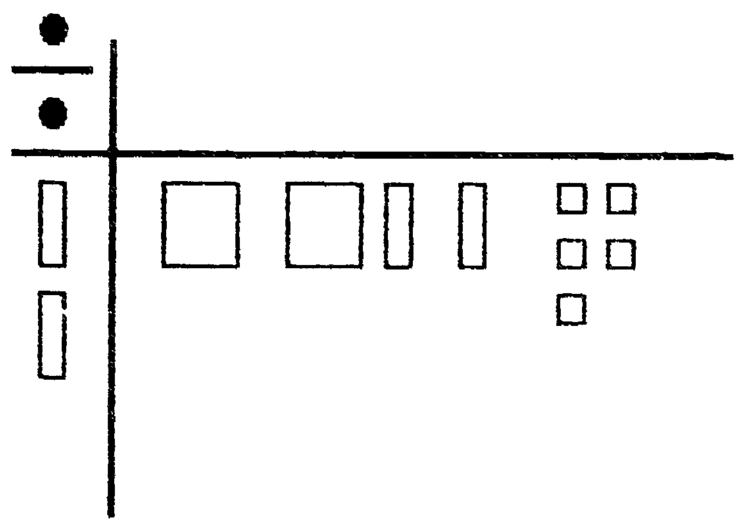
"One side is TEN. The rectangle is to be made of the hundred, tens and ones. How should we arrange these to make a rectangle?"



"We could fit only the hundred and the two tens. The other side is TWELVE and the four ones are a remainder."

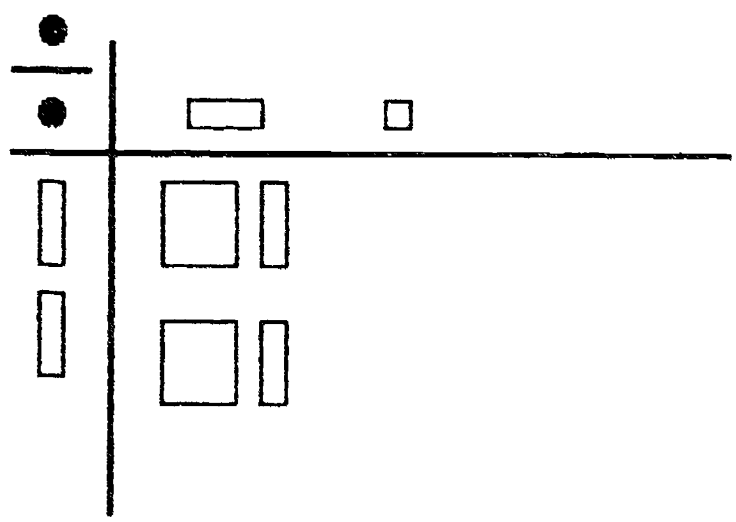
"This is the same result as we got earlier."

Do a second example of dividing by the base place.



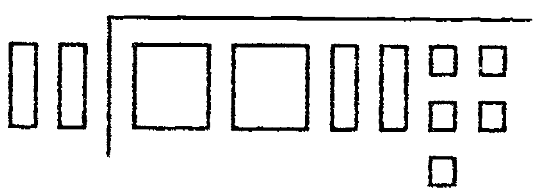
225 ÷ 20

"How do we arrange the base ten pieces so the rectangle has a side this long?"



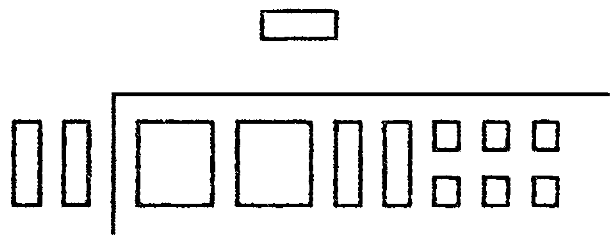
"The five ONES are a remainder." So: 225 ÷ 20 = 11 R 5

Put these base ten pieces in the computation form:

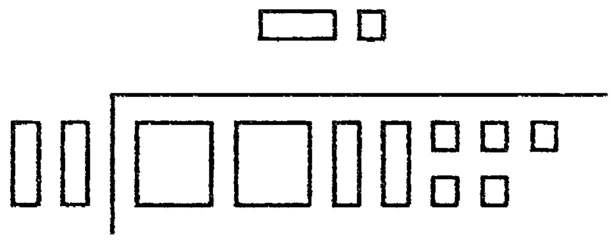


"What is two HUNDREDS divided by two TENS?" (ONE TEN)

"Two divided by two is one - and it is TENS."



"What place is TWO TENS divided by two TENS?.......How many?"



"Can we divide the ones by tens? - so the five ONES are a remainder."

Write:

```
      10 = 11 R 5
  2 | 225
     -200
       25
      -20
        5 remainder
```

Do as many examples as needed. Keep reminding children that:

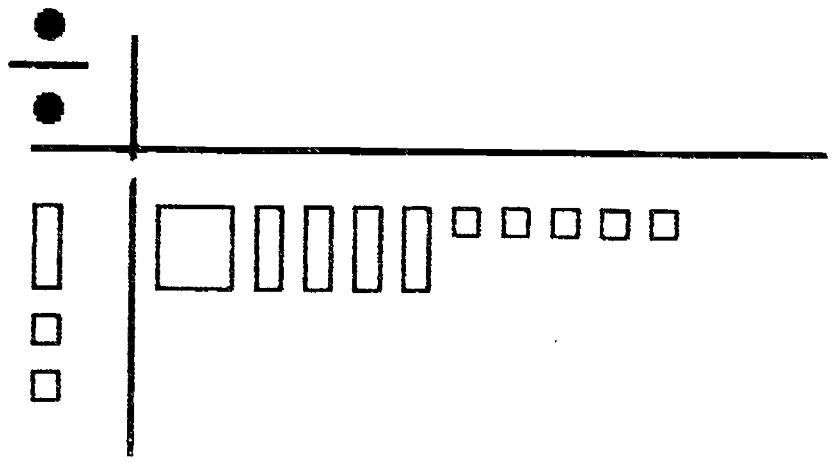
HUNDREDS ÷ TENS = TENS
TENS ÷ TENS = ONES
ONES ÷ TENS = Parts so we have a remainder

This is not easy, so go slowly and repeat as needed.

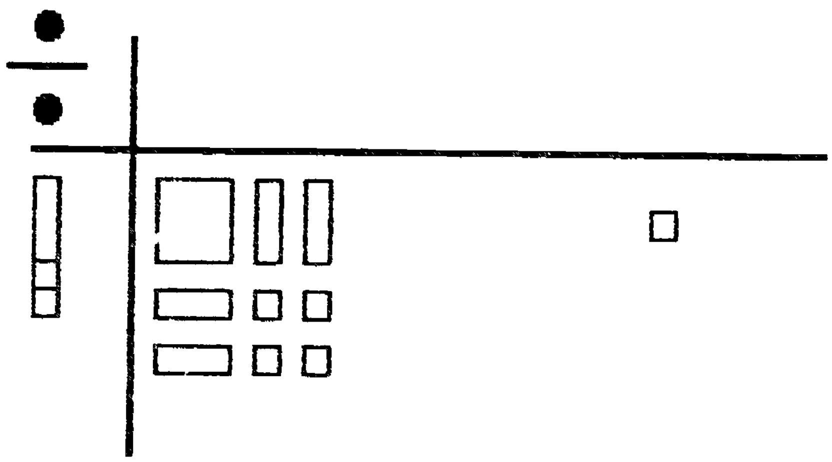
<u>Activity:</u> Have pairs of children use base ten blocks, the division work mat and the recording forms - monitor carefully.

<u>LESSON TWENTY-ONE: Two Digits</u>

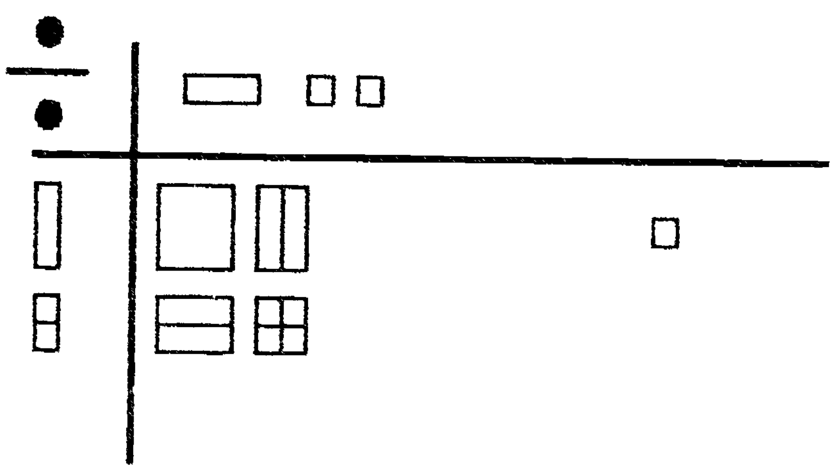
<u>Introduction:</u> This lesson extends division of numbers with hundreds to two digit divisors using the rectangle model. Put the following on the overhead. Have children follow you by arranging base ten pieces on their workmats:



"How should these be rearranged to make a rectangle?"



"Can we use the remaining unit anywhere to make the rectangle larger?" Put in the missing side:



Write: 145 ÷ 12 = 12 R 1 and

```
     12
12 | 145        or 12 R 1
    -144
       1 Remainder
```

Relate this to the computation form:

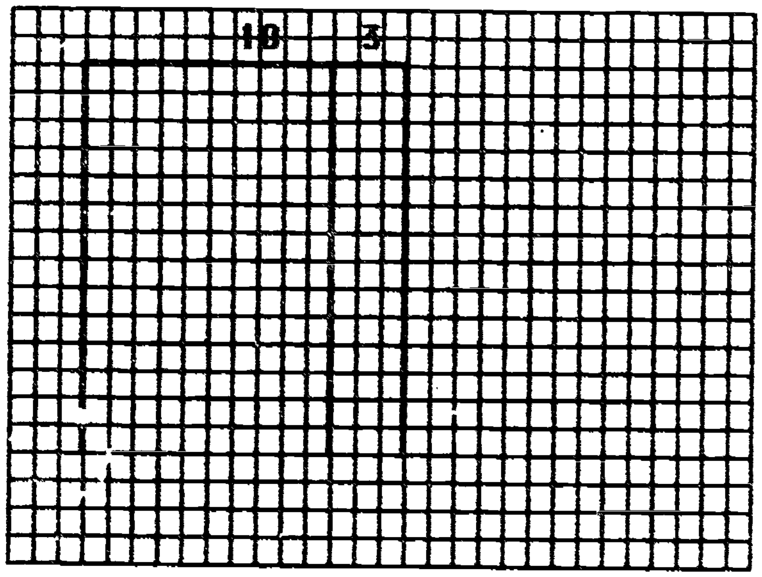
	Hundreds	Tens	Ones	
12 |	1	4	5	
-	1	0	0	(used in the HUNDRED'S piece)
		4	5	
-		2	0	(used in the lower two tens to extend the rectangle to the length needed)
		2	5	
-		2	0	(used in the two tens to extend rectangle missing side as far as possible)
			5	
-			4	(used to fill in the corner of the rectangle)
			1	Remainder

Do other examples as needed.

Activity: Have pairs of children use base ten pieces; division work mats to complete the recording forms. Carefully monitor the activity.

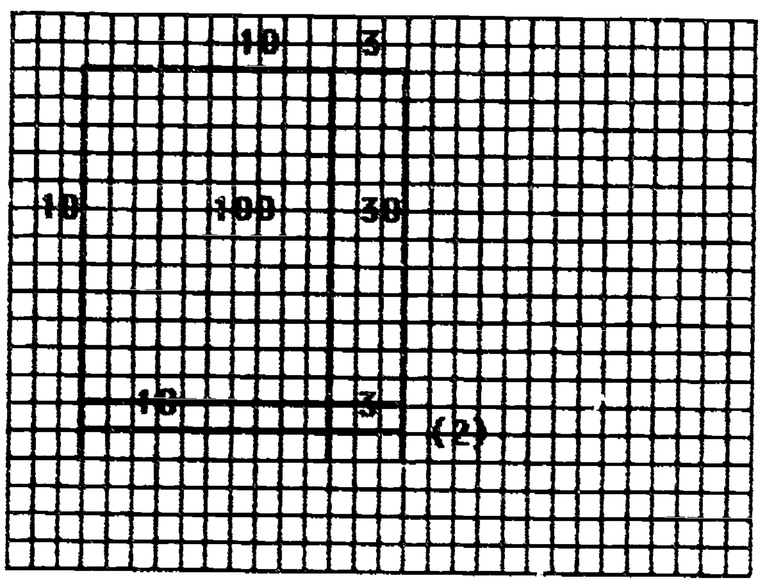
LESSON TWENTY-TWO: Graph Paper Division)

Introduction: Put the following on the overhead projector:



"We have one hundred forty-eight units available to make a rectangle with this side. What is a good estimate of how long the other side should be?"

"How many squares will this use up? Put the line in:



"Is there enough left to extend the side one more unit?"

Put in the second line. Write on the board:

```
 148
-130 (used first)
  18
- 13 (used next)
   5 remainder
```

"What are the five units remaining?" So

Write: 148 ÷ 13 = 11 R 5

```
and     10   = 11 R 5
   13| 148
     - 130
        18
      - 13
         5
```

Do a second rectangle, or use the divisor but a larger number of units available.

Activity: Give pairs of children the worksheets and graph paper and have them record the results of making rectangles on graph paper.

LESSON TWENTY- THREE: Rectangles

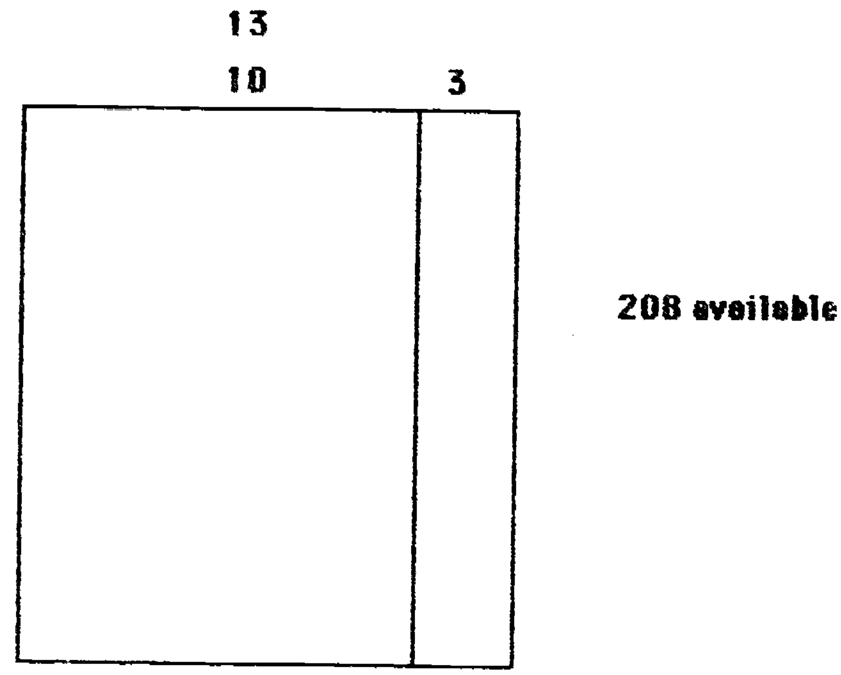
Introduction: When children have a good mastery of the rectangle model for division using graph paper and base ten blocks, they can use a more abstract model.

On the overhead projector write: 208 ÷ 13.

"What is the length of the rectangle we need?"

"How much is available for the area of this rectangle?"

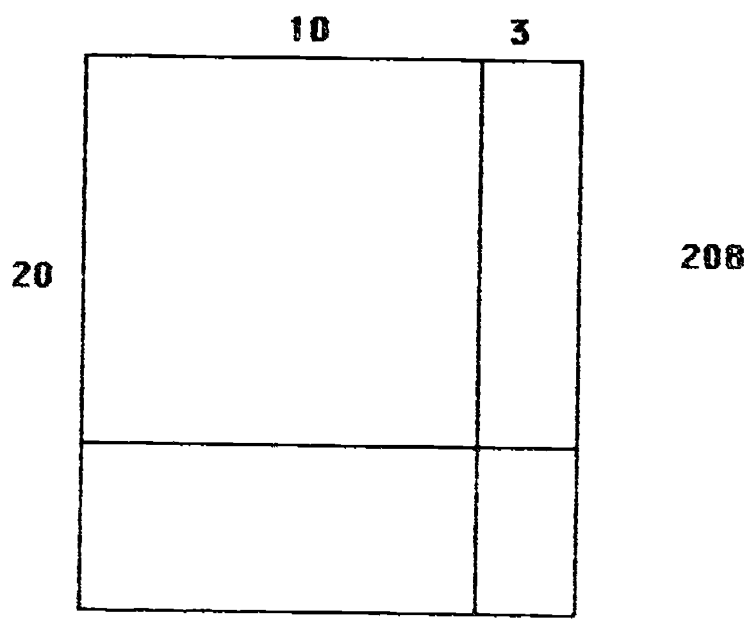
Draw:



"What side can we try for the rectangle?"

"An easy way is to mentally estimate this side as 200 ÷ 10."

Draw the line in:

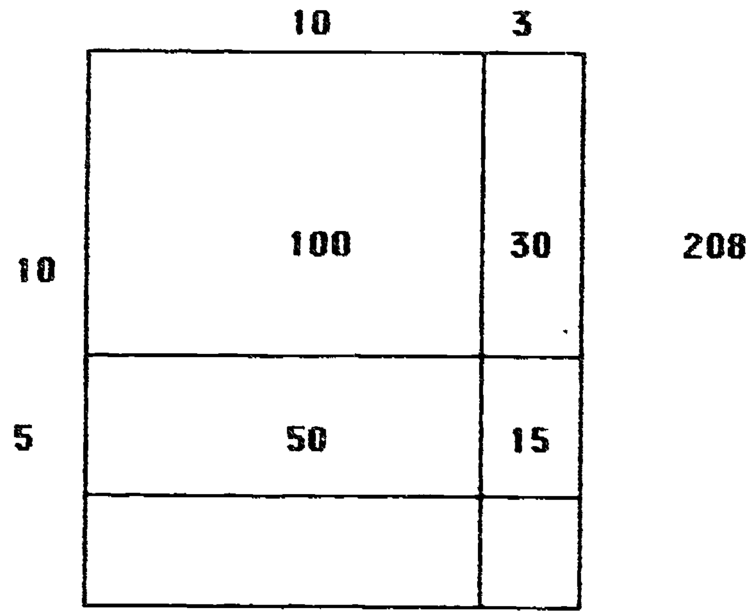


"How much is used by 20 x 10?" (200)

"How much is used by 20 x 3?" (60)

"Do we have 260?" (NO)

"How should we readjust the line?"

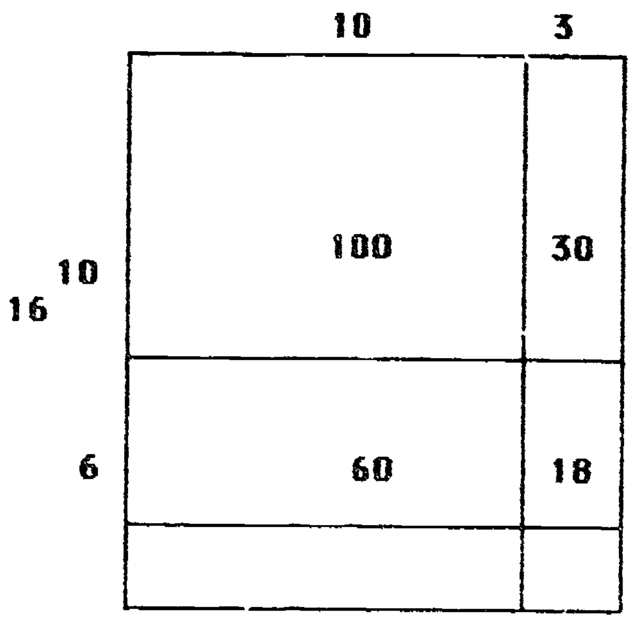


"How much does this use?" (195)

"Do we have 195?"

"How much is left now?" (13)

"We can make the side 16 and use all of these."



"How would we show this on the computat on form?"

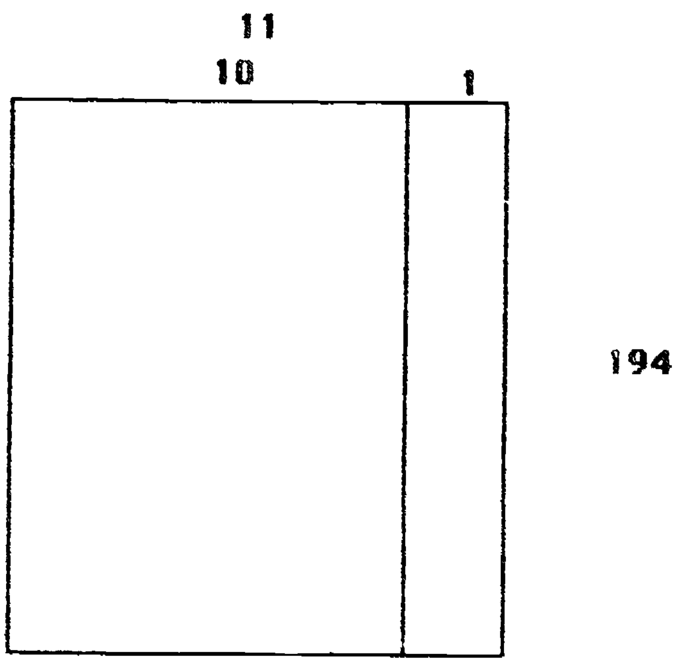
	Hundreds	Tens	Ones	
			6	
			101 = 16	
13 |	2	0	8	
	− 1	3	0	
		7	8	
	−	7	8	
			0	(comes out even)

Do a second example: 194 ÷ 11

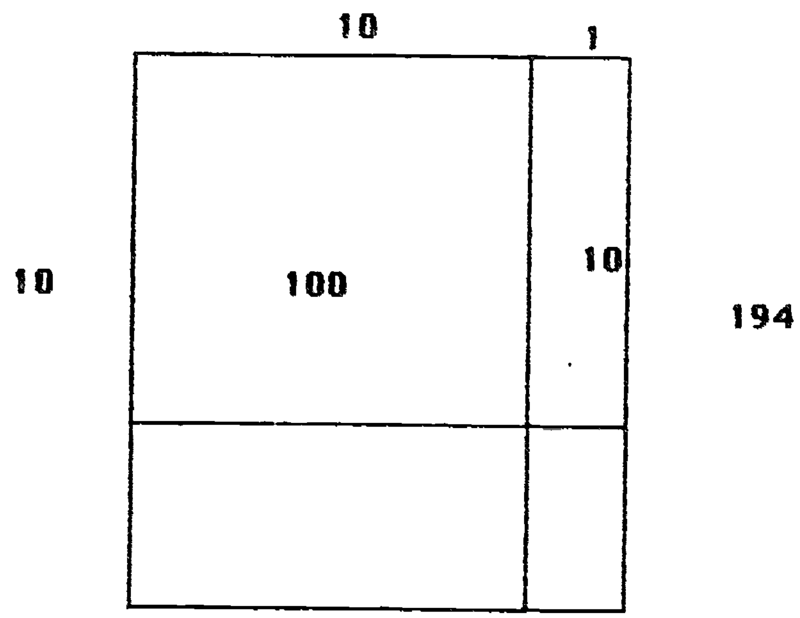
"How long is the given side?"

"How much is available for the area?"

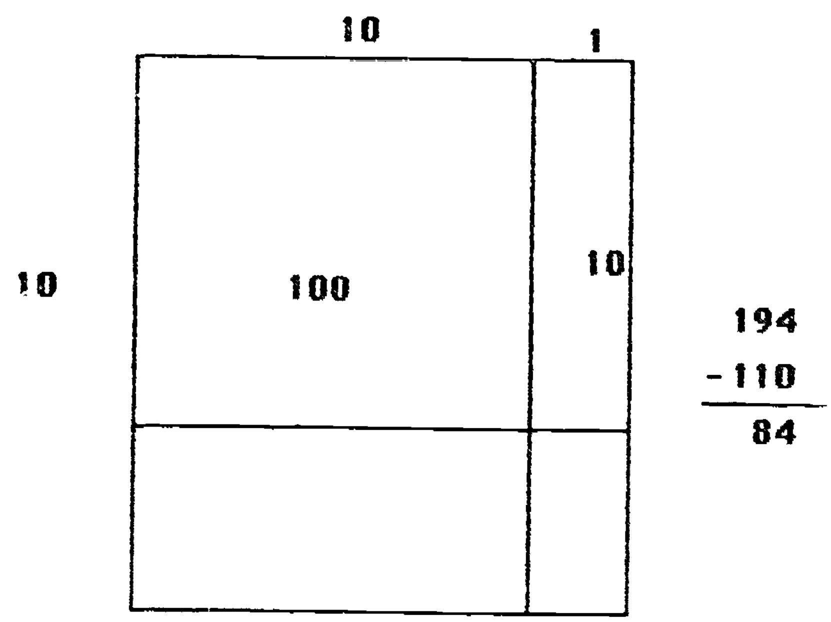
Put on the overhead:



"Try ten first. Why would 20 be too much?" (We don't have 200!)

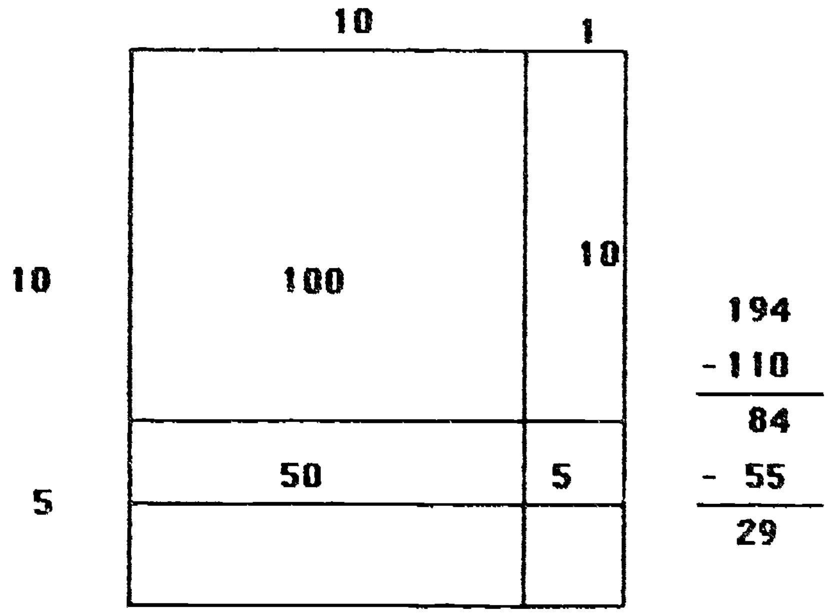


"How much does this use?"

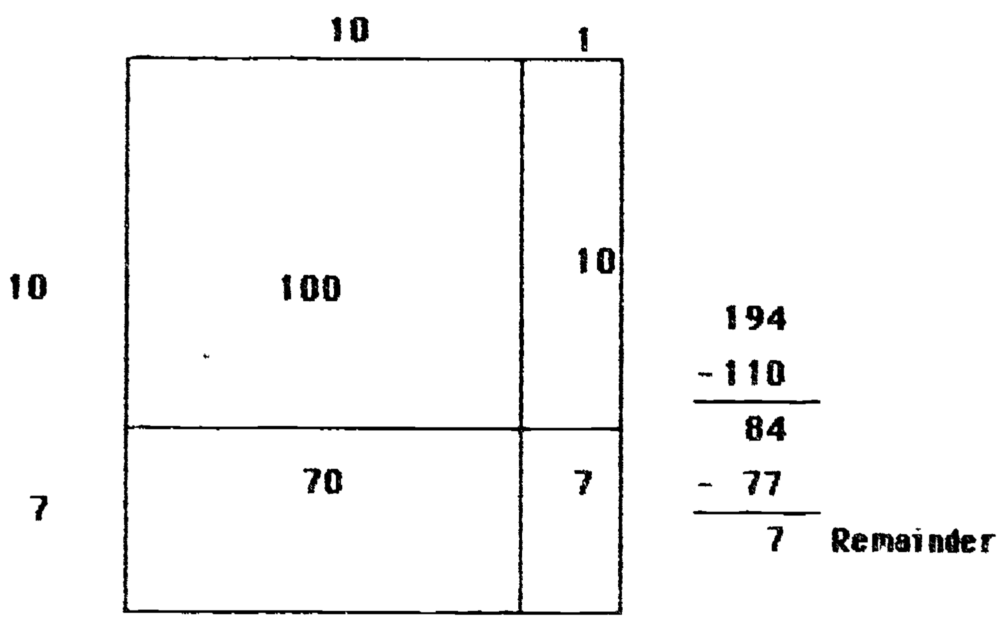


"How much more should we extend the side?"

Discuss how to estimate this in steps. "We'll go 5 more to be safe."



"How much more can we go?" Complete the diagram:



"7 is less than 11 so we have a remainder!"

Write: 194 ÷ 11 = 17 R 7

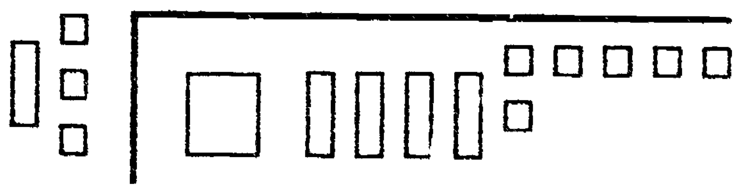
<u>Activity</u>: Pass out the activity sheets and have students first fill in missing parts, then find missing sides, then draw rectangles to do the division.

As you monitor the work, attend to the estimations of quotients. Encourage progression toward the quotient, based on what the children are confident will not require more than is available.

LESSON TWENTY-FOUR: Long Division Algorithm

Introduction: Although the long division algorithm is soon to go the way of the Model T Ford, textbook publishers and writers of standardized test items will continue to bedevil children with this. Change occurs slowly in education.

Use overhead transparency base ten blocks and set up the following: Have children do the same with base ten blocks:



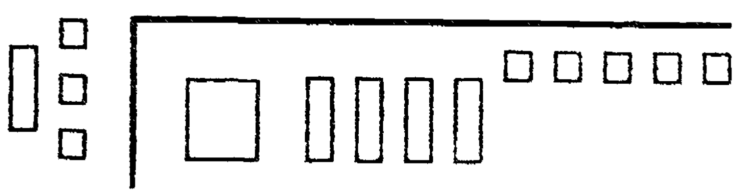
"This shows 13 | 146." (write out)

"In expanded form this would be:"

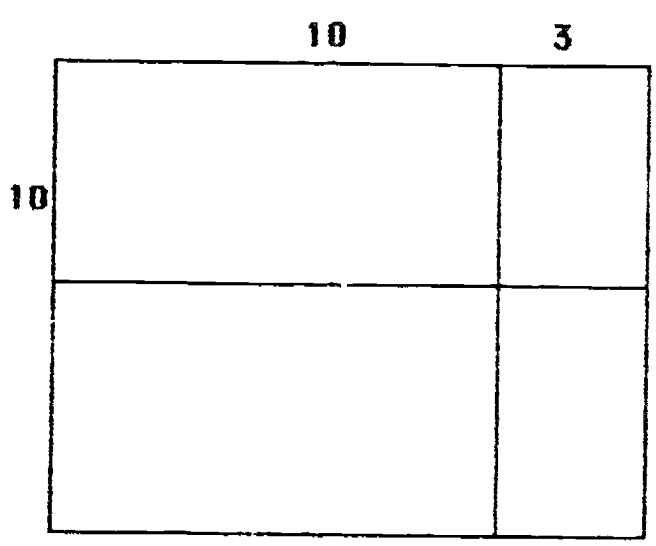
Write: 10 + 3 | 100 + 40 + 6

"This is what we have in base ten blocks."

"What is ONE hundred divided by ONE ten?" Arrange:



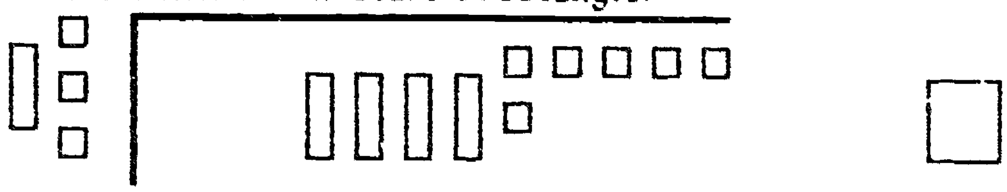
and draw



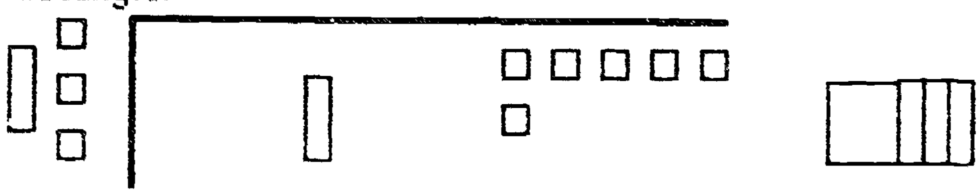
"Now we see how much this uses."

"ONE ten times ONE ten = ONE hundred."

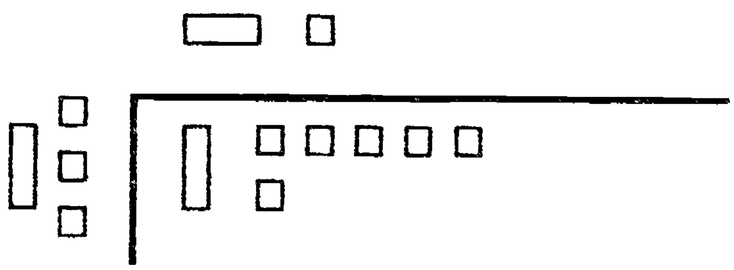
Move tne hundred to start a rectangle:



"ONE ten times THREEE ones gives THREE tens." Move these to extend the rectangle:

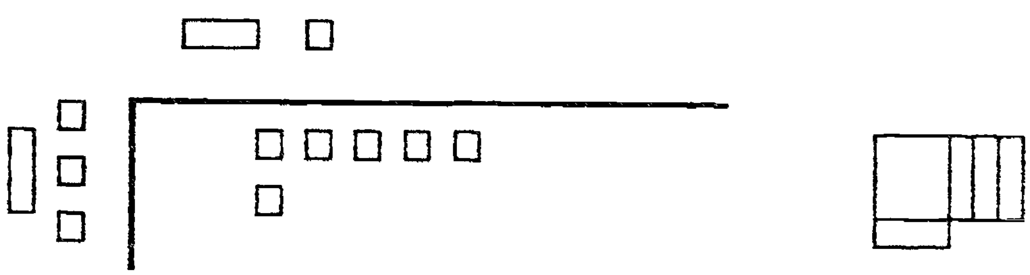


"What is ONE ten divided by ONE ten?" Arrange:

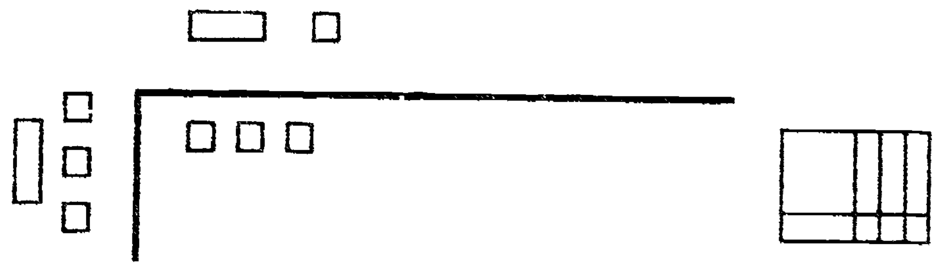


"How much have we used?"

"ONE times ONE ten gives ONE ten. Move this:



"ONE times THREE ones gives THREE ones." Rearrange and place in rectangle:



"What is the quotient?" (11)

"What is the remainder?" (3)

Write:

```
          1
         10  Quotient = 11
      13| 146
        - 130
           16
        -  13
            3  Remainder
```

"We can write this as: 146 ÷ 13 = 11 R 3

Do a second example in the same way

Activity Pairs:

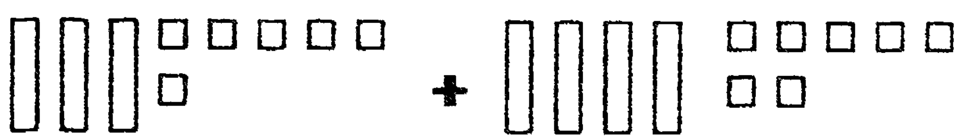
Give children base ten blocks and worksheets. Observe to see if they are following:

1. Dividing largest place by largest place in divisor
2. Multiply to see how much is used and remove this into the rectangle
3. Divides next largest place by largest place in the divisor and repeat 2
4. This should be followed until there is no remainder or the remainder is smaller than the divisor.

This process of repeating a sequence of steps is called an iterative process. The only value in teaching this algorithm is to give experience with an iterative process. These are common. In LOGO, for example, the REPEAT command accomplishes the repetition of a series of steps. In long division, the repetition of steps is on changing values. We use the end result of the last sequence of steps as the starting point to repeat the sequence. This generally considered as RECURSION - also a process with LOGO.

LESSON TWENTY-FIVE: Horizontal Form

Introduction: Review horizontal form for addition. Put base ten pieces on the overhead:



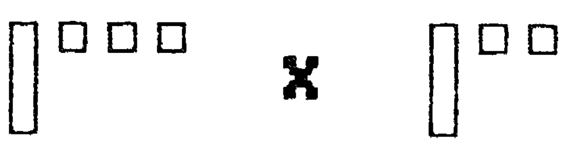
"Can we make a ten of the ones?"

"How many tens are there altogether?"

"How many ones are there after making the ten?"

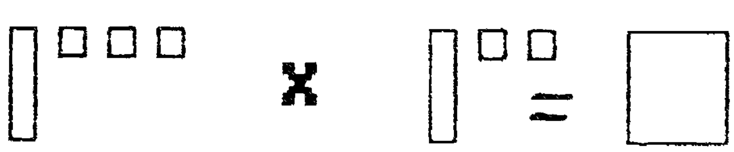
We write this as: 36 + 47 = 83

Put these on the overhead:

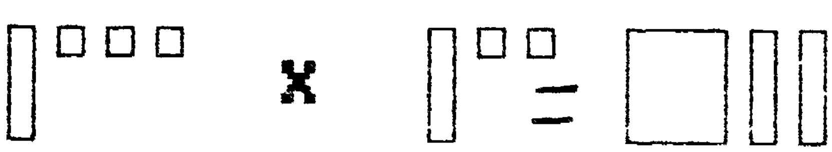


"What is the first TEN times the second TEN?"

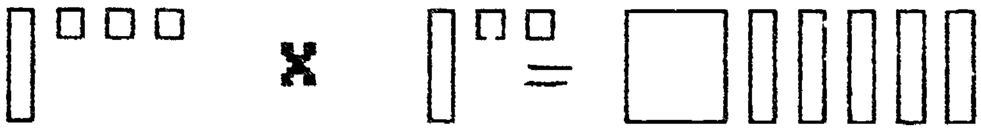
Put the HUNDREDS piece there



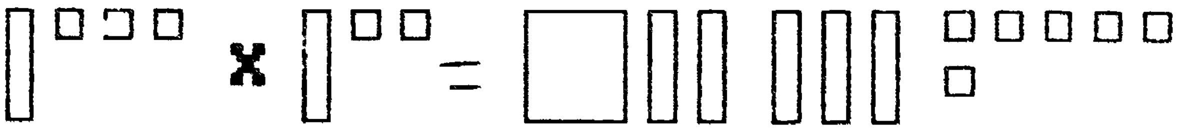
"What is the first ten times the two ones?" Put these there:



"Now we multiply by the first ones. What is the result of multiplying 3 ones times the ten?" Put the 3 tens.



"What is the result of multiplying 3 ones times the 2 ones?"
Put the 6 ones there also:



"We write this as: 13 x 12 or (10 + 3) (10 + 2)

"Multiply first by the left ten, then by the left ones:

13 x 12 = 100

13 x 12 = 100 + 20

13 x 12 = 100 + 20 + 30

13 x 12 = 100 + 20 + 30 + 6 = 156

Do several examples like this with simple 2 digit factors.

Activity: Have pairs of children use base ten blocks to complete the worksheet.

A LAST WORD ON COMPUTATION:

By the end of this level of schooling, children should have had enough experience with numeration, the computation algorithms in base ten and calculators to understand place value and how algorithms are designed so as to make the most effective use of calculators and computers. They should have proficiency with addition and subtraction and with 2 digit divisors and multipliers. They should be able to see how operations with whole numbers relate to operations with decimals. There is no need to develop paper and pencil computational proficiency beyond the level indicated in these lessons provided.

LEVEL FOUR

OPEN SENTENCES

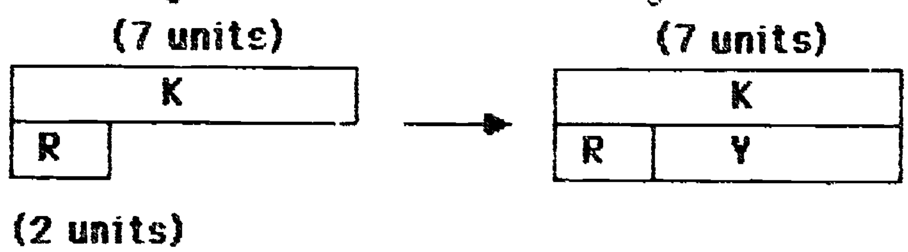
LESSON ONE

Introduction: Children can have concrete experiences with finding missing values in open sentences that are non-numeric. Children should have cuisenaire rods to follow your work at the overhead. Refer to the cuisenaire rod symbol chart in the room:

W = White
R = Red
G = Light Green
P = Purple
Y = Yellow
D = Dark Green
K = Black
N = Brown
E = Blue
O = Orange

Put: ☐ + R = K on the overhead or board.

"Find the rod that goes in the box." Then arrange rods on the overhead



Complete the open sentence: [Y] + R = K

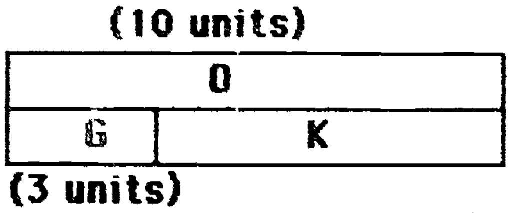
Point out how it is like finding missing numbers in number sentences like ☐ + 8 = 12.

Put a second sentence that uses the "-" sign on the overhead or board. Example:

O - G = ☐

"Find the missing rod for this sentence.

Put the rods:



This missing rod is K. We write:

O − G = [K]

Give open sentences that have the missing element in different locations for children to solve:

E − ☐ = D

☐ − G = P

P + ☐ = F, etc.

Activity: When children have had some experiences with several forms of open sentences, give pairs of children the worksheets and cuisenaire rods to use.

