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ABSTRACT

This project was designed to: (1) assess 7th-graders' metacognitive beliefs and processes and investigate how they affect problem-solving behaviors; and (2) explore the extent to which these students can be taught to be more strategic and aware of their own problem-solving behaviors. The primary assessment was conducted by analyzing video tapes of individual students and pairs of students working on multi-step problems and non-standard problems, and subsequently being interviewed about their performance. The interview probed the students' mathematical knowledge, strategies, decisions, beliefs, and affects. A variety of written protocol data, including pre- and post-tests, homework assignments, and in-class assignments were collected. The instruction was presented by one of the investigators to a regular-level and an advanced-level 7th-grade class about 3 days per week for a period of 12 weeks. Among the four categories of the cognitive-metacognitive framework, which included orientation, organization, execution, and verification, the orientation category had the most important effect on students' problem-solving. Finally, it appeared that instruction was most likely to be effective when it occurs over a prolonged period of time and within the context of regular day-to-day mathematics instruction. (Author/YP)

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A Study of Two Grade Seven Classes

Final Report

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Report Prepared by

Frank K. Lester, Jr.
Project Director

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Co-Principal Investigator

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We are deeply indebted to the many people who assisted us in this study from its initial conceptualization through the completion of this report. The study surely would not have been completed without their very able help. We wish to recognize the contribution of several individuals.

First and foremost, Barbara Willsey, the regular mathematics teacher of the two seventh grade classes that participated in the study, bent over backwards to make our work with her classes productive and enjoyable. Moreover, she provided us with insights about her students that would have been impossible for us to develop otherwise. Barbara is a real pro! We are certain that we benefited at least as much from our collaboration with her as she did from us.

We wish also to recognize the support given to us by Dale Glenn, Principal of Batchelor Middle School. His belief in our project gave us additional confidence that what we were studying was both timely and relevant.

The teachers at Batchelor deserve our thanks for allowing us to share lunch with them in the teachers' lounge. Teachers have all too little time to themselves during the school day. We were made to feel welcome at all times and for this we are grateful.

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Finally, this report is dedicated to the students in Barbara Willsey's fifth and sixth period mathematics classes. Their cooperation and willingness to assist us made our work a pleasure. We sincerely hope that they profited as much from the problem-solving project as we did.

FKL, JG, DLK
June 1989

ABSTRACT

This project was designed to study the role of metacognition (i.e., the knowledge and control of cognition) in seventh-graders' mathematical problem solving. More specifically, it was designed to: (1) assess seventh-graders' metacognitive beliefs and processes and investigate how they affect problem-solving behaviors, and (2) explore the extent to which these students can be taught to be more strategic and self-aware of their problem-solving behaviors. During the course of the study the assessment phases, both before and after instruction, were expanded in scope to include an investigation of students': (1) awareness and utilization of mathematical resources, (2) control and strategic decision-making processes, (3) beliefs and attitudes relevant to mathematics, and (4) beliefs, attitudes, and emotions relevant to their own mathematical performance. The primary assessment was conducted by analyzing video tapes of individual students and pairs of students working on multi-step problems (i.e., problems whose solutions require the application of two or more of the basic operations) and non-standard problems (i.e., problems that cannot be solved solely by the direct application of the basic operations) and subsequently being interviewed about their performance. The interviews probed the students' mathematical knowledge, strategies, decisions, beliefs, and affects. The analysis of the video-tape data provided us with a picture of students' mathematical performance, both before and after instruction, in sufficient scope and detail to enable us not only to better understand their mathematical cognition, but also to evaluate the effectiveness of the instruction by looking at specific changes in various aspects of mathematical performance. In addition to the video-tape data, we collected a variety of written protocol data, including pre- and post-tests, homework assignments, and in-class assignments.

The instruction was presented by one of the investigators to a regular-level and an advanced-level seventh-grade class about three days per week for a period of 12 weeks. The instruction consisted of three concurrent components: the teacher as external monitor, the teacher as facilitator of students' metacognitive development, and the teacher as a model of a metacognitively aware problem solver. The instructional components included many of the features of previous research on problem-solving instruction.

Results indicate that metacognitive decisions associated with each of four categories of our cognitive-metacognitive framework (viz., orientation, organization, execution, and verification) can be identified as contributing to students' success or non-success during problem solving. Moreover, the orientation category stands out as being the most important. Finally, it appears that instruction is most likely to be effective when it occurs over a prolonged period of time and within the context of regular day-to-day mathematics instruction (as opposed to being a special unit added to the mathematics program).

TABLE OF CONTENTS

LISTS OF TABLES AND FIGURES

List of Tables

List of Figures

Chapter 1

THE PROBLEM, BACKGROUND AND RATIONALE

THE PROBLEM

For generations, mathematics teachers have voiced concern about the inability of their students to solve any but the most routine verbal problems despite the fact that they seem to have mastered all the requisite computational skills and algorithmic procedures. Until recently, researchers have been content to attribute problem-solving difficulties almost exclusively to cognitive aspects of performance. However, there has been growing sentiment for the notion that a much broader conception is needed of what mathematical problem solving involves and what factors influence performance.

The rather elusive construct referred to as metacognition is among the factors that are currently considered to be closely linked to problem solving. Briefly, metacognition refers to the knowledge and control one has of one's cognitive functioning -- that is, what one knows about one's cognitive performance and how one regulates one's cognitive actions during the performance of some task. Metacognitive knowledge about one's mathematical performance includes knowing about one's strengths, weaknesses, and processes, together with an awareness of one's repertoire of tactics and strategies and how these can enhance performance. Knowledge or beliefs about mathematics that can affect one's performance are also considered metacognitive in nature. The control and regulation aspect of metacognition has to do with the decisions one makes concerning when, why, and how one should explore a problem, plan a course of action, monitor one's actions, and evaluate one's progress, plans, actions, and results. This self-regulation is influenced by one's metacognitive knowledge.

The research discussed in this report had two underlying goals and corresponding questions.

Goal I: To determine the influence of metacognition on the cognitive processes students use during mathematical problem solving.

Research Question I. What metacognitive behaviors do grade seven students exhibit when they attempt to solve certain types of mathematics problems and how do these metacognitive behaviors interact with the students' cognitive behaviors?

Goal II: To investigate the effectiveness of instruction designed to increase students' cognitive self-awareness and ability to monitor and evaluate their own cognitive performance.

Research Question II. What are the effects on students' problem-solving behavior of instruction that involves: practice in the use of strategies, training students to be more aware of the strategies and procedures they use to solve problems, and training students to monitor and evaluate their actions during problems solving?

Thus, the purposes of this study were to investigate the role of metacognition in mathematical problem solving among middle school students and to explore the extent to which they can be taught to be more self-aware of their problem-solving behaviors and to monitor and evaluate these behaviors. Data related to these research questions were gathered by means of pre- and post-instruction written problem-solving tests, pre- and post-instruction clinical interviews with selected students about various aspects of solving different types of problems, observations of them as they attempted to solve problems, analysis of their written classwork and homework, and observations of classroom instruction.

BACKGROUND AND RATIONALE FOR THE STUDY

All mathematics educators agree that problem solving is a very important, if not the most important, goal of mathematics instruction, and this view has been widespread for some time. Indeed, more than a decade ago the National Council of Supervisors of Mathematics (NCSM) stated that "Learning to solve problems is the principal reason for studying mathematics" (NCSM, 1977, p. 20). A few years later, the National Council of Teachers of Mathematics (NCTM), in its *Agenda for Action,* listed as its first recommendation that "problem solving must be the focus of school mathematics in the 1980's" (NCTM, 1980, p. 1). At about this same time, data from the NCTM's *Priorities in School Mathematics Project* (PRISM), a survey of the beliefs and reactions of mathematics teachers, mathematics educators, and certain lay educational professionals concerning mathematics curricula, indicated widespread support for the Agenda's recommendation on problem solving. The PRISM summary report states that "problem solving was consistently ranked high in priority for increased emphasis" (NCTM, 1981, p. 29). Most recently, NCTM, in its *Curriculum Standards for School Mathematics*, has reinforced its support for the importance of a problem-solving oriented mathematics curriculum by stating that "Problem solving should be the central focus of the mathematics curriculum" (NCTM, 1989, p. 23).

In addition to these reports, numerous journal articles and conference presentations have been devoted to calling attention to the importance of problem solving in school mathematics curricula. Moreover, in the past two decades research on problem solving has been one of the most popular areas of research in mathematics education (Lester, 1980, 1983, 1985). Unfortunately, despite all of the professional enthusiasm, expertise in problem solving remains the most difficult aspect of mathematical performance for students to develop. This lack of expertise is apparent at all levels of schooling. The Third National Assessment of Educational Progress (NAEP) revealed that although 9-year-old and 13-year-old students are fairly successful at solving simple routine one-step problems, they experience great difficulty with multi-step and non-routine problems (Lindquist, Carpenter, Silver and Matthews, 1983). The NAEP data also show that secondary students have trouble with non-routine problems (Carpenter, Lindquist, Matthews and Silver, 1983). Furthermore, there are a number of studies which show that many college students, even some majoring in mathematically demanding subjects, also have difficulty with various aspects of problem solving (Clement, 1982; Schoenfeld, 1985).

There are at least two reasons why problem-solving competence is so difficult for students to develop. First, problem solving is a complex cognitive activity. It requires much more than just the direct application of some mathematical content knowledge. Successful problem solving requires one to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine. Moreover, these cognitive actions are influenced by a number of non-cognitive factors. A number of mathematics education researchers have begun to realize that purely cognitive explanations of problem-solving behavior are insufficient since they do not encompass the various guiding forces that are involved (e.g., Lesh, 1982; Schoenfeld, 1985, 1987; Silver, 1982).

A second reason why so many students have trouble becoming proficient problem solvers is the fact that they are not given appropriate opportunities to do so. Since problem solving is so complex, students need to be given carefully designed problem-solving instruction and they need to be given extensive problem-solving experiences. Many do not develop expertise in problem solving because they are neither guided nor challenged to do so. Each of these reasons will be discussed briefly.

The Complex Nature of Problem Solving

Successful problem solving depends on both cognitive and non-cognitive factors. These factors can be placed in five broad, interdependent categories: knowledge, control, affects, beliefs, and contextual factors (Lester, 1987; Lester, Garofalo, and Kroll, 1989). A few words will be said about each of these factors.

Knowledge

This category consists of a wide range of resources that a problem solver can bring to bear on problems to be solved (cf., Schoenfeld, 1985). Included here are definitions (e.g., a prime number is a whole number with exactly two factors.), facts (e.g., 8 x 7 = 56), algorithms (e.g., long division), heuristics (e.g., looking for patterns), problem schemas (i.e., packages of information about problem types), and a host of routine, non-algorithmic procedures (e.g., techniques of integration). Of particular importance is the way individuals represent, organize, and utilize their knowledge (Silver, 1982). It is clear that many problem-solving deficiencies can be attributed to lack of knowledge or to "unstable conceptual systems" (Lesh, 1985).

Control

Control has to do with the regulation of one's behavior during problem solving. In particular, control has to do with the decisions and actions undertaken in analyzing and exploring problem conditions, planning courses of action, selecting and organizing strategies, monitoring actions and progress, checking outcomes and results, evaluating plans and strategies, revising or abandoning unproductive plans and strategies, and reflecting upon all decisions made and actions taken during the course of working on a problem (cf., Garofalo and Lester, 1985; Schoenfeld, 1985, 1987). Again, it is clear that lack of control can have disastrous effects on problem-solving performance.

The processes used to regulate one's behavior are often referred to as metacognitive processes, and these have recently become the focus of much attention within the mathematics education community. Metacognition is discussed in more detail in a later section of this report.

Beliefs

Schoenfeld (1985) refers to beliefs, or "belief systems," as the individuals' mathematical world view, that is, "the perspective with which one approaches mathematics and mathematical tasks" (p. 45). This "mathematical world view" includes beliefs about: mathematics, mathematical tasks, oneself as a doer of mathematics, and the environment or context within which one is doing mathematics. These last two classes of beliefs are tied to the first two and are often included in general discussions of belief systems, but for the sake of our categorization they might best be put into the next two categories respectively. Hence, this category might consist primarily of beliefs about mathematics and the nature of mathematical tasks. Examples of such beliefs include: all verbal problems can be solved by the direct application of one or more arithmetic operations (Lester and Garofalo, 1982), and mathematics problems are always solved in less than 10 minutes, if they are solved at all (Schoenfeld, 1985).

Affects

The affects that are believed to influence problem-solving performance range from emotions, or "hot" affects (e.g., anxiety, fear, frustration, and joy), to attitudes, or "cold" affects (e.g., confidence, interest, and motivation). Also included in this category are beliefs about oneself as a doer of mathematics. Much of the research in this category has been concerned with attitudes and has been primarily correlational. Attitudes such as willingness to take risks, resistance to premature closure, as well as confidence, interest and motivation have been shown to be related to mathematical performance (see Lester, Garofalo, and Kroll, 1989). Emotions, which involve more of a gut reaction than do attitudes, can have either a facilitating or debilitating effect on the individual, but negative emotions (e.g., frustration) are not necessarily debilitating nor are positive emotions (e.g., joy) necessarily facilitating. There is a growing body of research to support the notion that emotions and cognitive actions interact in important ways (Mandler, 1989). Most of this research has been restricted to the study of the conditions under which certain emotions occur or to the nature of an individual's behavior when in a particular emotional state. In the past few years, the role of affect in mathematical performance has been receiving an increasing amount of attention from the mathematics education research community (McLeod & Adams, 1989).

Contextual Factors

In recent years, the point has been raised within the cognitive psychology community that human intellectual behavior must be studied in the context in which it takes place (Brown, Collins, & Duguid, 1989; Neisser, 1976; Norman, 1981). That is to say, since human beings are immersed in a reality that both affects and is affected by human behavior, it is essential to consider the ways in which socio-cultural factors influence cognition. In particular the development, understanding, and use of mathematical ideas and techniques grow out of social and cultural situations. D'Ambrosio (1985) and Baroody (1987), for example, argue that children bring to school their own mathematical intuitions and informal procedures that have developed within their own socio-cultural environments. Furthermore, one need not look outside the school for evidence of social and cultural conditions that influence mathematical behavior. The interactions that students have among themselves and with their teachers, as well as the values and expectations that are nurtured in school, shape not only what mathematics is learned, but also how it is learned and how it is perceived (cf., Cobb, 1986). The point here is that the development of mathematical knowledge, the decisions to use control strategies, and the development of beliefs and attitudes, are all influenced strongly by the nature of the context in which they take place. These five categories overlap (e.g., it is not possible to completely separate affects, beliefs, and socio-cultural considerations), and interact in a variety of ways too numerous and complex to describe here (e.g., beliefs influence affect, and they both influence knowledge utilization and control; contextual factors impact on all of the other categories).

Problem-Solving Instruction

For students struggling to become competent problem solvers, the difficulty due to the complexity of problem solving is compounded by the fact that most of them do not receive adequate instruction, either in quality or quantity. Since problem solving is so complex, it is difficult to teach. We are not yet at the stage where we have a fool-proof, easily followed and implemented method of helping students to become good problem solvers. Because of this, and because many mathematics teachers have received little or no systematic training in problem solving when they were students, or when they were training to become teachers, not enough problem-solving instruction is taking place in today's classrooms. As mathematics educators very concerned with this state of affairs, we are interested in researching problem solving and in developing instructional methods and materials that will be useful to teachers. We see the need to use such developments in the training of pre-service teachers both to help them improve their own problem-solving performance and to help them learn and develop their capacity to help their students do the same.

In recent years there has been much research conducted on various approaches to mathematical problem-solving instruction, and here we give only a secondary overview of the associated literature. More detailed reviews can be found in Kilpatrick (1985) and Lester (1980, 1983, 1985). Further discussion of problem-solving instruction can be found in Silver (1985) and Schoenfeld (1987). Lester (1985) classified the instructional problem-solving research according to the kinds of skills or strategies that were taught. His four categories include: (1) instruction to develop master thinking strategies (e.g., creativity training), (2) instruction in specific tool skills (e.g., make a table), (3) instruction in specific heuristics (e.g., working backwards), and (4) instruction in the use of general heuristics (e.g., planning). None of these approaches has been shown to be substantially superior to the others. We believe that for problem-solving instruction to be truly effective it must involve some optimal combination of all of these skills. Such instruction will also necessitate students being exposed to a wide variety of problem types over a prolonged period of time. Furthermore, the instructional experience would have to be planned to address all of the factors discussed in the previous section.

A somewhat different categorization of the instructional research literature was given by Kilpatrick (1985). He categorized the research according to perspective or approach to instruction. Kilpatrick's five categories include: *osmosis*, *memorization*, *imitation*, *cooperation*, and *reflection*. The *osmosis* category includes methods of problem-solving instruction that advocate giving students lots of problems to solve, with the assumption that by working through many problems students will pick up and develop problem-solving techniques and strategies. As Kilpatrick notes, this is a necessary, but perhaps not sufficient, condition. The *memorization* category includes approaches that have students learn specific algorithms and heuristics to then be applied to classes of problems which can be solved by the memorized techniques. The obvious problem with this approach is that the methods can only be applied to certain types of problems, and when students are faced with different types of problems, they will be unprepared to deal with them effectively if at all. Some training in decision-making about the applicability of techniques is also needed. The *imitation* category includes approaches that have students compare their problem-solving attempts with those of a model student or teacher, with the hope that the student can pick up or adopt some of the techniques or attitudes of the model. The *cooperation* category includes approaches that use pair problem solving or group problem-solving sessions. The hope for such approaches is that these sessions will give students opportunities to share their ideas with each other, and will allow them to practice, discuss, and refine their skills. For this method to be successful, the right classroom and group atmosphere must be fostered. Kilpatrick's final category, *reflection*, includes approaches that have students reflect upon their problem-solving processes. Such approaches have students assess the effectiveness of their problem-solving procedures and actions, and reflect over the whole course of their problem-solving activity. Students often have difficulty with this type of approach at first because they are not accustomed to such activities. Kilpatrick (1985), points out, as does Lester, that, overall, the research shows that problem solving is learned "slowly and with difficulty" (p. 8).

There are a number of methodological, conceptual, and reporting problems with research on the teaching of mathematical problem solving. Silver (1985) identified four characteristics of the research that he found particularly disturbing. First, the research reports rarely described what the teacher actually did in the classroom when teaching problem solving. Second, the "teacher variable" was often too controlled. Third, the direct influence of the instruction on students' problem-solving behaviors was insufficiently assessed. And, fourth, many of the research studies were not guided by any theory of instruction.

A Cognitive-Metacognitive Framework for Mathematical Performance

The research discussed in this report is a continuation of work begun by us a number of years ago involving the role of metacognition in mathematical problem solving by young children (Garofalo & Lester, 1985; Lester & Garofalo, 1982). An important focus of this work was the development of a framework to guide the study of mathematical problem solving that incorporates both cognitive and metacognitive components.

The development of our framework began with a consideration of the efforts of others. The prototype which, until recently, most mathematical problem-solving research has been based upon is Polya's (1957) four-phase description of problem-solving activity. The four phases -- *understanding*, *planning*, *carrying out the plan*, and *looking back* -- serve as a means for identifying a multitude of heuristic processes that may foster successful problem solving. Unfortunately, his conceptualization considers metacognitive processes only implicitly.

We found that frameworks for analyzing particular kinds of mathematical performance have been devised by various researchers, most notably by Schoenfeld (1983, 1985). Schoenfeld devised a scheme for parsing protocols into episodes and executive decision points primarily for the purpose of analyzing problem-solving moves. Episodes include reading, analysis, exploration, planning, implementation, and verification. According to Schoenfeld, it is at the transition points between these episodes (as well as at other places) where metacognitive decisions, especially managerial ones, can have powerful effects upon solution attempts. Two years later Schoenfeld (1985) identified three qualitatively different levels of knowledge and behavior that he believed should be considered if an accurate picture of an individual's problem-solving performance is to be obtained. These levels include: (1) resources (i.e., knowledge that the individual can bring to bear on a particular problem); (2) control (i.e., knowledge that guides the individual's selection and implementation of resources); and (3) belief systems (i.e., perceptions about self, the environment, the topic, or mathematics that may influence an individual's behavior).

Recently, Kroll (1988) has extended Schoenfeld's scheme to account for monitoring moves and roles played by individuals during cooperative problem-solving situations. In particular, she categorized monitoring moves by statement type (i.e., type of comment made by one of the individuals working cooperatively on a problem) and by problem-solving function (i.e., orientation, organization, implementing, and verification). She specified four basic types of statements: self and partner reflections, and procedure and state assessments. Kroll defined reflections as verbal indications that a metacognitive decision had taken place and assessments as verbal indications that metacognitive regulation had taken (or was taking) place. Unfortunately, Kroll's scheme was developed after much of the analysis of the interview data had been completed.

A vital ingredient of any framework or scheme for analyzing mathematical performance is that it must allow for a very wide range of possible behaviors -- cognitive or otherwise. In particular, a framework should highlight aspects of the individual's performance where metacognitive actions are likely to be present or conspicuously absent. The work of Schoenfeld and of Kroll are big steps in this direction.

Our framework, which we refer to as a cognitive-metacognitive framework, is based primarily on Polya's model and Schoenfeld's scheme, but its development was also influenced by the work of Sternberg (1980, 1982) and of Luria (1973). This framework is directly relevant to performance on a wide range of mathematical tasks, not only to tasks classified as "problems." It is not a list of all possible cognitive and metacognitive behaviors that might occur; rather it specifies key points where metacognitive decisions are likely to influence cognitive actions.

The framework is comprised of four categories involved in performing a mathematical task: *orientation, organization, execution,* and *verification*. Table 1 provides a description of each category. It should be pointed out that the four categories are related to, but are more broadly defined than, Polya's four phases.

This framework served as a tool for analyzing metacognitive aspects of the seventh graders' problem-solving performance. More specifically, it was used in the selection of research tasks (for clinical interviews and written tests), the design of interview procedures, and the development of the instructional treatment. It also served as a means for organizing analyses and interpreting findings. Table 1 lists the phases of our framework together with the key points associated with each phase.

Table 1

A Cognitive-Metacognitive Framework for Studying Mathematical Performance
(Garofalo & Lester, 1985)

ORIENTATION: Strategic behavior to assess and understand a problem

A. Comprehension strategies
B. Analysis of information and conditions
C. Assessment of familiarity with task
D. Initial and subsequent representation
E. Assessment of level of difficulty and chances of success

ORGANIZATION: Planning of behavior and choice of actions

A. Identification of goals and subgoals
B. Global planning
C. Local planning (to implement global plans)

EXECUTION: Regulation of behavior to conform to plans

A. Performance of local actions
B. Monitoring of progress of local and global plans
C. Trade-off decisions (e.g., speed vs. accuracy, degree of elegance)

VERIFICATION: Evaluation of decisions made and outcomes of executed plans

A. Evaluation of *orientation* and *organization*
   1. Adequacy of representation
   2. Adequacy of organizational decisions
   3. Consistency of local plans with global plans
   4. Consistency of global plans with goals

B. Evaluation of *execution*
   1. Adequacy of performance of actions
   2. Consistency of actions with plans
   3. Consistency of local results with plans and problem conditions
   4. Consistency of final results with problem conditions

Chapter 2

PROCEDURES AND RESEARCH PLAN

The study consisted of four phases:

Phase 1: Design, collect, and pilot test suitable mathematics problems

Phase 2: Conduct clinical interviews, observe student problem-solving sessions, and administer written problem-solving tests.

Phase 3: Develop and present the instructional treatment.

Phase 4: Analyses of students' performance and of effectiveness of instruction.

PHASE 1: COLLECTION AND PILOT TESTING OF PROBLEMS

In January 1986, phase 1 was begun. This phase involved designing, developing and pilot-testing suitable mathematics problems to be used with the written problem-solving tests, the interviews, and the instructional treatments. The first step in this phase was to determine a categorization scheme for problems. Eventually, a scheme having nine categories was decided upon. Our plan was to identify problems that represented a wide range across all nine categories. The search for problems involved looking through problem books, elementary school textbooks, and books for elementary teachers, as well as acquiring copies of problem sets that various mathematics education faculty and graduate students had collected. As suitable problems were identified they were put on a computer database. The database was created to facilitate access to problems having characteristics specified by the nine categories. The resulting problem bank contained about 150 problems (initially there was some duplication of problems caused by the fact that three persons worked somewhat independently to compile the set of problems) each of which was considered appropriate for use with grade six students. The problem bank has been expanded considerably since its inception. Presently it contains problems from several grade levels and it has grown to more than three times its original size. A brief description of each of the nine categories comprising the categorizing scheme follows (see Appendix E for additional notes about the database):

(1) *Problem Type* -- included were complex computations. routine single-step story problems, routine multi-step story problems, process problems, and puzzle problems (each of these types is described elsewhere in this report;

(2) *Type of information provided in the problem statement* -- included were problems with insufficient, inconsistent, irrelevant, and strictly sufficient information. The subcategories within this category are not mutually exclusive.

(3) *Level of difficulty* -- three levels of difficulty were identified: easy, medium and hard. In addition, if a calculator would be an appropriate heuristic to solve a problem, the problem was designated as a calculator problem.

(4) *Strategy* -- problems were classified according to whether the following strategies could be used to solve them: guess and test, work backwards, look for a pattern, use equations, use logic, draw a picture, make an organized list, make a table, act it out, make a model, simplify, look for key words, and use resources (books, calculator, etc,).

(5) *Metacognitive phase* -- problems were classified as to the metacognitive phases likely to be tapped during the solution effort: orientation, organization, execution, verification (of orientation/organization or of execution).

(6) *Operations and types of numbers* -- included were the four arithmetic operations (+, -, x, ÷) and whole numbers, decimals, and fractions.

(7) *Mathematical content* -- numerous content are possible, among them are: ratio, money, measurement, logic, geometry, etc.

(8) *Number of solutions* -- problems were categorized according to whether they had no, one, or multiple solutions.

(9) *Isomorphism* -- problems were classified according to their underlying mathematical structure (e.g., 2 conditions and 2 variables). Two problems were regarded as being isomorphic if they had the same mathematical structure.

As work proceeded in the development of the problem bank we began to test the appropriateness of the problems for use with sixth graders. To do this we obtained the consent of a sixth grade teacher at a school that was not to be involved in phases 2 and 3 of the study. Throughout the Spring Semester (January - May 1986) we conducted interviews with students and observed them as they attempted to solve problems. As a result of this pilot testing of problems we were able to reduce the number of problems of interest to us and to identify problems that seemed to have promise for engaging students in a broad range of metacognitive behaviors.

PHASE 2: DEVELOPMENT OF INSTRUMENTS AND INITIAL TESTING AND INTERVIEWING

This phase commenced during the summer months of 1986 and continued until the end of the year. It was during this phase that we learned that it would not be possible to use sixth graders in the study. As we began to make arrangements with the central administration of the Monroe County Community Schools Corporation, the local school district, we found that a new instructional plan was to be implemented with all sixth grade classes during the 1986-87 school year. This plan called for the introduction of two new ideas: (1) tracking of all sixth grade students, and (2) teaching all sixth grade mathematics at the same time of day in each elementary school in the system. This new plan made it impossible for one instructor to teach two mathematics classes. Since it was essential for our purposes that all instruction be provided by the same instructor, the decision was made to switch from sixth to seventh grade. The problems we had chosen for the study seemed more appropriate for use with seventh graders than with fifth graders. Seventh graders were tracked in the system, but they were not all taught at the same time of day within a school. The cooperation of the principal and a seventh grade mathematics teacher at Batchelor Middle School, one of the two middle schools in the system, was obtained during the summer of

1986, barely four months before the first testing and interviewing were to begin.

Once arrangements had been made to conduct the study at Batchelor Middle School we began to decide upon the problems to be included in the written pre- and posttests as well as those to be used with the clinical interviewing. These instruments are described in Chapter 3 and the instruments themselves are included in Appendices A-1, A-2, and A-3. Finally, administration of the written pretest and the initial interviewing and observations took place during November and December 1986.

PHASE 3: INSTRUCTION

Our original plan was to develop two instructional treatments: a metacognition-based treatment and a control treatment. The metacognition-based treatment was to incorporate aspects of Brown and Palincsar's (1982) self-control training, and Charles and Lester's (1982, 1984) teaching strategy for mathematical problem solving. The control treatment was to involve much less than the metacognition-based treatment. In this treatment the teacher was to give students problems to solve, to work out solutions for the class in front of the room, and to answer students' questions. No attempt was to be made to make students aware of the value and significance of certain strategies, to provide systematic practice in the use of particular strategies, or to teach them how to monitor and control their performance. The purpose of including a control class of this type was to test the efficacy of the metacognition-based treatment by looking at pre-treatment and post-treatment performance.

However, we had also considered the idea of working with students of different abilities to be able to look not only at differences in their problem-solving abilities and in their reactions to the metacognition-based treatment, but also at the potential of the approach for use with students of different levels of mathematical ability. As it turned out, we were offered the opportunity to work with a "regular" and an "advanced" seventh grade class. (We were aware that the method used to track students was imprecise at best. Nevertheless, we suspected that, in general, the two classes would be very different in ability. We found this to be the case.) Consequently, we decided to abandon the notion of having a control class and instead to view the instruction that took place with the advanced class as a replication (with a different population) of the instruction with the regular class. The specific nature of the instruction is described in detail in Chapter 3 (in the section titled, *The Instruction as Implemented*). Instruction in the two seventh grade classes began in January 1987 shortly after students returned from the Christmas holidays and continued for 14 weeks. In May 1987 the written posttest was administered to all students and final clinical interviews and observations were conducted.

PHASE 4: FINAL DATA COLLECTION AND ANALYSIS

The final phase of this study involved the analysis of the data gathered from several sources: the written tests, the clinical interviews, the observations of individual and pair problem-solving sessions, students' class and home work, and video-tapes of classroom instruction. Much of this analysis proved to be very slow and difficult work. In particular, we found it difficult to make sense of the overwhelming amount of data gathered on each instructional session. Indeed, we have only scratched the surface in analyzing these data and expect to continue with additional analyses for some time to come.

Chapter 3

DESCRIPTIONS

DESCRIPTION OF THE SCHOOL AND THE STUDENTS

The students involved in the study were seventh graders in two mathematics classes in Batchelor Middle School, one of two middle schools in the Monroe County Community School Corporation in Bloomington, Indiana. Batchelor has about 800 students in grades 7 and 8, and a faculty of 50 teachers and other professionals (including 7 teachers of mathematics). Students in the school are fairly typical for Bloomington in ability, socioeconomic level, and ethnic background. A sizeable portion of the school population is made up of children of university faculty, while the remainder of the children are either from professional or from blue collar homes. There are relatively few minority students at Batchelor.

Seventh grade mathematics at Batchelor is taught in three tracks: *remedial* (small classes for those who need extra help), *regular* (who use the 7th grade Harbrace mathematics text, *Mathematics Today*), and *advanced* (who use the 8th grade *Addison-Wesley Mathematics* text). The two seventh grade classes involved in the study were a *regular* class (28 students) and an *advanced* class (37 students).

All seventh grade students at Batchelor take three tests, the results of which are used to determine their placement in 8th grade mathematics. These tests are the *Iowa Test of Basic Skills*, the *Stanford Diagnostic Mathematics Test*, and the *Hanna-Orleans Algebra Prognosis Test*. Students from the advanced class usually proceed to study algebra during the 8th grade if their scores on diagnostic tests indicate readiness. Table 2 shows the mean scores and the range of scores for each of the three tests for the two classes of students involved in the study.

A Comparison of the Regular and Advanced Classes

There was a big difference between the regular class (5th period) and the advanced class (6th period). In general, students in the advanced class were not only higher achievers in mathematics than the students in the other class, but they also had better study and work habits and were more lively and attentive during class. It was also the case that the sixth period class had nine more students (37 versus 28). These differences made it necessary to adjust instruction accordingly. For example, because many of the fifth period students seemed to be having difficulty identifying the relevant information in verbal problems, a special activity was designed for them that was not used with the advanced students (see day 7 lesson plan in Appendix B and Appendix C, day 7).

Table 2

Students' Performance on Three Standardized Mathematics Tests

	ITBS[a]	SDMT[b]	Alg. Prog.[c]
Regular Class (N=28)	65 (36-93)	56.5 (31-81)	43.4 (19%-74%)
Advanced Class (N=37)	90 (58-99)	86.4 (44-98)	58.2 (29%-96%)

[a] Iowa Test of Basic Skills
Mean Percentile (range of percentiles)

[b] Stanford Diagnostic Mathematics Test
Mean Percentile (range of percentiles)

[c] Hanna-Orleans Algebra Prognostic Test
Mean Percent (range of percents)

DESCRIPTION OF THE WRITTEN TESTS: PRE- AND POST-INSTRUCTION

Prior to the beginning of phase 3 of the study, written problem-solving tests were administered to all students in the two grade seven classes. An essentially parallel version of this test was administered to all students within a week after the end of the instructional phase. The purpose of these tests was to have a measure of pre-interview and post-interview changes on tasks that the students were likely to view as more school-like than the tasks included in the interviews. The test problems were chosen to include some routine problems like those commonly encountered in school, as well as some non-routine problems like those considered during the instruction of phase 3. The intent was to include some problems that students could not solve simply by means of the direct application of one or more arithmetic operations, and that would therefore require students to engage in strategic decisions and behaviors like those associated with the four categories of the cognitive-metacognitive framework discussed earlier. The tests included a one-step, a two-step, and three process problems (see Appendix A for a list of the problems included on the written tests). As mentioned above, the two tests were designed to be parallel versions of each other in terms of problem structure and difficulty.

Also included on the written tests were questions regarding students' judgments of the ease, familiarity, confidence, and enjoyment associated with each problem (see Appendix A for the set of questions included for each version). These questions were included to give us some further data from which to make pre-interview and post-interview comparisons, and also to give us further information to use in interpreting and explaining performance changes.

DESCRIPTION OF THE INDIVIDUAL AND PAIRED INTERVIEW PROCEDURES

Two weeks after the administration of the written pretest, interview sessions were held with several students in each of the two classes. A similar set of interviews was held with most of the same students shortly after the end of the instructional phase. Some of the interviews were conducted between an interviewer (either J. Garofalo or D. Kroll) and an individual student; others involved an interviewer and a pair of students who were encouraged to work together to solve problems.

Individual Interviews

The pre-instruction individual interviews occurred over two 45-minute sessions, held within a few days of each other. Students were given the Kennedy and Coin problems to solve and discuss during the first session, and the Atlas and <u>Handshake</u> problems during the second session (see Appendix A for a list of all problems used during the interviews). In these sessions, students had the opportunity to view themselves working and discussing their performance on video-tape, and to provide further commentary on their work. These viewing periods also allowed the interviewer to ask for clarification of previous responses and/or ask further questions.

The post-instruction individual interviews occurred during one 45-minute session. Students were asked to solve and discuss the <u>Felipe</u>, <u>Shuttlemeier</u>, <u>Luncheon</u>, and <u>NFL</u> problems. Students worked on and discussed all four problems (three problems in a few cases) during this one session. Students did not have an opportunity to view themselves on video tape during the post instruction interviews.

At each interview session, the student was asked to read the problem aloud first, then to proceed to solve it. The interviewer observed the student's performance and asked questions after the student finished working on the problem or gave up. In a few cases, when a student seemed to misinterpret a problem and get a very quick answer, or when a student got stuck on a problem for an extended period of time, the interviewer engaged the student in a discussion of the problem to get him or her going again.

The interviewer did not follow a script and there was no predetermined list of questions. It was hoped that by avoiding a script the interviews would be more conversation-like than interrogation-like. However, there was a predetermined set of aspects of problem-solving performance to ask about, particularly aspects related to metacognition. More specifically, the interviewers focused their questions on decisions and strategies related to each of the four categories in the cognitive-metacognitive framework.

Being fully aware that a set of routine verbal problems would not elicit the aspects of problem-solving performance of most interest to us, we chose a set of problems which were school-like, but which could not be solved solely by the direct application of one or two arithmetic operations. (The first problem in each set was an exception to this, but was included to ease the student into the interview). Our aim was to include a problem for which an organized list would be helpful, a problem involving quantities and relationships, and a problem whose structure is very different from those to which the students were accustomed. Most of the problem used in the interviews were taken from *Problem Solving Experiences in Mathematics: Grade 7* (Charles, Mason, & Garner, 1985).

Paired Interviews

The pre-instruction paired interviews also occurred over two 45-minute sessions. Students were given the Caravan problem during the first session, and the Susie problem during the second session. The post-instruction paired interviews occurred during one 45-minute session at which the students were given the Waitress and the Jules and Jim problems to solve. As with the pre-instruction interviews, these problems were also chosen for their non-routine nature. (See Appendix A for statements of the interview problems.)

The procedure for these interviews was similar to that of the individual interviews, but students were asked to work together as a team to solve the problems. They were encouraged to explain to their partner what they were doing or what they had done. They were also told that before they settled on an answer or decided to stop work on a problem they both had to be satisfied with what they had done. The reason for including paired interviews was to increase the amount of "thinking aloud" (verbalization) that would occur.

DESCRIPTION OF PROCEDURE FOR OBSERVATION OF INSTRUCTION

With only one exception, all instructional sessions were video- and audiotaped for both classes. (Frank Lester was the instructor for the instructional sessions. Diana Kroll operated video equipment and served as observer in each class.) During the first three weeks two video cameras were used, one in the front corner (near the exit), the other in the left rear corner of the room (see Figure 1). The camera in the rear of the room was focused on the instructor and followed him as he moved around the room. The other camera was focused with a view across the room from the left front to the right rear in such a way that most of the students were view. After the third week we decided that two cameras were not needed and that using two cameras simply caused unnecessary work for the observer who was responsible for operating them. Also, in order to pick up conversations between individual students or small groups of students and the instructor, the instructor wore a lavaliere microphone attached to an audiocassette tape recorder which was worn on his belt. The video and audio tapes were a primary source of data on the effectiveness of the instruction.

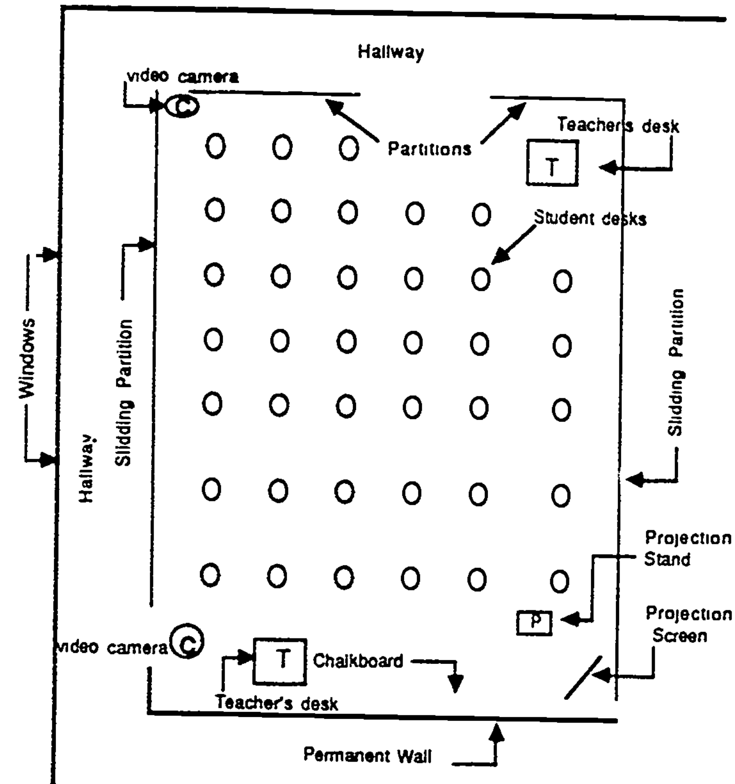

Figure 1. Classroom Floorplan

A standard practice followed on almost all occasions was for the observer to debrief the instructor shortly after a session ended. That is, the observer and instructor discussed how the session had gone, what had gone well (or not so well), and what might be done as a follow-up activity on subsequent days. On occasion the observer called the instructor's attention to something that he may not have noticed (e.g., a group of students who had not been attentive) or suggested an idea for modifying an activity.

In addition to the observer, the regular teacher sat in on about half of every class session. She never made comments or intervened during a lesson, but she did make several valuable suggestions to the instructor afterwards.

DESCRIPTION OF THE INSTRUCTION

The instructional phase as actually implemented was different from the instruction that was originally conceived. In this section we describe both our initial conceptualization of instruction and the instruction as it was actually carried out, and we discuss the reasons for the changes that were made.

The Instruction as Originally Planned

This section is a discussion of the philosophy and assumptions that guided our original planning of the instructional phase. It also includes a discussion of the factors that we identified as having an important influence on the effectiveness of mathematical problem-solving instruction.

Guiding Philosophy and Assumptions

We have been interested in mathematical problem solving instruction for several years. In fact, one of us, Frank Lester, has been involved in research in this area since the early 1970s. In 1981 when Joe Garofalo and Lester undertook a preliminary investigation of young children's metacognitive awareness as it relates to mathematical problem solving (Lester & Garofalo, 1982) they found the research of Ann Brown and her associates to be very helpful in planning their study (Brown, 1978; Brown, Campione & Day, 1981; Brown & Palincsar, 1982). In particular, Brown and Palincsar (1982) concluded from their research concerned with strategy training in the performance of memory and · ading tasks that any strategy training program should provide: (1) practice in the use of strategies (skills training), (2) instruction concerning the value and significance of strategies (awareness training), and (3) instruction concerning the monitoring and control of strategies (self-regulation training). These features are prominent ingredients of the problem-solving teaching strategy created in the mid-1970s by the Mathematical Problem Solving Project (MPSP) at Indiana University (Stengel, LeBlanc, Jacobson, & Lester, 1977) and refined by Charles and Lester (1982).

Perhaps the most important feature of the teaching strategy is that it identifies rather specifically a set of ten "teaching actions" to guide the teacher during classroom problem-solving lessons. (Table 3 lists the teaching actions and the purpose of each.) In a study designed to investigate the potential effectiveness of the teaching strategy, Charles and Lester (1984) found significant growth in students' problem-solving abilities with respect to comprehension, planning and execution strategies. Collectively, these results convinced us that problem-solving instruction should include attention to affective and metacognitive aspects of problem solving along with training in the use of a collection of skills and heuristics.

