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ABSTRACT

Analogical reasoning is frequently used in acquisition of mathematical concepts. Concrete representations used to teach mathematics are essentially analogs of mathematical concepts, and it is argued that analogies enter into mathematical concept acquisition in numerous other ways as well. According to Gentner's theory, analogies entail a structure-preserving mapping from a base or source to the target. Although concrete aids can provide valuable assistance to concept acquisition, their oft-noted failure to provide the anticipated benefits has been a source of some puzzlement. It is suggested that the reason for the failure may be the processing loads imposed by structure mapping. Some representative mathematical concepts are examined, together with typical concrete representations, and the nature of the processing loads analyzed. These loads can be reduced by recoding concepts into more abstract form, but it is argued that structure mapping also plays a role in abstraction. Analysis of this process provides insight into sources of difficulty, and recommendations are made for improving the efficiency of instruction. Thirty-nine references are listed. (Author/YP)

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Value and Limitations of Analogs in Teaching Mathematics

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Abstract

Analogical reasoning is frequently used in acquisition of mathematical concepts. Concrete representations used to teach mathematics are essentially analogs of mathematical concepts, and it is argued that analogies enter into mathematical concept acquisition in numerous other ways as well. According to Gentner's theory analogies entail a structure-preserving mapping from a base or source to the target. For example, in teaching mathematics using concrete aids, the concrete aid would be the source, and the concept to be taught the target. Although concrete aids can provide valuable assistance to concept acquisition, their oft-noted failure to provide the anticipated benefits has been a source of some puzzlement. It is suggested that the reason may be the processing loads imposed by structure mapping. Some representative mathematical concepts will be examined, together with typical concrete representations, and the nature of the processing loads analysed. These loads can be reduced by recoding concepts into more abstract form, but it is argued that structure mapping plays a role in abstraction also. Analysis of this process provides insights into sources of difficulty, and recommendations are made for improving the efficiency of instruction.

Value and Limitations of Analogs in Teaching Mathematics

Young children are confronted regularly by mathematical representations. Some of these are concrete analogs specifically designed for pedagogical purposes, such as the Cuisenaire rods or the multibase arithmetic blocks, while others are representations inherent in the discipline of mathematics, such as number lines and symbols. The purpose of this paper is to consider some of the psychological processes entailed in using mathematical representations, in order to explore their role in the development of new concepts.

The theory we will use for this purpose is part of a general account of cognitive development that is outlined more fully elsewhere (Halford, in press). It argues that there are two basic types of mechanisms in cognitive development. The first are essentially learning mechanisms, which lead to the gradual adjustment of mental models of the world through experience. These mechanisms are not fundamentally different from those that operate in other species. These mechanisms entail strengthening through experience, which applies to both declarative knowledge (mental models), and to procedural knowledge. When applied to declarative knowledge it means that when a mental model correctly predicts the environment it is strengthened, and when it does not it is weakened. A similar principle applies to acquisition of procedural knowledge; it is strengthened when successful, and weakened when unsuccessful. Learning mechanisms of this kind can account for a wide range of acquisitions (Holyoak, Koh & Nisbett, in press; Holland, Holyoak, Nisbett & Thagard, 1986).

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The second type of mechanism is concerned with recognition of correspondence between structures. It is involved in such processes as recognition of analogies, and with the selection and use of representations. It is argued that human beings have limited capacity to see correspondence between structures, but have much greater capacity to learn. It is suggested that this explains many cognitive developmental anomalies, such as ability to perform a task in one context but not in another. In this paper we are primarily concerned with ability to see correspondence between structures, and the effect that our limited ability to do this has on acquisition of mathematics. The argument is presented by reference to a few illustrative examples, but it can be applied to wide range of subject matter (Halford, in press).

Analogical reasoning, which is increasingly being seen as fundamentally important to human cognition, entails recognition of correspondence between one structure and another. Therefore we will begin by considering the theory of analogies, then we will broaden the issue to consider representations in general. According to Gentner (1983) an analogy consists of a mapping from one structure, called the source or base, to another structure, called the target. In the simple analogy "Man is to house as dog is to kennel" (see Figure 1), "Man is to house" is the source, and "dog is to kennel" is the target. Man is mapped into dog and house into kennel, and the relation "lives in" between man and house corresponds to the relation "lives in" between dog and kennel.

An important property of the structure mapping process in analogies is that it is selective. Attributes are not normally mapped at all, so that for example the attribute "wears clothes" associated with man is not mapped into, or attributed to, dog. Relations are also mapped selectively. In the present example, only one relation "lives in" is mapped, and other relations in the source such as "has mortgage on", or "repairs at weekends" are not mapped into the target. This means the mapping process selects those features of each structure that it shares with the other structure. As we will see, this has important implications for the formation of abstractions, because it means that structure mapping selects the features that are general to a particular class of structures, and eliminates the features that are specific to individual structures.

The theory of analogies is very close to the theory of representations. A cognitive representation consists of a mental model that is in correspondence to the segment of the environment that is represented (Halford & Wilson, 1980; Palmer, 1978). A cognitive representation is a mapping from a cognitive structure to an environmental structure. An analogy is a mapping from one mental structure to another (Holland, Holyoak, Nisbett & Thagard, 1986). Thus structure mapping theory can handle both analogies and representations.

Applying Gentner's structure mapping theory to mathematics, the concrete representation is the source and the concept to be taught is the target. The value of the concrete representation is that it mirrors the structure of the concept and the child can use the structure of the representation to construct a mental

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model of the concept.

It has been noted increasingly in recent literature in mathematics education that concrete representations often fail to produce the expected positive outcomes. Lesh, Behr and Post (1987) note that "concrete problems often produce lower success rates than comparable word problems. . " (p. 56). Dufour-Janvier, Bednarz, and Belanger (1987) also note the "... negative consequences that can be caused by the use of representations prematurely or in an inappropriate context. In fact this leads the child to develop erroneous conceptions that will subsequently become obstacles to learning." (p. 118).

There seems to be some mystification as to why concrete analogs sometimes aid and sometimes hinder acquisition of mathematics. We wish to propose that one reason why concrete analogs sometimes fail to live up to expectations is because of the processing load entailed in mapping a concept into an analog. Previous research by Halford and his collaborators (Halford, Bain & Maybery, 1986; Maybery, Bain & Halford, 1986) has shown that structure mapping imposes a processing load, the size of which depends on the structural complexity of the concepts.

Halford (1987, on contract) has defined four structure mapping levels as illustrated in Figure 2, the processing demands of which are known:

Element mappings. An element in one structure is mapped into an element in the other, on the basis of similarity or convention; e.g. an image or word representing an object or event.

Relational mappings entail mapping 2 elements with a relation between them; e.g. 2 sticks of different lengths to represent the fact that a man is larger than a boy. The mapping is validated by the similarity of the relation between the sticks to the relation between man and boy, and is independent of element similarity or convention. The man-house/dog-kennel analogy is a relational mapping, because it is validated by a similar relation in source and target.

System mappings are validated by structural correspondence, independent of similarity or convention. An example would be the representation of Tom > Dick > Harry by ordering the elements from left to right. Tom, Dick, Harry are mapped to left, middle, right respectively, and the relation ">" is mapped to "left-of". Mappings must be unique, and if a relation R in structure 1 is mapped into a relation R' in structure 2, the arguments of R must be mapped into the arguments of R'.

Multiple system-mappings are similar to system mappings except that they depend on a composition of structures that have three elements as arguments.

Processing loads.

The load imposed by structure mapping depends on the level of structure being mapped, and can be quantified by the information required to define that structure. Elements can be defined by one item of information (e.g. label), binary relations by two items, systems of binary relations by three items, and multiple systems by four items (Halford, 1987; Halford & Wilson, 1980) The mental process which checks validity of mappings must transmit sets of items no smaller than these values from the representation of one structure to the representation of another. The metric is similar to that used by Leeuwenberg (1969), and Simon (1972); the complexity of a pattern or structure is equivalent to its dimensionality, i.e. the number of independent signals that define it. This means that the level of a mapping depends on the amount of information that must be processed in parallel to validate a mapping, not on the total amount of information in a structure.

The 4 mapping rules increase in abstractness, but at the cost of higher processing loads. This effect has been empirically confirmed using dual-task load indicators (Halford, Maybery & Bain, 1986; Maybery, Bain & Halford, 1986). Also, our research has shown that children become capable of using progressively higher rules with age (Halford 1982, 1987, on contract; Halford & Wilson, 1980).

This implies that the level of structure mapping that children can use will be a function of processing capacity. The view that there is a maturationally-determined upper limit to cognitive processes has been a very unpopular one, at least partly because it is seen as having gloomy consequences for education (Carey, 1985). This is not a valid reason for rejecting the hypothesis however, for two reasons. First, our desire to accelerate cognitive development should not bias our acceptance of the scientific evidence. One consequence of such a bias would be that studies indicating children's inability to perform a given task would be subjected to much more rigorous scrutiny than studies indicating successful performance. There are in fact some well-known studies in the literature where authors have been permitted to report chance-level results as positive results (McGarrigle, Grieve & Hughes, 1978; Siegel, McCabe, Brand & Matthews, 1978), apparently in pursuit of the aim of showing that children can succeed on certain tasks. This question is discussed in more detail elsewhere (Halford, 1989). Second, the maturation hypothesis does not have uniformly gloomy implications, because most children are performing below their theoretical limit on some tasks, and more refined task analysis can result in very substantial improvements. Thus the maturation hypothesis is in no way incompatible with the goal of accelerating cognitive development. It simply implies that performance will be a function of processing capacity as well as experience. Therefore we will examine the capacity question next.

Capacity

The information required to validate structure mapping rules raises the question of the amount of information that can be processed in parallel. Our theory links work on chunking originating with Miller (1956) to current parallel distributed processing models (Rumelhart & McClelland, 1986). In the latter information is represented as a set of activation values over a large set of units. Each pattern of activations (module) can represent a large amount of information, but its output is restricted to one concept at a time. When this limitation is combined with a restriction on the number of pattern of activations that can be transmitted from one set of modules to another (Schneider & Detweiler, 1987) it

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provides an interesting theoretical basis for the observation that chunks can be of any size, yet only about four can be active simultaneously (Broadbent, 1975; Fisher, 1984; Halford, Maybery & Bain, 1988). A pattern of activations in one module can represent a chunk (information unit of arbitrary size) and since each pattern of activations can assume a range of values independent of other patterns, each pattern of activations represents a different dimension.

