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ABSTRACT

This paper presents an application of integration to the field of hydraulics. An integral relation for the time required to drop the fluid contained in a cylindrical tank from one level to another using a hole in the tank wall is derived. Procedures for constructing the experimental equipment and procedures for determining the coefficient of discharge are described. A student laboratory worksheet for experimental observations and theoretical predictions is provided. (YP)

INTEGRATION IN HYDRAULICS

by

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Abstract

An application of integration to the field of hydraulics is presented. An integral relation for the time required to drop the fluid contained in a cylindrical tank from one level to another using a hole in the tank wall is derived. The integral is

$$T = -\frac{A}{Ca\sqrt{2g}} \int_{h_i}^{h_f} \frac{dh}{\sqrt{h}}.$$

This expression may be easily evaluated by a first semester calculus student and the results experimentally validated using a coffee can filled with water.

Discussion

A simple physical process that involves integration at the first semester calculus level is the removal of fluid from a cylindrical tank through a hole (orifice) in its side. Consider the tank whose side view is depicted in figure 1.

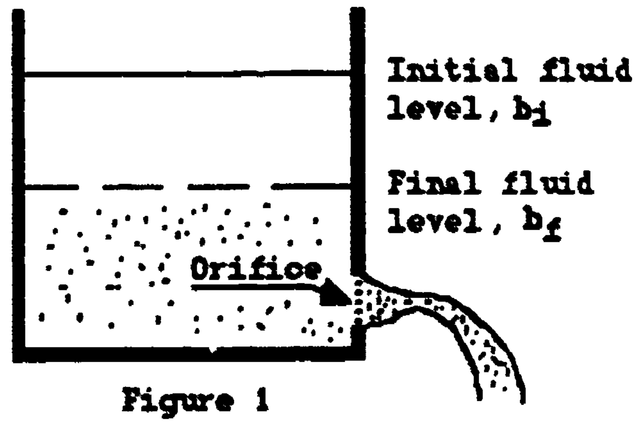

Figure 1
Side View of a
Cylindrical Tank

It is shown in the derivation below that the time, T, required to drop the fluid level from its initial level, h_i, to its final level, h_f, is given by

$$(1)\quad T = -\frac{A}{Ca\sqrt{2g}} \int_{h_i}^{h_f} \frac{dh}{\sqrt{h}}$$

where A is the cross sectional area of the cylindrical tank, a is the cross sectional area of the orifice, g is the acceleration due to gravity (i.e. the gravitational constant) and C is the coefficient of discharge. The coefficient of discharge varies with orifice geometry and depth of the orifice in the fluid. The depth component may be ignored in this application as insignificant if the change in fluid level is small compared to the actual fluid levels. The coefficient of discharge is determined either from a table of values or by experiment.

Derivation

The derivation of equation 1 presented here is essentially that given by King and Brater in the fourth chapter of Handbook of Hydraulics, 5th edition, McGraw-Hill, 1963.

Consider a cylindrical tank with an orifice (hole) in the side wall near the bottom as shown in figure 2.

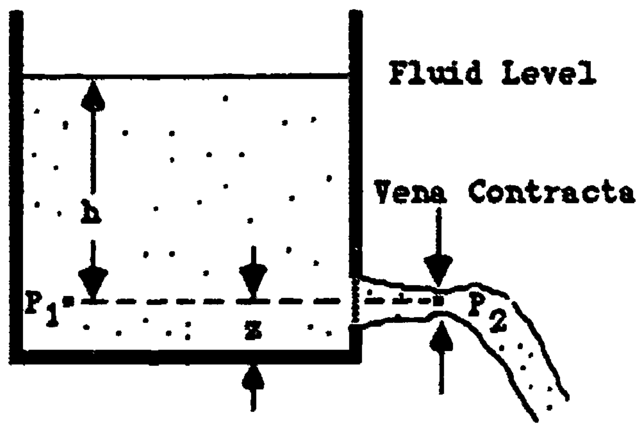

Figure 2
Side View of a Clindrical Tank with Definitions of h, z, P_1, and P_2

Suppose the tank has cross sectional area, A; the orifice has cross sectional area, a; h is the depth of the center of the orifice below the fluid surface; and z is the height of the center of the orifice above the bottom of the tank. As the fluid leaves the orifice, the diameter of the flow decreases to a minimum value at what is called the vena contracta (for a sharp edged circular orifice, this is located at a distance from the orifice about equal to the radius of the orifice). The contraction stops when the pressure of the fluid in the flow is equal to atmospheric pressure (0 psi gauge).