Table 3

Teaching Actions for Problem Solving

	Teaching Action	Purpose
	BEFORE	
1	Read the problem to the class or have a student read the problem. Discuss words or phrases students may not understand.	Illustrate the importance of reading problems carefully and focus on words that have special interpretations in mathematics.
2	Use a whole-class discussion about understanding the problem. Use problem-specific comments and/or the Problem-Solving Guide.	Focus attention on important data in the problem and clarify parts of the problem.
3	(Optional) Use a whole-class discussion about possible solution strategies. Use the Problem-Solving Guide.	Elicit ideas for possible ways to solve the problem.
	DURING	
4	Observe and question students to determine where they are in the problem-solving process.	Diagnose students' strengths and weaknesses related to problem solving.
5	Provide hints as needed.	Help students past blockages in solving a problem.
6	Provide problem extensions as needed.	Challenge the early finishers to generalize their solution strategy to a similar problem.
7	Require students who obtain a solution to "answer the question."	Require students to look over their work and make sure it makes sense.
	AFTER	
8	Show and discuss solutions using the Problem-Solving Guide as a basis for discussion.	Show and name different strategies sued successfully to find a solution.
9	Relate the problem to previously solved problems and discuss or have students solve extensions of the problem.	Demonstrate that problem-solving strategies are not problem-specific and that they help students recognize different kinds of situations in which particular strategies may be useful.
10	Discuss special features of the problem, such as a picture accompanying the problem statement.	Show how the special features of a problem influence may influence how one thinks about a problem.

Thus, as we began to conceptualize the instruction phase of this study, we decided to incorporate the features of Bro i's instructional approach with those of the MPSP problem-solving teaching strategy. In addition, we decided to consider having the teacher explicitly model strategic behavior and vocalize metacognitive thinking and decision making as he attempted to solve problems in front of the class. The notion of having the teacher serve as a model of a metacognitively-aware problem solver stemmed from Schoenfeld's (1983) recommendation that teachers should attempt to model good problem solving for their students.

It is safe to say that we did not restrict our thinking about the form the instruction should take to research results and the recommendations of authorities. Indeed, as we conceptualized the instructional phase of the study, we were guided by several assumptions (beliefs), some of which have little basis in research, about the relationship between problem solving and other forms of mathematical activity and the role of metacognition in mathematical performance. Among these assumptions, seven were particularly influential:

1. There is a dynamic interaction between mathematical concepts and the processes (including metacognitive ones) used to solve problems involving those concepts. That is, control processes and awareness of cognitive processes develop concurrently with the development of an understanding of mathematical concepts.
2. In order for students' problem-solving performance to improve, they must attempt to solve a variety of types of problems on a regular basis and over a prolonged period of time.
3. Metacognition instruction is most effective when it takes place in a domain-specific context (in the case of this study, problems were related to mathematics content appropriate for grade seven students).
4. Problem-solving instruction, metacognition instruction in particular, is likely to be most effective when it is provided in a systematically organized manner under the direction of the teacher.
5. Problem-solving instruction that emphasizes the development of metacognitive skills should involve the teacher in three different, but related, roles: (a) as an external monitor, (b) as a facilitator of students'

metacognitive awareness, and (c) as a model of a metacognitively-adept problem solver.

6. The standard arrangement for classroom problem-solving activities is for students to work in small groups (usually groups of four). Small group work is especially appropriate for activities involving new content (e.g., new mathematics topics, new problem-solving strategies) or when the focus of the activity is on the process of solving problems (e.g., planning, decision making, assessing progress).
7. The teacher's instructional plan should include attention to how students' performance is to be evaluated. We assumed that in order for students to become convinced of the importance of monitoring their actions and being aware of their thinking, it would be necessary to use evaluation techniques that rewarded such behaviors.

These assumptions led us to the view that if children are to learn how to take charge of their own problem solving, it is important to give direct attention in instruction to metacognitive aspects of the learning of mathematics -- from the initial introduction of concepts and procedures, through the development and, ultimately, the mastery of those concepts and procedures.

Factors That Influence Instruction

During the conceptualization of the instruction phase, six factors were identified as being of utmost importance: mathematics content, problem types, problem-solving strategies, types of metacognitive decisions, the teacher's role, and the amount and sequencing of instruction. Each factor is discussed in turn in the following paragraphs.

Mathematics Content

From the beginning we believed that instruction in problem solving and metacognition was likely to be most effective when it occurred in the context of learning or applying specific mathematics concepts and skills. For example, instruction about percent should include solving problems for the purposes of deepening students' understanding of percent and giving them experience in applying what they know about percent. Furthermore, problem solving was not considered by us to be a mathematics content topic. In our view, to suggest that problem solving is a content topic is to claim that problem solving is a body of knowledge to be mastered, not an activity in which one engages while doing mathematics. Finally, we wanted to avoid suggesting to the students that what they did on "problem solving" days was unrelated to (and, perhaps, not as important as) what they did when their regular ıcher (Ms. Willsey) was in charge. Thus, the initial intention was to coordina ur instructional activities with those of Ms. Willsey as far as possible and appropriate.

Discussion with Ms. Willsey indicated that she intended to consider chapters 9 - 15 in the text, *Mathematics Today* (Harcourt, Brace & Jovanovich, 1985, gr. 7), with the fifth period class and chapters 10 - 15 in the text, *Addison-Wesley Mathematics* (Addison-Wesley, 1987, gr. 8), with the sixth period class. Since we viewed the instruction in the two classes as replications of each other, we decided to minimize as much as possible the differences between the classes with regard to content. Consequently, we chose three topic areas to emphasize: ratio and proportion, percent, and measurement (viz., perimeter, area and volume). This focus was not maintained as the study progressed.

Types of Metacognitive Decisions

Prior to the start of this research we had developed our "cognitive-metacognitive framework" to be used as a tool for analyzing metacognitive aspects of mathematical performance (Garofalo & Lester, 1985; see Table 1). As was mentioned earlier in this report, the framework served as a guide in the selection of research tasks, the design of interview procedures, and the development of instructional activities. Furthermore it served as a means for organizing analyses and interpreting findings. As instructional activities were planned and developed, the framework was referred to frequently in order to insure that a wide range of types of metacognitive decisions were being addressed.

Problem Types

Problems used during class instruction and assigned for homework were of two broad types: routine and non-routine. Routine problems were exercises whose instructional purpose is to provide students with experience in translating verbal problems posed in real world contexts into mathematical expressions. Typical of this type of problem is the following:

Laura and Beth started reading the same book on Monday. Laura read 19 pages a day and Beth read 4 pages a day. What page was Beth on when

Laura was on page 133?

Three categories of non-routine problems were considered: process problems, problems with superfluous information, and problems with insufficient information. A process problem is one whose solution requires that the problem solver do something more than translate words to a mathematical expression, or apply an algorithm, or perform computations. Illustrative of this category of non-routine problems is the following:

> *A caravan is stranded in the desert with a 6 day walk back to civilization. Each person in the caravan can carry a 4 day supply of food and water. A single person cannot carry enough food and water and would die. How many people must start out in order for one person to get help and for the others to get back to the caravan safely?*

The other categories of non-routine problems are self-descriptive and so no examples are provided here.

All three types were included because each seemed especially suited to involve behaviors associated with particular categories of our cognitive-metacognitive framework. The routine problems always involved multiple steps for their solution and thus were included to elicit behaviors associated with *organization* and *execution*. Problems containing superfluous or insufficient information were included to elicit behaviors associated with *orientation* and *verification*. Finally, process problems were selected for their potential to elicit behaviors corresponding to both *orientation* and *organization*.

<u>Problem-solving Strategies</u>

Numerous problem-solving strategies are considered in recent elementary and middle school mathematics texts. Among these, we decided to give primary attention to the following: guess and check, look for a pattern, and work backwards. A lesser amount of attention was given to: draw a picture, make a table, and simplify the problem. These latter strategies were viewed by us as "helping strategies," that is, means by which to make good guesses, look for patterns, or work backwards. Since the purpose of the study was to investigate the metacognitive behavior of students during problem solving, we wanted to present problems to the students that were likely to elicit such behaviors. *Guess and check* was chosen for its potential to cause the problem solver to assess the adequacy (appropriateness) of her or his guesses and to make additional guesses based on this assessment. *Look for a pattern* seemed particularly well suited as a strategy because of the data *organization* demands required in order to use the strategy effectively. The strategy *work backwards* was chosen for two reasons. First the logical, cognitive demands made on the problem solver to "reverse" her or his thinking about a set of data are such that it is not until about grade 6 that students seem able to use this strategy successfully (Lester, 1980). (Thus, by including *work backwards* in the instruction we felt we would be able to investigate the early development of students' understanding of the strategy). Second, to be successful in working backwards most students need to *organize* the available information in a systematic manner and to proceed toward a solution in a step by step fashion. That is, the ability to do these things seems to involve a substantial amount of metacognition, especially behavior associated with *organization* and *verification*. Notwithstanding the foregoing discussion, to some extent this choice of strategies was an arbitrary one. However, we were guided in our selection by our experience with students of this age and by our belief in the fundamental importance of these strategies for solving problems.

Amount and Sequencing of Instruction

Instruction was originally planned to take place three days per week for 12 weeks. Each week was to involve either two full and one half period or two half periods and one full period. Thus, the original plan called for 36 sessions with each class for either 115 or 90 minutes per week (class periods lasted 45 minutes) and a total of 21.5 hours of instructional time over the 12 week period. (Actually, project instruction took place over a 14 week period of time, but one of those weeks was devoted to school-sponsored standardized testing and one week was spent on spring break). The total amount of instructional time allotted for mathematics instruction during this period was 45 hours. Thus, the plan was to devote about half (47.8%) of the instructional time to direct problem-solving/metacognition instruction. However, for reasons discussed in the section, *Instruction as Implemented*, this plan had to be drastically revised. Table 4 provides a comparison of the amount of instructional time planned and actually implemented. The table indicates that the actual problem-solving instruction that was ultimately provided involved a reduction in time of about 22%.

Table 4

Amount of Instructional Time: Planned vs. Implemented

	Planned	Implemented	Change
Number of weeks	12	12[a]	---------
Number of sessions	36	26	10 fewer sessions (28% reduction)
Average time per week	1 hr. 43 min.	1 hr. 20 min.	23 fewer minutes per week (22% reduction)
Total time: 12 weeks	20.5 hrs.	16.1 hrs.	4.4 fewer hours (22% reduction)
% of total math time for the 12 weeks	45%	35.7%	22% reduction

[a] 14 weeks elapsed from the beginning to the end of the instructional period. During this period one week was lost due to system-wide standardized testing and a second week was Spring Break.

The original organization of instruction was as shown in Table 5 (pp. 34-35). As indicated in this table, the teacher assumed the role of external monitor at least once per week and, as often as not, twice each week. The teacher served as a facilitator of students' metacognitive development during nine of the 12 weeks, the three exceptions coming during the first two weeks and the eighth week (the "catch-up" week). Modelling of good problem-solving behavior was done on five occasions (during weeks 2, 5, 8, and 11). Also, on a number of days (e.g., day 1 of week 5, day 3 of week 8, day 3 of week 10) the teacher gave the students a problem-solving activity to do and simply observed without commenting on their problem-solving performance.

The strategies guess and check (A), look for a pattern (B), and work backwards (C) were to be given approximately equal attention during the instructional period (A and C, seven problems each, and B, eight problems). A fourth strategy, D, was to be introduced during week 11, but was not identified prior to the beginning of the study. The intent was to choose a strategy during week nine or ten that was different from any the students had been exposed to during the preceding weeks. Also, strategies were revisited regularly. For example, during week eight strategy B was considered even though it had not been a natural strategy to use to solve problems since week five. Our belief was that by revisiting strategies introduced some weeks earlier the students would be forced to make decisions about the strategies to use. Further, students might be less likely to attempt to use only strategies they had been exposed to recently.

Teacher's Role

As stated in *Research Question II* (see chapter 1), we were interested in the effects on students' problem-solving behavior of instruction that involved: (1) practice in the use of strategies (skills training); (2) training students to be more aware of the strategies and procedures they use to solve problems (awareness training); and (3) training students to monitor and evaluate their actions during problem solving (self-regulation training). The instruction phase of this study incorporated all of these aspects and included three different but related roles for the teacher: (1) *external monitor*; (2) *facilitator* of students' metacognitive development; and (3) *model*.

Teacher as an external monitor. This teacher role consists of a set of ten teaching actions for the teacher to engage in. Specifically, the teacher directs whole-class discussions about a problem that is to be solved; observes, questions and guides students as they work either individually or in small groups to solve the problem; and, leads a whole-class discussion about students' solution efforts. The ten teaching actions and the purpose of each are described in Table 3. A more detailed discussion of the teaching actions can be found in Charles and Lester (1982, 1984), Lester (1983), and Stengel, LeBlanc, Jacobson and Lester (1977).

Teacher as facilitator of students' metacognitive development. When the teacher assumes this role, he or she: asks questions and devises assignments that require students to analyze their mathematical performance; points out aspects of mathematics and mathematical activity that have bearing on performance; and helps students build a repertoire of heuristics and control strategies, along with knowledge of their usefulness. One way in which we planned to direct students to reflect on their own cognition was to have them

Table 5

Originally Planned Organization of Instruction

Week	Class Sessions of the Week 1	2	3
1	Organization/ orientation	Small group problem solving	Small group problem solving (**external monitor**[a])
2	Small group problem solving (**external monitor**)	**Teacher modelling**[b] w/discussion	**Teacher modelling** w/discussion
3	Small group problem solving: strategy A[d] (**external monitor**)	Problem-solving skill activity (**teacher as facilitator**[c])	individual problem-solving strategy A (**external monitor**)
4	Small group problem solving: strategy B[e] (**external monitor**)	Problem-solving skill activity (**teacher as facilitator**)	Individual problem solving: strategy B (**external monitor**)
5	**Teacher modelling** strategy A	Small group problem solving: strategy B (**external monitor**)	I. Individual problem solving strategy B II. Problem solving skill activity (**teacher as facilitator**)
6	Small groups problem solving: strategy C[f] (**external monitor**)	Problem-solving skill activity (**teacher as facilitator**)	Individual problem solving strategy C (**external monitor**)
7	Small group problem solving: strategy C (**external monitor**)	small group problem solving: strategy A (**external monitor**)	I. Individual problem solving: strategy C II. Problem-solving skill activity (**teacher as facilitator**)

4

TABLE 5 (continued)

8	Small group problem solving: strategy B (**external monitor**)	**Teacher modelling** w/discussion strategy A	I. Individual problem solving: strategy C II. Catch up
9	Small group problem solving: strategy B (**external monitor**)	Problem-solving skill activity (**teacher as facilitator**)	Individual problem solving: strategy C (**external monitor**)
10	Small group problem solving: strategy A (**external monitor**)	Problem-solving skill activity (**teacher as facilitator**)	I. Individual problem solving: strategy A II. Individual problem solving: strategy B
11	Small group problem solving: strategy Dg (**external monitor**)	Problem-solving skill activity (**teacher as facilitator**)	I. Individual problem solving: strategy C II. (**Teacher modelling**) strategy D
12	Individual problem solving: strategy D (**external monitor**)	Small group problem solving: strategy B (**external monitor**)	Discussion about problem solving (**teacher as facilitator**)

Key: a: External monitor - Teacher: directs discussion about a problem to be solved, observes, questions and guides students as they work; and directs discussion about their solution efforts.

b: Teacher modelling - Teacher solves a problem at the chalkboard and models good problem so[illegible]ing behaviors and metacognitive awareness

c: Teacher as facilitator - Teache[illegible] [illegible]s questions that require students to analyze their performance; points out aspects of math t[illegible] [illegible]e a bearing on performance; and helps students develop control strategies.

d: Strategy A - Guess and check.

e: Strategy B - look for a pattern

f: Strategy C - work backwards

g: Strategy D - a differ[illegible] [illegible]rategy (yet to be determined) complete self-inventory sheets on which they listed their own strengths and weaknesses in doing mathematics. Another activity was to ask students to write short statements immediately after solving a problem about their thinking during their solution attempts.

Teacher as model. This role involves the teacher in explicitly demonstrating regulatory decisions and actions while solving problems for students in the classroom. The intent was to give students the opportunity to observe the monitoring strategies used by an "expert" as he solved a problem that he had never solved before. In addition, the teacher directs a discussion with the class about their observations of the expert's behavior.

Additional Comments About the Nature of the Planned Instruction

As it happens, the instructional approach we initially envisioned for this study is very similar to what Campione, Brown and Connell (1989) call "reciprocal teaching." Briefly, reciprocal teaching, which grew out of the earlier work of Brown and her colleagues mentioned above, is an approach to metacognition-based instruction. This method has six features that make it especially appropriate for mathematical problem-solving instruction: (a) instruction occurs in cooperative learning groups; (b) instruction takes place in the context of learning specific content; (c) the student's attention is focused on solving a specific problem, not on monitoring, regulating, or evaluating actions per se; (d) students are not protected from error in their solution efforts; (e) the teacher is allowed to be fallible; and (f) the teacher's role as a guide and model diminishes as students become more confident and competent. We mention reciprocal teaching and its similarity to our approach because it has been shown to be very effective in the contexts of reading and mathematics.

The Instruction as Implemented

The instructional phase of this study had to be revised rather drastically for a variety of reasons. In this section we elaborate on these reasons and describe in detail the actual plan that was implemented. Appendix B contains a complete set of the daily lesson plans that were actually followed during the 14 week period of instruction. Further, Appendix C contains copies of every class activity sheet and homework assignment.

Reasons for Revising the Instructional Plan

There were four principle reasons for having to alter the original instructional plan. The first reason has been discussed in chapter 2. To reiterate, we learned only a few months before the instructional phase was to begin that sixth grade students could not be involved in the study as was originally intended. Consequently, we decided to conduct the study with seventh graders. This change meant that some of the problems and problem-solving activities we had selected would not be suitable for the study. It also meant that we would have to adjust to the differences between elementary and middle schools with respect to flexibility in scheduling instructional sessions. More specifically, we were restricted to two 45-minute time periods occurring during the fifth and sixth periods at Batchelor Middle School (i.e., 5th period: 11:10 -11:35 and 12:05 - 12:25; 6th period: 12:40 - 1:15). The change to seventh grade also meant that classes with very different abilities would be involved. Because the two classes were so different (see *Description of the School and the Subjects*) it no longer made sense to consider having a control class. Further, although we intended to present essentially the same instruction to both classes, as instruction progressed it became necessary to "individualize" for each class. Thus, although the same general format for instruction was followed for both classes, on several occasions the specific nature of the class activities differed, as did the problems used.

A second reason for altering the instructional plan was that we found the students to be unprepared for some of the activities. For example, small-group cooperative learning was to be a central feature of the metacognition-based instruction. Unfortunately, the students in the two classes had had little or no experience in working cooperatively. Almost without exception, the students believed that mathematics class (perhaps any class) was a place where one listened to a teacher, completed exercises in the textbook, and worked silently and alone. Thus, although we knew we would have to devote some attention to establishing acceptable standards of behavior for small-group work and allowing students ample time to become used to working with their peers, we found that a major change in the students' beliefs about the nature of mathematics class would be needed. This realization caused us to lessen the amount of cooperative group work that took place.

We also found that many of the students had serious conceptual and procedural deficiencies in their mathematics knowledge (especially true in the regular class). Students with these deficiencies tended to avoid (or to shy away from) engaging in problem-solving activities that required the use of concepts or skills they were not comfortable with. For example, during the second week of instruction students in the regular class were asked to work cooperatively to solve a problem involving percents (refer to day 4 lesson plan in Appendix B and Appendix C, day 4). One group of four students refused even to try to solve the problems because, as they said, "We don't know how. We can't do percents." Their belief that they were incapable kept them from trying. Consequently, in such cases it was necessary to backtrack and teach them enough to help them get started. A belief that we came to accept was that if students have nothing to reflect on or if they have no behaviors to monitor, it is quite difficult to expect them to be reflective or to monitor their progress. That is, it is important that a problem solver must have something to be metacognitive about during problem solving . This, perhaps obvious, "truth" caused us to modify the planned instruction from time to time -- often rather drastically.

The third reason for revising the instructional plan was that we sensed that the regular teacher, Barbara Willsey, was becoming increasingly concerned that she would not be able to give adequate attention to topics presented in the mathematics textbook given the amount of time she had given over to me. This was of special concern to her for the fifth period class. Although she judged this particular section of "regular" students to be better than other classes of "regular" students she had taught, she felt that many of the students in this class were still deficient in fundamental computational skills with decimals, fractions and percent as well as lacking in basic understanding of concepts associated with decimals, percent, ratio and proportion, geometry and measurement. Consequently, the plan was revised to involve about 16 hours of instruction over the 12 week period, instead of the original 20.5 hours (a reduction of 22%). It should be noted, however, that even though Ms. Willsey had only about two-thirds as much time to devote to mathematics instruction as she did with her other (non-project) classes, she claimed to have been able to cover approximately the same amount of material. Furthermore, there was no evidence of any discernible difference in overall achievement among students in her classes (i.e., classes of the same level).

The final reason for the revision was the most important one. During the period of time that elapsed between the initial conceptualization and the implementation of the instructional phase, a philosophical change took place in our thinking about what sound problem-solving and metacognition instruction involves. We realized that a fixed schedule involving a systematic rotation among three types of teacher roles (viz., external monitor, facilitator, and model) had no theoretical basis (refer to Table 5). Furthermore, as was mentioned earlier in this report, much of the basis for our initial conceptualization came from work in reading (Brown, 1978; Brown, Campione, & Day, 1981; Brown & Palincsar, 1982; Campione, Brown & Connell, 1989) or with college-age students (Schoenfeld 1983, 1985). A consideration of these two observations led us to formulate two questions: (1) Are the metacognitive behaviors used in solving mathematics problems the same as those used to comprehend a passage of written prose? and (2) Will the instructional techniques shown to be effective in helping college students become better problems solvers also be effective with middle school students?

We decided that what was needed was an exploratory study to investigate the relative effectiveness of various teacher roles and the potential value of a wide range of types of problem-solving activities. Thus, as the instructional phase progressed we found ourselves making almost daily changes in our plans. Such changes were not simply made whimsically by the instructor. Instead, they resulted from discussions between the instructor (Frank Lester) and the observer (Diana Kroll) about the apparent effectiveness of a lesson (or group of lessons of the same type). In addition, it soon became obvious to us that the two classes were different enough in their knowledge of mathematics, study habits, and so on, that it was necessary to attempt to "tailor-make" activities for each class.

General Comments about the Implemented Instruction

The seven assumptions about problem-solving and metacognition instruction that were discussed earlier in this report continued to serve as guiding principles throughout the conduct of the instructional phase. Consequently, although the instruction as actually conducted was considerably different from what was initially envisioned, many of the original features were maintained. For example, students were asked to solve a wide variety of problems (assumption 2), they were regularly asked to reflect on their problem-solving efforts (assumptions 3 & 5), and evaluation of students' problem-solving attempts gave attention to the importance of students' monitoring behaviors and their awareness of their own thinking (assumption 7). Furthermore, the teacher's role remained essentially the same. That is, for each instructional session he assumed one or more of the three teacher roles (viz., external monitor, facilitator of metacognitive awareness, and model) identified in the initial conceptualization (assumption 5).

The actual instruction differed from the original plan in three ways: (1) no fixed schedule of activities was rigidly adhered to, (2) the relative emphasis given to each of the three teacher roles changed, and (3) more attention was given to skill development.

By not being locked into a set schedule of activities it was possible to explore the value of a broad range of types of activities and we were able to provide instruction that more nearly suited the abilities of the students. Also, by adopting a more flexible approach we were able to treat in much more depth and detail activities that were deemed to have been incompletely considered or that otherwise seemed to be in need of additional attention.

When the instruction began, we expected that the teacher's roles would not change significantly under the revised plan. That is, we thought that about one-half of the teacher's time would be devoted to serving as an external monitor, about one-fourth to being a facilitator of students' metacognitive awareness, and about one-sixth to modelling good problem solving (see Table 5). As instruction progressed it became apparent that the relative emphasis on each role would not be maintained. Furthermore, the distinction among the three roles became blurred. In particular, it became increasingly difficult to specify when the teacher was serving as an external monitor or a facilitator. A typical class session involved activities that required the teacher to serve as an external monitor for some students and as a facilitator for others. The situation was complicated further by the differences between the two classes. Many of the students in the fifth period class (regular group) seemed dependent upon the teacher as a facilitator in order for them to make progress, whereas the sixth period class (advanced group) needed less direct guidance from the teacher. Also, the advanced class did not appear to need the teacher to play the role of external monitor as often as the other class did.

A final difference between the initial and actual instruction stemmed fro the fact that many students (especially in the regular class) lacked various 'tool" skills that are essential to successful problem solving (Note: a tool skill is a fundamental, generic skill without which problem solving is practically impossible). For example, some students seemed incapable of identifying the relevant information in a story problem and others seemed unable to read information given in a chart or graph. It was necessary to provide these students with experiences that would assist them in acquiring such skills.

Sample Instructional Activities

In order to provide a clearer picture of the character of the instruction that actually took place with the two grade seven classes, three sample activities are described in this section. Each activity highlights one of the three teacher roles: external monitor, facilitator, and model.

Sample Activity I: Teacher as an External Monitor

A portion of the lesson for day 4 illustrates the teacher as an external monitor. For this lesson the students were organized into groups of four and a captain was identified for each group. After directing the students to put down their pencils/pens, the teacher displayed the following problem on the overhead projector.

> *Carla is the drummer in a band. On Tuesday she received her paycheck for work done during the past month. She spent 20% of it that day and 50% of what was left on Wednesday. She then had $50 left. How much did Carla receive in her paycheck?*

The teacher engaged in "teaching actions" 1, 2, and 3 in an effort to guide the students to a good understanding of the problem and to assist them in choosing plans of action (see Table 3). Specifically, the teacher read the problem aloud slowly and then asked questions of the following sort: "What does Carla do?" "What was she paid for doing?" "What did she do with her money on Tuesday?" "What does '50% of what was left' mean?" "What does the problem ask us to find out?" After this initial "understanding the problem" phase, the teacher solicited ideas from the class about possible solution strategies (students were asked to confer with the students in their group; no pencils/pens allowed yet). After a few ideas were generated, the students were directed to work with their partners to solve the problem. At this point, one copy of the problem statement was distributed to each group. As the groups worked on the problem, the teacher walked around the room, listening and watching. On occasion the teacher asked a group to tell him what they were doing and why. If a student asked the teacher a question, the question was directed to the other members of the student's group. If none of the students could answer the question, the teacher either answered the question or provided a hint. After a time, the teacher provided a focusing or direction-giving hint for groups who seemed to have made no progress. For example, if a group appeared to be at a complete loss for good ideas, the teacher made a suggestion like: "Can you make a guess about the amount of money Carla received and then check to see how close you are?" Another group might be asked a question like: "Carla had $50 left after she spent part of her money on Wednesday. How much money did she have just before she spent part of it on Wednesday?" Students were given about 15 minutes to work on this problem (students in the advanced class needed only about 10 minutes). When groups finished work on the problem, the teacher asked selected groups to share their approaches with the class. After a group finished explaining their solution, the teacher asked questions such as: "What did you do to help your group get started?" "Why did you do what you did?" "What did you learn about problem solving by solving this problem?" The key feature of this activity is that the teacher asked questions about the process of solving the problem with the aim of helping the students monitor what they were doing and evaluating their progress. As weeks passed, and students became more confident and proficient, the teacher asked fewer and fewer questions; that is, he played a less prominent role in monitoring the students' problem-solving efforts.

Sample Activity II: Teacher as Facilitator

The following activity is illustrative of the sort of activity in which the teacher acted as a facilitator of the students' metacognitive awareness. This activity was a part of the instruction for both classes on Day 20. Each student was given a "video-tape viewing guide" (see Appendix C, Day 11) to use as they watched a tape of an individual attempting to solve the following problem:

I am thinking of two numbers. When you multiply them you get 204. When you subtract them you get 5. What are the two numbers I am thinking of?

Diana Kroll was the instructor for this portion of the class. She began by explaining that she was in the midst of conducting her doctoral dissertation research which involved studying how adults solve math problems. In particular, she noted that she often videotaped students as they worked on problems. She mentioned that today she had brought a tape of a person working on a problem they had done a few days before. What the students were to do was to watch the tape and use the guide to make notes about what the person did that was good and not so good. Kroll emphasized that the students were to look for places where the person demonstrated (or failed to demonstrate) understanding, planning, and careful checking -- either along the way or at the end.

After the class had viewed the tape, Kroll asked questions such as: "What were some of the things this person did especially well in solving the problem? What were some of the things she did badly? What are some important questions for problem solvers to ask themselves while they are solving problems?" At the end she stressed that although it is important to ask yourself about the correctness of calculations, it is even more important to ask yourself questions about your overall plans (e.g., "What am I doing? Why a I doing this? How does it help me? Is it getting me anywhere?).

<u>*Sample Activity III: Teacher as a Model of a Metacognitively-aware Problem Solver*</u>

On Day 6 Frank Lester solved the following problem in front of the class (regular level class only):

> *A caravan is stranded in the desert with a 6-day walk back to civilization. Each person in the caravan can carry a 4-day supply of food and water. A single person cannot go alone for help because one person cannot carry enough food and water and would die. How many people must start out in order for 1 person to get to help and for the others to get back to the caravan safely?*

Before he began to solve the problem he mentioned that he was going to try to think out loud as he worked, so "you can hear what is going on inside my head." After reading the problem aloud slowly, he began to ask himself questions such as: "What's going on in this problem? How far is the caravan from civilization? How long will it take to reach civilization?" Then he said: "Well, it's obvious that one person can't make it alone, so let's see if two persons can do it." He then proceeded to give the two persons names (Jodi and Pete) and wrote the names on the chalkboard. He began to draw a sketch showing the progress of Jodi and Pete along the way from the caravan to civilization. Ultimately, he concluded that two people wouldn't be enough, so he abandoned two people and decided to try three people. Throughout his work he talked about what he was doing as if he were having a conversation with himself.

After he completed work on the problem, he asked the class about their reactions to his performance. Not only did he want the students to see that good problem solvers don't always go quickly and directly to a solution, but he also wanted them to see that good problem solvers ask themselves questions about what they are doing and thinking. Furthermore, he demonstrated that drawing sketches and making guesses were legitimate techniques used by good problem solvers.

The Final Schedule for Instruction

Table 6 gives the actual schedule for instruction for the entire 14 weeks. As has been mentioned, class sessions lasted 45 minutes. Thus, the percent of instructional time devoted each week to problem solving ranged from 31% (70 minutes) to 51% (115 minutes), with slightly more than 36% being devoted to problem solving over the course of 14 weeks (this is an estimate that does not account for fire drills and other unanticipated disruptions). Also, it would have been possible to include additional days of instruction had it not been for four days of school-wide standardized testing (5th week), two days of parent-teacher conferences (11th week), and three days lost due to attendance at the annual NCTM meeting by the instructor and observer (12th week).

Table 6
Final Schedule for Instruction

Week	Month	Monday	Tuesday	Wednesday	Thursday	Friday	Time[a]
1	Jan.	19	20 FULL[b]	21 HALF[c]	22	23 HALF	95 mins
2	Jan.	26 FULL	27	28	29	30 FULL	90 mins
3	Feb.	2 HALF	3	4 FULL	5	6	70 mins
4	Feb.	9 FULL	10	11 HALF	12	13 FULL	115 mins
5	Feb.	16 NO SCHOOL	17 TESTING[d]	18 TESTING	19 TESTING	20 TESTING	0 mins
6	Feb.	23	24	25 FULL	26	27 HALF	70 mins
7	Mar.	2 FULL	3	4	5	6 HALF	70 mins
8	Mar.	9 FULL	10	11	12 FULL	13	90 mins
9	Mar.	16 SPRING BREAK →	17	18	19	20	0 mins
10	Mar.	23 FULL	24	25 HALF	26	27	70 mins
11	Mar. Apr.	30 FULL	31	1 FULL	2	3	90 mins
12	Apr.	6 HALF	7 FULL	8	9	10	70 mins
13	Apr.	13 HALF	14	15	16	17 FULL	70 mins
14	Apr.	20 HALF	21	22 FULL	23	24	70 mins

NOTE:
a -- TIME refers to the total amount of instructional time devoted to problem-solving instruction.
b -- FULL indicates a full class period (45 minutes) was devoted to problem solving.
c -- HALF indicates a half class period (25 minutes) was devoted to problem solving.
d -- TESTING refers to system-wide standardized testing.

Chapter 4

RESULTS AND OBSERVATIONS

In this chapter we discuss the results of our analyses of data bearing on the two major research questions of this study: (1) What metacognitive behaviors do grade seven students exhibit when they attempt to solve certain types of mathematics problems and how do these metacognitive behaviors interact with the students' cognitive behaviors? (2) What are the effects on students' problem-solving behavior of instruction designed to increase students' cognitive self-awareness and ability to monitor and evaluate their own cognitive performance?

We begin with discussions related to the first question. More specifically, we provide commentaries based on an analysis of two different sources of data: student interviews and students' written work on tests, classwork and homework.

The second question is concerned with the effectiveness of instruction. To a great extent the student interviews and written work were a rich source of information about the effects of instruction. Consequently, the discussions of the interviews and written work also are relevant to question two. However, to limit our analysis to these data sources would not allow us the benefit of capitalizing on the fact that three very experienced teachers had been extensively involved in observations of the instruction. So, in addition to the other sources mentioned, we provide analyses of the instruction from the points-of-view of the special problem-solving instructor, the research assistant, and the regular mathematics teacher. Furthermore, we have included verbatim essays written by students who participated in the study.

RESULTS AND OBSERVATIONS WITH RESPECT TO STUDENT INTERVIEWS

As mentioned in chapter 3, pre- and post-instruction interviews were conducted with selected students in each of the classes. The interviews were of two types: individual and paired. In this section we provide a commentary on our analysis of the individual interviews with eight students (three from the regular class, five from the advanced class). Appendix D includes all the work of each of the eight students on each problem attempted. In addition, each paper contains a summary of the written work. Each summary was prepared after analyzing the videotape of the interview session. A list of all problems included in the interviews (both individual and paired) is given in Appendix A. (*authors' note: The results and observations provided in this section are based on only a small portion of the total number of interviews conducted. Analysis of other individual interviews and the paired interviews is currently under way.*)

Commentary on Students' Interview Work

The following commentary is intended as a summary of our analysis of the students' performance during the individual interviews. Attention in this commentary is focused upon the students' strategic decisions and behaviors during problem solving in interviews both before and after instruction. Little effort has been made in the discussion that follows to distinguish between pre- and post-instruction performances because there was not a dramatic difference between the two.

As expected, strategic decisions and behaviors (or their absence) associated with each of the four categories of the framework can be identified as contributing to students' success or non-success on the mathematical tasks given during the interviews. Although each of the categories proved to be important in this regard, the *orientation* category or phase stands out as being the most important, since much of what followed (or didn't follow) in the *organization*, *execution*, and *verification* categories was somewhat connected to or dependent upon a student's understanding or representation of the scope and structure of the tasks. Below is a discussion of how activities associated with each category contributed to students' problem solving performance. This discussion is based on the performances of eight students who were interviewed both before and after instruction.

Orientation

Approaches to Orientation

The decisions and actions students used to develop an orientation to, or understanding of, the mathematical tasks can be arranged into two qualitatively different kinds of approaches to orientation, each of which can be somewhat differentiated by degree into two levels. One kind of approach can be termed as *superficial*, since when students were using this approach, they focused primarily, but not exclusively, on surface level features of problems. The other kind of approach can be viewed as *meaningful*, since when students used it they tried to understand the meaning of problem conditions and questions. The first level of the superficial approach involved *no rereading* of the problem after the student read it aloud. In most cases it was apparent from the reading aloud that the student did not have a good understanding of the problem, but still the reading aloud was not followed by any rereading. The one student who exhibited this approach regularly, both before and after the instruction, claimed that she "just looked back to get the right numbers." Her explanation for not rereading the problem was "usually I can just look at a problem and tell." This student noticed such features as "it didn't give me a specific way," "the 2500 and the 4 caught my eye," and "it only has one number in it," but she often had little understanding of the meaning of the problem conditions, the relationships between the quantities in the problem, and the question being asked. Hence, she sometimes ignored important conditions, and sometimes lost sight of other conditions and the question while she was working on the problem. It is not surprising that this students' performance for the most part was unsuccessful, except for the two most straightforward problems--the Kennedy and Luncheon problems, and the NFL problem, which she recognized "right away" because she saw "something like this before."

What was surprising however, was that another student, who reread all of the problems during the pre-instruction interviews did not reread any of them after instruction. Before instruction, she claimed, when discussing her performance on one problem, "I wasn't quite sure what to do so I read it again . . . sometimes when I read aloud I don't exactly understand." But after the instruction, when it appeared that she was not rereading any of the problems, when asked whether she did or not, she replied "No, I used to before I came in here [before the instruction] . . . I got better . . . I can pick out information and keep it until I finish reading and when I'm done I sorta write it out." Reasons she gave for not rereading some specific problems included "I understood it the first time" and "we did one like it in class, so I knew what to do." These reasons were appropriate in some cases, but the no-rereading strategy persisted even when she admittedly did not understand a problem. For example, when working on the Shuttlemeier problem she stated "I still don't understand it," but still she did not go back to reread it! This student's post-instruction performance was not very successful, except for the Luncheon problem, and the NFL problem (on each of these she claimed to have "done a problem like this before").

The no-rereading approach seems to be sufficient for solving the simplest routine problems that are very much like the problems frequently encountered in school, but clearly it is not sufficient for problems with any more complexity nor for problems which are not recognized. However, in some cases, students who did not reread were able to get some understanding of the meaning of some problem conditions, and they sometimes came up with somewhat reasonable plans of attack. It is unclear exactly how this approach would be used in a classroom setting, since students do not have to read the problems aloud first. But it is probably safe to assume that the students who used this approach in this setting do not do very much rereading in school.

The next level of the superficial approach, and the two levels of the meaningful approach involve various types of rereading. The second level of the superficial approach involves a type of rereading which is in some sense surface-level. This approach can be called the *number consideration* approach because when students used it they focused their rereading only on certain parts of the problem statement, namely "the numbers and the question." It is clear that the no rereading approach also focuses attention on the numbers, but it is useful to differentiate between the two levels, because the number consideration approach involves rereading, and because it puts much more effort into examining the numbers. One student who regularly used the number consideration approach claimed that she reread "just certain parts . . . with the numbers . . . I looked at the numbers to figure out what to do with them." This student's over-focus on the numbers is forcefully demonstrated by her approach to the Shuttlemeier problem. She "sort of found a pattern to it . . . when you read it, it started making a pattern by itself." Another student remarked that she reread to "find the key words . . . the numbers and the problem." It is well documented that this strategy for figuring out what to do when attempting to solve a problem is used fairly commonly, and that it is often successful when used to solve school verbal mathematics problems. Students are aware that this method is often successful and efficient for many of the problems they encounter. For such problems, its success reinforces its use; however, as we have seen, it is often used to solve problems which require a deeper level of analysis and understanding.

It should be pointed out that students do not necessarily operate solely within one type of approach. For example, one student who fairly consistently used the number consideration approach remarked in relation to a few problems that she wants her ideas about a problem to "make sense." She also recognized the structure of the Felipe problem and the NFL problem as being like problems she had seen before. As with the no rereading approach, the number consideration approach can give a student some understanding of problem conditions and relationships, but often it is not sufficient, particularly with complex and non-routine problems.

This mention of problem understanding leads us into the next kind of approach, namely, the meaningful approach. As with the superficial approach, the meaningful approach can be viewed as having two levels, differing primarily on the degree of analysis of problem conditions and relationships. The first level of this kind of approach can be described as the *partial meaning* approach because when students used it they tried to get some understanding of the problem (beyond that of the students using a superficial approach) but not necessarily a complete understanding. In this approach students went somewhat beyond a consideration of numbers and tried to do some analysis of the problem conditions and/or question. For example, one student commented that she rereads because "you have to read it a couple of times before you know what it means." When working on the Coin problem she said she "read it five times, until I understood it," because she had difficulty understanding what "exactly the same" meant. On another problem she commented that on her rereading "I didn't have to [reread] . . . [I] already knew what it said, but I had to . . . to understand it." And on the NFL problem she stated "I don't understand . . . I don't know anything about telephone lines." It is clear that this student went beyond just examining the numbers and tried to develop some understanding of the problem, even though she didn't spend enough effort on analyzing the conditions and relationships.

Another student using this approach explained her rereading by stating "to make sure I understand it, when I read aloud I just read words." She also commented "because if I don't really understand, I keep reading and rereading until I understand a little bit." It is this "little bit" aspect that distinguishes the partial meaning approach from the next level. The partial meaning approach was observed most clearly in connection with problems that contained many quantities and conditions, namely the Atl s and the Shuttlemeier problems. Students using the partial meaning approach seemed to have a good understanding of some quantities and relationships, but a poor understanding of others. It was usually the quantities that appeared in the first part of the problem that were understood well and those appearing later on that were not. Clearly these students are going beyond the superficial, en if they don't often go far enough in analyzing the problem statement.

The second level of the meaning approach can be called the *full meaning* approach, because students who used it were very concerned about having a complete or very near complete understanding of the problem conditions before they began to plan out a method of solution. Their purpose in intensively rereading was to "find the meaning." It is interesting to note that the three students who regularly approached the problems in this manner all spent a considerable amount of time rereading, analyzing problem conditions and relationships, and planning before they began to work on paper. In many cases the amount of time spent on such activities was noticeably more that the amount of time spent by the other students.

One such student commented that he reread "to comprehend better." On the Coin problem he explained his rereading by saying "to get all the details . . . I thought it might have to do with if one person used a quarter then another couldn't . . . if they wanted that they would have said it better," and because "I wasn't at all sure what the question asked." He went on to say that he rereads "just to make sure I really understand . . . I can't do something hard without knowing what I'm doing."

A second student said that she tries to "get it fully." She claimed that she read one problem "three times, because it's not like the math I've been working with." She reread and thought about problems even when she felt that she understood them initially and/or recognized them. For example, when working on the Felipe problem she commented she "had a couple like that in class" and knew how to solve it, but she reread and thought about it anyway.

A third student who spent time rereading and analyzing problems explained that h reread one problem "twice because I had trouble with this in class. . . . I didn't read it through carefully."

These students spent a lot of time trying to develop a good understanding of problems before they proceeded to solve them.

Aids to Orientation

Students used a number of different strategies to assist them in becoming oriented to the problems. These included listing data, using pen movements, vocalizing, and discussing the problem.

Several students listed some of the data given in the problem statement on the side of their paper after reading the problem aloud or, in one case, after rereading. This strategy was used only in connection with the Atlas and Shuttlemeier problems. One student explained her use of the strategy by claiming "so I don't have to keep looking back at it." Another student stated "so I wouldn't forget . . . I knew that these were important" and "so I can sort it out in my head and don't have to read the whole paragraph." Both of these students might have benefited from looking back or rereading the problems because they were the two who did not reread the problems. It is interesting to note that none of those who attempted to develop a full understanding of the problem used this strategy.