Schneider & Detweiler (1987) propose a multi-capacity module in which there are a number of regions, representing separate functions such as speech, vision, motor processing etc. Each region contains up to four modules. They propose that working memory capacity can be increased by utilizing more than one region for difficult tasks. However, as mentioned above, only 4 patterns of activation can be transmitted from one region to another. This implies that only 4 patterns of activation can be processed in parallel. This in turn means that four-dimensional structures are the most complex that can be processed in parallel. This theory has been discussed in more detail by Halford (in press).

There is a link between the amount of information that can be processed in parallel and the level of structure mapping that can be achieved. Research indicating that adults process four chunks or dimensions in parallel implies that structures equivalent to multiple system mappings would be the most complex that can be processed in parallel. If children can process less dimensions in parallel, they would be restricted to lower level mappings, which would explain the difficulty they have with certain concepts (Halford, 1987).

Previous research (Halford, 1987) has shown that children can master element mappings at one year, relational mappings at two years, system mappings at five years, and multiple system mappings at 11 years (median ages). This has been used to explain the typical age of attainment of a variety of concepts. Table 1 shows representative concepts belonging to each level.

Segmentation and conceptual chunking

There are of course many concepts that contain more than four dimensions, but the empirical work discussed above suggests that only four dimensions are processed in parallel, even by adults. How then are more complex problems processed? The model proposes that problems too complex to be processed in parallel are handled either by segmentation or chunking. Segmentation entails decomposing the problem into components or segments, and processing these serially. Thus there is parallel processing within segments, but serial processing between segments. There is a limit on segmentation because some problems cannot be decomposed. For example, the minimum information in an addition problem is two addends. The answer to the sum "add 3" cannot be defined; we must know to what number 3 is added. The minimum information required to define arithmetic addition is a structure of the form "a,b --> c" (e.g. 2,3 --> 5). Binary operations cannot be defined on sets of less than three elements, and therefore are irreducibly three-dimensional concepts.

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Conceptual chunking reduces processing loads by recoding multiple dimensions into a single dimension, or at least into in less dimensions than the original. Conceptual chunks are similar to mnemonic chunks in that a number of formerly separate items of information are recoded as a single item, but there is more emphasis on structure. A good example of a conceptual chunk would be speed, defined by the dimensions distance and time, but it can be recoded as a single dimension, e.g. position of a pointer on a dial.

Once multiple dimensions are recoded as a single dimension, that dimension occupies only one chunk or module, and it can then be combined with up to three other chunks. This does not mean that processing limitations can be eliminated by recoding all concepts as a single chunk because a single dimensional representation only includes one combination. Alternate combinations become inaccessible, unless a return is made to the original dimensions, which entails the original processing load. Thus there tends to be a tradeoff between efficiency and flexibility. The other limitation is that conceptual chunking, like mnemonic chunking, is only possible with constant mappings of components into chunks. Nevertheless conceptual chunking is very useful in reducing processing loads, and permits us to master progressively more complex concepts. In a later section we will develop this argument further with reference to coding numbers in different bases.

The fact that processing loads can be reduced by segmentation and conceptual chunking does not make predictions about processing loads at each level of structural complexity untestable. It does mean however that hypotheses about the amount of information that can be processed in parallel must be tested using tasks that preclude segmentation or conceptual chunking. Segmentation can be precluded by devising tasks in which the dimensions that define the structure interact, so they cannot be processed serially. Conceptual chunking can be precluded by using tasks that require new structures to be generated, because conceptual structures can only exist if there has been previous experience with that structure. These methods have been used in our previous research on this topic (Halford & Wilson, 1980; Halford, Maybery and Bain, 1986; Maybery, Bain & Halford, 1986).

Analogs in mathematics

Concrete analogs have been especially popular in teaching mathematics, as the multitude of commercially available mathematical games attests. In fact the construction of concrete analogs for mathematical concepts has reached great heights of ingenuity, as is evidenced in the work of Dienes (). Some of the reasons why analogs are useful in learning are:

1. They reduce the amount of learning effort, and serve as memory aids.

3. They can provide a means of verifying the truth of what is learned.

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3. They can increase flexibility of thinking.

4. They can facilitate retrieval of information from memory.

5. They can mediate transfer between tasks and situations.

6. They can indirectly (and, perhaps, paradoxically) facilitate transition to higher levels of abstraction.

7. They can be used generatively to predict unknown facts.

On the other hand there are some potential disadvantages, including:

1. Structure-mapping imposes a processing load (discussed above), and this load can actually make it more difficult to understand a concept.

2. A poor analog can generate incorrect information.

3. If an analog is not fully integrated, and is not well mapped into the material to be learned or remembered, it can actually increase the learning or memory load.

We will explain these points by first using as an example the simple mathematical analogs in Figure 3. A popular way to teach simple addition facts is by using small sets of objects, as in Dienes multibase arithmetic blocks, or simply crosses on paper, to represent small numbers. Figure 3A shows such an analog in structure-mapping format, with a set of one object mapped into the numeral 1, a set of two objects mapped into the numeral 2, and so on. The use of the same analog to represent a simple arithmetic relationship, 2 + 3 = 5, is shown in Figure

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3B. Collis (1978) has shown how this analog can be used to represent some quite sophisticated mathematical notions, including addition and multiplication, commutativity, operations on ratios, and proportion. We will apply structure mapping theory to assessing this analog.

First, notice that in Figure 3A the mapping from sets to numerals is clear and easily verified. It is easy to recognize, by subitizing or counting, how many elements each set contains. In Figure 3B, the mapping of the numerals 2,3,5 into their respective sets is also clear and easily verified. The relation between the two addend sets and the sum set is also clear - the sum set includes all elements of the two addend sets, which have no common elements (are disjoint). This means that the structure of the base is clear and readily accessible (high base specificity in Gentner's terms). If we arrange sets in order of increasing magnitude as shown in Figure 3A, it is easy to see that each set contains one more element than its predecessor. This is one of many useful relationships that are contained in the analog, and which are readily available for mapping into the target.

Contrast this with another analog of elementary number facts, the Cuisenaire rods. In this case it is not so clear which rod should be mapped into each numeral. The longer rods are mapped into the larger numerals, but it is difficult to be sure precisely which numeral is represented by a rod of a given length. The rods are distinctive colors, to facilitate this differentiation, but as Figure 3C shows, the colors complicate the mapping process. There is a two-stage map from rod to color to numeral. The colors are arbitrary to some extent, so the mapping from rod to color, and the mapping from color to numeral, must be rote learned. Learning this arbitrary double-mapping greatly increases the load on the children. The relationships in the base are not as clear as in the sets analog. For example, it is not as clear that each rod represents one more unit than its predecessor.

The use of the Cuisenaire analog to represent 2 + 3 = 5 is shown in Figure 3D. Because of the two-stage mapping rod-color-numeral, which parallels the rod-numeral mapping, we can see that the structure-mapping is much more complex than the corresponding mapping in Figure 3B, based on sets. A structure-mapping analysis therefore predicts that the set analog would be more efficient than the colored rods analog. This analysis is intended to illustrate the application of structure-mapping theory to mental models of mathematics.

The sets analog also exemplifies the second advantage listed above, because it permits verification of the truth of what is learned. As Figure 3B shows, the sets analog provides a concrete model verifying that 2 + 3 = 5. Furthermore, it is a model which a child can learn to construct at any time so as to verify this relationship. The third advantage, facilitation of memory retrieval occurs because an analog can provide an additional retrieval cue. Siegler and Shrager (1984) have shown how this can occur with another popular small number analog, use of fingers. Even when children are able to retrieve number facts from memory, they might use fingers as an "elaborated representation", not to determine the answer by counting fingers, but as an additional retrieval cue.

The sets analog illustrates how flexibility of thinking can be increased. The analog in Figure 3B was constructed to show that 2 + 3 = 5, but it can be used equally well to verify that 3 + 2 = 5 (the commutativity property), and even that 5 - 3 = 2, and 5 - 2 = 3. Many good analogs can be accessed in several different ways, which makes it easy to examine a concept from a number of angles.

One reason why analogs facilitate transition to higher levels of abstraction is that they promote learning of integrated structures. For example, the analog in Figure 3A would facilitate the learning of numbers as an ordered set, whereas analogs such as that in Figure 3B would facilitate the learning of integrated sets of relationships such as 1 + 1 = 2, 1 + 2 = 3, 1 + 3 = 4,. . . 3 + 4 = 7, 3 + 5 = 8, etc. Structure-mappings can be made best when the base structure is well-learned (the property which Gentner, 1982 calls "base specificity"). When learning arithmetic using a concrete analog, the concrete material is the source or base and the arithmetic facts are the target. When the transition is made to a higher level of abstraction, the arithmetic becomes the base, and the algebraic relationships which mirror the arithmetic is the target. This is discussed by Halford (on contract, Chapter 8), and is also developed later in this paper. The better the arithmetic facts are learned, the better will be the base, and the better will be the structure-mapping used to learn algebra.

Complexity of concrete analogs

In this section we will analyze the complexity of some concrete analogs in terms of levels of structure mapping outlined earlier. Recognition of relations between numbers (or sets) would be a relational mapping, and recognition of binary operations (addition, multiplication, and their inverses, subtraction and division) would be system mappings. Laws relating to single operations, such as that of commutativity, also entail system mappings. Concepts based on compositions of binary operations such as the distributive property $a(b+c) = ab + ac$ entail multiple system mappings.

Simple analogs for the addition operation, $2 + 3 = 5$ and for the relation $7 < 8$ are shown as structure mappings in Figures 3B and 3C respectively. According to Halford's theory of levels, the mapping in Figure 3B is a system mapping, and that in Figure 3C is a relational mapping. The addition operation should impose a higher processing load than recognition of relations.

For young children the verification of arithmetic facts probably depends on reference to a concrete example, based on small sets as in Figure 3 (fingers make admirable sets up to 10), or on a number line. If the structure mapping is too difficult they will be unable to make this verification, and to that extent their understanding will be impaired. Because arithmetic operations entail system mappings whereas understanding relations between integers or sets entails a relational mapping, it follows that, other things being equal, the former should be more difficult and should be understood later.

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We will now apply structure mapping t..eory to some more complex arithmetic concepts taught in schools. The basic idea of base-10 Multi-base Arithmetic Blocks (Dienes,) is shown in Figure 4. With base-10 blocks, units are represented by small square blocks, tens are represented by blocks that are as long as 10 unit blocks (longs), and hundreds are represented by square blocks equal in area to 10 tens blocks (flats). The area relations between the blocks reflect the magnitude relations between quantities represented.