LESSON TWO:

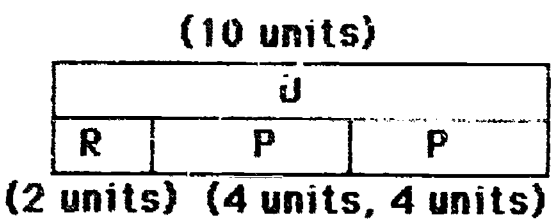
Introduction: This lesson introduces several of the SAME rod to complete a sentence.

Write: 2 ☐ + R = O on the overhead or chalkboard.

" 2 ☐ means 2 of the SAME rod must be used."

"Find 2 of the SAME rod that will make this sentence true."

Arrange rods:



"G was too short and Y was too long, but 2 P rods fit the open space."

Write: 2 [P] + R = O

Go back to the rods and point out that this is the same as **O - R = 2 [P]**

Again, have children solve several open sentences with the missing values in different places using cuisenaire rods:

D = O - 2[ ]

D - 2 [ ] = R

O = 2 [ ] +

2 [ ] - R = D, are some examples.

<u>Activity</u>: Have pairs of children do the worksheets, using cuisenaire rods.

<u>LESSON THREE</u>

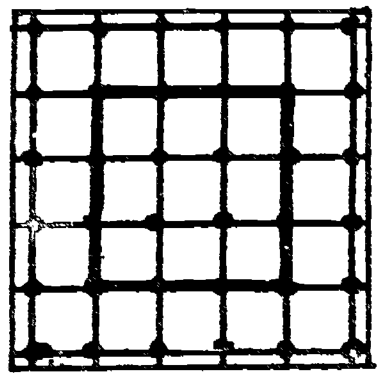
Introduce use of 3[ ] in the same way as 2[ ] and assign worksheets with 3[ ] open sentences upon completion of this.

LEVEL FOUR

GEOMETRY

LESSON ONE: Geoboards

Work with geoboards can be coordinated with work on dot paper. Shapes made on the geoboard can be recorded on dot paper. Graph paper can also be used in connection with geoboard activity. Use an overhead projector version of a geoboard to introduce activities:



"This square is 3 units on each side. What area is included in the square?" (9)

Talk about AREA as measured by UNIT SQUARES on the geoboard.

"What is the distance around this square?" (12)

Talk about PERIMETER as measured by LINEAR UNITS. Linear units are units of length that measure line segments.

Have children build shapes on their geoboards.

"Build a triangle with 4 SQUARE UNITS of AREA."

"Can you find the lengths of all of its sides using the UNITS of length on the geoboard?"

"How did you know the area is 4?"

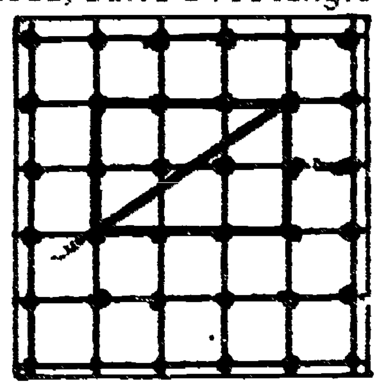
"Build a rectangle with 4 SQUARE UNITS of area."

"Can you find the lengths of its sides?"

"Is it possible to build a rectangle so you can't use the lines on the geoboard to find the lengths of its sides?"

"What is the distance around a rectangle whose sides match the lines on the geoboard?"

On the overhead, build a rectangle as shown:



"What is the area of the rectangle?"

"How many triangles is the rectangle divided into?"

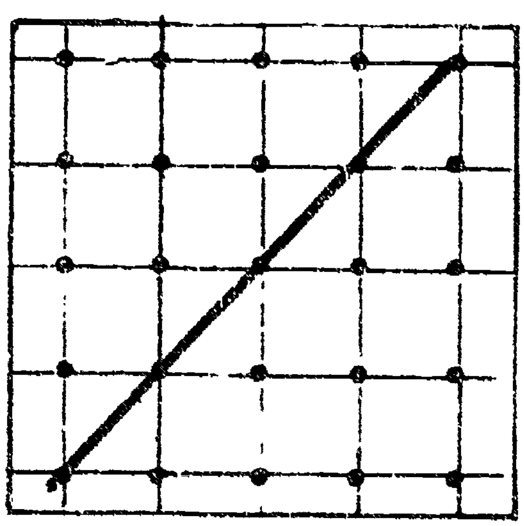
"What is the area of each triangle?"

<u>Activity:</u> Give each pair of children a geoboard, a worksheet and a piece of recording form material. Monitor the work carefully.

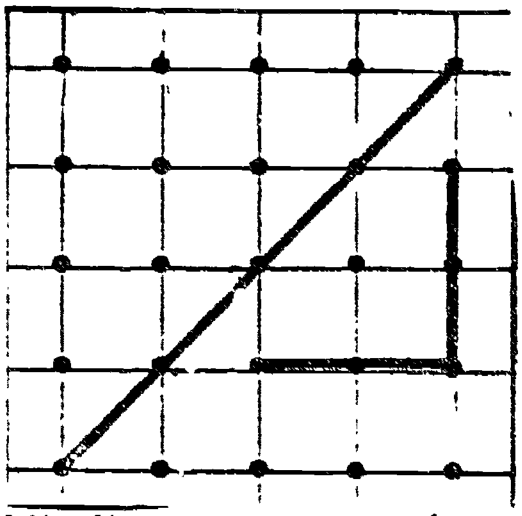
<u>LESSON TWO:</u>

<u>Introduction:</u> In this lesson, symmetry on the geoboard is emphasized.

On the overhead projector, place the following geoboard display:



"I shall build a shape on one side of this line."

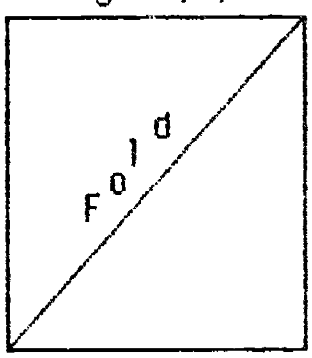


"Think of the line as a mirror."

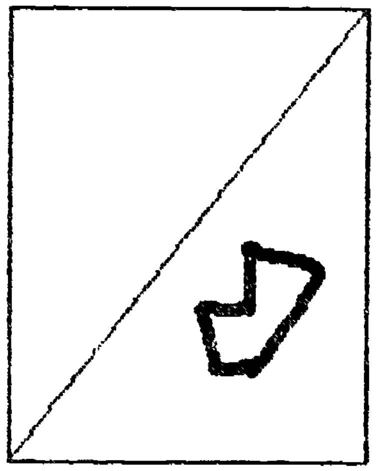
"Would someone put a shape on the other side of the line so it is a 'mirror image' of the first."

If no one can do this, pass out sheets (8 1/2 x 11) of white paper.

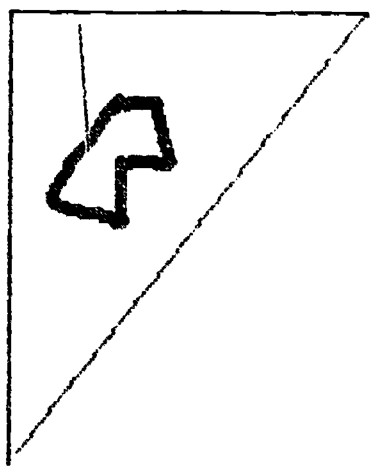
"Fold your paper corner to corner." (demonstrate)



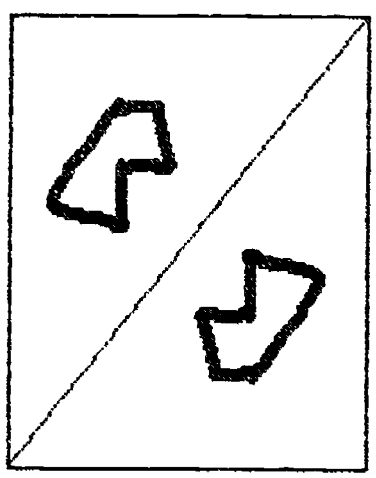
Use a dark color crayon to make a shape on one side of the line:



Fold the sheet. Trace the shape you see through the paper.



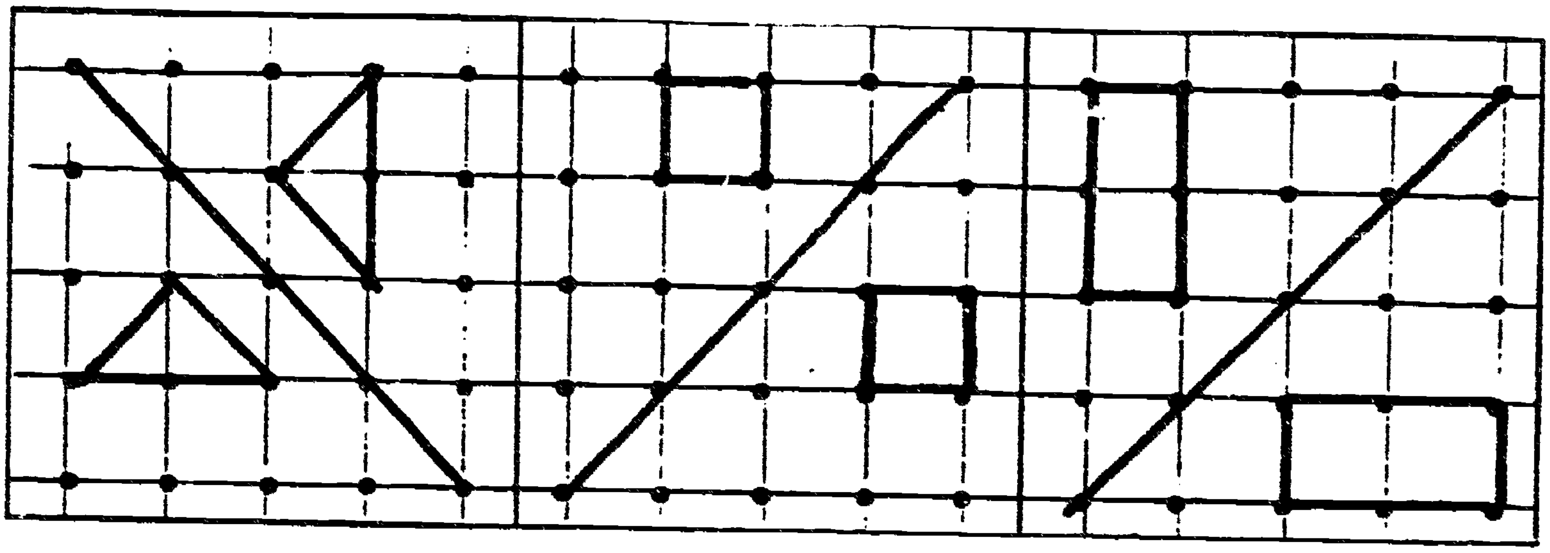
Unfold your sheet and trace that shape on the other side of the line from the first shape you made.



The second shape is a "mirror image" of the first.

Now go back to the geoboard and have a "mirror image" shape made.

Do two or three of these on the geoboard. Point out how points on the shape seem as far from the line on both sides of the line:



Activity: Give pairs of children geoboards and recording forms. Keep reminding them as they work that points must be the same distance from the line on one side as on the other.

LESSON THREE: Tangrams 1

Introduction: Put the tangram square on the overhead projector.

"This is to be ONE unit of area."

"What is the area of each of the six other tangram pieces?"

"If I set the area of this square to be 1/3 square unit, what is the area of each of the other pieces?"

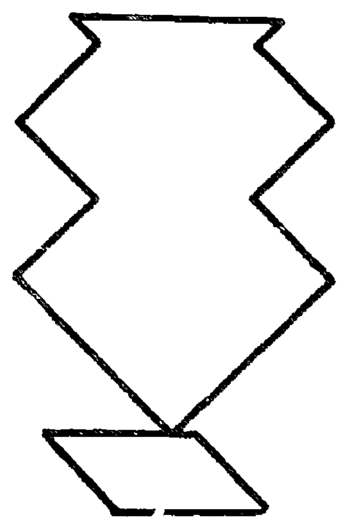
Activity: Use the worksheet with all shapes across the top and a chart to complete. Each pair of children should have a set of tangrams and individual copies of the form to fill in.

LESSON FOUR: Tangrams 2

Introduction: This lesson involves finding the areas of larger shapes made from tangrams using two different "UNITS" to measure.

On the overhead projector place a transparency of a shape that can be covered with tangrams.

(An example, but reduced in size)



"I shall use the area of the square as ONE UNIT."

Cover the shape with tangrams.

"Since the areas of all of the other tangrams can be found when I know the area of the square, I can find the area of this shape."

"The area is eight UNITS."

"You have several shapes to find the areas of using tangrams."

"First cover the shape with as many tangrams as needed. Then find the area of all of these tangrams using the area of the small square as one unit."

"Let the r a of the small triangle be one UNIT. Now, what is the area of this shape?"

"This area should now be sixteen UNITS since the triangle is one-half of the square."

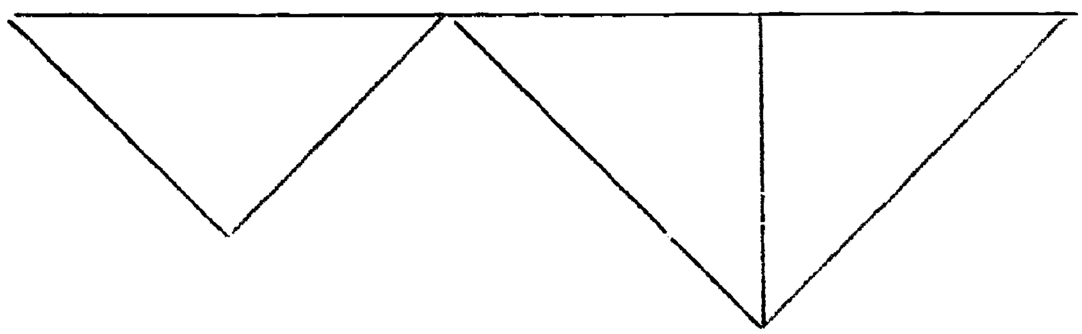
Activity: Assign pairs of children the worksheets. They are to use tangrams and the recording forms. Watch for the use of 1/2 for the small triangle when the square is 1. They may have some trouble seeing this. You may have to remind them there are sixteen of the small triangles represented in the tangram set.

LESSON FIVE: Tangrams 3

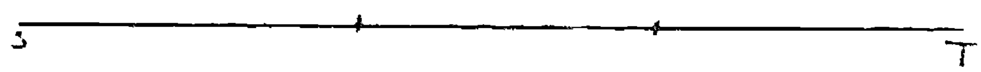
Introduction: This lesson will emphasize PERIMETER as the distance around a shape.

Place one of the large triangles from the tangrams on the overhead projector. Have a line on a transparency as shown:

Lay each side of the triangle in turn on the line marking the side as a segment on the line:



The result will be as shown:



The resulting long segment labelled ST is the perimeter of the shape. This is the sum of the lengths of the sides of the shape.

Do this again, using the tangram parallelogram.

Activity: Give each pair of children a worksheet, recording forms, and a set of tangrams.

LESSON SIX: Geoblocks

Introduction: Hold up the R5 and T9 blocks (see attached labels for the 24 geoblocks attached.)

"How are these alike?"

"How are these different?"

Point to a face of one:

"This is a face."

"Are any faces of one block like the faces of the other blocks?"

"How do you think the amount of wood in one compares with the amount of wood of the other?"

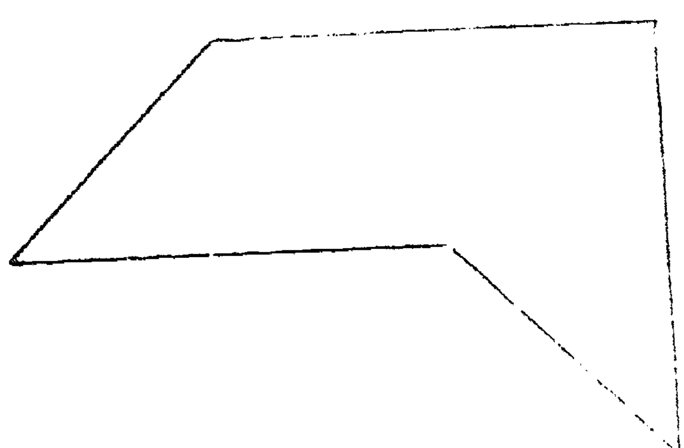
"How many of the TRIANGULAR blocks would be needed to make the other one?"

Activity: Put a set of geoblocks at a center. Prepare worksheets from sources available commercially, or use some that are attached. Give children an opportunity to explore a variety of topics using these.

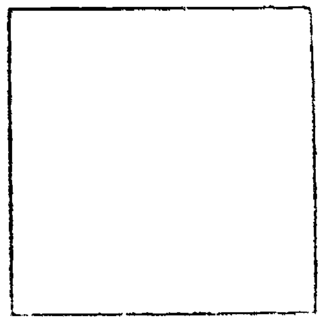
LESSON SEVEN: Tangrams 4

Introduction: This lesson is to give children experience with finding areas and perimeters of unusual shapes.

On the overhead, place a transparency of a shape made from tangrams such as this:

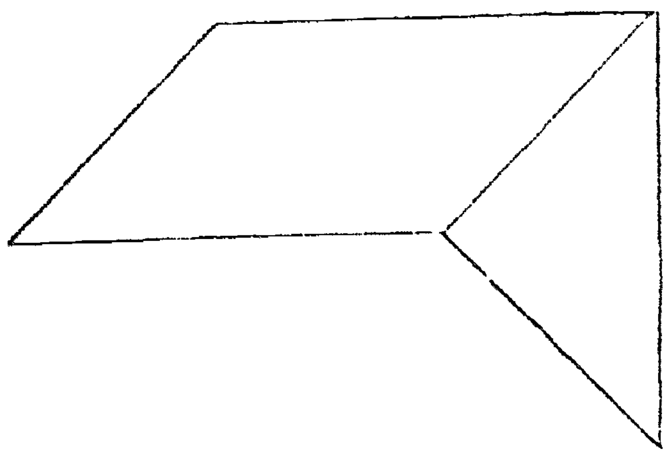


"How would we find the area of this shape using:



as ONE?"

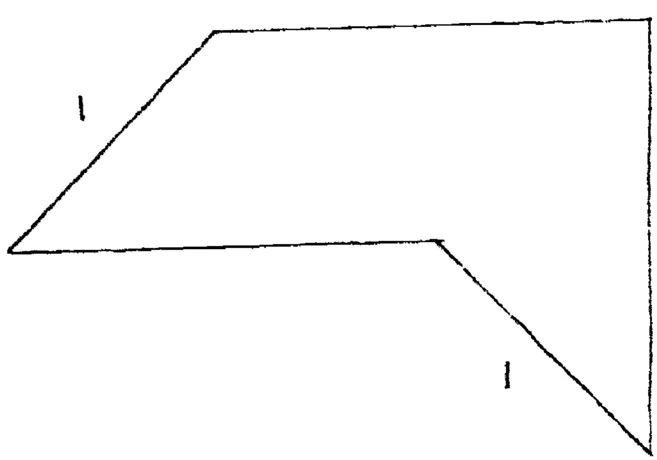
After consideration of student responses, fit the tangrams on the shape:



Emphasize the idea of the value of the triangle as 1/2

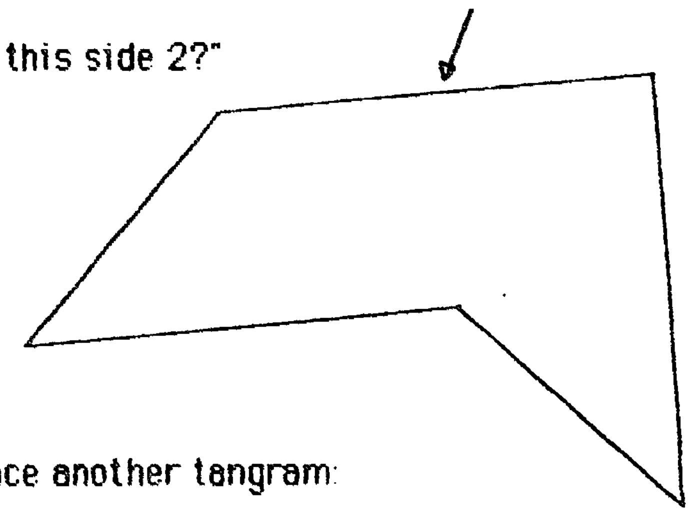
"The area of this is 1 1/2."

"If a side of the square is a unit of length, we can find the PERIMETER of this shape."

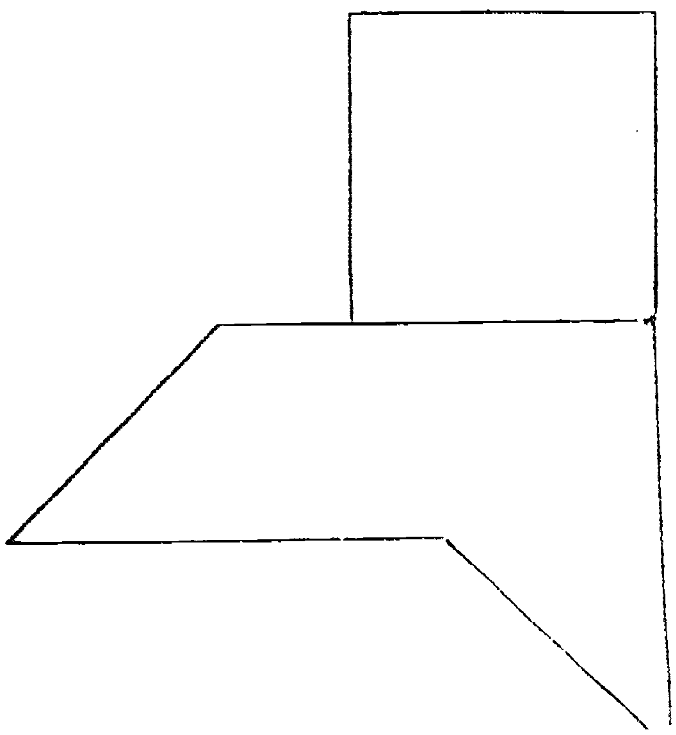


"We know the sides labelled, but have to ESTIMATE the last side. Is it more than one?"

"Is this side 2?"



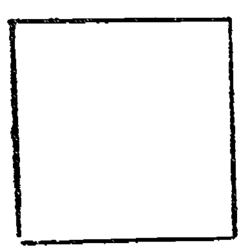
Place another tangram:



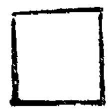
"It looks like a little less than 1 1/2. How many sides like this do we have? The perimeter is 1 + 1 + 3 + almost one or a little less than 6."

Remind the children that area and perimeter are measured in units that are chosen by us. It is important to compare things by using the same units, whatever they are. Some common units of length are inches, feet, centimeters, meter. Area measures are squares with these lengths as sides. We have square inches, square centimeters, square feet, etc.

"This is a square inch:"



"This is a square centimeter:"

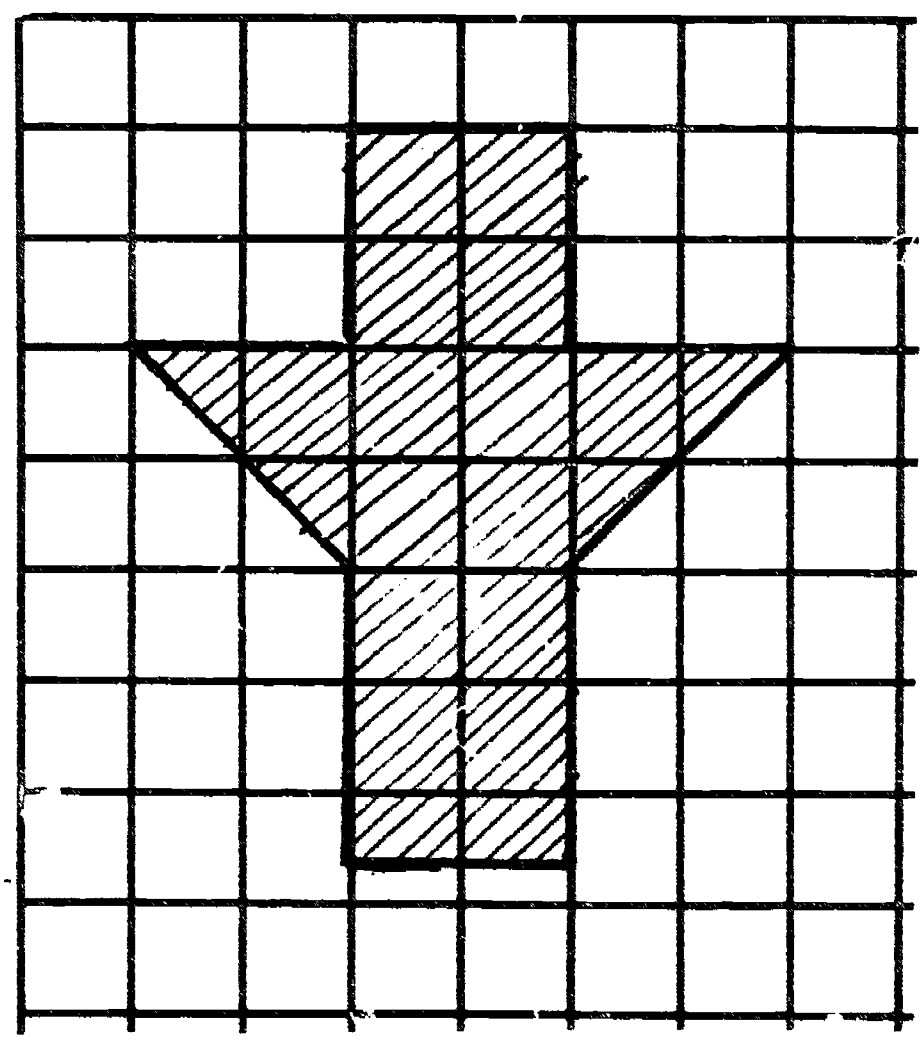


Make these out of graph paper. Show how many square centimeters will fit on a square inch.

Activity: Give pairs of children the shape sheets, recording forms and tangrams. Monitor their work closely.

LESSON EIGHT: Geoblocks Surface Area

Introduction: Use an appropriate graph paper transparency. While the children watch, trace each face of a geoblock onto the graph paper. The SURFACE AREA is the sum of the areas of all exposed FACES of the solid:



This is the surface area of a T-1 geoblock ~ 17 square units. Each of these is 1/2 x 1/2 = 1/4 square inch, so the surfaces are ~ 4 1/4 square inches.

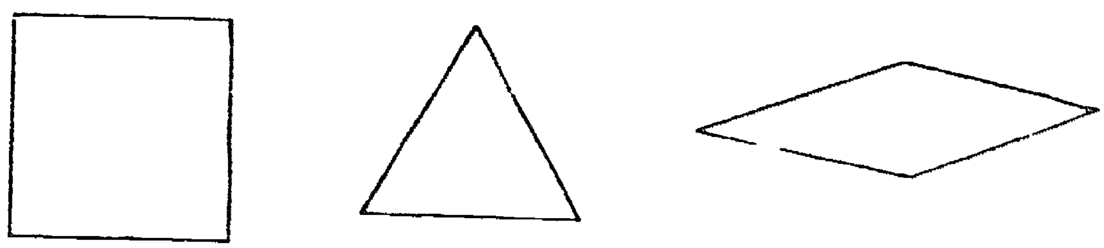
This is the area that would have to be painted or covered if the block were to be painted or covered with fabric.

At the geoblock center, provide the kind of graph paper appropriate for the geoblocks and have the children make layouts like that provided for all of the geoblocks at some time during the year.

<u>LESSON NINE: Pattern Blocks</u>

<u>Introduction:</u> Children will have had considerable experience with pattern blocks. This lesson is to focus on the geometric properties of the blocks. Give the children one period of free play with the blocks to remind them of past lessons.

On the overhead projector, place 3 of the pattern blocks in transparency form.:



"How many different lengths are on the sides of these blocks?" (one)

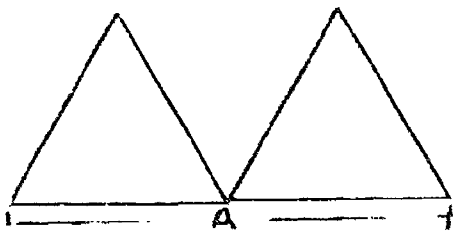
"Do the squcre and diamond shape have the same perimeter?"

"How can you compare the areas of these?"

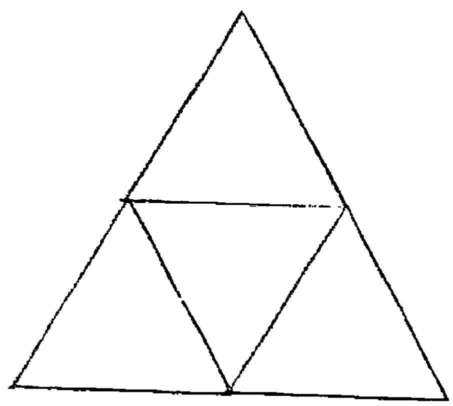
"Which shape has the largest angle?"

"Are any angles doubles or halves of any other angles?"

Put two shapes together:



"This length (point to A) is twice the length of the side of the triangle."
Complete the shape:



"This triangle has sides that are how many times as long as the side of one smaller triangle?"

"The area of this larger triangle is how many times the area of a smaller triangle?"

Put a triangle on top of a diamond.

"Are any of the angles of the triangle the same as some angle of the diamond?"

"Are any of the angles of the diamond the same as any angles of the square?"

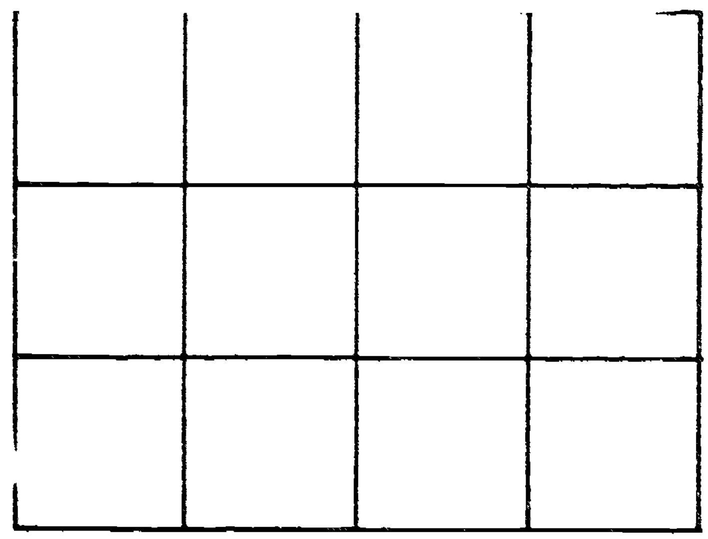
"Which shape has the largest angle and the smallest angle on the same shape?"

Activity: Give pairs of children pattern blocks and the recording forms. As you monitor the work, keep referring to the angles on the shapes, the sides of the shapes and the areas of the shapes.

LESSON TEN: Multilinks

Introduction: This lesson is to give children more experience with building rectangular solids to see the cubic units in them. Use multilinks (or 2 cm. on edge), or wooden cubes 3/4" or 1" on edge.

Put the transparency of a "floor" on the overhead:



"Build a floor like this out of your multilinks."

"How many multilinks did you use?"

"How many will be needed for the second floor."

"Build the second floor."

"How many cubes have you used altogether?"
"Build a third floor."
"Without counting, tell how many cubes are in the 'house'."
"How many cubes would be in a four story house like this?"

Activity: Give pairs of children multilinks or cubes that are even numbered centimeters on the edges. They should have copies of the "floors" and the recording form to use. If they don't have enough multilinks to make the higher numbers of floors, have them use the information they have found to extend into these higher numbers.

LEVEL FOUR

COMPUTATION WITH MONEY

LESSON ONE

Introduction: Review addition and subtraction with multidigit representations. Be certain the operations of "carrying" and "borrowing" are well in hand. Use base ten blocks with those who still need them.

Put the following on the overhead projector and work through it step-by-step, asking frequent questions of the children.

	Check
432	211
-221	+221
221	432
234	37
-197	+197
37	234

Emphasize lining digits up in the right place value "columns." Do as many examples as needed of both addition and subtraction.

Activity: Have children do the worksheets. Watch for the "carrying" and "borrowing", particularly in going from the tens to the hundreds place. Remind the children of the inverse nature of addition and subtraction and of use of one operation to check the results of the other.

LESSON TWO

Introduction: This lesson is to give children experience with the money forms. They need not understand decimals for this since what is to the right of the decimal point is "cents" and to the left is "dollars." Do some examples of computation:

\$ 4.32	\$ 5.00
+\$ 1.04	-\$ 2.98
\$ 5.36	\$ 2.02

Emphasize:

1. The use of the decimal point to separate dollars and cents;
2. Use of the $ sign
3. Correct "borrowing"
   Correct "carrying"
4. Rounding to check neighborhood of answers.

Example:

$$\begin{array}{r} \$5.00 \\ - \underline{2.98} \end{array}$$

"$2.98 is almost $3.00 so the answer should be close to $2.00." Money is a good topic to emphasize rounding to estimate since so many prices end in 9 or 8.

<u>Activity:</u> Pass out the worksheets for children to work with. If you have "play money", allow the use of this for those who need it.

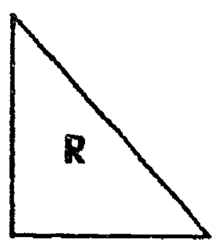
LOGIC

Background: If-then logic rests upon a foundation of recognizing similarities and differences, distinguishing between AND and OR and understanding how NOT separates a collection into 2 parts. Students will have had some experience with classification, but these lessons place emphasis on the above.

LESSON ONE: Similarities and Differences

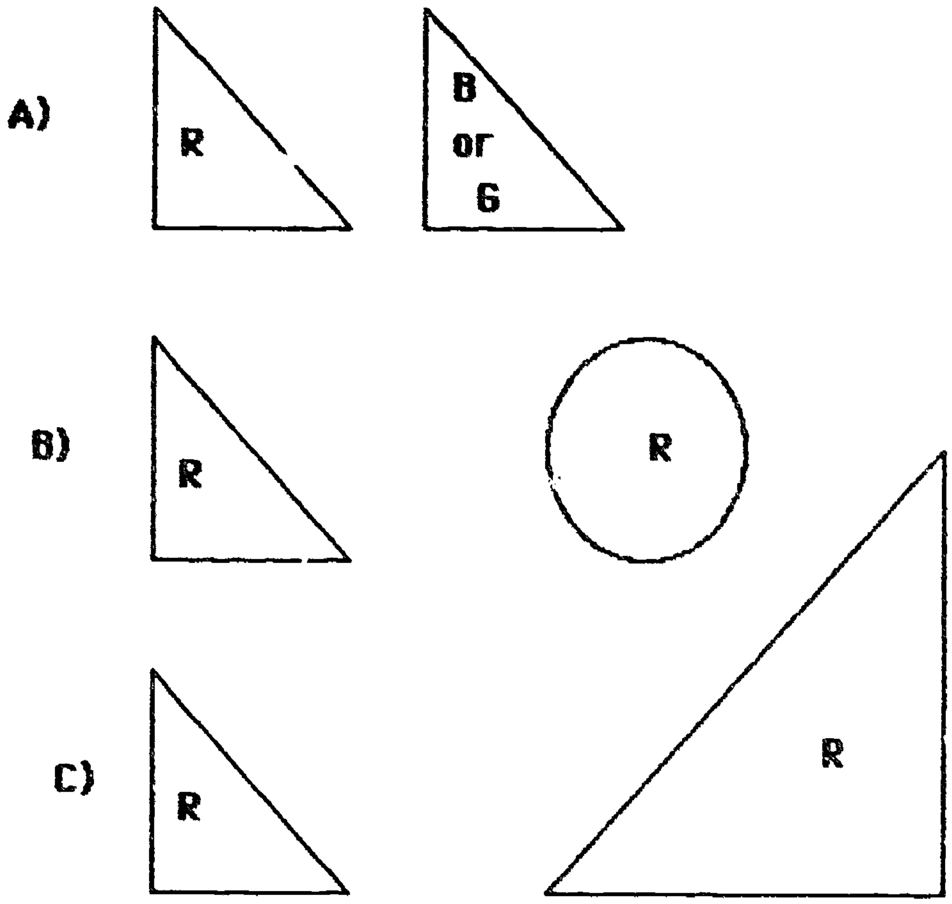
Introduction: You can use logic blocks, ESS type attribute blocks or attribute blocks made from the template provided. Use an overhead transparency version of the template provided that has been made from colored film.

Children should have a set of the logic blocks. The template can be used to make four different colors of the blocks. That gives you a set with 2 sizes, 4 shapes and 3 colors - 24 in all.



"Hold up a piece that differs from this in one way. (Large triangle of same color [size], small triangle of a different color [color], small square, rectangle or circle [shape])."

Put the piece suggested THAT DOES DIFFER ONLY IN SIZE OR SHAPE OR COLOR alongside the original. You could get any of these arrangements:



If A), ask for another piece that differs from that triangle in exactly one way.

Continue until you have a line of 4 or 5 pieces, each differing from its neighbors in <u>exactly one way.</u>

<u>Activity:</u> Have children work in pairs to arrange all of the 24 pieces in a circle so that each piece differs from its neighbors in EXACTLY ONE WAY.

<u>LESSON TWO:</u>

Just as in Lesson One, except each piece differs from the piece next to it in exactly TWO ways.

<u>LESSON THREE:</u>

Just as in Lesson One, except each piece differs from the piece next to it in exactly THREE ways.

In each case have children record by drawing the arrangement of blocks, to size and colored correctly with crayons. They can arrange the blocks on a large piece of white wrapping paper, trace them in place, then color the shapes as they are.

LESSON FOUR: And

Introduction: AND indicates the joint presence of two or more properties, actions, etc. Children acquire a multiple classification schema slowly, but most should be at that stage of development at this level. This lesson emphasizes this idea. Children should have the attribute materials.

Put a transparent attribute piece on the overhead projector:

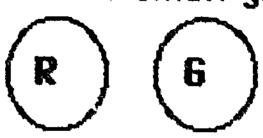
"Describe this using AND." (small and red and circle)

"Do you have any red circles that are not small?"

"Do you have any small pieces that are not red? not circle?"

"Do you have any circles that are not red?"

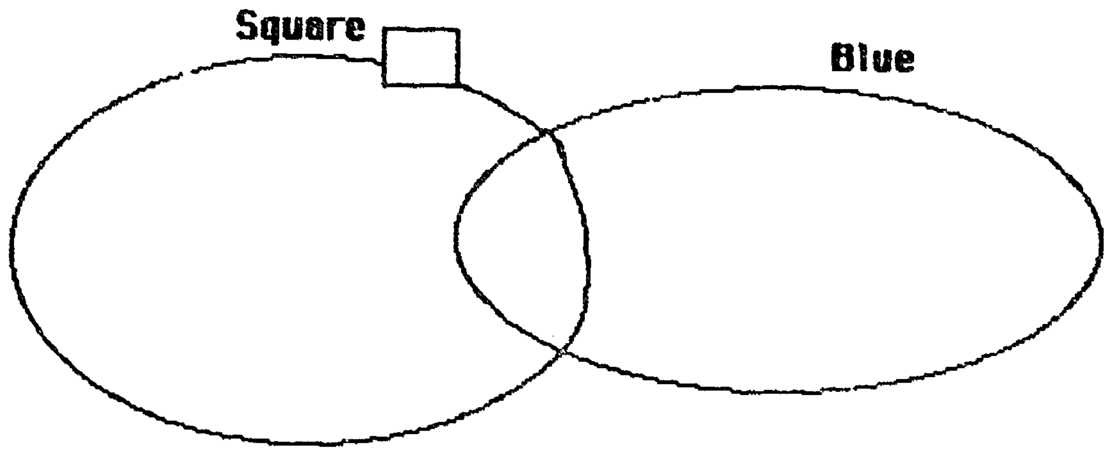
Place a small green circle on the overhead:



"How can you describe both of these using AND?" (small and circle)

"Why can't both pieces be included in a description with a specific color?"

Place a transparent version of the workmat on the overhead with the closed curves labelled as shown:

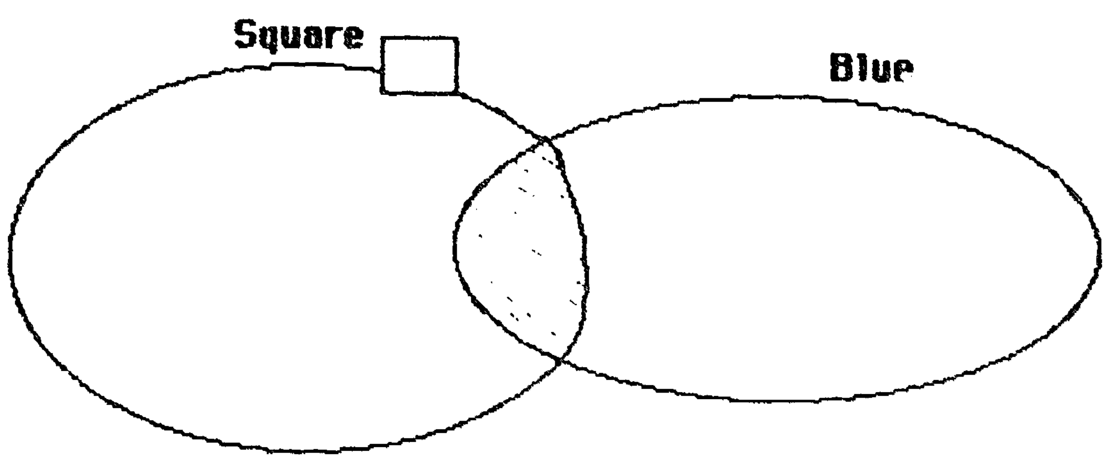


"All squares are to be placed inside the loop labelled □ "

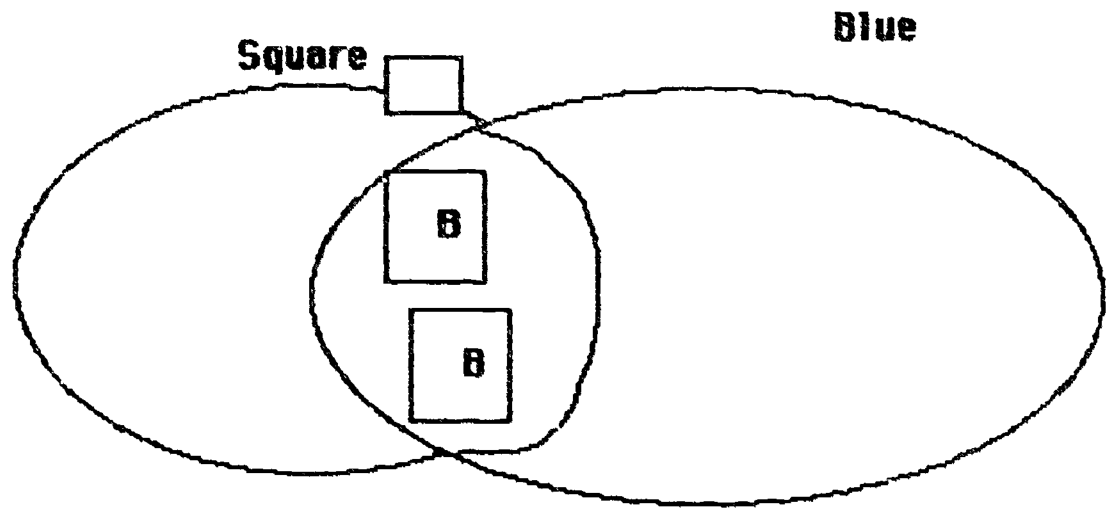
Point to that loop and trace it with your finger. "All blue pieces are to be placed inside the loop labelled BLUE."