Another strategy employed by a number of students was that of using their pen or pencil to guide their reading or rereading of problems. When asked why they did this, one student commented "it helps me follow the words better . . . pick out the information I need"; another claimed he did it "to help me keep where I am"; and a third replied he used his pencil "to make it clearer to me which words I am reading." All of these students reread the problems; none of the non-rereaders used this strategy when they read the problems aloud.

A third helping strategy used by some students was that of vocalizing while they reread the problems. One student explained that she does this to "help it sink in." The two students who did this both took a meaningful approach to understanding the problems.

A fourth aid to orientation was discussing the problems with the interviewer. In most cases this was not a strategy initiated by students, because the discussions usually resulted from questions posed by the interviewer. These questions and the ensuing discussions often influenced students to go back and reread problems, assess their understanding of the problems, and reanalyze problem conditions. Students sometimes asked for clarification of conditions they were unsure of, and often they eventually achieved a better understanding of problems. Students made assessment comments such as "This is definitely wrong"; they asked and reflected on clarification questions such as "Can they mix up the change or do they have to be the same?"; and they had realizations such as "They are not going to shake [hands] with themselves." The students who benefited most from such discussions were those who did not previously attempt to get the full meaning of problems. It was these students who needed the discussions to influence them to engage in the kinds of reflective and analytic activities that the others engaged in without external suggestions to do so. However, one student who usually reread problems to get a full understanding did find a discussion of one problem revealing and commented "I forgot . . . wish I would have thought about it more." He realized that he had not given the problem conditions enough consideration in this case.

Organization

Approaches to Organization

Four different levels of organization were identified, and they seem, to a limited extent, to parallel the four levels of orientation just discussed. Like those in orientation, these levels are useful ways of characterizing students' activity. And, also like those levels, these are not completely distinct from one another, and are often used in conjunction with one another.

The levels of organization can be described as *guessing*, *eliminate operations*, *partially meaningful*, and *fully meaningful*.

The guessing approach is exactly what its name implies. A student taking this approach tries something without much of a rationale. For example, after having obvious difficult[illegible] reading the Atlas problem aloud, a student did not reread it and immediately divided 4 into 225. She stated "I'd try it first, I wasn't sure if I did it right" and could offer no other rationale. Another student who did not reread a problem said this of her solution strategy: "I'm not su[illegible] . . I think it's right . . . but I don't think it's right . . . I'm not sure what else to do." This student did not guess as randomly as the previous student, and did have some very limited sense of the correct operation. Anot[illegible] student had a general strategy that combined the guessing approach with the next level, the eliminate operations approach. This strategy could be described as the "try all" strategy.

It involved trying "all of them instead of choosing" -- all possible operations, or at least several (usually three) different operations, and choosing the result that "sounds right." Her strategy for deciding which was the correct result was described as "common sense . . . if it sounds right and you come out with the numbers in the actual problem (when you check using the inverse operations)." This strategy was illustrated in this student's approach to the Atlas problem where she added, multiplied, and divided various quantities in the hope that some result "made sense." She also used it when working on the Luncheon problem. There she started out with a partially meaningful strategy -- multiplying amount by cost, but after she assessed her progress by number considerations, she began to multiply and divide quantities almost indiscriminately. As was the case with these latter two students, other students used a guessing approach in combination with some of the others described below.

This was also true of the eliminate operations approach. This approach is when an operation is chosen because others have been eliminated or ruled out. For example, a student who did not take a fully meaningful approach to orientation described her solution to the Luncheon problem by saying "just a guess . . . how much it was," and in particular when subtracting the expenses said "you couldn't divide, couldn't add." In this case the student also had a partial understanding of the problem. Another student, in discussing her work on the Atlas problem said "I knew it wasn't division. I decided to multiply." And later she said she multiplied amounts by costs to get "how much money they made of one type . . . then I did the rest of them." She eliminated division but also had an understanding of the meaning of some of the problem conditions and relationships between them -- her decision to multiply was not based solely on the fact that she eliminated other operations. This student's work can also be described as partially meaningful. This approach to organization is when the student's plan to solve the problem is based on some understanding of the problem conditions -- quantities, relationships between them, and the questions being asked, but not a complete or thorough understanding.

Another example of this approach is one student's approach to the Coin problem. Her strategy was to "see what numbers made 50." This was meaningful, but sh[illegible] not have the full meaning of the conditions -- she overlooked the fact that she was dealing with coins and used numbers which did not correspond to coin values. In addition, her listing of combinations was unsystematic, and she did not conduct a thorough check for completeness. Her approach to the Felipe problem was similar. Her approach to the Shuttlemeier problem was also partially meaningful. She did realize that she should multiply amounts by costs to determine "how much it cost," but she did not completely understand the meaning of the pricing conditions, and she was unsure about what to do afterwards She said of her work "just guessed . . . if you try something and it doesn't work you can try something else." Another student who had a partial meaningful approach to the Atlas problem said she multiplied "300 times 40 to get the total cost," and added "I subtracted 2500 from the total it cost the factory to convert, I added this up (unit prices) and subtracted it from that." Here she had some elements of a meaningful plan, but it was not fully meaningful because she was unsure what to do with some of the numbers. A final example of the partial meaningful plan involves the Shuttlemeier problem. Here the student calculated costs correctly, then added them up. After she looked at the total she remarked "what will they have to charge? . . . at least 100 . . .." She was confused and had no idea what to do at that point. In general, students whose organization was partially meaningful had either a plan that was meaningful but incomplete, or meaningful but ignored some important condition. In the case of the Coin and Felipe problems, students did list combinations and possibilities but usually the lists were unsystematic, incomplete, or both.

This stands in marked contrast to the lists of students whose organization was fully meaningful. Students using a fully meaningful plan had a good understanding of what they were doing and where they were going before they actually started carrying out their plan. It is not surprising that these students were the same students whose approaches to orientation were fully meaningful. Their plans were almost always successful. For example, one student, after he thought it through, verbalized his thoughts as he worked on the Atlas problem, saying "gotta multiply the number of tons by how much a ton cost . . . now I got to add the answers together . . . now I have to find out how much it cost to make it." Another student described his approach in this way "how much it would cost them to process and buy the steel . . . then see how much they would make altogether, then subtract." This student's plan for the Felipe problem was to "break up the span into groups of ten, put the number of keys in each, then total up."

The students who took a fully meaningful approach to orientation, and thus to organization, developed more organized lists than the others when working on the Coin and Felipe problems. The fact that the lists were more organized made them easier to check for completeness, which was also a concern of these students. One student commented about his work on the Coin problem "I started with a dime and ended with a dime . . . I forgot to begin it with all nickels." Another looked over her very organized work on the same problem many times to see if she could "think of any more." A third made comments such as "I wanted to make sure I had all of them," and "to check I had all possibilities."

Aids to Organization

As was the case with orientation, students' organization and plans were greatly aided when students recognized the problems. This was especially true in the case of the Felipe problem and the NFL problem. Examples of comments related to the relationship between recognition of the problems and plans to solve them included "I had a couple like that in class. . . . I knew how to solve it before I came in here," "I've done one just like that in class so I knew what we were doing," "I've done a problem like this before . . . it was a streamer and pole problem," "I've seen something like this before . . . with crepe paper." These last two comments point out that students did not merely recognize cover stories, but -- more importantly -- recognized something about the structure of the problem. It must also be pointed out that recognition of a problem's type did not necessarily guarantee success in solving it, but it did guarantee at least a partially meaningful plan.

One student commented about the Felipe problem "I knew what to do . . . we've done something like this in class with charts." This student, and some of the others, used charts to help organize their listing and counting, and they used diagrams, like dots and circles to represent cities, and lines to connect them to each other. These new strategies, which had been addressed in the instruction, seemed to especially benefit some of the weaker students. Students seemed to realize their usefulness, and for the most part, use them appropriately. One student who did not always use her newly found heuristics appropriately commented "I saw problems like this before . . . when you don't know what to do, draw." This student, who relied heavily on a number consideration approach to orientation, also liked the strategy of looking for a pattern. She noticed that the numbers in the Shuttlemeier problem "sort of made a pattern," and began working on verifying and using the "pattern" without any consideration of the meaning of the problem conditions. Her final solution had almost no relationship to the problem statement.

Execution

Students used a variety of strategies and actions to help implement their solution plans. One such strategy was to use their pen or pencil to help them carry out a number of actions. Students used pen movements to help count numbers, to help keep track when they were adding, and to count other items (e.g. telephone lines drawn, combinations of numbers adding to 50 . . .). One student used her pen to draw in guidelines when lining up columns while multiplying. She explained this by saying "I write crooked . . . if I don't put the lines, I screw up the calculations." Some students used such pen or pencil movements when they were assessing or checking their implementation "to make it easier." These students used their pens to quickly run through their calculations or coin combinations. One student explained the use of his pen to check combinations by stating "instead of just thinking in my head, I'm doing something physical . . . it helps but it slows me down."

Another popular strategy to help in calculations, particularly in the addition of a long column or series of numbers, was that of subtotalling. Some students added numbers in pairs and then added the pairs, while others just kept running subtotals. These strategies had a number of different variations.

Students also used various types of labeling or numbering strategies. For example, in connection with the coin problem, one student numbered his combinations "to make them definite, separate them, so I know they're different." Another explained that she labeled her combinations by putting d's, n's, and q's next to them to help check for repeats or "same ones." Another numbered her combinations to help "from getting 25 and 2 mixed up" because "all of the numbers are jammed together."

Another strategy used to help implementation was that of tallying. In solving the Shuttlemeier problem, for example, students tallied the number of papers or "copies made so far," to help keep track of how many papers were still needed. One student explained she did this to "know where I am."

Some students, more precisely those who were very concerned with having fully meaningful plans, were not very concerned with the correctness of their calculations and did not check them. However they had other strategies to compensate for their lack of checking. Two of the three students worked their calculations very slowly. One mentioned "I usually check along the way." The other stated "I know I have a plan to follow, I block out everything . . . instead of analyzing every aspect to get the answer . . . right calculations . . . I'm not so much interested in that . . . more interested in knowing how to do it." The third student vocalized her calculations rather than checking them. This vocalizing helped keep her "mind on it." She explained "when I think it, it doesn't sink in . . . I have to think harder . . . when I'm thinking there are lots of other thoughts in my head." This student also vocalized through many of her actions.

The last execution action to be mentioned is that of trying to keep from getting "messed up" because there was too much written on the worksheet. A few students were comfortable working on a sheet which was filled with calculations and other writings, but some wanted to cross out discarded calculations "not to get messed up." Others just wanted to move their work to "a clean spot," or just to start all over. This strategy, of course, can be helpful; but it was not always used wisely. One student crossed out combinations of coins so she "wouldn't use them again," not realizing that once combinations were crossed out, she would not be able to tell if she had used them before.

Verification

Evaluation of Orientation and Organization

In many instances it is not possible to separate evaluation of orientation from evaluation of organization. This is because very often the two were so closely tied together that they were evaluated simultaneously. Some attempt will be made to discuss these separately, but only when it is possible and reasonable to do so.

It is not surprising that those students who did not reread problems did not go back at any time to assess their understanding of them. One student who did not reread a problem did go back to it at one point, but only to "make sure it's 100 not 1000."

Students who did reread problems initially, but who did not take the fully meaningful approach to orientation, were inconsistent with respect to going back to the problem at some point to evaluate their understanding of it. For the most part these students did not go back to the problems in any obvious way to check their understanding. It seems likely that, at least in some cases, these students felt that they had a sufficient understanding, possibly because they spent some time evaluating while they were rereading and analyzing it initially. However, in a few cases students did go back to the problem to reread it. For example, one student working on the Kennedy problem went back to reread it after she obtained an obviously wrong quotient. She explained "I knew it was wrong so I went back to the problem to see if I was doing it right . . . to make sure I had the numbers copied right." Note the emphasis on the numbers. A second student reread a different problem because she wanted to see if she "was doing it right, if it was division." Another went back to the Coin problem to determine "can they mix up the change or do they have to be the same?" A third reread a problem "just to make sure it's division," and later explained "to see if I got the right numbers and if it made sense."

The students who took the fully meaningful approach often assessed their understanding while they were initially rereading and analyzing the problems. Some of their comments about rereading seem to illustrate this. However, two of these students also did go back to the problem statement after working on a solution. One claimed she "went back to think about the question" and to "see if it makes sense." A second student, who did an extensive amount of evaluating, made comments such as "to make sure I did the right thing," "just to make sure that's what I should have done," "to make sure I was supposed to divide," "to check if I didn't leave anything out and that the method was correct," and "anything I missed before, I didn't understand well."

Some of this student's comments point out that evaluation of orientation and organization take place simultaneously. Making sure that one "did the right thing" is evaluating both. However, there are some aspects of evaluation which seem to be somewhat more directed to organization, even if not entirely so.

The majority of students, on a majority of problems, did not do a sufficient evaluation of their plans. Most did not check the reasonableness of what they did or what they were doing, or where they were heading. However, some students, on some problems, did undertake some strategies for evaluation, even if they were not always based on a criteria of meaningfulness.

One strategy for evaluating one's plan of action is that of *assessing a plan by carrying it out*. Some students assessed their plan, and to some degree their understanding by carrying the plan out and then seeing if the results "made sense." This is certainly the type of assessing that was used in conjunction with the "try all" strategy described under organization, where a number of operations were carried out and the one that "sounds right" is chosen as the correct one. This evaluation strategy was not limited to the "try all" plan but was also used in conjunction with other types of plans. One student commented "if it doesn't work, I'll try another." This assessment strategy was used primarily, but not exclusively, by students who did not have fully meaningful plans.

This strategy was also used for assessing part of a plan rather than the whole plan. One student, who had a meaningful plan and preferred to assess it by evaluating its meaningfulness, was unsure about one part of it and assessed this part by carrying it out, only because she "didn't want to sit there a long time doing nothing."

Another evaluation strategy can be called *assessing by number considerations*. This strategy, which can be considered a form of the above strategy, judged the appropriateness of a plan by the numbers which result from calculations made in carrying out the plan. This is a very reasonable strategy, because it can alert a student to an unreasonable result which might be due either to the incorrect choice of an operation or to an incorrect calculation. But this evaluation strategy can sometimes mislead a student who uses it to replace an assessment based on the reasonableness of the plan. This strategy was used by a number of students working on the Luncheon problem. When students calculated the total income of $52,050 and compared it to the expenses of $5,000, some of them paused. One student, who analyzed the problem conditions and had a meaningful plan, commented "not really sure about the multiplication . . . 52,050 and there's only 5,000 . . . there's a big difference." He assessed the reasonableness of the operation and proceeded to subtract. Two other students, whose plans were partially meaningful but who were not very confident in them, were thrown off course by these same numbers. One commented "520.50 doesn't sound right, maybe 5,205 or 52,050, but that seems too much." This student subtracted the correct quantities but thought the resulting difference sounded "too much" and abandoned this method for one that was not reasonable. A third student started out with a partially meaningful plan, but when she compared the 52,050 and 5,000 commented "it's not anything close to it," and also abandoned her meaningful plan for the "try all" method.

A different kind of evaluation, related to the *elegance of the method*, was made by all three of the students who had fully meaningful understanding and plans. They each made a comment on the listing strategies that they used to solve the Coin problem. In commenting about her listing one student said "it's awkward doing it this way . . . write down all of the combinations you can make . . . hard to think what else might fit into it." A second stated "I was just trying to figure out an easier way to do it than listing, but I couldn't think of one . . . some clue to figure out a formula to make it easier. The third student commented "isn't very good, there's a better way . . . like a chart, so after you're done you can check it over . . . more self-contained." None of the students who did not have a fully meaningful understanding of the problem made any remarks like these.

One final evaluation strategy, which can also be considered a form of the carrying out strategy, deserves mention here. It is one student's assessment of her initial solution to the Handshake problem. She thought it might be appropriate to multiply 10x10, but didn't seem completely comfortable with that. She raised her pen into the air and carried out an imaginary calculation of 10x10, looked at the spot where it would be and said "no . . . not 10 times 10 because people don't shake 10 times, they shake 9 times."

Evaluation of Implementation

In many instances the evaluation of implementation was implicit in the implementation itself, and hence the section on implementation above does discuss some aspects of evaluation. However, some of these aspects will also be mentioned here.

Most students did not check their calculations at all, although a few carried out evaluation procedures on most problems. Some students checked their work sometimes, and sometimes did not, and they used a number of strategies to evaluate their implementation, particularly their calculations. A few calculations were checked by being repeated, a few others were checked by their inverse calculations, but more often (although not very often) calculations were checked by students quickly running their pen or pencils through the original calculation. Very often, however, calculations were not checked and a good number of errors were not found. As mentioned above, some students, particularly those more interested in having a good plan, were not very concerned about the correctness of their calculations. Some students used their pen or pencil to run through their combinations of coins, or other listings to check over them for completeness. Students explained this by stating "making sure I put the right amount of nickels," "making sure I hadn't missed something like a 10 there or a 5 here," and "making sure I had the right number of teams."

Some Pre-Post Instruction Differences and Similarities

The above observations were all based on student performances both before and after instruction. Because there was not a dramatic difference between pre- and post-instruction interview performances, little effort was made in the preceding discussion to distinguish between the two. However, there were some interesting differences and similarities. Below are some observations related to these.

Helpful Changes

Some post-instruction changes that aided student performance were:

1. Most students recognized some problem types. They saw structural similarities between post-instruction problems and problems given to them earlier (e.g., Handshake and NFL). They also remembered elements of successful solution strategies and were able to implement them.

2. Some students spent more time and effort trying to analyze problem conditions after instruction than they did before instruction.

3. Some students were able to use appropriately some newly learned heuristics, namely using charts and tables, using diagrams, and looking for patterns.

4. Some students were somewhat more systematic and organized. This is due largely to 1, 2, and 3 above.

Detrimental Changes

Two observed post-instruction changes were detrimental to student performance. Fortunately these were limited to one student each.

1. One student who reread problems before instruction became overconfident

and did not reread any problems after instruction.

2. One student, who approached problems with a strong emphasis on number considerations, rather than on an understanding of conditions, became hooked on looking for patterns and did not always do so appropriately.

Resistant to Change

Some undesirable aspects of student performance did not change significantly with instruction. These include:

1. The student who did not reread any problems during the pre-instruction interviews did not do so after instruction either, even though problem rereading and understanding were emphasized during instruction.

2. Even though it was mentioned above that some students did spend more effort on analyzing problem conditions, some of these same students (and some others as well) still did not do enough analysis.

3. In connection with 2, some students still relied on previously used strategies (e.g., number considerations, "try all") that were unrelated to analyzing conditions for meaning.

4. Even though it was emphasized during the instruction, there was insufficient assessments of progress and reasonableness of results on the part of many students.

Possible Reasons for the Lack of Significant Changes

There are a number of possible explanations that can be offered for the lack of more significant pre-instruction and post-instruction changes in student performance. These include:

1. There may not have been enough time spent on problem-solving instruction to influence more significant changes. It seems likely that twelve weeks of intermittent instruction cannot turn all unsuccessful problem solvers into successful ones.

2. Since the students were still getting regular mathematics instruction between the periods of problem-solving instruction, their previous approaches and strategies were still being confirmed and reinforced. This situation may also have given some students the impression that problem solving and mathematics are somehow different from each other.

3. Students may have viewed the regular instruction as being more important than the problem solving instruction because grades on tests of the

regular material were being counted more towards their report card grades than scores on problem-solving assignments.

4. The problem-solving instruction might not have emphasized all that needed to be emphasized and/or it might not have emphasized aspects effectively.

RESULTS AND OBSERVATIONS WITH RESPECT TO STUDENTS' WRITTEN WORK

The following discussion of students' performance on written work is divided into four sections: pretest and posttest performance, classwork performance, homework performance, and case studies of the written work of three students.

Pretest and Posttest Performance

A description of the content and intent of the pre- and posttests is provided in Chapter 3. Copies of the tests themselves are provided in Appendix A. The discussion that follows here focuses on the students' performance on these tests rather than on the tests themselves.

Performance on Solving Problems

In brief, both the pretest and the posttest contained five matched word problems to be solved. Each of the five posttest problems was a simple variant (i.e., structurally equivalent) to a problem on the pretest. (In addition to the five problems matched to problems on the pretest, the posttest also contained a sixth and seventh problem -- students were to choose one of these additional problems to solve.) For each problem on the pre- and posttests, students were to show all their work and to record their answer at the bottom the page on which the problem was done. The students' work on each problem was scored 0, 1, or 2, yielding a maximum total of 10 on each test for the 5 matched problems. Thus, the pre- and posttests were designed to yield evidence on the students' progress after a semester of work in the problem-solving class.

As can be readily seen from Table 7, both the regular class and the advanced class showed an overall gain in total score from pre- to posttest. It is interesting to note that the pretest scores of the advanced class exceeded the posttest scores of the regular class. Four students in the advanced class scored a perfect 10 on the pretest (only two scored a 10 on the posttest), whereas no regular class student scored 10 on either the pre- or the posttest. The presence in the advanced class of several perfect scores, and the fact that only two students in the advanced class scored lower than 8 on the posttest may indicate a ceiling effect on these tests for this class. It appears that more challenging tests would probably have provided better indicators of problem-solving ability and growth in this class. The students in the advanced class may well have learned more about problem solving than their test results indicate. On the other hand, the tests did not seem to be too easy for the regular class, and it was not possible for us to know beforehand how much difference in ability there would be between the two classes.

Table 7

Student Performance on Written Tests by Class

Class	Pretest[a]		Posttest[a]	
	Mean	Range	Mean	Range
Regular Level (N = 22)	4.73	1 -- 9	6.00	2 -- 9
Advanced Level (N = 34)	6.74	3 -- 10	8.38	6 -- 10

[a] maximum possible score = 10; minimum possible score = 0

An interesting and perplexing result is seen in the breakdown of students in each class according to whether their scores increased, remained the same, or dropped from pretest to posttest (see Table 8). In each class, the group of students whose scores decreased from pre- to posttest had higher scores on the pretest than the group whose scored increased, with the reverse being true on the posttest. Both groups of students whose scores increased out-performed those whose scores decreased on the posttest. The disordinal nature of this interaction leads us to suspect that it represents more than simply regression toward the mean.

It may be the case that many of those students whose scores decreased had been using successful control strategies prior to instruction, and that these strategies were interfered with by the instruction. These students may have tried to give up their previously useful strategies to adopt the newly taught ones; or they may have tried to combine some of each; or they may have been more deliberate in using their strategies after instruction to the point where it inhibited their performance. Whatever the case, it appears as though they were hampered by the instruction.

Table 8

Types of Change in Performance on Written Tests by Class

Class	Pretest Mean[a]	Posttest Mean[a]
Regular Level		
Students whose scores:		
increased (N = 15)	4.13	6.53
remained same (N = 3)	4.33	4.33
decreased (N = 4)	7.25	5.25
Advanced Level		
Students whose scores:		
increased (N = 25)	6.00	8.67
remained same (N = 2)	7.50	7.50
decreased (N = 7)	9.14	7.59

[a] maximum possible score = 10; minimum possible score = 0

The group whose scores increased, on the other hand, seemed to have benefited from the instruction. Perhaps they now had strategies to use, whereas prior to the instruction they did not. At this point it seems that the instruction was more appropriate for the weaker students in each class. We are not sure why this would be the case, or whether or not the increases and decreases would endure over time.

Perhaps the major observation to make about the pre/post test results is simply that they point up an often discussed and much debated difficulty in problem-solving instruction: the difficulty of finding straightforward, easily scored assessment measures for such a multi faceted skill.

Responses to Self-Inventory Items

Both the pretest and the posttest also contained a series of four multiple choice self-inventory items accompanying each of the problems to be solved (see Appendix A). Each item offered three response options. These self-inventory items were designed to collect information about the students' personal awareness of their thinking, of their abilities, and of their feelings related to problem solving (i.e., the items were designed to tap certain aspects of the students' metacognitive awareness). The four areas asked about were (1) self-assessment of the difficulty level of the problem under consideration [a) easy, b) medium, c) hard], (2) student's confidence in his or her solution to the problem [a) sure it was right, b) might be right, c) sure it was wrong], (3) familiarity with this type of problem [a) never seen before, b) seen before but don't remember how to solve, c) seen before and do remember how to solve], and (4) interest in solving problems like the one under consideration [a) like doing problems like this, b) don't mind doing problems like this, c) dislike doing problems like this]. Whereas these items were assessed only by multiple choice questions in the pretest, in the posttest students also encountered, after each multiple choice question, a probe question that asked why they had responded as they did and that provided space for an open-ended explanation. Responses to the multiple choice self-inventory items were awarded points as follows: 3 points for an answer "a," 2 points for an answer "b," and 1 point for an answer "c." Note that since there were five test items, students could score a maximum of 15 on each item.

The first self-inventory item asked students for their assessment of the ease of each problem (i.e., higher scores indicated easier, lower scores harder). The regular and advanced classes had mean scores of 11.12 and 12.32 respectively on the pretest on this measure of ease. On the posttest, these means were 12.00 and 12.54 respectively. In spite of the fact that quite a number of students, particularly in the regular class, had difficulty solving the test problems correctly, the students apparently felt -- both before the instruction and after -- that the test items were relatively easy.

The second self-inventory item asked students to judge whether they thought they had gotten each problem right or not. Mean scores on the pretest for the regular and advanced classes were 11.61 and 12.76 respectively, and for the posttest were 11.77 and 12.88 respectively. Students were for the most part not very good judges of their skill in problem solving, since these scores indicate that many believed they either "got the problem right" or "might have gotten the problem right" even when they actually made major errors in solving.

The third self-inventory item asked students to indicate how familiar they felt they were with each problem. Mean familiarity scores on the pre- and posttests were 11.53 and 10.65 respectively for the regular class and 12.65 and 12.43 respectively for the advanced class. Although the pre/post differences in means are quite small for both classes, it is nevertheless interesting to note that these differences are negative. On the whole, students in both classes felt less familiar with the problems after the instruction than before. We believe this finding is probably due to the fact that students were exposed to many new and different types of problems during the problem-solving instruction. As a result of this exposure they began to be much more aware of subtle differences among problems. Thus, problems that on the pretest might have been judged familiar because they contained a somewhat familiar context or a relatively familiar mathematical operation, were labeled unfamiliar after the instruction because the students had worked many more problems and could see that the problems on the test were not "just like" any that they'd done before. In other words, students may well have applied stricter criteria in judging familiarity after the instruction than they did before.

In the fourth self-inventory item, students indicated their level of enjoyment in doing each problem. Pre- and posttest means for the regular class for the "enjoy" construct were 10.41 and 11.72, respectively. For the advanced class they were quite similar: 10.88 and 11.42 respectively. Thus, over time, students in both classes indicated a small increase in enjoyment in solving the types of problems on the tests.

No significant correlations were found for either class among scores on the various metacognitive items, nor between those scores and problem-solving scores on the tests.

Classwork Performance

Copies of all the problems assigned for classwork for both the regular and the advanced class appear in Appendix C, organized by day on which they were presented. Students frequently worked cooperatively in small groups on in-class assignments, although they occasionally were asked to work independently, and there was always some large-group follow-up and discussion after any classwork (either group or individual) had been completed.

Classwork was not graded, so it is not easy to generalize about students' classwork performance. The rationale for not grading classwork stemmed from the instructor's belief that class time should be a time for students to experiment with new techniques, to practice new strategies and skills, and to learn how to solve new types of problems. The instructor did not want to penalize students for taking a long time to think about problems, or for trying new, unperfected methods in class. After problems had been completed and discussed in class, he sometimes assigned similar or related problems for homework -- and these homework problems were graded. For the most part, students participated willingly in class problem solving. There were few discipline problems, although there were a few students (especially in the regular-level class), who put forth a minimal amount of effort and who often had to be cajoled into working during classtime.

Homework Performance

Each of the classes in the study (regular and advanced) was given 8 homework assignments over the 12 week period of the study. These assignments were not always the same for both classes (because the classwork of the two classes was often different). (Copies of all homework assignments are included in Appendix C.) Homework assignments were graded with a total of 10 points possible. If there was just one problem to be solved on the assignment, 4 points were given toward evidence of *understanding* the problem, 4 points were given toward evidence of *making a reasonable plan and carrying it out*, and 2 points were given for the *answer*. (The 10-point scoring scheme is elaborated in Appendix A-5.) When more than one problem was included in an assignment, the same categories of points were used, but the points were assigned proportionally to each problem (e.g., if there were two problems, they would each be graded 2 2, and 1 for understanding, planning & execution, and answer).

A disappointment in the study was that most students were not as conscientious about turning in homework as the instructor had expected they would be. In particular, many students in the 5th period (regular) class seemed to take the homework assignments rather lightly. Table 9 shows the percent of students who turned in each of the eight homework assignments in both the classes, and Table 10 shows the mean scored obtained by those students who turned in homework. After the second homework assignment, which 86% of the

5th period class turned in, the percent of students in that class who turned in their homework declined steadily. Only 30% of the 5th period class turned in the final (8th) assignment. Students in the 6th period (advanced) class were somewhat more consistent and more conscientious about doing their homework.

Table 9

Percent of Students Who Turned in Homework

	Homework Number								
	1	2	3	4	5	6	7	8	Overall
Regular Class (N=28)	68	86	75	59	44	37	37	30	55
Advanced Class (N=37)	62	76	65	81	78	54	76	54	68

Table 10

Mean Score[a] of Students Who Turned in Homework

	Homework Number							
	1	2	3	4	5	6	7	8
Regular Class	8.3	7.1	7.7	6.1	5.4	5.9	4.6	5.5
Advanced Class	9.3	7.1	8.5	6.2	5.1	..4	6.6	5.6

[a]10 points possible for each homework assignment

Nevertheless, the largest percentage of homeworks turned in for the advanced class was still only 81% (assignment #4), while the lowest was 54% (assignments #6 and #8).

Case Studies of the Written Work of Three Students

When we decided, for the purpose of this final report, to go beyond merely reporting test results by including an in-depth look at the written work throughout the semester of several students, we encountered the problem of how to choose the students who would be the subjects of these case studies. In order to assure a choice of students who reacted in very different ways to the problem-solving instruction, we decided to make our selection based on pre- and posttest scores and on responses to the self-inventory items. It was not difficult to identify students whose scores increased (or decreased) from pretest to posttest. But in looking at patterns of scores on the self-inventory items, we noted that the questions concerning interest and difficulty yielded information that was necessarily quite closely linked to the specific problems used on the pre- and posttests. Consequently, students' scores on these items were difficult to interpret in a more global fashion. On the other hand, even though the self-inventory items concerning confidence and familiarity were also presented along with specific problems, these questions seemed to have more potential for providing evidence about the students' confidence in and familiarity with more general problem-solving situations.

Thus, upon closer examination of the data from the pre- and posttests and pre- and post-self-inventories, we decided -- if possible -- to choose one student for a case study from each of four categories:

1) Category +/+: those who showed an increase in test score and in both confidence and familiarity,
2) Category +/-: those who showed an increase in test score, but a decrease in both confidence and familiarity,
3) Category -/+: those who showed a decrease in test score, but an increase in both confidence and familiarity, and
4) Category -/-: those who showed decreases in test score and in both confidence and familiarity.

There were no students who fell into the third category, -/+. Therefore, the three students discussed in the case studies that follow are students chosen as representatives of categories 1, 2, and 4. In reading the introductions to each student's case study, keep in mind that the maximum possible score on the problem-solving section of the pre- and posttests was 10. For each of the scores relating to self-awareness (confidence, familiarity, interest, difficulty), the maximum possible score was 15.

Student Representative of Category +/+: NG

NG -- a student in the regular class -- scored 2 points higher in the posttest than in the pretest in her problem-solving score (5 vs. 3), 3 points higher in confidence (13 vs. 10), and 3 points higher in familiarity (13 vs. 10). She also scored higher in ease (13 vs. 9) and in enjoyment (10 vs. 9).

NG was a very quiet student. attentive and hard working in class and diligent in her homework. At the end of the semester, Frank Lester awarded her a "B" as her grade for problem solving, and 10 points (of 10 possible) for effort. NG's grades represented quite good performance in the regular class. For both classes, the range of final problem-solving grades was intentionally designed to be consistent with the standards set by the regular classroom teacher, Barbara Willsey (i.e., neither more lenient nor more strict). NG was one of only 4 students in the regular class who earned 10 points for effort. Grades in the regular class included 1 A, 7 Bs, 9 Cs, 6 Ds, AND 3 Fs.

On the pretest, NG got only problem #1 (a 1-step computation problem) completely right. She labeled the problem medium in difficulty, and was similarly neutral in her other ratings of the problem (thought she "might have gotten the problem right"; she had "seen problems like it before, but forgotten how to solve them"; she doesn't "mind doing problems like this one"). She also got the matching problem on the posttest correct. This time, however, she labeled it easy in difficulty "because all that was to be done was multiplication." On the posttest she was more certain of her solution, marking "I'm sure I got the problem right" and she felt the problem was more familiar, marking "I've seen problems like this before and I remembered how to solve them." This increase in perceived familiarity is interesting because very little work was done during the problem-solving instruction with simple 1-step computation problems.

NG got 0 points on each of problems #2 and #3 on the pretest. Although she wrote some calculations on her paper, they did not consist of work that was useful in leading toward a problem solution. She improved to 1 point for each of problems #2 and #3 on the posttest, where she still failed to get a correct answer but nevertheless showed appropriate work for partial solutions to the problems

NG's first three labels for problem #2 (a 2-step computation) were the same for both pre- and posttest (easy, very confident, remembered how to solve), but her enjoyment in solving this type of problem decreased (from "like doing problems like this one" to "don't mind doing problems like this one"). It's quite possible that NG was not so enthusiastic about problem #2 on the posttest because she had worked on many other, more interesting, problems during the course of the instruction.

NG judged problem #3 (a process problem) hard on the pretest and medium on the posttest ("because I wasn't really sure how to do it."). On the pretest she felt certain she had gotten it wrong, whereas on the posttest she was unsure (marking "I might have gotten the problem right"). Although she marked that this type of problem was "totally new to me" on the pretest, she recognized it on the posttest -- but couldn't remember how to solve it. On both tests she noted that she disliked this type of problem.

On problem #4, NG got 0 points on the pretest (where she made just a few futile attempts at random calculations) and full credit (2 points) on the posttest (where she efficiently found the solution by use of a table). Her labels of the problem changed as follows: from hard to medium, from certainty that she was wrong to uncertainty ("I might have gotten the problem right"), from unfamiliarity ("totally new to me") to moderate familiarity (on the posttest, after circling the choice "I've seen problems like this before, but I don't remember how to solve them" she added the words "very well"). Most interestingly, her enjoyment went up considerably -- from "dislike doing problem like this one" to "like doing problems like this one." It seems that possession of a useful strategy made all the difference in NG's attitude toward this type of problem.

Finally, NG received 1 point on both the pre- and posttest versions of problem #5 (another process problem). In both cases she drew a diagram and performed logical calculations, but failed to monitor her work carefully enough to detect a difficulty inherent in her solution. On the pre- and posttests, her labels for this problem were identical except for difficulty. She labeled it of medium difficulty on the pretest and easy on the posttest. Otherwise, she was very confident, she found it very unfamiliar, and she was rather neutral in enthusiasm ("don't mind doing problems like this one").

Recall that NG not only improved her problem-solving scores from pretest to posttest, but also went up in confidence and familiarity. Examination of her classwork and homework provides some possible reasons why she may have found the problem-solving class such a positive experience. NG was more conscientious than other students in her class about completing her classwork and about turning in her homework. And she obviously put quite a bit of effort into her work, especially the work she did at home. Although her class papers generally sh[illegible] very neat, but average, work (including com[illegible]nts from the instructor such as "good start, but you misread a part of the pro[illegible]m" and "close, but not quite," NG's homework papers were far better than those of her peers. Seven homework papers were graded (on a 10-point scale) in NG's class. She turned in five of the seven homework assignments, receiving scores of 10, 8, 9, 6, and 9 on the papers (compared with mean scores for the class of 5.6, 6.1, 5.8, 3.6, and 2.4 on the same assignments, respectively). NG's homework was almost always carefully done, and showed influences of the problem-solving strategies she'd seen in class.

For example, as the semester wore on, NG used more organized lists and carefully labeled tables in her problem solutions. Use of these techniques was by no means second nature to NG. Even after she had done a number of problems (both in class and at home) in which numbers were carefully l[illegible]ed in a logical order, NG drew various configurations of stamps in nearly totally random fashion on her paper when she attacked a homework problem asking how many different configurations are possible for five attached postage stamps. Apparently NG did not immediately see the relationship between strategies for making logically organized lists of numbers, and strategies for making a logically ordered set of geometric figures. But once she had been shown in class that there was a logical way that the stamp drawings could be organized, NG incorporated some of these suggestions into her drawings of various configurations of card tables in a later homework problem that asked how many ways five tables can be arranged together. Although her systematic approach broke down after a while, NG showed that she was now much more aware of the need for organization in her work, and that she had some familiarity with how to achieve it.

NG's papers do not show any particular evidence that she improved in self-awareness or in monitoring skills. Her written responses to the self-inventory items that were included with a number of class and homework assignments were quite minimal throughout. Perhaps she was simply not very fluent in writing. But it is clear, both from her test scores and from examination of her written problem-solving work, that NG not only improved in problem-solving performance, but also went up in her assessments of ease, confidence, familiarity, and enjoyment. Increased familiarity with problem types and with problem strategies seem to have contributed to NG's increases in enjoyment and confidence. In this respect NG seems rather typical of many other students involved in the study.

Student Representative of Category +/-: AR

From the pretest to the posttest, AR -- a student in the advanced class --scored 4 points higher in her problem-solving score (5 on the pretest vs. 9 on the posttest), 1 point lower in confidence (13 vs. 12), and 2 points lower in familiarity (12 vs. 10).

AR is a very interesting student to study because her performance on the pre - and posttests seems counterintuitive: her problem-solving score improved considerably, but her confidence and familiarity decreased. However, upon closer examination of her work, this pattern begins to make some sense. AR was, in general, quite a good student -- a better student than NG. She was very intelligent, and reasonably well motivated. She earned a B in the problem-solving instruction. (In her advanced math class, there were 4 As, 11 Bs, 16 Cs, 4 Ds, and 2 Fs). No separate grades were given for effort in the advanced class, since motivation was not as much of a problem as in the regular class.

On the pretest, AR got only problems #1 (a 1-step computation) and #2 (a multi-step computation) completely right. She got 0 points for problems #3 and #4, and 1 point for problem #5. By contrast, on the posttest AR got all of problems #1-#4 completely right, and she again received partial credit (1 point) on problem #5. AR seemed to have made quite an improvement in her ability to solve nonroutine problems.

The first two problems on both tests were routine computation problems, and AR got them both right on both tests. But her self-inventory answers to these problems were somewhat different and interesting. She rated the first problem easy on both tests, explaining on the posttest "it only involves x [multiplication] and the plan to solve it is obvious." However, she was less confident about her work on problem #1 on the posttest, choosing to mark "I might have gotten the problem right, " even though she had been certain of herself on the pretest (marking "I got the problem right"). Even more interestin-ly, although AR claimed on the pretest, problem #1, that she "remembered problems like this and how to solve them," on the first probl m on the posttest she marked "I've never seen problems like this before." On both versions of the test, the problem was a straightforward 1-step multiplication word problem. AR must have been using some much more stringent criteria for judging familiarity for her to say she had never seen such a p oblem before.

Problem #2 on both tests was a two-ste- roblem that involved finding profit and AR got both problems right. However, she rated the problem on the posttest harder (marking "easy on the pretest and "medium" on the posttest), and explaining on the posttest that "it took some planning and a correct strategy." From her c nment it seems apparent that AR had become more aware of the importance of planning and of choosing strategies, but it is not clear why this would cause her to judge the difficulty level of problems differently. On both versions of the test AR was confident (ar correctly so) that she had gotten problem #2 correct. But her judgments of familiarity and interest were different on the two tests. On the pretest, she marked that she'd "seen problems like this before but forgotten how to solve them," whereas on the the posttest she was famil r both with the problem type and also with its solution. On the pretest she had been neutral in her preference for this type of problem (marking "don't mind doing problems like this one), but she rated the corresponding posttest problem more interesting (marking "I like doing problems like this one" and explaining in her own works "because it takes careful planning."). Clearly, AR had become very aware of, and interested in, the planning aspect of problem solving.

The third problem on both tests was a nonroutine problem that could be described as involving two linear relationships with two unknowns. Of course, AR did not know algebra, so she could approach the problem only in a less sophisticated manner. On the pretest she primarily made numerous random guesses, and received 0 points for her random efforts. Her paper shows some evidence of checking her guesses, but no evidence of a systematic method for generating guesses. By contrast, AR made a three-column table for her work on problem #3 on the posttest: two columns in which to record her guesses for the two variables, and one column for checking these guesses against one of the constraints given in the problem (that the values for the two variables had to sum to 18). AR's work on the posttest problem is not completely straightforward, but it is much more planful and better organized. Her self-inventory responses make it clear that she had learned some things in the course of the problem-solving sessions that helped her to solve problem #3 on the posttest. She marked the problem "hard" on both pretest and posttest, explaining on the posttest that "guess and check takes a lot of thinking so you don't end up where you started." In other words, AR not only recognized that a strategy she had learned during the course could be applied to this problem, but also recalled a label for that strategy and was insightful about difficulties involved in use of that strategy. Whereas AR was certain that she'd gotten #3 wrong on the pretest (and she had), she was confident that she'd gotten it right on the posttest (and she had). The problem had been "totally new" to her on the pretest, but on the posttest she marked that she had "seen problems like it before and remembered how to solve them." The posttest self-inventory asked students to provide an explanation whenever they marked that they had seen problems before, and AR complied by describing one of the guess-and-check problems from the class instruction. AR had marked that she disliked problems like #3 on the pretest (and this is not surprising since she was unable to solve it). After the instruction, her attitude had improved somewhat: she marked the problem medium in preference ("I don't mind doing problems like this one"), explaining her lukewarm endorsement by writing "because I don't like guessing and checking."

Problem #4 on both versions of the test involved a number pattern. On the pretest, AR made a neat list of numbers according to the pattern outlined in the problem, but made both a calculational error in listing and a logical error in interpreting the list. Consequently she received a score of 0. On the posttest, AR made a very similar list, but labeled it as a table. This time she made no errors in making her list, and she interpreted her work logically to arrive at the correct answer, so AR received 2 points for problem #4 on the posttest. On problem #4 of the pretest, AR was apparently unaware of her errors because she it labeled it "easy" and marked that she thought she "got the problem right." On the posttest, she similarly marked problem #4 easy ("just make a chart and add on"), but was more cautious in her confidence rating (marking "I might have gotten the problem right"). AR, who was generally a very good mathematics student, may well have become more cautious about her confidence ratings as a result of encountering challenging problems during the problem-solving instruction. Many of the better students commented that they had rarely encountered such challenging problems before. AR marked problem #4 as totally new to her on the pretest, but moderately familiar on the posttest ("seen problems like this before but I don't remember how to solve them"). Interestingly, AR's enjoyment rating for problem #4 decreased even though her proficiency increased: on the pretest she had marked "I don't mind doing problems like this one" but on the posttest she marked "I dislike doing problems like this one." The reason for her change in preference is interesting. AR did not like to bother with problems that she considered too easy; she preferred to be challenged. She explained her rating on the posttest by observing that "you just make a chart and add to the number -- they're too easy."