Consider point P_1 in the tank and point P_2 at the vena contracta (see figure 2). These points were chosen to be at the same height above the bottom of the tank as the center of the orifice. If a particle of fluid of weight W pounds at point P_1 is considered, the total energy of the particle can be expressed as the sum of its potential and kinetic energy. The potential energy term has a component attributable to the position of the particle above the bottom of the tank, Wz, and a component due to fluid pressure, Wh (where h is the pressure head or depth of the fluid at P_1). The kinetic energy is given by

$$K.E. = \frac{1}{2}\frac{W}{g}v_1^2$$

where v_1 is the velocity of the fluid at P_1. The total energy at P_1 is

$$E_1 = Wz + Wh + \frac{1}{2}\frac{W}{g}v_1^2$$

and consequently,

$$\frac{E_1}{W} = z + h + \frac{v_1^2}{2g}$$

We can now look at point P_2. Since this point is located at the vena contracta the pressure is 0 psi gauge and thus the pressure head is zero. Consequently,

$$\frac{E_2}{W} = z + 0 + \frac{v_2^2}{2g}$$

Because no energy is being added to the system (i.e. it is not being heated or moved), the energy at P_1 is equal to the energy at P_2 plus a small energy loss term, ε. The loss term accounts for the energy given up to friction in moving from P_1 to P_2. Thus

$$\frac{E_1}{W} = \frac{E_2}{W} + \varepsilon$$

$$z + h + \frac{v_1^2}{2g} = z + 0 + \frac{v_2^2}{2g} + \varepsilon$$

$$h + \frac{v_1^2}{2g} = \frac{v_2^2}{2g} + \varepsilon \qquad (2)$$

For a large tank the fluid away from the orifice will move fairly slowly. Thus the kinetic energy term is overshadowed by the potential energy term and can be neglected. Equation 2 thus reduces to the following

$$h = \frac{v_2^2}{2g} + \varepsilon$$

$$\therefore \qquad v_2 = \sqrt{2g(h - \varepsilon)}$$

$$= c_v\sqrt{2gh}$$

where the energy loss term is expressed as a multiplicative coefficient ($0 < c_v \leq 1$) called the coefficient of velocity.

The flow, Q, of the fluid from the tank can be calculated at the vena contracta (since this is where the fluid velocity, v_2, is known).

$$Q = a_2 v_2 = a_2 c_v \sqrt{2gh}$$

where a_2 is the cross sectional area of the flow at the vena contracta. This area, a_2, can be related to the cross sectional area of the orifice, a, via a coefficient of contraction, C_c, that depends on the orifice geometry.

$$a_2 = C_c\, a$$

$$\therefore \qquad Q = C_c a c_v \sqrt{2gh}.$$

The product of the coefficient of velocity and the coefficient of contraction is called the coefficient of discharge, C.

$$\therefore \qquad Q = Ca\sqrt{2gh}$$

Values of C, the coefficient of discharge, may be found from experimentation or from tables as given by King and Brater (1963:4-28).

As the fluid leaves the tank, the fluid level in the tank drops. The fluid flow out of the tank, Q, is also equal to the cross sectional area of the tank times the velocity of the fluid at its surface. This velocity is given by the derivative - dh/dt and the negative sign is necessary to insure a positive fluid flow when the height of the fluid is decreasing).

$$Q = A\left(-\frac{dh}{dt}\right) = Ca\sqrt{2gh}$$

$$dt = -\frac{A}{Ca}\frac{dh}{\sqrt{2gh}}$$

$$T = \int_{t_i}^{t_f} dt = -\frac{A}{Ca\sqrt{2g}} \int_{h_i}^{h_f} \frac{dh}{\sqrt{h}}$$

Consequently,

(3) $$T = -\frac{A}{Ca\sqrt{2g}} \int_{h_i}^{h_f} \frac{dh}{\sqrt{h}}$$

where t_i, the initial time, corresponds to the time when the orifice was first opened and the fluid level was at its initial value of h_i; the orifice was closed at the time t_f and this corresponds to the time the fluid level reached the final level, h_f; the total time necessary for the fluid level to drop from the initial to final position was T.

Construction of Experimental Equipment

The only component of experimental equipment that needs to be constructed for a student laboratory exercise based on the process described above is the cylindrical tank with an orifice. A simple one pound coffee can with a circular hole drilled near the bottom with a hand held power drill works well for this purpose. The student should be provided with the coefficient of discharge, C, corresponding to the tank. This value can be determined using equation 3 above and a few practice runs.