Resnick and Omanson (1984) found that the children could write numerals to represent numbers, correctly using the place-value notation, and could construct valid representations using the concrete analogs, Dienes blocks or coins. They could also validly represent recompositions, such as changing 34 from 3 tens and 4 units to 2 tens and 14 units. However they were not able relate this understanding to the decomposition procedures in add'tion and subtraction. Furthermore an attempt to train the children to map their concrete representations into the arithmetic procedures was not particularly successful. We can begin to understand why children would have difficulty mapping these concrete representtions into decomposition procedures, and why relatively brief mapping training might not remedy the problem, if we define the mappings involved more completely.

Figure 5 shows the structure mapping for a simple trade operation, where 324 is changed to 200 plus 110 plus 14. In the concrete representation, 324 is represented as 300 hundred blocks, two ten blocks and four unit blocks. The

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first point to notice about this mental model is that it really entails a two-stage vertical mapping. The three hundreds blocks are first mapped into the quantity 300, but this in turn has to be mapped into the 3 digit in the hundreds column in accordance with the place-value notation. That is we have mappings from concrete analog to quantity to notation.

Moving horizontally we have a quantity conserving change in which the original representation is replaced by two hundreds blocks, 11 tens blocks, and 14 units blocks. To appreciate the value of the concrete representation, the child must recognize that this is a quantity conserving change. This is not easy to see because we have to sum 300 + 20 + 4 and recognize that it is equal to 200 + 110 + 14.

On the right hand side we again have a two-stage mapping from concrete analog to quantity to place-value notation. The value of the concrete analog is lost unless it is realized that there is a quantity conserving change at all three levels. All in all, this is a very complex structure mapping, but it is only part of the mapping that is required to understand the decomposition procedure in subtraction, as we will soon see.

The structure mapping required to show how 324 minus 179 can be understood in terms of a concrete analog is shown in Figure 6. The decomposition procedure is illustrated in the left side of the figure, as in Figure 8.10. The subtrahend, 179, is shown as concrete analog, as quantities 100 plus 70 plus 9, and in place value notation, 179. The resulting quantity, 145 is shown in the same way.

Note that the structure mapping diagram is designed to show relations between elements of the representation, corresponding relations between the things represented, and the mapping from one to the other. It is not designed to show the sequence of steps in the subtraction procedure. Consequently, the decomposition procedure is shown to the left of subtraction, but this is not intended to convey that one occurs before the other. Structure mapping is a way of analyzing the relations that are inherent in the structure of a concept and revealing their complexity. It is not a substitute for a process model.

To realize how the concrete analog justifies the subtraction procedure the child must recognize several sets of relationships;

1. The vertical mappings from each concrete display to the quantity represented, and then to the place value notation.

2. There is a quantity conserving change at all three levels from the initial representation, 324, to the representation with decomposition, 200 plus 110 plus 14.

3. The subtraction process yields the same relationships at all three levels. For example, at the top level, when we remove a hundreds block from a set of 2 hundreds blocks, the result is 1 hundreds block. Similarly, at the next level, when we subtract 100 from 200, the result is 100. Similarly again, at the lowest level,

subtracting a 1 in the hundreds column from a 2 in the hundreds column yields a 1 in the hundreds column. Thus the same relationships obtain at each of the three levels. This is also true for tens and units. It is the fact that the same set of relations hold at all three levels that provides the justification for the arithmetic procedure. The problem is that children will not recognize the justification unless they can see this complex set of relationships. If the justification is not understood, the concrete analogs may be worse than useless, because they are extra things to learn, they take time to manipulate, and cause distraction.

Taken over all, there is a very complex set of relationships. It is really a composite of numerous lower level mappings. It entails more information than even an adult could process in parallel if the capacity theory outlined above is correct, so no adults could make the complete mapping in a single step. For both adults and children it would have to be learned, component by component. When we see the complexity of the mapping task, it becomes obvious why processing loads entailed in making the mapping could be impossibly high. The already complex mapping is further complicated by the fact that in this structure mapping there are two levels of representation, the concrete level and the quantity level. There is also a mapping from one to the other so that, for instance, 3 hundreds blocks represents the quantity 300, which is then mapped into the numeral 3 in the hundreds column. As we have seen, structure mapping imposes a processing load, and if this load is excessive it will constitute a barrier to understanding. Some way must therefore be found to reduce the processing load so the concrete analog can be useful.

There are at least two ways that the processing load can be reduced. One is by prelearning the mappings. For example children can be taught that a hundreds block (flat) represents hundreds, and relates to the hundreds column. Knowing this so it can be retrieved automatically from memory removes the processing load entailed in making the mapping. Much practice is requir~d however to make this retrieval automatic. The other way to reduce processing loads is to recode the relationships into more abstract form. As Biggs (1968) has noted, the Multi-base Arithmetic Blocks were intended to teach abstract concepts such as power and place value. The problem however is that abstraction is not a process that can be taken for granted, but must itself be explained. Therefore we will consider how abstractions might arise from experience with concrete analogs in the next section, and we can assess the processing loads this entails.

Structure mapping and abstraction

The processes by which abstractions are developed out of experience is a major problem at the very cutting edge of our discipline. For example Holland, Holyoak, Nisbett and Thagard (1986) present a sophisticated model of induction, the process by which general rules are acquired through experience with specific instances. Another major problem is to explain how people progress from representing constants to representing variables. This, and the recoding issue generally, are discussed by Clark (unpublished), Karmiloff-Smith (1987), and Smolensky (1988). We will not summarize this issue here, except to say that the problem of how abstractions develop is far from solved. However we will try to indicate how mapping from one structure to another might contribute to the development of abstraction.

We will develop the argument by reference to the distributive law of multiplication with respect to addition; a(b + c) = (a x b) + (a x c). We would propose that children, and most adults, understand this rule primarily in terms of specific examples. That is, they do not understand the rationale that is provided by pure mathematicians, but have a more pragmatic, experience-based rationale. This hypothesis is consistent with the virtually ubiquitous finding that natural human reasoning is not based on formal principles of general validity, but on pragmatic schemas that have some degree of generality, but are not universal (Cheng & Holyoak, 1985; Halford, in press; Shaklee, 1979).

A child, or for that matter an adult, might recognize the validity of the distributive law by testing it against a specific example. They might note that, for instance, 3(2 + 1) = (3 x 2) + (3 x 1). Understanding the validity of the law means recognizing the correspondence between the law and one or more specific examples. This is tantamount to recognition of structural correspondence; that is, it amounts to recognizing the structural correspondence between the example and the law.

Structure mapping analyses are a conceptual tool for expressing structural correspondences. The process of recognizing the correspondence between the law and an example can be expressed by the structure mapping diagram in Figure 7G. In terms of analogy theory, the example becomes the source (shown in the top line of the mapping) and the law is the target (in the bottom line of the mapping). The fact that the law can be mapped into a number of examples, and corresponds to those examples, is the major reason for regarding the rule as justified. It is therefore understood by analogy, but it is an analogy between a general rule and one or more examples of that rule. This might not be a conventional way to use the term analogy, but the structure mapping processes are those of analogies.

The only additional step that is likely to be made is to check for counter-examples; the rule is accepted as valid if no example can be retrieved that does not fit it. To illustrate, we might recognize that commutativity of subtraction, (a - b) = (b - a), is not valid because (3 - 2) =/ (2 - 3). That is, we can produce a counter-example, or a case that cannot be mapped into the rule. As Johnson-Laird (1983) has pointed out, seeking counter-examples is one of the more sophisticated aspects of natural reasoning processes.

The process of learning the general algebraic rule is partly a matter of replacing constants by variables. That is, the specific example 3(2 + 1) = (3 x 2) + (3 x 1) is replaced by a(b + c) = (a x b) = (a x c), in which each constant is replaced by a variable. But, as we said before, this has proved to be one of the most difficult processes for cognitive psychologists to explain, and we cannot take it for granted. We suggest however that structure mapping can play a role in this process. This can be demonstrated in a very general sense, and also in terms of specific examples.

At the general level, structure mapping means that specific examples of structures can be mapped into one another. This is illustrated in Figure 7E, where two specific instances of the distributive law are mapped into one another. The mapping is valid because the two structures are isomorphic, and mapped in such a way that they correspond. Correspondence is defined by consistency; two structures correspond if each element in one structure is mapped into one and only one element in the other structure, and if relations between elements in one structure correspond to relations between the image elements in the other strucure. More generally, a predicate P in structure A corresponds to a predicate P' in strucure B if and only if the arguments of P are mapped into the arguments of P' and vice verse (Halford, in press).

When two structures are mapped into one another, the structure itself remains constant, but the elements vary. As we see in Figure 7E, there are two identical structures, but the specific elements are different. Therefore structure mappings can simulate variables, because they permit a structure to be maintained while the instantiation of each part of it changes. A mapping such as Figure 7E does not literally contain variables, but it can certainly simulate the use of variables in at least some contexts, and can be a step towards the acquisition of variables, as we will see. Furthermore structure mapping processes are understood at quite a deep level. Holyoak and Thagard (in press) have produced a computer simulation of structure mapping based on parallel constraint satisfaction mechanisms which explains structure mapping in terms of the very basic processes of excitation and inhibition. Whereas abstraction per se remains something of a mystery to cognitive science, and is therefore a poor basis for explanation, structure mapping is much better understood, and provides a much more solid foundation on which to build explanations.

Another reason why structure mappings aid the abstraction process is that only the common aspects of the structures tend to be mapped, and surplus attributes and relations are deleted. When discussing analogy theory earlier we pointed out that in the man-house:dog-kennel analogy attributes of man are not mapped into dog, and only certain relations between man and h[illegible] are mapped into dog-kennel. Thus structure mapping is inherently selective in a way which is useful in creation of abstractions.

Now let us trace through a possible sequence of steps that might be entailed in acquiring the distributive law through structure mapping. Some hypothetical steps are shown in Figure 7. As mentioned earlier, we propose that the law is understood by recognizing the correspondence that it has to some specific examples. But there are knowledge prerequisites for this understanding, and these are briefly sketched in Figure 7.