The fifth pair of problems to be compared on the pre- and posttest were isomorphic -- with the very same numbers and just the context changed. They were the caterpillar problem and the coconut problem, respectively. On the pretest, AR did something that many weaker students did not do -- she made a drawing. Unfortunately her drawing of a jar was mislabeled, she did not show any written evidence of keeping track of the caterpillar's progress up her graduated jar, and her final answer was incorrect, so she received 1 of 2 points for the problem. On the posttest coconut problem she started out by drawing a series of boxes to represent the different nights when coconuts were collected and stolen, then changed her method to keeping a table labeled with "#night" and "coco's left." Once again, her answer was incorrect, but AR's organized efforts earned her 1 of 2 points on the problem. AR marked problem #5 medium on the pretest and easy on the posttest (noting "chart -- boring!"). She was certain of her (incorrect) work on the pretest, but more skeptical and noncommital on the posttest (marking "I might have gotten the problem right"). On both versions of the test, AR claimed that this type of problem was entirely new to her (but she clearly felt that she probably should have recognized it on the posttest, because she added a note: "I don't remember!"). Finally, her interest rating was the same (medium -- "don't mind doing problems like this one") on both tests. She explained her answer on the posttest: "It's ok because it's easy -- but boring." AR clearly preferred a challenge.

AR was most motivated by interesting, different problems. When the class was asked to rate according to interest a list of 9 problems that had pre ously been done in class, the reason she gave to explain her choice for the most interesting was "because I like working backwards and I enjoy working with number puzzles." When asked what things make a problem interesting, she answered that interesting problems are those that are "out of the ordinary, challenging." On the other hand, according to AR, boring problems involve, for example, "just adding -- a 'no brainer'."

AR seemed very aware of the strategies she was using, and of the need to be planful in her work. For example, on the fifth day of class, after solving an in-class problem, she replied to the self-inventory question, "why was this problem hard/easy for you? by writing "it took me awhile to plan what I was going to do -- but it was easy from then on." On a videotape viewing guide, when she made notes about the good and bad points that she observed in the problem solver on the tape, she noted as a good point that the problem solver "thought it out, understood it, made a plan."

AR was also well aware of -- and proud of -- her intelligence. And she had confidence in herself, believing that she was a good problem solver. On several occasions on her classwork, when she was asked to "circle the strategies you used," she declined to circle any of the strategies that had been discussed in class, choosing instead to circle the more open-ended option "another strategy" and to write in a "strategy" such as "multiplication and brains."

Certainly, AR was intelligent, and a good problem solver too, but her written work was often quite messy, and this messiness sometimes led to errors. Several points that AR missed on the pretest were due to messiness or carelessness. AR was also quite verbal. For example, she did much more writing on homework papers, on classwork, and on self-inventories than NG. However, there is also some evidence of a lack of complete effort on AR's part. On several problem-solving homework papers, she turned in jus an answer, with no supporting work. There is no way of knowing whether she had actually done the problems on another paper, or perhaps simply copied the answers from someone e. AR also turned in two optional challenge problems that were only partially complete. The papers were returned to her with a handwritten note from the instructor, "you are so close, why not try to finish it?" But she didn't.

Even though AR was a already a good problem solver at the outset of the instruction, she seemed to have profited considerably from the problem-solving instruction. Her work on the pre- and posttests attests to her progress. But, in spite of the progress she made, AR's confidence seems to have gone down -- perhaps simply because that confidence was so high to begin with. AR thought she already know a lot about problem solving -- and she found out that there was much she didn't know. AR thought she was already familiar with most problem types -- and she found out that there were problems she hadn't seen before and problems she couldn't do easily. In other words, as she matured as a problem solver, AR simultaneously became less naive about, and more skeptical of, her ability to recognize and solve problems.

Student Representative of Category -/-: SS

From the pretest to the posttest, SS scored 3 points lower in his problem-solving score (5 vs. 2), 1 point lower in confidence (13 vs. 12), and 8 points lower in familiarity (15 vs. 7).

SS, a student in the regular class, was quite typical of many of the weaker students in the study (especially those in the regular class). He was a slow, often quite quiet student, who did not seem either as intelligent or as motivated as either of the two girls discussed above. He lacked proficiency in problem solving before the classes began, and he did not improve much, if at all, in the 12 weeks of instruction. His scores on homework assignments were 7, 10, 6, 2, 4, and there were three papers that he did not turn in. Several of his homework papers show very careless work, and any of his papers are very messy, even when his large immature handwriting is discounted. He received an overall grade of D for his work in problem solving, and a rather average effort grade (7 of [illegible]). In general, he was a very ordinary student: neither antagonistic nor bored, but also neither energe[illegible] nor enthusiastic.

SS's five points on the pretest came from getting the first and fourth problems correct (2 points each), and partial credit on the second problem (1 point). Unfortunately, on the posttest he got only the fourth problem correct (2 points).

The first problem on each test was a relatively straighforward 1-step multiplication word problem. In both versions, however, the problem contained an extraneous number in addition to the two numbers that needed to be multiplied to answer the question. SS got the first problem right on the pretest (although it is evident from his paper that he had at first multiplied the wrong numbers, then crossed out that calculation and performed the correct multiplication). On the posttest, SS multiplied the wrong numbers, and did not catch his error. Thus, he received two fewer points on problem #1 on the posttest than on the pretest. SS' self-inventory answers for problem #1 were identical for the two tests: he thought the problem was "easy," he was sure he "got the problem right," he "remembered problems like this and how to solve them" and he liked doing problems like this one. On the posttest, he explained quite simply (and with innovative spelling) why he likes this kind of problem: "simpile and I love to multipily."

The second problem on each test was a multi-step problem requiring students to find how much profit was made when a number of items were bought in a group for one price and later sold individually for another. On the pretest, SS's work showed that he understood that finding profit involves subtracting one price from the other, but he overlooked several other important points in the problem, and thus obtained the wrong answer. On the posttest, SS correctly divided to find the buying price for each item, before finding the profit for each item. Unfortunately, the question asked for the profit for the entire transaction. He received 1 point on the pretest and 0 points on the posttest. SS's self-inventory answers on problem #2 were very different on the two tests. On the pretest, his answers were exactly the same as they had been for problem #1 (the problem was easy; he was very confident; he recognized the problem and its solution; and he liked this type of problem). On the posttest, SS's different answers were apparently caused by his suspicion that he had done the problem incorrectly. He apparently wondered if he should have used division (and in fact, he need not have -- an easier way to approach the problem was to work with the entire group of items rather than with individual items). SS may have been made more uneasy by the fact that the division he performed involved decimals. He marked the problem medium in difficulty; he thought he "might have gotten the problem right"; he claimed he had "never seen problems like this before"; and he was lukewarm about doing such a problem (marking "I don't mind doing problems like this one"). He explained his responses by writing, "hard, do not like to divide."

On both the pretest and the posttest, SS got 0 points on the 3rd problem when he performed a division to attempt to answer these problems involving two unknowns. His use of division is totally illogical but ot unusual (Lester & Garofalo [1982] noted that an illogical approach involving division was common for this problem among the 5th graders they interviewed). The question on the pretest asked, given certain other constraints, how many chickens (2-legged animals) and how many pigs (4-legged animals) there are on a farm. SS's answer (neatly boxed in) of "2 legs per animal" makes absolutely no sense at all. On the posttest, the question was how many 2-point problems and how many 4-point problems were solved correctly on a test. This time, although his answer is incorrect, at least SS's appears to have understood what the question was: he answered "13 (4 pts) 6 (2 pts)." SS judged #3 on the pretest "medium" in difficulty, but #3 on the posttest "easy." On both tests he thought he "might have gotten the problem right." On the pretest, he claimed to recognize both the problem and how to solve it, but on the posttest he claimed never to have seen such a problem before. Finally, he was lukewarm in his interest rating for both problems (marking "I don't mind doing problems like this one"), and adding a 1-word explanation to his rating on the posttest: "channelagine" (challenging?).

It is very interesting to note that SS got problem #4 correct on both tests, because this nonroutine problem was designed to be more difficult than the routine problems (#1 and #2) on the tests. On the pretest, SS made a very orderly list, and obtained the right answer. On the posttest, his list is similarly orderly, but he also drew vertical and horizontal lines to make the list look like a chart, and labeled the columns of the chart. This action certainly seems to reflect the instruction he had received in making tables and charts. On the pre- and posttest, SS marked his self inventory for problem #4 with the following choices respectively: medium and easy in difficulty, certain and possibly certain in confidence, and very familiar and very unfamiliar in familiarity. On the pretest he marked "I don't mind doing problems like this one." (He failed to mark anything on that question on the posttest, except to write as explanation: "challen...").

Finally SS got no points on problem #5 on either the pretest or the posttest. These were nonroutine problems that were best solved by making a orderly list or table, or by drawing a picture. On the pretest, SS did a few very logical calculations, but was not careful in checking the implications of his work and consequently received no credit. On the posttest, he attempted to make a rather complicated 4-column table. He filled it in partially, then scratched out that work, and simply filled in an (incorrect) answer: 4 nights. SS failed to mark a level of perceived difficulty for problem #5 on the pretest, but he thought the problem was easy on the posttest. He was confident of his answer on the pretest, and only moderately confident ("I might have gotten the problem right") on the posttest. On the pretest he claimed to remember problems like this and how to solve them, but on the posttest claimed never to have seen problems like it before. Finally, he liked doing the problem on the pretest, but was lukewarm about it on the posttest, explaining that he thought it was "easy."

SS seemed to have a very limited sense of excitement in problem solving --he preferred doing familiar, routine, safe problems. When his class was asked part-way through the semester to look at a list of problem they had solved thus far, and to rank them according to interest, SS's most interesti problem was "find the simple interest on $600 at 13% for 1/2 year." Why? He ought this problem was interesting "because there is a reasonable amount of numbers to pick from." Like many of his classmates, SS apparently preferred a safe, familiar type of problem to a challenging, unusual problem. On the other hand, SS made claims that contradict this observaton when he wrote that "topic" and "challenge" make a problem interesting for him. Furthermore, he claimed that a problem that is "too easy" is the type of problem that's boring for him. SS's claims did not jibe with his choices of interesting problems.

Instruction in the problem-solving class was designed to expand the students' horizons by presenting them with new and different types of problems. In SS's case, the instruction may have also overwhelmed him with apparently different types of problems. He may well have been unable, in 12 weeks of instruction, to consolidate his new vision of problems, and to see that there were commonalities among them. Before the instruction began, he apparently thought he was familiar with problems such as those on the pretest. After the instruction he had become much more s ptical about his familiarity with problems, somewhat less confident, and also less proficient. It would be interesting to s e what SS would have gotten out of a problem-solving course that was twice as long as this one was. At the end of the course he appeared to have regressed in almost every aspect. Perhaps this only a temporary phenomenon as he attempted t consolidate new problem-solving skills and ideas wit his old, less efficient methods. It is interesting to speculate whether his performance might have turned about, swung back up, and perhaps surpassed his pretest performance if he had had more time to digest and practice the new skills to which he had been introduced.

Observations about Students' Performance on Written Work

What can be concluded about the written work of the students in the problem-solving class and the written assignments to which they were responding? First, it is worth mentioning once again, that the instruction seems to have been most effective with the average students. NG is an example of such a student. She had relatively few problem-solving skills when the semester began, and she showed considerable gain in the 12-week period. Her positive attitude and motivation may have played a large part in her receptiveness to instruction. AR, on the other hand -- in spite of her relatively low pretest score -- was apparently already fairly profici nt at problem solving even before the instruction begain. The instruction seems to have consolidated and honed skills she already had. The instruction did have the effect of making AR a bit more skeptical and less confident about her ability to solve problems. At the start of the study, AR may well have been somewhat overconfident because she was a good student and had rarely experienced anything but success in school before. Finally, the problem-solving instruction seemed to have had very little positive effect on SS, one of the students with less ability. SS did poorly on the pretest, and even more poorly on the posttest. Furthermore, he became less confident, and far less able to accurately judge the familiarity of problems.

A second observation about the written work is that the problem-solving homework assignments were not as successful as the instructor had expected. In chapter 3, lack of clear expectations is given as one reason for students' failure to complete homework assignments. Another distinct possibility is that the students had different expectations from the instructor for what mathematics problem-solving homework assignments should be. The students apparently were used to homework assignments that involved relatively routine practice on low level skills, whereas the homework assigned from the problem-solving class was rarely routine, and rarely involved low level thinking. The assignments were never merely short answer or multiple choice, and they were never simply marked right or wrong. Students were expected to really think about difficult problems, and to reflect on their own thinking. It appears that for many students this type of homework assignment was confusing because it violated their expectations. And, since the consequences of not completing homework assignments were often unclear, many of the students did not expend much effort on their problem-solving homework assignments.

Finally, examination of the written classwork of the two classes serves to underscore the difficulty of developing problem-solving instructional materials. The tasks used for classwork succeeded, in large part, because they were chosen day-by-day to fit the situation at hand, and because of the experience of the instructor and the attention he gave to individuals during class. Lesson plans were revised when students' reactions indicated a need for more practice with a particular skill or for more challenging and interesting problems. Class assignments had to be designed differently for the regular and the advanced class. Good problem-solving assignments are difficult to devise, and they cannot be teacher proof.

RESULTS AND OBSERVATIONS WITH RESPECT TO INSTRUCTION

Our approach to analyzing the instruction was to provide written accounts of the instruction from the point-of-view of three "major players" involved with the instruction. Since the instructional component of the study was exploratory in nature, we decided that it made sense to attempt to describe the instruction as completely as possible and from different vantage points. Consequently, in this section we present results and observations through the eyes of the problem-solving instructor (Frank Lester), the instructor's assistant (Diana Kroll), and the regular teacher (Barbara Willsey). In addition, essays written by several students who had participated in the instruction are also presented. Each view offers some insights into the nature of the instruction that are different from the others. Furthermore, each view aided in developing a reasonably clear sense about the effectiveness of the instruction and the potential value of the approach to problem-solving instruction exemplified by what took place.

The View of the Project Instructor

I began my first real job as a teacher (that is, the first time I was paid to teach an entire classroom of students) nearly 24 years ago in a large urban junior high school in Jacksonville, Florida. My assignment included teaching three seventh grade and three eighth grade classes in six different classrooms. In retrospect, I must say that that first year produced as much frustration as satisfaction and as much confusion in my mind as clarity. Nevertheless, having survived the experience, I decided that teaching was for me. I learned quite a lot about teaching during this period and even more about young teenagers (and near teenagers). Consequently, it came as something of a surprise that I found teaching two of Barbara (Barb) Willsey's seventh grade classes such a challenge. The students had not really changed very much. In the main, seventh graders in 1965 were worried about the same sorts of things as seventh graders in 1987 and, as I recall, students in 1965 had about the same mathematical abilities as their 1987 counterparts.

Apparently, over the years I was the one who had changed. In particular, I had become used to teaching university-age students, which meant that I had learned to be less concerned about classroom management, discipline, and a whole host of other aspects of teaching that come to be second nature for a veteran middle school teacher. For example, I did not constantly remind the students that a homework assignment was due tomorrow and I tended not to hold them immediately accountable for turning in assignments. With university students my attitude is that, once they have been informed that assignments are worth some percent of their course grades, it is the students' responsibility to decide whether or not to complete assignments. My seventh grade students expected to be reminded. again and again. And, when they were not reminded, many of them assumed that they did not have to take homework seriously. This belief may account for the low rate of completion of homework among students in the regular class.

On the whole my style of teaching contrasted in some very important ways from that of Barb. I tended to answer a student's question with a question rather than giving a direct answer or an explanation, and, in general, I was less directive in my instruction than she was. I was less concerned about "noise level" in the room than she was and, unlike her, I encouraged students to work with each other on class assignments. This is not to suggest that she should have been more like me; on the contrary, I found her to be a very effective and well respected teacher. I mention the differences in our styles because these differences provide perspective for explaining the lack of effectiveness of some of the instructional activities in which the classes engaged during the time I taught. An example may illustrate this point.

The original instructional plan called for the students to work cooperatively in small groups on most class activities. Although I had expected that there would be a need to give some attention to helping the students become accustomed to working in cooperative learning groups, I had hoped that no more than a few class periods would have to be devoted exclusively to this (after all, I had only 26 sessions with each class, and many were half period sessions). It soon became apparent that, if the students were going to learn how to work effectively in small groups, it would be necessary to devote several class periods over a number of weeks to social aspects of group interaction and cooperation. This need stemmed from the fact that the vast majority of the students had never before worked in small groups, at least not in math classes. As mentioned in the discussion of the as[illegible]tions that guided the development of the instruction, it was our belief that th[illegible]dard arrangement would be for students to work on class activities in small cooperative learning groups. Despite this belief, I decided to allow students to choose with whom they would work, if anyone, and not to attempt to develop a routine for group work (e.g., identifying a group captain, establishing a common set of rules for group work). It is possible that the impact of the instructional intervention might have been much more pronounced had cooperative learning been a regular part of class activity (But, see my comment about cooperative learning groups later in this commentary).

What I Have Learned from My Experience

As mentioned in chapter 3, seven assumptions (beliefs) guided the development of the instruction. These assumptions are restated here followed by a brief comment concerning the extent to which I presently accept these assumptions in light of my experience with these students.

1. There is a dynamic interaction between mathematical concepts and the processes (including metacognitive ones) used to solve problems involving those concepts. That is, control processes and awareness of

cognitive processes develop concurrently with the development of an understanding of mathematical concepts.

2. In order for students' problem-solving performance to improve, they must attempt to solve a variety of types of problems on a regular basis and over a prolonged period of time.

3. Metacognition instruction is most effective when it takes place in a domain-specific context (in the case of this study, problems were related to mathematics content appropriate for grade seven students).

4. Problem-solving instruction, metacognition instruction in particular, is likely to be most effective when it is provided in a systematically organized manner under the direction of the teacher.

5. Problem-solving instruction that emphasizes the development of metacognitive skills should involve the teacher in three different, but related, roles: (a) as an external monitor, (b) as a facilitator of students' metacognitive awareness, and (c) as a model of a metacognitively-adept problem solver.

6. The standard arrangement for classroom problem-solving activities is for students to work in small groups (usually groups of four). Small group work is especially appropriate for activities involving new content (e.g., new mathematics topics, new problem-solving strategies) or when the focus of the activity is on the process of solving problems (e.g., planning, decision making, assessing progress).

7. The teacher's instructional plan should include attention to how students' performance is to be evaluated. We assumed that in order for students to become convinced of the importance of monitoring their actions and being aware of their thinking, it would be necessary to use evaluation techniques that rewarded such behaviors.

I have essentially the same level of belief in assumptions 1, 3 and 4 and I believe even more strongly in assumptions 2 and 7. However, some doubt has developed in my mind about assumptions 5 and 6. Let me first discuss the doubts that have been raised in my mind about assumptions 5 and 6, then I will comment on assumptions 2 and 7.

Difficulties in Implementing the Three Teacher Roles (Assumption 5)

Although for the most part the instruction proceeded well, I did experience a number of difficulties in remaining "true" to the three roles specified by the instructional plan. One difficulty with the "Teacher as External Monitor" component was that in many cases the students were weak in basic skills. This caused me to spend much of the "external monitoring" time explaining how to do calculations or how to reason logically. One observation at this point is that teachers should expect to provide instruction in basic skills simultaneously with instruction in ntrol strategies and heuristics.

A concern also arose with some of the assignments related to the "Teacher as Facilitator" role. The problem was that the weaker students, in general, had trouble completing these assignments. For example, many of the regular-level students (fifth period) could not think of anything to record on self-inventory sheets where they were to have listed their strengths and weaknesses in problem solving. The advanced-level students had much less difficulty with this assignment. Similarly, when assigned a problem and then asked to write a narrative describing their thought processes as they had solved it, the advanced-level students complied with about a paragraph each. The regular-level students, almost without exception, failed to turn in a narrative. I suspect a combination of factors contributed to this outcome: the weaker mathematics students may have also suffered from weaker language skills to the extent writing such a narrative was beyond them; the weaker students were not as conscientious about completing the assignment; and the weaker students may have had more difficulty reflecting on their own thought processes.

Two concerns arose about the "Teacher as Model" role. First, it was no mean feat to come up with problems to use for this purpose. The difficulty stemmed from the fact that I wanted to be as authentic as possible as I modelled solving a problem. That is, I wanted to be truly engaged in solving a real problem, not merely pretending at times to be perplexed or struggling. Unfortunately, most tasks that really are problems for me are beyond the ability of seventh graders to solve. It would have been easy for me to "dazzle the students with my brilliance" by solving a problem that would have been far too difficult for them to solve, but I wanted to avoid this at all costs. This concern alone is reason enough to warrant giving this teacher role a second thought regarding its place in problem-solving instruction.

A second concern with modelling had to do with the difficulty I had in maintaining the role of expert. Of course, whenever I attempted to model the solution of a problem, I pointed out the importance of rereading to clarify understanding, I discussed why a particular strategy was chosen, I openly checked my calculations, and I always pointed out the importance of comparing the final answer with the conditions of the problem. However, my efforts to go a step further and have students observe all the control strategies used by an "expert" problem-solver as he solved a problem that he had not solved before were less than successful at first. I fell quickly into the role of teacher, and soon was explaining rather than modelling. Not surprisingly, the students found it hard to focus on me as an expert model, rather than as a teacher-explainer. Some of their notes on what the "expert" did well, and not so well, concerned the effectiveness of what they apparently considered a teaching demonstration rather than a demonstration of expert problem solving. They wrote such observations as "talked too fast," "wrote big and neat on the board," "didn't explain clearly," etc. However, a modification of the teacher-expert modelling procedures, in which the students viewed a video-tape of Diana solving a problem at a desk worked much better. The students had no expectations that she should "write neatly" or "explain clearly," since she was obviously just writing while talking to herself, not to them. Thus, they were better able to concentrate on observing the control strategies that she used.

In summary, it appears to me that considerably more thinking needs to be done about the role(s) the teacher should play during problem-solving instruction.

Are Cooperative Learning Groups Really Best? (Assumption 6)

I must admit to attempting to be somewhat provocative by posing this question. But, I think I have two good reasons for raising it. First, despite the large, and to some impressive, body of research evidence supporting the use of small cooperative learning groups in instruction, I think we must be careful not to infer from these results that problem-solving instruction can only be effective if students are organized in small groups. In fact, I can imagine situations in which forcing students to work "cooperatively" might have a negative effect on the performance of some students. We simply know far too little about the conditions under which cooperative learning groups truly enhance student problem solving.

The second reason for questioning the value of cooperative learning groups is more personal. When students are organized into small groups the teacher has far less control over the instructional activity and a fair bit of her or his attention tends to be taken away from mathematics instruction and given to classroom management matters. In the case of this study, I was confronted with students who had had little or no previous experience with cooperative learning in mathematics class (perhaps in any class). I simply was not prepared to devote the amount of time necessary to make small group learning work.

The point of my preceding comments is that, as a result of my experience with the two seventh grade classes, I am less strongly convinced of the validity of assumption 6.

The Importance of Long Term, Regular Instruction (Assumption 2)

If one hopes to become an accomplished pianist (or dancer, or basketball player, or writer, or painter, etc.), he or she must expect to devote many, many hours to playing the piano. The same is true of the individual who wants to become a better problem solver. It has long seemed axiomatic to me that problem-solving skill will improve only if the individual is willing to attempt to solve a wide range of problems on a regular basis over an extended period of time. Furthermore, such attempts cannot occur sporadically, they must take place on a regular basis. I have long held this belief and it was further reinforced by my experience teaching the two seventh grade classes. In a typical week I taught the class for slightly more than an hour (70 minutes was the modal time per week). During the first several weeks, many students in the fifth period class seemed not to be benefiting from the instruction. It appeared that there was very little, if any, carry over from one class to the next. My assessment of this is that they simply were not being exposed often enough or long enough to the kind of thinking needed to be successful problem solvers. Many of them had developed beliefs, attitudes, and habits about doing mathematics that were resistant to change. It was only toward the end of the 12 weeks of instruction that some of them began to demonstrate any discernible growth and a few had not changed even then. Foremost among my goals as the teacher in this study was to motivate students to be willing to make attempts at solving problems. For several students (especially in the regular class) my biggest challenge was to get them to make an attempt. Over the course of several years of school, a few of these students had come to believe that they could not solve mathematics problems, particularly if the problems involved topics such as percent, decimals, fractions, or ratio. Among those students who seemed to benefit little, or not at all, from the instruction, there appeared to be a direct connection between previous lack of success (and subsequently, failure to develop many basic mathematical skills) and willingness to engage in problem solving. The instructional approach implemented in this project did not appear to have had much, if any, effect on the problem-solving performance of these students.

Further, the problem-solving performance of some of the advanced students actually declined (a few students got lower scores on the written posttest than they had on the pretest). Although this is not surprising (it is common for performance on many sorts of tasks to decline when new techniques or skills are being learned), it serves to illustrate that short term problem-solving instruction can even have deleterious effects. For example, one student, whose score dropped from a maximum of 15 on the pretest to a score of 10 on the posttest, also experienced a 33% decline in her confidence in her answers from pre to posttest. Despite the fact that she was one of the most conscientious and brightest students in the advanced class, her exposure to a wider range of types of problems appeared to have made her more skeptical of her ability to solve difficult problems.

Another kind of complication arose for some other students. Some were becoming much more sophisticated as problem solvers, but they had not yet become comfortable with their newly developed skills and ways of thinking. For example, during one of the last sessions I had with the students I asked them to solve a problem that was very similar to one they had solved several weeks earlier. As I was walking around the classroom, I noticed one student staring intently off into space. When I asked him what he was thinking about, he said: "I'm pretty sure I can solve it, but I'm looking for a systematic way to do it." On another occasion a different student told me that he was no longer satisfied simply with getting a correct answer. As he put it: "I can do the problem, but I want to find a good way." These students began to be interested in elegance of a sort. But, as a result, at times they were less successful than they had been before the instructional intervention began.

The message in this is that it is difficult to make confident claims about the effectiveness of problem-solving instruction unless it takes place over a long period of time and engages students in real problem-solving activity on a regular basis. I suspect that if there had been an additional 12 weeks of instruction we would have observed some very different kinds of behaviors in both the weaker and the stronger students.

Evaluation: An Important Aspect of Instruction (Assumption 7)

Every teacher knows that students are experts at detecting what teachers consider important and unimportant. If homework is assigned but never collected, students soon catch on and stop preparing it. If the teacher claims to expect all work to be shown but marks papers only on the basis of final answers, students soon stop writing out their work. Students internalize as important those aspects of problem solving that the teacher emphasizes and assesses regularly. I made the students aware from the very beginning that I was just as interested in their thinking as their answers. In order to convince them that I was serious about this I took some care on several occasions to go over the analytic scoring scheme that was used to evaluate their problem-solving efforts (see Appendix A-5 for a description of the scoring scheme). Overall I think the importance in successful problem solving of developing good understanding, careful planning, etc. was reinforced by using an evaluation technique that emphasized these processes. However, there was evidence that the impact may not have been as strong as I had hoped.

At the end of one class period toward the end of the 12 weeks of instruction I gave as a homework assignment the task of writing problems that the students thought were interesting to them. During the next meeting I collected their problems and in a subsequent class session I asked the students to rate each of these problems on a scale from 1 to 5 (very boring to very interesting). After they had completed this task, I asked them to solve any one of the problems that they had rated. About 75% of them chose to solve a boring or very boring problem. This was true of students in both the regular and the advanced classes. When I asked why they had chosen to solve a boring problem, the most common reply was that they wanted to be sure that they "got it right." It appeared that the desire for a good grade was stronger than the desire to avoid solving a boring problem, and what was viewed as important was the answer.

Evaluation methods communicate to students what is considered important. The assessed curriculum strongly influences what students are taught, and what they value. But, just as problem-solving ability develops slowly over a long period of time, changes in perceptions of what is valued also change slowly. I would go so far as to say that the evaluation methods used in the classroom are a driving force in the development of students' beliefs and attitudes. Consequently, great care must be taken to insure that what is evaluated is consistent with what is intended to be learned.

The View from the Back of The Class

(authors' note: The following observations were made by Diana Kroll, the research assistant, who videotaped each of the problem-solving lessons from the back of the classroom.)

Experienced teachers generally know their own students and their own strengths and weaknesses in teaching better than anyone else possibly could. However, experienced teachers also realize that they can gain valuable insights about their teaching from observations made by an outsider observer in the classroom. In one sense, these are the observations of an outsider, since my main function as research assistant during each class was to stand apart from the action of the classroom, behind the camera lens. But in another sense, these are not the unbiased, uninformed observations one might expect from an outsider, since I took part in many of the aspects of the instruction that occurred outside the classroom doors: the planning and organization of the classes, the construction of questionnaires and interview questions, the choice of problems, and the grading of homework. Furthermore, I knew the students better than a casual observer ever could: first, because my presence in the classroom every day that problem-solving instruction took place meant that I got to know them at least as well as the problem-solving instructor, Frank Lester, and second, because I was one of the two interviewers during the pre-instruction and post-instruction interviews of selected individuals and small-groups. As a result, the observations that follow constitute a different, but not an unbiased, view of what went on in Barb Willsey's 5th and 6th period classes during their 12 weeks of problem-solving instruction.

How Instruction in the "Project" Class Differed from that in a "Normal" Class

One important way in which the problem-solving classes differed from normal mathematics classes was that Frank made a particular point of bringing to the surface a number of "non-mathematical" problem-solving issues. For example, the classes involved not only extended instruction on use of various strategies for solving problems (e.g., guess and check, make a table, draw a picture, work backwards, etc.), but also discussions of attitudes and feelings toward mathematics and toward various types of problems, beliefs about problem solving and about self, and awareness of goals and progress toward them during problem solving.

Another way in which the classes differed from normal classes was that Frank made it very clear that he was more interested in the methods that students used than in the answers they obtained. Some students became quite frustrated when lengthy discussions about how to solve a problem failed to yield a definitive solution to the problem. Their tolerance for ambiguity was sorely tested. But they began to understand that Frank was genuinely more interested in the process than in the product. It should probably be noted that the emphasis on process rather than answers was probably much more feasible than it would have been in a "normal" class because the problem-solving class was not constrained by the usual requirement of giving students a grade. (Although a grade was eventually assigned to each student, to be averaged in with their test grades from the regular work of the class, these grades were not the focus of the instruction, as is usually the case in a more normal teaching situation.)

A third difference between the problem-solving classes and usual mathematics classes was in the amount of time spent on individual problems. Released from the dual constraints of "covering" a certain amount of material and awarding grades, Frank was able to devote much more time to each individual problem than would normally be the case. Students often worked in small groups for 25 minutes or more on a single problem. And whole-class discussion of their solutions ("looking back") often took an equal amount of time.

Differences between the Two "Project" Classes

Perhaps the most salient observation is that from very early on there were very definite differences in the way the two project classes were handled. The original plans for the instruction did not call for differences in curricula for the two classes, even though it was known that the 5th period class was a "regular" class and the 6th period class "advanced." However, lesson plans were kept quite flexible and open to change since one of the goals of the project was to adapt the instruction as much as possible to the needs of the students.

The implemented curricula for the classes began to diverge as early as Day 7. By that time it had become apparent that the 5th period class needed work on very fundamental problem-solving skills (such as locating and understanding the question, identifying important and irrelevant information, etc.), whereas the 6th period class was more ready for encountering challenging problems and for instruction in more subtle and metacognitive aspects of problem solving. But the observation that the two classes were taught quite differently includes observation of differences in more than just the content of the instruction.

The contrast between the two classes also involved differences in discipline (Frank was somewhat more strict in the regular than in the advanced class), differences in expectations (considerably higher in the advanced class), and a marked difference in atmosphere in the classroom (which was much more relaxed and jovial during 6th period than it ever was during 5th period). Frank seemed much more at ease when working with the advanced class, perhaps because it was easier in that class to implement the plans that he actually had in mind. In the regular class there were more motivation problems, and it quickly became apparent that difficulties with basic skills (e.g., facility with percents or fluency with reading) and problems with motivation caused these students at least as much difficulty with problem solving as did their inexperience with heuristics or their inability to monitor their own work.

Strengths and Weaknesses of the Instruction

Enthusiasm and Interest in Mathematics and Problem Solving

Frank's biggest strength as a teacher of mathematical problem solving to 7th graders was his enthusiasm in the endeavor. Frank's interest in mathematics and in problem solving was readily apparent to all the students, and his enthusiasm ignited the interest of many of them. One very weak student in the regular class became so interested in the work of the problem-solving class that she begged each day we were there for another "challenge problem" to take home. At the end of the semester she announced her decision to become a mathematics teacher when she grows up. Frank inspired many of the weak students, and simultaneously managed to appeal to the better students too. One particularly gifted student in the advanced class began the semester by cornering Frank before each class to discuss topics such as "black holes," or "infinity," or "the theory of relativity." By the end of the semester, this student was more interested in discussing different ways he had thought of for solving the most recent class problem. He became interested in multiple approaches to problems, and in striving for elegance in his solutions, and he obviously was grateful to have a teacher who was just as fascinated with mathematics as he was.

Of course, not every student in the two classes was ignited by Frank's enthusiasm. Some remained as aloof and passive about mathematics as they had been before we set foot in their classroom. But, for the most part, Frank served as a role model of an interested problem solver -- a role that we suspect few of the elementary teachers these students had encountered in their previous six years of schooling had been comfortable playing.

Maturity as a Problem Solver and as a Teacher of Problem Solving

Another important strength of Frank's problem-solving instruction was that he was comfortable not only with problems and solutions, but also with the types of questions that the students tended to ask. His experience as a teacher of problem solving was evident in the classroom. On the one hand, he knew what kinds of questions to ask to help floundering students think for themselves; on the other hand, he was always ready with an interesting (but do-able) extension problem when students finished a problem quickly or easily. He was infinitely patient -- never ridiculing students when they had difficulty. He was always willing to try to help. At the same time, Frank skillfully and naturally wove into his everyday lessons frequent attention to metacognitive concerns. For example, as he led a discussion while students looked back at the work they'd done on a problem, he might ask, "What was the hardest thing about this problem? I'd like you to start thinking about your strengths and weaknesses. Ask yourself: What am I not very good at?" On another occasion he might suggest that students focus their attention on the question "What makes a problem interesting?" In the process, he would encourage the students to begin thinking not only about their own personal likes and dislikes, but also about characteristics of typical problem situations.

Further evidence of Frank's strengths as a problem-solving instructor are included in the section below entitled "Observations about the Three Types of Problem-Solving Instruction." But I'd like to turn now to some comments about weaknesses I observed from the back of the classroom.

Difficulty with Clarity of Expectations

One weakness of the instruction was that expectations were not always clearly elucidated. Frank frequently seemed either unsure, or unwilling, to require effort and compliance from the students -- a situation that probably led to (or at least exacerbated) the difficulties that he experienced with some students not completing assignments on time, and with many students not taking problem-solving homework seriously. Frank seemed to expect more maturity from the students than they possessed.

For example, although rules for small group work were established during the first class meeting, posted on a chart in the front of the room, and reiterated during the second class meeting, Frank failed as early as the second class meeting to insist that students adhere to these rules. One such rule was that no questions could be asked of the teacher unless everyone in the small group had the same question. Yet, when Frank circulated through the class as small groups worked on problems, he enforced this rule only sporadically at first, and increasingly less and less throughout the semester.

Although homework was assigned in the problem-solving lessons, many students either failed to do it at all, or worked on it but failed to turn it in. The fact that Frank was not in the classroom every day made it difficult to establish definite times and places for the homework to be collected. Nevertheless, homework assignments could have been more definite and due dates more precise. His vague assignments (e.g., Day 2: "Try to bring in a problem. Try to get one you think will be really tough for me to solve. If you don't bring one in, that's all right.") often failed to elicit effort from students who were more used to having definite directions and due dates.

Furthermore, Frank never made clear to the students exactly how their work in problem solving would be incorporated into their regular mathematics grade (in fact, it had not been clearly agreed upon between him and Barb Willsey). Neither was it explicitly discussed with the students how this extra work on problem solving could be expected to have any effect upon their ability to do the work they clearly expected to be the most important, the regular classroom work from their textbook -- the work from which they would be awarded their quarterly mathematics grade. This divorcing of the problem-solving work from the grading scheme of the class probably had both positive and negative consequences. Certainly, the students grew throughout the semester to be much more willing to take risks, much more willing to concede that they had difficulty with a problem, and much less likely to become frustrated if they were unable to find the answer to a problem in a matter of a few minutes. If their grades had depended upon their success in solving problems, they might not have been so willing to experience such difficulties. However, it also appeared at times that the students could have put forth more effort but did not always do so, perhaps because expectations were unclear or because the carrot of a good grade was not dangling in front of them.

Difficulty with Classroom Management

In general, Frank was not good at classroom management. In particular, he often ignored idle chatter (and even, on one occasion, group choral singing!) -- off-task behaviors that were going on while the students were supposed to be working in small groups. The noise level in Frank's problem-solving class was much higher than most middle school teachers would tolerate.

Frank frequently ran out of time without being able to adequately wrap up his lessons. More used to the somewhat flexible scheduling of college classes (where students have 15 minutes before their next class) than to the strictly timed bell ringing of the public school (where pupils must run in 5 minutes not only to their next class, but often to the bathroom on the way), Frank was sometimes guilty of keeping the class overtime while he quickly finalized his comments. Admittedly, the regular class had a particularly difficult schedule because their half-hour lunch period fell right in the middle of their 45 minute mathematics class. This made teaching them a coherent problem-solving lesson especially challenging.

In spite of his expertise as a teacher of problem solving, Frank was not experienced in managing small group work of children in a classroom. This inexperience, coupled with the students' own unfamiliarity with working cooperatively, made small group work especially difficult to manage. For example, on one occasion Frank purposely distributed only one copy of a problem to each small group because he wanted to encourage them to collaborate in a group solution to the problem. But before dismissing them to work together on the problem he attempted to have a whole-class discussion about their understanding of the problem. A problem arose because not every student had a copy of the problem to refer to during the discussion, and the pupil who read it aloud to the class read so timidly that not everyone could hear. Similarly, whole -class discussions of small group solutions were frequently not optimal because the groups' explanations were either too quiet to be heard or too confusing to be useful for discussion. Difficulties with classroom management of small group activities lessened as the semester progressed. However, it was clear from the back of the room that classroom management during cooperative problem-solving instruction is no easy matter, even for a very experienced teacher.

Observations about the Three Types of Problem-Solving Instruction

Teacher as External Monitor

In his role as external monitor, Frank provided commentary that helped students to become aware of the types of monitoring that they should be doing while they solved problems. One way that he provided this commentary was by skillful use of problem-solving "teaching actions" (see Table 3). Through his use of these actions, students soon became accustomed, for example, to assessing their understanding of problem conditions before beginning work, to attempting to relate new problems to problems they'd seen before, or to looking back at their solution after they'd completed a problem. Frank accomplished his goals as an external monitor in a number of other ways: through whole-class discussions about problems, through useful comments as he circulated around the room while students worked in small groups, through written comments on student homework papers, and through use of a 10-point grading scheme that communicated to students the relative importance he placed on various parts of the problem-solving process.

Teacher as Facilitator

In his role as facilitator, Frank structured assignments and activities so that students would become more reflective about their own problem-solving activities, and -- in particular -- about their work during various phases of the cognitive-metacognitive framework: orientation, organization, planning, execution, verification (see chapter 1). For example, to encourage more attention to orientation, he chose problems that provided opportunities for students to think about how problems related to one another (as when he assigned variations on similar problems -- for example the stamp problem and the card table problem, or the locker problem and the Ima Poet problem).

Frank included in his lessons numerous problems that required students to think carefully about organization -- to be more systematic in their problem solving (for example, problems that required making tables or making organized lists). Unfortunately, such skills seem to require quite a bit of time to develop. Although most of the students became more aware of the need to be systematic, many were still unable to actually do so at the end of the course.

I know that Frank had intended to be quite organized about making sure that the problems he used would feature difficulties spanning the categories of the cognitive-metacognitive framework. For example, he wanted to have some problems that were primarily difficult to understand, other problems in which organization was the most difficult aspect, and others in which execution or verification would be the most challenging phases. However, this turned out to be very difficult to accomplish in practice, and as a result the lessons were not as well balanced across the categories of the framework as he had intended. Many other problem variables (mathematical content, strategies, difficulty level, time constraints, interest level, etc.) had to be considered when choosing problems for the classes. Consideration of these other variables often impinged on the goal of choosing problems that would span the cognitive-metacognitive framework.

Finally, Frank helped students to develop self-awareness as problem solvers by accompanying many of their problem-solving assignments with various types of student self-reports (eg., short-answer belief or attitude self-inventories, questions about strategies used to answer in several sentences, reflective essays about thinking processes, etc.). In completing these self-reports, students were expected to think about their own likes and dislikes, their own strategies, their own thinking processes, their own strengths and weaknesses, etc. To the extent that students took these assignments seriously, the self-reports were quite useful in encouraging self awareness.

Teacher as Model

In his role as model, Frank tried to demonstrate for students the types of behaviors expected of good problem solvers. Of course, whenever he explained a solution to a problem he was careful to point out things that a good problem solver would do (e.g., reread, consider alternative strategies, verify work throughout the problem, compare answer with problem conditions, etc.). But he attempted to go further than most teachers do with such modeling, by providing students with opportunities to actually witness problems in the process of being solved. Such modeling met with mixed success.

During the first few weeks of the course, students were encouraged to bring in problems for Frank to solve in front of the class. It proved impossible to get the students to come up with problems that were simultaneously difficult and interesting enough for Frank to solve in a genuine way, and easy enough for the students to understand and appreciate. This effort to use Frank as a model problem solver was soon abandoned.

A more interesting modeling session resulted when Frank worked in front of the entire class to solve a problem (chosen by me) that he had never seen before. The problem was chosen carefully -- to be a bit different and challenging for Frank, but also quite understandable for the students. Frank started off full of confidence, and the class watched with no more interest than in any other lesson. But when he began to flounder, the students suddenly became animated and engaged in the demonstration. This modeling lesson clearly had considerable potential. Unfortunately, another -- quite unexpected -- difficulty arose. Frank found it difficult to maintain his role as problem solver. He fell back rather quickly into the role of teacher -- explaining for the students rather than thinking aloud so they could see a real problem solver in action. He called on students who were madly waving their hands with suggestions (and then was obligated to make some effort to try those suggestions, rather than simply modeling the ideas he would have tried if he had been solving the problem on his own). The modeling lesson deteriorated into a teacher-directed discussion. Written notes that the students made about what Frank had done well and what he had done poorly make it clear that they viewed Frank as teacher (modeling teaching behaviors), not as expert problem solver (modeling problem-solving behaviors). In many cases, they commented on good and bad teaching actions they had observed, rather than on good and bad problem-solving actions. Frank did not try again during the semester to directly model the actions of an expert problem solver.