Trace that loop with your finger. "Where would the pieces that are blue AND square go?"

Point to that region marked //// below, which is AND



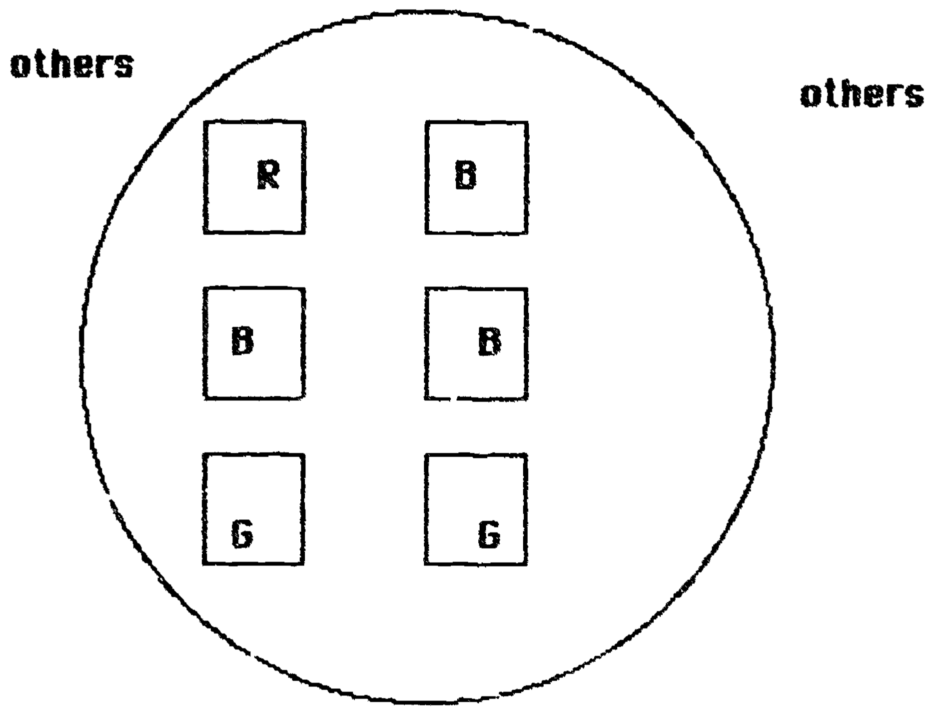
Put those pieces in that region:



Activity: Workmats with two closed regions are provided. Use these to make larger, more permanent versions that can be laminated. Give pairs of children a workmat, a set of 24 pieces and the activity sheet. Use the template to make additional worksheets like those furnished.

LESSON FIVE: Not

Introduction: NOT separates a collection into 2 groups - those that satisfy a requirement, possess a property, etc., and those that do not. Place single loop on the overhead. Place ALL squares inside it. Place the rest of the pieces around the loop and outside it.



"How can we describe the pieces outside the loop using TWO words?"

Several responses might come. Children could try describing those outside in detail, i.e., "triangles and circles and rectangles and small and red, etc."

Reinforce the response: NOT SQUARE

Discuss how NOT shows the absence of something.

Put all green pieces inside the loop, the others outside and ask for a two word description of those outside.

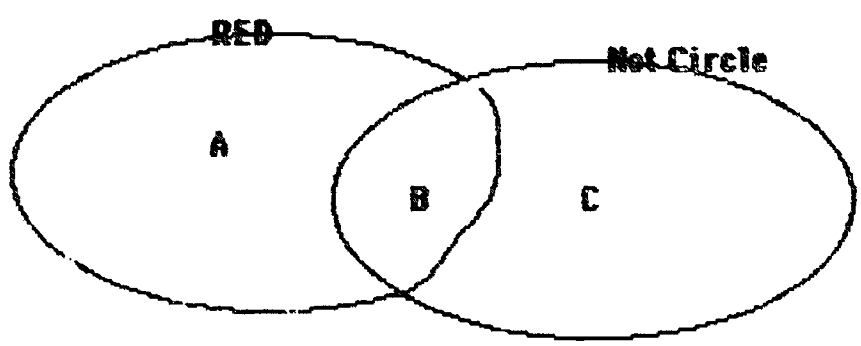
Repeat with others, such as "small", "0", etc., until children can use NOT easily to show the absence of the specified property.

Activity: Give pairs of children workmats with a single loop, a set of 24 attribute pieces and the recording form. Use the template to make additional worksheets like those furnished.

LESSON SIX: AND with NOT

Introduction: Using AND and NOT together presents a challenge to children. They have trouble seeing that NOT NOT (property) means HAVING the property so that NOT NOT SQUARE is SQUARE. This was presented in the NOT worksheets. Be sure that idea is clear before using this lesson.

Use the transparent two loop workmat and the transparent attribute pieces. Label the two loops as shown. Label the regions A, B & C.



"Where should I put the small red triangle – in A, B or C?"

"Where should I put the small red circle?"

Continue this way until all pieces have been sorted. Go slowly enough to deal with any confusion about how the NOT CIRCLE label influences where pieces are placed. Keep discussing the placement of pieces because "it IS ______," or it is NOT ______." Pay attention to those that go outside (D).

Do a second case but ask the children for pieces that go in A, then into B, then into C, then outside BOTH loops (D).

Activity: Give pairs of children two loop workmats, a set of 24 attribute pieces, and the worksheets. Watch the sorting very carefully to be sure the NOT is being used correctly. Children WILL have some problems with this, but they must be presented with the challenge. Use the template to make additional worksheets like those furnished.

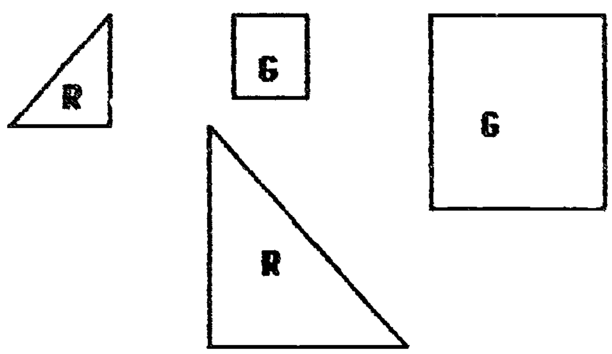
LESSON SEVEN: Extras for Experts

Introduction: This lesson extends the idea of multiple classification to THREE requirements to be satisfied. This can be tried with the whole class by introducing it on the overhead projector as in previous lessons. It probably is better used with the more advanced students. Use the attached worksheets.

LESSON EIGHT: IF-THEN

Background: After children have mastered AND, NOT and a combination of AND with NOT, they should be familiar enough with the materials to be introduced to if-then reasoning.

Introduction: Place a small collection of the attribute materials on the overhead projector:



"If a piece that I see is a triangle, what is its color?"
"We can then say, 'If triangle, then red.'"

Write:

If Triangle, then red.

"If a piece that I see is green, what must the shape be?"

Write: If green, then square.

"What other if-then statements like these are TRUE about the shapes on the overhead?"

Write all of these down.

If red, then triangle
If triangle, then red
If green, then square
If square, then green

Put a box around one

If square, then green

"What piece in our set can I put on the overhead so ONLY this statement will no longer be true?" (any non-green square). Put one up.

"Which part - if or then - must be made false in order to make the whole statement false?"

Remove the non-green square so the original collection is there. Highlight one of the other statements.

"What piece in our set can I put on the overhead to make ONLY this statement FALSE?"

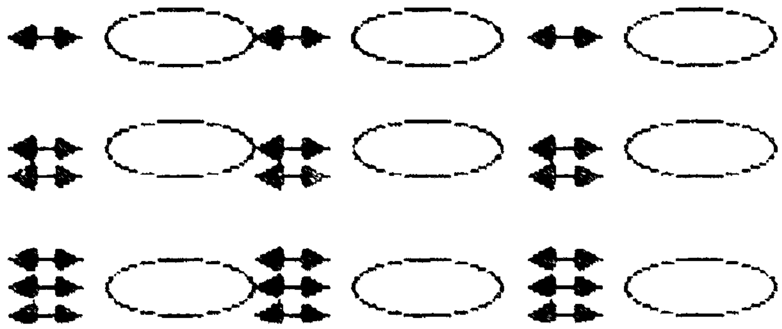
Repeat this activity with different subsets of the 24 piece attribute set until children are adept at (1) finding statements true of the collection, and (2) finding a piece to negate one of the true statements as selected.

Activity: Give pairs of children attribute materials, and the recording forms. As you monitor the work, keep asking how a true statement that is written might be negated by adding a piece to the collection used.

ACTIVITIES FOR LOGIC LESSONS

Difference Activities:

Have students place the blocks in lines in the following ways:

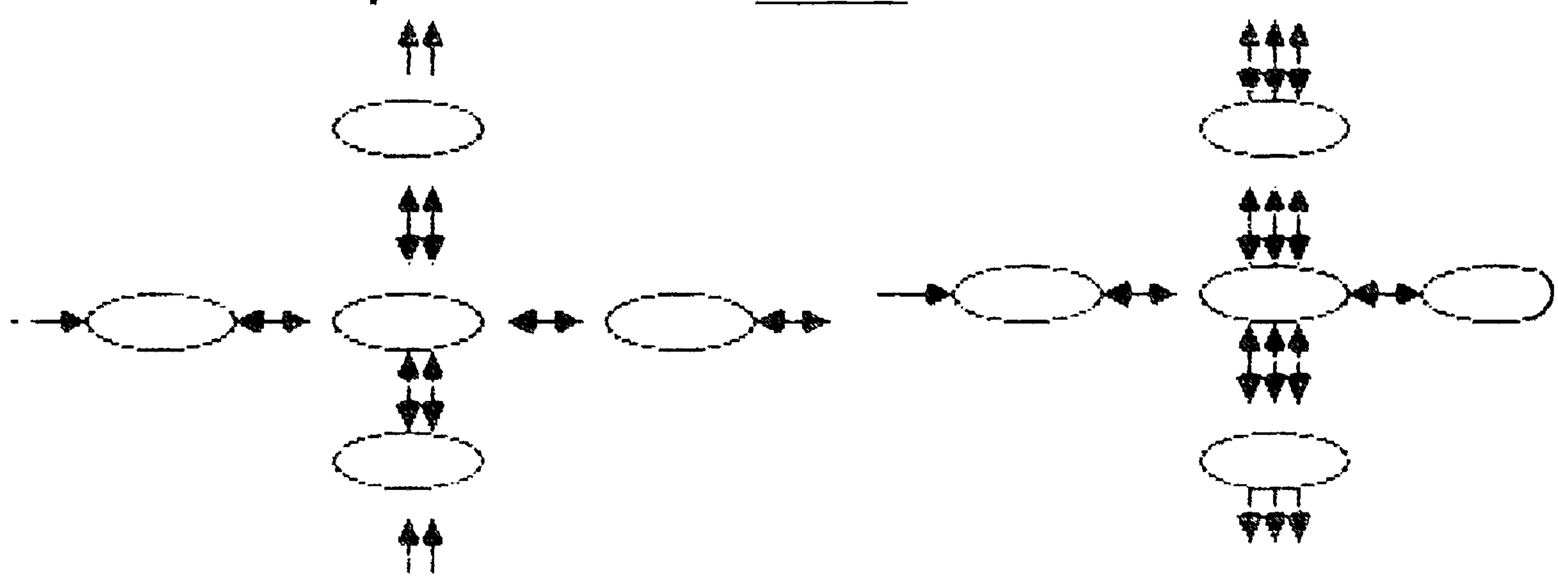


(Differs in One Way).

(Differs in Two Ways)

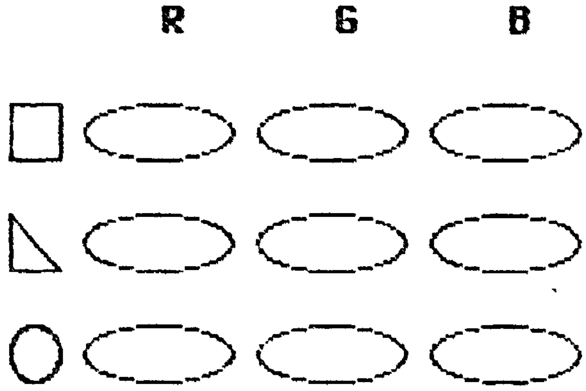
(Differs in Three Ways)

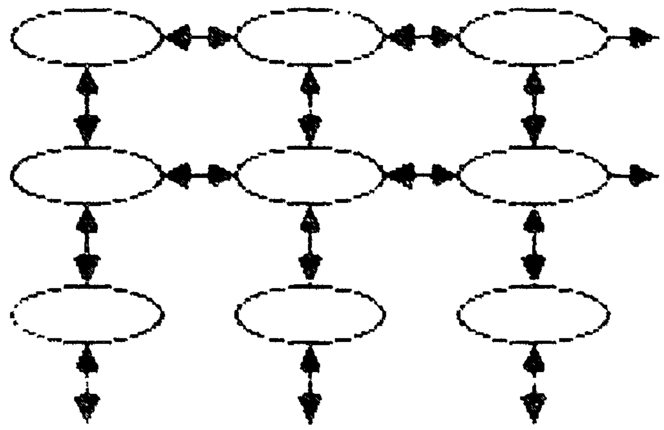
Have the students place the blocks in crosses

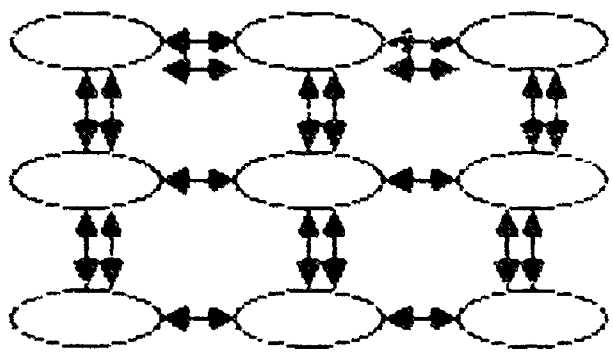


etc.

Have the students place the blocks in arrays of rows and columns.

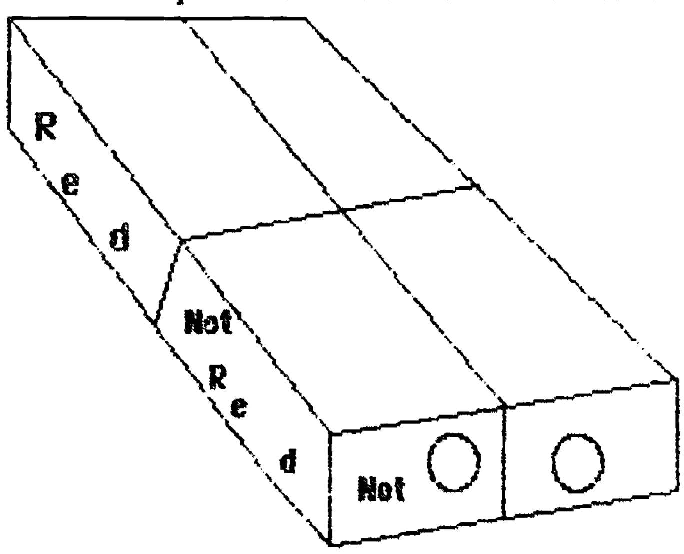




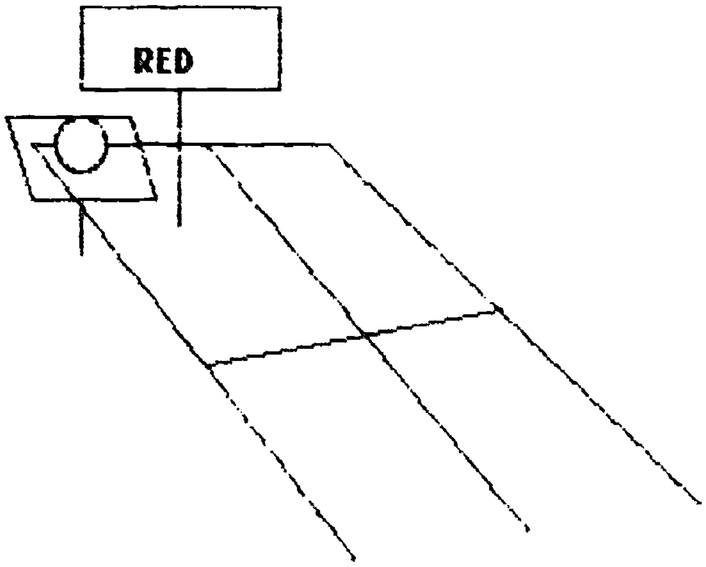


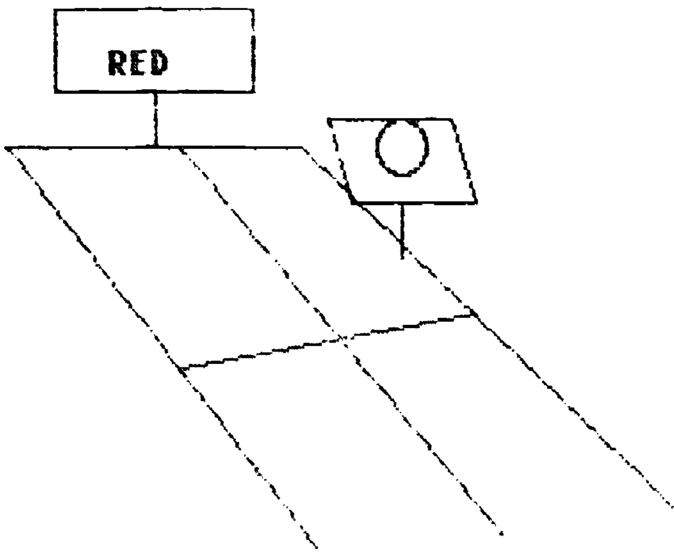
Using NOT:

Sort the pieces into a box or on a sheet



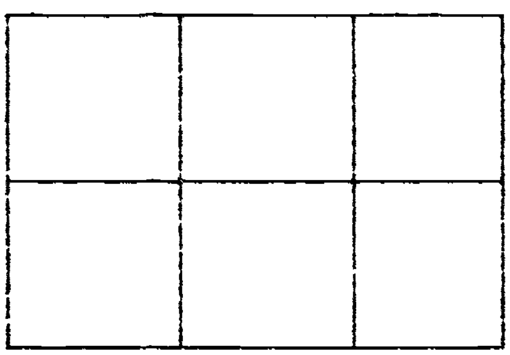
Put pieces in the right Parking places



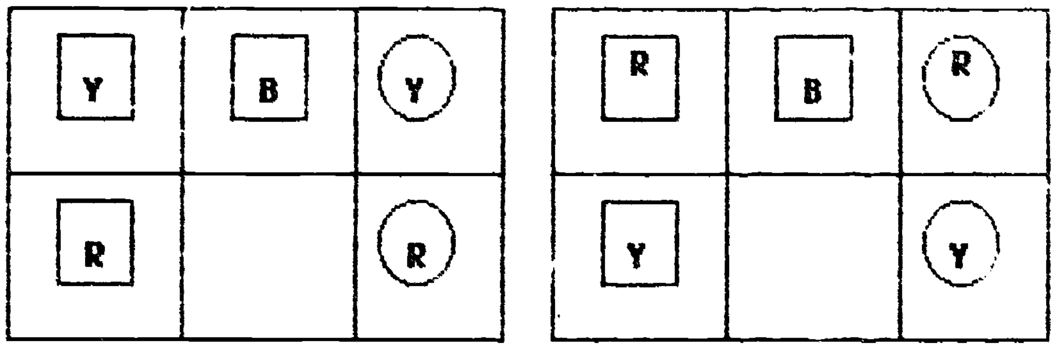


Using AND, NOT

Use the six-celled workmat

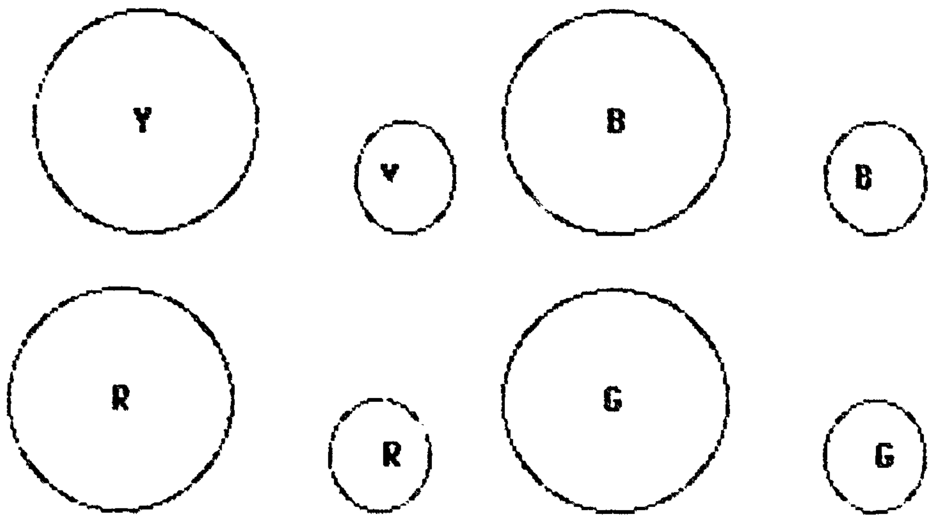


Place 5 pieces on it, for example:

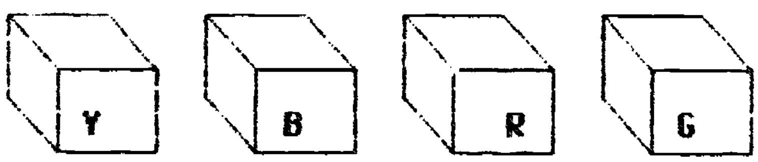


By sliding one block at a time into the <u>neighboring</u> empty square, reverse the positions of:

1. The Y and R circles, or
2. The Y and R squares



Two circles are in each box:



Which circles are in each box, if:

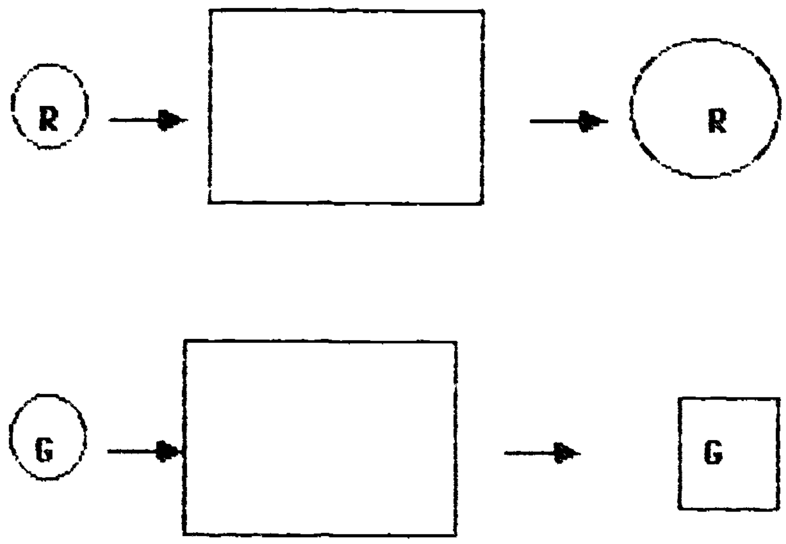
1. None of the circles is in the box of its colors.
2. There is <u>no</u> B circle on the R box.
3. There is one Y circle and one G circle in either the B box or the R box.
4. There is a B and a G circle in the Y box.
5. There is a Y circle in the B box.

Guess My Rule:

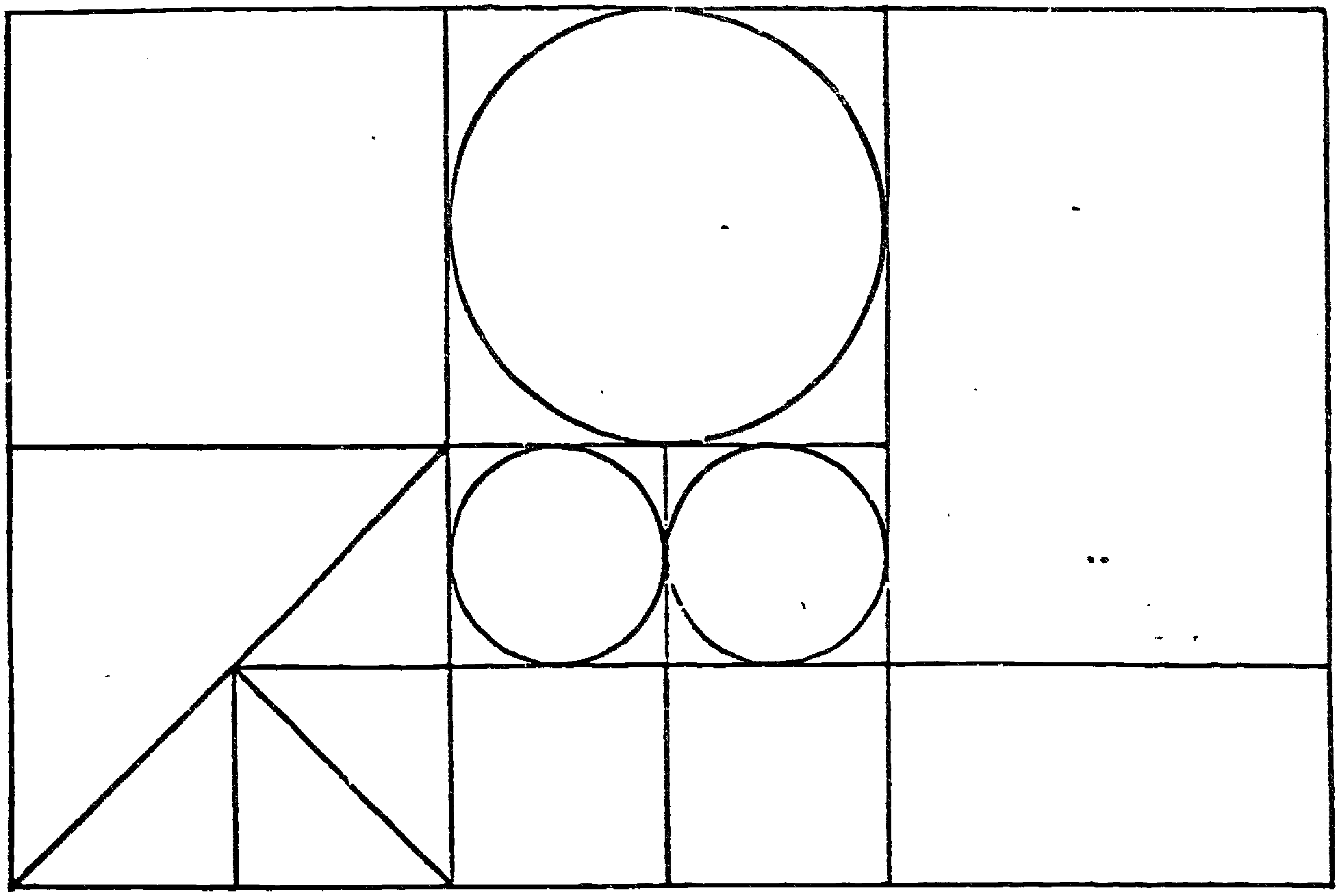
Select a rule such as "Change Size Only."

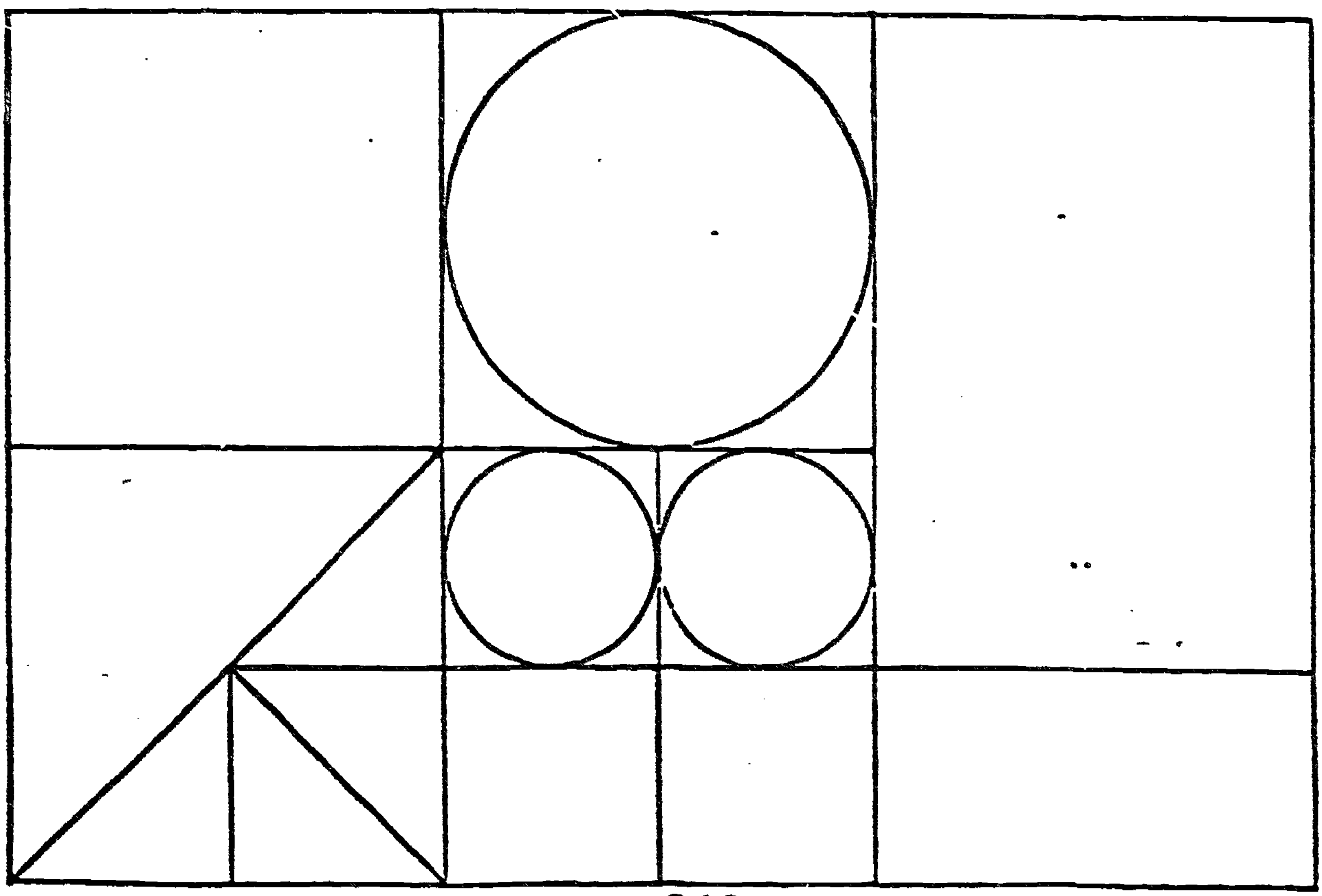
Put a barrier (piece of tagboard up in front of the table.) Ask the students for a piece to go "in" Output on the other side of the piece resulting from the rule.

Do enough examples until students can successfully predict which pieces will come out: Example:



Other ideas can be found in *Attribute Games and Activities*, by Creative Pugblications; *Attribute Acrobatics*, by Activity Resources.

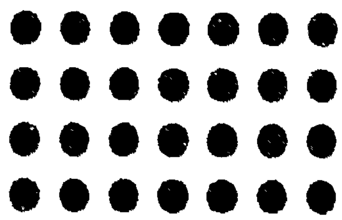




The Distributive Property

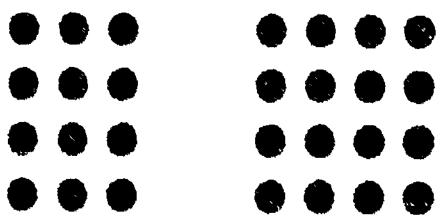
Background: Using the fact that multiplication distributes over addition and subtraction has always given problems to junior high school and high school students, it is important that this be introduced early to children.

Introduction: Use colored chips on the overhead:



"What are the dimensions of this array? "How many chips are in this array?" Write: $4 \times 7 = 28$.

Separate the array into two parts:



"Are there still just as many chips in these two arrays?" "What are the dimensions of the two arrays?"

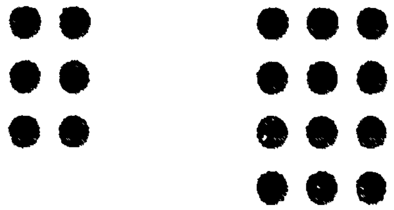
Write $4 \times 3 + 4 \times 4 = 28$

"Note that they must have ONE dimension the same." "We can add the two arrays to get the same number (28) as in the first array."

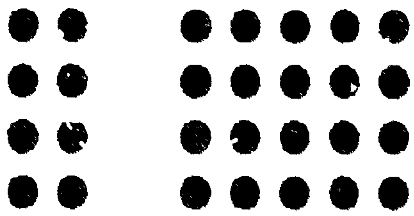
Write: $4 \times 7 = 4(3 + 4) = 4 \times 3 + 4 \times 4 = 28$

"The 7 has been rewritten as 3 + 4, but the result is the same."

"Here are two arrays."



"Can these be pushed together into a single array?" "Why or why not?"
"Consider these arrays."



"Do these have ONE common dimension?" "How many chips in the first? in the second?

Write: 4 x 2 = 8   4 x 5 = 28

4 (2 + 5) = 4 x 2 + 4 x 5 = 28

"What are the dimensions of the single array when they are pushed together?" Write: 4 x7 = 28

Write: 3 (3) = 24

How many ways can we write the 8?"

3 (6 + 2) = 3 x 6 + 3 x 2 = 18 + 6 = 24
3 (7 + 1) = 3 x 7 + 3 x 1 = 21 + 3 = 24
3 (3 + 5) = 3 x 3 + 3 x 5 = 9 + 15 = 24
3 (4 + 4) = 3 x 4 + 3 x 4 = 12 + 12 = 24
3 (9 - 1) = 3 x 9 - 3 x 1 = 27 - 3 = 24
3 (10 - 2) = 3 x 10 - 3 x 2 = 30 - 6 = 24

Calculate each one.

"Do this for 4 x 7 = 28 by writing the 7 in different ways." Give the assignment sheet to pairs of students to work on. Give them blocks or chips to use if needed.

LEVEL FOUR

RATIO

LESSON ONE: Unifix Cubes

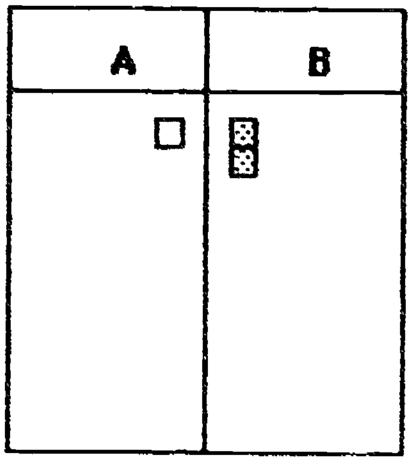
Introduction: This lesson is a review of past activity. Children should have unifix cubes of two different colors and a recording form.

Use colored squares that are like unifix cubes on the overhead projector along with a split board transparency.

Designate one of your colors as "A" and the other as "B".

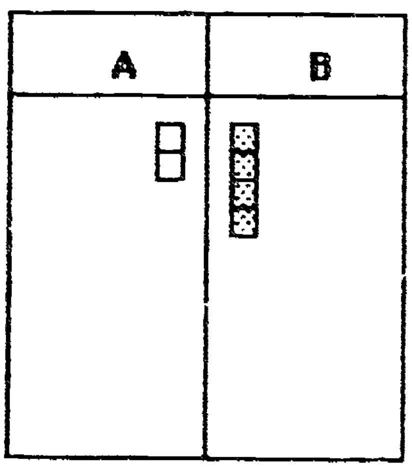
"Chose an 'A' color and a 'B' color for your unifix cubes."

"Put a group of ONE A and another group of TWO B on the split board:



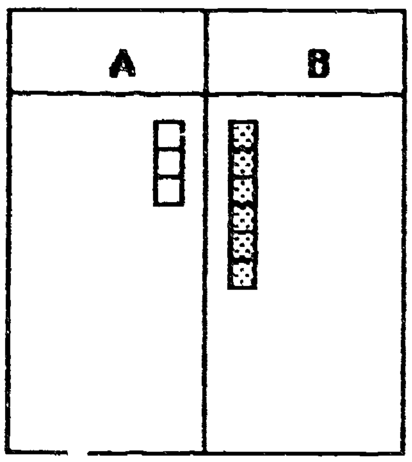
"Record this on your form - one link of each color - one A and 2 B in the links."

Make a second group of each kind and add to the first on the overhead:



"Now we have TWO of each group. You have a link of TO A and FOUR B."

"Could we get the groups back that we started with – TWO each of ONE A and TWO B?" "If I make another group of each kind, how many A will there be? how many B will there be?"



Children should have on the records:

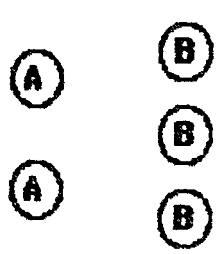
No. of Links	A	B
1	1	2
2	2	4
3	3	6

"Continue doing this until your recording form is completed."

Activity: Give pairs of children 40 unifix cubes of two different colors and the recording form. Repeat the lesson by using a recording form requiring different ratios.

LESSON TWO: Equal Ratios - Unifix

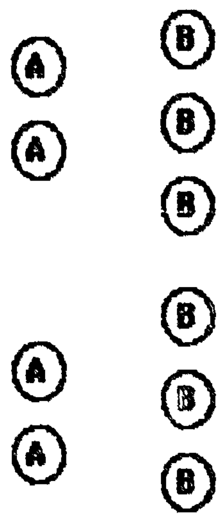
Introduction: Put two colored chips or squares of color A and three of color B on the overhead projector.



Write 2:3 on the board or projector.

"We use this symbol to show the comparison of 2 to 3."

Put a second set of 2A and 3B on the overhead projector:

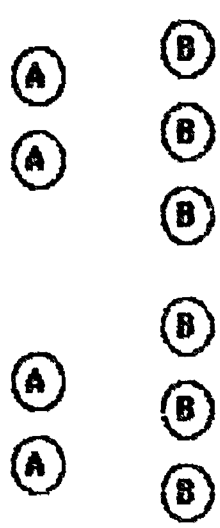


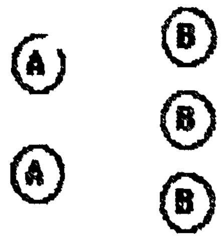
"Although there are 4A's and 6B's, there are still 2A's for each 3 B's."

Point to the two cases of 2:3.

"Since these show the same comparison we write: 2:3 :: 4:6

Put a third set of each on the overhead projector:





"Now we have six A's and 9B's."

"This is three of the 2:3 comparisons collected together."

"We can write this as: 3:3: :6:9"

Write that on the board.

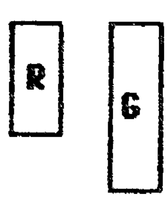
"We can also write – 4:6:: 6:9 since these both are multiples of the 2:3 comparison. We could say that 4:6 and 6:9 belong to the 2:3 family."

<u>Activity:</u> Give pairs of children unifix cubes of 2 different colors – about 20 of each – and the recording forms.

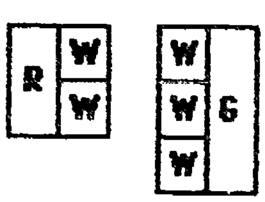
<u>LESSON THREE: Cuisenaire Rods</u>

<u>Introduction:</u> Use transparency versions of cuisenaire rods. Children should have cuisenaire rods to use.

Put the following cuisenaire rods on the overhead projector:

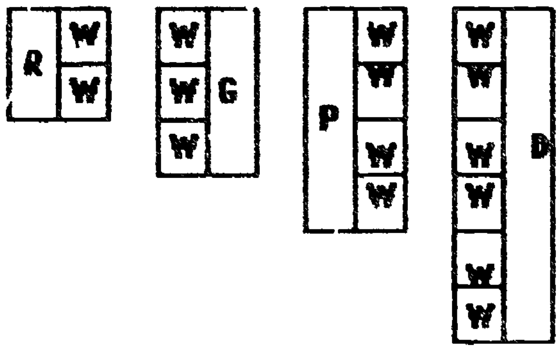


Place whites next to each as shown:



Point out that the comparison of R:G is 2W:3W or 2:3 and that G = 1 1/2R or R = 2/3G.

Without removing these, put the following rod arrangement:



Point out P:D is 4W:6W or 4:6. Remind the children this comparison is in the 2:3 family and that 4 = 2/3 of 6 or 6 = 1 1/2 x 4, so the longer is still 1 1/2 x the shorter. "What is the next pair of rods in this family of 2:3?"

"D:E = 6W:9W or 6:9." Again point out 6 is 2/3 of 9 and . . 1 1/2 x 6.

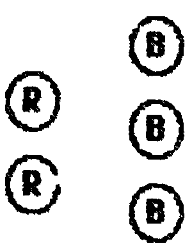
Activity: Pass out cuisenaire rods to pairs of children and have them work on the worksheet.

LESSON FOUR: Using Ratios

Int. uction: On the overhead projector, use colored chips to represent the materials.

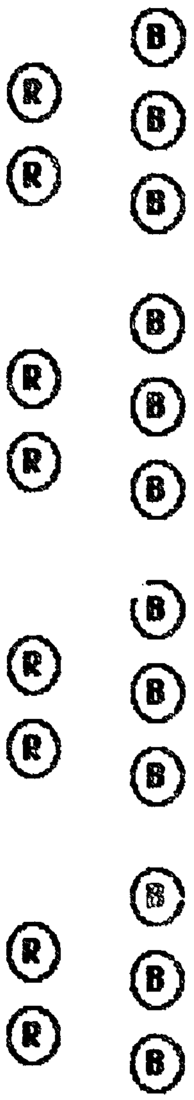
"In the Aeloian Club there were 2 men for every 3 women. In 1924, 8 men belonged. How many women belonged?"

Put the following on the overhead projector:



"This represents the comparison. How many of these must we put so there will be 8 men?"

Put the four sets:



"4 times 2 F 8, so we need 4 times the 3. "There were twelve women."

"The number sentence is: 2:3 B:

"8 is 4 times 2, so what is in the box must be 4 times 3."

2:3 8: 12

Do a second using a simple ratio comparison.

<u>Activity</u>: Give pairs of children either unifix cubes, cuisenaire rods or colored chips to use to work the problems on the sheet.