Determination of the Coefficient
Discharge, C

Solving equation 3 yields the time, T, necessary for the fluid level to drop from h_i to h_f and is

$$T = \frac{2A}{Ca\sqrt{2g}}\left(\sqrt{h_i} - \sqrt{h_f}\right).$$

Solving for C yields

$$C = \frac{2A}{Ta\sqrt{2g}}\left(\sqrt{h_i} - \sqrt{h_f}\right) \quad (4)$$

Since A, g, a, and h_i are constants, these values may be put into expression 4. The coefficient of discharge is left as a function of T and h_f. Filing the tank with water and allowing fluid to drain out of the orifice will yield values for t and h_f that can be used with equation 4 to solve for C.

As an example,of the computation of C for a particular cylindrical tank, consider a standard one pound coffee can with a one-quarter inch hole drilled in the cylindrical wall near the bottom. The diameter of the can was 3.88 inches and the cross sectional area, A, found to be approximately 0.0821 square foot. The diameter of the orifice was actually 0.26 inch and the cross sectional area, a, found to be 0.00037 square foot. The gravitational constant, g, was assumed to be approximately 32.2 feet per second per second. The initial height of the fluid above the center of the orifice, h_i, was chosen to be five inches or 0.4167 foot. Substitution of these values into equation 4 resulted in the following relationship

$$C = \frac{55.305}{t}\left(0.6455 - \sqrt{h_f}\right) .$$

Several trials were conducted with varying values of discharge time, T, and the resulting final height of the fluid above the center of the orifice, h_f. These values were observed and used to calculate C This information was placed in Table 1.

Table 1

Determination of the Coefficient of Discharge

T	h_f	C
5 seconds	0.3385 foot	0.7046
10 seconds	0.28125 foot	0.6371
15 seconds	0.22396 foot	0.6353
20 seconds	0.1667 foot	0.6562

Based on these data the value of the discharge coefficient was chosen to be approximately 0.65.

Student Laboratory Worksheet

The material below the dotted line represents an example of the actual written materials a student might be given in support of a laboratory exercise using the process of fluid discharge from a cylindrical tank.

Discussion

A physical process that involves integration is the removal of fluid in a cylindrical tank through a hole (orifice) in its side. Consider the tank whose side view is depicted in figure 1.

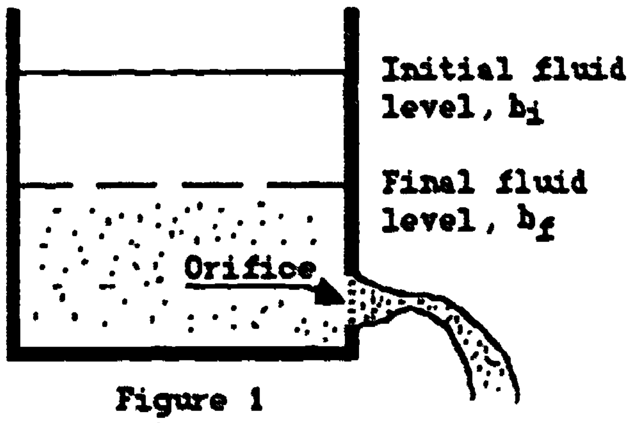

Figure 1
Side View of a Cylindrical Tank

It can be shown that the time, T (measured in seconds), required to drop the fluid level from the initial level, h_i (measured in feet), to the final level, h_f (measured in feet), is given by

$$T = -\frac{A}{Ca\sqrt{2g}} \int_{h_i}^{h_f} \frac{dh}{\sqrt{h}}$$

where A is the cross sectional area of the cylindrical tank (measured in square feet), a is the cross sectional area of the orifice (measured in square feet), g is the acceleration due to gravity (i.e. the gravitational constant and measured in feet per second per second) and C is the coefficient of discharge. The coefficient of discharge varies with orifice geometry and depth of the orifice in the fluid. The depth component may be ignored in this application as insignificant if the change in fluid level is small compared to the actual fluid levels. The coefficient of discharge for the cylindrical tank and orifice you will use is 0.65.

Using the ruler provided determine A and a in square feet. For the gravitational constant, g, use 32.2 feet per second squared. Fill the tank with water to a point, h_f, five inches (0.4167 foot) above the center of the orifice. Open the orifice, and allow the water to flow until the level reaches a point four inches (0.3333 foot) above the center of the orifice. Record the time necessary for this to occur in the table below. Refill the tank and repeat this exercise, using other values for h_i, two more times and record your results below. Using your knowledge of calculus solve the integral given above for T in terms of h_f. Calculate the theoretical values of T for the particular values of h_f observed and place these values in the table. Calculate the percent error between the observed and predicted times and discuss possible reasons for this error.

Experimental Observations and Theoretical Predictions

Final Fluid Level	Observed Time	Predicted Time	Percent Error

Discussion of Possible Error Sources.