A much more successful method of modeling problem-solving behaviors was use of videotapes of problem solvers. We showed the students two videotapes: one in which I sat at a desk thinking aloud while solving a problem (serving as a model of an expert problem solver) and a second in which another graduate student modeled faulty monitoring during problem solving. The students found it much easier to note good and bad problem-solving actions from the modeling videotapes than during live modeling in the classroom. On the negative side, however, I observed that the whole process of showing a videotape, having students make notes about it, and then discussing and analyzing student reactions takes quite a bit of time. Because of time constraints in the class, Frank was not able to allow enough time for extensive note taking or for thoughtful reflection. Modeling expert (and novice) problem solving behaviors sounds like a good way to make students more aware, but it is much more difficult to pull off in the classroom than it sounds.

The View of the Regular Teacher

Several months after the teaching experiment had been completed, Diana Kroll returned to Batchelor for a talk with Barb Willsey, the regular teacher of the two 7th grade classes involved in the project. Although Barb had already discussed many of her observations on an informal day-to-day basis as the instruction was being implemented, this interview provided her with an opportunity to present a more detailed and organized picture of her point of view concerning the problem-solving instruction that her classes had undergone. The interview took place during one of Barb's planning periods, and was held in a small teacher preparation room adjacent to Barb's classroom. The discussion was informal and candid.

Barb Willsey's Background

Barb first described her background and her view of herself as a mathematics teacher. She actually sees herself as an elementary teacher--the position for which she was originally trained--although she explained that she was trained as a generalist, she developed rather early in her career a special interest in teaching mathematics. Her interest in teaching mathematics was aroused when she took college-level teaching methods classes over 25 years ago; she was disappointed that so many of these courses concentrated primarily on content, and not on other aspects of teaching. By contrast, her mathematics methods instructor spent time discussing classroom management, flexibility in teaching approaches, problem solving, and other topics she found very interesting. From this positive experience with a mathematics educator who demonstrated broader interests than in his content area alone, her special interest in teaching mathematics grew. After graduation from college she served as an elementary teacher in a rural school system for two years before transferring to the Bloomington community to become a teacher who specialized in teaching math and science at the upper elementary level. After five years teaching elementary school math and science, she moved to the middle school level where she has taught mathematics (in several different schools) for 14 years. In spite of her expressed interest in teaching mathematics, Barb did not have any advanced training in mathematics or in mathematics education per se.

Comparison of the Two Project Classes

Barb provided her views about how typical the two classes involved in the study actually were, compared with the many other 7th grade classes she has taught. And, at the same time, she explained how she managed to cover all of the regular 7th grade material with these classes, even though the classes met with her, on the average, only three days each week rather than the usual five days per week during the entire spring semester because they were involved with the problem-solving project on the other two days. She had very different answers for the two classes.

In retrospect, she felt that the 5th period (regular-level) class that took part in the project had been, on the whole, brighter and quicker than a normal regular-level class. During the first semester (before the project began), they had consistently been ahead of her other regular sections. Although she had worried at times about being able to cover all the textbook material during the second semester because the special problem-solving lessons associated with the project took a considerable amount of time, she did not feel that she had rushed at all to complete the normal amount of material with this class. The only way that she changed what she normally would have done was to put less emphasis on problem solving with this class because they were already receiving instruction in problem solving two days per week through participation in the project..

On the other hand, Barb did not feel that her 6th period (advanced class) had been in anyway exceptional (for an advanced class). Nevertheless, she managed to complete the work expected in that class, but only by modifying what she did with them in several ways. During the first semester she attempted to get the 6th period class somewhat ahead of another advanced class that she taught because she anticipated that participation in the problem-solving project would put the 6th period class behind during the second semester. Once the second semester was underway, and the advanced class that was not involved in the project caught up (in the text) to the 6th period class, Barb tried from that point on to keep the two classes moving at the same pace through the material. She said it was more convenient to have the two classes studying the same material at the same time and to give tests to both classes on the same days since many of the students in the two classes were friends. Because she met with the project class only 3 days per week, she simply worked them harder during those class periods. She allowed them fewer time-outs and breaks; she left out many of the optional, enrichment, and fun things in the text; and -- as with the 5th period (regular-level) project class -- she spent less time on problem solving with the 6th period (advanced-level) project class than she did with her advanced-level non-project students.

The Problem-Solving Approach Compared with Barb Willsey's Approach

Barb described her perception of how the problem-solving instruction that the students received through the project differed from that which they normally received. She pointed out that although many of the strategies that Frank taught in the project were included somewhere in the students' texts, he spent much more time explaining and emphasizing them than she would have. She explained that when she encounters a page of (word) problems in the text, her problem-solving instruction to the class usually consists solely of the instruction to "solve these problems any way you can." After the students have worked for a while, she asks for volunteers to outline solutions. Given one correct solution, she usually asks whether anyone else solved the problem in a different way. Or, if she knows of a better way, she might demonstrate it. But she reiterated that she generally tells students to solve in whatever way they can, whereas Frank gave them specific instruction in how to solve problems. In particular, he had names for the strategies he recommended.

Other differences that Barb observed between the project problem-solving instruction and her own were that Frank never utilized textbook problems and that he emphasized cooperative rather than individual work in class. While Frank always supplied his own problems, reproduced on purple ditto masters, Barb never assigns problems from outside the text. She readily admitted that she relies on the textbook to guide her concerning what types of problems to assign. Frank often had the students work in groups, whereas she very rarely does this. She might assign different rows of students different problems to work and then to present to the class, but students would probably work primarily on their own, perhaps checking their completed solutions with others in their row.

Barb explained that she would be more comfortable with a more direct approach than the problem-solving lessons generally entailed. She likes to be able to explain to students exactly how they should solve a problem or do a calculation, not to let them experiment with different methods or alternative approaches. For example, she critiqued the estimation lessons in the Addison-Wesley texts for this very reason. She objected to the fact that the texts presented so many different ways to estimate (rounding to various place values before calculating, using front-number estimation, using compatible numbers, etc.) She felt that the students were confused by this multiplicity of approaches. In subsequent lessons, she herself felt confused when the students asked "which way" they should estimate in order to get the "right" estimate. Barb's comments about estimation seemed to exemplify a larger concern of hers, that mathematical problem-solving is often quite nebulous and difficult to teach.

However, Barb felt that Frank's instruction was, on the whole, quite appropriately targeted for the students in the 5th and 6th period classes. She observed that he often tried different types of problems and styles of teaching, and that he seemed to be constantly modifying his lessons to fit the needs of the students. For example, she observed that when some of the problems did not especially interest the students, Frank had the students themselves make up what they considered interesting problems. From this exercise he noted, for example, that problems involving pizza were very popular -- and he used this and other similar observations in designing subsequent lessons. She commented that she did not remember very many routine multi-step problems being assigned, noting that these are always a major source of difficulty and that she would probably recommend including more of this type of problem. She also mentioned the need to teach pupils of this age to <u>read</u> mathematical word problems -- including showing them how to find the question explaining how to reread to find which information is important and which is unnecessary, and encouraging them to make a mental estimate before beginning a problem. Although she recalled a lesson in which Frank conducted a discussion about how problem solvers must be aware, when reading a problem, that problems may be able to be interpreted in several different ways -- and that depending upon problem interpretation the solution might differ -- she did not recall any lesson in problem reading per se. (There were, in fact, several such lessons. Perhaps she simply was not present to observe these.) The one type of problem-solving instruction that Barb felt was significantly lacking in the project work was instruction in the use of formulas. She feels that 7th graders have a great deal of difficulty not only choosing appropriate formulas, but also just in substituting values into formulas which are given them. Because problems involving formulas were not included in the project work, she included this type of instruction in her regular classroom work with the project students. But she seemed to feel that work with formulas could have been part of the project's domain.

How Project Work Fit with the Regular Work of the Classroom

There were some days when project instruction took only half the class period, and Barb taught the classes from the text in her normal way during the other half of the class. Barb was asked how this split mode of instruction worked, and whether there were any noticeable problems with shifting gears in this way. She felt that there really were very few problems with adapting to the split teaching mode.

The 5th period class had a split mathematics class anyway: they attended each day for 25 minutes, then had 30 minutes lunch, and then returned for another 20 minutes of math instruction. For this class, it made very little difference to Barb's instruction during the second half of the class period whether the first half of the class had been project instruction or her own textbook-based instruction. The lunch period provided a natural break in the routine.

On the other hand, the 6th period class was scheduled for 45 minutes of continuous mathematics instruction. Barb admitted that when project instruction took only half the class period she sometimes encountered difficulties with changing gears in the middle of a lesson. But she felt these difficulties were minimal. In fact, she offered the opinion that changing gears seemed to be more easily accomplished when there were two different teachers doing the two parts of the class. She ventured the guess that if she had tried, on her own, to do two quite different things during the class period, that she would have had more difficulties getting the class to change gears.

Observations Concerning Discipline

Barb observed that Frank seemed to have a higher tolerance for noise and chaos in the classroom than she does. She wondered if it was because he had to put up with the noise for only 4 to 6 class periods per week, or if she was, perhaps, a bit more conscious than he was about the potential of noise to disrupt other classes in the open classroom situation. Her classroom is one of three which share a common front wall and are separated from one another by moveable curtain-style dividers which pull out perpendicular to the front wall. None of the classrooms has a back wall -- they are all open to a common corridor at the back. During the project instruction there was only one occasion on which a neighbor teacher objected to the noise emanating from the problem-solving class. But there were numerous days on which the discussion among students generated a level of noise higher than that observed in most of the other classrooms in this open school setting.

Barb observed that Frank was more patient with the students. He sometimes allowed the class to continue working on a problem when only two or three students were still working. In these situations, many students were finished quite early. As a consequence, some students acted up, causing discipline problems and noise. Barb usually calls an end to work time and goes ahead with the lesson if just one or two students are stuck. But, she claimed that she could understand Frank's position. She felt he had the luxury of more class time and fewer curricular constraints than she, as the regular teacher, has.

Effect on Her Future Teaching

Since Barb observed most of Frank's problem-solving lessons, it would be reasonable to expect that she might have adopted or adapted some of his methods for her own use. The interview with Barb took place during the semester after the project was completed, so it was possible to ask her whether she was, in fact, doing anything with her new classes that she might have picked up from her project observations.

She noted that she had always, in the past, simply assigned students to solve word problems by using any method they thought best. After observing Frank labeling strategies (e.g., guess-and-test, work backwards, etc.), she had tried to follow up on his work by using those labels in her work with the project classes during the rest of the semester. She then continued to use the labels occasionally during her subsequent instruction in problem solving in non-project classes as well.

She also observed that Frank often had the students work in groups, whereas she almost always has seated them in strict rows and required them to work alone. She would like to try more group work. She confided that she had recently purchased some books on cooperative learning and that she had enrolled for a workshop on group work. She hoped the workshop will give her advice and confidence to try more cooperative work in her classroom. But she was still somewhat reluctant to try it, primarily because she is afraid -- in her open school situation -- of the noise level that might result.

Barb's final comments concerned her admiration for Frank's teaching style. She characterized it as much more open and more flexible than she generally expects from a secondary-level or college teacher--more like the approach of an elementary teacher, which is how she envisions herself even though she has taught middle school for most of her teaching career.

The Views of the Students

During the year after the two NSF-sponsored problem-solving classes were taught at Batchelor Middle School, an English composition teacher at the school agreed to ask students in her classes who had participated in the project to write -- as an optional extra-credit assignment -- a description of what they had experienced during Dr. Lester's special problem-solving classes. The assignment included the following instructions:

> Last year when you were a seventh grader in Mrs. Willsey's math class, you were part of a project. In the project, Dr. Frank Lester, from the IU Math [Education] Department, taught mathematical problem solving two classes each week.
> The researchers on the project, Dr. Lester and Ms. Kroll, are now writing up their reports and they need information from you. What the researchers want to know is what you remember from the project and what you thought about the teaching.
> . . . [the composition teacher gave a list of instructions concerning generating ideas, writing a thesis sentence, writing a rough draft, and revising and editing.] [Your] final draft will go to Dr. Lester and Mrs. Kroll to use in their final report to the National Science Foundation in Washington, D. C.

Here are some unedited excerpts from the students' compositions. These commentaries indicate rather clearly that, for these students at least, there was some long-lasting residue from the experience. These excerpts are presented without further comment.

W.C.
One way it [was] fun was because you got to learn new kinds of problems and new ways of solving those problems. Another way you have fun while your solving problems is because you learn to take your time. Taking your time can be very important. When your in the math problem solving class you get encourged to use the problem and Mr. Lester and Mrs. Kroll did a very good job in teaching problem solving.

J.B.
I learned alot last year in math. I learned to do math equations create my own and help others too. Every week Mr. Lester and another lady would both take 2 people to work with them. They would take us in these small rooms and give us an equation. Then they would record us on camera and then we'd watch ourselves. And discuss how we went about doing the problem. And the next week they'd take four more people and so on and so fourth.

D.B.
We learned a lot from Mrs. Kroll and Mr. Lester. They taught us how to solve math problems we hadn't encountered before. They also taught us how to select the right technique to tackling story problems. Some examples are algabracics, backwards, trial and error, or logic if we really didn't know how to do it. I enjoyed what they taught us. As it happens, I had a head start in algebra [in eighth grade], because of what they taught us. Some of the problems they gave us were almost identical to ones in our algebra book.

J.C.
The math problem solving class was interesting and fairly well set up. Filming a student while he is working a problem is an interesting idea that she used to ask me questions about what I was doing at certain times. The questions were fairly interesting and they required a lot of careful' reading and thought, and sometimes the answer that seemed obvious was not the right one. Although I don't think I learned much of anything new it was good practice in sharpening my problem solving skills.

B.P.
The classes helped me learn better how to think through and solve difficult word and story problems. There were many things I enjoyed about the classes. The fact that the whole class was invited to participate and volunteer was nice. Another thing was the way it was presented, the people were very nice, the problems were about things that are interesting to young people. Also the way they described the steps used to solve the problems, made me want to put them to use in solving the problems. By the end of classes everyone was getting into the discussions and were thinking of new ways to solve the problems. I liked that.
There was one thing I didn't really care for though was that a video camera was used to tape our solving problems. I was shook up by it and since I was only in front of the camera once I really couldn't do my best under the pressure. Other

than that though I really enjoyed the classes, and I think they are a very good idea.

J.E.
Many other people were in this class. I'm not sure they learned the same skills I did, but at least one person learned something. That something was how to look at a story problem, find the information to solve and do it correctly. One way that helped me was to get into groups. If one of us didn't understand maybe another one would. And if they didn't we could ask for help. Another way that helped me was the folders. If I couldn't remember how to do a problem, I could look back in my folder, find one like that problem & understand it. I did very much enjoy this class. I now understand how to do more difficult problems and what to look for, as in signs and information to what numbers to use when, where and how. As you can see, I did very much enjoy this class and learned a lot from it. Also I think it would be great if this happened to any children when they are in school. I learned a lot and I'm sure they would also.

T. C.
Last year in seventh grade our math went through a special program with IU. One day out of every week Dr. Lester and Mrs. Kroll would teach our class how to solve story problems. They showed us different ways and techineces of solving problems. During class we would get into groups and work on solving problems. Some of the problems were easy, but most of them ere hard. Dr,. Lester and Mrs. Kroll showed our class many different ways of solving the problems like drawing a picture, making a graph, or making a list. Every once in a while Mrs. Kroll or Dr. Lester would take a group of two students in a room, give them a story problem to solve, and video them solving the problem. I thought that was a lot of fun. Especially being able to watch ourself on video tape. Dr. Lester and Mrs. were both very nice, helpful and extremely patient.

J.B.
I felt that last year's Math Problem Solving class was a good class that was very beneficial to me. It taught me to think more clearly. In my opinion the techniques used were very effective and should be used more often. Even though the problems were always rather difficult and I never got the full 10 points, I still learned a lot. A few things that I remember that we had a certain point scale. We got 4 points for effort, 4 points for the work shown, and 2 points or the right answer, making a total of 10 points for each problem. We almost always got videotaped when we worked in groups but only a couple of times we got to watch the tapes. One thing that I really liked was that one time we got to watch someone else work the problem we had done for homework the previous night. It really helped to see someone else do the problem. The only other thing I can remember about the class is that after each problem we had to fill out an evaluation sheet about how we think we did on the problem and how difficult the problem was.
Overall I think that it was good class and I'm glad I had it.

A. R.
Last year in math Dr. Frank Lester and Mrs. Kroll came to teach us problem

solving. It was fun and informative. We did problems concerning mice breeding, money, age, and valentine candy. Even today when I'm in algebra I understand my problems better. I wish we could have this class again but this time using algebra skills.

J. S.
In my math class last year, we were involved in a special project with Mrs. Kroll and Dr. Lester, from Indiana University. The class was a good experience because I learned a lot. It seemed as if I learned more than in a regular class because we worked in groups and learned from each other. We were put in pairs to work on problems and were video taped to be studied of how we did it. Dr. Lester and Mrs. Kroll seemed very experienced as teachers and instructors. They explained the problems and how to do them very well. They were always ready to help if the students needed it. I enjoyed the class because we were able to work together and there was more of a chance to get the problem right than wrong. I learned more about and how to do story problems as I have less trouble this year with them in algebra.

T. B.

Studying your faults and habits while solving mathematical problems are very interesting and helpful to your future math classes. There are so many ways for you to study your faults and habits but, here are some of examples Dr. Lester and Mrs. Kroll used. They gave each student a folder that they kept with them for our work. Our work that had to be done were worksheets concerning mathematical problems. We kept our work in our folder. They put each of us in groups of four. WE did group work solving difficult story problems. They would sometimes make a game out of solving the problems. After each problem they would always answer each question any student had over the problem(s) and they would sometimes get the whole class to make a discussion about the problem(s). That would make us understand very well. To make us understand our habits and faults even better, they taped us individually or in pairs on video tape solving math problems. While you were solving the problem, you thought aloud so they could know what you were thinking. You had no certain time limit so that took a lot of pressure off. What was fun was seeing how silly you looked on television solving a math problem and seeing habits you never realized before. If it took you longer than it should have solving the problem, they would show you shortcuts that were easier and more fun. I used to think story problems were such a bore but, after they showed you the easier ways to solve them, they suddenly became more fun. I learned a lot from their project and it was very educational in the mathematical field. I discovered how easy and fun math problems can be!

K. R.
I learned a lot from Dr. Lester and Mrs. Kroll, in the math problem solving classes. We learned differnt ways to go about solving story problems, many of which I still use and find helpful. Dr. Lester taught us to do things like draw a picture to figure out a story problem, and he also taught us to think about a problem logically before we began to try and solve it. I enjoyed the homework assignments because I found them challenging and interesting. I remember

when Dr. Lester and Mrs. Kroll first began coming, Dr. Lester said to do the problem but not to worry about getting the correct answer. He said to concentrate on a problem solving strategy, like drawing a picture or diagram. Gradually as we became more familiar with problem solving strategies, it became more important to have a correct answer. I enjoyed the classes taught by Dr. Lester and Mrs. Kroll, I always looked forward to them coming twice a week. I also feel that I learned a great deal from the classes and it has helped me a lot this year in math. I'm very glad I had the chance to participate in the math problem-solving classes taught by Dr. Lester and Mrs. Kroll.

Chapter 5
REFLECTIONS

In this final chapter we reflect on what we have learned from having conducted this study. In addition to discussing the insights we have gained about the two major research questions, we comment on some general issues related to research on metacognition and other important aspects of mathematical problem solving.

SEVENTH GRADERS' METACOGNITIVE BEHAVIORS

The first research question concerned the metacognitive behaviors seventh grade students use during problem solving and the extent to which these behaviors interact with cognitive behaviors. In chapter 1 we indicated that metacognition refers to the control of one's thinking and the knowledge about one's cognitive functioning. That is, metacognition involves two related features: self-regulation and awareness.

In general, it seems that the more successful problem solvers in our study were better able to monitor and regulate their problem-solving activity than the poorer problem solvers, with the difference being most apparent during the *orientation* phase. As was pointed out in chapter 4, the good problem solvers tended to be concerned with developing a *meaningful* sense about the conditions and questions in problems, whereas the weaker problem solvers tended to be content with *superficial* understanding. That is, the good problem solvers were concerned about structural features of problems, while poor problem solvers focused on surface level features of problems. This observation is, of course, consistent with the preponderance of the research on expert-novice problem solving (cf. Nickerson, 1988).

However, it is not clear from our observations that awareness of one's thinking is directly related to problem-solving success. This is not to say that metacognitive awareness was found to be unimportant. Rather, we simply found little evidence to support any position on the role that knowledge of one's thinking plays in problem solving. It seems reasonable to attribute our failure to identify any clear relation between metacognitive awareness and problem solving to two possibilities: (1) Certain behaviors and decisions may have become automatic to good problem solvers in much the same way that good tennis players do not stop to think, even for a split second, before hitting a backhand shot. Indeed, conscious attention to what one is thinking about during problem solving may have a negative effect on performance (again, in like fashion, good tennis players tend to be much less successful when they stop to think about strokes that have become automatic for them). (2) Awareness of one's cognitive processes may be very closely related to one's attitudes, preferences, beliefs, etc. For example, knowledge about one's strengths and weaknesses in mathematics seems to be very clearly tied to one's beliefs about oneself as a problem solver and about mathematics. Thus, evidence that a problem solver is aware of her or his thinking may be masked by the presence of other, perhaps dominant, factors.

The interaction of metacognition with affective and belief factors does not seem to be limited to awareness of one's cognition. As we began to conduct interviews, it became apparent to us that it would be very difficult to study students' metacognitive behaviors without also considering various noncognitive factors. As a result, we decided to collect data about students' interest in various types of problems, their self-confidence in solving these problems, their perceptions of problem difficulty, and their beliefs about the nature of mathematics and mathematical problem solving (a more detailed discussion of the sometimes dominant influence of noncognitive factors is given in Lester, Garofalo and Kroll [1989]). For example, we found that: lack of confidence can render students helpless in solving certain types of problems, beliefs about problem solving can dominant metacognitive behavior, and monitoring one's problem-solving progress can be greatly influenced by lack of basic mathematics knowledge and skill.

Our work in this project, as well as some of our other research work (Kroll, 1988; Lester & Garofalo, 1982), have led us to posit certain beliefs related to the interrelationships among affects, beliefs, metacognition and mathematical performance. These beliefs are discussed in some detail in Lester, Garofalo, and Kroll (1989). We list them here without further comment:

(1) An individual's beliefs about self, mathematics, and problem solving play a dominant, often overpowering, role in his or her problem-solving behavior.

(2) Effective monitoring requires knowing not only *what* and *when* to monitor, but also *how* to do so. Students can be taught what and when to monitor relatively easily, but helping them acquire the skills needed to monitor effectively is more difficult.
(3) Metacognition training is likely to be most effective when it takes place in the context of learning specific mathematical concepts and skills.
(4) Persistence is not necessarily a virtue in problem solving (Lester, Garofalo, & Kroll, 1989, pp. 85 - 86).

METACOGNITION AND INSTRUCTION

Research question two concerned the effects of instruction on students' problem-solving behavior. More specifically, it concerned instruction involving practice in the use of strategies, training students to be more aware of the strategies and procedures they use to solve problems, and training students to monitor and evaluate their actions during problems solving.

In chapter 4 we discussed various strengths and weaknesses of the instruction. In particular, we pointed out that it was difficult to maintain a sharp distinction between the teacher's external monitoring and facilitating roles, and that the modelling role is especially difficult at the seventh grade level. We did not point out that we developed a greater appreciation for the obvious importance of providing students with interesting and motivating activities. We were struck by the fact that, for more than a few students, willingness to attempt to solve a problem was significantly influenced by the context in which it was posed. Interest in a problem's context often was more important to success than any other single factor. This was particularly true of students in the regular class. We also did not point out the importance of "problem-solving skill" activities (e.g., see Appendix C, days 7, 8, and 10). It is essential that problem-solving instruction give direct attention to developing such skills as selecting information needed to solve a problem, making and reading tables, identifying subgoals, and determining if an answer i reasonable. Many students do not possess these skills and may not develop them unless specific attention is given to their development in instruction.

As we have begun to think about our next steps in studying problem-solving instruction we have realized that, due to the exploratory nature of our study, we gained very little insight about the specific relationship between teacher roles and student growth as problem solvers, or about which classroom activities were effective or ineffective. What we did develop were some feelings or sense about what to look at next. For example, we sensed that the role of teacher as facilitator needs much more attention. More specifically, as was mentioned earlier, we now suspect that considerably more attention should be given to problem-solving skill activities. We also sensed that a closer look should be taken at students' beliefs about the teacher's role and expectations. The point is that we now want to begin to identify the specific aspects of classroom instruction that result in student growth.

SOME GENERAL REFLECTIONS

As we begin to bring this report to a close, it seems appropriate to present some reflections of a rather general nature that are based on the insights we have developed as a result of conducting this study and related research over the past eight years. The first reflection concerns the relationship between metacognition and mathematics learning and the other four deal with issues of methodology.

(1) Two fundamental premises of our study were that metacognitive processes develop concurrently with the development of an understanding of mathematical concepts (assumption 1) and that metacognition instruction is more likely to be effective if it takes place in the context of learning mathematics (assumption 3). This is not simply another way of saying that metacognition instruction should be domain specific. Instead, we are suggesting that as students are learning new mathematical concepts, facts, skills, and so on, they should also learn how to manage and regulate the application of this new knowledge. Although the instruction we provided did take place in the context of learning mathematics (assumption 3), it typically was not consistent with assumption 1. More particularly, the instruction was largely isolated from the regular mathematics curriculum, and it probably did not take place over a long enough period of time. For the most part, the problem-solving sessions had little or no direct relation to the regular mathematics instruction and many students did not view them as being a central part of their mathematics class. Any future effort of the sort undertaken in this study should insure that the instruction was truly consistent with its guiding principals.

(2) Inter-task variability with respect to metacognitive processes is very high. When problems are chosen, it is imperative that consideration be given to their potential for eliciting behaviors associated with the aspects of metacognition that are of interest. For example, problems with superfluous information might be included for their potential for requiring metacognitive behaviors associated with the identification of important information (an aspect of developing an adequate representation of the problem). It was our intention to select problems for interviews, testing, and instruction with great care, and in many cases we feel our choices were successful. In spite of this it appears that some of the problems we chose were not as appropriate for our purposes as desired.

(3) Inter-person variability with respect to metacognition is also very high. The differences between the students in the two classes amply illustrate this point. These differences suggest that metacognitive skills may be closely tied to mathematical ability. It is important that researchers describe the characteristics of their subjects (e.g., instructional history, previous mathematics achievement, beliefs, attitudes) as completely as possible. Although we did collect some information about the students' backgrounds (see chapter 3, *Description of the School and the Students*), we knew very little about their beliefs and attitudes, and we knew nothing about the nature of their previous experiences with problem solving or cooperative group work. Prior knowledge of this sort would have aided us tremendously in planning the instruction.

(4) Asking problem solvers to think aloud, keep written records of their thinking, or work cooperatively with a partner, proved to be less successful than we had hoped. For some students, thinking aloud during problem solving was unnatural and sometimes had a debilitating effect on their performance. Written accounts of one's thinking also provided little information for us. This may have been due in part to the students' inexperience with this sort of activity. Cooperative work in small groups has been cited as a natural way to get students to talk aloud and to share their ideas openly. Unfortunately, our experience was that most students were unwilling or found it difficult to do this. We suspect that this reticence was due to the students' belief about appropriate classroom behavior and to an atmosphere of competition that had been fostered by many of their teachers since grade 1.

(5) The validity of self reports as a source of data about metacognitive awareness is an issue that has been indirectly alluded to earlier in this report. Does the fact that a student is unable to write cogently about her thinking mean that she is unaware of her thinking? At the same time, is a nicely worded statement evidence of good awareness, or might it simply indicate that the student is trying to write what she thinks the teacher/researcher wants to read? A similar difficulty exists in attempting to analyze students' written work. Consider the case in which a student works on a problem but does not appear to have used a particular skill or strategy. What can be concluded about this students' behavior? That she does not know how to use the strategy? Or did not recognize that the strategy could be used? Or used the strategy but did not record it on her paper? Or simply chose not to use the strategy? To complicate matters further, if the written work on a paper indicates that a particular strategy was begun but abandoned in favor of another, is it reasonable to claim that the student had decided that the first approach would lead nowhere (a metacognitive decision) and so gave up on it in order to pursue a different strategy? Our experience indicates that the credibility of self reports would increase as students gain experience with writing them.

SEVERAL POSSIBLE NEXT STEPS

We have only scratched the surface in our analysis of the very large amount of data collected through interviews, classroom observations, and students' written work. Data analyses reported here were performed on a very restricted set of data. In the following paragraphs we review the analyses that were conducted and we suggest possible next steps in studying the additional data that are available.

Individual Interview Data

Analysis of individual interview data was limited to eight students from among 12 students who were interviewed individually. Further analysis of a similar nature as was done with these eight students could be undertaken, and different sorts of analyses (e.g., using Schoenfeld's [1985] or Kroll's [1988] scheme) could be conducted with the data of these eight students as well as with the other four students who participated in the individual interviews.

Paired Interview Data

Although pre- and post-instruction interviews were conducted with 12 pairs of students, no analysis was done with the data resulting from these interviews. An important next step would be to analyze these data in detail. Because Kroll's (1988) protocol analysis scheme was specifically designed to analyze the efforts of problem solvers working in pairs, it would seem to be particularly appropriate for studying our data.

Written Pretest and Posttest Data

Each problem on both the pretest and the posttest was scored by awarding 2 points (correct answer and work shown indicated good understanding and an appropriate plan), 1 point (incorrect answer based on a computational or other relatively error, or correct answer but little or no work shown), 0 points (incorrect answer, work shown indicated fundamental misunderstanding, and an inappropriate plan). This scheme was adequate for ou purposes, but it is not sensitive enough to allow for in-depth analysis of students' strategic behavior. One alternative approach that we could use would be to give sets of papers to knowledgeable individuals/experts (e.g., colleagues who have been active in problem-solving research) to sort into groups according to strategic behaviors exhibited. Each set of papers would include written work of several students on problems from both the pretest and the posttest. The experts would not be apprised as to which papers represented pretest or posttest work. The results of the experts' sorts would give us an independent assessment of the effectiveness of the instruction.

Classwork and Homework Papers

The analysis of classwork and homework was limited to three students from among a total of 65. In the future, it would be natural to analyze the work of all 65 students. Furthermore, it would be appropriate to study these papers in a more systematic and focused manner than was done for this report. For example, it would be interesting to identify students whose performance or affects changed significantly from the pretest to the posttest, and to analyze their work with an eye to identifying points at which changes began to take place. Another possibility would be to look for the presence or absence of specific kinds of metacognitive actions in the students' written work (e.g., indications of analysis of problem conditions, evidence of global/local planning, evidence of evaluation of progress).

Problem-Solving Instruction

The analyses of the problem-solving instruction were limited to personal accounts of three individuals: the problem-solving instructor, the research assistant, and the regular teacher. As valuable as their reports were to helping us establish some sense about the strengths and weaknesses of the instruction, they are essentially general impressions. All instructional sessions were videotaped (with one exception). Consequently, it would be possible to undertake much more systematic and thorough analyses of various facets of the instruction. Moreover, additional insights into the effectiveness of the instruction could be gained from further analyses of the data from the other sources (viz., interviews, written test papers, classwork, and homework)

A FINAL WORD

Problem solving, metacognition, beliefs, attitudes! Each of these is multifaceted; each is extremely complex. Collectively, we have been involved in the study of these and other aspects of learning and doing mathematics for over thirty years, yet we have only just begun to scratch the surface of what there is to know. At present, what we believe about the role of metacognition and other noncognitive factors in mathematical problem solving is still based more on our reflections about our own experiences as teachers and learners of mathematics than on the results of carefully and systematically conducted research. As valuable as our experiences have been to us, we intend to subject our beliefs to closer scrutiny in the next phases of our investigation of this, the most intriguing area of mathematical activity -- problem solving.

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Appendices

Appendix A:	Instruments
Appendix B:	The Daily Lesson Plans
Appendix C:	Instructional Materials
Appendix D:	Students' Interview Work
Appendix E:	Problem-Solving Data Bank
Appendix F:	Publications and Papers Related to the Project

Appendix A

INSTRUMENTS

APPENDIX A - 1

Individual Interview Problems

Pre-Instruction

1. Kennedy collected 225 tape cassettes and 4 old shoe boxes to put them in. If he puts the same number of cassettes in each box, how many extra cassettes will there be?

2. The sixth grade math teacher did an experiment with her students. One at a time, students were to give her change for a 50 cent piece without using pennies. No student could use the same set of coins as someone else. How many students were able to give her change?

3. Atlas Steel makes 4 different types of steel. From a shipment of 300 tons of raw steel, the factory produced 60 tons of type I, which sold for $60 a ton; 75 tons of type II, which sold for $65 a ton; 120 tons of type III, which sold for $72 a ton; and 45 tons of type IV, which sold for $85 a ton. Raw steel costs $40 a ton. It costs the factory $2500 to convert every 300 tons of raw into the 4 types. How much profit did Atlas make on this shipment?

4. There are 10 people at a party. If everyone shakes hands with everyone else, how many handshakes will there be?

Post-Instruction

1. Felipe's typewriter sticks when the 7, 8, or 9 key is typed. If he types each number from 100 to 200, how many times will his typewriter keys stick?

2. There were 347 people at a $150-a-plate luncheon to raise money for charity. Expenses were $5000. How much actually went to the charity?

3. Mr. Shuttlemeier's English class is printing a newspaper for the school and giving the proceeds to student government. The local newspaper is charging them $0.15 a paper for the first 100 papers printed, $0.10 each for the next 200, and $0.06 for each paper thereafter. The class has orders for 625 papers. What will they have to charge for each paper in order to give student government at least $100?

4. There are 16 football teams in the National Football League, each in a different city. To conduct their annual draft, each team has a direct telephone line to each of the other teams. How many direct telephone lines must be installed by the telephone company to accomplish this? Suppose the league expands to 24 teams?

APPENDIX A - 2

Paired Interview Problems

Pre-Instruction

1. A caravan is stranded in the desert with a 6-day walk back to civilization. Each person in the caravan can carry a 4-day supply of food and water. A single person cannot go alone for help because one person cannot carry enough food and water and would die. How many people must start out in order for 1 person to get to help and for ther others to get back to the caravan safely?

2. Susie liked to go shopping with her Uncle Louie because he would always lend her money if she needed it. Saturday they wereshopping and Susie saw a $12.00 sweater. "If you will lend me as much money as I have in my wallet, I will buy that," she told him. He agreed and lent her the money. She made the same bargain with him in the next store and bought a $12.00 blouse. In the third store she did the same thing and bought a $12.00 scarf. Then she was broke. How much did she start out with?

Post-Instruction

1. Three waitresses, Jane, Alice, and Tina, put all their tips in one jar. Jane went home first and took 1/3 of the money as her share. Alice, not knowing that Jane had taken her share, took what she thought was her share. Tina, unaware that the others had already taken what they thought were their shares, took 1/3 of the remaining money. There was $8 still left in the jar. How much did the waitresses have in tips in the beginning?

2. Jules and Jim have to travel 80 km. They want to start and arrive at the same time, and they have a bicycle that can carry only one of them at a time. One will ride a certain distance then leave the bicycle for the other, and continue walking until the other catches up. If they both walk at the same rate and they both ride at the same rate, where should the bicycle be left so that each walks twice and rides twice?

APPENDIX A - 3

Pre-Instruction Written Test

Problems:

1. Wilhelm loads boxes at a department store. He loads 150 boxes a week. Each box weighs 50 pounds. How many boxes does he load in 36 weeks?

2. Juanita bought 3 pairs of doves for $50 and sold them for $20 per pair. How much profit did she make?

3. Tom and Sue visited a farm and saw chickens and pigs. Tom said, "There are 18 animals." Sue said, "Yes, and there are 52 legs." Can you tell me how many there are of each kind of animal?

4. Martin's exercise program requires that he do 1 push-up the first day, 4 push-ups the second day, 7 push-ups the third day, 10 the fourth day, and so on until he does 30 push-ups each day. How many days must Martin exercise to reach his goal of 30 per day?

5. A caterpillar is at the bottom of a jar that is 8 inches tall. Each day the caterpillar goes up the side of the jar a total of 4 inches, but at night it slides down 2 inches. At this rate, how many days will it take the caterpillar to reach the top of the jar?

Pre-Instruction Written Test -- page 2

Questionnaire accompanying each problem:

For each question below, circle the response that is most appropriate for you.

1. I think that this problem was: EASY, MEDIUM, HARD.

2. I think that:
   - I got the problem right.
   - I might have gotten the problem right.
   - I got the problem wrong.

3. I believe that:
   - this problem was totally new to me.
   - I've seen problems like this before, but I've forgotten how to solve them.
   - I remember other problems like this and how to solve them.

4. I:
   - like doing problems like this one.
   - don't mind doing problems like this one.
   - dislike doing problems like this one.

APPENDIX A - 4

Post-Instruction Written Test

Problems:

1. At the Kent High School graduation last week there were 374 graduates. The principal awarded 15 scholarships worth $2500 each. How much money was given?

2. Yolanda bought 4 pairs of gerbils for $62 and sold them for $20 per pair. How much profit did she make?

3. At a math contest, 18 problems were given. Some problems were worth 4 points each and some were worth 2 points each. Harriet's total score was 52. How many of the 2-point problems and how many of the 4-point problems did she get correct?

4. Terry's swim program requires that he swim 10 laps the first day, 13 laps the second day, 16 laps the third day, 19 laps the fourth day, and so on until he does 50 laps each day. How many days must Terry swim laps to reach his goal of 50 per day?

5. Sinbad the Sailor was shipwrecked on a desert island. Each day he gathered 4 coconuts and piled them beside his grass hut. But each night a monkey came and stole 2 coconuts. At this rate, how many days will it take Sinbad to collect a pile of 8 coconuts?

6. Choose one of the following problems to solve.

A. A cube that is 3 inches by 3 inches by 3 inches is dipped in a bucket of red paint. After the paint is dry, the cube is cut into 27 smaller cubes, each measuring 1 inch on each edge. Some of the smaller cubes have paint on 3 faces, some on 2 faces, some on only 1 face, and some have no paint on them at all. Of the 27 smaller cubes, how many have exactly 2 faces painted red?

B. Mindy, Ned, Opal, and Paul were skipping rocks in a lake. Paul's rock skipped 8 more times than Mindy's. Mindy's skipped 3 more times than Ned's. Ned's rock skipped 1/2 as many times as Opal's. Opal's rock skipped 8 times. How many times did Paul's rock skip?

(At the bottom of the problem sheet) Why did you choose to solve this problem instead of the other one? Write your answer on the back of this sheet.

Post-Instruction Written Test - page 2

Questionnaire accompanying each problem:

For each of questions 1 - 4 below, circle the letter of the choice that is most appropriate for you.

1. A. I think this problem was EASY.
   B. I think this problem was MEDIUM.
   C. I think this problem was HARD.

Why do you think this?

2. A. I am sure I got the problem right.
   B. I might have gotten the problem right.
   C. I am sure I got the problem wrong.

3. A. I've never seen problems like this before.
   B. I've seen problems like this before, but I don't remember how to solve them.
   C. I've seen problems like this before and I remembered how to solve them.

If you have seen problems like this before, what were they like? Please explain.

4. A. I like doing problems like this one.
   B. I don't mind doing problems like this one.
   C. I dislike doing problems like this one.

Why? Please explain.

5. What strategies did you use to solve this problem?

APPENDIX A - 5

Ten Point Analytic Scoring Scheme Used for Classwork and Homework

Understanding the Problem -- 4 points

0 Complete misinterpretation of the problem *or* no work shown to indicate understanding.

1 Only one relevant piece of information is used in a problem having more than one relevant pieces of information.

2/3 More than one piece of information is used but not all (the distinction between 2 & 3 points lies in the extent to which appropriate relations among pieces of information are evident; also, irrelevant information may be used).

4 All relevant information is used and no irrelevant information is used. Also, appropriate relations among data are evident.

Planning a Solution -- 4 points

0 No evidence of planning (e.g., no work shown or apparently random work).

1 Procedure used does not fit the data of the problem, but there is evidence of some planning having taken place.

2 Partly correct procedure based on part of the problem being interpreted correctly.

3 Same as for 2 points except the procedure is "more" correct.

4 Plan could lead to a correct solution if implemented correctly and efficiently.

Answering the Question(s) -- 2 points
(executing the plan, performing computations)

0 No answer or wrong answer based on an inappropriate plan.

1 Copying error, computational error, partial correct answer for a problem with multiple answers, no answer given for one or more questions in problems having more than one question, or answer labelled incorrectly.

2 All questions answered correctly and all answers labelled correctly.

Appendix B

THE DAILY LESSON PLANS

DAILY LESSON PLANS USED IN INSTRUCTION

This appendix contains a set of the daily lesson plans for the two classes. Except where noted, the same lesson plan was followed with both classes.

Week 1

Day 1 (January 20) - Full period (50 minutes)

I. Introductions and Organization [10 minutes]
   A. Introduce self and DLK
   B. Explain why we are here, when we will be here, and for how long
   C. Mention why all lessons will be video-taped and that no one will see the tapes except me, DLK and JG.
   D. Mention that DLK will take photos of them today so that I can learn their names quickly
   E. Remind Ss of problems they solved in November and December. Link what I will do with them to solving problems of that type.
   F. Describe the types of activities we will work on:

      (i) Solving problems in small groups
      (ii) Solving problems on their own
      (iii) Watching me solve problems in front of the class and analyzing what I did. (Note: Make an assignment for tomorrow - Give a math problem to BW. You can make up one or find one somewhere. I will try to solve one or two on Friday)
      (iv) Discussing what you think about when you solve problems.
      (v) Discussing your attitudes toward and beliefs about solving math problems (say: what you feel and what you believe)

II. Refer back to small group work. [25 minutes]
   A. We are going to do an activity in groups of four. We need to establish some rules to follow

      RULES FOR GROUP WORK [large poster; adapted from Meyer & Sallee, 1983]]

      I. You are responsible for your own behavior.

      II. You must be willing to help anyone in your group who asks for help.

      III. You may not ask the teacher for help unless all of you have the same question.
   B. Play Color-square game [1 sheet/group] (Appendix C)
   C. Play digit-place game (Appendix C)
   D. Discuss working in groups
      (i) What I saw
      (e.g., Not talking with team mates, blurting out questions without raising hand, ---)
      (ii) Ss' impressions - Was it easy to work with your teammates? What was the hardest thing to do, etc.?

