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ABSTRACT

This paper presents a laboratory exercise in which an integration problem is applied to cinematography, without the need for apparatus. The problem situation is about the oscillation control of a camera platform to attain the contrast angular rate of objects. Wave equations for describing the oscillations are presented and an expression for calculating the position error is derived. For solving the equations, an approximation method is used and the solution is compared with the analytic solution. Wave forms of the equations are illustrated. (YP)

An Application of Calculus to Cinematography

by

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Introduction

Students introduced to the processes of antidifferentiation and integration in first semester calculus often have difficulty keeping a clear picture of the physical significance of the homework exercise solutions. They quickly become proiicient at manipulating the various rules and theorems but lose, with equal speed, any idea of what it is that they are calculating. Presented in this paper is a laboratory exercise without the need for apparatus (i.e. a "dry lab") in which an integration problem is applied to cinematography. Completion of this exercise should help keep the student's concept of integration from drifting to that of an exercise in algebra.

Problem Description

You have been hired to operate a camera in a remake of the great movie western "Wagon Train." During the Indian attack scene you are to operate a camera located at the center of the circled wagons and keep the Chief of the circling (attacking) Indians as close to the center of your field of view as possible. You take your position behind the camera as instructed by the director but, during the long delay until the scene is actually shot, fall asleep. You are awakened by your assistant after the shooting has begun and just as the chief passes through the field of view of the camera. In your surprise you give a high angular rate command to the camera platform in an effort to catch up with the Chief. The chief moves with a constant angular rate and it is your decision to match his rate as soon as you can and then hold steady at that rate. You thus have no means of correcting for the fact that the center of the camera is not pointed at the Chief (this is called a position or pointing error). You struggle with the camera platform controls to attain the constant angular rate of the chief.

An example of the oscillations typical of the platform rate as the attempt is made to match the rate of the Chief is shown in figure 1. Eventually your rate will be the same as the Chief's. At this time the position error will no longer change, as shown in figure 2. We wish to know your position or pointing error after you have achieved the desired rate.

Problem Analysis

It is assumed that the Chief has a constant angular rate given by ω_c deciradians per second. The camera platform rate is assumed to be a decaying oscillation of the form given by equation 1 and graphed in figure 1:

(1) $\dot{\theta} = \omega_c + Ae^{-\sigma t}\cos(2\pi f_n t - \alpha)$ deciradians/second where $t \geq 0$ and α is chosen such that $\dot{\theta}(0) = 0$ since the platform operator was initially asleep and thus the camera was stationary. Thus

$$\dot{\theta}(0) = 0 = \omega_c + A\cos(-\alpha) \text{ or}$$

$$-\frac{\omega_c}{A} = \cos(\alpha) \text{ and}$$

$$\alpha = \cos^{-1}\left(-\frac{\omega_c}{A}\right) = \tan^{-1}\left(\frac{-\sqrt{A^2 - \omega_c^2}}{\omega_c}\right) + \pi.$$

Note that for α to be real it is required that $\omega_c \leq A$. The position error can be found by evaluating the following integral:

$$\text{Error}(t) = E(t) = \int_0^t (\dot{\theta}(\tau) - \omega_c)\,d\tau$$

$$= \int_0^t Ae^{-\sigma t}\cos(\omega_n \tau - \alpha)\,d\tau$$

$$= Ae^{-\frac{\sigma\alpha}{\omega_n}}\int_0^t e^{-\sigma(\tau - \frac{\alpha}{\omega_n})}\cos\omega_n(\tau - \frac{\alpha}{\omega_n})\,d\tau$$

With the substitution $x = \tau - \frac{\alpha}{\omega_n}$, the expression for Error(t) becomes the following:

$$E(t) = Ae^{-\frac{\sigma\alpha}{\omega_n}}\int_{\frac{\alpha}{\omega_n}}^{t-\frac{\alpha}{\omega_n}} e^{-\sigma x}\cos\omega_n x\,dx.$$