In Figure 7A, we represent the child's knowledge that $3(2 + 1) = 9$. This knowledge must be acquired through calculation, and the child must learn to interpret and manipulate parentheses and operation symbols in arithmetic expressions. There is therefore procedural knowledge that must be acquired. Our concern here however is primarily to express the conceptual knowledge that underlies the procedural knowledge. We can express this conceptual knowledge that 3(2 + 1) = 9 corresponds to 3 x 3 = 9; i.e. process the operation in parentheses, which yields 3, then process the operation represented by the numeral which precedes the parentheses. This knowledge that 3(2 + 1) = 9 corresponds to 3 x 3 = 9 can, like other structural correspondences, be repesented as a structure mapping, as shown in Figure 7A. Note that, once again, structure mapping is a conceptual tool for analysing structural correspondences, and does not represent a process model as such.

The next step is for the child to recognize that (3 x 2) + (3 x 1) = 9 corresponds to 6 + 3 = 9. This is represented as a structure mapping in Figure 7B. This is essentially similar to the process in Figure 7B. It is a major step from there however to recognize that 3(2 + 1) = (3 x 2) + (3 x 1). Understanding this depends on recognizing that it corresponds to 3 x 3 = 6 + 3, which is shown as a structure mapping in Figure 7C. The child already knows that 3 x 3 = 9 = 6 + 3, because of previous experiences of the kind shown in Figure 7A and 7B. Therefore the known relationship, 3 x 3 = 6 + 3 can serve as a mental model that enables the child to understand 3(2 + 1) = (3 x 2) + (3 x 1). For this understanding to occur, the child must recognize the structural correspondence between the kinds of expressions, as shown in Figure 7C.

The next step is probably to acquiie further examples of this correspondence. Another example is shown in Figure 7D. Furthermore Figure 7E expresses the correspondence between a new example and the original example. The idea here is that a child might adopt one prototypical example and compare it with other examples, recognizing the correspondence between the prototype and numerous other examples. The prototype then becomes a kind of template for the general rule. A further example of this process is shown in Figure 7F.

The final step occurs when the child recognizes the correspondence between the prototype arithmetic example and the general rule. An additional process is required here, because the child must know that letters can be used to represent unknown numbers. This fact would normally be taught in other ways, such as showing children how to drawn a container representing an unknown number of objects, then teaching them how to write a letter to represent the unknown number of objects. Assuming the child has already learned to represent unknown numbers by letters, the step in Figure 7G can be taken once the correspondence the algebraic law and the arithmetic example can be recognized.

The fact that letters can represent unknown numbers is a component of the domain knowledge that is required to learn the algebraic law, but it does not explain how the algebraic rule is understood. The point that we have wanted to illustrate through this extended example is that understanding depends on recognition of the correspondence between the algebraic rule and one or more reference examples. Structure mapping analyses of this correspondence shows that it depends on a series of multiple system mappings. The processing loads and therefore quite high, and that is the next subject we must consider.

Abstraction, structure mapping and processing loads

If our analysis is correct, acquisition of an abstraction entails quite high processing loads, because it entails recognizing the correspondence structures that exemplify that abstraction. In our example based on the distributive law, it is necessary to see the correspondence between different instantiations of the law, and also between one prototypical instantiation and the algebraic expression of the law. Evidence mentioned earlier indicates that humans have limited capacity to recognize correspondence between structures, and adults can probably only process in parallel correspondences between four-dimensional structures, equivalent to one quaternary operation. Children of one year can probably only process structures based on one dimension in parallel, children of two years on two dimensions, children of five years on three dimensions. This subject has been discussed in detail elsewhere (Halford, in press).

Because we can recognize correspondence between structures of only limited complexity, we have other ways of processing structures. One way, as noted above, is to learn correspondences; i.e. we learn which component of one structure maps into which component of another structure. Once these mappings are learned they no longer impose a processing load. The other way is to recode the correspondences in a more abstract form. This reduces the processing load once the abstraction is achieved but, as we have seen, the processing loads can be high during acquisition because of the correspondences that must be recognized.

In order to reduce this load it is critically important that each correspondence is learned before progressing to the next. That is, the correspondence in Figure 7A must be learned before progressing to the one in Figure 7B, which must be learned before progressing to the correspondence in Figure 7C, and so on. Furthermore the learning must be such that retieval is automatic, so that no load is imposed. The load imposed by one structure mapping must be reduced to zero before the next structure mapping is undertaken, otherwise the cumulative load will become excessive.

Conceptual chunking can also be used to reduce processing loads. What we call an abstraction is often better conceptualized as a conceptual chunk. For example, the complex relationships in Figure 6 can be recoded as a conceptual chunk. The chunk consists of the idea of a number, to which decomposition can be applied, resulting in an equal number but differently configured, then subtraction is applied yielding a new number. This is a very simple set of relationships, and in itself it imposes quite a low processing load. It is equivalent to two successive relational mappings. It produces great gains in processing load by constraining more complex mappings. For example, number is mapped, or can be decomposed, into hundreds, tens, and units. The decomposition relation between two numbers at the abstract level constrains the operations that are performed on the hundreds, tens, and units; if the tens are reduced by 1, the units must be incremented by 10, and so on. The fact that the abstract concept of decomposition constrains us to adjust tens and units in this way can be learned, and when it is learned the conceptual chunk greatly reduces the processing load. This reduction does not come about automatically however, but only by learning some complex relationships. Once acquired it produces massive gains in efficiency.

Abstraction of place-value

The multibase arithmetic blocks were designed partly to facilitate understanding of power and place value, and therefore different bases were used. Figure 8 shows the correspondence between base 10 and base 2 blocks. In each case the relation between a unit and a long is an increase from the zero to the first power. The relation between a long and a flat is an increase from the first to the second power. The same relationship occurs in base-ten and base-two blocks. Notice that this correspondence is easily expressed as a structure mapping diagram, and doing so shows that it is much simpler than the correspondence involved in the subtraction algorithm. Recognition of correspondence between the structure of base-ten and the structure of base-two (or other bases) is an important component of abstraction. "Raising to the next power" is the relation that is common to both structures, and this concept can be extracted by seeing the correspondence between the structures. This is another illustration of the point made earlier that analogies are useful for promoting abstractions, because they entail selectively mapping those relations that are common to both structures.

Multiple embodiment

One principle which Dienes (1964) has advocated is multiple embodiment. The general idea is that the same principle is instantiated in different materials. This helps abstraction because it leads to focusing on the common features of the instantiations, to the exclusion of idiosyncratic features. A structure mapping analysis can help to explain this process, and also leads to some insights as to how it should be employed.

A technique which has been observed in schools is to replace multibase arithmetic blocks by paddle-pop sticks. Units are represented by individual sticks, and tens by bundles of 10 sticks bound together. A stick then corresponds to a unit block, and a bundle of sticks to a long. In both cases the representations are raised from the zero to the first power. A structure mapping analysis shows that such multiple embodiments are only useful if the child sees the correspondence between the two structures. Putting it another way, the child must recognize the analogy. If the analogy is not recognized then the extra embodiment is worse than useless, because it is actually a distraction. Thus play, or the use of manipulative materials, may not achieve the desired acquisitions. Analogy theory, particularly as applied to children (see Halford in press, for a review) can be used to predict the conditions under which recognition of the analogy is most likely to be achieved.

Structure mapping theory and pedagogy

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We have presented a number of examples designed to show how structure mapping theory can be a useful way of analysing what needs to be understood, the loads imposed in such understanding, and ways of reducing those loads. Structure mapping is a conceptual tool, a major value of which is that it draws attention to the correspondences which children must be taught to recognize. However we cannot emphasize too strongly that this does NOT imply that children should be taught to draw structure mapping diagrams. Structure mapping theory is for the theorist and educator to use in analysing tasks, and its presentation to children would only load them with useless and impossibly burdensome information.

The correct way to use these analyses is to gain insights into the correspondences that need to recognized, then to use an assortment of pedagogical tools to present these correspondences to pupils. This paper is about analysing concepts rather than about teaching methods per se, but by way of illustration we can consider briefly how the correspondence in Figure 7A might be taught. The idea would be to show children how to compute the number inside the parentheses, then multiply it by the number outside the parentheses; i.e. "2 + 1 = 3, 3 x 3 = 9". Then point out that 3(2 + 1) = 9 is the same as 3 x 3 = 9. Children would need multiple exercises with this relationship, until they could retrieve it automatically.

Correspondence between abstractions

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Structure mapping can be used to represent correspondence between abstractions. For example recognition of correspondence between equations can be represented as a structure mapping, as shown in Figure 11. The equation AX = b corresponds to the equation A(X + b) = c, if (X + b) in equation 2 is mapped into X in equation 1.

This illustrates the general point that structure mapping is not only used to represent correspondences between concrete analogs and arithmetic. It can be used to represent correspondences between any two isomorphic structures. The type of structure depends on the domain. When teaching arithmetic relations, concrete analogs are useful models. When teaching elementary algebra, previously learned arithmetic relations are useful models. With more advanced algebra, previously learned algebraic concepts are useful models. The appropriate mapping is between a previously-learned model, treated as source, and the new concept, treated as target. Thus we are proposing an inductive concept of mathematics learning, in which previously learned concepts are used as mental models of new, higher level concepts. This induction process depends heavily on recognition of correspondence between the mental model and the new concept. We use structure mapping to analyse the correspondences that are required, and to provide estimates of their complexity.

Concrete analogs and criteria for good analogies

Some insights can be obtained by assessing the structure mapping in Figure 6 according to the criteria for a good explanatory analogy devised by Gentner (1982). The first of these is base specificity, which corresponds to the degree to which the structure of the base is understood. In Figure 6 this means that the structure of the block analog must be well understood, including the size relations between the blocks (the fact that a hundreds block is 10 times a tens block which is 10 times a unit block). It is therefore essential that it be reduced as much as possible. One way to do this is to ensure that all information about the base can be retrieved without effort, so the retrieval process does not impose a processing load.

This mapping can be prelearned, so that it does not need to be constructed when the child has the additional load of learning the subtraction algorithm. That is, children can learn that 3 hundreds blocks represent the quantity 300. Much practice should be given in mapping quantities to concrete analogs before the mapping to the subtraction notation is introduced. Probably the only way the quantity representation can be encoded is verbally, so that the verbal quantity labels for each concrete analog would need to be learned.

Resnick and Omanson (draft 87) found that children who had facility with the verbal quantity labels profitted more from mapping training. This is quite consistent with the structure mapping analysis, because it would mean that the mappings from the verbal quantity labels to the concrete analogs were well known. This would facilitate completing the rest of the structure mapping for the

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subtraction algorithm.