III. Problem-solving folders (5 minutes)

A. One folder for each student - personalized
(you may decorate it, "designer folder")
B. Keep all your class work, class notes and homework in your folders
C. Folders will be distributed every time I am here (kept in file boxes - show them).

Day 2 (Jan. 21) - Half period (25 minutes)

I. Go over "rules for group work" again [refer to poster] - [5 minutes]

II. Group problem solving (15 minutes)
   A. Give each group one sheet with a problem to solve on it. Work with your teammates! (Appendix C)
   [Note: No discussion of the problem before. Give hints as needed. Have variant ready for early finishers]
   B. Discuss routine - each group must choose a representative to say how the group solved the problem.

III. Remind Ss to give BW a problem for me to solve on Friday (this was due today). [5 minutes]
[Note: Stress this-I will not have seen the problem beforehand-stump the "expert"]

Day 3 (Jan. 23) - Half period (25 minutes)

I. Have students watch me solve one of the problems they submitted (10 minutes)
   A. DLK will choose a problem from among those submitted
   B. Ss should watch me and jot down notes about: What I am doing that you usually do? What I am doing that you usually don't do? Your own impressions! (Problem chosen is in Appendix C)

II. Discuss their reactions to my effort as time allows.

Week 2

Day 4 (Jan. 20) - Full period

I. Have Ss form groups of 4 and choose a captain (5 minutes)

II. Distribute problem (1 sheet per group) - (Appendix C)
Discuss problem statement (all pencils down) (5 minutes)
Ask questions such as:
   1. What does Carla do?
   2. What did she do to get paid?
   3. What did she do with her money on Tuesday? How much?
   4. What did she do with her money on Wednesday?
   5. What does "50% of what was left" mean?
   6. What do we want to find out about Carla?

III. While groups are working on the problem provide hints such as: (15 minutes)
      1. Suppose Carla had $10. How much did she spend on Tuesday? On Wednesday?
      2. Can you make a reasonable guess about the amount of her paycheck

and then check to see how close you are?
3. Carla had $50 left after she spent part of it. How much did she have just before she spent part on Wednesday?

IV. Give problem variant to early finishers (Appendix C)

V. Choose groups to show their solutions (15 minutes)
A. Captains describe solutions written on board
B. Ask: What did you do to help your group get started? Why did you do that? What did you learn?
C. Discuss variant with those who worked on it

VI. Summarize what went on today (10 minutes)
(Developing good understanding, Guess & Test, Work Backwards)

Day 5 (Jan. 30) - Full period

I. Same procedure as on Day 4 except Ss work individually today (20 minutes)
A. Show problem on overhead projector and discuss the information. (all pencils down) (Appendix C)
B. Ask questions such as:
1. What happened in this story?
2. Why did the goldfish die?
3. How many died?
4. What does "bought as many goldfish as he had left" mean?
5. What part (%) of the goldfish did he have after he divided them among himself and his employees?
6. How many goldfish did the manager get? Each employee?

C. While Ss work on the problem provide hints such as these as needed.
1. Can you make a good guess and then check it out?
2. Suppose he has 30 goldfish to begin. How many did he buy?
3. The manager got 25 goldfish. How many did the 3 employees get?

D. Give problem variant to early finishers. (Appendix C)
E. Choose 2-3 Ss to show and discuss their solutions.
F. Relate solutions to Monday's problem (Carla).

II. Have Ss complete questionnaire about their work on the problem (Appendix C) (5 minutes)

III. Grading procedures I will use (15 minutes)
A. Mention that what we are doing together will count as part of their math grade.
B. I am interested in how you do your work more than your answers. (stress)

C. Discuss scoring scheme to be used on all homework and selected class work 10 points total:
1. 4 points - How well did you understand what the problem was about?
2. 4 points - How good was your plan and the work you showed?
3. 2 points - Did you get a completely correct answer?
Stress - If you want credit for each part, you must show your work.
D. Give 2 examples (contrived) of S work.

IV. Pass out S folders (10 minutes)
Have Ss put their names on the folders and put their papers of today in them. Let them

"personalize" them as they wish without tearing or folding. (Collect them before they leave)

Week 3

Day 6 (February 2) - Half period (5th period class only)

Note: Objective is for me to model a solution to the "caravan" problem (Appendix C) with student involvement. Focus on <u>understanding</u> (getting a good sense of what the problem is about). In particular, stress the following:

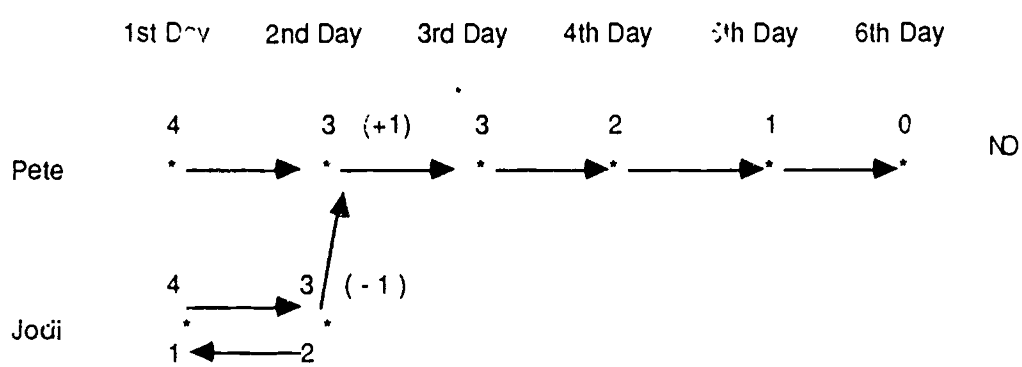
1. Identify what I know and what I want to find and write down this information.
2. Ask myself questions along the way about my work and about my progress.
3. At the end, summarize what I have done and decide if my answer is reasonable.
4. If time, get class to tell me that I did.

I. (a) Mention: "Today I will solve a problem. I am going to try to think out loud as I work so you can hear what is going on in my head."
(b) Read the "caravan" problem (Ss each have a statement of it).

II. Use teaching actions - especially <u>understanding</u> and <u>planning</u>.
<u>Understanding</u>: (1) What's going on in the problem?
(2) How far is the caravan from civilization?
(3) How long will it take to reach help?
(4) How many days could 1 person travel alone?
(5) Can 1 person make the trip alone? - etc.

<u>Planning</u>: Explore the information - try adding one more person at a time.

A. Suppose 2 people start out together, could they <u>both</u> make it to civilization? (No)
Give names to 2 people: Jodi and Pete.
1. Could Jodi give Pete some of her food? (Yes, but Pete can only carry 4 days' supply)
2. If 2 people start out, how much food would each have at the end of the first day? (3 days' supply each)
3. How much could Jodi give Pete and still make it back to the caravan? (1 days' worth)
4. Could Pete then make it to civilization? (No)
So - Conclusion! A person can't go back after 1 day. (Draw a picture)



We also know because of our answer to question (3) that Jodi can't travel with Pete for more than 1 day and still make it back to the caravan.

B. Try 3 people: Jodi, Pete and Robin. Draw picture as shown - don't erase original. Let Ss try to complete the picture.

	1st Day	2nd Day	3rd Day	4th Day	5th day	6th day
Jodi						
Pete						
Robin						

Mention: check work and make sure information used correctly. The picture helps you to organize the information and to keep track of what you are doing.

Day 6 (Feb. 4) - Half period (6th period class only)

I. Give Ss the "caravan" problem and read it with them

II. Ask "understanding" questions (see 5th period)

III. Let them solve the problem in small groups.

Day 7 (Feb. 4) - Full period - (5th period class only)

I. Summarize what I did to solve the "caravan" problem
   A. Use word "analyze" - What does it mean "to analyze" the information?
   B. Get Ss to tell me what they saw me do that was helpful.
      [Exs: (1) Read problem carefully, (2) Wrote down important information, (3) Simplified by trying 1, then 2, then 3 people; (4) Drew pictures; (5) checked my work; (6) asked myself questions along the way]

II. Give Ss "Problem Solving Tips" sheet and discuss it with respect to my work on the "caravan" problem. (Appendix C)

III. Have Ss complete the "Honing Problem Solving Skills" sheet. (Appendix C)
   A. Discuss Problems 1, 2 and 3 - Ask questions such as: (1) Why is ___ important? (2) Why isn't ___ important?
   B. Discuss Stories 4, 5 and 6 - Ask questions such as: (1) What information did you use to formulate your question? (2) Do you have to use all the information to answer your question?
   C. Have Ss answer the questions they wrote.

IV. Put "Tips" sheet and "Honing" sheet in problem solving folders.

Day 7 (Feb. 4) -Full period (6th period class only)

I. Summarize Day 6 work on the "caravan" problem as done with 5th period - difference: Ask them, "What did you do to help you solve this problem?"

II. Give Ss "Problem Solving Tips" sheet and discuss it with respect to their work on the caravan problem.

III. Give Ss 2 problems to solve in small groups ("Hector hiking" and "Cough medicine" problems - Appendix C).
   A. Observe their work and provide "hints" as needed. [Expected strategies: #1 (a) scale drawing; (b) Use tangent table; #2: Make a table]
   B. As time allows discuss their attempts with whole class.

Week 4

Day 8 (Feb. 9) - Full period (5th period class only)

I. Have Ss get their folders

II. Give Ss "Writing Questions for Math Stories" sheet (Appendix C).
   A. Have Ss write a question for each of the 2 stories.
   B. Have Ss exchange papers with a neighbor and then solve the neighbor's 2 problems - Observe their work.
   C. Discuss selected questions and solutions

III. Solve "chocolates" problem in small groups (Appendix C)
   A. Ask "understanding" questions before Ss begin work
   B. Observe groups

IV. Put papers in folders

Day 8 (Feb. 9) - Full period (6th period class only)

I. Have Ss get their folders

II. A. Discuss solutions to "Hector Hiking and "Cough medicine" problems - show 2 different "reasonable" solution strategies. (one using tangent table; one using a careful scale drawing and a protractor)
   B. Discuss a "wrong" solution that some groups used. (Appendix C)

III. Give Ss "Wily Willie" problem to solve in small groups. (Appendix C)
   A. Ask "understanding" questions.
   B. Use "Teaching Action 3" to focus their attention or possible strategies (see Appendix *: Teaching Actions).
   C. Observe groups as they work.

IV. Have Ss add "Make a table/chart" to their "Problem Solving Tips" list.

V. Put papers in problem-solving folders

Day 9 (Feb. 11) - Half period (5th period class only)

I. Have Ss get their folders

II. Go over Day "skills" page ("ice cream" & "distilled water" stories) (15 minutes)
   A. Discuss questions Ss wrote, in particular,
   <u>Story I (Ice cream)</u>:
   How much is 1 scoop? (Can this be answered? Why not?)
   (Erika's question) - Is it cheaper to buy 18 scoops of ice cream at <u>Penguin</u> or at

Flavors Galore?
[Note: Uses all information and requires understanding]
(Stephanie's question) - If you bought some ice cream from one of these places, where would you get the better deal?
[Note: Uses all information and requires the problem solver to make a decision]
Story II (distilled water):
(Amanda's question) - Did Jackie's solution turn out OK even though she experimented a little with it?
[Note: Shows that she understands the situation]
(Kris J's question) - How many ml of distilled water would you need to make cleaning liquid, if you have 21 gr. of salt to use?
[Note: Does not use all the information. Problem is harder - Why?
100/4 = x/21]
(Scott's question) - How many grams of salt do you need for 1 l of water?
[Note: Must know that 1000 ml - 1 l]

III. Discuss the "chocolates" problem using things that Ss did (from Day 9=8) (10 minutes,
   A. Why is the division shown below not good enough?

```
    1 9
6 ) 1 1 4
    6
    5 4
    5 4
```

Emphasize the need to check your answer with "What you want to find" and "What you know."
   B. Show 2 strategies - Guess & check & Draw a picture (based on their attempts - Appendix C)

Day 9 (Feb. 11) - Half period (6th period class only)

I. Discuss "Wily Willie" problem (15 minutes)
   A. Look at Jeri & Trisha's attempt (Appendix C)
   Point out that good planning is very important.
II. Have Ss solve the "chocolates" problem individually (10 minutes)
   (Observe and give hints as needed; do careful audio taping to allow me to analyze hint giving)

Day 10 (Feb. 13) - Full period (5th period class only)

I. Discuss new folder distribution system (pick up your own as you enter the room)

II. Talk about Guess and check as a strategy -
   A. Refer to "Tips" sheet.
   B. Refer to use of Guess and Check to solve the "chocolates" problem.
   C. Have students complete the "Learning to Make Good Guesses" sheet (Appendix C)

III. "Model" solving a "Guess & check" problem provided by Diana (Note: I have not seen the problem - "Dolphin Swim Club") (Appendix C)
   A. Try not to erase anything, think "out loud," write everything down.
   B. Ask students to watch what I do carefully.

II. Homework - Due Feb.17 (Give to BW) (Appendix C)

Day 10 (Feb. 13) - Full period (6th period class only)

I. Discuss folder system as with 5th period.

II. Discuss "chocolates" problem solutions briefly - refer to selected solutions (Toby, Anna)

III. Have Ss complete "Learning to Make Good Guesses" sheet. Point out items at the bottom of the sheet. (Appendix C)

IV. Model the problem Diana gave me during 5th period.

V. Homework - Due Feb.17 (Give to BW). [Appendix C]

Week 5

No Mathematics Classes

Week 6

Day 11 (Feb. 25) - Full period (5th period class only)

I. Discuss answers to "Learning to Make Good Guesses" sheet. Emphasize good, reasonable guesses, not correct answers.
   A. Ask: "Why is 5 kg a reasonable first guess? What would another reasonable first guess have been? Why?
   B. "By checking our work we found that 5 kg is much too heavy. What is a reasonable second guess? Why?"
   C. Go through the same sorts of questions with the math contest pro problem

II. Discuss the homework (Sunny Weather) in detail.
   A. Note the importance of reading information carefully and of showing work on one paper.
   B. Ask specific students to explain how to read parts of the chart (e.g., What was the percent of sunny weather in Washington, D.C. during November, 1986?
   C. Regarding item #2, ask if anyone can think of a way to answer the question without figuring the averages? (Everyone found averages, but month-by-month comparisons are all that are needed)
   D. Show TL's solution for item #4. [34% > 1/3 and 1/3 of 30 = 10, so 34% gives more than 10 days]
   Ask: "Why is this a good (reasonable) way to solve this problem?
   E. Regarding item #5, ask if anyone was puzzled by the question. If so, why?"
   F. Discuss grading of homework. Go over scoring scheme again.

III. Direct Ss to put both the 'Good Guesses" and "Sunny Weather" papers in their folders.

IV. Direct Ss to solve the following problem (stated orally):
   I am thinking of two numbers. When they are multiplied you get 1610.
   When they are added, you get 93. What are the numbers?
   Ask: Why is it better to investigate numbers that give a product of 1610 before making a guess?

Day 11 (Feb. 25) - Full period (6th period class only)

I. - III. Same as for 5th period class.

IV. Discussion of my solution of the "Dolphin Swim club" problem.

A. Give Ss the answer (There were 16 men, 4 women and 80 children).
B. Give Ss the "Video-Tape Viewing Guide" to use as they watch my attempt to solve the problem (Appendix C).
C. Procedure - show pieces of the tape, then pause and have Ss write down what they saw that was good and what they saw that was not so good.

(i) Understanding (about 3 minutes)
(ii) Planning & Solving (about 7 minutes)
(iii) Checking - stop tape at 40:45, 41:30, 43:00 and 45:08 and ask Ss to remark on my checking of my work and progress.

D. Ask certain Ss to read what they wrote down for various sections. Ask: "Why?" (have them elaborate).
E. If Ss do not comment on this, point out my initial failure to recognize that there had to be either 70 or 80 children.

V. Give Ss homework (Appendix C)

Day 12 (Feb. 27) - Half period (5th period class only)

I. Discussion of my solution of the "Dolphin Swim club" problem.

A. Give Ss the answer (There were 16 men, 4 women and 80 children).
B. Give Ss the "Video-Tape Viewing Guide" to use as they watch my attempt to solve the problem (Appendix C).
C. Procedure - show pieces of the tape, then pause and have Ss write down what they saw that was good and what they saw that was not so good.

(i) Understanding (about 3 minutes)
(ii) Planning & Solving (about 7 minutes)
(iii) Checking - stop tape at 40:45, 41:30, 43:00 and 45:08 and ask Ss to remark on my checking of my work and progress.

D. Ask certain Ss to read what the wrote down for various sections. Ask: "Why?" (have them elaborate).
E. If Ss do not comment on this, point out my initial failure to recognize that there had to be either 70 or 80 children.

II. Assign homework (Appendix C). Stress the importance of showing your thinking and reasoning.

Day 12 (Feb. 27) - Half period (6th period class only)

I. Collect homework

II. Discuss laws of exponents, in particular:

A. What does 2^3 mean? Is $2^3 = 2 \times 3$? Is $2^3 = 3^2$?

B. What is $2^3 \times 2^4$? (write in expanded form)
$2^3 \times 2^4 = 2^7$ - Can you look at the left-hand side and decide what the right-hand side will be? What is $3^2 \times 3^1$?

C. What is 1^5? 1^{10}? 1^{100}?

D. What is $2^4 \div 2^3$?

E. Try to use the rules to figure out these:
(1) $1^5 \times 1^8$; (2) $2^2 \times 2^2$; (3) $3^4 \times 2^2$

III. Laws: $2^a \times 2^b = 2^{a+b}$
$2^a \div 2^b = 2^{a-b}$

IV. Have Ss form groups of 3 or 4 and give them a problem to solve (Appendix C)
A. Focus camera on one or two groups exclusively
B. Observe group work and provide hints based on their progress.

V. Assign homework (Appendix C). Stress the importance of showing your thinking and reasoning.

VI. Give "challenge" problem to those who are interested (Appendix C).
Mention that it is entirely optional.

Week 7

Day 13 (March 2) - Full period(5th period class only)

I. Collect & discuss homework ("thermostat" problem)
A. Let Ss explain their solutions.
B. Discuss what Ss think made this problem difficult for them.
C. Mention the value of deciding on what a "reasonable" answer might be before actually solving the problem.

II. Introduction to "look for a pattern" strategy
A. Present the basketball tournament problem (Appendix C)
B. Model a solution for the class as follows: (think aloud as much as possible)

<u>WHAT I KNOW</u>	<u>WHAT I WANT TO FIND</u>
1. 1 on 1 B-ball tournament 2. 8 players entered 3. Every player plays 1 game against each other player	How many total games were played

Give names to the eight players; e.g., Arlo, Bert, Carey, Delbert, Edna, Fortescue, Gafney, Hector.

A plays B, C, D, ..., H: <u>7 games</u>
B plays C, D, ..., H: <u>6 games</u> more
C plays D, E, ..., H: <u>5 games</u> more
D plays D, E, ..., H: <u>4 games</u> more
How about E? F? G? H?

Mention that there is a pattern to it.
7 + 6 + 5 + 4 + - + - + -
Total number of games? 28

C. Pose this problem: What if there were 15 players entered in the tournament? 100?
[14 + 13 + 12 + - + - + - + = 120 games] [99 + 98 + - + - + = 5050 games]

D. (Skill activity - Appendix C) - Have Ss work individually on the activity. Put the sheets in their folders.

Day 13 (March 2) - Full period (6th period class only)

I. Collect and discuss homework ("thermostat" problem)
(Briefly discuss their solutions)

II. Discuss "2^{64} remainder" problem.

A. Ask: "Was this problem easy or hard"? " Why"?

B. Ask: "Which of you decided to multiply out 2^{64} and then divide by 3 to find the answer"? "What happened"?

C. Ask: "Which of you decided to use a calculator to determine the answer"? "What happened"?

D. Ask: "Who tried something else"? (Let 1 or 2 discuss what they tried Note: some tried to use laws of exponents.)

E. Tip: When I want to solve a problem that involves a lot of messy or long computations, I ask myself: "Is there some sort of pattern or easier way to think about this problem?" Maybe there is a pattern!

H. Build a table as follows on the board with the class:

$2^{\square}$	Value	Remainder
2^0	1	1
2^1	2	2
2^2	4	1
2^3	8	2
2^4	16	1
2^5	32	2

Is there a pattern? What is it? (even powers - remainder is 1)

I. What is the remainder when 2^{101} is divided by 3? (2)

J. Reminder: Don't do anything hard unless you can't find an easier way.

III. Give Ss the "Basketball tournament" problem (same as 5th period). Extend to 50 players.

IV. Skill activity - (same as 5th period)

V. Put all work in Ss' folders.

VI. Pose challenge problem (Appendix C)

Day 14 (Mar. 6) - Half period (5th period class only)

I. Ask Ss to find these papers in their folders:
   A. March 2 class assignment
   B. February 27 homework (it was due on March 2) - this has been graded and put in their folders

II. Review the grading procedure using the 4-4-2 analytic scale.

III. Go over the March 2 class assignment. Analyze each of the 3 problems.
   A. For each problem ask: "Is this problem like any others you have done?" "How are they alike?" "How are they different?"
      Ex: #1 is similar to the basketball tournament problem.
      #2 is similar to many "Pattern" problems
      #3 may be viewed as being like the basketball tournament problem because both follow patterns.

IV. Assign homework (due March 9; Appendix C)

V. Give Ss progress reports to take home to show their parents (Appendix C)

Day 14 (Mar. 6) - Half period (6th period class only)

I. Ask Ss to find the class assignment for March 2 in their folders.
   A. Analyze each of the 3 problems on the assignment.
   B. For each problem ask: "Is this problem like any others you have done?" How are they alike? How are they different?"

II. Discuss the "Lake Lemon Campgrounds" problem (several Ss attempted this challenge problem). Again ask: "Is this problem like any others you have done?"

III. Assign homework (due March 9; Appendix C)

IV. Give Ss progress reports to take home to show their parents (Appendix C).

WEEK 8

Day 15 (March 9) - Full period (both classes)

I. Collect homework but do not discuss - will discuss on 3/12

II. Distribute "Immature mice" problem for students to work on in groups (Appendix C).
   A. Discuss problem statement before they begin work:
      1. What is an "immature" mouse?
      2. When does a mouse become mature?
      3. How long does it take a pair of mice to produce a new pair of mice?

4. were the pair of mid that got loose the same sex?

B. Observe g oups and comment as needed.
C. Have Ss put their papers in their folders.

III. (5th period after lunch)
A. give Ss video-tape viewing guides (Appendix C)
B. Ask Ss to watch the video-tape of DLK attempting to solve the problem
1. Stop the tape at regular intervals and ask Ss to write down "what they saw that was good" and "what they saw that was not so good" for each section of the guide.
2. After viewing the entire tape discuss what they wrote down and why.

IV. Assign homework - (due March 12 - Appendix C)

Day 16 (Mar. 12) - Full period (both classes)

I. Collect homework but do not discuss - will discuss on March 17.

II. Discuss homework that was turned in on 3/9.
A. "Math teachers convention."
1. What do we know about solving problems like this one that might help us? (make a table and look for a pattern)
2. Mention that some Ss gave as an answer the number of teachers who entered the convention hall the 20th lime the door opened (39 teachers entered). Is this what the problem asks you to find? (No).
3. Show solutions of two Ss (Wendy and Rachel)

B. "Molly's vacation."
1. Was there something about this problem that made it confusing or difficult to understand? (5th period)
2. What would be a good 1st step to take to solve the problem? (Find 75% of $500)
3. What next? etc.
4. Show solutions of two Ss (Nikki & David)

III. Work on missing class and homework pages.
A. Mention that several Ss have not completed one or more pages (for various reasons). This is a catch-up day.
B. Pages that have not been completed are in Ss' folders. They are to work on them now.

WEEK 9

[Spring break: March 9 - March 13]

WEEK 10

Day 17 (**Mar. 23**) - Full period (both classes)

I. Give Ss the new "Problem Solving Tips' sheet (Appendix C).
A. Discuss the value of drawing a picture or diagram to help you understand a problem.

B. Discuss the new strategies that have been added: make a table and look for a pattern.

II. Group work on the "squares in the grid" and "fruit drink" problems. No discussion beforehand. (Appendix C)

III. Have Ss complete the sheet of questions about their solution efforts. (Appendix C)

IV. Homework: (due March 25) Make up or find a math problem that you think is interesting. You must provide the answer but you don't have to show how you got it.

Day 18 (Mar. 25) - Half period (both classes)

I. Collect homework and discuss 2 questions:
What types of problems are interesting to you?
What topics are interesting to you?

II. Using the overhead projector show Ss 8 problems, one at a time (they worked on all of them at some point during the past 2 1/2 months).
A. Using the sheet provided (attached) Ss are to indicate the level of interest each problem had for them.
B. After all 8 problems have been rated, ask Ss to choose the most interesting and most boring problems. (Why?)

III. Homework: Solve the "postage stamps" and "triangle" problems. (Appendix C). Suggest that they should try to be systematic when they count all the ways.

WEEK 11

Day 19 (Mar. 30) - Full period (both classes)

I. Collect homework and discuss their solutions but do not give the answers today. (this is brief)

II. Have Ss complete the rating of the problems they created for homework (see Mar. 25 homework) (rating sheets and problems in Appendix C)

III. If time permits, discuss their ratings and why.
A. Ask: What makes a problem interesting?
B. R some of the comments of 6th period Ss regarding what makes problems interesting. Get Ss reaction to these comments.

IV. Have students solve the problems, then rate again. (Did they change their minds?)

V. No homework - try again on the postage stamp problem.

[Note. The Ss were not given the answers to the homework problems. Instead I discussed how they might proceed to solve the postage stamp problem.]

Day 20 (Apr. 1) - Full Period (both classes)

I. Begin with a discussion of the stamp problem.
A. Did anyone try to solve the stamp problem again?
B. If no response, show a solution and answer--63 ways (do not discuss in

detail)

II. Discuss Ss' answers to the triangle problem.

A. (5th period) Ask AM or AW or AD to explain how they got their answers

B. (6th period) Ask SS or KW or RP or DF to explain their solutions.

III. Viewing of video tape of a college student working on a problem (Appendix C).

A. DK will take over and discuss what the Ss are to do as they watch the tape.

B. Ss complete viewing guides as they watch the tape.

[Note: The person on the tape was a doctoral student, BD. in mathematics education who was asked to model typical errors made by 7th graders. However, the Ss were not told that she was consciously making errors.]

C. DK will recap with the class some of the things BD did as she solved the problem. She will highlight the importance of asking yourself questions as you try to solve a problem.

IV. As time allows:

A. (5th period) Ss will work on the "Igor and Prunella" problem (Appendix C).

B. (6th period) Discuss the stamp problem in detail.

WEEK 12

Day 21 (Apr. 6) - Half period (both classes)

I. Use overhead projector to display the "Igor and Prunella" problem. [Note: This was not done with 5th period on Apr. 1.] Do not show the class the question sentence.

A. Ask the following with the projector turned off:

1. What is special about the card game?
2. Which card is a 1?
3. What was the sum of Igor's 3 cards?
4. What questions can you answer with this information? [Solicit several replies.]

B. Let Ss work in groups to try to find all the ways.

C. Discuss the group solutions.

1. Ask: What was the first combination you formed? Why did you start with that one?
2. Let Ss share their methods of solution.
3. Show all 8 combinations.
   a. Explain my <u>system</u> (start with the highest card)
   b. Ask why I can stop at certain points.

D. <u>Extension:</u> "I have 3 cards. No card is higher than 6. Which 3 cards could I have?"

E Homework: due Apr. 7 (Appendix C)

Day 22 (Apr. 7) - Full period (both classes)

I. Discuss homework briefly.

A. Emphasize the <u>system</u> (organized list) for completing the table for problem 1. (e.g., I began by listing the <u>most</u> numbers of quarters and d: Why? Then, the next most, etc.)

B. ... out that the system helped me to:

1. Make sure I listed <u>all</u> the possibilities, and

2. Find a pattern if there was one

C. (Square Tables Problem) Is this problem like any you have ever seen before? Which one(s)?

1. What system did we have for the postage stamp problem? (Find basic arrangements, then rotate.)

II. Discuss Ss' ratings of problems (interesting - boring).

A. Mention the problems that were considered to be "most interesting" by Ss.

B. Ask why Ss tended to choose the "most boring" problem to solve.

III Lead into "rock groups" activity from II.B by asking: How did we determine "most interesting?"

A. Have Ss work in groups to complete the "rock groups" activity (Appendix C)

B. Discuss how they interpreted what "most popular" and "least popular" means. (Introduce <u>mean</u> and <u>mode</u>.)

WEEK 13

Day 23 (Apr. 13) - Half period (both classes)

I. Ask Ss to find their "Problem Solving Tips" sheet in their folders. They should refer to it as they work on the following problem.

II. Ss are to work <u>individually</u> on the "Felix locker" problem. (Put names on the papers) (Appendix C)

A. Ss are to show all their work but <u>not</u> show their answer on their paper

B. When all are finished have them exchange with each other and try to determine the answer by looking at the work shown.

III. Questioning before Ss begin work:

A. How many digits did Felix use? What is a digit?

B. How many digits does 143 have? 232?

C. What question do you want to answer?

D. Refer to your "Tips" sheet. What strategies might help you?

IV. After Ss determine an answer for the "exchanged" paper, they should rate the quality of the solution (1-2-3). (Put the papers in problem-solving folders.)

V. Homework (due Apr. 17) "Fiona locker" problem (Appendix C)
[Note: Answer questions that follow the problem.]

Day 24 (Apr. 17) - Full period (both classes)

I. Begin class by discussing the second problem on the homework page for 4/6 (square tables). Ss can find their homework in their folders.

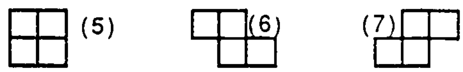
A. Use overhead projector to demonstrate (model) a solution for 4 tables only (original problem had 5)

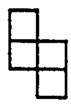
B. Stress being systematic (having a plan) in order to be sure you have found all the arrangements.

1. My system (plan)

4 in a row 

3 in a row + 1 

2 in a row + 2 

2 in a row + 1 + 1 
(same as #7)

2. Demonstrate turn symmetry of arrangements using squares cut from cm grid paper.

3. Ask: So, there are 7 arrangements for 4 tables. Will this system work for 5 tables.

II. Collect homework and discuss the "Fiona locker" problem.

A. Use what was learned about the "Felix locker" problem. Point out that 492 digits gave 200 lockers. Can we begin here to find out how many more are needed?

B. Show a system for keeping track.

Range of #s	How many?	Digits in each no.	Total no. of digits
1-9	9	1	9
10-99	90	2	180
100-199	100 (be careful)	3	300
...	..	...	...

III. Class work (in groups)

A. (5th period) Work on "Ima Poet" problem but not the challenge at the bottom of the page. The challenge is extra credit - do at home (Appendix C).

B. (6th period) Work on "Ima Poet" problem and the challenge problem at the bottom of the page. (Appendix C)

WEEK 14

Day 25 (Apr. 20) - Half period (both classes)

I. Go over "Ima Poet" problem. Ss should look at their solutions. (work is in their folders)
   - A. (5th period) Discuss a system for solving the problem. Relate this problem to other, similar problems. (How did we solve those?)
   - B. (6th period) Discuss a system for solving this problem - note its similarities and differences to other problems.
     1. Ss' systems - solicit and discuss.
     2. My system:

	No. of 2s	No. of 5s
1-digit numbers (1-9)	1	1
2-digit numbers (10-99)	10 + 9	10 + 9
3-digit numbers (100-199)	0 + 10 + 10	0 + 10 + 10
3-digit numbers (200-239)	40 + 10 + 4	0 + 0 + 4

II. Discuss the homework that will be assigned today (due of Apr. 22). (Appendix C)
   - A. Stress the value of <u>trying</u> and using a system.
   - B. Urge them to solve all but at least 2.

Day 26 (Apr. 22) - Full period (WRAP UP)

I. Mention that Ss who want copies of everything in their folders must tell me.

II. Collect homework (5 problems)
   - A. Ask Ss to discuss what they did to solve them.
   - B. Discuss "5 consecutive numbers" problem in detail (a few examples aren't enough)
   - C. Show a method for solving #5 on the overhead projector.

III. Homework (due Apr. 27 - Appendix C)

1C

Appendix C

INSTRUCTIONAL MATERIALS

Day 1

The Color Square Game

Directions for playing the game:

The game is played on a 3 x 3 square grid (more complex games can be played on 4 x 4, etc. grid). Each of the 9 squares that make up the grid is colored with one of three colors (red, blue , or green) so that 3 are red, 3 are blue, and 3 are green. However, all of the squares of one color are joined together along sides. The diagram on the next page shows some permitted and forbidden arrangements for a single color

To begin a game one person colors all 9 squares on a grid according to the rules above, but no one is allowed to see how they have been colored. The other players use a blank 3 x 3 grid and try to reconstruct the color pattern that has been created. In order to get information, players may ask to know the total number of squares of each color in any row or column. The player who created the color pattern must respond truthfully, but it is not necessary that he/she indicates the order in which the colors appear in the row or column asked about.

The object of the game is to determine the color pattern by filling in the grid. The game is scored on the basis of the number of questions asked before the grid is accurately recreated (note: in a 3 x 3 game 3 questions are sufficient to completely determine a color pattern).

The Digit Place Game

Directions for playing the game:

One player thinks of a 2-digit number (larger numbers can be used to make the game more complex) and writes it down. The other players try to determine what the number is. These players name a 2-digit number and are then told the number of correct digits and the number of correct places in their number.

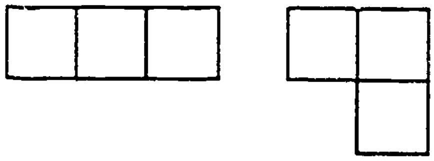
For example, suppose a player writes down "35." Suppose further that the first guess by the other players is "52." This guess has one correct digit ("5") and no correct places. Suppose the second guess is, say, "45." This guess has one correct digit and one correct place.

Play continues until the number can be determined without guessing and the players know what it is (as opposed to guess what it is).

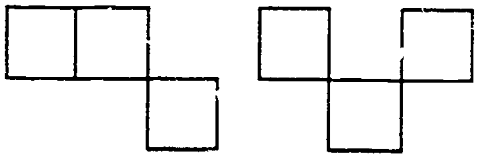
Day 1

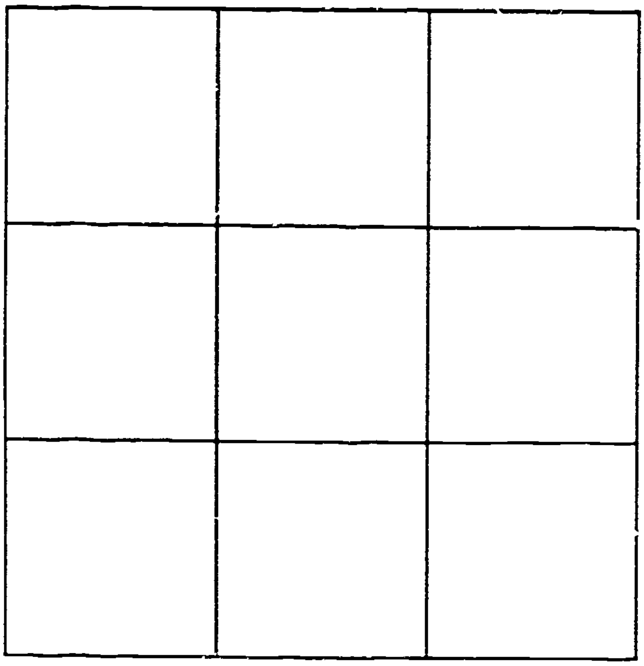
The Color Square Game

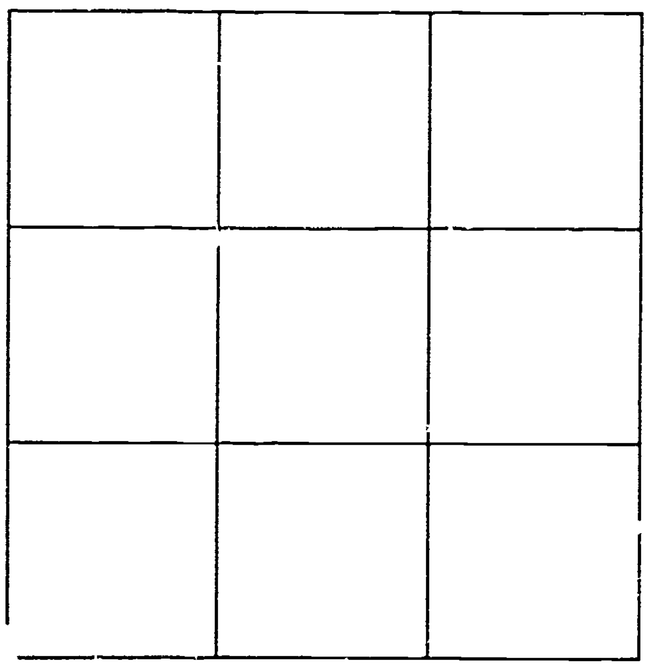
Each of these is permitted.



Each of these is forbidden:







(Taken from Mayer & Sallee, 1983, pp. 23 - 26)

Day 2

Four students decided to throw a party to celebrate the end of school and to share the expenses equally. Amy bought a cake for $8.50, Randy bought $5.25 worth of ice cream, Sarah spent $2.30 on green and red crepe paper to hang, and Dave got $3.25 worth of soft drinks. In addition, Dave paid $5.50 to rent a giant popcorn popper. To be fair, who owes money to whom?

Hints: (1) How much money was spent in all?
(2) How much is each person's share?

Problem variant: Pat joined the group at the last minute and only brought $1.50 worth of peanuts. Now who owes who?

Day 3

Problems Created by Students for the Teacher to Solve in Front of the Class

Fifth Period Problems:

1. (created by SP) There were 3 girls. Two of them had \$6.24 more than the 3rd girl. Altogether they had \$18.48. How much less does the 3rd girl have?

2. (created by TB) Archie owned a puppy circus. Ten puppie can do flips. Five puppies run through hoops. Twenty-five puppies dance. Sixteen puppies dance and do flips. Twenty-six puppies jump through hoops and do flips. Thirty-one dance, run through hoops, and do flips. Ten puppies jump over 3 boxes at a time. How many puppies does Archie have in his circus?

Sixth Period Problems:

The following problems were submitted by students and were read in class. They were not solved in class because the mathematics required to solve them was judged to be beyond the knowledge and sophistication of the studnets in the class. The first problem was posed by the sister of one of the students. At the time she was taking a precalculus course in grade 12. The second problem was posed by the father (a science professor at the university) of a student. The third problem was posed by a student who was a science buff (he seemed always to have a physics or astronomy book with him). This student was regarded by the others as being very bright, but rather unusual. The fourth problem was posed by a student in the fifth period class. The first three problems were solved by the teacher and their solutions were given to the students sho has submitted them. The fourth problem was solved in class and the teacher's method of solution was discussed.

1. (submitted by AA) A hemispherical bowl has an inside radius of 8 inches. There are 2 inches of water in the bottom of the bowl. Through what angle may the bowl be tilted before the water spills out?

2. (submitted by AZ) In Milton's Paradise Lost, Satan fell from heaven to earth. It took 24 hours for him to fall. How high is heaven? (Note the following information)

$$\frac{d^2r}{dt^2} = -g(r) = -9.8\text{m/sec}^2\left(\frac{2\times10^7}{\pi}\text{m}\right)^2 r^{-2}$$

where r is the distance from the center of the earth, g(r) is the acceleration due to gravity as a function of r, and dr/dt = 0 when t = 0. Furthermore, the radius of the earth (i.e., r at t = 24 hrs.) is

$$\frac{2\times10^7\text{m}}{\pi}$$

16

3. (created by TT) Points A and B are 25 light years apart. They both travel at one half the speed of light in the same direction relative to all surroundings. If point A sends a radio wave to B, how long will it take the signal to reach B relative to a stationary observer?

4. (submitted by AD) One hundred people entered a tennis tournament. The rules specified that every player would play one match against every other player. How many matches were played before the tournament was finished?

Carla is the drummer in a band. On Tuesday she received her paycheck for work done during the past month. She spent 20% of it that day and 50% of what was left on Wednesday. She then had $50 left. How much did Carla receive in her paycheck?

Extension: Carla saved as much as she spent on Tuesday and Wednesday. How much money did she save in a year if she was paid every month?

Day 5

One really cold winter night the furnace failed at the "Tropicana Fish Shop". Because of this the temperature dropped in the shop and 20 goldfish died. To replace the fish the manager bought as many goldfish as he had left. He then divided all the goldfish equally among himself and 3 employees. The manager got 25 goldfish. How many goldfish were in the shop before the night that the furnace failed?

BE SURE TO SHOW ALL YOUR WORK ON YOUR PAPER

Your answer: ______________________________

Day 5

Think about the problem you just worked on, then answer these questions by circling what you think:

1. How sure are you that your answer is right?

ABSOLUTELY SURE	PRETTY SURE	SORT OF SURE	NOT SO SURE	I KNOW I GOT IT WRONG

2. How hard was this problem for you?

VERY, VERY HARD	PRETTY HARD	SORT OF HARD	NOT SO HARD	REALLY EASY

3. Answer this question only if you think your answer is right.

Why was this problem easy for you?

4. Answer this question only if you think your answer is wrong.

Why was this problem hard for you?

5. Have you ever solved a problem like this one before? If so, can you describe that problem?

16

Day 5

Variant of "Tropicana Fish Shop" Problem:

Suppose the manager got half as many goldfish as the other three employees. How many goldfish did the shop have before the furnace failed?

A caravan is stranded in the desert with a 6-day walk back to civilization. Each person in the caravan can carry a 4-day supply of food and water. A single person cannot go alone for help because one person cannot carry enough food and water and would die. How many people must start out in order for 1 person to get to help and for the others to get back to the caravan safely?

Day 7

February 4,1987

PROBLEM SOLVING TIPS

Understanding the Problem

* Read the problem carefully; often you should read it two or more times.
* Be sure you understand what the question is asking; ask yourself: "Do I understand what I am trying to find?"
* Write down all the important information and the question; these are called: What I know and What I want to find.