This integral can be evaluated following reference to a standard set of integral tables

$$E(t) = -A\left(\frac{\sigma\cos\omega_n x - \omega_n \sin\omega_n x}{\sigma^2 + \omega_o^2}\right) e^{-\sigma(x + \frac{\alpha}{\omega_n})} \Bigg|_{-\frac{\alpha}{\omega_n}}^{t-\frac{\alpha}{\omega_n}}$$

If $\sigma = B\cos\beta$ and $\omega_n = B\sin\beta$ then

$$\sigma^2 + \omega_n^2 = B^2 \text{ and } \beta = \tan^{-1}\left(\frac{\omega_n}{\sigma}\right).$$

Consequently $\left(\frac{\sigma\cos\omega_n x - \omega_n \sin\omega_n x}{\sigma^2 + \omega_n^2}\right) = \frac{\cos(\omega_n x + \beta)}{\sqrt{\sigma^2 + \omega_n^2}}$

and thus

$$E(t) = -\frac{A}{\sqrt{\sigma^2 + \omega_n^2}} e^{-\sigma(x + \frac{\alpha}{\omega_n})} \cos(\omega_n x + \beta) \Bigg|_{-\frac{\alpha}{\omega_n}}^{t-\frac{\alpha}{\omega_n}}$$

and finally, the error, expressed in deciradians, is given by equation 2 as follows and an example plotted in figure 2:

$$(2) \quad E(t) = -\frac{A}{\sqrt{\sigma^2 + \omega_n^2}}\left(\cos(\beta - \alpha) - e^{-\sigma t}\cos(\omega_n \tau - (\alpha - \beta))\right)$$

$$\text{where } \alpha = \tan^{-1}\left(\frac{-\sqrt{A^2 - \omega_c^2}}{\omega_c}\right) + \pi \text{ and } \beta = \tan^{-1}\left(\frac{\omega_n}{\sigma}\right).$$

The following quantities can now easily be computed:

i) The Chief's angular rate -- a constant ω_c deciradians per second.

ii) The Chief's angular position -- $\omega_c t$ deciradians

iii) The angular rate of the camera platform -- given by equation 1.

iv) The angular position error of the camera platform -- given by equation 2.

v) The angular position of the camera -- given by $\omega_c t$ + the platform angular position error.

Each of these quantities can be plotted if the system parameters are specified. For example suppose: ω_c is 2 deciradians per second, A is 20 deciradians per second, f_n is 2 Hertz, and σ is 2 second^{-1}. In figure 1 are plotted the angular rates of the Chief and platform. The platform position error is plotted in figure 2. The steady state value for this error is 1.52 deciradians or 17.42 degrees. The angular position as a function of time for the camera platform and Chief are plotted in figure 3.

Solution Techniques

Students in first semester calculus are not prepared to come up with the analytic solution presented in the last section but they can find a reasonable approximation for the solution numerically. The position error is found by evaluating the following integral

$$E(t) = \int_0^t (\dot{\theta}(\tau) - \omega_c)\,d\tau$$

and this can be approximated by summing the "signed areas" between the angular rate of the platform and the Chief. The "signed area" is positive when the platform rate exceeds the

Chief's rate and negative if the reverse is true. We can approximate these signed areas by plotting the two rates using the same axes on quadruled paper (as done in figure 1) and counting the number of squares in each region between crossing points of the two rates. The results of performing this operation on the curves of figure 1 that correspond to the example discussed in the last section are displayed in table 1.

Table 1

Signed Areas for the Nine Regions between the Chief's Rate and Platform Rate as Displayed in Figure 1

Region Number	Number of small Squares
1	+ 375
2	− 212
3	+ 135
4	− 83
5	+ 52
6	− 28
7	+ 19
8	− 10
9	+ 5

The algebraic sum of the square counts is + 253. The height of each small square is .25 deciradian/second. The length of each small square is 0.025 second. Thus the area of a small square is .00625 deciradians. The net signed area between the curves and thus the approximate value of the integral is 0.00625 * 253 deciradians = 1.58 deciradians = 18.1 degrees. This is a reasonable approximation for the exact result, 17.42 degrees, calculated in the previous section.