RATIO PROBLEMS

Pencils are 2 for a quarter. How many can I buy with 3 quarters?

John put 18 cans of tomato juice on the grocery cart. The clerk rang this up as $6.00, saying, "That's 6 times the sale price." What was the sale price of the tomatoes?

The recycle center pays 7¢ for every 10 aluminum cans brought in. How much did Jerry get for his bag of 100 aluminum cans?

The Consumer's Tipster said 2 of every 25 West Coast apples were shipped green. Karen found 6 green apples in a box of apples. How many apples were probably in the box?

Giant Grocers sold melons at 2 for 89¢. How much did Carol pay for 8 melons?

Audio tapes were on sale at $6.67 for 3 tapes. How much should a clerk charge for 4 tapes?

If guitar strings sell at 5 for $4.00, what should 10 guitar strings cost?

Jane's doctor prescribed 4 tablets of an antiobiotic each 24 hours. How many should the pharmacist put into a week's supply?

Which is a better buy - 99¢ for 16 oz. loaf of bread or $1.39 for a 20 oz. loaf?

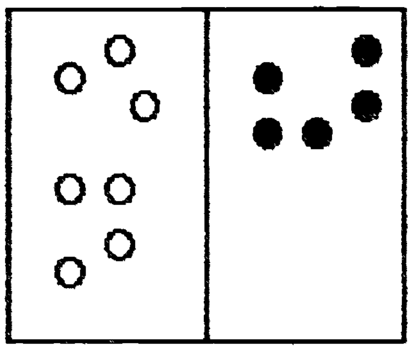
LEVEL FOUR

SIGNED NUMBERS

Background: Children have some exposure to the idea of below zero through weather reports, going in the hole in terms of money, etc. These lessons have children work with negative numbers to understand them.

LESSON ONE:

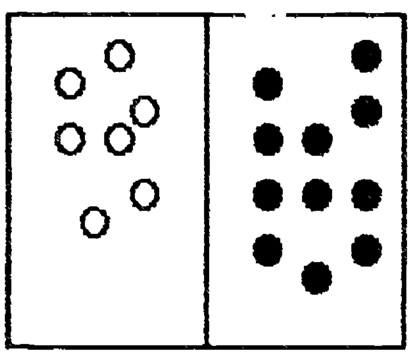
Use a split board on the overhead projector with 2 colors of chips:



"Black represents positive and white represents negative. Is this more positive or negative?"

"How much more negative is it?"

"We write this as $^-2$." Rearrange as:



"Is this more positive or negative?"

"How much more positive is it?"

"We write this as $^+4$."

Leave that arrangement on the overhead.

"How can we make this arrangement more positive?"

Likely response - add positive.

"Is there another way to have more black chips than white chips there?" Desired response: Subtract negative

"The two ways to make a number more positive are to add positive or to subtract negative."

"How could we make the arrangement of chips show more negative?" Desired response: add negative or subtract positive

Summarize as:

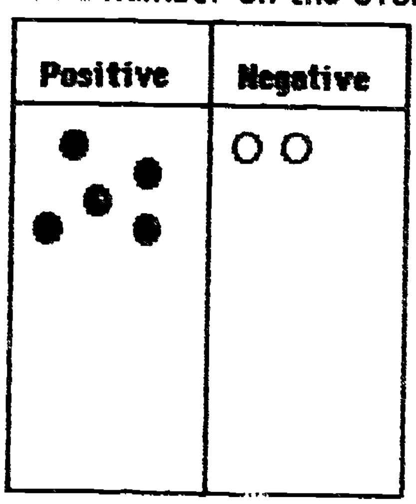
More positive	+ positive $+{}^{+}2$
	- negative $-{}^{-}2$
More negative	+ negative $+{}^{-}2$
	-positive $-{}^{+}2$

<u>Activity:</u> Pairs of children should have several chips of each of 2 colors, a split board and the worksheets.

<u>LESSON TWO: More Positive</u>

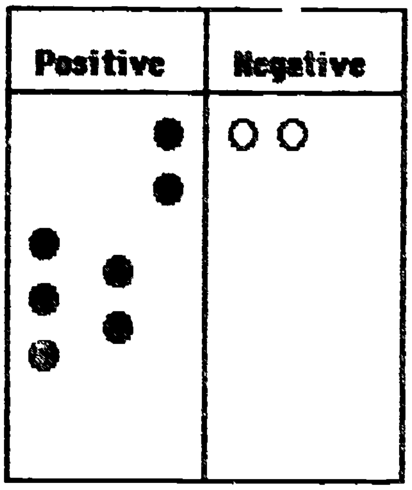
<u>Introduction:</u> This lesson is to emphasize how to make numbers more positive by (1) adding positive, and (2) subtracting negative.

Put a number on the overhead with chips on a split board:



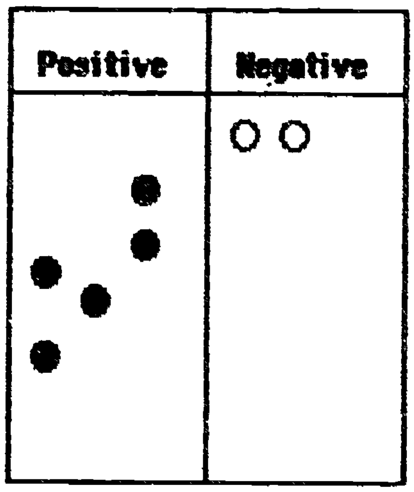
"What number does this represent?" ($^{+}3$)

"What should I add to increase this to $^{+}5$?)



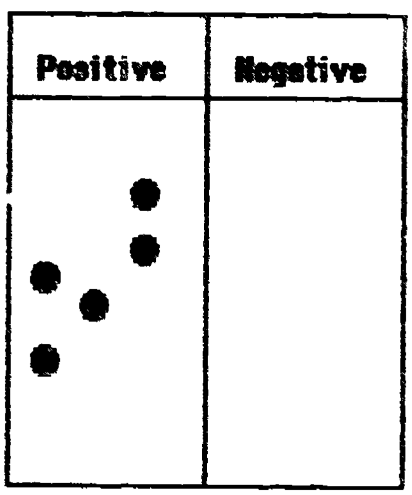
Point out how a black and a white cancel each other out so the 5 black chips show $^{+}5$.

Restore to $^{+}3$



"What can I subtract to make this $^{+}5$? ($^{-}2$)

Remove 2 whites:



"The board will show $^+5$ as long as there are 5 more black chips than white chips."

"What are some other ways to show $^+5$?"

Make any suggested, e.g.

Black	White
5	0
6	1
7	2
8	3
9	4
10	5
11	6, etc.

Write: $^+3 + {}^+2$

"This shows adding positive. Will the result be more positive? What is it?"

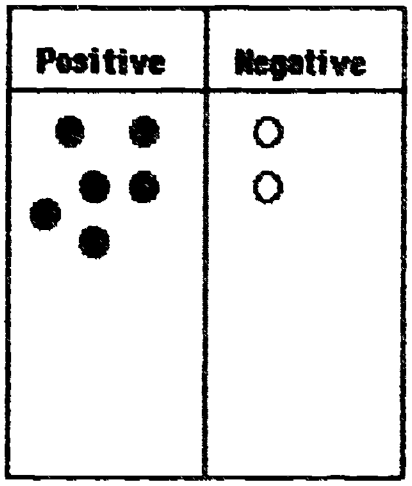
Write: $^+3 - {}^-2$

"This shows subtracting negative. Will the result be more positive? What is it?"

Complete each:

$^+3 + {}^+2 = {}^+5$
$^+3 - {}^-2 = {}^+5$

Do this example:



Number shown? $^{+}4$

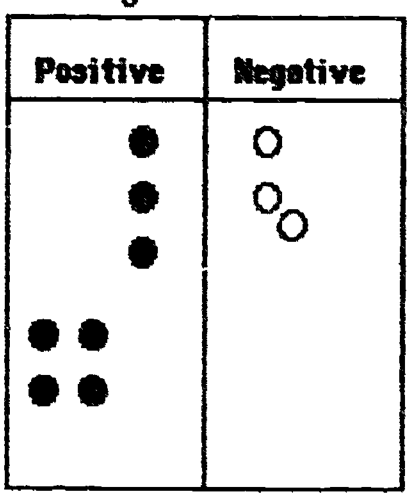
Want $^{+}7$

Adding positive $^{-}3$

Subtracting negative - not enough!

How to change? - one more of each kind

"As long as + and - are added or subtracted in equal amounts, the number won't change."
Show by:



Still shows - +4

Remove 3 whites

Desired +7

<u>Activity:</u> Give pairs of children split boards, 2 colors of chips and the recording forms. Monitor the recreating of numbers so that enough negative is there to subtract from. This is just like "borrowing in subtraction" so that there are enough ones to be able to subtract.

LEVEL FOUR

CALCULATORS

Background: Calculators should be an instructional tool to help children learn basic mathematical concepts. It should free children from computation "drudgery" in problem solving so they can solve more realistic problems. It should NOT be used to check the results of paper-and-pencil computation.

The class should have available a set of four function calculators that are alike. You should have an overhead projector calculator so you can show students how to use these. Most calculators will use the SECOND number entered in an addition or subtraction as the repeater, so the = button can be used to add or subtract the same number repeatedly. Most also use the FIRST number entered as a factor as a repeated multiplication, while using the divisor entered in a division repeatedly when pushing = .

Familiarize yourself with the calculator the children will use. Use either long-life or solar cell calculators. Also be sure the calculators have an automatic power shut off property.

Introduction: A sample activity has been given for the many ways a calculator can be used to reinforce basic concepts.

Skip Counting: Pick a starting number - enter. Press + and a second number to count by. Each time = is pressed, that much will be counted on.

Example:

KEY SEQUENCE:	2 +	5 =	=	=	=	=	=	=	=
DISPLAY	2	7	12	17	22	27	32	37	42

To count back. Start with a large number and - . Counting back far enough will give negative number entries. Some put the "-" sign far to left to be ahead of any number, others put it behind the number.

Estimation:

Have children estimate through rounding and place value use the results of computations. Use the calculator to do the computations. Make this a game by giving points for closest estimates.

Example:

143.5	ESTIMATE		500
281.6	CALCULATION	515.8	
90.7	ERROR	15.8	

(Reducing this error is the goal)

Using data: The World Almanac gives a wealth of data to use for problem solving.

Example: The table gives the mean distance from the sun of the planets in our solar system, unity being the distance of the Earth ~ 93 million miles. Thus the distance of Mars ~ 171.7 million miles.

Mercury	.3871
Venus	.7233
Earth	1.0000
Mars	1.5237
Jupiter	5.2028
Saturn	9.5388
Uranus	19.1910
Neptune	30.0707
Pluto	39.5

1 Have the children pose questions to be answered relative to these distances.

2. Use calculators to find these answers.

Decode multiplication and division

Have students enter multidigit numerals and multiply successively using "10". To avoid always getting decimals, have students enter multidigit numerals followed by 0's and divide by "10".

Examples:

KEY SEQUENCE: [Enter Numeral] [×] [10] [=] [=]

DISPLAY: [485] [4850] [48500]

KEY SEQUENCE: [Enter Numeral] [÷] [10] [=] [=]

DISPLAY: [30500] [3050] [305]

<u>Common Multiples:</u>

Use calculators to complete tables of multiples and circle common multiples.

Example:

Multiples of 2	Multiples of 3
KEY SEQUENCE:	
[0] [+] [2] [=] [=]	[0] [+] [5] [=] [=] [=]
2	3
4	(6)
(6)	9
8	(12)
10	15
(12)	(18)
14	
16	
(18)	

Discuss the results.

<u>Shopping:</u>

Use copies of an advertisement from a shopper or newspaper and cost out different purchases.

Item	Price	Purchase
Socks	.99	3 socks + 1 pr. running shoes
Running shoes	23.95	
Running shorts	11.49	
T-shirts	3.95	
Sweat bands	1.39	

<u>Patterns:</u>

Have students look for patterns in additions, subtractions, multiplications, divisions.

Examples:

21 x 9 = 189
21 x 90 = 1890
21 x 900 =- 18900
487 ÷ 6 = 81.166666...
487 ÷ 60 = 8.1166666
487 ÷ 600 = .8116666

Activities like these arouse students' curiousity about repeating digits in decimals, etc.

<u>Squares and Square Roots:</u>

KEY SEQUENCE: [5] [×] [5] [=]

[14] [×] [12] [=]

[20] [×] [20] [=]

Look for patterns in the digits and the last digits in particular.

Example:

5 x 5 = 25
15 x 15 = 225
25 x 25 = 625
35 x 35 = 1225
45 x 45 = 2025

Resources:

Some excellent resources are available commercially with calculator activities. Some examples are: (not exhaustive)

1. *Calculate: Whole number operations*

2. *Keystrokes* (a 4 volume series)

3. *Calculator Activities For the Classroom: Book 1*

(these are available from Creative Publications)

4. Texas Instruments has a set of Instructional Materials available for the TI-108 - a simple calculator suitable for this level of student.

LEVEL FOUR

USING LOGO

Children using this program should have had considerable experience using the following LOGO commands: ST HT PU PD PE SETPC SETBG RT LT FD BK REPEAT and the use of TO to define procedures consisting of several commands.

At this level, they should review these and use the REPEAT command in procedures extensively.

Some examples are:

Making polygons of differing numbers of sides and sizes.

"Make a triangle with side 50"
REPEAT 3[FD 50 RT 120]

"Make a square with side 50"
REPEAT 4[FD 50 RT 90]

(the number of times the command is repeated increases while the angle to be turned decreases)

"Make a pentagon with side 50"
REPEAT 5[FD 50 RT 72]

After the children have made several of this through discovery, make a table:

Times to REPEAT	∠ Turned
3	120
4	90
5	72
6	60

"What is the pattern in this table?

Times to repeat x ∠ turned = 360 degrees

Children will notice that unless the side is shortened, the figures keep using up more of the screen!

After they recognize the "total trip" theorem and the above idea, give them the following:

"Make a circle"

One way is: REPEAT 30[FD 1 RT 1]

Have them experiment with making circles in different ways.

As children experiment with the REPEAT command and later with recursions, they are likely to make errors that result in non-ending or too long loops. CTRL-G stops any procedure or command in its tracks. Show them how to use it.

Another powerful tool is the nesting of REPEATS. Have the children execute this command.

REPEAT 3[REPEAT 4[FD 20 RT 90]RT 10]

Discuss the result and walk command by command through the complex command. Emphasize:

Commands are included in brackets to show what is being done in what order. The above is telling the computer to:

1.
FD 20 RT 90
FD 20 RT 90
FD 20 RT 90
FD 20 RT 90
RT 10

4

2.
FD 20 RT 90
FD 20 RT 90
FD 20 RT 90
FD 20 RT 90
RT 10

```
      ⎧ FD 20 RT 90
      ⎪ FD 20 RT 90
   3. ⎨ FD 20 RT 90
      ⎪ FD 20 RT 90
      ⎩ RT 10
```

Have the children experiment with one repeat nested with another. Then have them do:

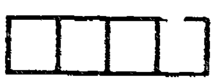
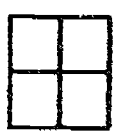
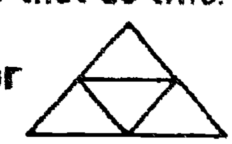
```
REPEAT 3[Repeat 4[Repeat 3[FD 60 RT 60]RT 10]FD 10]
```

Have children add PU, PD and HT commands along with color setting commands to create designs.

Have children give names to designs and create procedures to produce them. For example:

```
TO STARBURST
REPEAT 6[REPEAT 4[FD 30 RT 90]RT 60]
PU
FD 50 RT 95
PD
REPEAT 36[FD 9 RT 10]
RT 85
PU
FD 51
PD
REPEAT 36[PU FD 60 PD FD 10 PU BK 20 RT 10]
HT
END
```

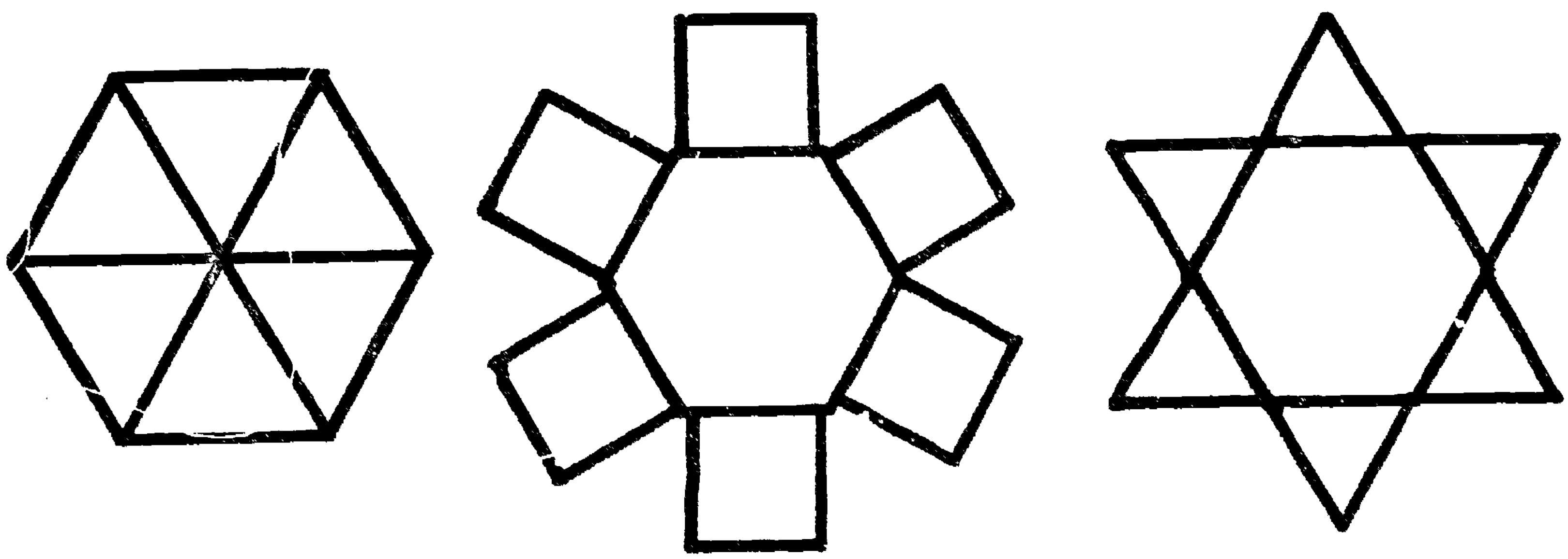
Have children try to make designs that do this:

 or  or 

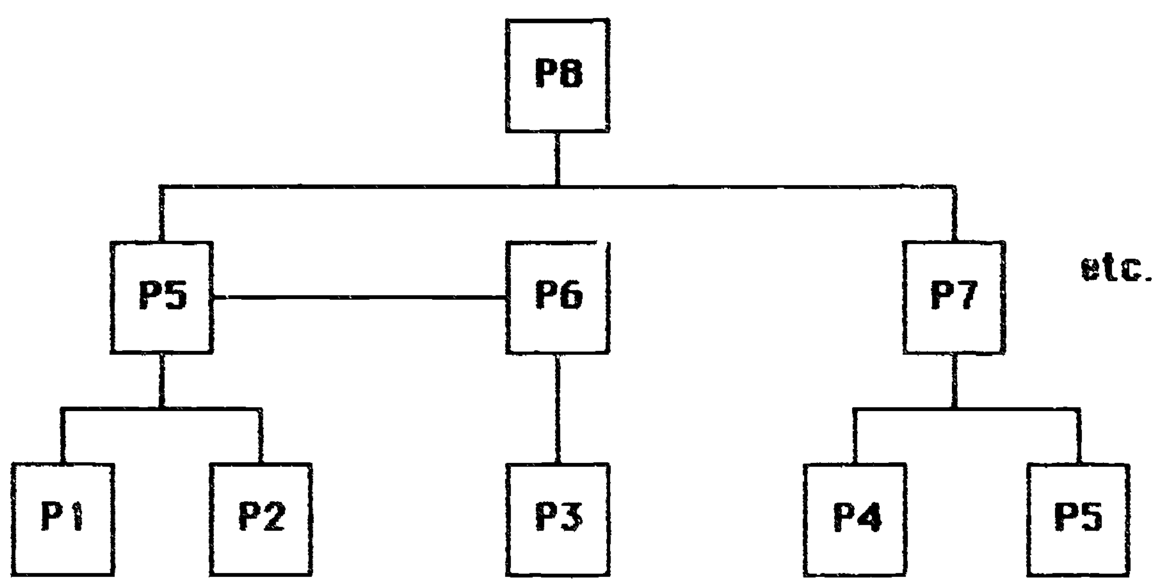
After they work with Pattern Blocks, it is easier for them to duplicate Pattern Block kinds of designs using LOGO. When trying to decide how to turn RT or LT to a new position to start a design, have them use physical objects such as Pattern Blocks to do it first.

Always remind children to observe the direction in which the turtle is pointed. Is this the direction the next line must go in?

Some suggested designs:



The third powerful idea to introduce is that of nesting procedures, or using procedures within procedures. This is what makes LOGO a structured language. A host of small, simple procedures can be developed and put together in combinations to make new ones:



An example is:

sub procedure

```
TO SIDE
FD 50 RT 90
END
```

a second using this

```
TO SQUARE
REPEAT 4[SIDE]
```

note the procedure name replaces a series of commands in the [ ]

a third using these two:

```
TO DESIGN
REPEAT 18[BOX FD 20 RT 20]
```

To do DESIGN, BOX must be already created and available.

To do BOX in DESIGN, SIDE must be created and available.

Have students use this process to make houses, faces, trucks, etc. Have them go back to designs made with nested REPEATS and do them again using nested procedures.

Verbal Multiplication and Division Problems: Some Difficulties and Some Solutions

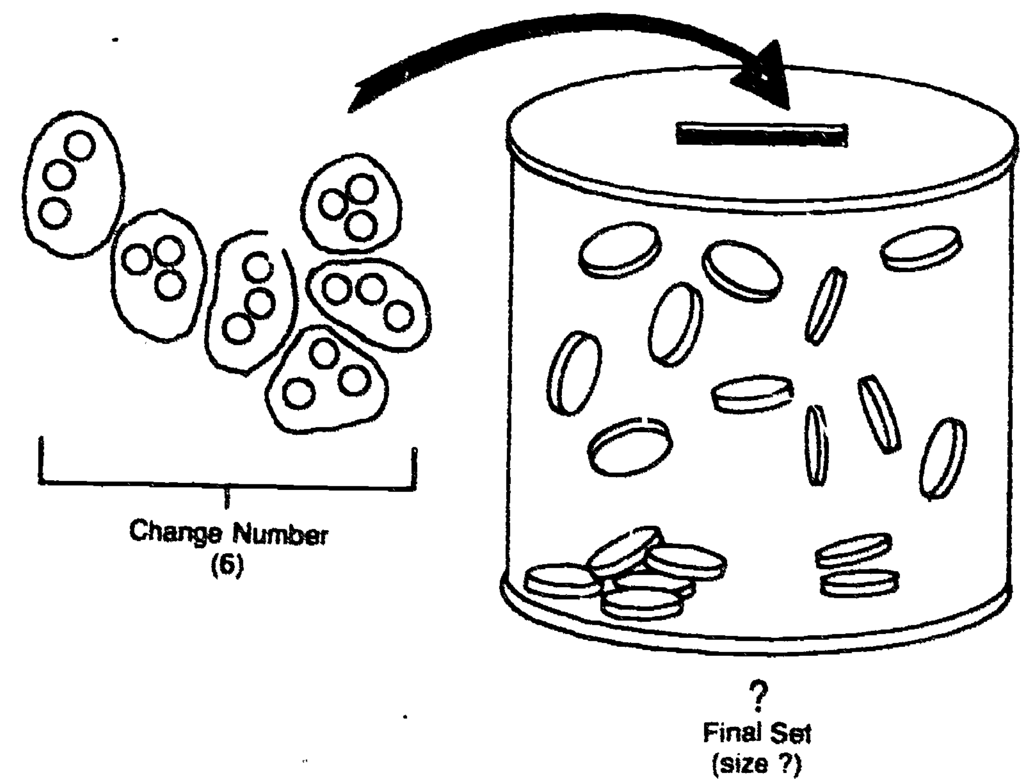
By A. Dean Hendrickson

Verbal problems that involve multiplication and division are difficult for children to solve. Many of these difficulties arise because of their limited understanding of these arithmetic operations. Their experience with the different kinds of situations that call for these operations is also limited. At the same time, these problems cannot be categorized easily because the situations that require these operations are varied. Nonetheless, multiplication is often taught only as "repeated addition" and division only as "repeated subtraction." Children must have specific instruction in *all* the situations that require multiplication and division as arithmetic operations if they are to apply them successfully to verbal problems.

Change Problems

Extensions of the "change problems" for addition and subtraction can lead to multiplication and division. In this particular kind of problem we have an initial set, a change number, and a final set. Given an initial set of small size and a change number that describes how many of this size set are joined, we find the size of the larger final set by multiplication. These problems are *change 1*, or repeated addition, problems. Here is an example (fig. 1):

Fig. 1 An example of a *change 1* or "repeated addition" problem: John put 6 handfuls of 3 tokens each into a bank. How many tokens did John put in the bank?



John put 6 handfuls of 3 tokens each into a bank. How many tokens did John put in the bank?

Change 2 problems result when a large initial set is given along with the size of a smaller final set, and a change number needs to be found that describes *how many* sets of that size can be made from the initial set. This problem represents the *measurement*, or repeated-subtraction, interpretation of division. Here is an example (fig. 2):

Susie has 24 cookies. She gives 3 cookies to each of the children on the playground. How many children are on the playground?

A child who can reverse the "putting together" transformation can relate a measurement interpretation of the division of countable materials to the repeated-addition kind of multiplication. In some ways the division is easier, since the child must retain only the final set size and count the number of sets that can be made. The count is constructed in the process and the size of the initial set is not important, since the count stops whenever the process runs out of objects. In repeated addition, both the count num-

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ber and the size of the initial set must be retained mentally along with the result at the end of each successive joining.

Change 3 problems involve a large initial set and a known change number; the size of the final, *equal* sets that can be made from the initial set must be found. This is the *partition* interpretation of division. An example follows (fig. 3):

> Susie has 24 cookies. She gives an equal number to each of her 4 friends. How many cookies does each friend get?

Change 2, or measurement division, is easier, since only the size of the set being formed repeatedly must be retained and a count of these sets kept as they are made. Change 3, or partition division, requires a strategy to assure the equality of the sets being made and hence is more difficult.

Comparison Problems

Questions involving "less than" or "more than" lead to addition and subtraction problems. These problems involve a comparison set, a difference set, and a referent set. When we compare two sets and the comparison involves questions of "how many times as many" or "what part of," we use multiplication and division. Such problems involve a comparison set, a referent set, and a correspondence other than a one-to-one correspondence between these sets. In figure 4, if the question is asked, "*A* has how many times as many as *B*?" then *A* is the comparison set, *B* is the referent set, and the correspondence of *A* to *B* is sought.

Compare 1 problems result when the referent set and a many-to-one correspondence are given and students are asked to find the comparison set. The following is an example (fig. 5):

> Iris has 3 times as many nickels as dimes. She has 4 dimes. How many nickels does she have?

Multiplication is used to find the answer: $3 \times 4 = 12$.

Compare 2 problems occur when the comparison and a many-to-one

Fig. 2 An example of a *change 2* problem, measurement or repeated-subtraction interpretation of division: Susie has 24 cookies. She gives 3 cookies to each of the children on the playground. How many children are on the playground?

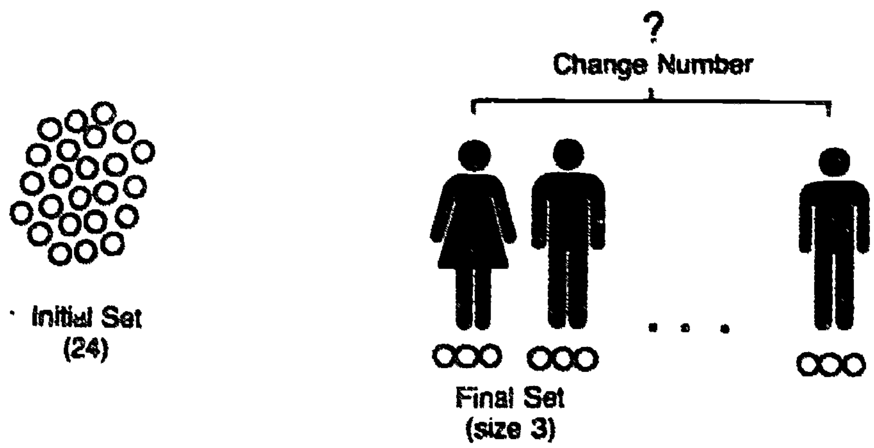


Fig. 3 An example of a *change 3* problem, a partition interpretation of division: Susie has 24 cookies. She gives them in equal numbers to her four friends. How many cookies does each friend get?

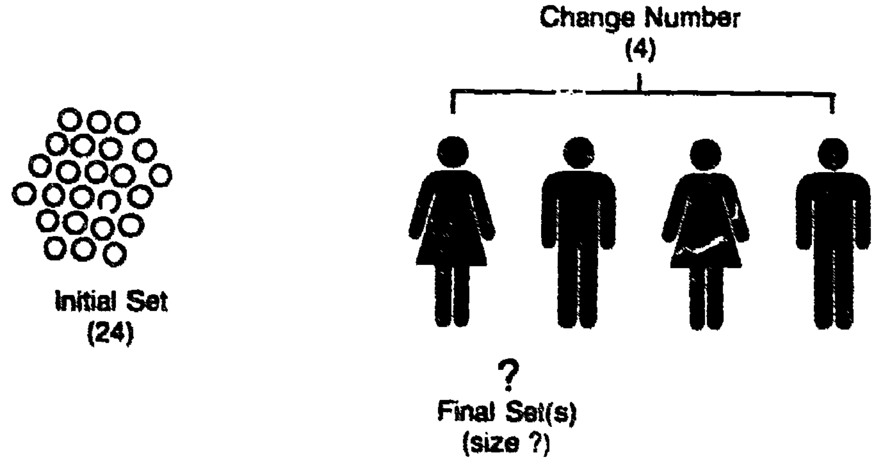


Fig. 4 A comparison problem: Find the correspondence of *A* to *B*. *A* has how many times as many as *B*?

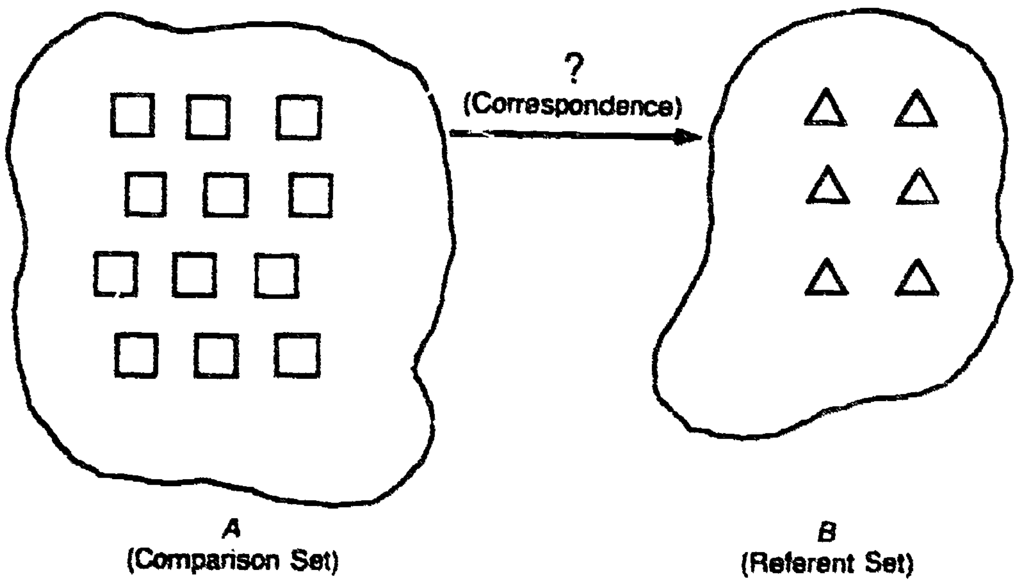


Fig. 5 A *compare 1* problem: Iris has 3 times as many nickels as dimes. She has 4 dimes. How many nickels does she have?

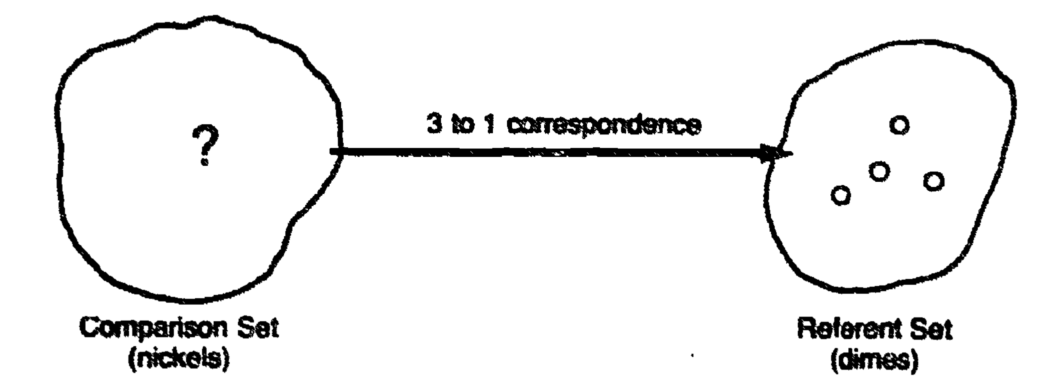


Fig. 6 A *compare 2* problem: Iris has 15 nickels. She has 3 times as many nickels as dimes. How many dimes does Iris have?

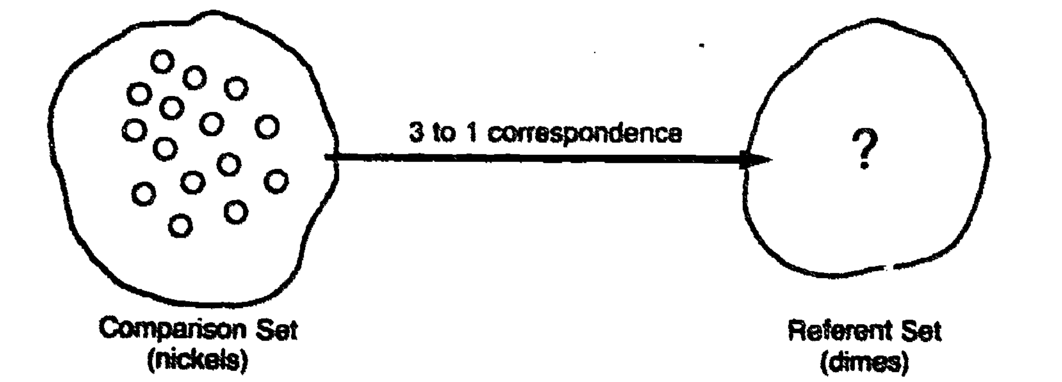


Fig. 7 A *compare 3* problem: Frank has 24 nickels and 8 dimes. He has how many times as many nickels as dimes?

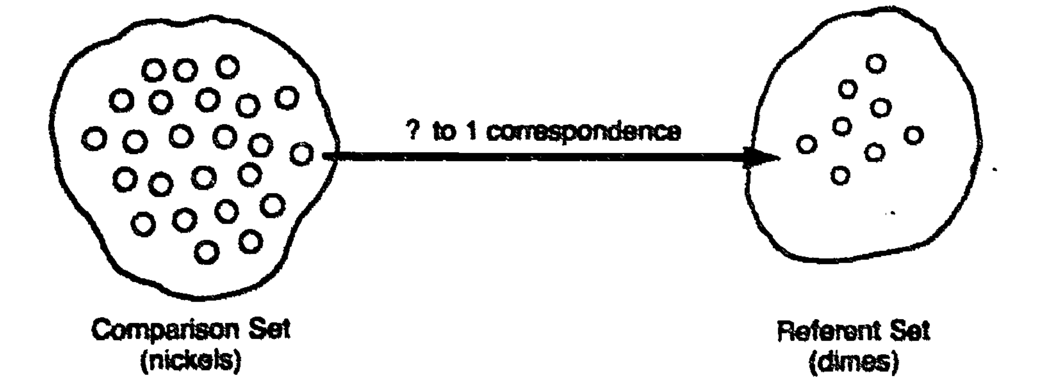
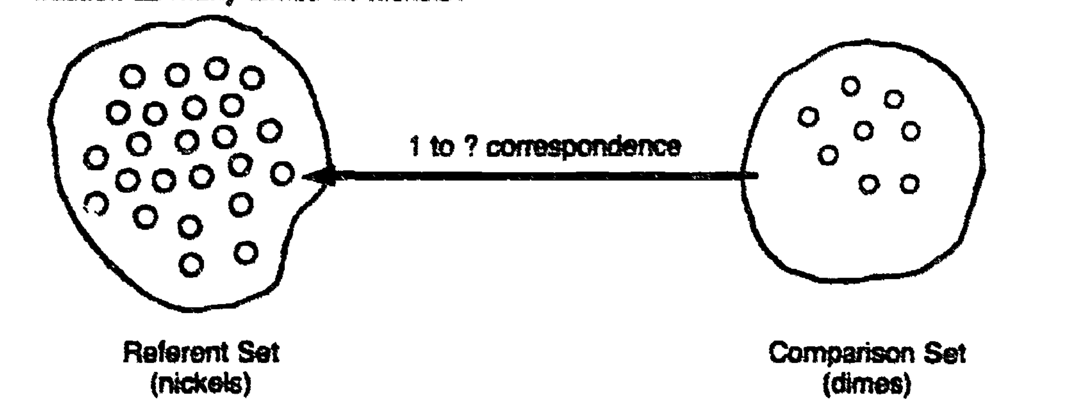


Fig. 8 A *compare 4* problem: Frank has 24 nickels and 8 dimes. He has what fraction as many dimes as nickels?



correspondence are given and the referent set must be found. Here is an example (fig. 6):

> Iris has 15 nickels. She has 3 times as many nickels as dimes. How many dimes does Iris have?

Division is used to find the answer: $15 \div 3 = 5$.

Compare 3 problems result when the comparison set and referent set are known and a many-to-one correspondence must be found (fig. 7):

> Frank has 24 nickels and 8 dimes. He has how many times as many nickels as dimes?

Division is used to find the answer: $24 \div 8 = 3$.

Compare 4 problems occur when a comparison set and a referent set are given and a one-to-many correspondence is sought. In this case, the comparison set is the smaller of the two. Here is an example (fig. 8):

> Frank has 24 nickels and 8 dimes. He has what fraction as many dimes as nickels? (or, Frank's dimes are what fractional part of his nickels?)

The result is division of a smaller by a larger number or formation of a rational number, usually expressed as a fraction: $8 \div 24 = 1/3$.

This kind of question puts a child's concept of *fraction* being equal parts of a whole into conflict with this ratio situation. What other language can be used to ask for this correspondence? Because of the difficulty of finding suitable language, questions related to finding this correspondence are seldom found in textbooks.

Compare 5 problems arise when the comparison set and the referent set are given and a many-to-many correspondence is sought (fig. 9):

> There are 12 girls and 16 boys in the room. How many times as many boys are there as girls?

One divides to find the answer ($16 \div 12 = 4/3$). Here again a fraction tells how many times as much, although a ratio correspondence is made in the thinking.

Compare 6 problems occur when the comparison set is smaller than the referent set and the correspondence is sought (fig. 10):

There are 12 girls and 16 boys in a room. The number of girls is what part of the number of boys?

The result is found by division again, 12 ÷ 16 = 3/4, and the same conflict between ratio and fraction results.

Compare 7 problems result when the larger comparison set and the many-to-many correspondence are given and the size of the smaller referent set is sought (fig. 11):

There are 16 boys in a class. There are 4/3 as many boys as girls. How many girls are there?

The answer is found by dividing: 16 ÷ 4/3 = 12.

Compare 8 problems arise when the smaller referent set is given along with a many-to-many correspondence. The size of the larger comparison set is sought (fig. 12):

There are 12 girls in the room. The number of boys is 4/3 the number of girls. How many boys are in the room?

The answer is found by multiplying: 4/3 × 12 = 16.

The compare problems that involve many-to-many correspondences are difficult, since they bring into conflict the child's recognition of a fraction as comparing a given number of equal parts to the whole and the idea of ratio as a correspondence. The use of the same symbolism for both fractions and rational numbers compounds this difficulty.

Thinking in ratios, equating ratios, and applying ratios to situations involve formal operational thought. Very few elementary children are capable of this kind of reasoning. In fact, few eighth and ninth graders can think through the Mr. Tall–Mr. Short problem:

	Mr. Tall	Mr. Short
Measured in match sticks	9	6
Measured in paper clips	12	?

Fig. 9 A *compare 5* problem: There are 12 girls and 16 boys in the room. How many times as many boys are there as girls?

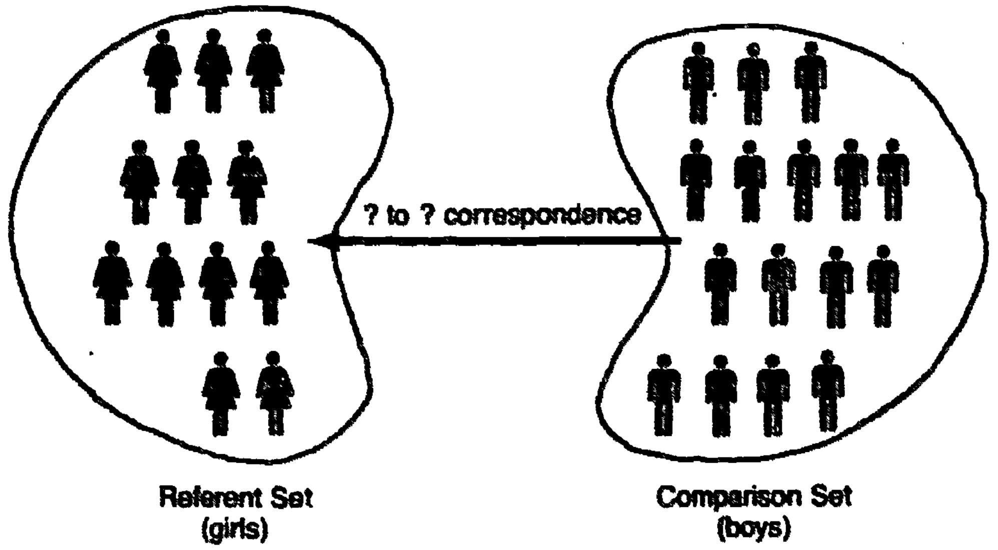


Fig. 10 A *compare 6* problem: There are 12 girls and 16 boys in a room. The girls are what part of the boys?