Solving the Problem

* Explore the problem to get a good "feel" for what the problem is about.
* Don't do anything hard until you have tried easy ideas first; if easy things don't help, then you may need to do something more complicated.
* When you don't have any idea of what to do, try to make a good guess and then check it out with the important data.
* Use the strategies that you have learned; for example:

DRAW A PICTURE

SIMPLIFY THE PROBLEM

GUESS AND CHECK

WORK BACKWARDS

Getting an Answer and Evaluating It

* Be sure to check your work along the way, not just at the end; you may be able to avoid some unneccesary work by finding a mistake early.
* Be sure that you used all the important information.
* Write your answer in a complete sentence; this makes it easier to decide if the answer is reasonable.
* Ask yourself: "Does my answer make sense?"

Day 7

Feb 4, 1987

Honing Problem Solving Skills

Problems 1, 2 and 3 contain unnecessary information. Underline the information you need to solve these problems.

1. A local radio station is on the air from 8:00 a.m. to 10:30 p.m. 365 days a year. It carries an average of 2 hours of network news and it plays rock music for 8 hours every day but Sunday. On Sunday it broadcasts church services for 3 hours. How many hours a year is the station on the air?

2. Jeff's brother has a pickup truck. It gets 9 miles per gallon in town and 12 mpg on the highway. The fuel tank holds 18 gallons. Regular gas costs 78.8 cents per gallon at the local pump. How much does a full tank of regular gas cost Jeff at the local pump?

3. Indiana is a leader in above ground coal production. Reported coal production for one year was 84.7 million tons taken from 250 mining sites located in 32 of the state's 67 counties. The largest company, Amex, mined 8,697,631 tons. The smallest, Borem, produced 54,933 tons. What per cent of the state's coal production did the largest and smallest company produce together?

Three stories are stated below. For each story write a question that can be answered using the information in the story.

4. Hanna's family is making a trip to Denver, Colorado from home in Bloomington. They planned to make the 1258 mile trip in 4 days. The first day they drove 327 miles. They drove 338 miles the second day. On the third day they drove 60% of the remaining miles.

5. Jon's father purchased the video game Space Wizards for $48.95. Jon had been playing that game 8 times a week at the corner store, where it costs 25 cents to play 1 game.

6. Heloise is going to repaint the ceiling of a 15-by-18 ft. recreation room. She is painting over a darker shade of paint, so it will need 2 coats. A gallon of paint covers 3560 square feet.

Day 7

Hector decided to spend the day hiking around the area near his campground. With him he took a backpack that weighed 4.8 kg. He first hiked 6.6 km east in a straight line from the campground. Then he turned due south and walked 2.4 km. Hector had had a good breakfast but all the hiking helped him work up a really big appetite. Since he was so hungry he decided to take the shortest route possible back. At what angle from due north should he walk in order to walk directly back to the campground?

Carrie and her brother, Paul, both have very bad colds and they decided that they should stay home from school on Tuesday. On Monday evening their mom bought a 18 ounce bottle of a very powerful colorless cough medicine.

She said: "When you go to bed tonight I'll give each of you a 1 ounce dose. Then 6 hours later I'll give you another. I think both of you will be fine tomorrow and you can go to school."

After Paul had taken his first dose, he moaned: "That's the vilest stuff I've ever tasted."

Carrie replied: "Paul, you're such a baby. It can't be that bad."

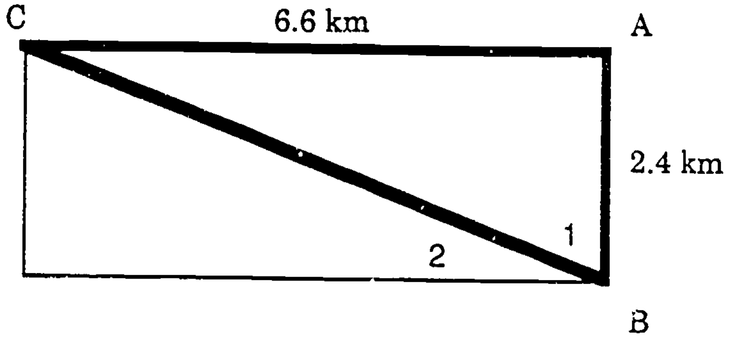
But, after taking her dose she admitted she was wrong. It was even worse than Paul had said. Without telling the other both Carrie and Paul began to formulate a plan. An hour after everyone had gone to bed Carrie went to the bathroom, poured 4 ounces of the medicine down the drain, and replaced it with 4 ounces of water. "Now it won't taste quite so awful", she thought.

An hour later Paul got up and he also poured out 4 ounces of the mixture in the bottle and replaced it with water.

What is the ratio of cough medicine to water after they did this?

An Incorrect Solution to the "Hector the Hiker" Problem:

The teacher showed the following diagram on the overhead projector:



CB is a diagonal of the rectangle. So, CB divides the angle in half. So, angle 1 = angle 2.
Therefore, angle 1 = 45 degrees

What is wrong with this solution?

Day 8

Writing Questions for Math Stories

Read the two math stories below then write a question for each story that can be answered using the information in the stories.

Story One:

The Penguin Ice Cream Parlor and the Flavors Galore Ice Cream Shop are trying to attract new customers by selling ice cream at special prices. Both stores sell exactly the same kinds of ice cream. Two of the specials are shown in the pictures.

Penguin Ice Cream Parlor

Special!! Special!!

3 scoops of any flavor only $1.50 while supply lasts.

Flavors Galore Ice Cream Shop

Once in a lifetime offer !!

2 scoops -- $1.10

Any flavor in stock

Story Two:

It takes 100 milliliters of distilled water and 4 grams of salt to make a solution for cleaning contact lens. Jackie used 150 milliliters of water and 5 grams of salt to make her own solution.

Day 8

February 9, 1987

Robert bought several packs of chocolate to give to his friends on St. Valentine's Day. Some of the chocolates were in packs of 4 and some were in packs of 6. Robert bought 114 pieces of chocolate. How many packs of 6 did he buy, if he bought a total of 24 packs of chocolate?

Two Solutions to the "Valentine's Day Chocolates" Problem

I. Guess and Check

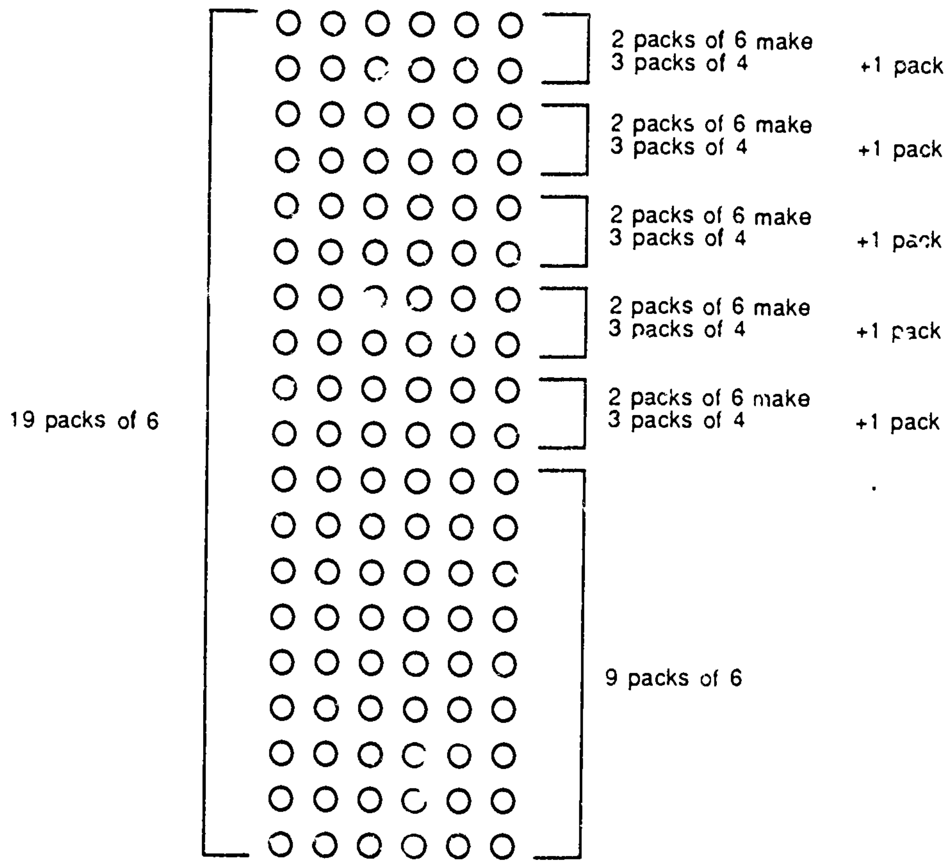
If all the chocolates were in packs of 6, Robert would have 19 packs
(114 /6 = 19)
But, there are 24 packs and some are packs of 4.
So, I need fewer packs of 6.
Guess 1. 12 packs of 6 and 12 packs of 4
Check 1. 12 x 6 = 72; and 12 x 4 = 48; and 72 + 48 = 120. This is 6 too many.
I need to get rid of 6 chocolates.

Guess 2. Add 3 more packs of 4 and remove 3 packs of 6 (This is 6 fewer).
Check 2. 9 x 6 = 54; and 15 x 4 = 60; and 54 + 60 = 114. Correct!

Answer There were 9 packs of 6 and 15 packs of 4.

II. Draw a Picture/ Use Logical Reasoning

If all the chocolates were in packs of 6, there would be 19 packs. But, Robert bought 24 packs of chocolates -- some packs of 6, some packs of 4.
Two packs of 6 = three packs of 4. So, in order to get the additional 5 packs I must trade in packs of 6 for packs of 4. This is shown in the picture below:



Day 8

February 9, 1987

Wiley Willie is applying for a job that pays $5 an hour. He tells his prospective boss that he will work for 1 cent the first week, for 2 cents the second week, for 4 cents the third week, and so on. Willie's boss agrees to this amazing offer and hires him on the spot. If Willie works 40 hours per week, how many weeks must he work before he is making the same weekly salary as he would have made earning $5 an hour? At the end of 20 weeks how much will Willie have earned altogether up to that time?

17

Day 8

Solution Effort of Two 6th Period Students on the "Wiley Willie" Problem

	Pay	Total so far
1	1 ¢	1 ¢
2	2 ¢	3 ¢
3	4 ¢	7 ¢
4	8 ¢	15 ¢
5	16 ¢	31 ¢
6	32 ¢	63 ¢
7	64 ¢	$1.27
8	$1.28	$ 2.55
9	$2.56	$ 5.11
10	$5.12	$ 10.23
11	$10.24	$ 20.47
12	$20.48	$ 40.95
13	$40.96	$ 81.91
14	$ 81.92	$ 163.83
15	$ 163.84	$ 327.67
16	$ 327.68	$ 655.35
17	$ 655.36	$ 1310.71
18	$ 1310.72	
19	$2621.44	
20	$[illegible]242.88	

Day 10

February 13, 1987

LEARNING TO MAKE GOOD GUESSES

Problem 1: Four boxes labelled A, B, C and D together weigh 40 kg. Box B is 3 times heavier than box A. Box C is 3 times heavier than box B. Box D is heavier than box C. Also, boxes B and C together weigh 12 times as much as box A. What are the weights of the 4 boxes?

Guess 1: 5 kg for box A, so the others weigh 15, 45, and 135 kg.
Check 1: 5 kg + 15 kg + 45 kg + 135 kg is more than 40 kg. Too heavy!

A GOOD SECOND GUESS WOULD BE:

Problem 2: At a math contest, 20 problems were given. Each correct answer was worth 5 points, but 2 points were deducted for an incorrect answer. Harriet's score was 72. How many correct answers did she have?

Guess 1: Correct 10 Incorrect 10

Check 1: 10 x 5 = 50 and 10 x 2 = 20 ; 50 - 20 = 30
Harriet's score is 30. No, much too low.

A GOOD SECOND GUESS WOULD BE:

****
List 2 strengths you have in solving math problems.

Day 10

Problem Use·' to Model Good Problem Solving:

The Dolphin Swim Club's 12-year-old-and-under division had an undefeated season. To celebrate, the adult division sponsored ı all-you-can-eat chicken bar-be-cue. Admission for men was $5, for women was $3, and children 12-and-under paid only 10 cents. Total attendance at the bar-be-cue was 100, and $100 was collected. How many men, women, and children attended?

Day 10

February 13, 1987

MATH PROBLEM SOLVING HOMEWORK

Solve the problem below and turn it in to Mrs. Wilsey on Tuesday (Feb. 17)
You will notice that the problem has more than one part. Answer each part.

Problem:

Monthly Percents of Sunny Weather**

	Bloomington, IN	Washington, D.C.
19[illegible] overall	55%	50.5%
October, 1986	50%	51%
November, 1986	29%	34%
December, 1986	28%	43%
January, 1887	35%	data unavailable

**Actual data compiled by the National Climatic Center, Asheville, NC

1. Did Bloomington or Washington have sunnier weather, on average, in 1985?
2. Was the weather during the last three months of 1986 sunnier, on the average, in Bloomington or in Washington?
3. About how many days did the sun shine in Bloomington in October, 1986?
4. Were there more, or less, than 10 days of sunshine in Washington in November, 1986?
5. Were there more sunny days in Washington in December, 1986 or in January, 1987?

****

SHOW YOUR WORK HERE AND ON THE OTHER SIDE OF THIS PAGE. BE SURE THAT EACH OF YOUR ANSWERS IS CLEARLY INDICATED.

Day 11

VIDEO-TAPE VIEWING GUIDE

Understanding:

What did you see that was good?

What did you see that was not so good?

Planning and Solving:

What did you see that was good?

What did you see that was not so good?

Checking: (checking along the way and checking the final answer)

What did you see that was good?

February 25, 1987

Problem: Frank said: "I am thinking of two numbers.
When you multiply them you get 204.
When you subtract them you get 5.
What are the two number I am thinking of?"

WHAT I KNOW

WHAT I WANT TO FIND

Circle the strategies you used:

DRAW A PICTURE
GUESS AND CHECK
WORK BACKWARDS
MAKE A TABLE
ANOTHER STRATEGY

YOUR ANSWER:

18

Day 12

February 27, 1987

Name: ____________________

Work with your group partners on the following problem:

What is the remainder when the number 2^{64} is divided by 3?

(Be sure to show your work)

Your Answer: ____________________

Day 12

NAME: ______________________________ February 27 1987

HOMEWORK: Due Monday, March 2.

BE SURE TO SHOW YOUR REASONING !!

Problem: Annual Savings on Heating Costs

Percent of the Heating Costs Saved by Setting Back at Night		
	Setback of 3°C	Setback of 6°C
New York	8%	12%
Milwaukee	5%	10%

Fuel can be saved if the thermostat in a house is set back at night. The table above shows the approximate savings in 2 different cities.

The Abrahms family, who live in New York, usually keep their thermostat at 17°C at night. At this temperature their annual heating bill would be $1200. The Abrahms plan to set back their thermostat to 14°C at night. How much money will they save in a year?

The Stephenson family lives in Milwaukee where annual heating costs for an average house are 15% higher than in New York. If the Stephensons set their thermostat back 6°C each night, how much higher will their heating bill be than the Abrahms' heating bill?

Circle the strategies you used:

DRAW A PICTURE GUESS AND CHECK
WORK BACKWARDS MAKE A TABLE
ANOTHER STRATEGY

YOUR ANSWERS:

Day 12

NAME: ______________________________ February 27, 1987

CHALLENGE PROBLEM: Due Monday, March 2.

This problem is optional! Try it if you are interested.

Problem: Larry, Moe, and Curly played a game three times. For each game one person was a loser and the other two were winners. The loser had to double the points that each winner had by taking points away from his own point total. Each man won twice and lost once and they each had 40 points at the end. How many points did each man have to start? [Note: They did not necessarily start with the same number of points.]

Day 13

Problem: There are eight players in a one-on-one basketball tournament. Each player must play each other player one game. How many games will be played in the tournament?

Day 13

NAME: ______________________________ March 2, 1987

Class Assignment: Answer each part. Use what we talked about today in class.

1. Read this problem, then answer the question below:

   Six friends exchanged cards on Valentine's Day. How many cards were exchanged if each person gave a card to each of the others?

   Sue thinks the answer is 30 cards. Pete thinks the answer is 15 cards.

   WHO IS CORRECT: SUE OR PETE ? YOUR ANSWER: ______________

2. The physical education teacher at Wilma's school uses the following table of bonus points for students who do exercises at home. Wilma broke the record and got 33 bonus points. How many times did she do exercises at home? (Hint: Complete the table)

Number of times a udent does exercises	10	11	12	13	14	15	16
mber of bonus points	0	3	6	9	12		

   YOUR ANSWER: ______________

3. In art class Sally learned to make 2-color silk-screen prints. She had a choice of 8 colors for a project. How many 2-color combinations can she make with these 8 colors? (Hint: Finish the table and look for a pattern)

Number of colors	2	3	4	5	6	7	8
Number of 2-color combinations	1	3	6				

   YOUR ANSWER: ______________

4. Name another strategy for solving problems that we have studied.

   YOUR STRATEGY: ______________________

Day 13

NAME: ______________________________ March 2, 1987

Challenge Problem: Optional!: Solve it only if you are interested.

The roof of the pavilion at the Lake Lemon Campgrounds is supported by 15 posts, each of which is 12 feet high. Carl and Hank plan to decorate the pavilion for a party. One thing they want to do is to connect each post to all the others with crepe-paper streamers. How many streamers will they need to buy?

1C.

Day 14

Name: ______________________________ M rch 6,1987

Homework: DUE MONDAY, MARCH 9,1987

Read the following instructions carefully!!!

1. Solve both of the problems stated below.
2. Show your work for problem A. on the front of this page.
3. Use your own paper to write a description of your reasoning for what you did to solve problem A.
4. Show your work for problem B. on the back of this page.
5. Use your own paper to write a description of your reasoning for what you did to solve proble B.

PROBLEM A: The math teachers were having a convention and they arrived in a very orderly way. The first time the door opened 1 person came into the convention hall. Each time the door opened after that a group entered that had 2 more people than the previous group. After all the teachers had a someone mentio l that the door had opened 20 times. How many math teachers attended the convention?

PROBLEM B: Molly wanted to earn at least $500 during the summer. She wanted to reach 75% of her goal by July 4 so she could take a vacation. By June 25 she had reached 2/3 of her July 4 goal. How much money had she earned by June 25?

Day 14

MATHEMATICAL PROBLEM-SOLVING PROJECT
Progress Report: March 6, 1987

Project Description

The aim of this project is to help students learn how to use their math skills to solve problems. Students participating in the project learn how to be more reflective about what they are doing when they solve problems. In particular, they learn techniques for analyzing information given in a problem statement, how to use certain problem-solving strategies, and how to evaluate their solution efforts. More specifically, four aspects of problem solving are emphasized: understanding the information in a problem, planning a solution, executing the plan, and evaluating the solution. Evaluation of student work includes not only an assessment of the correctness of the answer to a problem, but also of the quality of the work done with respect to the four areas that are emphasized. This project is directed by Frank Lester of Indiana University and is sponsored by the National Science Foundation.

Sample Activities

The following are a few of the various kinds of activities that have been included in the instruction so far.

1. Underline the information you need to solve this problem: A local radio station is on the air from 8:00 a.m. to 10:30 p.m. 365 days a year. It carries an average of 2 hours of network news and it plays rock music for 8 hours every day but Sunday. On Sunday it broadcasts church services for 5 hours. How many hours a year is the station on the air?

b. Write a question that can be answered using the information stated in this story: Jon's father bought the video game Space Wizards for $48.95. Jon has been playing that game 8 times a week at the corner store, where it costs 25 cents to play one game.

c. Solve this problem: Robert bought several packs of chocolate to give to his friends on Valentine's Day. Some of the chocolates were in packs of 4 and some were in packs of 6. Robert bought 114 pieces of chololate. How many packs of 6 did he buy, if he bought a total of 24 packs of chocolate? [students applied the "Guess-and-Check" strategy.]

d. Solve this problem: In art class Sally learned to make 2-color silk-screen prints. She had a choice of 8 colors for a project. How many 2-color combinations can she make with these 8 colors? [students applied the "Make a Table" and "Look for a Pattern" strategies]

e. Solve this "challenge" problem: The roof of the pavilion at the Lake Lemon Campgrounds is supported by 15 posts, each of which is 12 feet high. Carl and Hank plan to decorate the pavilion for a party. One thing they want to do is connect each post to all the others with crepe-paper streamers. How many streamers will they need to buy. [This was an optional, challenge problem for the students.]

f. Other problems: Included among the many other kinds of problems that the students have solved are problems involving percent, ratio and proportion, and extracting information from a chart.

March 6, 1987

MATH PROBLEM-SOLVING PROGRESS REPORT FOR MARK LEWIS[1]

TO: Mark's Parents
FROM: Frank Lester, I.U. Dept. of Math Education
phone: 335-0860

Seven weeks ago I began a special math problem solving project with Mark's class. During this time I have been teaching the class about twice a week as a supplement to the regular math class and I thought you would be interested in receiving a progress report. This report is not the same as the math grade, but work in the project will be considered when math grades are determined. In addition to a report on Mark's progress, I have prepared a page which includes a brief description of the project and examples of some of the kinds of activities that have been completed to date. You may want to ask Mark about the project if you are interested in what it involves.

GRADE: B.

Mark is usually very attentive in class, but he has not completed some of the homework assignments. His progress has been good. If he would do his homework on a more regular basis, it is likely that his grade would continue to improve.

[1]The student's name is fictitious. There was no one named Mark Lewis in either class.

Day 15

Name:______________________________________ March 9, 1987

Problem: A pair of immature mice got loose in the basement. It takes mice 1 month to mature and another month to produce a new pair of mice each month. The new pairs of mice grow and reproduce at the same rate. If no mice die, how many pairs of mice will there be after 8 months?

Day 15

Name: ________________________________ March 9, 1987

HOMEWORK: DUE MARCH 12, 1987

BE SURE TO SHOW YOUR REASONING FOR YOUR ANSWERS.

Problem 1: There is a pattern for each series of numbers shown below. Look for the pattern in each series and then fill in the blanks.

A. 1, 4, 9, 16, 25, 36, 49, ___, ___, ___

B. 1, 3, 6, 10, 15, 21, 28, ___, ___, ___

C. 3, 9, 27, 81, 243, 729, ____, ____, ____

D. 1, 3, 7, 15, 31, 63, 127, ____, ____, ____

Problem 2: A large pumpkin weighs 3 times as much as a smaller pumpkin. Their total weight is 48 pounds. How much does each pumpkin weigh?

Your answer to problem 2: ______________

Day 17

March 23, 1987

PROBLEM SOLVING TIPS

Understanding the Problem

* Read the problem carefully; often you should read it two or more times.

* Be sure you understand what the question is asking; ask yourself: "Do I understand what I am trying to find?"

* If you aren't sure you understand the problem, draw a picture or diagram of the information.

* Write down all the important information and the question; these are called: What I know and What I want to find.

Solving the Problem

* Explore the problem to get a good "feel" for what the problem is about.

* Don't do anything hard until you have tried easy ideas first; if easy things don't help, then you may need to do something more complicated.

* When you don't have any idea of what to do, try to make a good guess and then check it out with the important data.

* Use the strategies that you have learned; for example:

DRAW A PICTURE	GUESS AND CHECK	LOOK FOR A PATTERN
MAKE A TABLE	WORK BACKWARDS	SIMPLIFY THE PROBLEM

Getting an Answer and Evaluating It

* Be sure to check your work along the way, not just at the end; you may be able to avoid some unneccesary work by finding a mistake early.

* Be sure that you used all the important information.

* Write your answer in a complete sentence; this makes it easier to decide if the answer is reasonable.

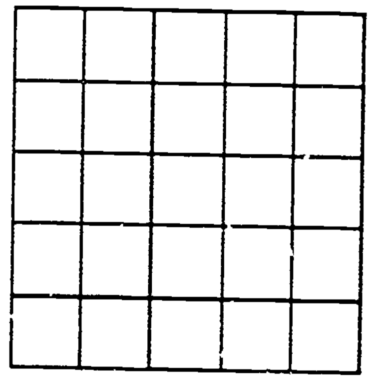
* Ask yourself: "Does my answer make sense?"

Day 17

Name: ______________________________ March 23, 1987

Class Assignment: Solve both problems, then complete the attached page.

Problem 1: How many squares are in the grid?



Your answer: ______________

Problem 2: Kathryn is buying fruit drink for her party. She can purchase a six-pack of 16 ounce cans for $1.92 or three 25 ounce bottles for $1.35. Which is the better buy?

Your answer: ______________

Day 17

Name:________________________________ March 23. 1987

Answer these questions about the 2 problems you just solved.

Problem 1

1. Was this problem hard for you? Why or why not?

2. Was this problem interesting to solve? Why or why not?

3. Have you ever solved a problem like this one before? If you have, can you describe it?

4. What strategies did you use to solve this problem?

Problem 2

1. Was this problem hard for you? Why or why not?

2. Was this problem interesting to solve? Why or why not?

3. Have you ever solved a problem like this one before? If you have, can you describe it?

4. What strategies did you use to solve this problem?

Day 18

Problem 1:

One really cold winter night the furnace failed at the "Tropicana Fish Shop". Because of this the temperature dropped in the shop and 20 goldfish died. To replace the fish the manager bought as many goldfish as he had left. He then divided all the goldfish equally among himself and 3 employees. The manager got 25 goldfish. How many goldfish were in the shop before the night that the furnace failed?

Problem 2:

Wiley Willie is applying for a job that pays $5 an hour. He tells his prospective boss that he will work for 1 cent the first week, for 2 cents the second week, for 4 cents the third week, and so on. Willie's boss agrees to this amazing offer and hires him on the spot. If Willie works 40 hours per week, how many weeks must he work before he is making the same weekly salary as he would have made earning $5 an hour? At the end of 20 weeks how much will Willie have earned altogether up to that time?

Problem 3:

Robert bought several packs of chocolate to give to his friends on St. Valentine's Day. Some of the chocolates were in packs of 4 and some were in packs of 6. Robert bought 114 pieces of chocolate. How many packs of 6 did he buy, if he bought a total of 24 packs of chocolate?

Day 18

Problem 4:

Frank said: "I am thinking of two numbers.

When you multiply them you get 204.

When you subtract them you get 5.

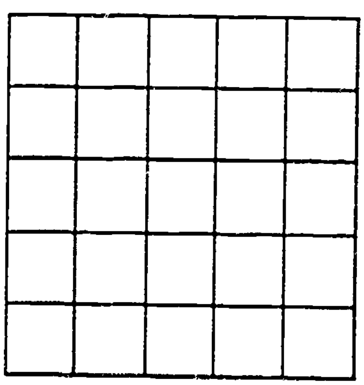
What are the two numbers I am thinking of?"

Problem 5:

A pair of immature mice got loose in the basement. It takes mice 1 month to mature and another month to produce a new pair of mice each month. The new pairs of mice grow and reproduce at the same rate. If no mice die, how many pairs of mice will there be after 8 months?

Problem 6:

How many squares are in the grid?



Day 18

NAME: ______________________ March 25, 1987

Read each problem carefully!
Circle the number that you think goes with each problem.

Problem	Very Interesting	Sort of Interesting	OK	Sort of Boring	Very Boring
1.	5	4	3	2	1
2.	5	4	3	2	1
3.	5	4	3	2	1
4.	5	4	3	2	1
5.	5	4	3	2	1
6.	5	4	3	2	1
7.	5	4	3	2	1
8.	5	4	3	2	1

Which problem was the most interesting? Why?

Which problem was the most boring? Why?

Day 18

Problem 7:

What is the remainder when 264 is divided by 3?

Problem 8:

Annual Savings on Heating Costs

Percent of the Heating Costs Saved by Setting Back at Night		
	Setback of 3°C	Setback of 6°C
New York	8%	12%
Milwaukee	6%	10%

Fuel can be saved if the thermostat in a house is set back at night. The table above shows the approximate savings in 2 different cities.

The Abrahms family, who live in New York, usually keep their thermostat at 17°C at night. At this temperature their annual heating bill would be $1200. The Abrahms plan to set back their thermostat to 14°C at night. How much money will they save in a year?

The Stephenson family lives in Milwaukee where annual heating costs for an average house are 15% higher than in New York. If the Stephensons set their thermostat back 6°C each night, how much higher will their heating bill be than the Abrahms' heating bill?

Day 18

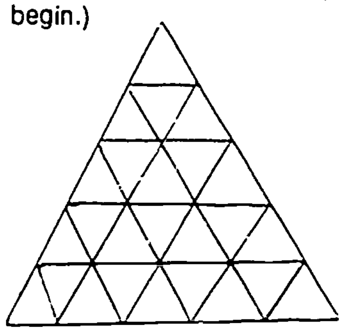
Name: ______________________ March 25, 1987

HOMEWORK: Solve these problems and be sure to show your reasoning! Use the back of the page if necessary.

Due: Monday, March 30

Problem 1: When you buy stamps at the post office, they are usually attached to each other. How many different ways can you buy 5 attached stamps?

Problem 2: One equilateral triangle can be divided into many smaller equilateral triangles. The triangle here has been divided into many smaller ones. Count only point-up triangles of all sizes. How many are there? (Be sure you understand what the problem means before you begin.)



Day 20

Igor and Prunella played a special card game using only 9 cards, numbered: 1, 2, 3, 4, 5, 6, 7, 8, 9. Prunella dealt Igor a hand of 3 cards.

Igor said: "The sum of the 3 cards you gave me is 15."

What are all the possible hands Igor could have?

Day 21

Name: ______________________________ April 6, 1987

HOMEWORK: Due Tuesday, April 7

Solve the problems below. Use the back of the page if you need more room to show your work.

Problem 1: Tony found $1.25 in change under the cushions of the sofa in his home. When he counted the coins, he noticed all of them were either nickels, dimes, or quarters. How many different combinations of coins could there be?

[NOTE: 3 COMBINATIONS ARE SHOWN IN THE TABLE; THERE ARE MANY MORE]

QUARTERS	DIMES	NICKELS	TOTAL
5	0	0	$1.25
4	2	1	$1.25
4	1	3	$1.25

Problem 2: Ruth and Abe are in charge of setting up tables for refreshments after the school play. There are 5 square tables and they are all the same size. The tables have to be arranged so that the sides are touching. In how many different ways can the tables be arranged?

Day 22

Name: ______________________ April 7, 1987

TWELVE PEOPLE WERE ASKED TO RANK 5 WELL-KNOWN ROCK GROUPS.
ANSWER THESE QUESTIONS ABOUT THEIR RANKINGS:
1. WHICH GROUP IS THE MOST POPULAR?
2. WHICH GROUP IS THE LEAST POPULAR?

Name: Ruth
5 AC/DC
2 Bon Jovi
3 Iron Maiden
1 Motley Crue
4 Run DMC

Name: Zena
3 AC/DC
2 Bon Jovi
5 Iron Maiden
1 Motley Crue
4 Run DMC

Name: Hector
4 AC/DC
1 Bon Jovi
3 Iron Maiden
5 Motley Crue
2 Run DMC

Name: Cuthbert
4 AC/DC
3 Bon Jovi
2 Iron Maiden
1 Motley Crue
5 Run DMC

Name: Roy
3 AC/DC
2 Bon Jovi
5 Iron Maiden
1 Motley Crue
4 Run DMC

Name: Irene
5 AC/DC
2 Bon Jovi
4 Iron Maiden
1 Motley Crue
3 Run DMC

Name: Eloise
4 AC/DC
2 Bon Jovi
3 Iron Maiden
1 Motley Crue
5 Run DMC

Name: Carmen
4 AC/DC
2 Bon Jovi
3 Iron Maiden
5 Motley Crue
1 Run DMC

Name: Ron
5 AC/DC
2 Bon Jovi
4 Iron Maiden
1 Motley Crue
3 Run DMC

Name: Hargraves
4 AC/DC
2 Bon Jovi
5 Iron Maiden
1 Motley Crue
3 Run DMC

Name: Egbert
4 AC/DC
[illegible] Bon Jovi
[illegible] Iron Maiden
5 Motley Crue
2 Run DMC

Name: Hortense
3 AC/DC
2 Bon Jovi
4 Iron Maiden
1 Motley Crue
5 Run DMC

YOUR ANSWERS: 1. ______________ 2. ______________

Day 23

Name: ______________________ April 13, 1987

CLASS ASSIGNMENT

PROBLEM: Felix was asked to put numbers on all the lockers in a new school. The lockers were to be numbered consecutively beginning with 1. He was supposed to put the numbers on the lockers one digit at a time. When he finished the job, Felix noticed that he had used 492 digits in all. How many lockers were in the new school?

IT IS VERY IMPORTANT THAT YOU SHOW ALL OF YOUR WORK !

Day 23

Name: ______________________ April 13, 1987

HOMEWORK: Due Friday, April 17. Turn in early to Ms. Willsey if you wish.

PART A: Solve the following problem.

Problem: Fiona was asked to put numbers on all the lockers in a new school. The lockers were to be numbered consecutively beginning with 1. She was supposed to put the numbers on the lockers one digit at a time. When she finished the job, Fiona noticed that she had used 642 digits in all. How many lockers were in the new school?

PART B: Answer the following questions about your work on the problem above.

1. What strategy did you use? (refer to your Problem-Solving Tips sheet)

2. How did you decide to use that strategy? Did something in the problem help you decide?

3. Was this problem like any other problem that you have solved? Which one?

4. How sure are you that your answer is correct?

5. What did you do to convince yourself that your answer is correct?

6. Make up a problem that is similar to the locker problem that does not involve lockers. (Use the back of this page.)

Day 24

Classwork: April 17, 1987 Name __(5th period class only)____________

Ima Poet was typing the final manuscript of her book. The book was 239 pages long. How many digits did she have to type as she typed the page numbers on the bottom?

Show all your reasoning here. Be systematic!

Answer ____________

Challenge: Extra Credit

If the "2" key on Ima's typewriter was broken, she would have to write in all the 2's by hand. How many 2's would she have to write?

Show your work on the back of this paper. Answer: ____________

Day 24

Classwork: April 17, 1987 Name _(6th period class only)_________

Ima Poet was typing the final manuscript of her book. The book was 239 pages long. Unfortunately, the "2" and the "5" keys on Ima's typewriter were broken and so she had to write in all the 2's and 5's by hand. How many digits did she have to write in by hand?

Show all your reasoning here. Be systematic!

Answer _________

Challenge: Extra Credit

When the famous German mathematician Karl Gauss was only 9 years old, he was asked to add up all the counting numbers from 1 to 100. He quickly added 1 + 100, 2 + 99, 3 + 98, 4 + 97, and so on. Each pair of numbers added up to 101, and there were 50 of these pairs. So, his (correct) answer was 50 x 101 = 5050.

Can you add up all the digits in the whole numbers from 0 to 100? [Hint: Be systematic -- find a pattern -- don't just try to do it by brute force. It's too messy.]

Show your work on the back of this paper. Answer: _________

21

Day 25

Name: ______________________________ April 20, 1987

HOMEWORK: DUE WEDNESDAY, APRIL 22

Here are some problems for you to try. YOU MUST DO AT LEAST TWO. Follow these tips to solve them.

A. Read over each problem very carefully to be sure you understand what it is about.

B. Draw pictures, make tables, and use any other of the strategies we have talked about.

C. Don't give up too quickly! If you get stuck, stop and see if there may be a different way to do it.

D. BE SYSTEMATIC. Look for patterns, organize the information.

Problem 1: The product of 1089 and the first few numbers produce some interesting patterns. Describe one of these patterns. Will this pattern continue if 1089 is multiplied by 5, 6, 7, 8 and 9?

1 x 1089 = 1089
2 x 1089 = 2178
3 x 1089 = 3267
4 x 1089 = 4356
5 x 1089 =

Problem 2: Pete had a pizza party after the basketball game. He ordered a Colossus Supreme pizza with sausage, green peppers, and mushrooms. He made exactly 5 straight cuts completely through the pizza. What is the maximum number of pieces of pizza he could have obtained by doing this?

Problem 3: Add any five consecutive whole numbers. Can the sum always be divided evenly by 5? (This means there is no remainder.)

Problem 4: A bottle and a cork together cost $1.10. If the bottle costs a dollar more than the cork, how much does the cork cost?

Problem 5: The angles of a triangle measure 180°. The angles of a square measure 360°. How many degrees are there in the angles of a figure that has 12 sides? [You can assume that all the sides are the same length and none of the sides intersect except at endpoints.]

Day 26

Name: ______________________________ April 22, 1987

HOMEWORK: DUE MONDAY, APRIL 27

BE SURE TO SHOW YOUR REASONING ON YOUR PAPER !!

This problem was invented by a French mathematician who lived in the 19th century. It has been changed to make it more up to date.

<u>Problem</u>: Every hour from noon until midnight on weekdays a Trailways bus leaves St. Louis headed for Chicago. Also, every hour from noon until midnight a Trailways bus leaves Chicago bound for St. Louis. All buses travel along exactly the same roads and highways and the trip takes 7 hours each way. If a bus leaves St. Louis at 1 p.m., how many buses travelling from Chicago to St. Louis will it meet during its journey?

<u>HINT</u>: To solve this problem will require some thinking on your part. You may find it helpful to draw some sort of picture to help you decide how to solve it. Also, THE ANSWER IS <u>NOT</u> 7 BUSES.

Your Answer: ________________

Appendix D

STUDENTS' INTERVIEW WORK

Kennedy collected 225 tape cassettes and 4 old shoe boxes to put them in. If he puts the same number of cassettes in each box, how many extra cassettes will there be?

56 R1
4)225
-20
25
-24
1

Amanda read and reread because "you have to read it a couple of times before you know what it means". She thought "it was division" because "if he puts the same number in each box, you have to divide them up". Had right idea. She thought that with multiplication "it would come out too many" and with subtraction "you wouldn't get a remainder". Some assessing of other operations. She calculated correctly, no check, no looking back.

The sixth grade math teacher did an experiment with her students. One at a time, students were to give her change for a 50 cent piece without using pennies. No student could use the same set of coins as someone else. How many students will be able to give her change?

25
10
5

25
25

10
5

5
5
5
5
5
5
10
10
5

10
10
10
10
5

5
5
10
25

10
10
10
5
5
5
5

10
10

$$\begin{array}{r} 16.6 \\ 3\overline{)50.000} \\ -3 \\ \hline 20 \\ -18 \\ \hline 20 \\ -18 \\ \hline 2 \end{array}$$

Amanda read and reread "about 5 times until I kinda understood it". She really didn't; in discussion she was confused about "exactly the same" condition (some analysis), and also "I didn't know how many students there are". Not enough analysis. She was "unsure how to do it". After some DISCUSSION, she began listing and "added in my head", saying to herself "5,10,15..". Not systematic. During listing she did check for repeats, and found one ("I've already written that one"). Feeling that she had too many more to list and "it would take too long and I thought there was an easier way", she looked back at the problem, then began to divide 3 into 50, because "if you divide how many coins there are into how much you needed, then that would give you how many ways". Assessed progress and revised, but no analysis or other check for reasonableness, although it seemed reasonable to her. She got 16.6 and rounded to 17, but felt unsure because 17 seemed too many (but she did this method because she thought there were a lot). Later she felt "it is probably right". No check of calculations, no look back, no later check for reasonableness.

Atlas Steel makes 4 different types of steel. From a shipment of 300 tons of raw steel, the factory produced 60 tons of type I, which sold for $60 a ton; 75 tons of type II, which sold for $65 a ton; 120 tons of type III, which sold for $72 a ton; and 45 tons of type IV, which sold for $85 a ton. Raw steel costs $40 a ton. It costs the factory $2,500 to convert every 300 tons of raw steel into the 4 types. How much profit did Atlas make on this shipment?

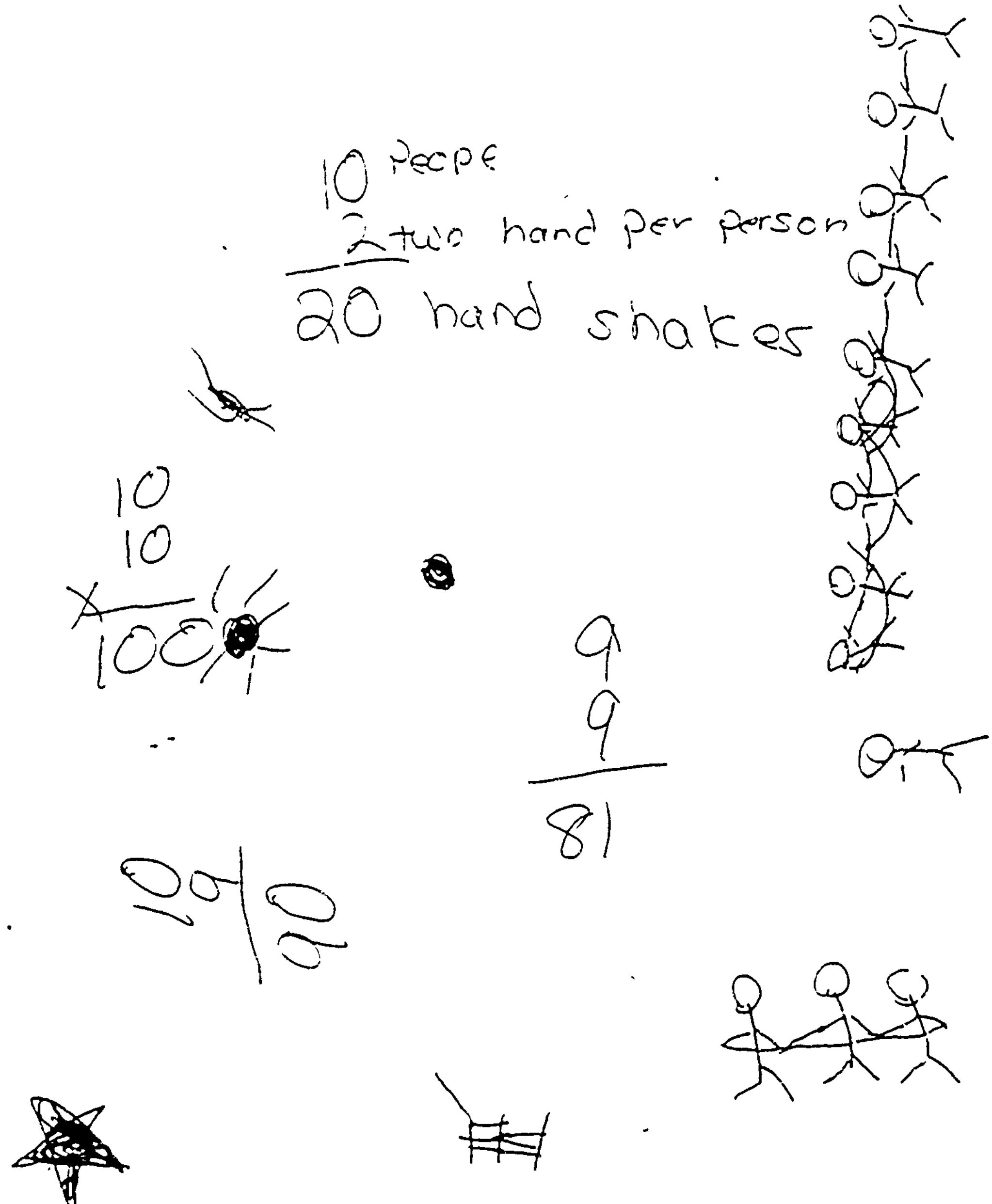
```
  300        75        120
x  60        65         72
-----      ----       ----
  000       375        240
18000      4500       8400
-----      ----       ----
18,000     4,875      8,640
```

```
  45         18,000
  85          4,875
----          8640
 225          3825
3600        -------
----        $35340
3,825         2500
            -------
             37840
```

Amanda

Atlas Amanda read and reread "because I didn't understand the 1st time". She had idea, she thought about multiplying: "multiply tons by cost for each ton, then added up the money...because it said how much profit and profit is money, so you add". She thought this after the 2nd reading. This was her plan. While carrying this out, she thought "added how much to convert...I was going to leave it out, it said to convert, split it up I guess". She ignored the 40 and whole idea of costs. Not a full analysis. She put 300 times the 1st 60, but realized during discussion she made a copying error, no attempt to change it. Not enough analysis of conditions, eg costs, some assessment of plan before carrying it out, also she did look back while carrying out to check "if I was doing it right , or if it was division". Plan did follow her understanding. She did not check the multiplication, but looked back at problem, missed 300. Not enough analysis to get relationships.