The structure mapping rates quite well according to Gentner's next criterion, clarity, which means that there are no ambiguous mappings. It also rates well according to richness, because of the large number of mappings that are made, and according to scope, because of many applications of the analogy.

The structure mapping rates very highly in systematicity, because the many relations between elements at the same level do provide a coherent overall structure. This is both a strength and a weakness, however. The value of the analogy lies mainly in the complex set of relationships that are represented, but this also makes it very complex and difficult to recognize. An important point is that children need to recognize, not only the vertical mappings between elements at the different levels, but also the corresponding relations between levels. That is, it is not sufficient to know the mappings from 3 hundreds blocks to the quantity 300 and then to the 3 in the hundreds column, plus the other vertical mappings of this type. It is necessary also to realize that, for example, the quantity conserving change between the two block arrangements is mirrored in a quantity conserving change between the quantities represented at the next level, and that this in turn is mirrored in a quantity conserving change at the notation level. The training procedure used by Resnick and Omanson might have been more successful in teaching the vertical mappings, and might not have taught children to recognize the corresponding relations at the different levels.

An efficient technique for teaching the mappings shown in Figure 8.10 has been developed by Champagne and Rogalska-Saz (1984). The Dienes blocks are represented by computer graphics, which obviates the classroom management problems associated with blocks, and facilitates manipulation. A graded series of lessons teaches children the mappings from blocks to notation, how to trade higher demonination blocks for lower demonination blocks, and how this quantity conserving trade is mirrored in the decompositiodecomposition notation.

Conclusions

We have used analogy theory and the theory of cognitive representations to analyse some problems in mathematics education. Analogy theory and representation theory both depend on structure mapping theory, because both depend on mapping one structure into another. Thus structure mapping is really the generic concept, of which analogies and representations are specific cases.

Concrete aids that exemplify mathematical conepts are technically analogs, and they can be analysed by specifying the structure mapping from external analog to mental representation of the concept. This leads to some useful insights into the reason why come analogs are likely to be more efficient than others. Furthermore it makes it possible to analyse the processing loads that use of such analogs can impose. Perhaps most important of all, it emphasizes that analogs of any kind are useless unless children see the correspondence between the analog and the concept.

Research into structure mapping in other structures shows that humans have limited capacity to recognize correspondence between two structures. Adults can probably only process four-dimensional structures in parallel, and children can process structures of less dimensionality than adults. Our research indicates that the dimensionality of structures that children can process in parallel increases from one at age one year, two at age two years, three at age five years, and the adult ability to process four-dimensional structures is acquired at 11 years. The wider implications of this for cognitive development are considered elsewhere (Halford, in press). In most contexts this limitation is overcome by using prelearned correspondences between structures, by recoding structures so they are defined over fewer dimensions, or by segmenting problems and using a mixture of serial and parallel processing. The limitation only affects performance where one of these strategies cannot be used. This occurs where at least one of the structures is new, and cannot be decomposed.

Much of mathematics learning entails acquisition of progressively more abstract concepts. Abstraction reduces processing loads, but we propose that abstractions are acquired by induction from examples. For this to occur, children must be able to see the correspondence between different examples of the same abstraction, and also between an example and the abstract rule. Unless this correspondence is recognized the rule is not really understood. Structure mapping can be used to analyse these correspondences. In general structure mapping is important in acquisition of abstractions because it simulates the use of variables, and leads to selection of attributes and relations that are common to different examples of the same concept.

We propose that mathematics is learned by using previously acquired concepts as mental models for later, more abstract concepts. Elementary number concepts are probably learned using concrete external experiences with sets as mental models. Elementary algebraic concepts are acquired by using previously-learned number concepts based on constants as mental models. Some higher level algebraic concepts are acquired using previously-learned algebraic concepts as mental models. This progression from concrete experiences to increasingly abstract concepts depends, at each step, on recognition of correspondences between earlier concepts and later ones. Therefore recognition of correspondences between structures, which we analyse in terms of structure mapping theory, is central to mathematics learning at all levels.

REFERENCES

Biggs, J.B. (1968). *Information and human learning*. Melbourne: Cassel.

Broadbent, D.E. (1975). The magic number seven after fifteen years. In A. Kennedy & A. Wilkes, *Studies in long term memory*. (3-18). London: Wiley.

Carey, S. (1985). *Conceptual change in childhood*. Cambridge, MA: MIT Press.

Champagne, A.B., & Rogalska-Saz, J. (1984). Computer-based numeration instruction. In V.P. Hansen, & M.J. Zweng (Eds.), *Computers in mathematics education, 1984 yearbook* (pp. 43-53). : .

Cheng, P.W., & Holyoak, K.J. (1935). Pragmatic reasoning schemas. *Cognitive Psychology, 17*, 391-416.

Clark, A. (1989). *Connectionism, non-conceptual content and representational re-description*. Unpublished paper, University of Sussex, Brighton.

Collis, K.F. (1978). Implications of the Piagetian model for mathematics teaching. In J.A. Keats, K.F. Collis, & G.S. Halford (Eds.), *Cognitive development: Research based on a Neo-Piagetian approach*. (249-283). London: Wiley.

Dienes, Z.P. (1964). *Mathematics in the primary school*. Melbourne: Macmillan.

Dufour-Janvier, B., Bednarz, N. & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In Janvier, C. (Ed.), *Problems of representation in the teaching and learning of* *mathematics* (pp. 109-122). Hillsdale, NJ: Erlbaum.

Fisher, D.L. (1984). Central capacity limits in consistent mapping, visual search tasks: four channels or more?. *Cognitive Psychology, 16*(4), 449-484.

Gentner, D. (1982). Are scientific analogies metaphors?. In D.S. Miall (Ed.), *Metaphor: Problems and perspectives* (pp. 106-132). Brighton, England: Harvester Press.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science, 7*, 155-170.

Halford, G.S. (1982). *The development of thought*. Hillsdale, NJ: Erlbaum.

Halford, G.S. (1987). A structure-mapping approach to cognitive development. *International Journal of Psychology*, *22*, 609-642.

Halford, G.S. (1989). Reflections on 25 years of Piagetian cognitive developmental psychology, 1963-1989. *Human Development, vol?*(num?), pp?.

Halford, G.S. (in preparation). *Children's understanding: the development of mental models*. Hillsdale, N.J.: Erlbaum.

Halford, G.S., Maybery, M.T., & Bain, J.D. (1986). Capacity limitations in children's reasoning: A dual task approach. *Child Development, 57*, 616-627.

Halford, G.S., Maybery, M.T., & Bain, J.D. (1988). Set-size effects in primary memory: An age-related capacity limitation?. *Memory and Cognition, 16*(5), 480-487.

Halford, G.S., & Wilson, W.H. (1980). A category theory approach to cognitive development. *Cognitive Psychology, 12*, 356-411.

Holland, J.H., Holyoak, K.J., Nisbett, R.E., & Thagard, P.R. (1986). *Induction: Processes of inference, learning and discovery.* Cambridge, MA: Bradford Books/MIT Press.

Holyoak, K.J., Koh, K., & Nisbett, R.E. (in press). A theory of conditioning: inductive learning within rule-based default hierarchies. *Psychological Review.*

Holyoak, K.J., & Thagard, P. (1987). *Analogical mapping by constraint satisfactions.* Unpublished manuscript, University of California, Los Angeles, CA.

Johnson-Laird, P.N. (1983). *Mental models.* Cambridge: Cambridge University Press.

Karmiloff-Smith, A. (1987). Beyond modularity: a developmental perspective on human consciousness. Paper presented at the Annual Meeting of the British Psychological Society, Sussex, April.

Leeuwenberg, E.L.L. (1969). Quantitative specification of information in sequential patterns. *Psychological Review, 76*, 216-220.

Lesh, R., Behr, M. & Post, T. (1987). Rational number relations and proportions. In Janvier, C. (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 41-58). Hillsdale, NJ: Erlbaum.

Maybery, M.T., Bain, J.D., & Halford, G.S. (1986). Information processing demands of transitive inference. *Journal of Experimental Psychology: Learning, Memory and Cognition, 12*, 600-613.

McGarrigle, J., Grieve, R., & Hughes, M. (1978). Interpreting inclusion: A contribution to the study of the child's cognitive and linguistic development. *Journal of Experimental Child Psychology, 26*, 528-550.

Miller, G.A. (1956). The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological Review, 63*, 81-97.

Palmer, S.E. (1978). Fundamental aspects of cognitive representation. In Rosch, E. & Lloyd, B.B. (Eds.), *Cognition and categorization* (pp. 259-303). Hillsdale, NJ: Erlbaum.

Resnick, L.B., & Omanson, S.F. (1984). Learning to understand arithmetic. In R. Glasser (Ed.), *Advances in instructional psychology* (Vol. 3, pp. ???). University of Pittsburgh: Erlbaum.

Resnick, L.B. & Omanson, S.F. (1987). *Learning to understand arithmetic.* Unpublished manuscript, Learning Research and Development Centre, University of Pittsburgh, Pittsburgh, PA.

Rumelhart, D.E., & McClelland, J.L. (Eds.) (1986). *Parallel distibuted processing: Explorations in the microstructure of cognition..* Boston, MA: MIT Press.

Schneider, W. & Detweiler, M. (1987). A connectionist/control architecture for working memory. *The Psychology of Learning and Motivation, 21*, 53-119.

Shaklee, H. (1979). Bounded rationality and cognitive development: Upper limits on growth?. *Cognitive Psychology, 11*, 327-345.

Siegel, L.S., McCabe, A.E., Brand, J., & Matthews, J. (1978). Evidence for the understanding of class inclusion in preschool children: Linguistic factors and training effects. *Child Development, 49*, 688-693.

Siegler, R.S., & Shrager, J. (1984). Strategy choices in addition and subtraction: How do children know what to do?. In C. Sophian (Ed.), *Origins of cognitive skills* (pp. 229-293). Hillsdale,NJ: Erlbaum.

Simon, H.A. (1972). Complexity and the representation of patterned sequences of symbols. *Psychological Review, 79*, 369-382.

Smolensky, P. (1988). On the proper treatment of connectionism. *Behavioral and Brain Sciences, 11*(1), 1-74.