Giving the students an opportunity to visualize this problem and solve it by counting the squares in the signed areas in a group setting as a dry laboratory activity should significantly increase their understanding of the process of integration.

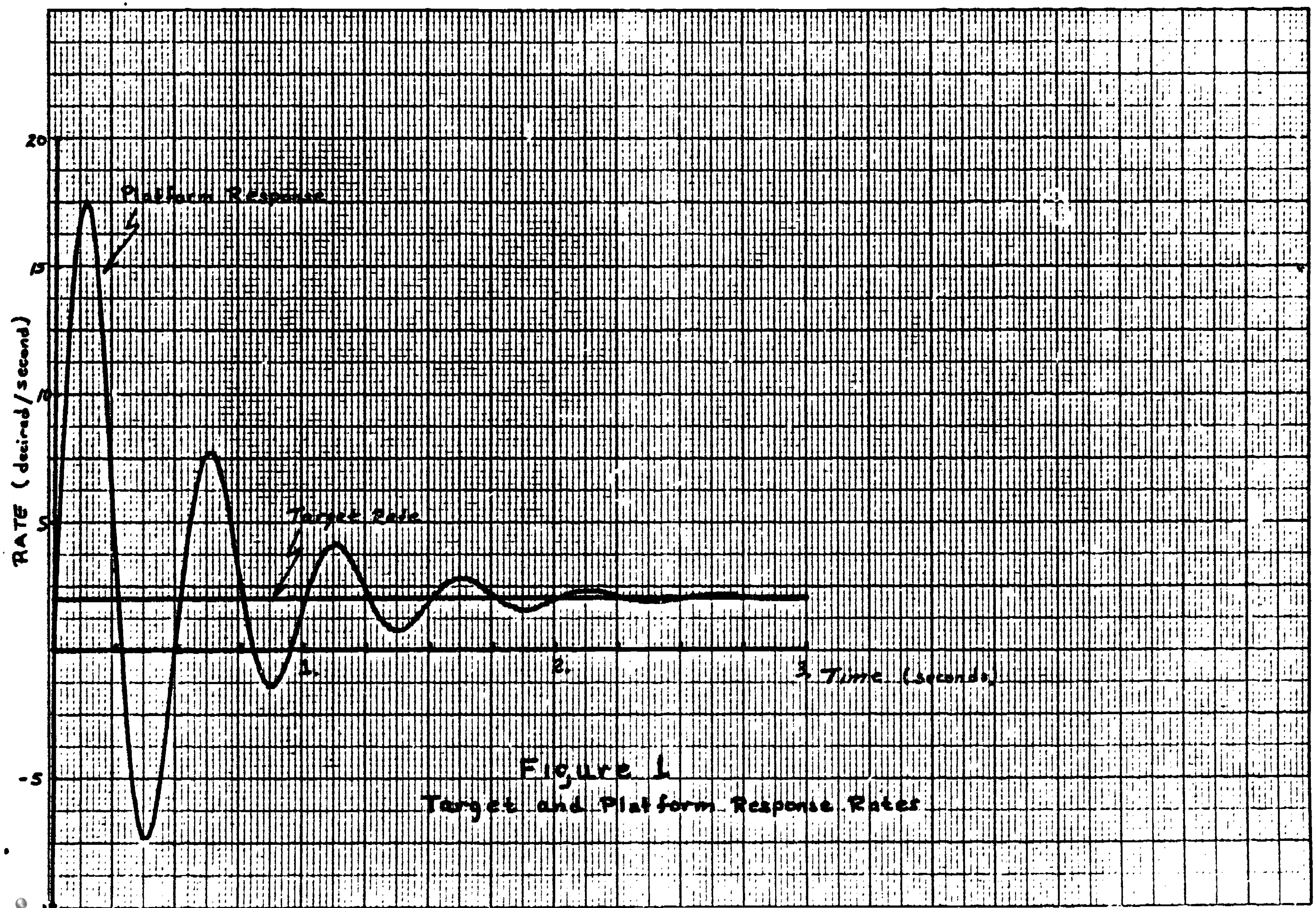

Figure 1
Target and Platform Response Rates