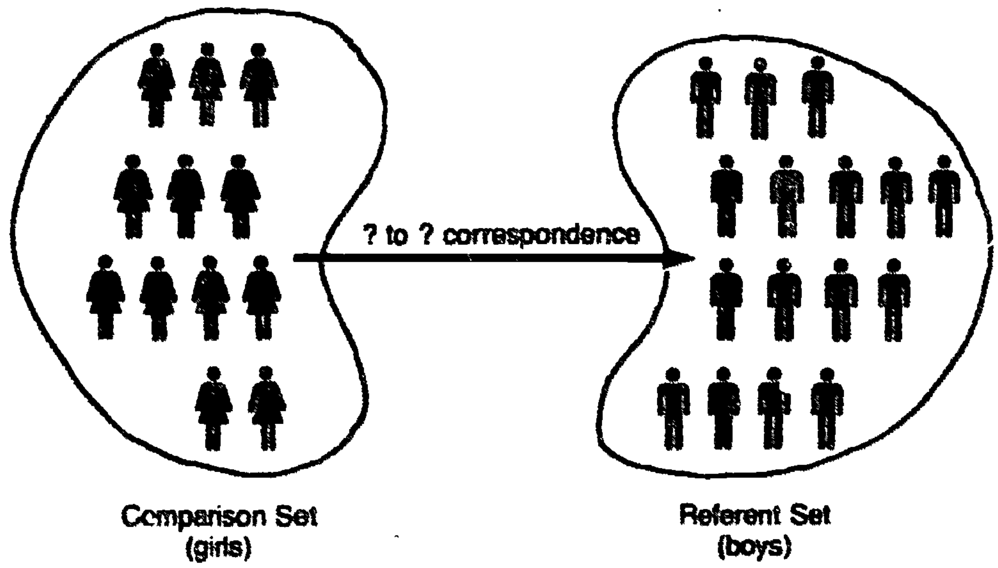


Fig. 11 A *compare 7* problem: There are 16 boys in a class. There are 4/3 as many boys as girls. How many girls are there?

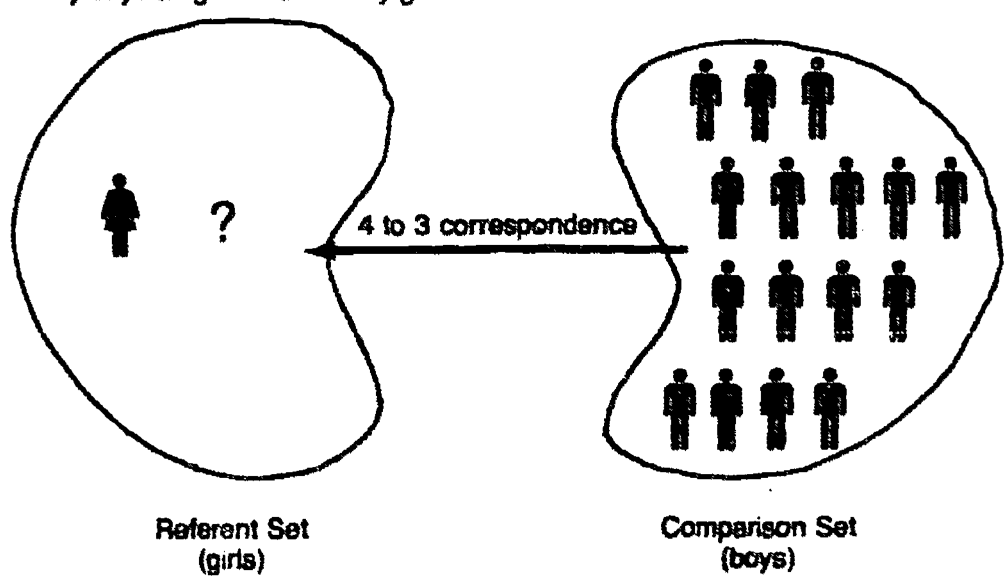
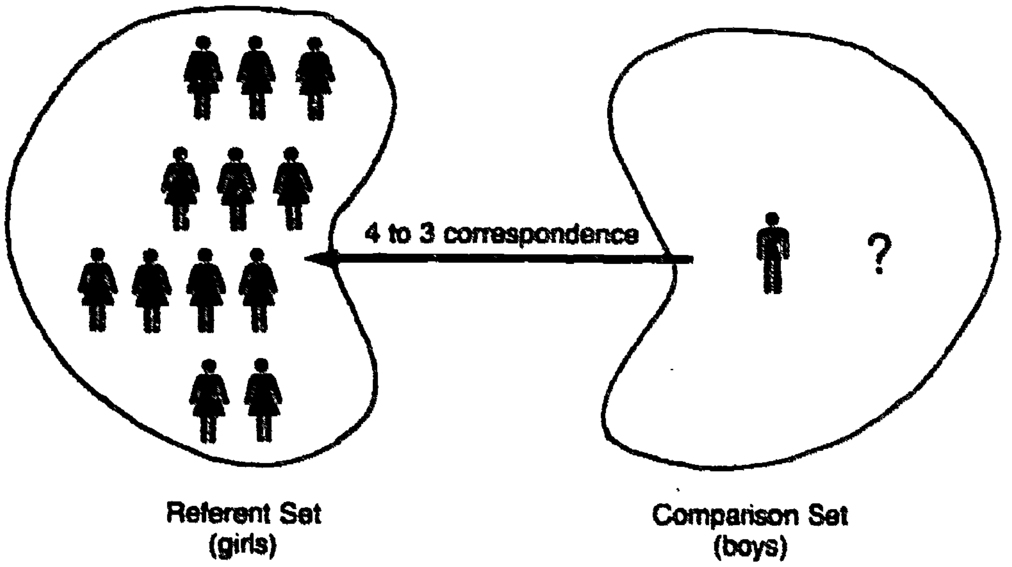


Fig. 12 A *compare 8* problem: There are 12 girls in the room. The number of boys is 4/3 the number of girls. How many boys are in the room?



Rate Problems

The kind of proportional reasoning used in equating ratios is also involved in thinking about rate problems. These are commonly found in intermediate textbooks. A rate problem involves two variables—one independent and one dependent—and a rate of comparison between them. An example is distance (miles) = rate (miles per hour) × time (hours). Here the number of hours is the independent variable, the distance in miles (a total) is the dependent variable, and the ratio of miles to hours is the rate.

Some common rate examples are these:

Fig. 13 A *rate 1* problem: Fred pays $12.00 a square yard for outdoor carpeting. How much will 16 square yards cost?

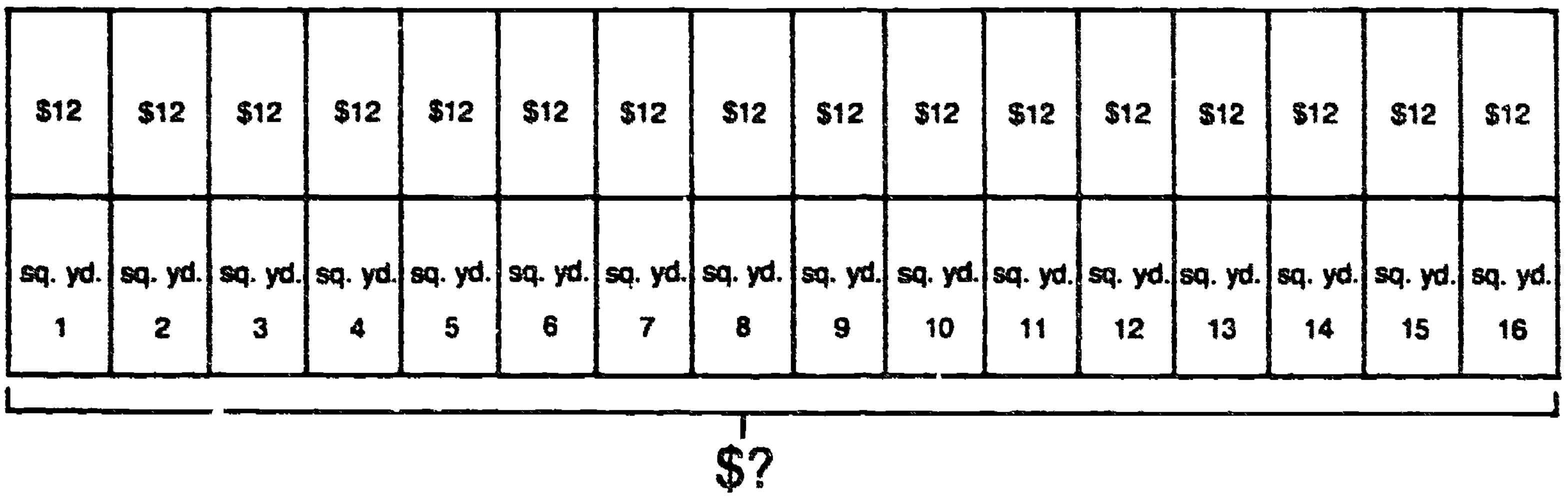
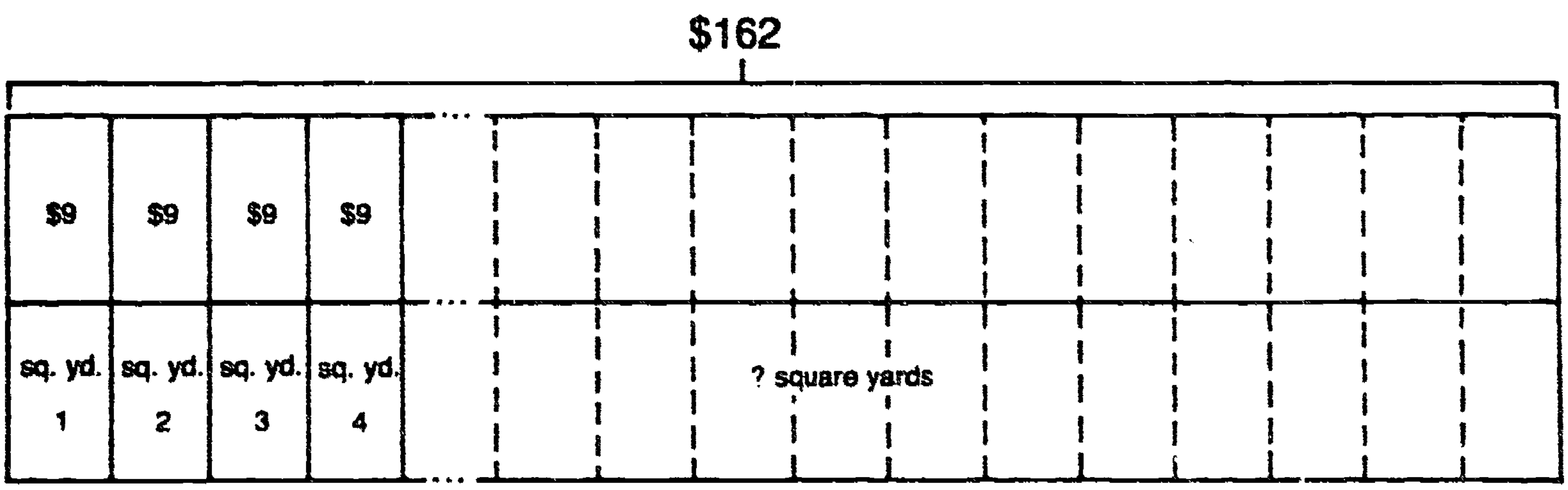


Fig. 14 A *rate 2* problem: Jane pays $162 for carpeting at $9 a square yard. How many square yards did she get?



- feet per second
- dollars per pound
- pounds per cubic foot
- gallons per minute
- cents per kilowatt hour
- parts per hundred

Children who are unable to think about rates and ratios will have difficulty doing these problems in any way other than substituting numbers into memorized formulas. Problems dealing with percentages are probably the best example of this difficulty.

Rate 1 problems result when the rate and the value of independent variable quantity are given (usually in units of measurement) and the value of the dependent variable, usually a total, must be found (fig. 13):

> Fred pays $12 a square yard for outdoor carpeting. How much will 16 square yards cost him?

The resulting application,

total cost
= cost/sq. yd. × number of sq. yd.
= $12/sq. yd. × 16 sq. yd. = $192,

is the easiest of the rate situations to use.

Rate 2 problems result when the rate and the value of the dependent variable are given and the value of the independent variable is sought (fig. 14):

> Jane pays $162 for carpeting at $9 a square yard. How many square yards does she get?

We have

$$\$162 = \$9/\text{sq. yd.} \times \square \text{ sq. yd.}$$

or

$$\text{sq. yd.} = \frac{\$162}{\$9/\text{sq. yd.}} = \boxed{18}.$$

Rate 3 problems result when the values of the dependent and independent variables are given and the ratio or comparison rate is sought (fig. 15):

> Peter paid $342 for 200 eight-foot two-by-fours. What was the cost in dollars of each two-by-four?

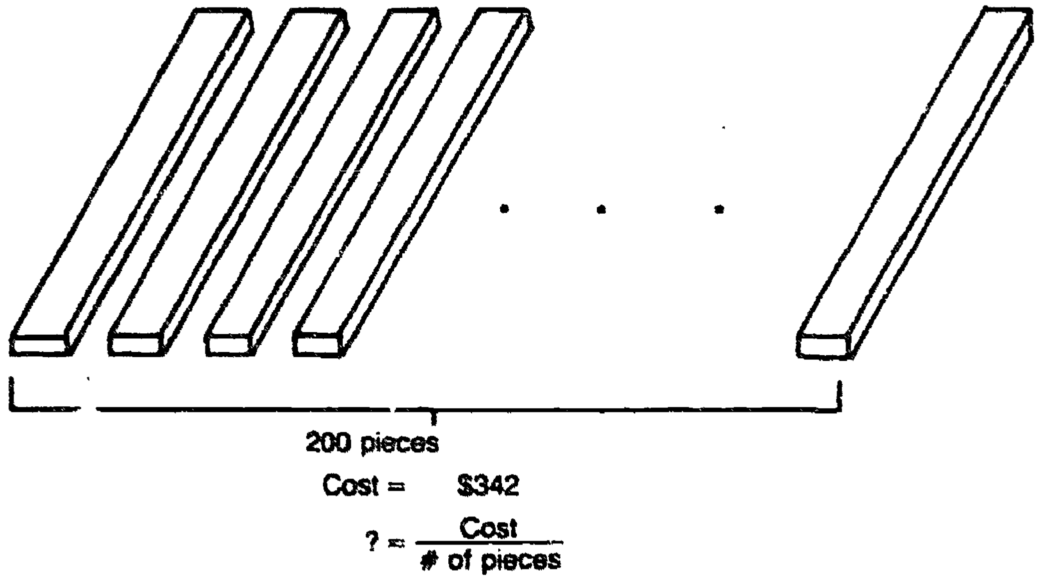
We have

$$\$342 = \boxed{\$\ \ }/\text{board} \times 200 \text{ boards}$$

or

$$\$ \text{ cost/board} = \frac{\$342}{200 \text{ boards}} = \$1.71/\text{board}$$

Fig. 15 A rate 3 problem: Peter paid $342 for 200 eight-foot 2 x 4's. What was the cost in dollars of each 2 x 4?



Selection Problems

Among the most difficult problems are those that require multiplication. These belong to a more general group of selection problems.

Selection 1 problems involve simple ordered pairs where the choice sets for each element of the ordered pair are given and the number of ordered pairs possible is sought. The pairs are ordered in the sense that one choice set is associated with one element and a second choice set with the other. No ordering occurs in the writing or selection. In the following example, (skirt, sweater) is not different from (sweater, skirt). See figure 16.

> Amy has 3 sweaters with different patterns. She also has 5 different skirts. How many outfits consisting of a sweater and a skirt are possible?

The pairs can be determined from a matrix (table 1) or from a "factor tree." Either way, multiplication is used: 3 × 5 = 15 outfits.

Table 1
A Matrix to Record the Pairs in Figure 16

	Skirts				
Sweaters	1	2	3	4	5
A	A, 1	A, 2	A, 3	A, 4	A, 5
B	B, 1	B, 2	B, 3	B, 4	B, 5
C	C, 1	C, 2	C, 3	C, 4	C, 5

Selection 2 problems result when one choice set and the number of pairs are given and the other choice set is sought. These problems are similar to selection 1 problems.

Selection 3 problems involve triples, quadruples, or other extended n-tuples ($n > 2$) and the choice sets for each place in the n-tuple.

> Frank has 5 sport coats, 3 vests, and 5 pairs of trousers, all of which are color compatible. How many different outfits consisting of a sport coat, vest, and pair of trousers are in his wardrobe?

Here a 3-tuple must be formed (sport coat, vest, trousers) where ordering is not important. Finding the total number of 3-tuples uses the multiplication principle: 5 × 3 × 5 = 75.

Selection 4 problems give the number of n-tuples and the sizes of all but one choice set, which is sought. An example follows:

> Frank can make 24 different outfits consisting of a sport coat, vest, and trousers. He has 3 sport coats and 4 pairs of trousers. How many vests does Frank have?

Fig. 16 A *selection 1* problem: Amy has 3 sweaters with different patterns. She also has 5 skirts of different colors. How many outfits, consisting of a sweater and a skirt, are possible?

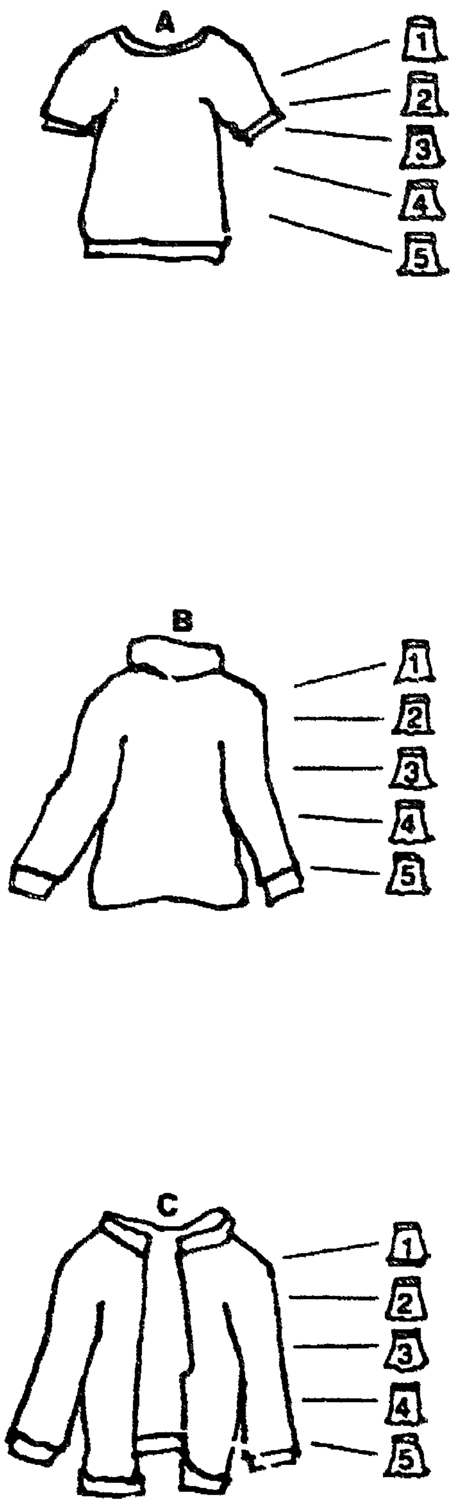


Fig. 17 Ceramic tiles can be used to link the repeated-addition idea of multiplication to area: Make 4 rows of 6 tiles each. How many tiles are used?

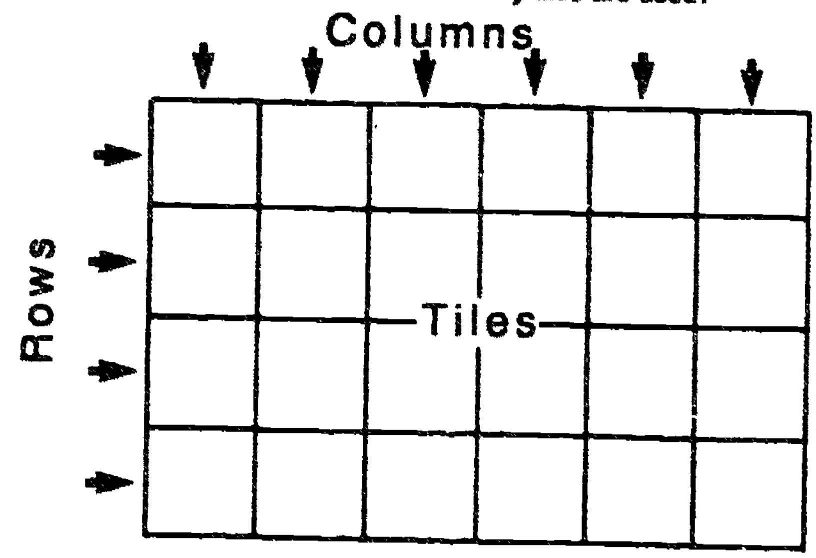


Table 2
Change Problems

Problem title	Sample problem	Characteristics
(Change 1) Repeated addition	Fred has 3 boxes with 4 cars in each box. How many cars does Fred have?	Initial (smaller) set sizes and change number known; question about final (larger) set.
(Change 2) Repeated subtraction (measurement)	Jean had 12 cookies. She gave 3 cookies to each of her friends. How many of her friends got cookies?	Initial (larger) set and final (smaller) set sizes known; question about change number.
(Change 3) Partitioning into equal sets	Paul had 24 marbles that he gave away to 4 friends. Each friend received the same number of marbles. How many marbles did each friend get?	Initial (larger) set and change numbers known; question about the size of final (smaller) sets.

This is a two-step problem: first multiply and then divide, or successively divide.

The selection group of problems involves the multiplication principle or one aspect of what Piaget calls combinatorial reasoning—the ability to consider the effect of several variables simultaneously. Selection 1 problems can be thought of as cells in a matrix. The thinking needed to solve them is similar to that used to solve area problems, such as being given two dimensions and finding the area and being given the area and one dimension and finding the other dimension.

Overview

If students are going to apply multiplication and division to everyday situations, they must have experience with materials that represent these different situations.

The change situations that involve joining and separating can be introduced with materials that can be joined, separated, and arranged. Unifix cubes can be used to illustrate repeated additions and repeated subtractions as well as measurements. Ceramic tiles can also be used to link the idea of repeated addition to area (fig. 17). The measurement concept of division can also be introduced with tiles. The following kinds of questions can be asked:

- Given 24 tiles, how many rows can be made with 4 tiles in each row?
- Make 4 rows of 6 tiles each. How many tiles are used?

Beans and paper cups can be used to give experience with the partition interpretation of division as well as to the repeated-addition and repeated-subtraction interpretations of multiplication and division. Some examples

Table 3
Compare Problems

Problem title	Sample problem	Characteristics
Compare 1	Joellen has 3 pairs of sandals. She has 4 times as many pairs of shoes. How many pairs of shoes does she have?	Referent set and many-to-one correspondence known: question about the comparison set.
Compare 2	Irene has 30 pennies. She has 5 times as many pennies as Pat has. How many pennies does Pat have?	Comparison set and many-to-one correspondence known; question about the referent set.
Compare 3	Donald has 5 marbles. Peter has 15 marbles. Peter has how many times as many marbles as Donald?	Comparison set and referent set given; question about kind of (many-to-one) correspondence.
Compare 4	Bonnie has 16 white blouses and 4 colored blouses. Her colored blouses are what (fractional) part of her white blouses?	Comparison set and referent set given: question about kind of (one-to-many) correspondence.
Compare 5	Our class has 16 boys and 12 girls. There are how many times as many boys as girls?	Comparison set and referent set given: question about the (many-to-many) correspondence.
Compare 6	Our class has 16 boys and 12 girls. The girls are what (fractional) part of the boys?	Comparison set and referent set given; question about many-to-many correspondence.
Compare 7	Fred has 25 baseball cards. He has 5/4 as many cards as Jim has. How many baseball cards does Jim have?	Comparison set and many-to-many correspondence given; question about referent set.
Compare 8	Erica has 25 stickers. Peggy has 4/5 as many stickers as Erica. How many stickers does Peggy have?	Referent set and many-to-many correspondence given; question about comparison set.

Table 4
Selection Problems

Problem title	Sample problem	Characteristics
Selection 1	Paula has 3 kinds of cheese and 2 kinds of sausage. How many different cheese-and-sausage pizzas can she make?	Number given from which to select for each pair element; question about number of pairs possible.
Selection 2	Frank makes 18 different cheese-and-sausage pizzas. He has 6 kinds of cheese. How many kinds of sausage does he have?	Number in one choice set and number of pairs given; question about number in other choice set.
Selection 3: extended *n*-tuple	Dave has 3 different-sized sets of wheels, 4 kinds of bodies, and 3 different motors. How many different cars with wheels, a body, and a motor can he put together?	Number given from which to choose for each portion in *n*-tuple; question about number of *n*-tuples possible.
Selection 4: extended *n*-tuple	Dave has 3 different-sized sets of wheels and 4 kinds of bodies; he can make 96 different cars with wheels, bodies, and motors. How many different kinds of motors does he have?	Number given from which to choose for all but one position in *n*-tuple and also number of *n*-tuples; question about remaining position.

Table 5
Rate Problems

Problem title	Sample problem	Characteristics
Rate 1	Lisa buys 18 cans of polish at $0.72 per can. What is the total cost?	Given the rate and the independent variable value; question is about the dependent variable.
Rate 2	Peter buys a suit on sale. The price, after a 25% discount, is $90. What was the original price?	Given the rate and the dependent variable value; question is about the independent variable.
Rate 3	Corrine runs 200 meters in 72 seconds. What is her average speed in meters per second?	Given the values of the dependent and independent variables; question is about the rate.

are the following:

- Given 21 beans, put 3 beans in cups until the beans are gone. How many cups did you use?
- Given 35 beans, put an equal number of beans into each of 5 cups. How many beans are in each cup?
- Given 4 cups, put 5 beans in each cup. How many beans were needed?

The *ratio comparison* situations can be introduced with two different shapes, two different colors of chips or cubes, or any other materials that can be put into sets and compared using the multiplication- and division-related questions in the examples.

The *selection* ideas can be introduced best with colored cubes or several geometric shapes in different colors, forming pairs and triples of these materials. Subsequently using situations that involve items from the students' experience, such as stickers, pizza toppings, clothing, and record labels, can help children apply these basic ideas of multiplication to the real world.

Rate problems should probably be introduced after establishing the idea of a constant rate of change in two related variables. This introduction must be done slowly and carefully and timed to the stage of cognitive development of the students. The demands are primarily on the proportional-reasoning capability of the students.

Introducing problems involving such relationships as *distance = time × rate*, *cost = cost/unit × units*, and *percentage = percent × base* should be within the more general context of rate of change. Otherwise students may substitute values into formulas without understanding the processes involved.

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BLACK LINE MASTERS

for many of the

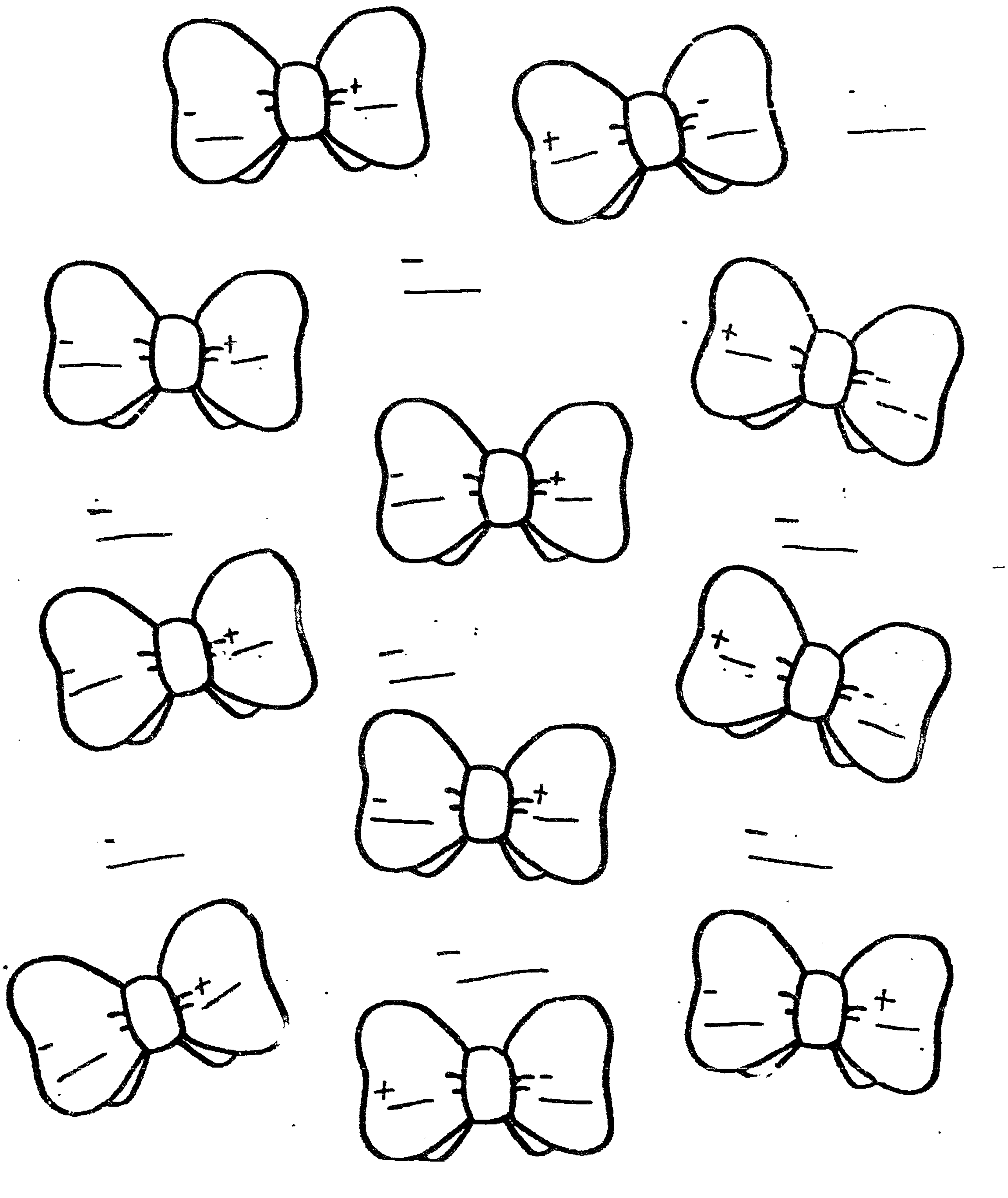
WORKSHEETS

and

RECORDING FORMS

MATHEMATICIAN: ______________________________

"I used the addition fact given to subtract."



Mathematician: ______________________

"These are my number sentences for the NUMBER OF THE DAY."

NUMBER ________	NUMBER ________
Number Sentences	**Number Sentences**
=	=
=	=
=	=
=	=
=	=
=	=
=	=
=	=
=	=
=	=

Mathematician: ______________________________

"I did these multiplication facts in __________ minutes."

X										

MATHEMATICIAN: ____________________

"I put an X on all numbers which are multiples of ________
I put a circle around all numbers which are multiples of ____________ I drew a line through four circles or four X's that are in a row."

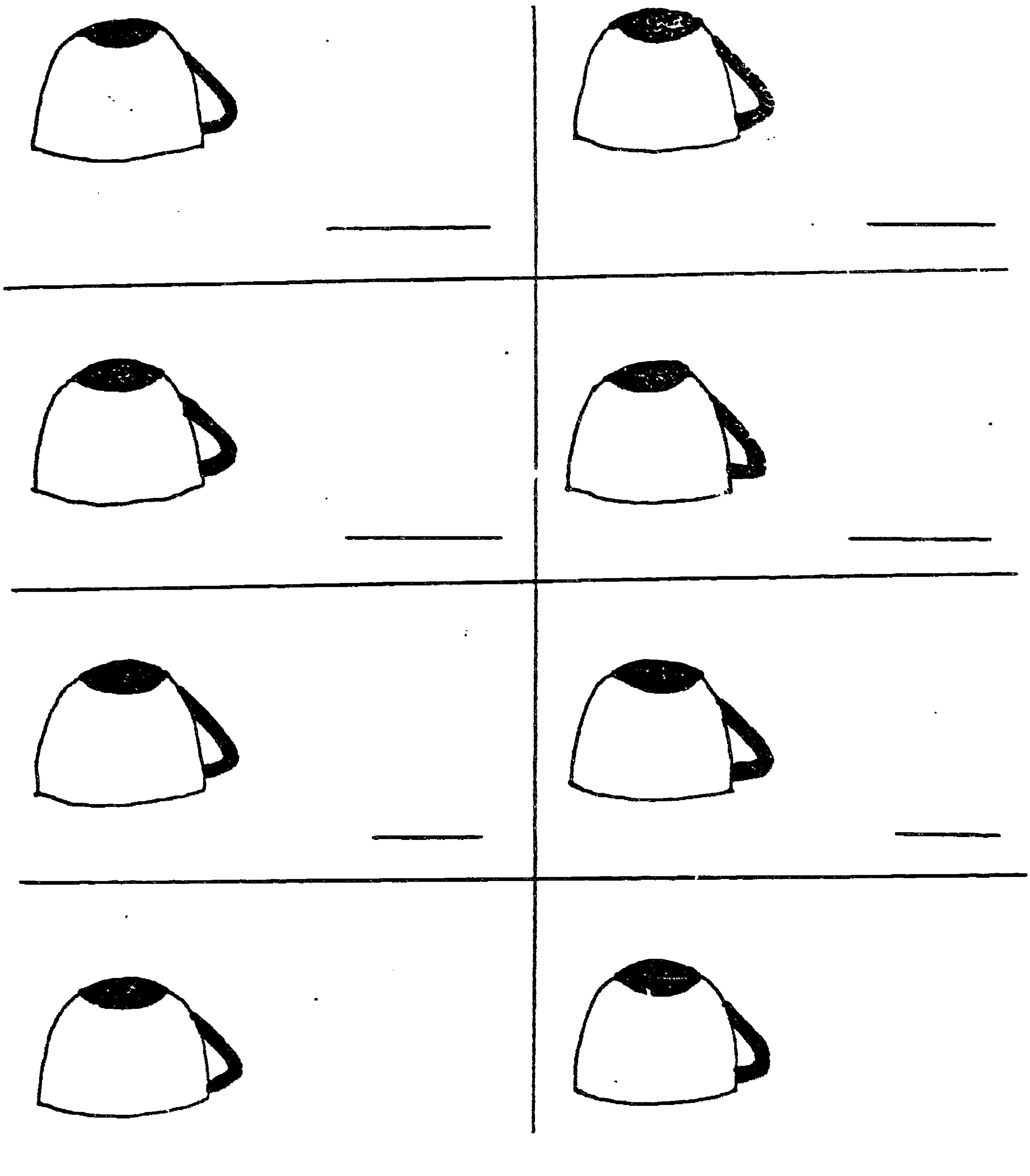
MATHEMATICIAN: ______________________________

"I completed the addition table in __________ minutes."

+										

MATHEMATICIAN: ______________________

"I wrote how many were still in the cup."



MATHEMATICIAN: ______________________________

"I finished finding these answers in __________ minutes."

5 − 0	7 − 2	4 − 1	9 − 0	6 − 1	3 − 2	1 − 1	2 − 0	8 − 2	9 − 1
6 − 2	7 − 0	10 − 1	2 − 1	3 − 0	4 − 2	8 − 1	9 − 2	5 − 1	6 − 0
11 − 2	4 − 0	3 − 1	8 − 0	7 − 1	1 − 0	5 − 2	0 − 0	10 − 2	1 − 1
8 − 7	9 − 9	11 − 8	5 − 4	7 − 5	6 − 6	9 − 8	7 − 4	8 − 8	10 − 9
9 − 7	5 − 3	5 − 5	8 − 6	4 − 3	6 − 4	3 − 3	12 − 9	4 − 4	6 − 3
6 − 5	10 − 9	9 − 6	2 − 2	7 − 6	11 − 9	7 − 7	10 − 7	8 − 5	7 − 3
13 − 6	12 − 8	11 − 3	17 − 8	12 − 5	10 − 6	14 − 5	9 − 3	16 − 8	11 − 4
18 − 9	10 − 3	12 − 6	15 − 9	13 − 8	12 − 3	9 − 5	14 − 6	13 − 4	11 − 6
12 − 7	8 − 4	16 − 9	13 − 7	12 − 4	15 − 7	11 − 5	14 − 9	16 − 7	10 − 4
9 − 4	15 − 8	13 − 5	15 − 6	14 − 7	13 − 9	10 − 5	17 − 9	14 − 8	11 − 7

MATHEMATICIAN ______________________________

"I finished finding these answers in ____________ minutes."

7 + 1	0 + 5	6 + 2	1 + 3	9 + 0	2 + 3	2 + 1	5 + 3	3 + 8	0 + 2
1 + 9	3 + 1	0 + 8	2 + 6	8 + 3	4 + 1	5 + 0	3 + 2	9 + 1	1 + 0
4 + 2	3 + 0	1 + 4	2 + 5	8 + 1	4 + 3	5 + 1	8 + 0	2 + 7	9 + 3
1 + 6	1 + 1	3 + 5	1 + 7	2 + 0	8 + 2	1 + 2	0 + 3	1 + 5	3 + 4
6 + 0	0 + 1	7 + 2	6 + 3	0 + 9	1 + 8	0 + 0	2 + 4	0 + 7	9 + 2
4 + 0	3 + 9	2 + 8	3 + 3	3 + 7	0 + 6	3 + 6	6 + 1	5 + 2	7 + 0
7 + 3	0 + 4	2 + 2	2 + 9	6 + 7	8 + 4	8 + 9	5 + 7	6 + 4	5 + 9
8 + 8	4 + 7	9 + 9	6 + 6	9 + 6	8 + 5	5 + 4	6 + 8	4 + 9	6 + 5
7 + 5	4 + 4	9 + 7	7 + 6	4 + 8	7 + 8	5 + 6	9 + 5	7 + 9	4 + 6
4 + 5	8 + 7	5 + 8	6 + 9	7 + 7	9 + 4	5 + 5	9 + 8	8 + 6	7 + 4

GENERALIZATION: ______________________________

"I added or subtracted to put the right number in the ◯."

1 + 4 = ◯

2 + 3 = ◯

3 + 4 = ◯

◯ = 3 + 5

◯ = 4 + 6

2 + 3 = ◯ + 4

7 + 2 = 5 + ◯

8 + 1 = 3 + ◯

6 + ◯ = 5 + 5

7 - ◯ = 2 + 1

9 - ◯ = 3 + 2

◯ = 8 - 5

10 + 3 = 6 + ◯

◯ + 5 = 8 + 7

3 + 4 = 9 - ◯

9 - 2 = ◯

5 + 6 = ◯ - 1

3 + 2 = ◯

◯ = 8 - 3

◯ = 9 + 3

8 + 5 = ◯

7 + 2 = 4 + ◯

8 + 2 = ◯ - 2

◯ - 3 = 7 + 9

◯ - 1 = 8 + 3

6 + 4 = 7 + ◯

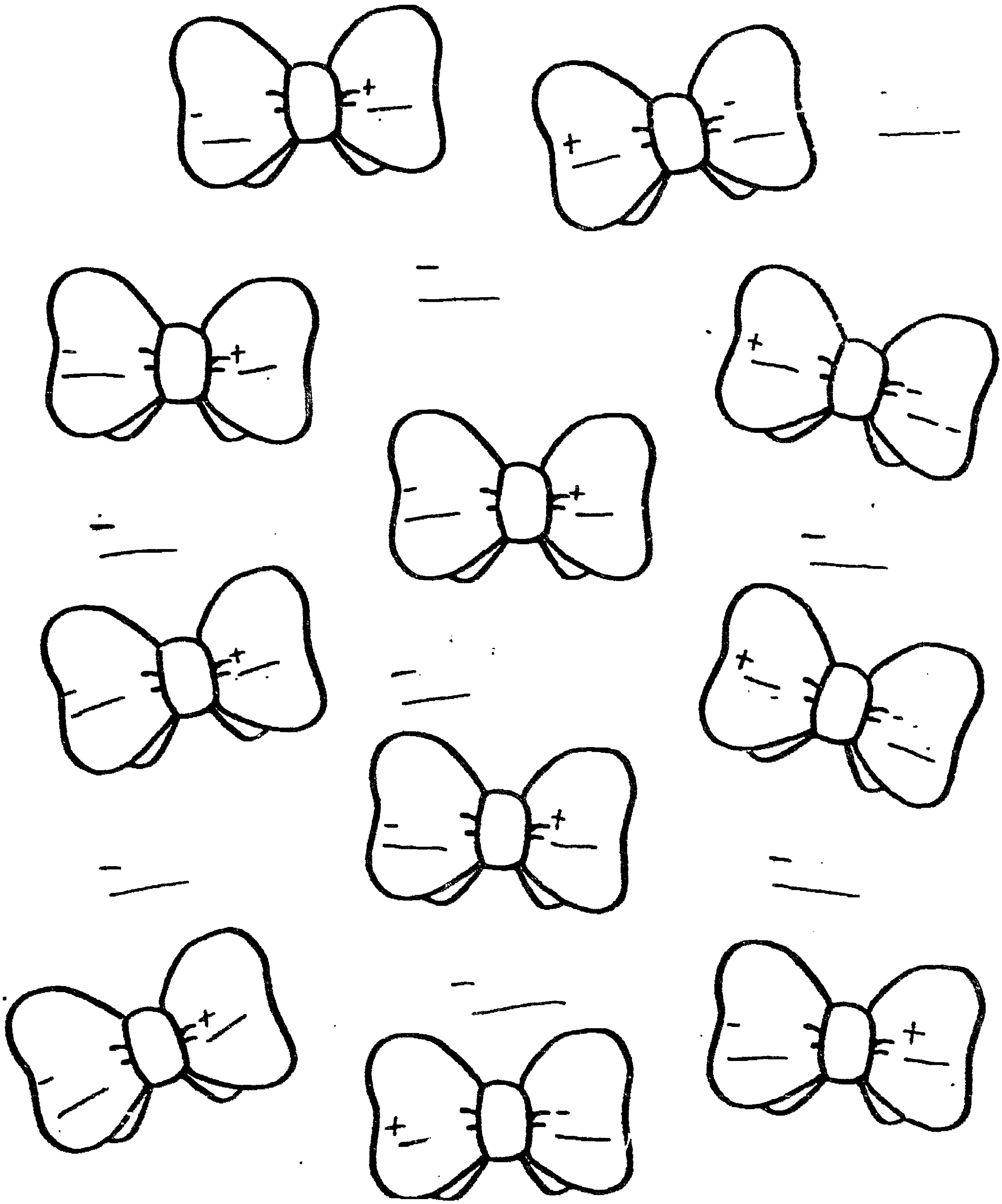
5 - 2 = ◯ + 1

12 - 3 = ◯ + 4

1 +4	8 -3
2 +3	7 +6
3 +5	8 +5
7 +2	9 +7
9 -3	8 +6
8 -2	7 +9
9 -2	8 +9

MATHEMATICIAN: ______________________________

"I used the addition fact given to subtract."



Mathematician:______________________________

"This is my record of numbers for 'Guess My Rule.'"

Numbers from the class (□)	Numbers by the teacher (△)	Numbers from the class (□)	Numbers by the teacher (△)

Rule △ = _____ □ _____

Rule △ = _____ □ _____

Mathematician:____

"I remembered to circle the answer in the number sentence."