Figure captions

Figure 1. A structure mapping analysis of a simple analogy

Figure 2. Four levels of structure mapping

Figure 3. Structure mapping analysis of some concrete aids

Figure 4. Structure mapping analysis of a place-value analog

Figure 5. Stucture mapping analysis of a decomposition analog

Figure 6. Structure mapping analysis of a subtraction analog

Figure 7. Structure mapping analysis of acquisition of the distributive law

Figure 8. Correspondence between base-10 and base-2

Figure 9. Correspondence between two equations

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Fig. 1

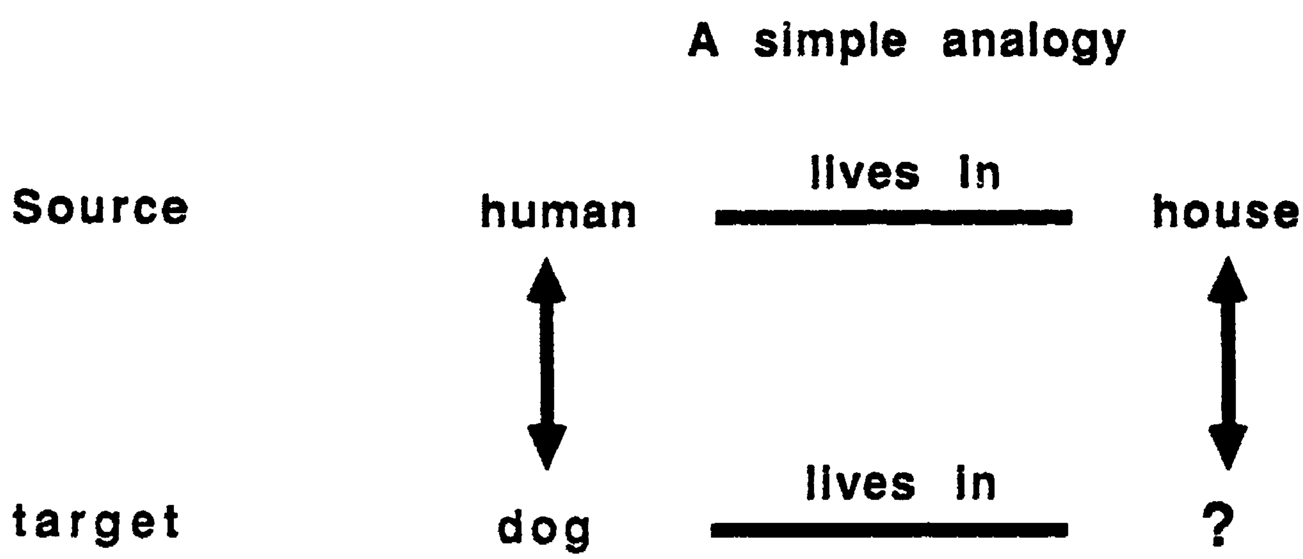
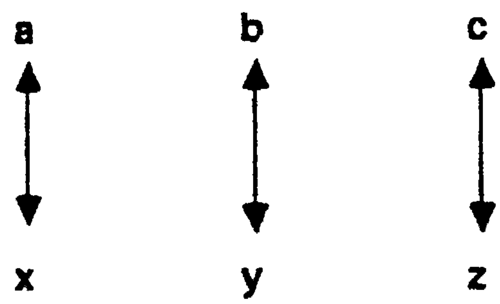

A simple analogy
Source
human
lives in
house
target
dog
lives in
?

Fig. 2

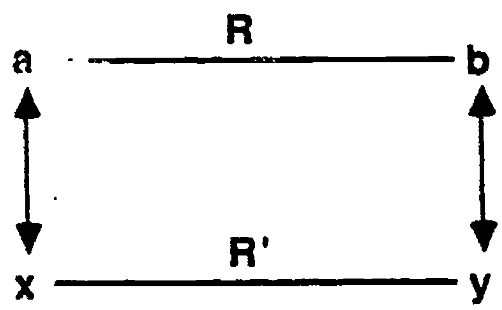
ELEMENT MAPPINGS



Elements mapped individually so one element only considered in each mapping decision.

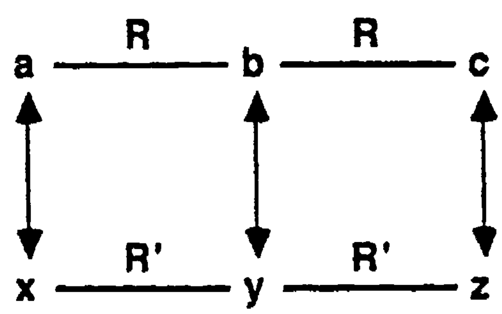
RELATIONAL MAPPINGS

Two elements considered in each mapping decision.



SYSTEM-MAPPING

Three elements considered in each mapping decision.



MULTIPLE-SYSTEM MAPPING

Four elements considered in each mapping decision.

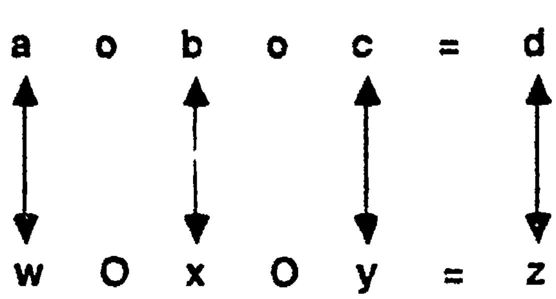
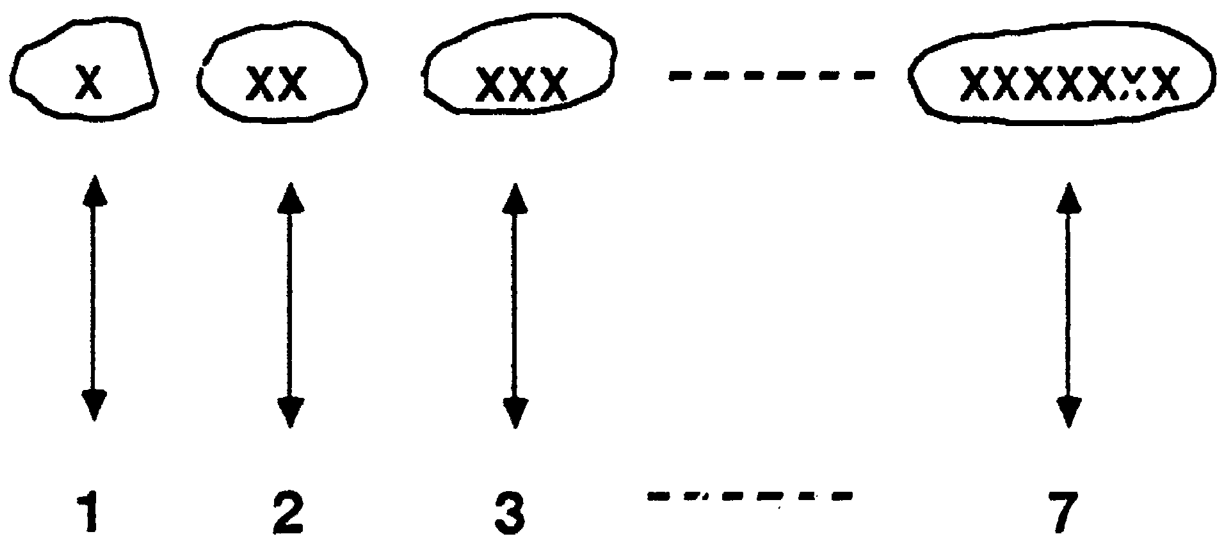


Fig. 3

A



B

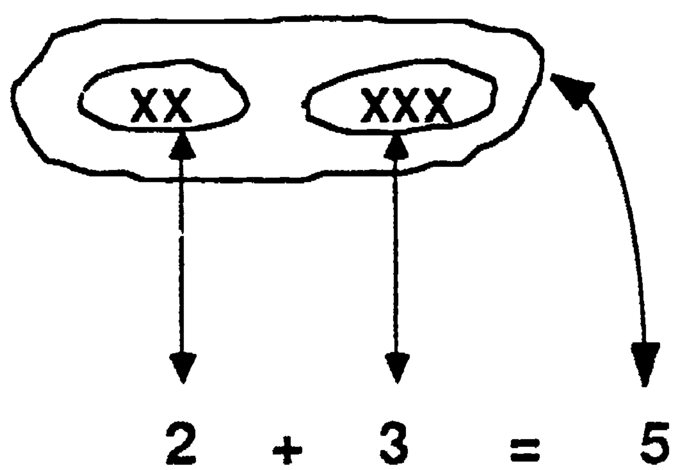
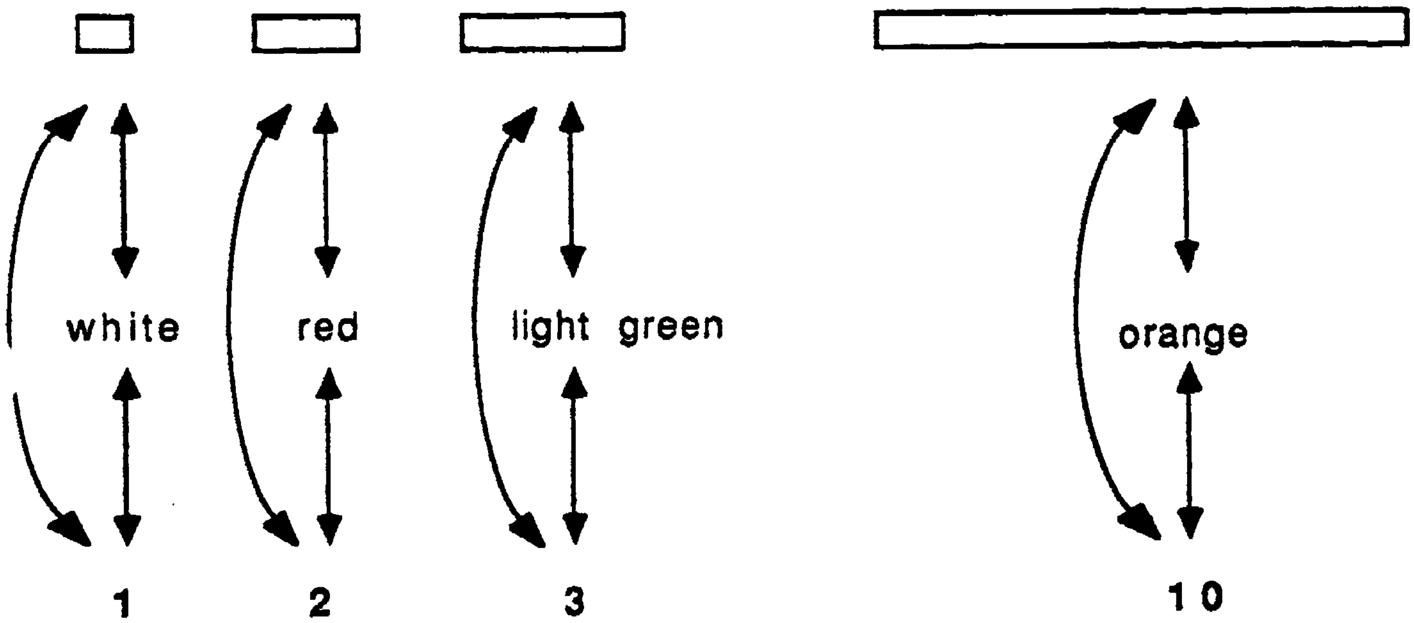


Fig. 3 ctd.