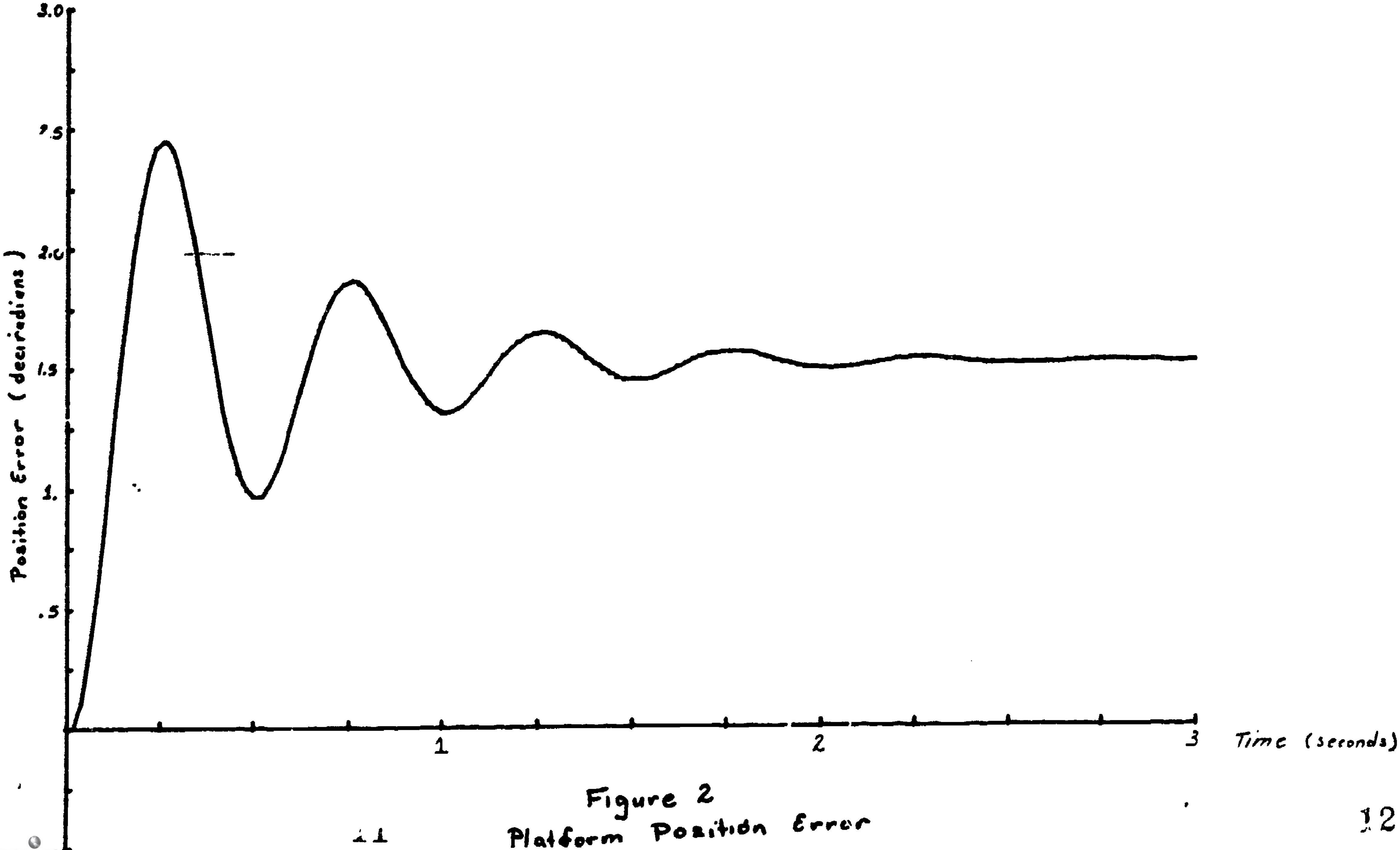

Figure 2
Platform Position Error

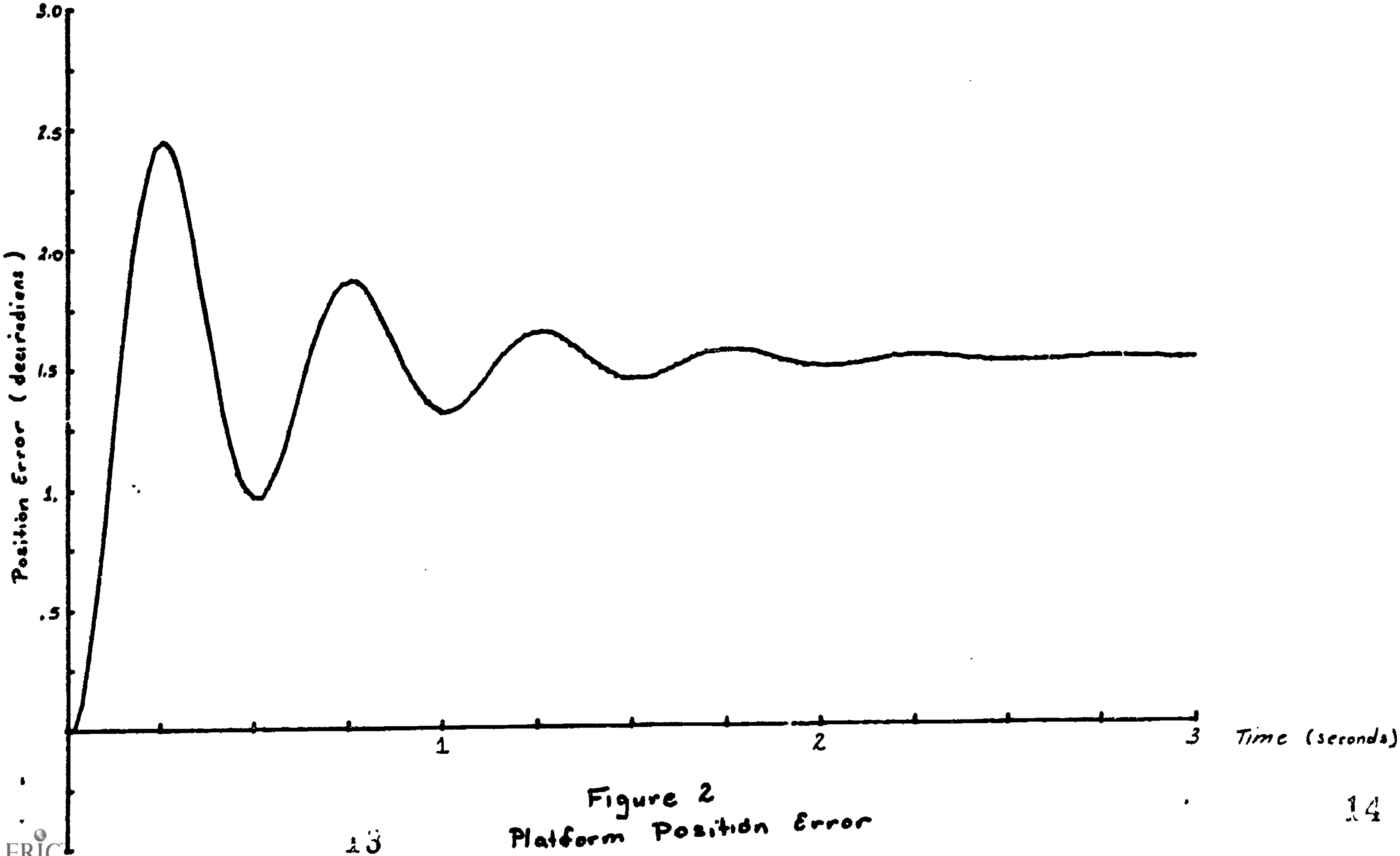

Figure 2
Platform Position Error