PROBLEM	ANSWER	NUMBER SENTENCE

Mathematician:____

"Our group measured the area of different shapes using centimeter squares."

SHAPE	ESTIMATE of Area	MEASUREMENT of Area

98803

Mathematician: ______________________________

"I found volumes for the blocks given."

BLOCK SIDES SIDE 1	SIDE 2	SIDE 3	VOLUME OF THE BLOCK IN CUBIC UNITS

Mathematician:____

"Our group made MULTILINK cubes blocks the same size as the blocks we were given to use."

LOCK SIDES MEASURED /ITH MULTILINKS			NUMBER OF MULTILINKS TO MAKE THIS BLOCK	VOLUME OF THE BLOCK
ide 1	Side 2	Side 3		

Mathematician:________________________________

"I made 'houses' out of ______________________ on the floors given and recorded how many cubes were in the house."

FLOOR SIZE L	W	X NUMBER OF STORIES	= TOTAL ROOMS
		1	
		2	
		3	
		4	
		5	
		1	
		2	
		3	
		4	
		5	
		1	
		2	
		3	
		4	
		5	
		1	
		2	
		3	
		4	
		5	

THE ROD CODE

WHITE = W

RED = R

GREEN = G

PURPLE = Y

YELLOW = Y

DARK GREEN = D

BLACK = K

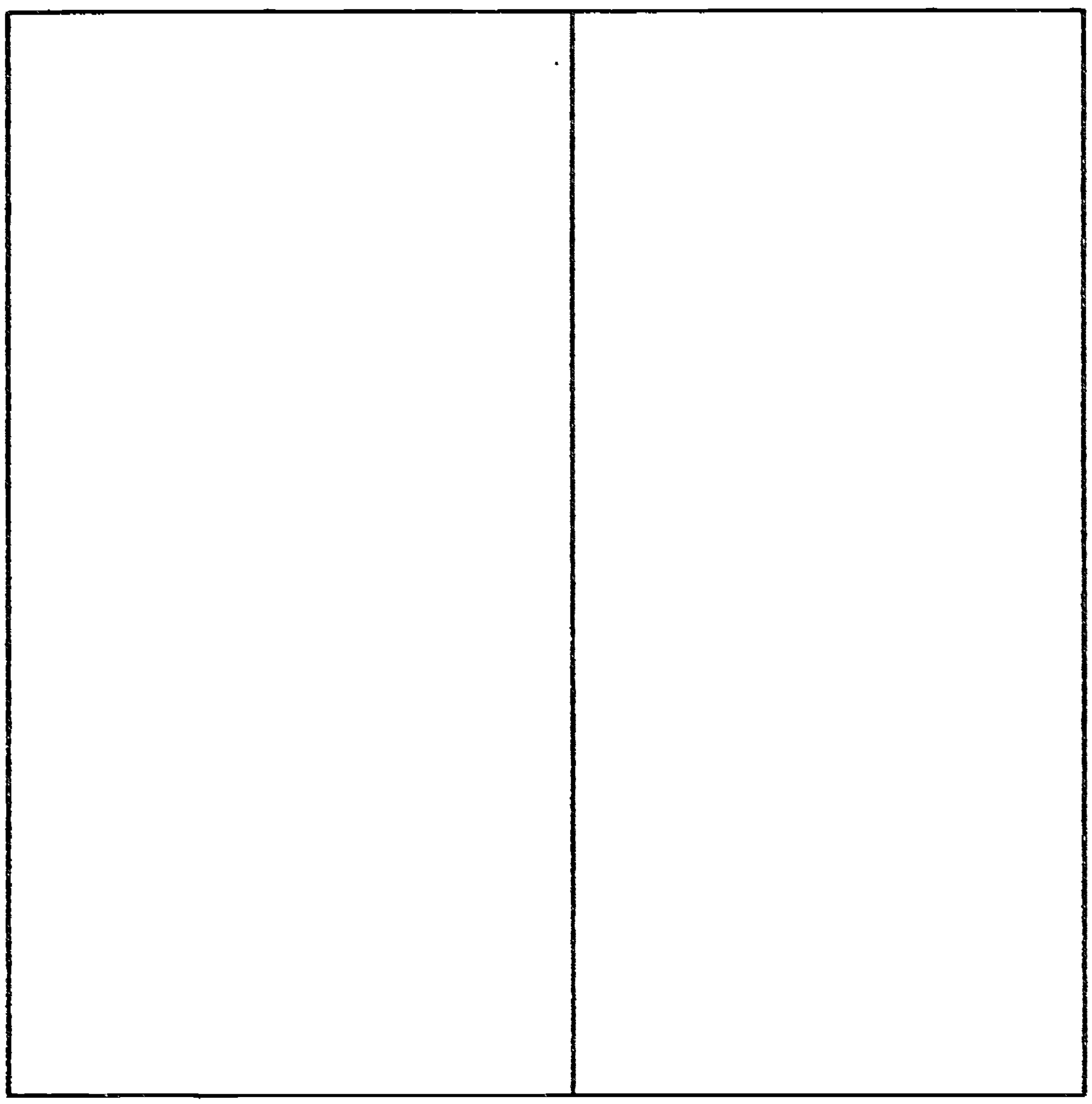
BROWN = N

BLUE = E

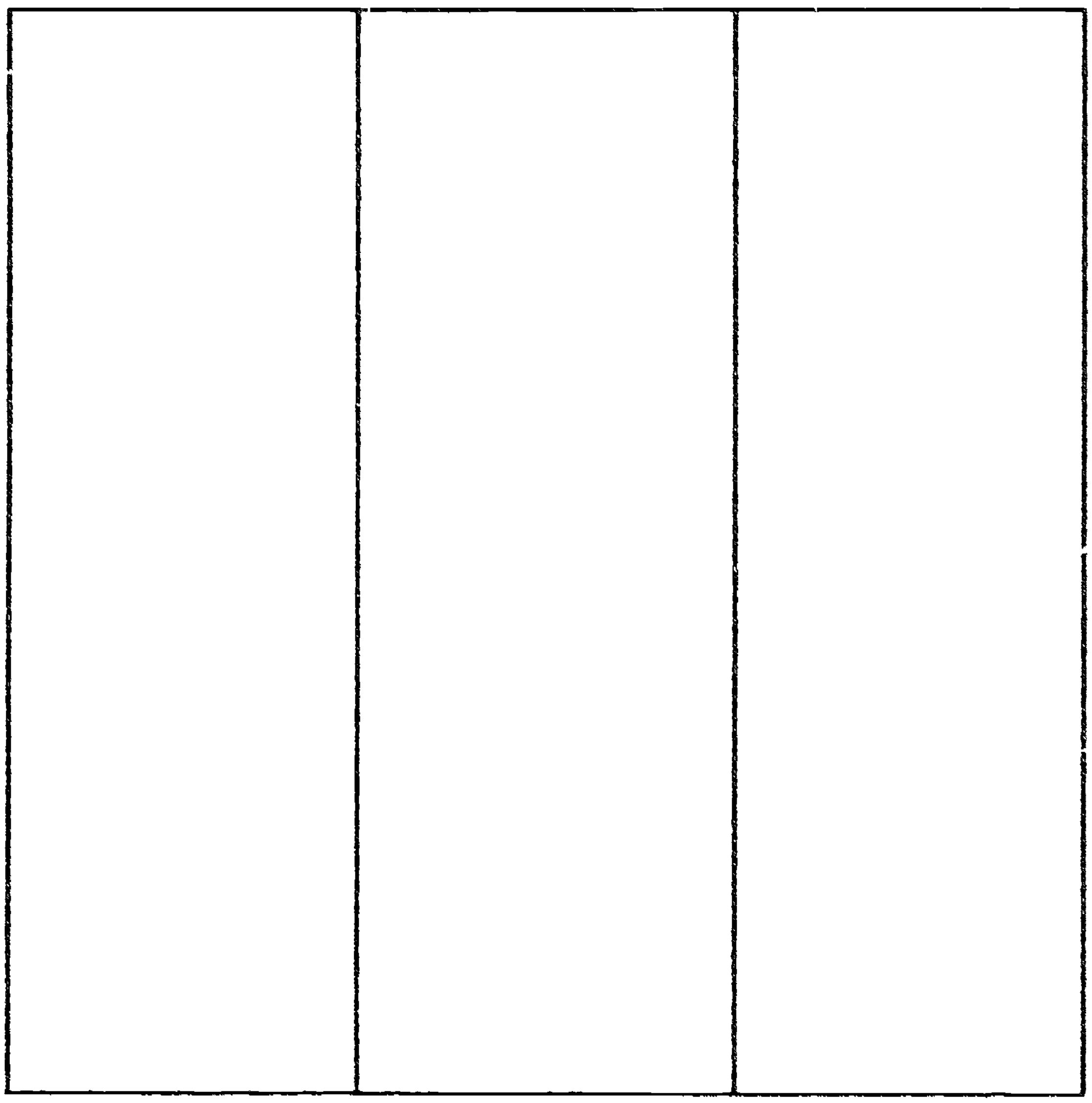
ORANGE = O

O R ORANGE + another rod in a train = O+

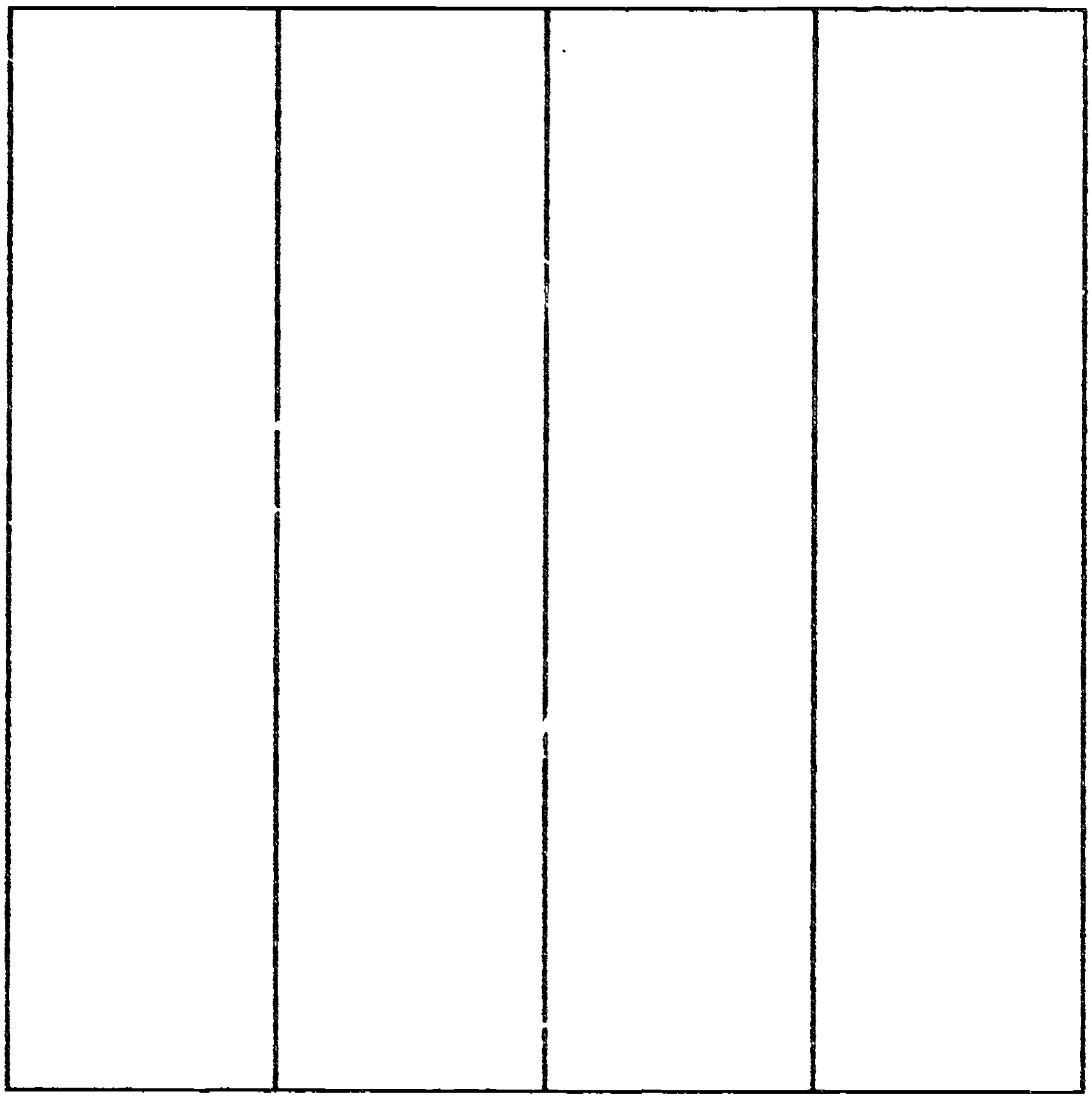
TEMPLATE FOR 1/2



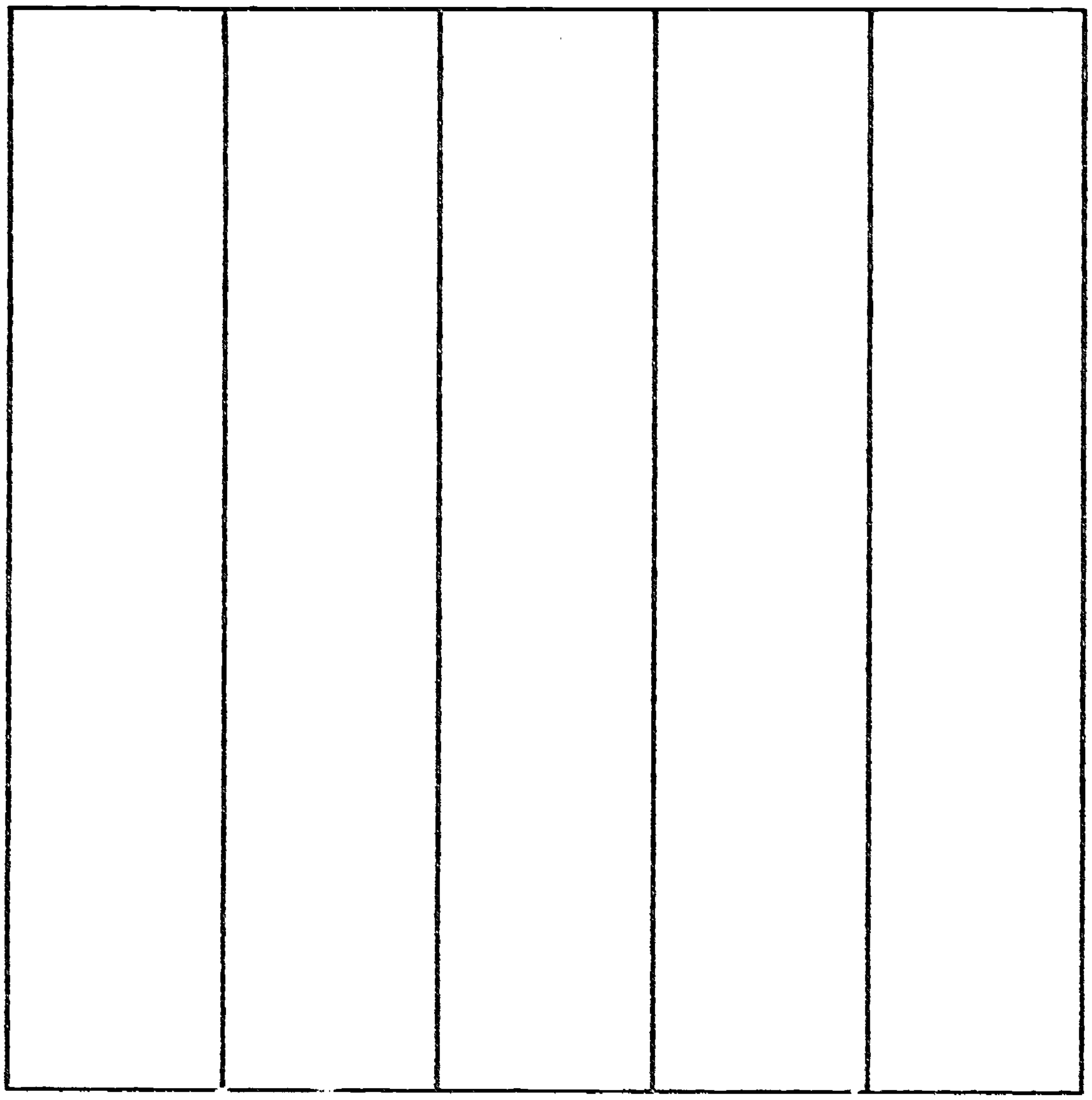
TEMPLATE FOR 1/3



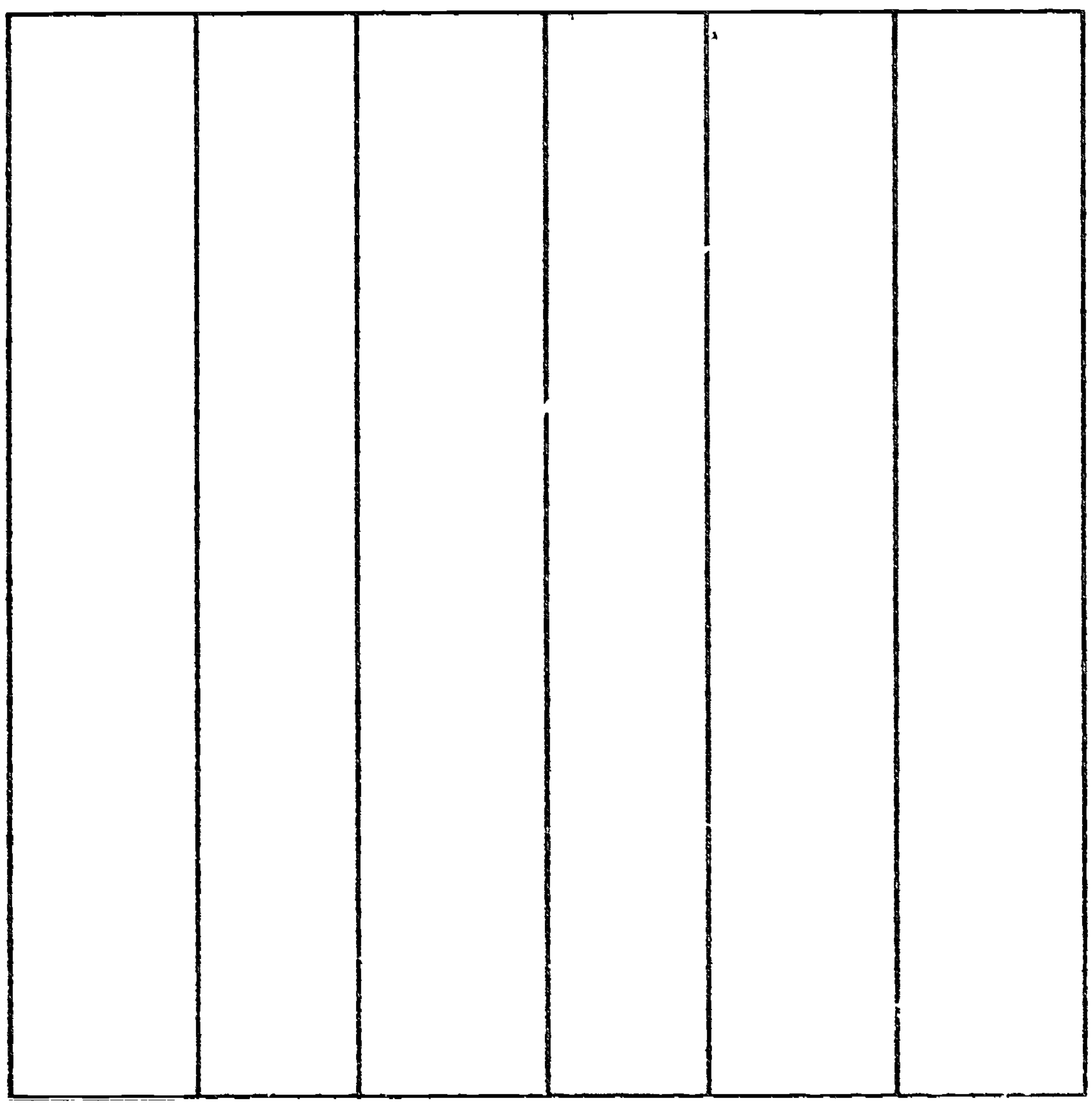
TEMPLATE FOR 1/4



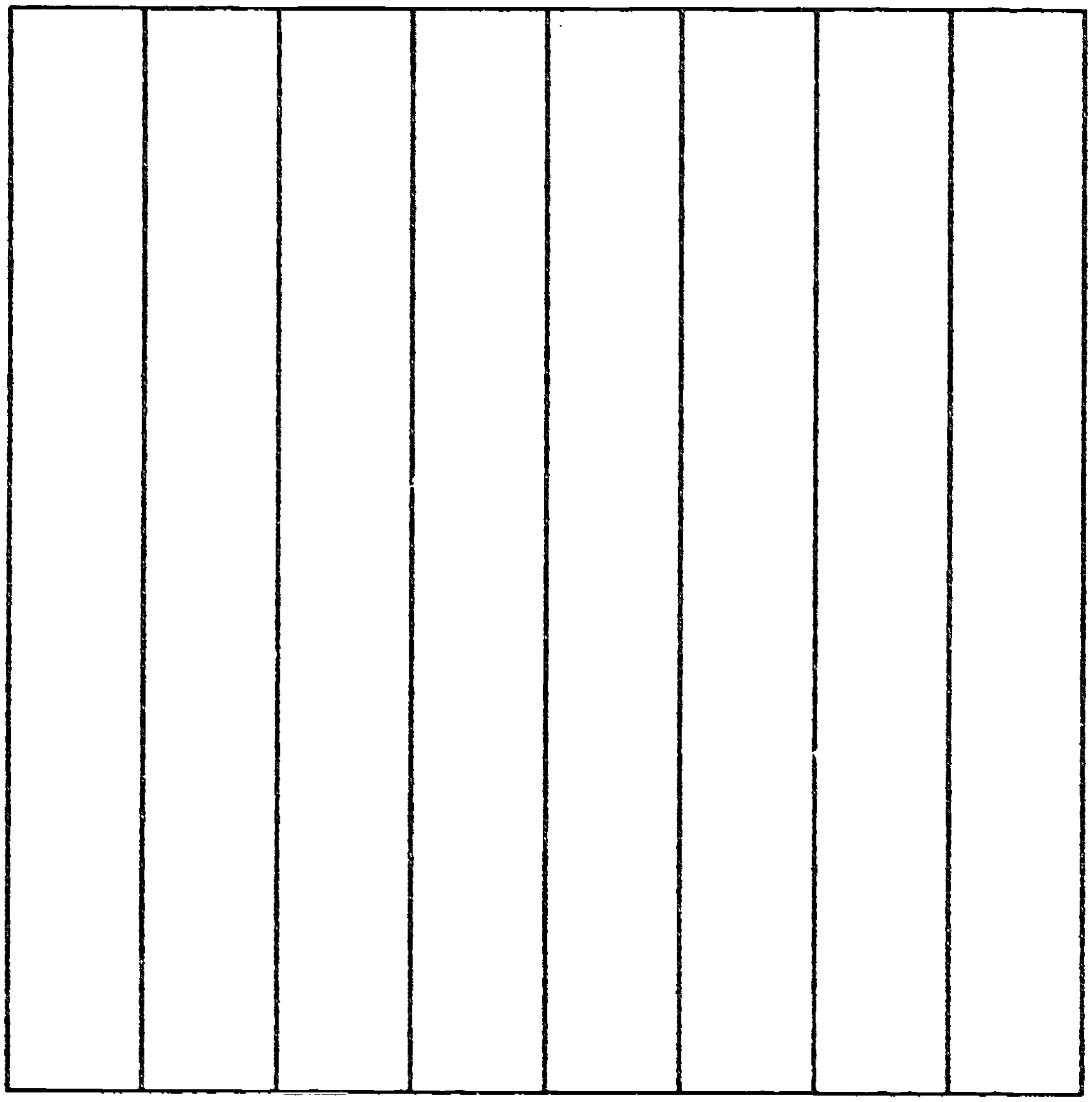
TEMPLATE FOR 1/5



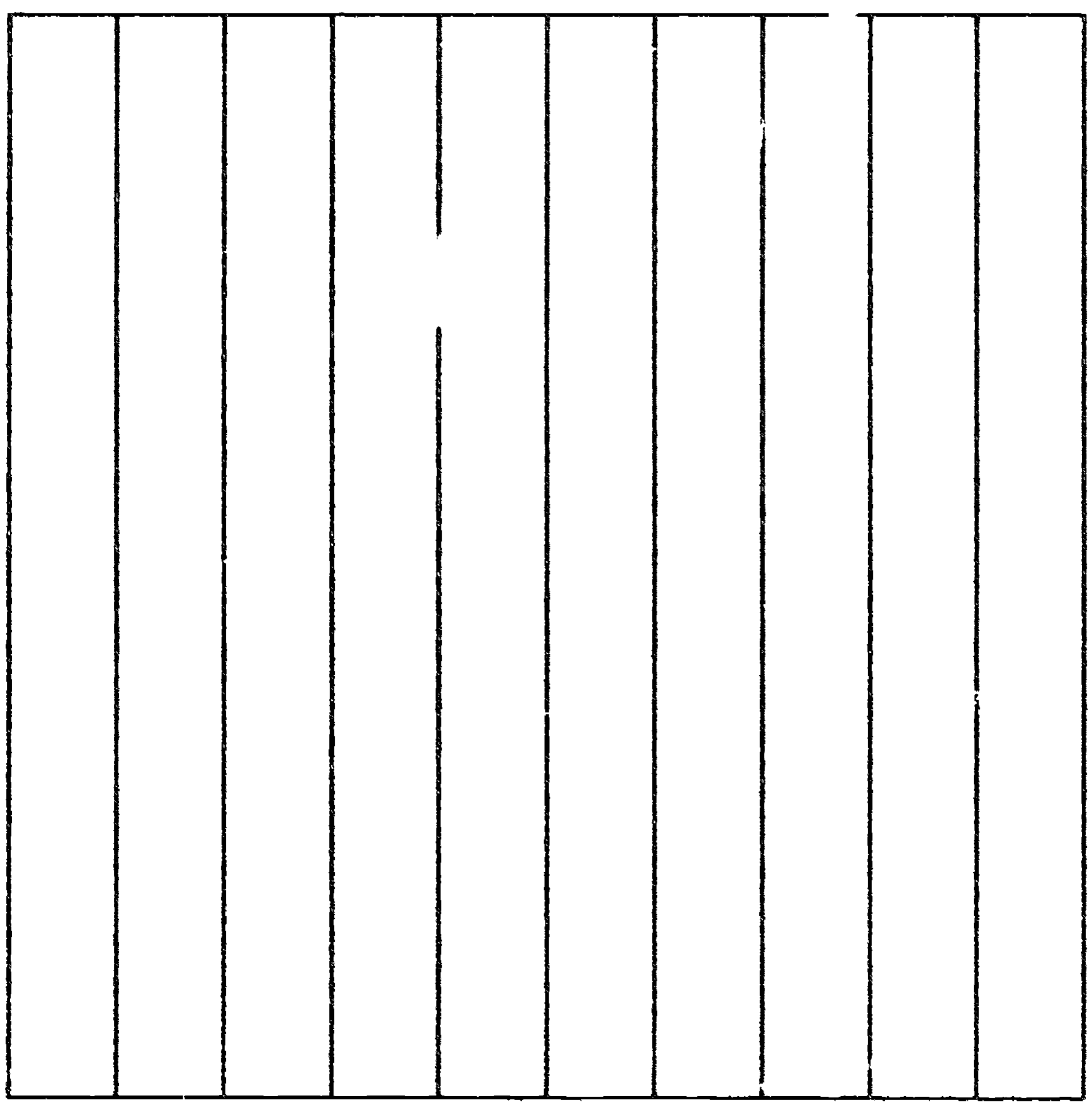
TEMPLATE FOR 1/6



TEMPLATE FOR 1/8

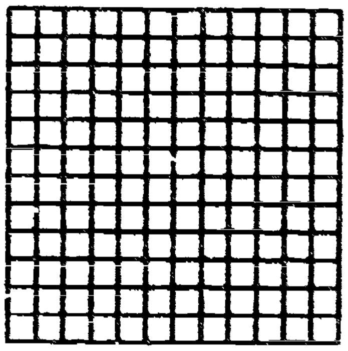
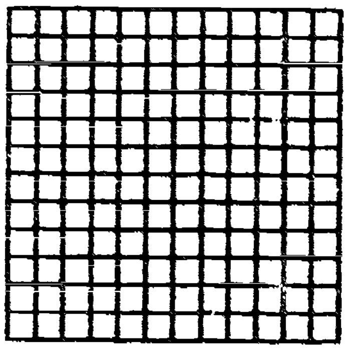
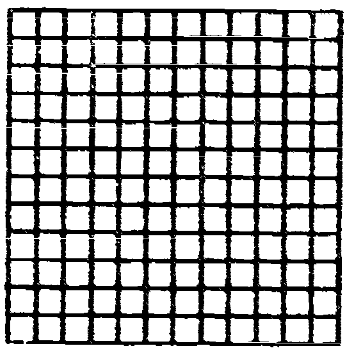
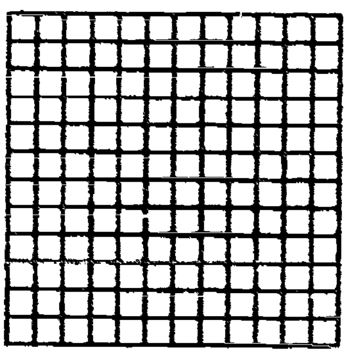


TEMPLATE FOR 1/10



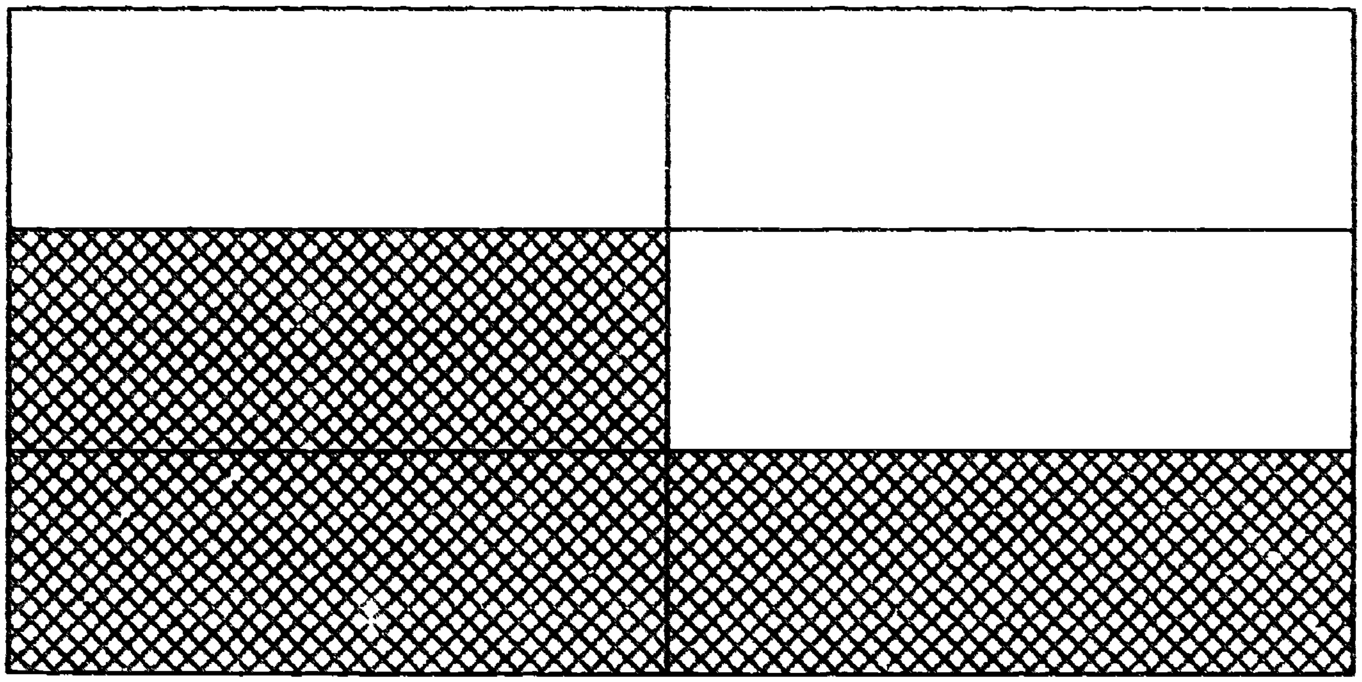
Mathematician:____

"I made rectangles to do these fraction multiplications."

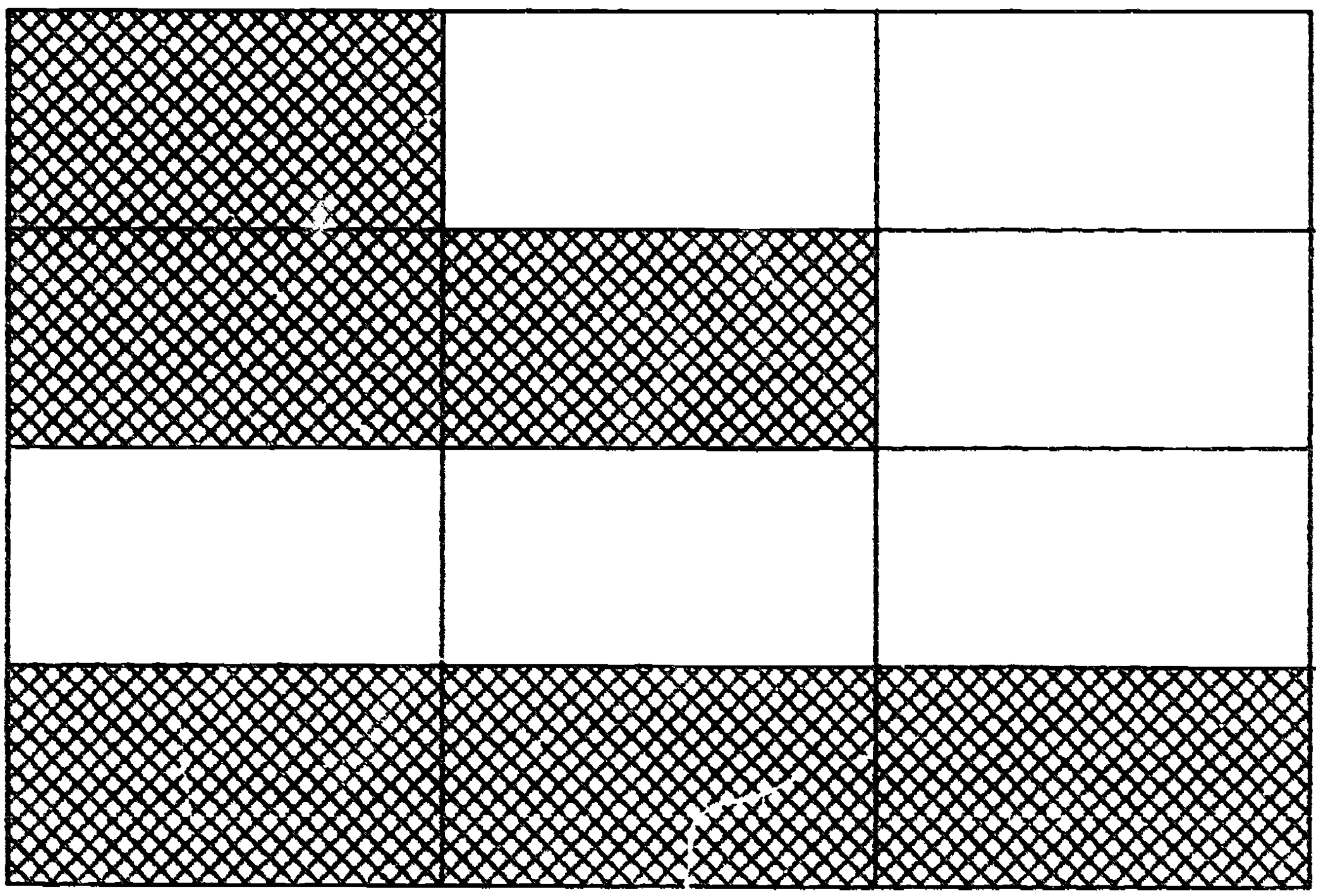
Fractions	Rectangles	Product
		
		
		
		