C

Cuisenaire rods



D

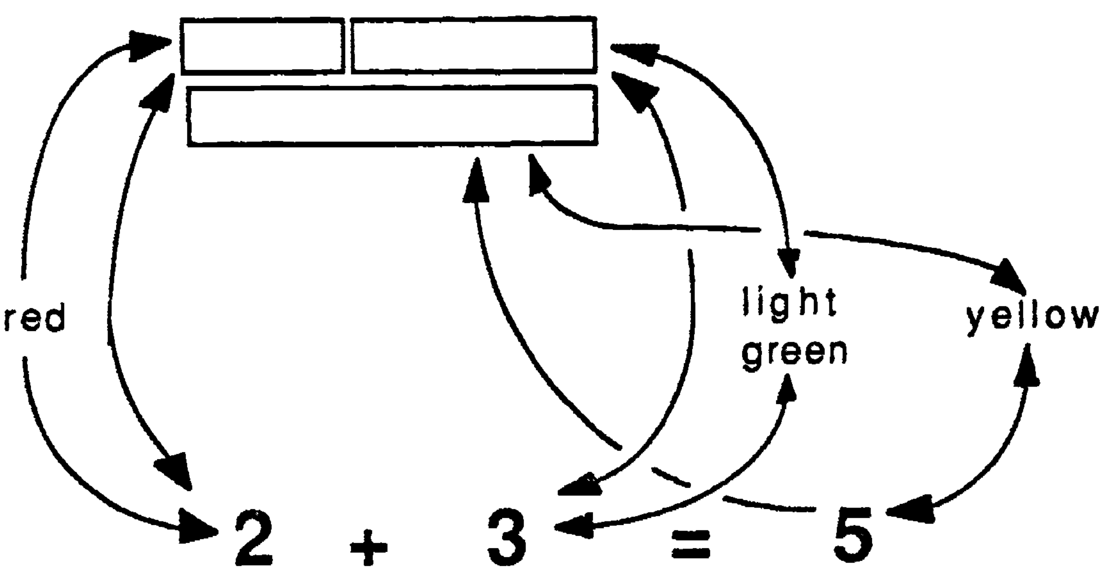


Fig. 3 ctd.