FRACTION BARS

GREEN



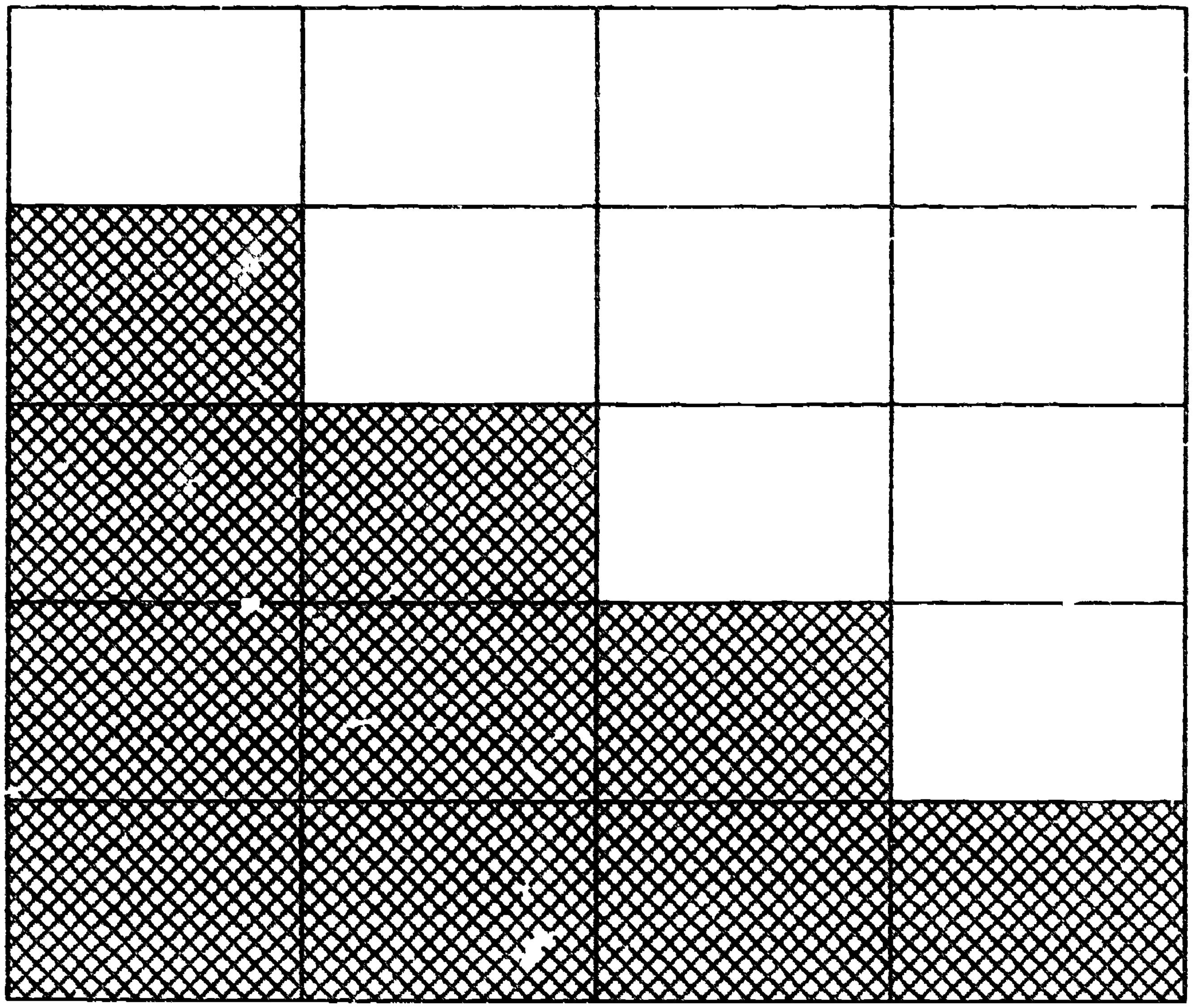
YELLOW



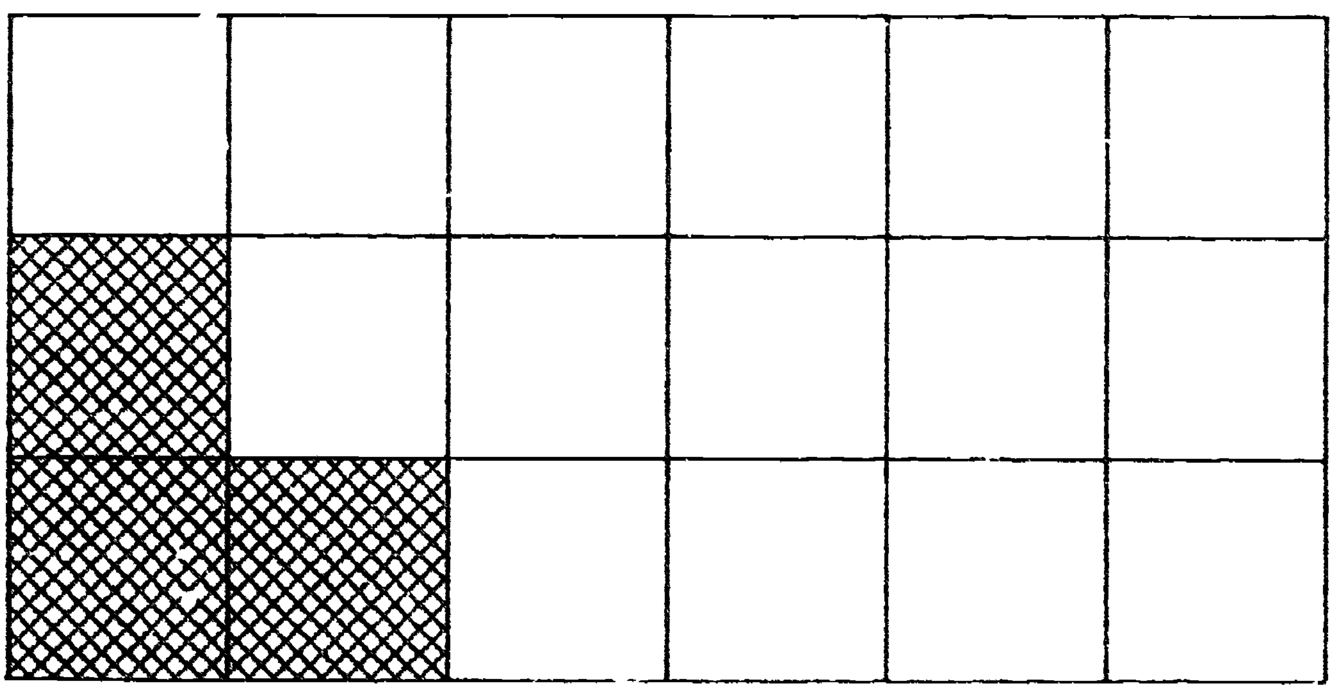
98814.1

FRACTION BARS

BLUE



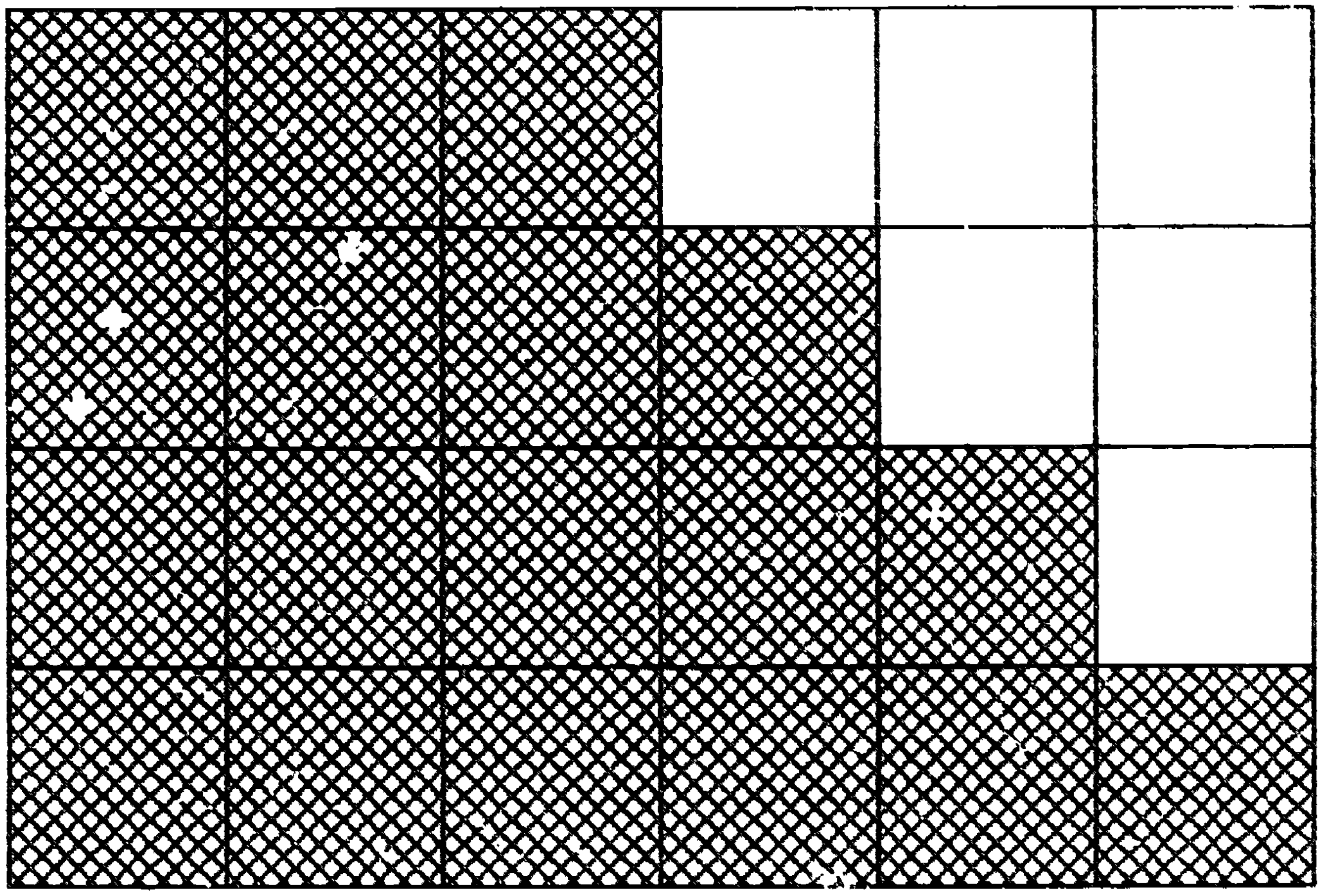
RED



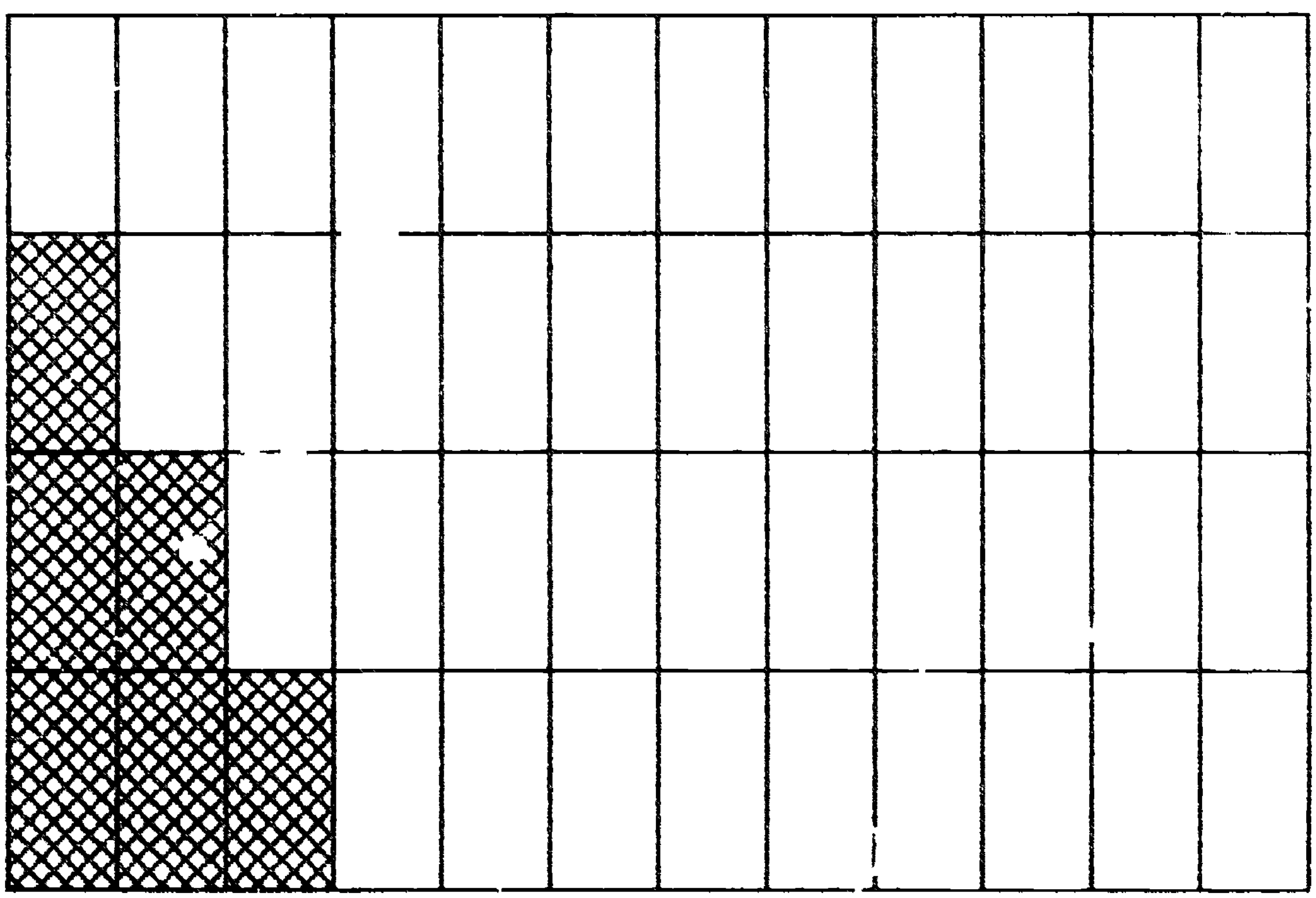
98814.2

FRACTION BARS

RED



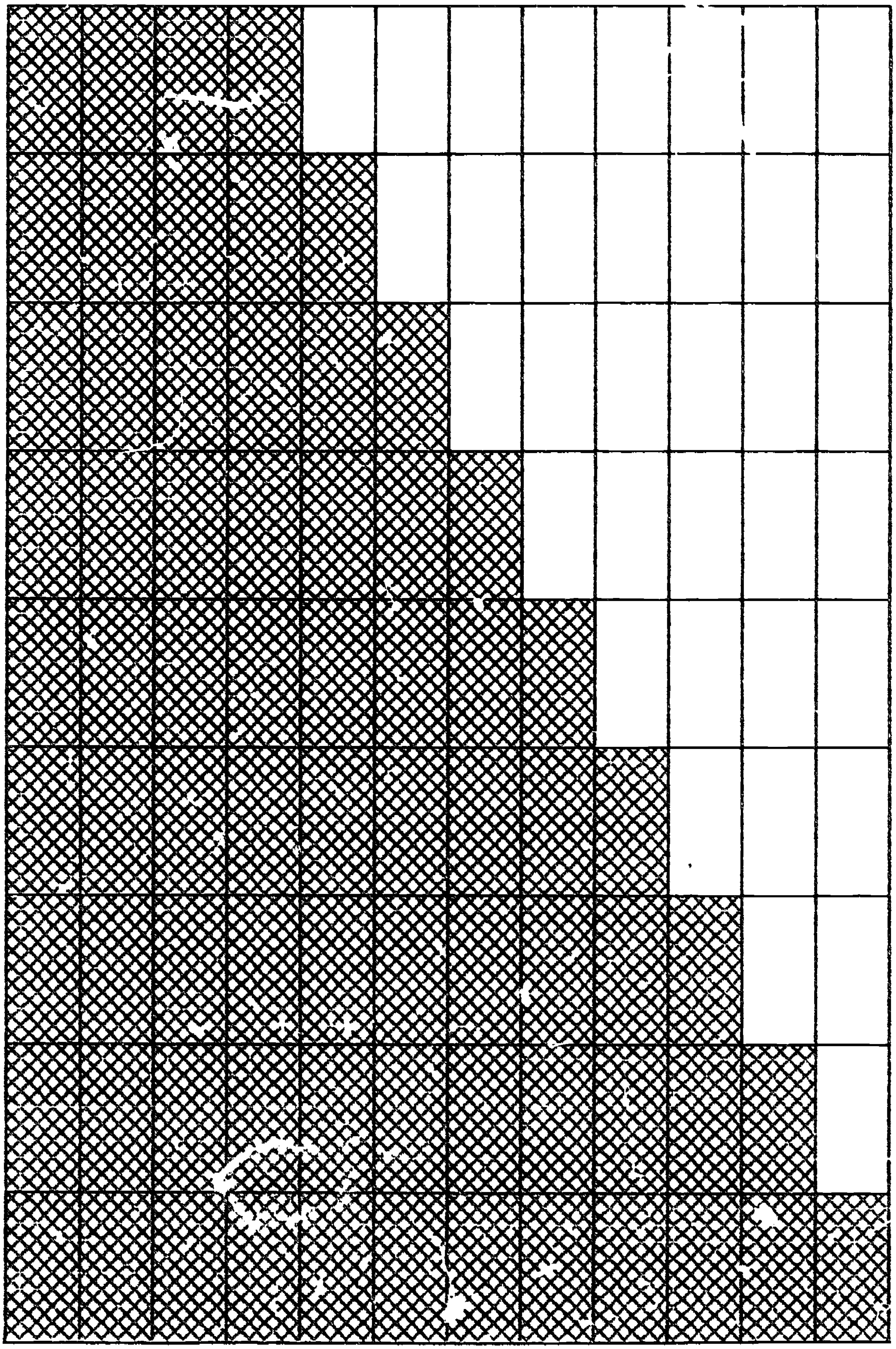
ORANGE



98814.3

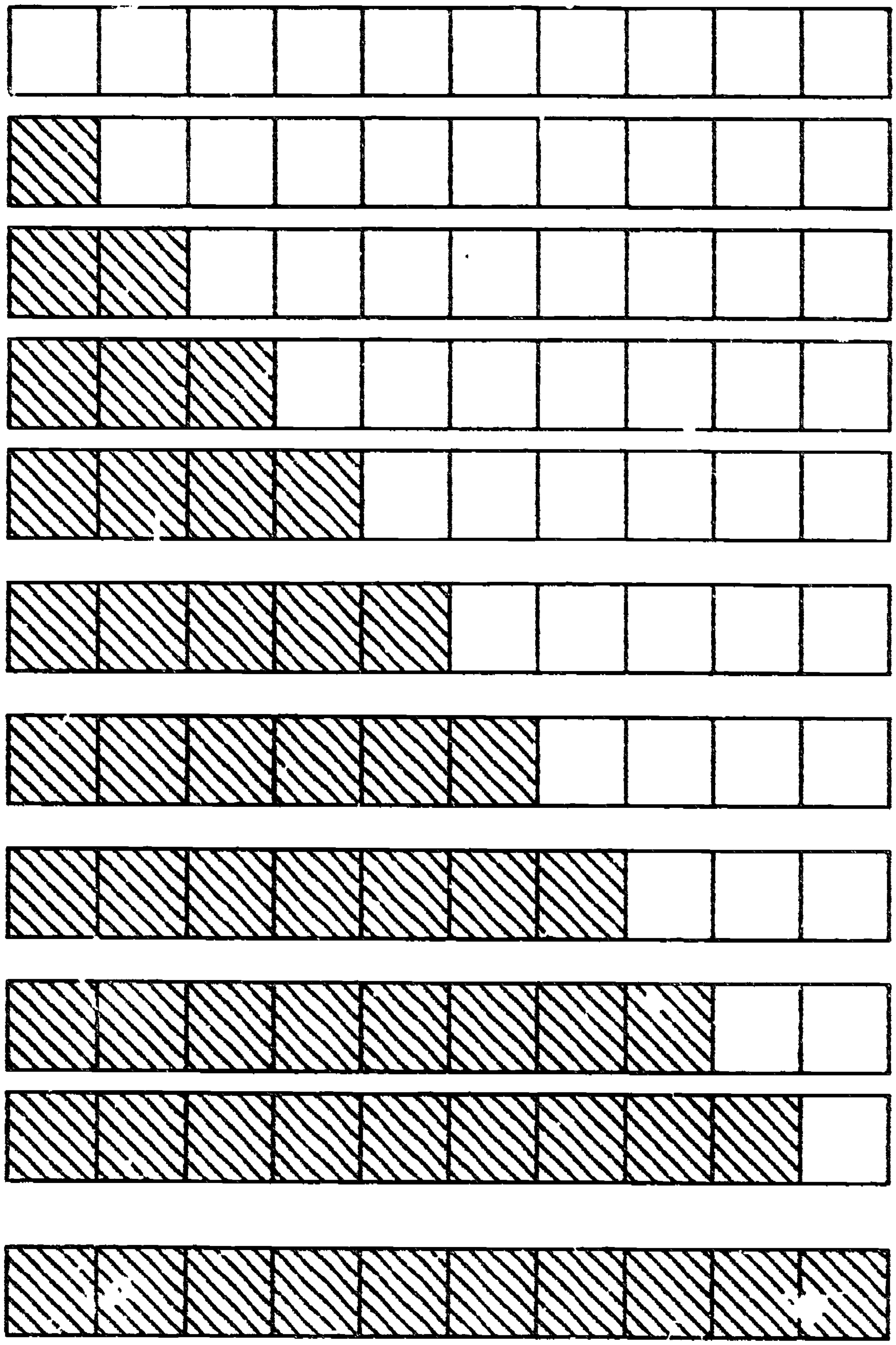
FRACTION BARS

ORANGE



98814.4

DECIMAL FRACTION BARS



Mathematician______________________________

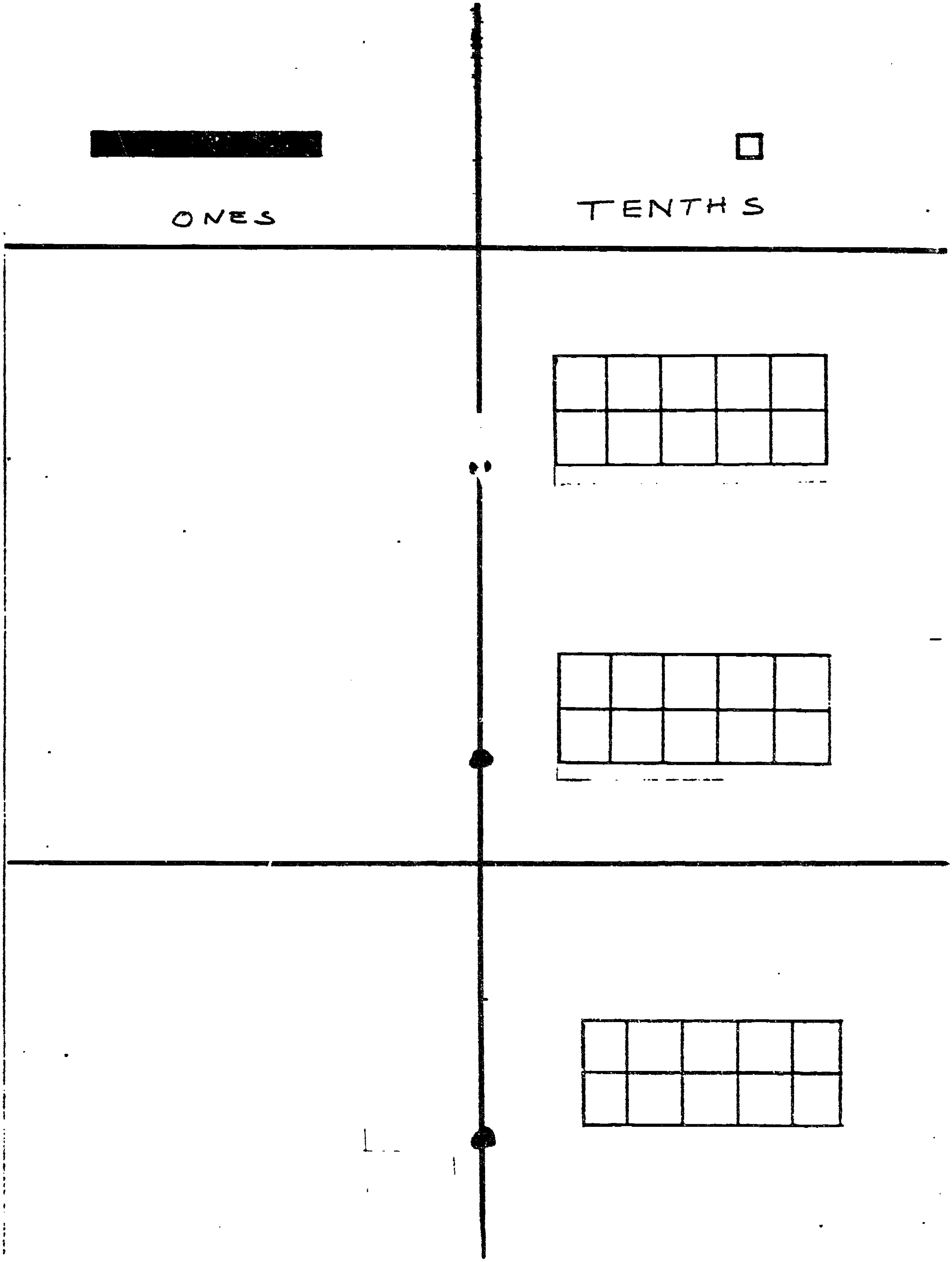
"I use [   ] color fraction bars and COMPARED ALL pairs of shaded parts and JOINED together ALL pairs of shaded parts.."

Number of shaded parts	Differences between shaded parts	Sum of shaded parts

Mathematician__________________________________

"I use [ ] color fraction bars to see what fractions can be made from other fractions."

Pairs of bars	Larger ÷ smaller	Smaller ÷ larger


ONES
TENTHS

Mathematician:______________________________

"I used base ten blocks with the long as ONE and the units as TENTHS to work these."

□□□□□□□□□□ = ONE

□ = TENTH

ONES	TENTHS	ONES	TENTHS

Mathematician:________________________________

"I used base ten blocks to complete this chart."

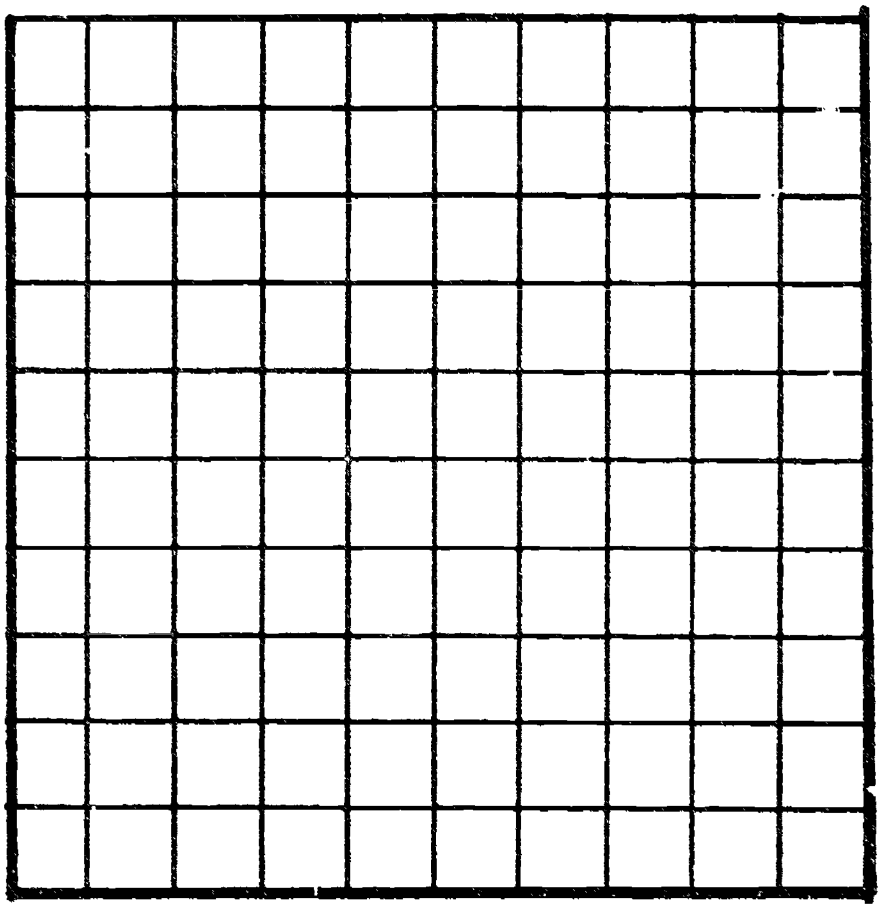
WORDS	NUMERALS
______ TENTHS ______ HUNDREDTHS	
______ TENTHS ______ HUNDREDTHS	
______ TENTHS ______ HUNDREDTHS	
______ TENTHS ______ HUNDREDTHS	
______ TENTHS ______ HUNDREDTHS	
______ TENTHS ______ HUNDREDTHS	
______ TENTHS ______ HUNDREDTHS	

Mathematician: ______________________________

"I used base ten pieces to complete this decimal chart."

BASE TEN BLOCKS	DECIMAL NAME	NUMERAL
	____ONES AND ____HUNDREDTHS	
	____ONES AND ____HUNDREDTHS	
	____ONES AND ____HUNDREDTHS	
	____ ONES AND ____HUNDREDTHS	
	____ONES AND ____HUNDREDTHS	

TEMPLATE FOR OVERHEAD AND/OR CARDBOARD
BASE TEN BLOCKS



Use a paper cutter to get ONES out of TENS and TENS out of HUNDREDS.

Mathematician ________________________________

"I build numbers in 3 different ways from base ten thousands, hundreds, tens and ones."

Number	Thousands	Hundreds	Tens	Ones

Mathematician:________________________________

"I wrote the numbers represented by the base ten blocks as they appeared."

Times the Cover was Moved	Words	Numeral
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

Mathematician:____________________________________

"I did these subtractions and checked by adding."

SUBTRACTION

CHECKS

Mathematician:____

"I did these additions and checked by subtracting "

ADDITION

CHECKS

÷

Mathematician:

"I multiplied the number given by each number down the side by arrangement base ten blocks on the workmat."

X		
1		
2		
3		
4		
5		
6		
7		
8		
9		

Mathematician: ______________________________

"I built rectangles from base ten blocks on the workmat to do these multiplications."

PROBLEM	RECORD H	T	O	PROBLEM	RECORD H	T	O

Mathematician:________________________________

"I divided a rectangle up to show the partial products, wrote all of the partial products for the multiplications and added them to get the total product."

MULTIPLICATION | RECTANGLE

	Thousands	Hundreds	Tens	Ones
1.				
2.				
3.				
4.				
5.				
6.				
Total				

	Thousands	Hundreds	Tens	Ones
1.				
2.				
3.				
4.				
5.				
6.				
Total				

Mathematician:____________________________________

"I made rectangles on the graph paper to do the multiplications. I recorded all partial products and added them to find the product."

	HUNDREDS	TENS	ONES
A =			
B =			
C =			
D =	+		

A =			
B =			
C =			
D =	+		

A =			
B =			
C =			
D =	+		

	HUNDREDS	TENS	ONES
A =			
B =			
C =			
D =	+		

A =			
B =			
C =			
D =	+		

A =			
B =			
C =			
D =	+		

Mathematician:________________________________

"I built rectangles from base ten blocks on the workmat to do these multiplications."

Problem	Picture of Rectangle made	Computation H	T	O
		+		
		+		
		+		
		+		

Mathematician: ______________________________

"I made rectangles on the graph paper to do the multiplications. I recorded all partial products and added them to find the product."

	HUNDREDS	TENS	ONES
A =			
B =			
C =			
D =	+		

	HUNDREDS	TENS	ONES
A =			
B =			
C =			
D =	+		

	HUNDREDS	TENS	ONES
A =			
B =			
C =			
D =	+		

	HUNDREDS	TENS	ONES
A =			
B =			
C =			
D =	+		

	HUNDREDS	TENS	ONES
A =			
B =			
C =			
D =	+		

	HUNDREDS	TENS	ONES
A =			
B =			
C =			
D =	+		

Mathematician: ______________________________

"I used base ten blocks on graph paper to do these multiplications. I made a picture of the rectangle made showing partial products. I completed the computation form and wrote the multiplication in expanded form."

Problem	Partial Products	Expanded Forms

Mathematicien:________________________________

"I used base ten blocks or graph paper to make rectangles.
I recorded ALL FOUR partial products and found the total product.

	THOUSANDS	HUNDREDS	TENS	ONES
1.				
2.				
3.				
4. +				
TOTAL				

	THOUSANDS	HUNDREDS	TENS	ONES
1.				
2.				
3.				
4. +				
TOTAL				

	THOUSANDS	HUNDREDS	TENS	ONES
1.				
2.				
3.				
4. +				
TOTAL				

	THOUSANDS	HUNDREDS	TENS	ONES
1.				
2.				
3.				
4. +				
TOTAL				

Mathematician:____

"I divided a rectangle up to show the partial products, wrote all of the partia products for the multiplications and added them to get the total product."

MULTIPLICATION | **RECTANGLE**

	Thousands	Hundreds	Tens	Ones
1.				
2.				
3.				
4.				
5.				
6.				
Total				

	Thousands	Hundreds	Tens	Ones
1.				
2.				
3.				
4.				
5.				
6.				
Total				

Mathematician: ____________________

"I used the given number of tiles to make rows of five different lengths.
I wrote the computation form and the number sentences for these."

TILES	ROW LENGTHS	DIVISION FORM	NUMBER SENTENCE

Mathematician: ______________________________

"I used the graph paper to find the QUOTIENT (missing side) for these divisions."

DIVISION	COMPUTATION				NUMBER SENTENCE
	Divisor	HUNDREDS	TENS	ONES	

Mathematician: ______________________________

"I used the given number of tiles to make rows of five different lengths.
I wrote the computation form and the number sentences for these."

TILES	ROW LENGTHS	DIVISION FORM	NUMBER SENTENCE

Mathematician:________________________________

"I made rectangles to do the divisions, drew these and labelled the sides. I also completed the computation form."

Division	Rectangle	Computation Form
		Hundreds | Tens | Ones
		Hundreds | Tens | Ones
		Hundreds | Tens | Ones

Mathematician:____

"I used the graph paper to find the QUOTIENT (missing side) for these divisions."

DIVISION	COMPUTATION				NUMBER SENTENCE
	Divisor	Hundreds	Tens	Ones	

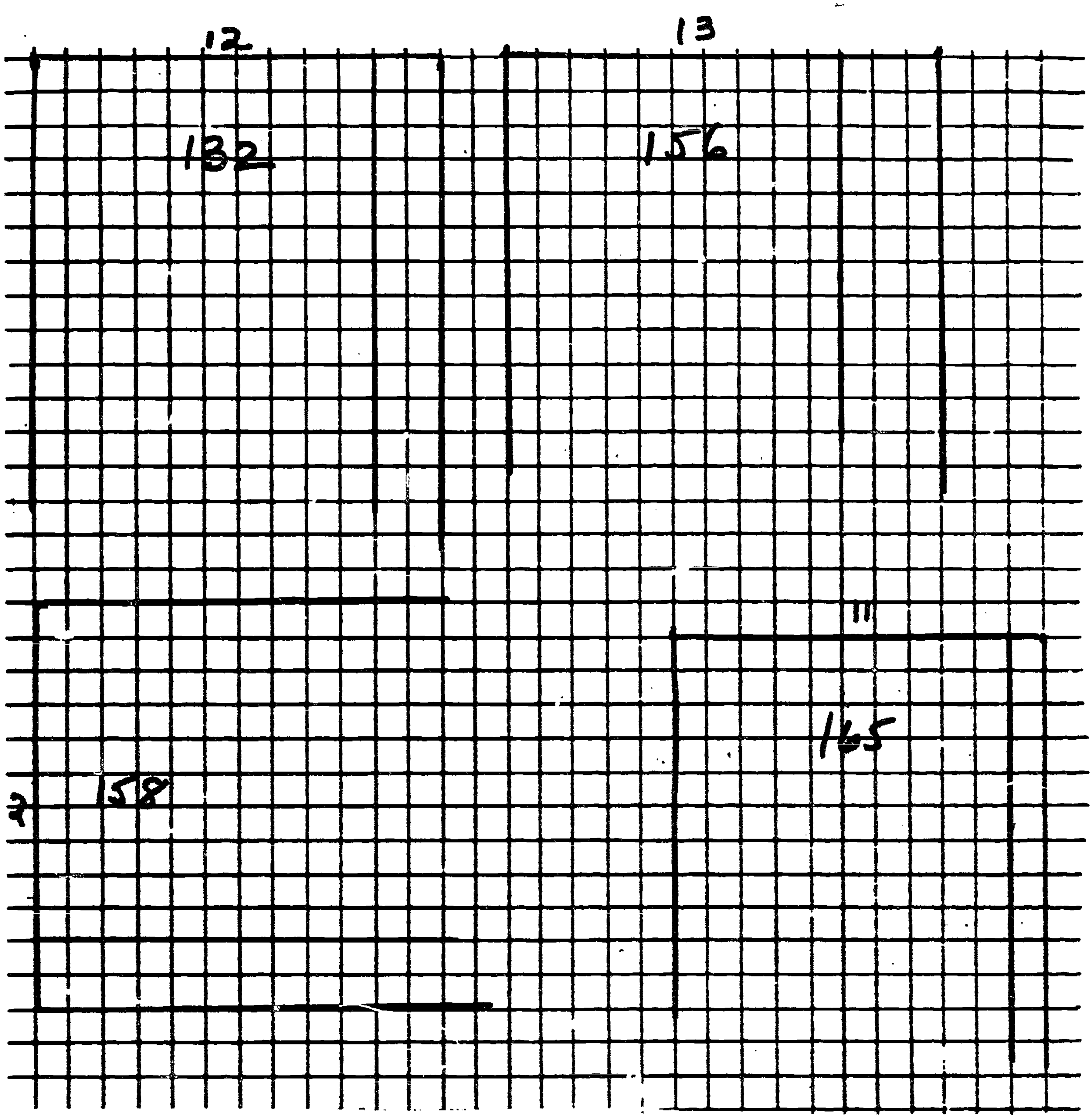
Mathematician:____

"I made the largest rectangles I could on the graph paper using the square units given, the side given, and completed the computation form."

Squares given	Side given	Side found	Computation form	
			TENS	ONES

Mathematician:______________________________

"I drew a line on the graph paper to make the largest rectangle. I circled the remainder. I wrote the missing side on the paper after I found it."



Mathematician: ____________________

"I use base ten blocks to find the missing sides ot he largest rectangle I could make from a given number of base ten pieces."

BASE TEN BLOCKS		SIDE GIVEN	SECOND SIDE OF LARGEST RECTANGLE	COMPUTATION FORM	
TENS	ONES			TENS	ONES

Mathematician:____

"I made rectangles to do the divisions, drew these and labelled the sides. I also completed the computation form."

Division	Rectangle	Computation Form
		Hundreds | Tens | Ones
		Hundreds | Tens | Ones
		Hundreds | Tens | Ones

Mathematician:______________________________

"I did these divisions by making rectangles on the workmat. I showed the results by completing the computation form."

PICTURE OF WORKMAT	COMPUTATION FORM		
	HUNDREDS	TENS	ONES
	HUNDREDS	TENS	ONES
	HUNDREDS	TENS	ONES
	HUNDREDS	TENS	ONES

Mathematician:____________________________________

"I use base ten pieces to make rectangles on the division mat. I completed the computatio form."

Side Given	Base Ten Pieces Available	Side Found	Remainder	Computation Form		
				HUNDREDS	TENS	ONES
	____ Hundreds ____ Tens ____ Ones					
	____ Hundreds ____ Tens ____ Ones					
	____ Hundreds ____ Tens ____ Ones					
	____ Hundreds ____ Tens ____ Ones					

Mathematician:______________________________

"I made rectangles to do the divisions, drew these and labelled the sides. I also completed the computation form."

Division	Rectangle	Computation Form
		Hundreds / Tens / Ones
		Hundreds / Tens / Ones
		Hundreds / Tens / Ones

Mathematician: ______________________ ______________________

"I used base ten blocks to do these. I divided the largest places first. I arranged the base ten blocks subtracted into a rectangle."

DIVISION	BASE TEN BLOCK PICTURES	COMPUTATION FORM
		98810

Mathematician:____________________________________

"I used base ten blocks to do these multiplications. I found the partial products one at a time."

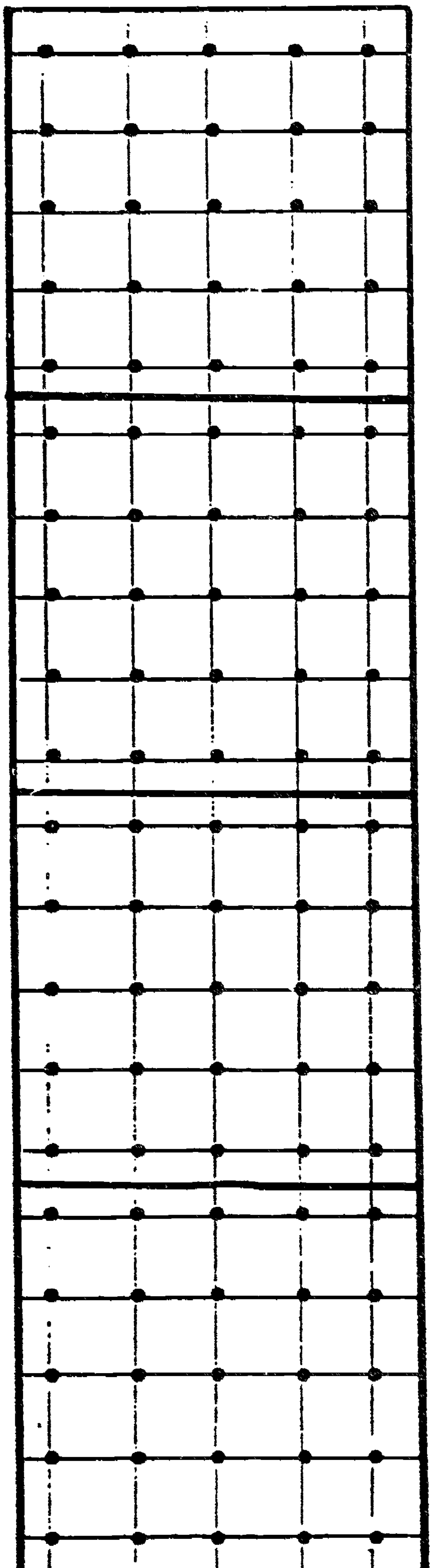
MULTIPLICATION	PARTIAL PRODUCT SUM	PRODUCT

98811

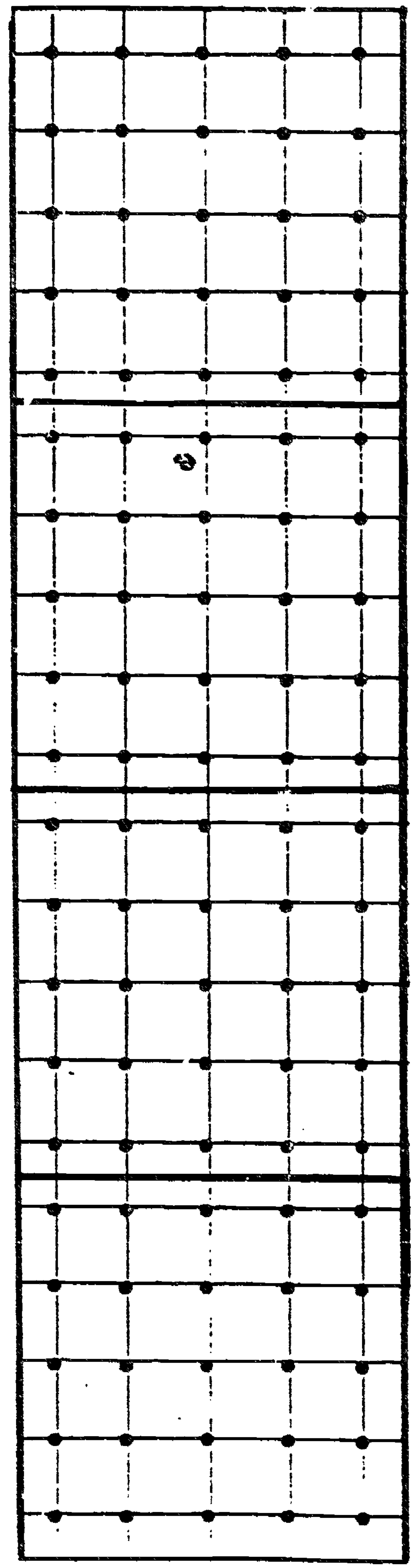
Mathematician:____

"I used a geoboard with ________ to make mirror images of shapes.
We took turns making the first shape. Here are our shapes."

I made the first shape



________ made the first shape



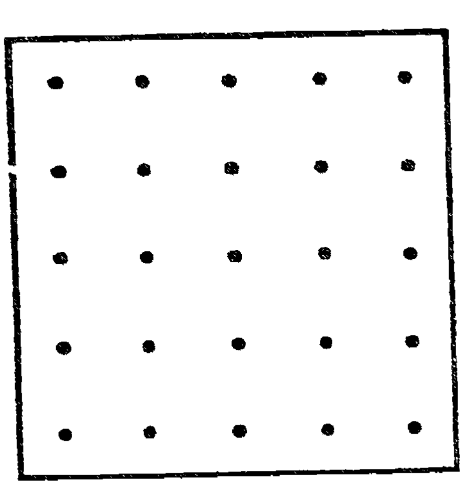
Mathematician:____

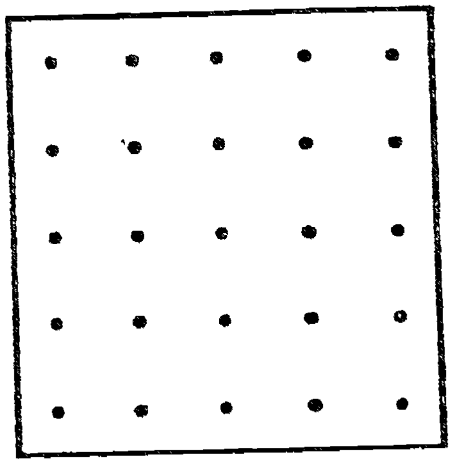
"I made mirror images of the pattern block pattern given."

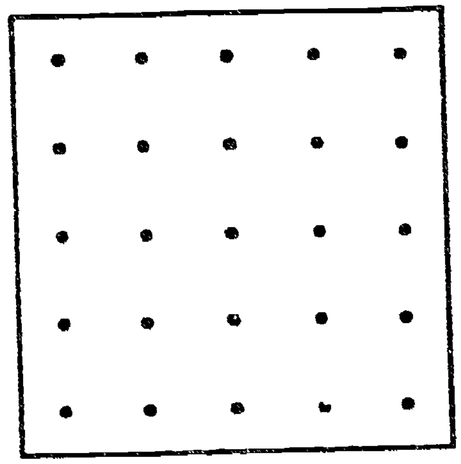
PATTERN	MIRROR IMAGE

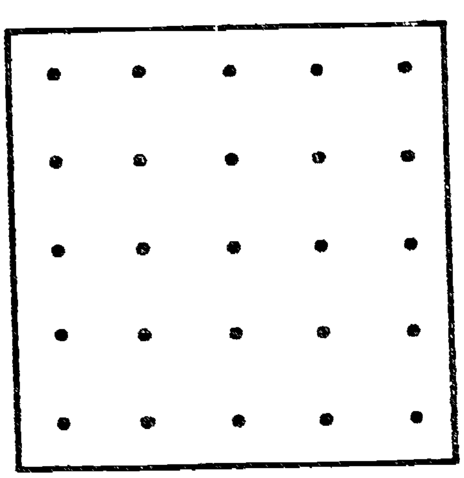
Mathematician:________________________________

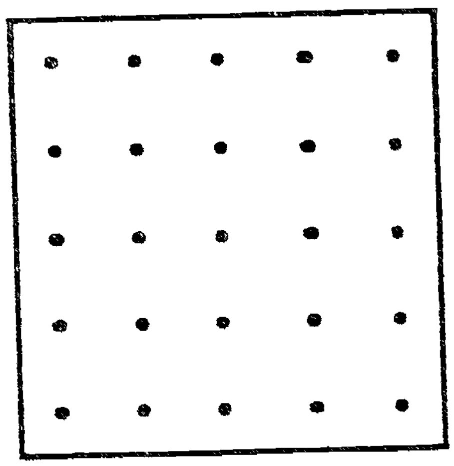
"I found the perimeter of each shape and wrote it below the shape for each geoboard."