E

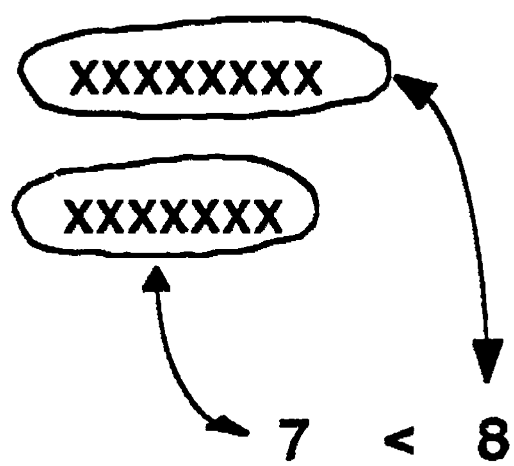


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Fig. 4

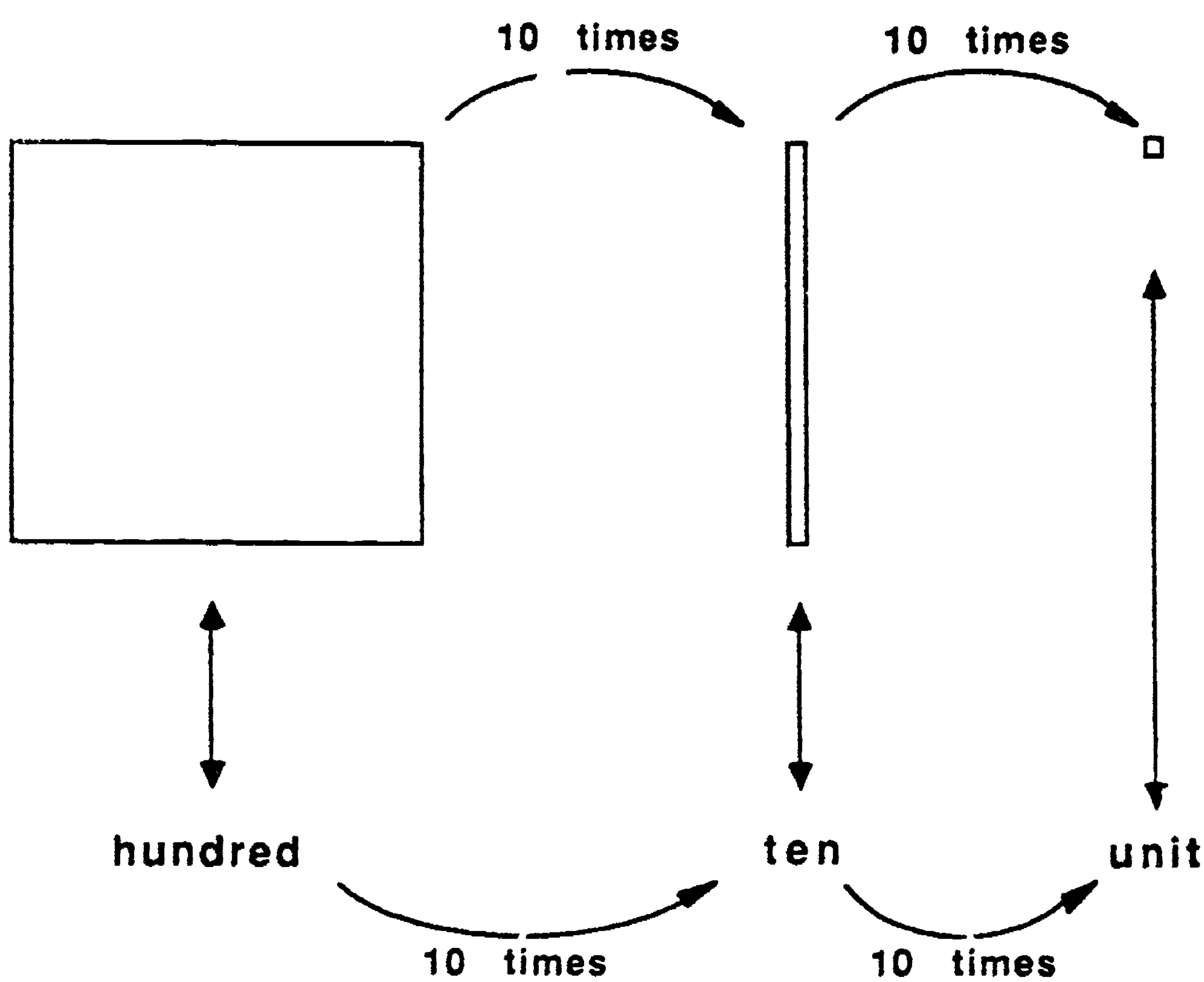


Fig. 5

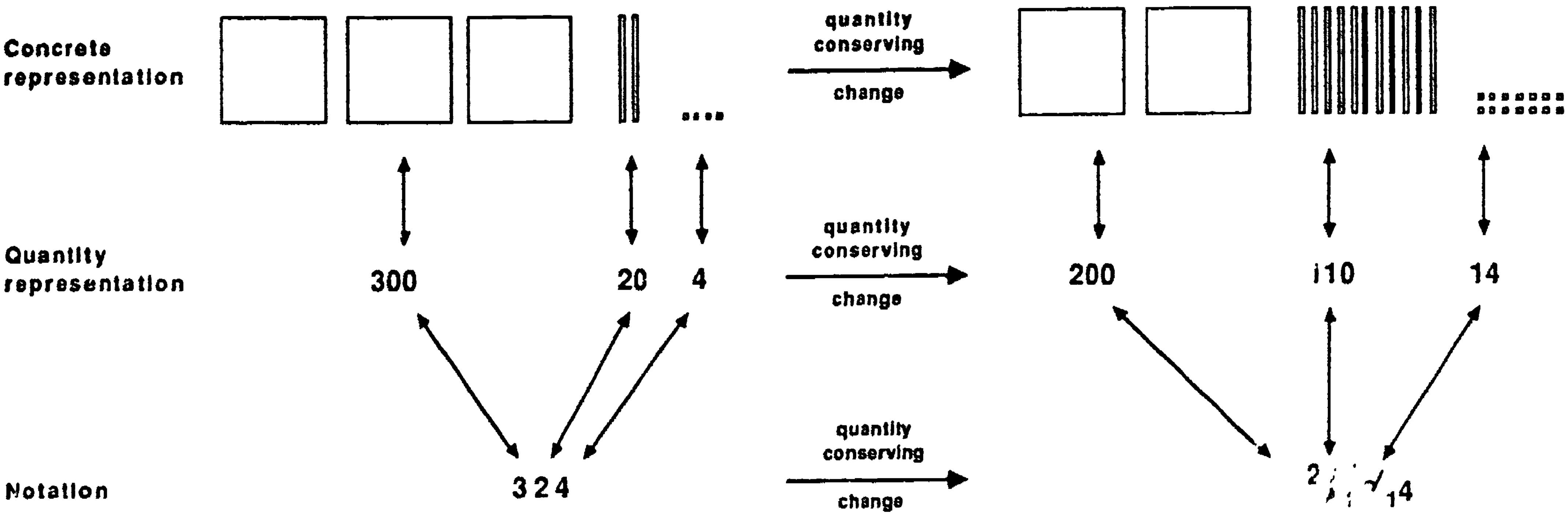
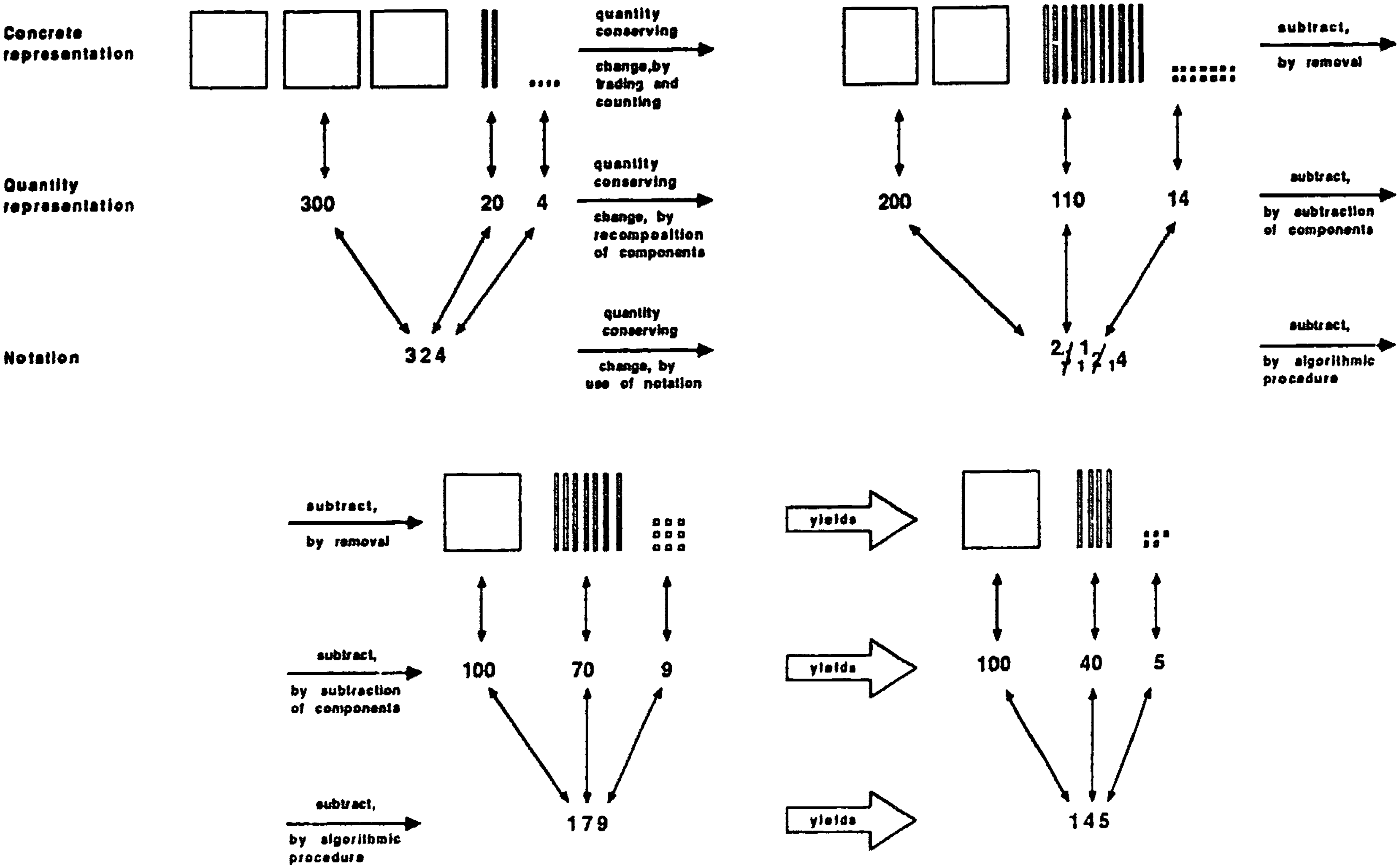

Concrete representation
quantity conserving change
Quantity representation
300
20
4
quantity conserving change
200
110
14
Notation
324
quantity conserving change
14

Fig. 6



Concrete representation
quantity conserving
change, by trading and counting
subtract,
by removal
Quantity representation
300
20
4
quantity conserving
change, by recomposition of components
200
110
14
subtract,
by subtraction of components
Notation
324
quantity conserving
change, by use of notation
subtract,
by algorithmic procedure
subtract,
by removal
yields
subtract,
by subtraction of components
100
70
9
yields
100
40
5
subtract,
by algorithmic procedure
179
yields
145

A

$$3\,(2 + 1) = 9$$
$$3 \times 3 = 9$$

B

$$(3 \times 2) + (3 \times 1) = 9$$
$$6 + 3 = 9$$

C

$$3\,(2 + 1) = (3 \times 2) + (3 \times 1)$$
$$3 \times 3 \qquad 6 + 3$$

D

$$4\,(2 + 3) = (4 \times 2) + (4 \times 3)$$
$$4 \times 5 = 8 + 12$$

E

$$3\,(2 + 1) = (3 \times 2) + (3 \times 1)$$
$$4\,(2 + 3) = (4 \times 2) + (4 \times 3)$$

F

$$3\,(2 + 1) = (3 \times 2) + (3 \times 1)$$

↕ ↕ ↕ ↕ ↕ ↕ ↕

$$5\,(10 + 7) = (5 \times 10) + (5 \times 7)$$

G

$$3\,(2 + 1) = (3 \times 2) + (3 \times 1)$$

↕ ↕ ↕ ↕ ↕ ↕ ↕

$$a\,(b + c) = (a \times b) + (a \times c)$$

Fig. 8

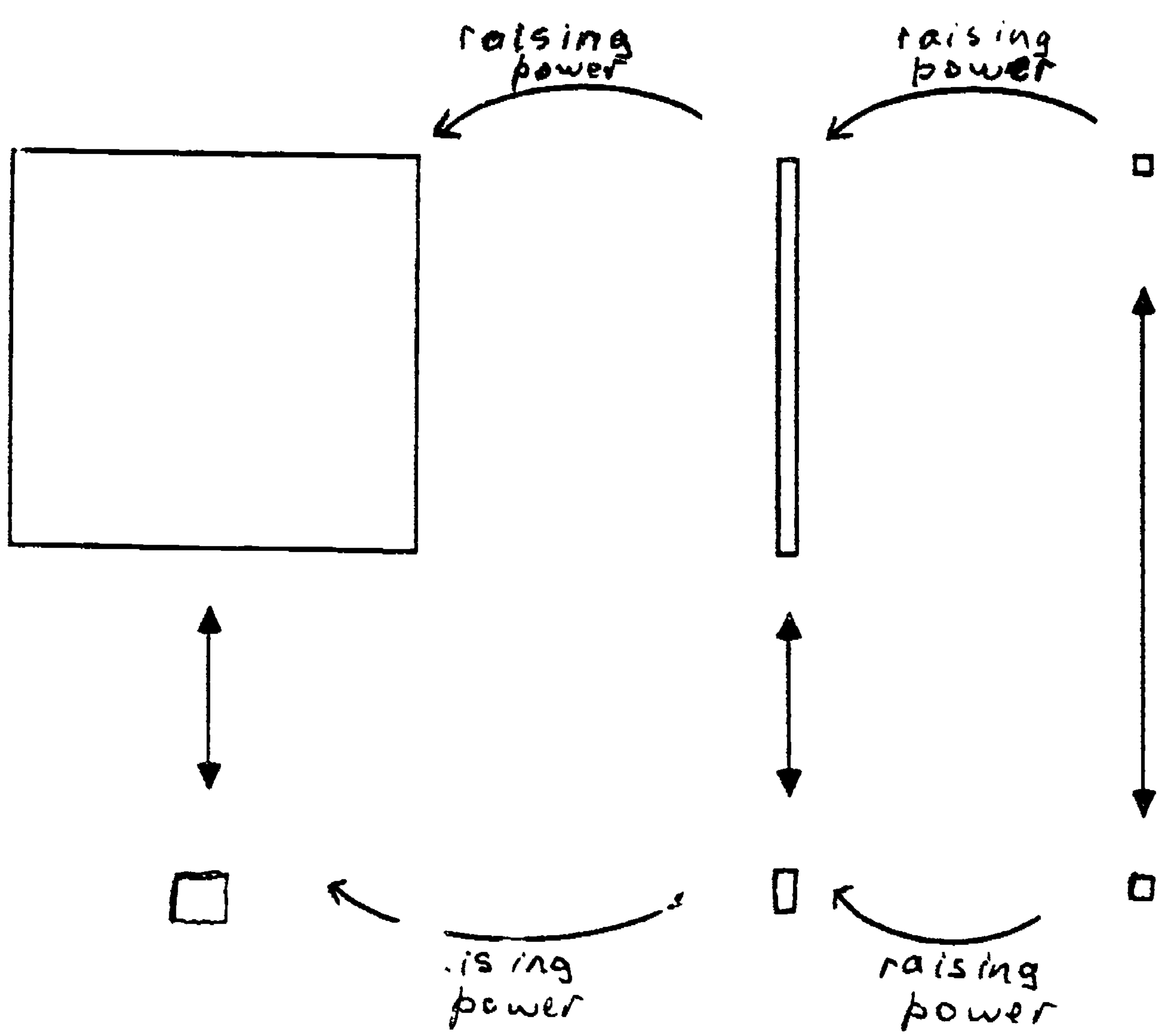


Figure 9

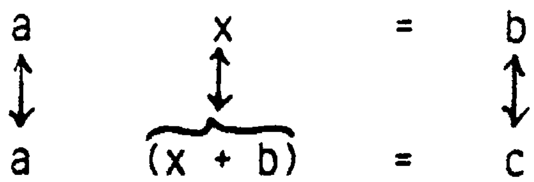


Table 1
Examples of concepts at each level of structure mapping.

Concepts that require element mappings
Simple categories (dog,house)

Concepts that require relational mappings
Concepts based on simple binary relations (more than, bigger than)
Simple oddity
Simple analogies

Concepts that require system mappings
Transitivity and ordering
Class inclusion
Multiple classification
Dimension-abstracted oddity
Systematicity in analogies
' lypothesis testing in affirmation concepts (dimension checking and more sophisticated strategies)
Hypothesis testing in the attribute identification paradigm, with conjunctive, disjunctive and conditional rules.
Interpretation of simple algebraic expressions containing arithmetic operations

Concepts requiring multiple system mappings
Hypothesis testing in the attribute identification paradigm, with the biconditional rule
Hypothesis testing in the rule identification paradigm, for all rules other than affirmation
Interpretation of algebraic operations containing compositions of arithmetic operations