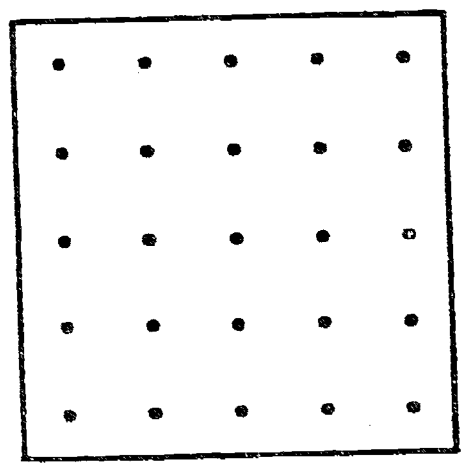


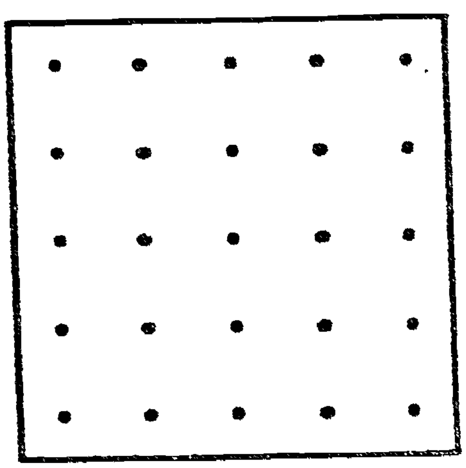


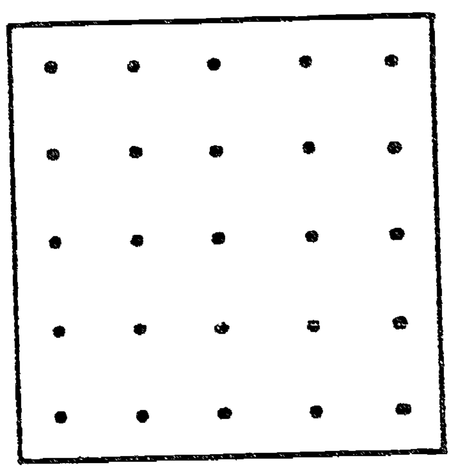


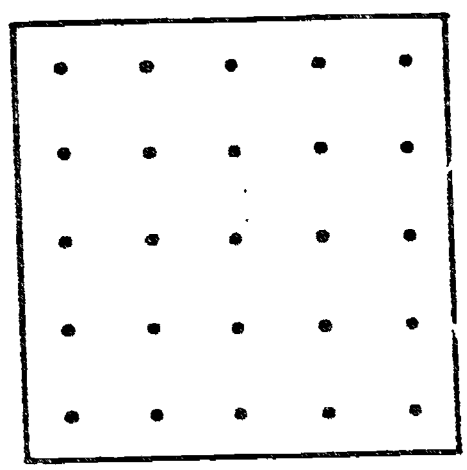






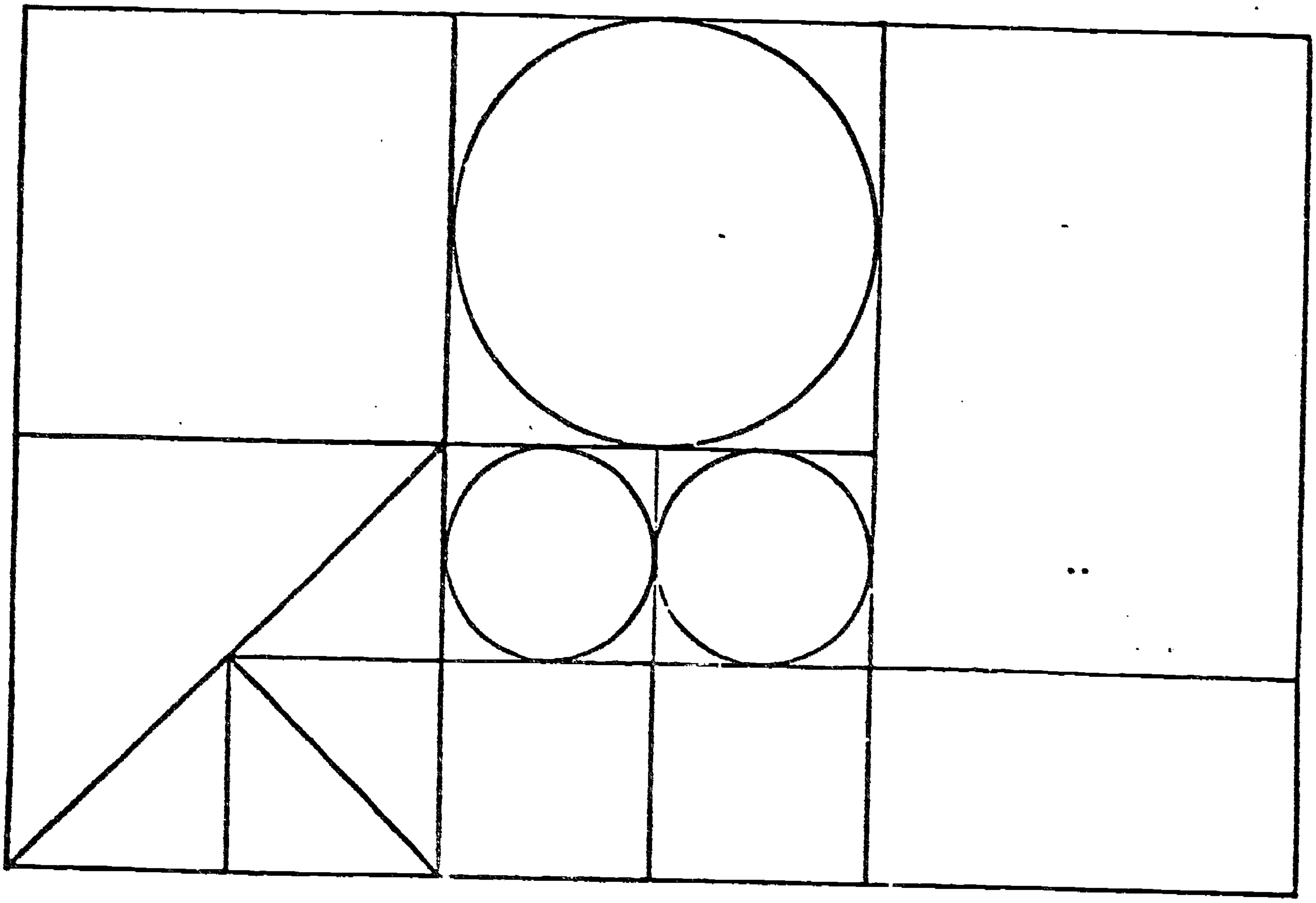


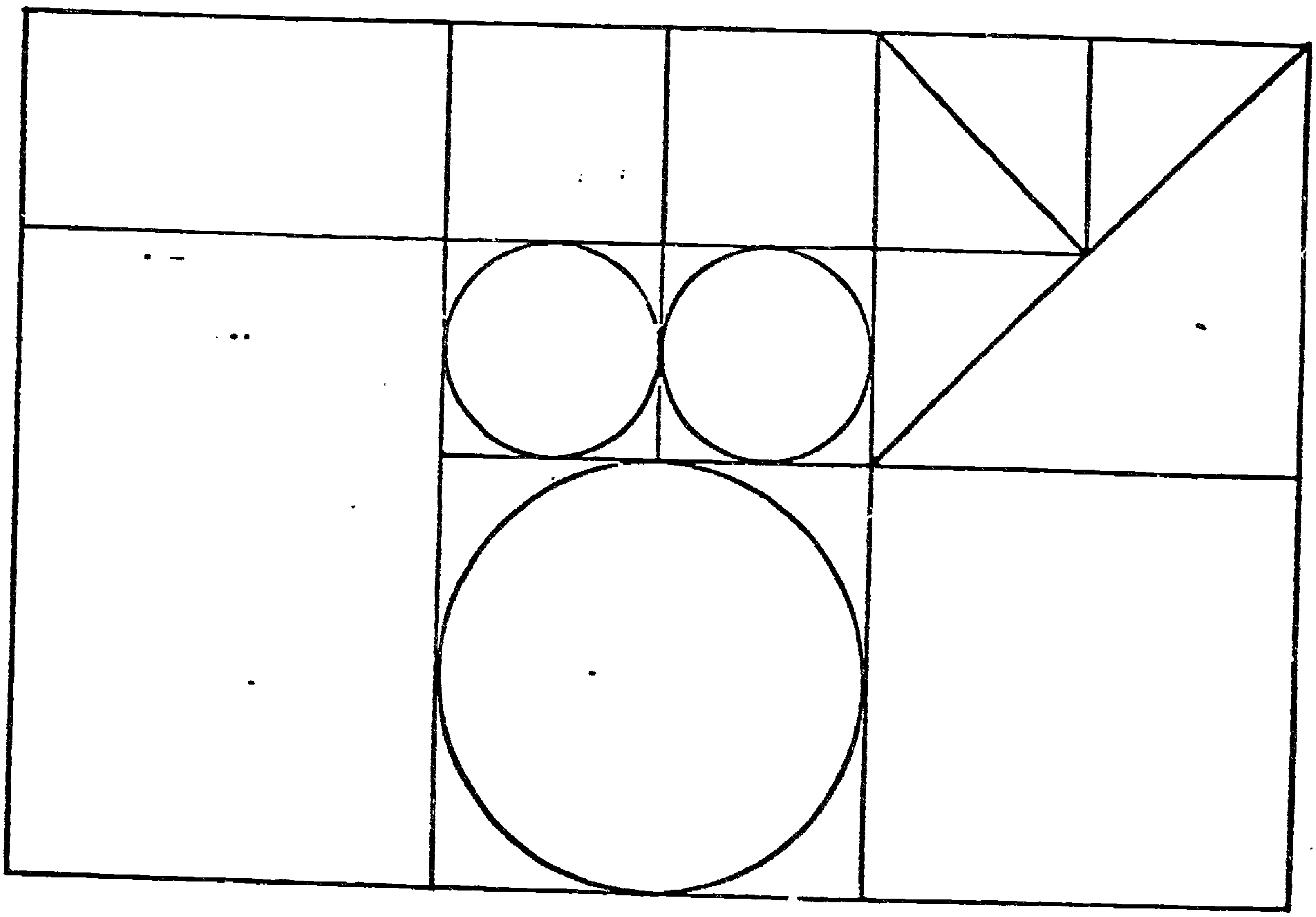




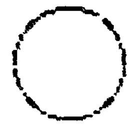
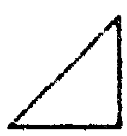
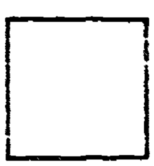
Mathematician:____

"I covered each of the larger shapes with several of the smaller shapes. I wrote on each larger shape the number used to cover it."





LABEL CARDS

RED	CIRCLE 
BLUE	RECTANGLE 
GREEN	LARGE
TRIANGLE 	SMALL
SQUARE 	

"NOT" LABEL CARDS

NOT BLUE	NOT RECTANGLE
NOT GREEN	NOT CIRCLE
NOT RED	NOT LARGE
NOT TRIANGLE	NOT SMALL
NOT SQUARE	

Mathematician:____

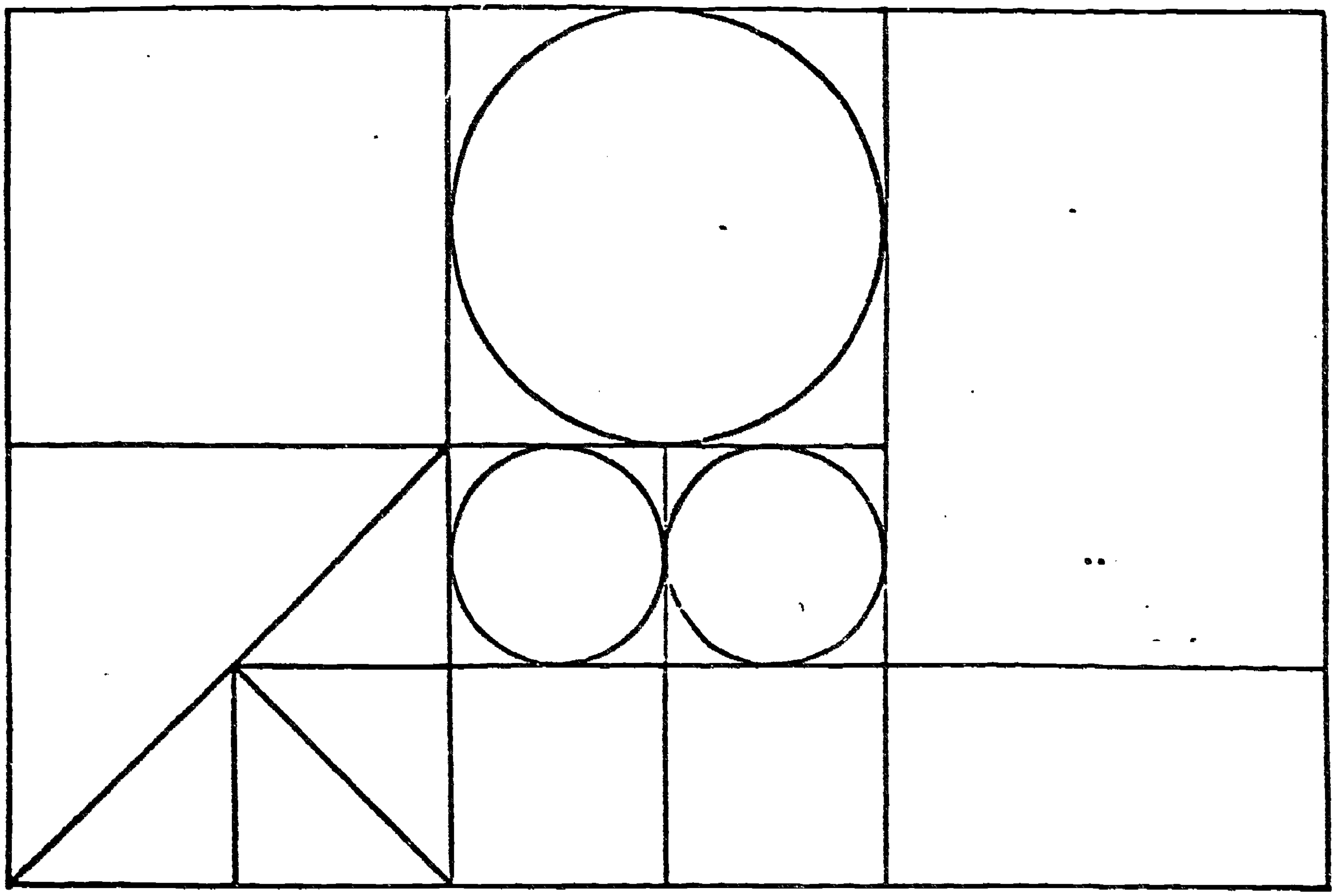
"For the TRUE statement given for the collection, I found a piece that would make the statement false. I drew that piece and colored it."

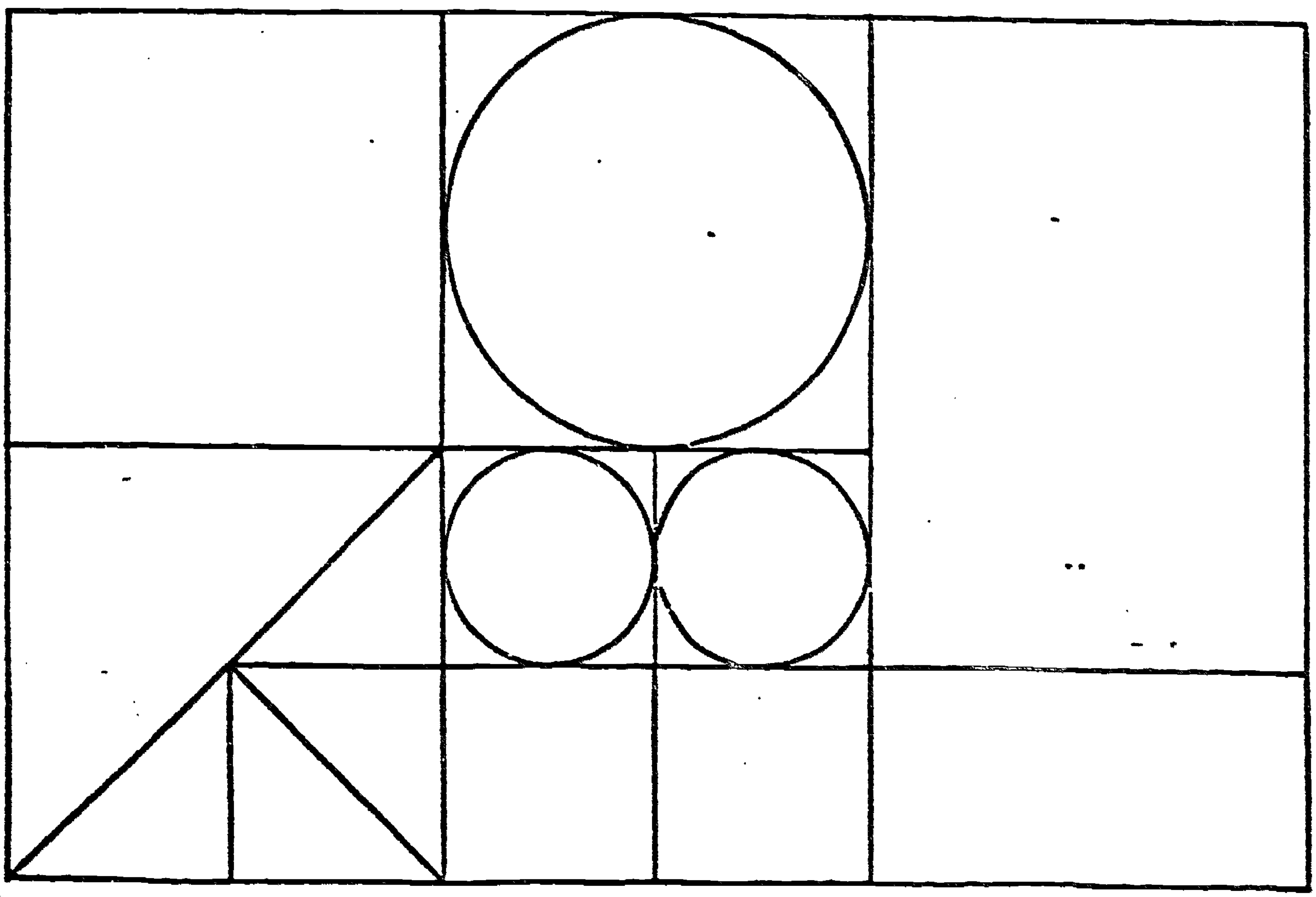
PIECES USED	GIVEN TRUE STATEMENT	PIECE TO MAKE IT FALSE

Mathematician:________________________________

"I found total costs and made change for the following. I wrote $ to show money for all amounts."

DOLLARS	CENTS	DOLLARS	CENTS	DOLLARS	CENTS
$		$		$	





Mathematician:____

"I made links of 2 different colors of unifx cubes in the RATIO given to complete this form."

Number of unifix links	A Color	B Color	A : B ___ : ___
1			
2			
3			
4			
5			
6			
7			
8 9			
10			

Mathematician: ______________________________

"I used cuisenaire rods to complete the following chart."

RODS COMPARED	WHITES COMPARED	FAMILY

Mathematician:____________________________________

"I used cuisenaire rods to find the right number to put in the box in the ratio sentence."

RATIO SENTENCE	RATIO FAMILY

Mathematician:____

"I used cuisenaire rods to find the right number to put in the box in the ratio sentence."

RATIO SENTENCE	RATIO FAMILY

Mathematician: ____________________

"I wrote the number shown by each split board."

NUMBER SHOWN	SYMBOL	NUMBER SHOWN	SYMBOL
Positive / Negative		Positive / Negative	
Positive / Negative		Positive / Negative	
Positive / Negative		Positive / Negative	
Positive / Negative		Positive / Negative	

98806

Mathematician:____

"For each split board, I changed the number given to the new number asked for by adding or subtracting chips. I did this in TWO different ways."

Number Given	Number Added	New number (1)	New Number (2)

98801

Mathematician: ____________________

"For each split board, I changed the number given to the new number asked for by adding or subtracting chips. I did this in TWO different ways."

Number Given	Number Subtracted	New number (1)	New Number (2)

98813

Verbal Addition and Subtraction Problems: Some Difficulties and Some Solutions

By **Charles S. Thompson** *and* **A. Dean Hendrickson**

Many of the difficulties that children have in solving verbal (story) problems involving addition and subtraction arise because of their limited understanding of the arithmetic operations that are involved. They don't know when to use addition or subtraction because they lack specific knowledge regarding the various situations that give rise to these operations. Often, children are taught addition only as "putting together" and subtraction only as "taking away," but many other settings involve addition and subtraction operations. Children need to receive specific instruction in different contexts if they are to become good solvers of verbal addition and subtraction problems. This article describes the contexts and then explains a successful sequence of activities that teach verbal problems.

Categories of Verbal Problems

In elementary school mathematics, three categories of verbal problems suggest addition and subtraction operations. These categories—Change, Combine, and Compare—are described by Nesher (1981). Various types of problem situations exist within each category.

Table 1
Change Problems

Problem title	Sample problem	Characteristics
Change 1	Bill has two pencils. Jean gives him three pencils. How many pencils does Bill have then?	Increase, initial set and change set known, question about final set
Change 2	Bill has five pencils. He gives three to Jean. How many pencils does he have left?	Decrease, initial set and change set known, question about final set
Change 3	Bill has two pencils. Jean gives him some more. Now he has five. How many did Jean give him?	Increase, initial set and final set known, question about change set
Change 4	Bill has five pencils. He gives some to Jean. Now he has two. How many did he give to Jean?	Decrease, initial set and final set known, question about change set
Change 5	Bill has some pencils. Jean gave him two more. Now he has five. How many did he begin with?	Increase, change set and final set known, question about initial set
Change 6	Bill has some pencils. He gave three to Jean. Now he has two. How many did he begin with?	Decrease, change set and final set known, question about initial set

Let's look first at the Change category. Change problems involve increasing or decreasing an initial set to create a final set. One sample Change problem is a familiar "putting together" situation (fig. 1).

> Bert has two books. On his birthday he gets three new books. How many books does Bert have then?

All Change problems have three quantities: an initial set, a change set, and a final set. In the problem given, the *initial set* is two books, the *change set* is three books, and the *final set* is unknown. The unknown quantity in Change problems can be any one of the three sets, yielding three kinds of problems. Furthermore, the change can be either an increase or a decrease, thus yielding two problems for each of the three kinds, for a total of six types of Change problems. These problems are described and characterized in table 1.

The second category of problems is called Combine, or part-part-whole. Combine problems describe an existing, static condition involving a set and its several component subsets. A major difference between Change and Combine problems is that no action is involved in Combine problems. A sample problem is as follows:

> Consuelo has five buttons. Three are round and the rest are square. How many are square?

See figure 2.

A typical Combine problem has three related quantities—one subset, the other subset, and the whole set. These yield only two types of problems. In our example, the whole set and one subset are known. In the

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Fig. 1 Change problems involve increasing or decreasing an initial set to create a final set.

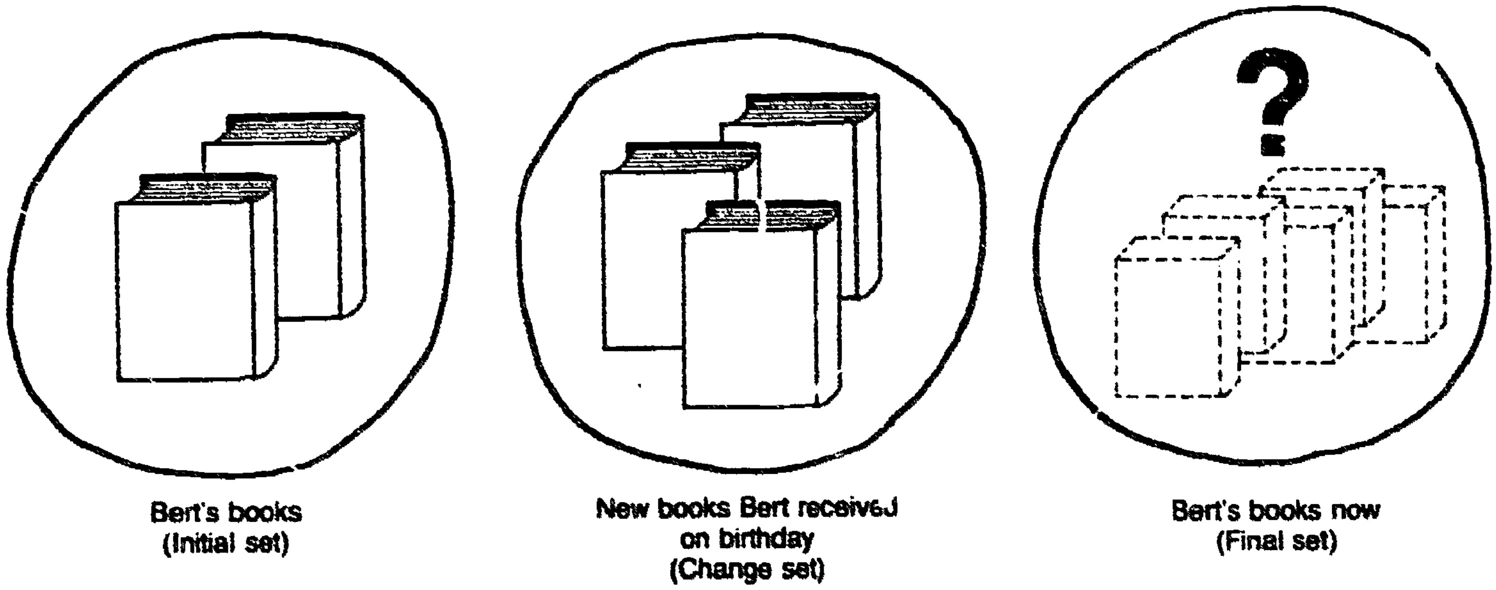


Fig. 2 "Combine" problems describe an existing condition involving a set and its several component subsets.

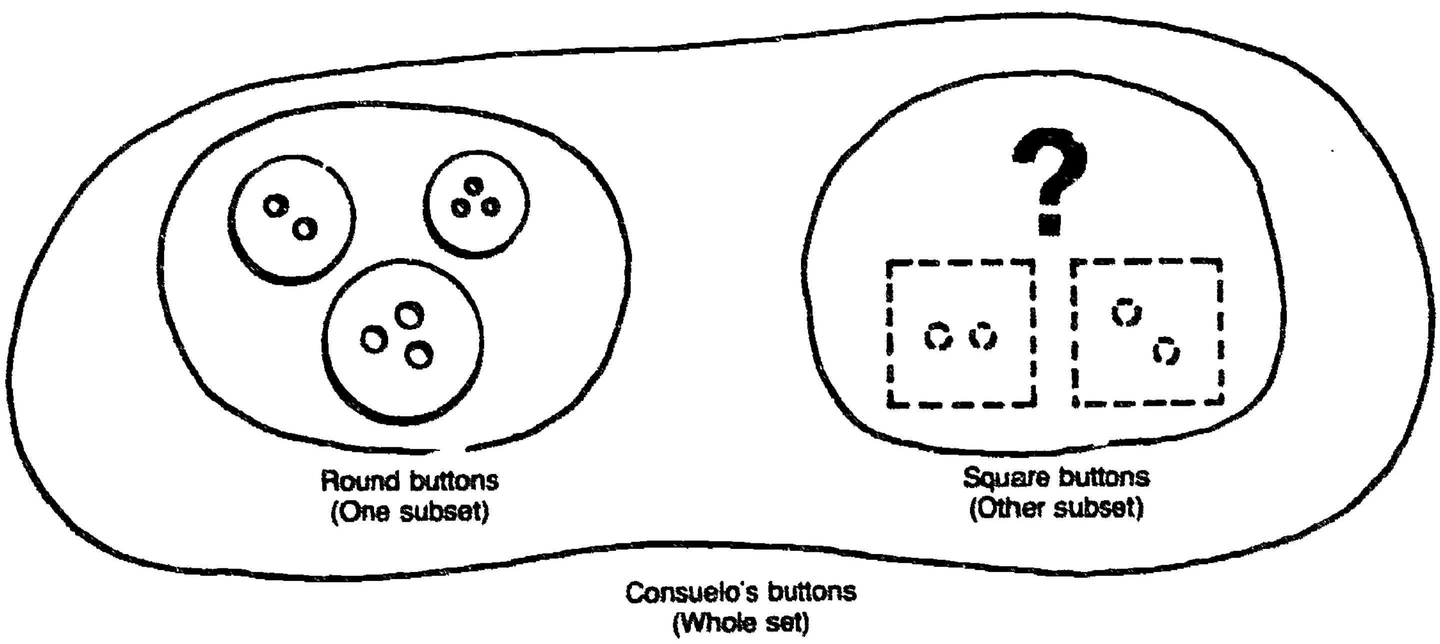
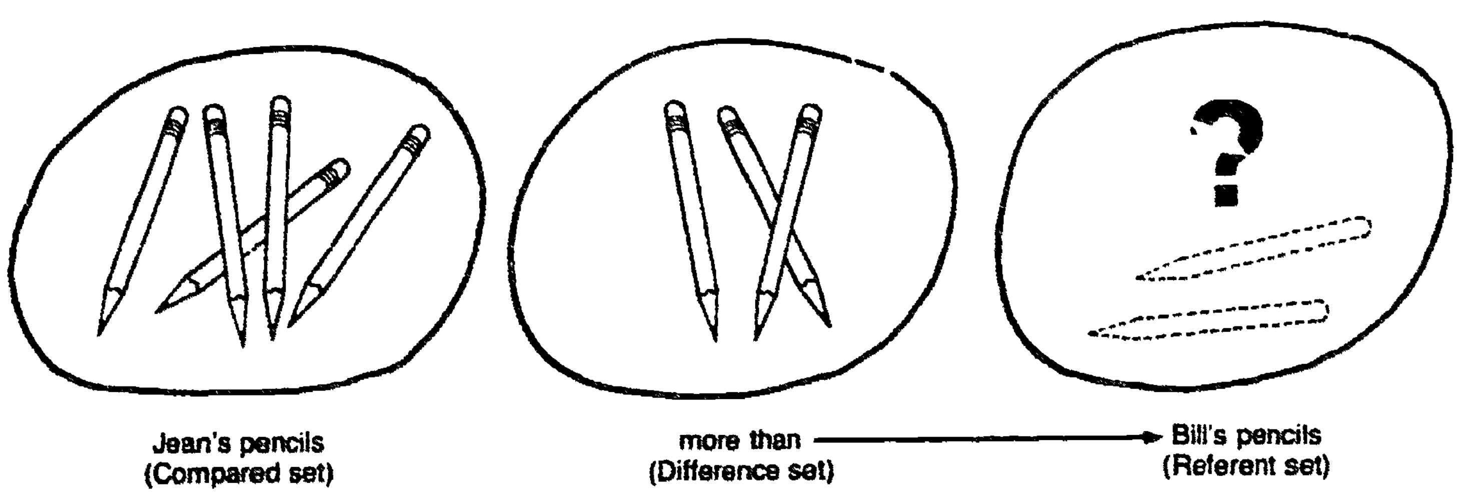


Fig. 3 Compare problems involve a comparison of two existing sets.



other type of problem, both subsets are known and the whole set is unknown. Table 2 summarizes these Combine problems.

Other Combine problems involve more than two subsets and the whole set. These problems typically involve a two-step process and are not discussed here.

The third category of problems is called Compare. Compare problems, which involve a comparison of two existing sets, are probably the most ignored type of problem in school curricula. Yet many children's experiences involve comparisons. Here is a sample problem:

> Jean has five pencils. She has three more pencils than Bill. How many pencils does Bill have?

See figure 3.

Each Compare problem has three expressed quantities—a referent set, a compared set, and a difference set. The referent set is the set to which the comparative description refers. In the sample problem, Bill's pencils compose the referent set, since Jean "has three *more* pencils *than Bill*." The compared set is the set being compared to the referent set. In the sample problem, Jean's set of five pencils (the compared set) is compared to Bill's set (the referent set). The difference set is the difference between the referent set and the compared set.

There are six types of Compare problems. The unknown quantity can be the referent set, the compared set, or the difference set. For each of these three possibilities, the comparison can be stated in two ways: (1) the (larger) compared set is *more than* the (smaller) referent set, or (2) the (smaller) compared set is *less than* or *fewer than* the (larger) referent set. Table 3 summarizes and gives examples of the six types of Compare problems.

Table 2
Combine Problems

Problem title	Sample problem	Characteristics
Combine 1	Bill has three red pencils and two green pencils. How many pencils does Bill have all together?	Two subsets are known, question about whole set
Combine 2	Bill has five pencils. Three are red and the rest are green. How many are green?	Whole set and one subset are known, question about other subset

Table 3
Compare Problems

Problem title	Sample problem	Characteristics
Compare 1	Bill has two pencils. Jean has five. How many more does Jean have than Bill?	Comparison stated in terms of "*more*," referent set and compared set known, question about difference set
Compare 2	Bill has two pencils. Jean has five. How many fewer pencils does Bill have than Jean?	Comparison stated in terms of *less (fewer)*, referent set and compared set known, question about difference set
Compare 3	Bill has two pencils. Jean has three more than Bill. How many pencils does Jean have?	Comparison stated in terms of *more*, referent set and difference set known, question about compared set
Compare 4	Jean has five pencils. Bill has three fewer pencils than Jean. How many pencils does Bill have?	Comparison stated in terms of *less (fewer)*, referent set and difference set known, question about compared set
Compare 5	Jean has five pencils. She has three more pencils than Bill. How many pencils does Bill have?	Comparison stated in terms of *more*, compared set and difference set known, question about referent set
Compare 6	Jean has two pencils. She has three fewer pencils than Bill. How many pencils does Bill have?	Comparison stated in terms of *less (fewer)*, compared set and difference set known, question about referent set

Relative Difficulties of Verbal Problems

Examination of the various types of problems and observations of children solving these problems lead to the conclusion that some types of problems are more difficult to solve than others. In general, it appears that the inherent structure of the problem is the crucial factor in determining its difficulty. For example, Combine-1 problems are structurally straightforward (table 2).

> Combine 1. Bill has three red pencils and two green pencils. How many pencils does Bill have all together?

The two subsets are given. Children can count those subsets separately. Then, they must simply recount the entire collection of objects to determine the solution to the problem. Or, depending on instruction they have received, they might use "all" or "all together" to transform it to a Change problem.

Combine-2 problems, by comparison, are not straightforward. The sets to be considered are not separate from one another.

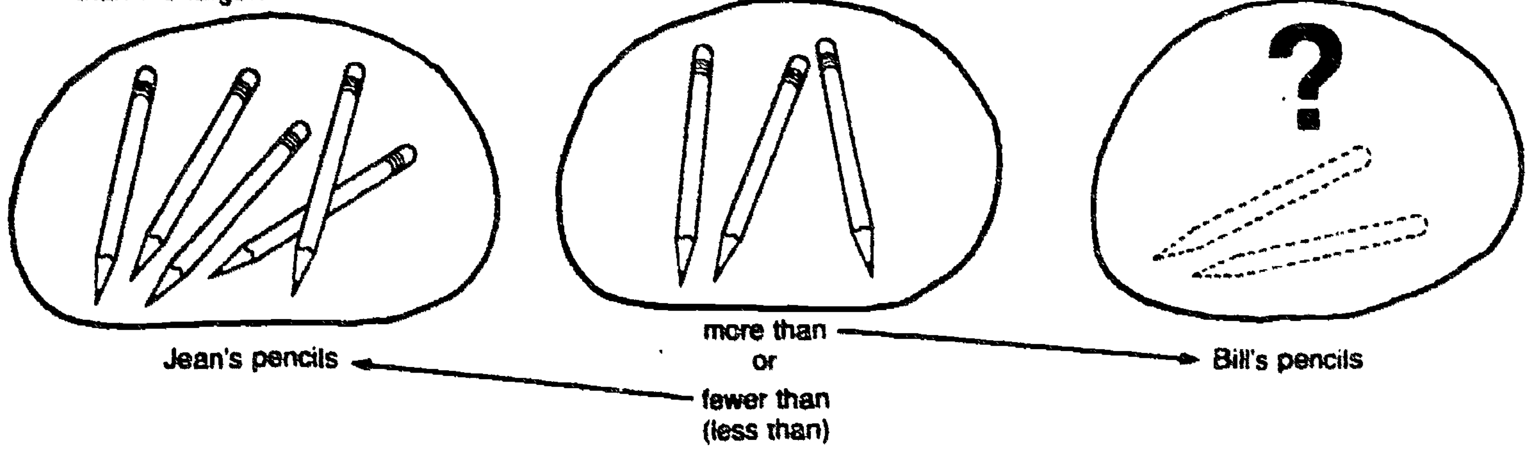
> Combine 2. Bill has five pencils. Three are red and the rest are green. How many are green?

The children must have a well-developed part-whole understanding. The whole set and one subset are given. To solve this kind of problem, children must know that the given subset is contained within the whole set mentally or physically to separate that subset from the whole set and then count the other subset. This problem can be transformed correctly into a Change-2 problem by many children. Other children transform it incorrectly into a comparison of the two subsets.

Another major factor affecting the difficulty of a problem is its semantics. How the relationships between the sets are expressed determines, to some extent, which cognitive structures must be used by the child to solve the problem. For example, study the following Compare-4 and Compare-5 problems:

Compare 4. Jean has five pencils. Bill

Fig. 4 The child must understand that the larger set is three more than the smaller set and the smaller set is three less than the larger.



has three pencils *fewer than* Jean. How many pe^cils does Bill have?

Compare 5. Jean has five pencils. She has three *more* pencils *than* Bill. How many pencils does Bill have?

See figure 4.

In each problem the larger set, of the two being compared, and the difference set are given. The child is to determine the smaller set. In the Compare-4 problem the expression used to relate the larger and smaller sets is "The smaller set is three pencils *fewer than* the larger (known) set." To solve this problem, the child might simply create what is described, by removing three pencils from the larger set to create the smaller set of two objects. This behavior transforms the problem into a Change-2 problem. In the Compare-5 problem, however, the statement used to relate the larger and smaller sets is, in effect, "The larger set is three pencils *more than* the smaller (but unknown) set." In this problem the child must use a different cognitive structure to determine what to do. Three pencils cannot be added to the smaller set, since its quantity is not known. The child must understand that if the larger set is three *more than* the smaller set, then the smaller set is three *fewer than* the larger. The child must have a well-developed cognitive structure called *reversibility*. The child must understand that the statement "*x* is *a* more than *y*" is equivalent to "*y* is *a* less than *x*." Only then will the child know that removing objects from the larger set will create the "more than" relationship expressed in the verbal problem. This same reversibility enables some children to transform Combine-2 problems into Change-2 problems.

Another factor affecting the difficulty of Compare problems is that in Compare-3, 4, 5, and 6 problems, the difference set must be mentally constructed by the child. It is not actually part of the compared set or the referent set. Furthermore, after the difference set is mentally constructed, the child must mentally add it to, or subtract it from, one given set to determine the unknown set.

Another difficulty is the varying use of the expressions *more than, less than,* and *fewer than*. The phrase *fewer than* is common in these fourteen types of problems, since discrete, countable sets are involved. *Fewer than* suggests counting strategies more readily than does *less than*. However, *more than* is used to express relationships between either countable or noncountable quantities. Further, the word *more* is often used in Change problems in another way, as in "John gave Frank four more."

The relative difficulties of all fourteen types of verbal problems have not yet been fully determined. But informal observations of children solving these problems, careful analysis of the problems' structures and semantics (Nesher et al. 1982), and analysis of research results (Carpenter and Moser 1981; Nesher 1981; Riley 1981; Steffe 1971; Tamburino 1981) provide preliminary information about the difficulty of problems. Currently available information indicates four levels of difficulty:

Easiest: 1. Change 1 & 2, Combine 1
2. Change 3 & 4, Compare 1 & 2
3. Combine 2, Change 5 & 6, Compare 3 & 4
Hardest: 4. Compare 5 & 6

Instructional Procedures

We have been working in a conceptually oriented, materials-based elementary mathematics program. The children in first, second, and third grades have received instruction in solving verbal problems of the fourteen types that have been described. The following general instructional sequence has been followed over a period of weeks:

1. Problem situations are presented orally to children. The children use countable materials that can be grouped, linked, and separated to aid them in solving problems. Their answers are expressed *orally*.

2. Children use countable objects to explore combinations of numbers that make larger numbers. For example, they separate five counters into two subsets in different ways and describe the results orally, such as "three and two" or "one and four."

3. Children use prepared numeral cards (0–9), and cards with the "+," "−," "=," and "□," in conjunction with activities similar to those previously described in step 2. They con-

struct number phrases and sentences with the prepared sign cards to represent the objects being used. This task helps them to connect the signs to the concepts involved. For example, if a child uses five counters and covers two of them, then a partner can create the open sentence 3 + □ = 5 then insert a "2 card" to complete the open sentence.

4. The problem situations are presented orally to children as in step 1. They use countable objects to solve the problems and now use the *prepared cards to construct number sentences* to represent the objects used and the conditions of the problem. For example, consider the following problem:

Change 1. Bill has two pencils. Jean gives him three pencils. How many pencils does Bill have now?

To solve this problem, children frequently make separate links of cubes to represent the two sets, join the two links, and arrange cards as shown:

[2] [+] [3] [=] [5]

5. Children use countable materials to solve orally presented problems and then *write number sentences* to indicate how they interpreted the problems. In particular, children circle their answers in the number sentences. In many problem situations several possible number sentences can be written. Consider this problem:

Compare 1. Jean has five pencils. Bill has two pencils. How many more pencils does Jean have than Bill?

Some children will interpret this as an addition problem and write 2 + ③ = 5. Others will interpret it as subtraction and write 5 − 2 = ③. Both interpretations are correct.

6. Open sentences in written form are given to children, who use countable materials to solve them.

7. Materials are not used, and children solve written verbal problems mentally while writing the corresponding number sentences.

8. Children solve open sentences (not directly tied to verbal problems) in written form without the use of countable materials.

From a broad perspective, the sequence has used the following steps: (1) develop concepts using materials, (2) connect signs to the concepts, (3) construct symbolic forms (number sentences) using prepared symbols, (4) write symbolic forms, and (5) interpret prepared symbolic forms. This sequence has resulted in students being able to interpret these problems and translate them into number-sentence models.

In conjunction with these activities, children participate in numerous counting exercises. They learn to count on from any given number and to count back from any given number. Counting on is useful in many problems, particularly in part-whole situations, in which one subset and the whole set are known, and in compare situations, where equalizing of the two sets is the strategy to be used. Counting back is also used frequently, especially in Change problems. For example, in Change-2 problems the children often count back from the larger (initial) set to create the smaller (final) set.

Instructional Results So Far

The instructional sequence described seems to be effective in enabling children in the primary grades to solve verbal problems. Of crucial importance seem to be the use of countable materials, the use of the prepared numeral and sign cards, and the practice of circling answers when writing number sentences.

Using the countable materials enables the children to create or model the conditions presented in the problems. The children can then determine which sets to count, compare, separate, or join to solve the problems. The use of the prepared cards allows the children quickly to attach numerals to the quantities represented and to construct the corresponding number sentences. We have found that children who have not used numeral cards experience greater difficulty in writing number sentences corresponding to a verbal problem. The practice of having children circle answers when writing number sentences helps teachers understand how the children are thinking about the verbal problems. Indeed, for many of the types of problems, either an addition or a subtraction number sentence is appropriate. These practices also help teachers to recognize when children are successfully using the class-inclusion relation, reversibility of both actions and relations, and equalization of two sets.

In summary, we have learned that children can become good solvers of verbal problems. What they need is an instructional program that proceeds from the concrete to the symbolic *and* the opportunity to encounter the various problem situations that occur in real life.

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