There are 10 people at a party. If everyone shakes hands with everyone else, how many handshakes will there be?



Amanda read silently, aloud, reread. "I didn't have to...I already knew what it said, but I did it anyway... I had to, to understand it". She thought each shook hands with each hand, two at a time, like one big handshake. Not enough analysis. She "just thought to multiply" 10x2, labelled it, said "I don't know if there would be 20". No final check or evaluation. After I asked about conditions (1 hand), she did not reread , just responded "10" then "20", after I asked about shaking with self, she responded "90", then 1"- again no rereading, no analysis, no evaluation.

Felipe's typewriter sticks when the 7,8, or 9 key is typed. If he types each number from 100 to 200, how many times will his typewriter keys stick?

7 8 9 10's 20's

10's 20's 30's 40's

100

1.7,8,9, 17, 18,19, 27,28,29 37,38,39 47,48,49

57, 58, 59 67,68, 69, 21

21
39
60

~~70, 71, 72, 73, 74, 75, 76, 77, 78, 79~~

70, 71, 72, 73, 74, 75,76,77,78,79, 80.
80, 81, 82, 83,84, 85, 86, 87,88,89, 90
90, 91, 92, 93, 94, 95, 96, 97,98,99.

Amanda read silently, aloud with pen, then silently. "It's hard". After a brief DISCUSSION with Diana, she drew a "little table" to organize her listing of numbers. While filling in, she started over (to make more room?), then started over again to include the 100's digit previously omitted. She didn't realize (overlooked) the 10's digits counted other than the doubles (eg 77..). Her plan to list fit her understanding. The list was organized but incomplete. She counted the digits with her pen and got 39, claiming "I didn't know any other way to do it". No check or verification or looking back. No full analysis of digit conditions. No check for completeness but probably felt confident. DIANA pointed out 70. Amanda wrote down the 70 and claimed the answer was "69...OK there are 10 in the 70's..10 in the 90's, plus all of the...". She just added 30 on to the 39, recounting the 10's in the doubles. No analysis of digit condition until she explaned. No evaluation. While explaning, she realized the overlap, separated those decades with 3 from those with 10 and claimed "51", " added all those that had 3 in every 10, then theres 70,80,90's"[21 (thru 60's) and 30 (thru 90)]. No check or evaluation. DIANA questioned her, so she stayed with this idea but wrote out all 70...99, crossing out previous numbers in the way, and counted digits individually with pen (to be certain?), "39 and the 21 here...60". No final count or check over- probably last counting served as verification. Needed discussions.

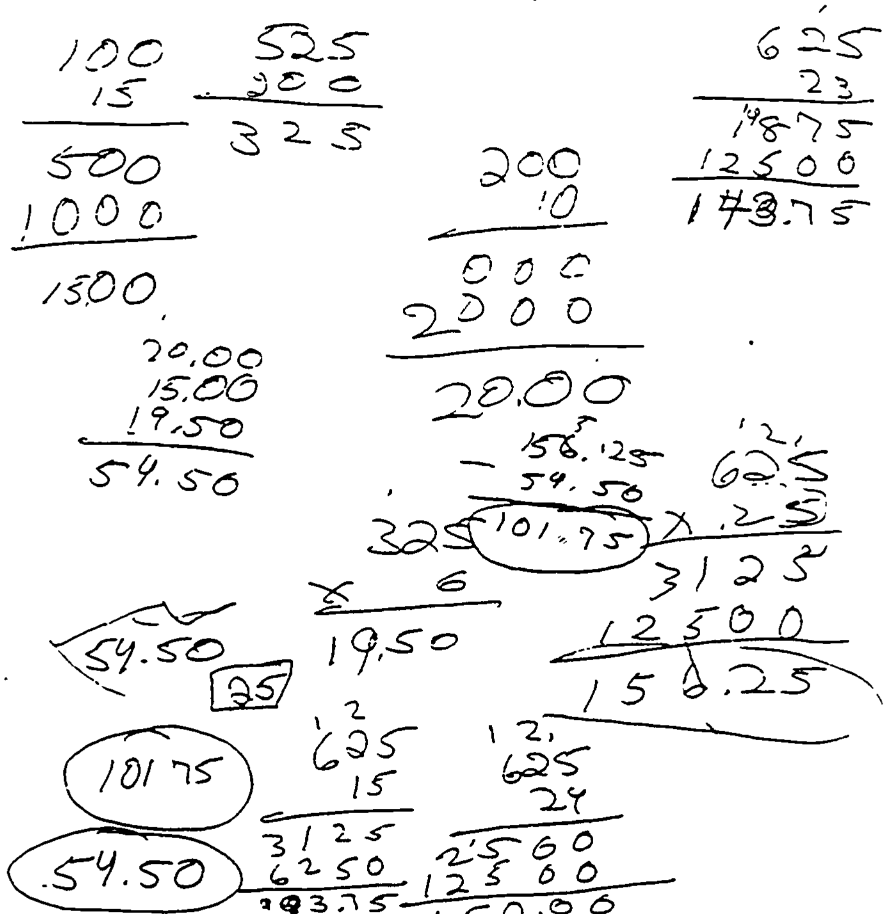
There were 347 people at a $150-a-plate luncheon to raise money for charity. Expenses were $5000. How much actually went to the charity?

2
~~150~~
~~347~~
~~00~~

150
347
1050
6000
+ 45000
$52,050
5000
47,050

Amanda read silently, aloud, reread "a couple of times". Listed 347 and 150, almost added, crossed out, multiplied, catching mistake on-line: "I was looking at the 3" (instead of 4). She "timesed the money by the number of people...got that answer and subtracted". After multiplying, she looked back at problem and asked "does expenses mean food", then subtracted. Seemed to understand, and confident, not much analysis needed (except expenses), had plan. No check of calculations, nor final evaluation.

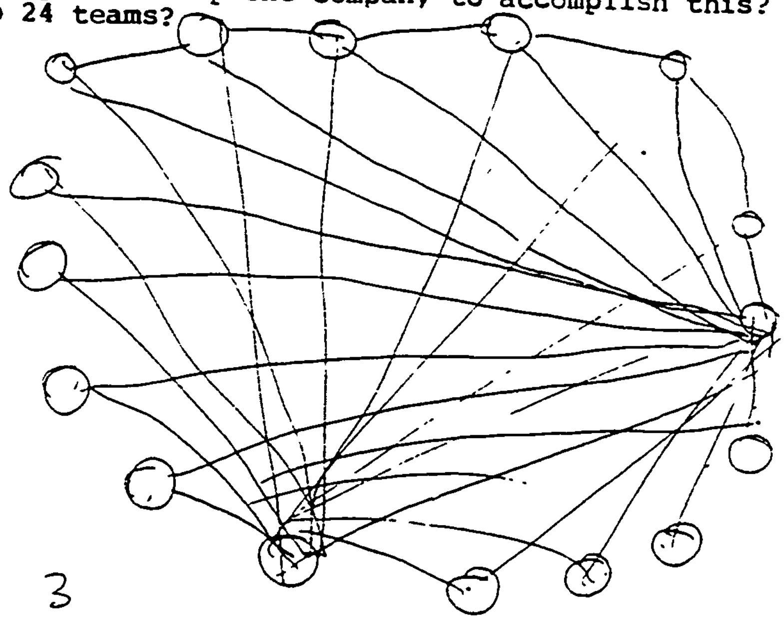
Mr Shuttlemeier's English class is printing a newspaper for the school and giving the proceeds to student government. The local newspaper is charging them \$0.15 a paper for the first 100 papers printed, \$0.10 each for the next 200, and \$0.06 for each paper thereafter. The class has orders for 625 papers. What will they have to charge for each paper in order to give student government at least \$100.00? 25¢



Amanda read and reread and said "I don't understand this, you can easily add up to get 100 dollars", "I didn't know what to do at 1st". She looked at Diana, a lot of DISCUSSION of conditions. She seemed to understand the cost per, but not how costs related to the other conditions. Not enough analysis. She then "just thought that the 1st 100, you would have to times by 15, the 2nd...". She multiplied costs by number, tallied total left after the 200, multiplied remaining and added them up.."to the local newspaper". She understood those conditions, but then asked "so how many papers,...after they paid". Didn't understand full picture, didn't have full plan, unsure how amount of papers fit in. Not enough analysis. After some thought, she had idea, tried multiplying 625 by 25, "just an easy number" guess. The product was too high - she was trying to hit the $100 condition but got "too much". She forgot the cost of the papers that she had previously calculated. She then tried a smaller number, 15, the product came out higher! She caught the discrepancy, "that one's more", and mistake. After DISCUSSION she realized "they would have to sell \$154 worth". She looked back at 25, realized "I would need a little bit less, systematically tried 23, and 24 "that one's more and that one's less". She was trying to hit 154.50 exactly, forgot "at least" condition. Confused, no rereading to check. After further DISCUSSION, she went back to 25 and subtracted the 54.50 "101.75 for the student government, and 54.50 for the paper". No final check, or check of calculations (even when didn't hit exactly). Trouble putting all conditions together - not enough analysis.. Needed discussion.

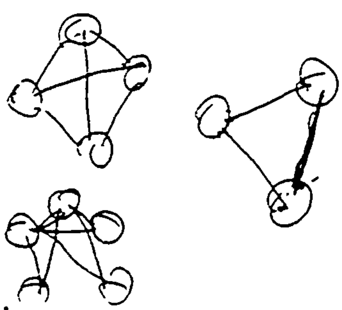
Name ______________________

There are 16 football teams in the National Football League, each in a different city. To conduct their annual draft, each team has a direct telephone line to each of the other teams. How many direct telephone lines must be installed by the telephone company to accomplish this? Suppose the league expands to 24 teams?



3

16
15
———
160
160
———
240



23/6

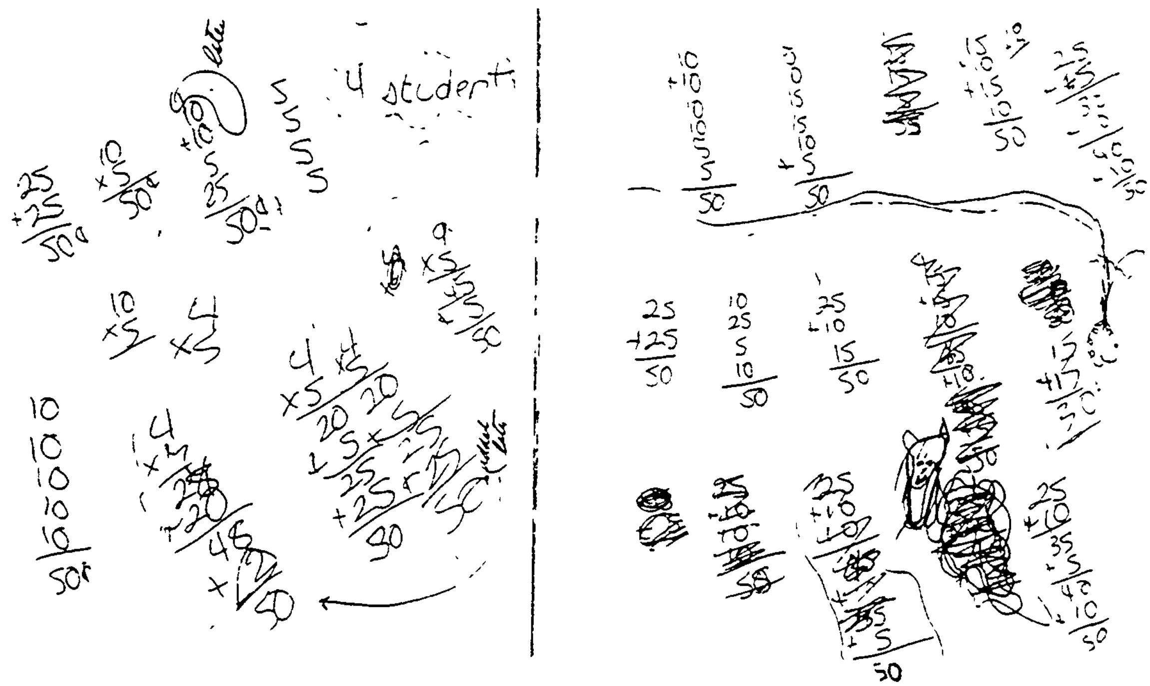
Amanda read aloud, having some trouble. She reread and said "I don't understand.. don't know anything about telephone lines". She didn't recognize problem or method. She asked "should I make a little place for each team". Drew diagram. Seems she is using Diana as a resource or check - she did this also with the other post-problems. She drew circles, paused and asked "each one has to be connected to the others - some analysis, but used Diana to do the clarifying for her. Not enough on her own. Maybe Diana spoke too much/too soon on previous - Amanda is taking advantage. She got an idea, then started connecting all to one, counted up the lines aloud and with pen and said "there's 15 first". Calculated "15x16=240", because "if you have 15 for one, and there's 16, you must multiply 15 times 16 and get 240 - they have to be all counted". No check of calculation, no look back, no check for reasonableness beyond her initial reasoning - it made sense to her. She saw the "not to self" aspect, but not the double. DISCUSSION about 3 teams -> she drew 3 circles and counted, connected lines "only need 3". Further DISCUSSION showed confusion between connections and lines..clarifying discussion followed, but she still thought 240. Didn't see doubles. Needed some more analysis.

Kennedy collected 225 tape cassettes and 4 old shoe boxes to put them in. If he puts the same number of cassettes in each box, how many extra cassettes will there be?

$$
\begin{array}{r}
56 \text{ R.1} \\
4 \overline{)225} \\
-20 \\
\hline
25 \\
-24 \\
\hline
1
\end{array}
$$

Tessa did not reread the problem, she just looked back at it to get the "right numbers". She knew it was division because "you can't times it cause it would be more". She divided and gave 56 R1. During discussion she commented that she sometimes talks to herself when she doesn't understand because "it straightens me out so I know what I'm doing". No analysis, little or no assessment, no check of anything - she was confident.

The sixth grade math teacher did an experiment with her students. One at a time, students were to give her change for a 50 cent piece without using pennies. No student could use the same set of coins as someone else. How many students will be able to give her change?



Tessa did not reread the problem, she looked at it briefly - "it didn't give me a specific way...it didn't come to me...didn't have another number". She started calculating sets, some adding, some multiplying. She seemed to have some understanding, "see what numbers made 50". She got 4 sets - "all I can think of". She did not look back, was not very organized, and had little concern for completeness. I asked if there could be any more "yeah...probably 10 or something". SECOND TRY. She started getting new sets, multiplying and adding together with running totals, using pen movements to "make sure it's right". She lost sight of coin condition (using 4, 7, 9) - just remembered she had to get 50 (the "big scene"). I had her reread it, she only caught that she used pennies, not that she used non-coin amounts. I REWORDED - THIRD TRY. She continued along, with no systematic plan, "I would use coins", just trying different combinations, but again lost sight of coin condition (using 15). No rereading, no analysis, no concern for completion, little monitoring, not systematic.

Atlas Steel makes 4 different types of steel. From a shipment of 300 tons of raw steel, the factory produced 60 tons of type I, which sold for $60 a ton; 75 tons of type II, which sold for $65 a ton; 120 tons of type III, which sold for $72 a ton; and 45 tons of type IV, which sold for $85 a ton. Raw steel costs $40 a ton. It costs the factory $2,500 to convert every 300 tons of raw steel into the 4 types. How much profit did Atlas make on this shipment?

593 R 2 dollars

4) 2500
-20
50
-36
14
-12
2

625 dollars

4) 2500
-24
10
-8
20
-20

2,500

18
x4
72

15
x4
60

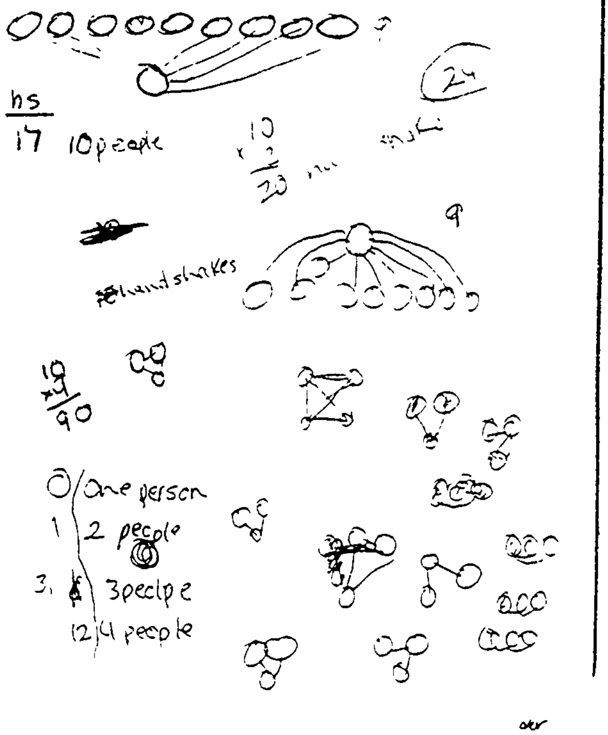
14
x4
56

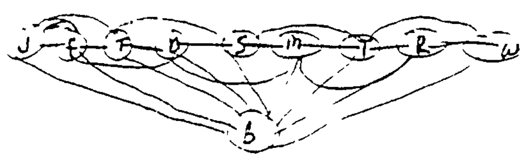
13
x4
52

12
x4
48

Tessa did not reread the problem, so she did no analysis of conditions, and had no understanding. She just started dividing 4 into 2500 because "the 2500 and the 4 caught my eye, I'd try it 1st, I was not sure I did it right" - but did no assessment or looking back. She divided, using side multiplications to help out, made a mistake but thought it was correct. She acted as fast as she did because "usually I can just look at a problem and tell, if it doesn't work, I'll try another" - an assess by carryout strategy, but she didn't assess this one. She had little concern for reasonableness, and ignored much important data. No final check on anything.

There are 10 people at a party. If everyone shakes hands with everyone else, how many handshakes will there be?





$3 \times 3 = 9$

Tessa did not reread the problem, but remarked "its only got one number in it". No analysis of conditions, no thinking, she just multiplied 10x2=20. No checking of anything. SECOND TRY - later when discussing, she drew 4 circles, connected them, got "12". She then got 6 for three people, but got 3 when modelled. When pointed out to her she said "but you can't do that, shake both hands" - she was lost even with a lot of discussion and modelling.

Name ______________________

Felipe's typewriter sticks when the 7,8, or 9 key is typed. If he types each number from 100 to 200, how many times will his typewriter keys stick?

3-107 17 27 | 37 47 | 57 | 67 | 77 87 97 | 200
13- 8 | 18 28 38 48 | 58 | 68 | 78 88 98 | 100
13- 9 | 19 29 39 49 | 59 | 69 | 79 89 99 | 50

39 ~~23~~ times
3
42

60

90
70
80

107 108 109 110 111 112 113 114 115 116 117 118
119 120 121 122 123 124 125 126 127
128 129 130 131 132 133 134 135 136
137 138 139 140 141 142 143 144
145 146 147 148 149 150 151 152 153
154 155 156 157 158 159 160 161 162
163 164 165 166 167 168 169 170
171 172 173 174 175 176 177 178
179 180 181 182 183 184 185 186 187
188 189 190 191 192 193 194
195 196 197 198 199 200

Tessa did not reread the problem, just "scanned it". She began writing out numbers in rows, counting with her pen. No analysis - her plan was to list and count. She had no consideration of completeness, some organization, some tapping while counting. No assessment, no final check. During discussion, she thought she made a mistake (100-200), assessed it, OK. When omissions were pointed out (70,...doubles...), she gave it a SECOND TRY. "Better start over" not to "get messed up". No rereading or analysis of conditions. Her new plan was to list out all numbers (assess: too many problems with shortcut?). She listed them in order, counted with pen "to make it easier". No check, no final evaluation. Later mentioned she doesn't feel good when she gets many wrong.

There were 347 people at a $150-a-plate luncheon to raise money for charity. Expenses were $5000. How much actually went to the charity?

```
   23
   347
 × 150
 ------
  , 000
 735
347
 ---------
5,2050
- 5000
 ---------
47,050
```

Tessa did not read the problem, she just looked back for the numbers ("took out parts"). She understood what to do - she multiplied "just a guess...how much it was" then subtracted expenses - "couldn't divide...couldn't add". She had the correct idea, but no analysis or assessment or check.

Name ____________________

Mr Shuttlemeier's English class is printing a newspaper for the school and giving the proceeds to student government. The local newspaper is charging them $0.15 a paper for the first 100 papers printed, $0.10 each for the next 200, and $0.06 for each paper thereafter. The class has orders for 625 papers. What will they have to charge for each paper in order to give student government at least $100.00?

625
500
100

100 - 5¢
200 - 10¢
whatever

525
500
~~300~~ copies made so far

100 × 5 = 500, 100, 1500

200 × 10 = 000, 200, 2000

2000 + 1500 = 3500
+ 2000 = 5500

200 × 10 = 2000; 15 ... 1650 ... 1500 ... 6850

100

6,950 final cost

− 100 dollars
from 7050

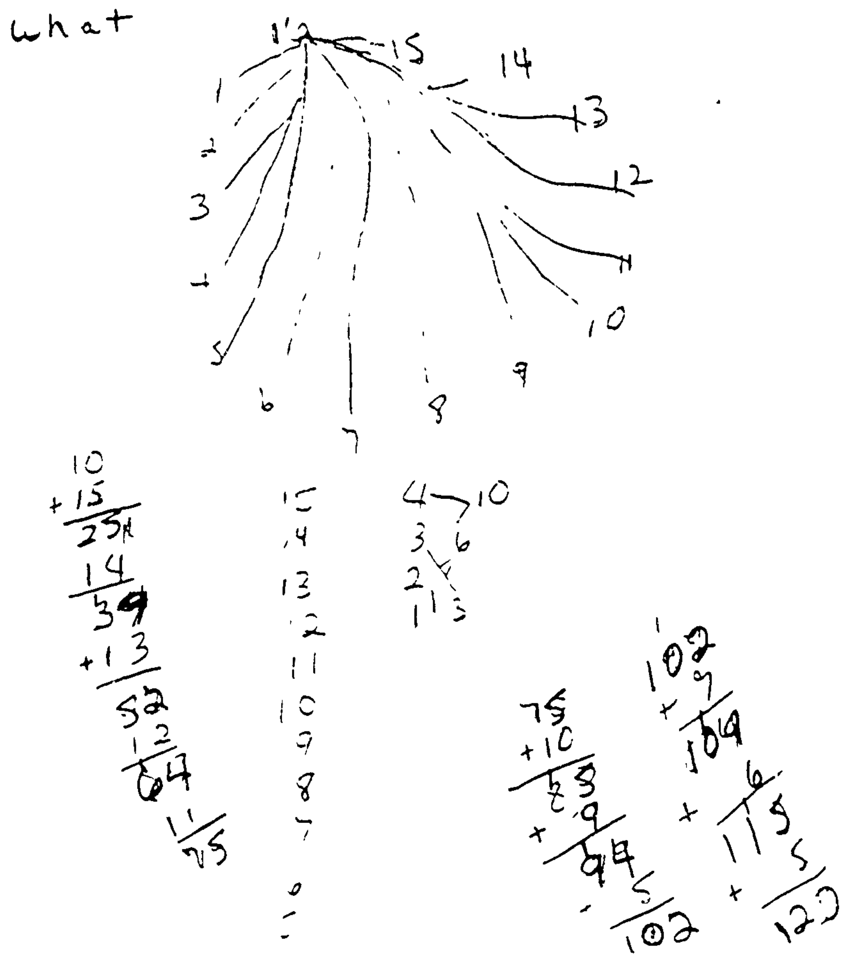
35 × 6 = 150

100 × 15 = 500, 100, 1500

Tessa did not reread the problem, she looked back to list numerical data on the side "so I don't have to keep looking at it". (Frank told them to consider "what I know"...). No analysis of conditions/meaning of data. She had some understanding/plan - multiplied amounts x costs to get "how much it cost", this was done semi-systematically but not according to exact conditions (100, 200). She kept track of "copies made so far" so she would "know where I was". She was a little unsure of her strategy - she "just guessed...if you try something and it doesn't work you can try something else". At least some reason in method. Added in 100, but lost sight of question asked - never tried to find cost of each paper, never looked back to check. Assess by carry out, with no assessment. No analysis, no check.

Name ______________________

There are 16 football teams in the National Football League, each in a different city. To conduct their annual draft, each team has a direct telephone line to each of the other teams. How many direct telephone lines must be installed by the telephone company to accomplish this? Suppose the league expands to 24 teams?



Tessa did not reread the problem. "I looked back at 16...it didn't tell me how many different cities it was in" - little analysis. She did recognize the problem/type and solution "right away" because "I've seen something like this before...with cray paper". She began listing numbers in a circle, drew lines "if I wasn't sure", counted with pen, began listing 15 14 because "16 goes to 15, 15 goes to one short", added 15 14....1, with subtotals, made error in first subtotal, didn't realize. When I informed her, she looked back, pointing and vocalizing, found it and carried through correction through each subtotal. She never got around to the second part of the question. No final assessment.

Kennedy collected 225 tape cassettes and 4 old shoe boxes to put them in. If he puts the same number of cassettes in each box, how many extra cassettes will there be?

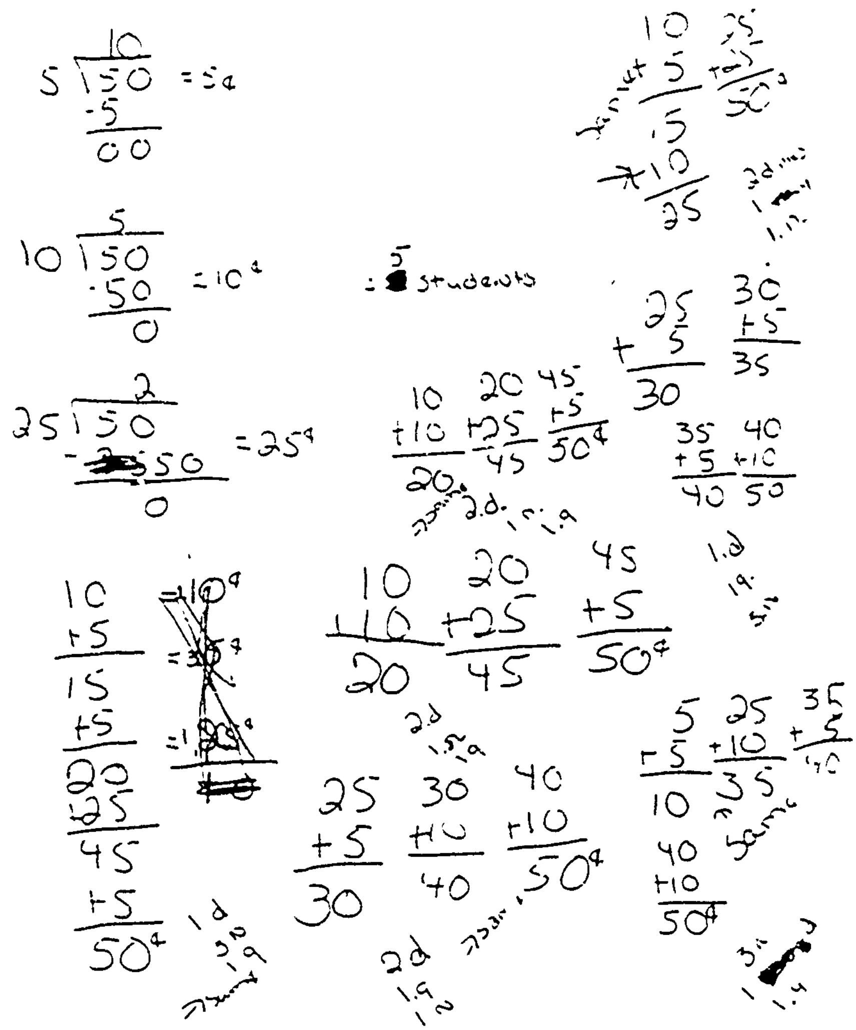
$$
\begin{array}{r}
56 \\
4\overline{)225} \\
-20 \\
\hline
25 \\
-24 \\
\hline
1
\end{array}
\qquad
\begin{array}{r}
256 \\
\times 4 \\
\hline
224
\end{array}
$$

56 in each box

1 left over

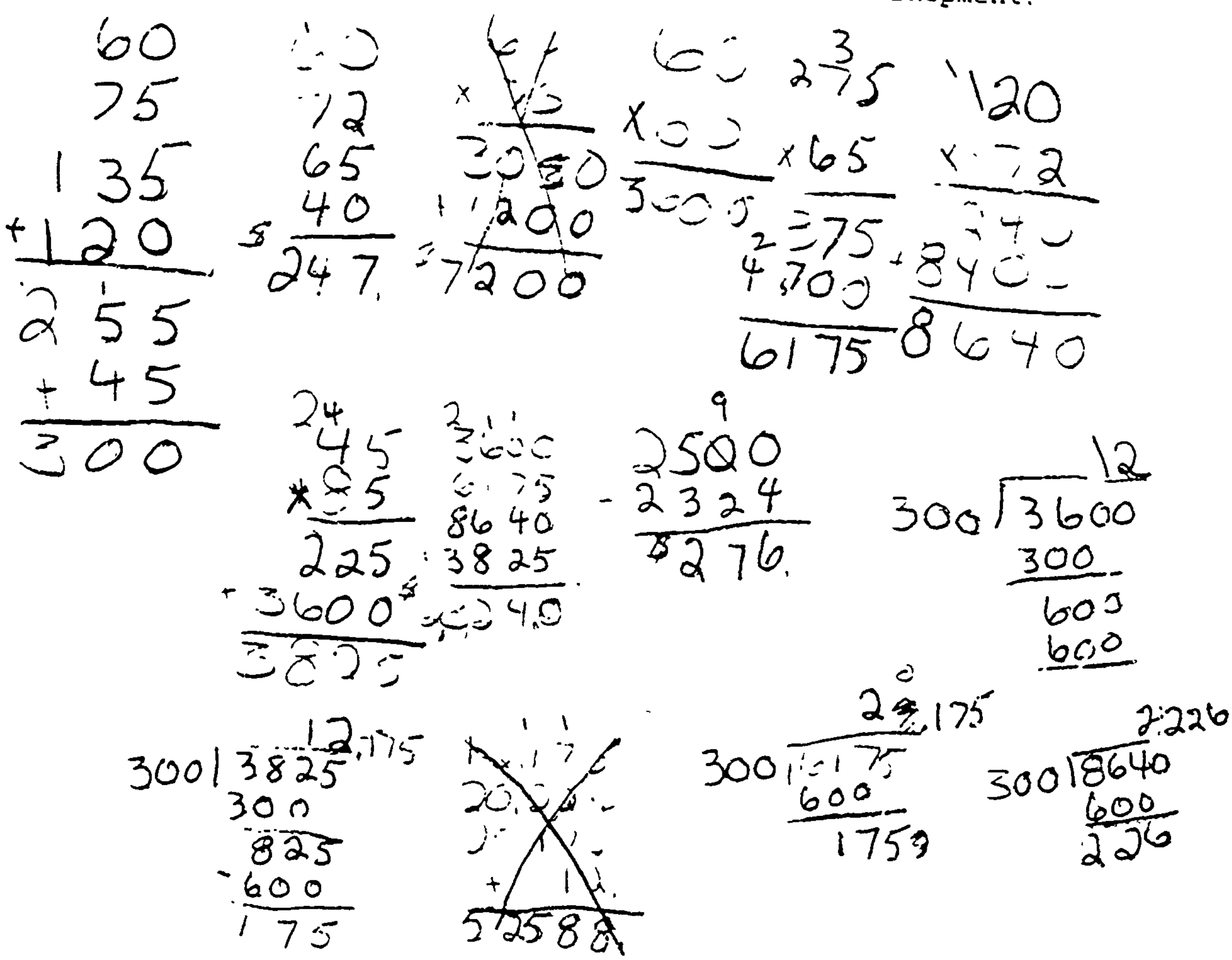
Wendy reread it and it "made sense". She almost started, but reread it again "just in case". She sensed it was division, was a little unsure and assessed this by carrying it out to see if the answer "makes sense". She divided, reread "just to make sure it's division", multiplied "to get it right", reread it , and wrote out the answwer. Wendy described a general strategy she uses when unsure what to do - she tries "all of them" and decides which operation is correct by using "common sense...if it sounds right and if you come out with the numbers in the actual problem".

The sixth grade math teacher did an experiment with her students. One at a time, students were to give her change for a 50 cent piece without using pennies. No student could use the same set of coins as someone else. How many students will be able to give her change?



Wendy reread it, with her lips moving because it helps by "hearing yourself". She didn't analyze the conditions well enough because she assumed all of the same coin. She divided to get sets quickly, counted the sets, cheched her calculations with her pen, looked back at the problem and gave "3". After being told the correct condition, she jumped into action - no rereading, analyzing, or planning apparent. She knew she "couldn't divide" so began with a vertical running total, realized checking and tallying this way was difficult, and changed to horizontal running totals. Her plan was to (unsystematically) get sets by adding, check for repeats at the end because she would get some "same ones", then tally up. She did some on-line checking for repeats because maybe she was "doing the same thing", but did so very ineffectively. After she put down as many as she could think of, she labelled each, and used the labels to check for "same ones " and totals. She did not check for completeness nor for reasonableness.

Atlas Steel makes 4 different types of steel. From a shipment of 300 tons of raw steel, the factory produced 60 tons of type I, which sold for $60 a ton; 75 tons of type II, which sold for $65 a ton; 120 tons of type III, which sold for $72 a ton; and 45 tons of type IV, which sold for $85 a ton. Raw steel costs $40 a ton. It costs the factory $2,500 to convert every 300 tons of raw steel into the 4 types. How much profit did Atlas make on this shipment?



$27600

Wendy reread several times, mainly "just certain parts..with the numbers". She "looked at the numbers to figure out what to do with them". Unsure, she tried adding the weights, then the prices "just in case". She had some plan to "take this times the money ,it shows how much it would be", so she started multiplying weights times cost, looked back to make sure , found a copying error, so extracted the rest of the numbers with her pen. She added the 4 products then adjusted the decimal places of the sum to give 4 digits, subtracted this from 2500, got 276, but "thought it might not be right". After a long pause, she started dividing by 300, added up the quotients, crossed this out went back to the 276, looked over her calculations, wrote out 276, then gave up. She did not use all of the needed data, didn't analyze conditions, assessed by carrying out her try all strategy and was lost.

There are 10 people at a party. If everyone shakes hands with everyone else, how many handshakes will there be?

$$\cancel{\begin{array}{r} 10 \\ \times 1 \\ \hline 10 \end{array}} \quad \cancel{\begin{array}{r} 10 \\ \times 10 \\ \hline 1{,}000 \end{array}} \text{ handshakes}$$

$$\begin{array}{r} 9 \\ \times 1 \\ \hline 9 \end{array} \quad \begin{array}{r} 10 \\ \times 9 \\ \hline 90 \end{array} \text{ handshakes}$$

$$\begin{array}{r} 45 \\ 2\overline{)90} \\ 8 \\ \hline 10 \end{array}$$

$$\begin{array}{r} 2 \\ \times 1 \\ \hline 2 \end{array} \quad \begin{array}{r} 3 \\ \times 2 \\ \hline 6 \end{array}$$

Wendy reread, visualized people handshaking, said "it depends if they shake with both hands or shake twice", and seemed to do no further analysis. She reread and multiplied 10 times 1, 10 times 10, got 1000 (didn't catch), no assessment or look over, and was done. During discussion she realized "they are not going to shake with themselves" and multiplied 9 times 1, and 9 times 10, got 90 but was unsure and confused. During modelling she realized they wouldn't shake twice, then got 45.

Felipe's typewriter sticks when the 7,8, or 9 key is typed. If he types each number from 100 to 200, how many times will his typewriter keys stick?

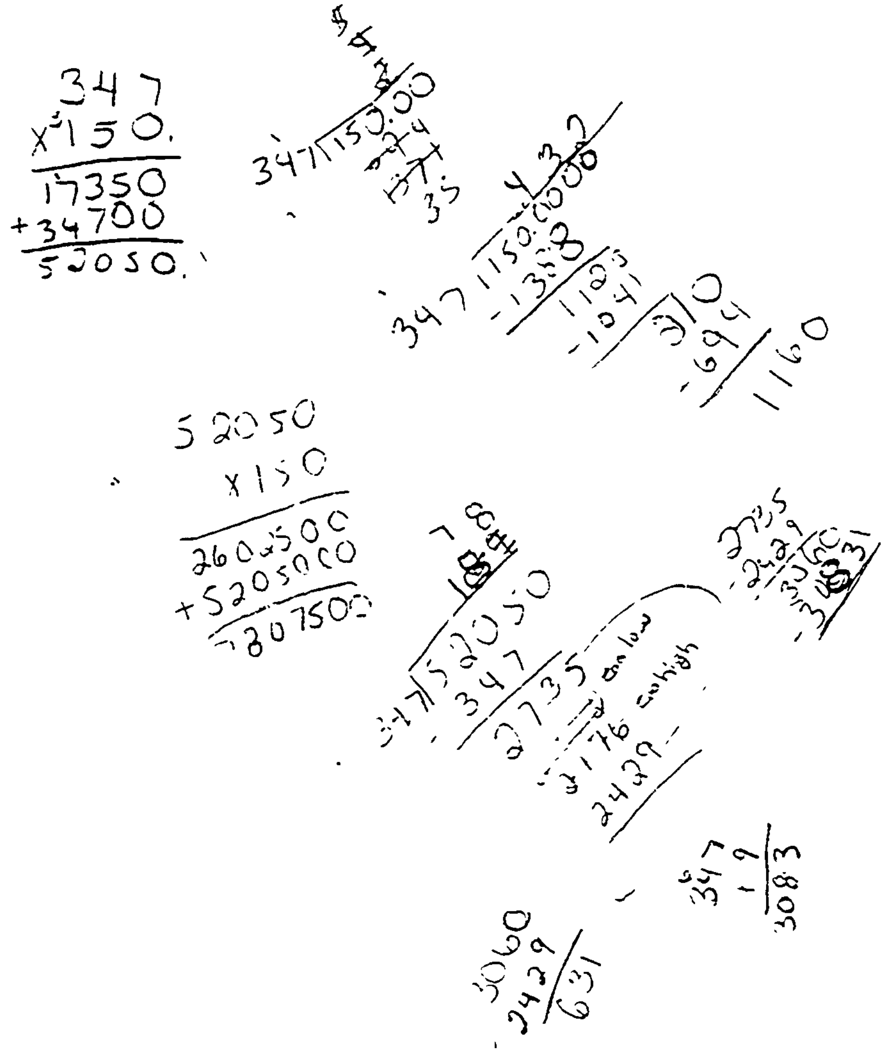
# of keys	total times stick	
100- 110	3	-33
110- 120	3	30 13 to many
120- 130	3	27
130- 140	3	24 21
140- 150	3	18
150- 160	3	15
160- 170	3 4	-12
170- 180	13	-8
180- 190	13	-5
190- 200	+ 12	
	60	

60 times

Because she "knew what to do", she reread quickly to "pick up the info I need quick". She then drew a chart because "we've done some in class with charts". She filled out the chart, added up the keys, using subtotals to help, then checked the addition again and caught an error, corrected it and was finished. No assessment or looking back.

Name ____________________

There were 347 people at a $150-a-plate luncheon to raise money for charity. Expenses were $5000. How much actually went to the charity?



Wendy reread the problem focussing on the "numbers in the question". She had some appropriate understanding/plan - "multiplication sounded right" then subtraction, but was unsure what from what. She assessed this by carrying out the multiplication, checked it, then looked at the problem to "see if I got the right numbers and if it makes sense". She abandoned the plan because the product 52,050 was "not anything close to it...way over the number(5000)". She then started dividing because "since the answer was too big, maybe division would make sense". She then divided other numbers, then multiplied some, then gave up. This was her old strategy - try all when lost, but she didn't add or subtract. She did little analysis of conditions, and focussed on the numbers.

Mr Shuttlemeier's English class is printing a newspaper for the school and giving the proceeds to student government. The local newspaper is charging them $0.15 a paper for the first 100 papers printed, $0.10 each for the next 200, and $0.06 for each paper thereafter. The class has orders for 625 papers. What will they have to charge for each paper in order to give student government at least $100.00?

# of papers	Price per Newspaper
100	.15
200 = 300	.10 -5
300 = 400	.06 -4
400 = 500	.03 -3
500 = 600	.02 -2
600 = 625	.01 -1
	.37

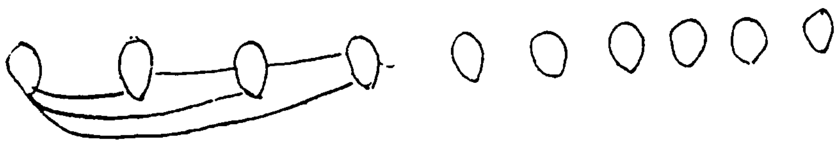
100
x15
500
+1000
1500.

100
x10
1000.

1000.00 = $.10 per paper

She reread, thought and "sort of found a pattern to it ...when you reread it, it started making a pattern by itself". She drew a chart, labelled it, and started filling it in. While working on it, she checked differences to verify the pattern, added up prices(?), multiplied amounts by price until she got 1000(?), looked at the problem, and wrote out the answer. Wendy likes to use patterns, even when not appropriate. Method was not reasonable. No analysis of conditions and meaning. No assessment of reasonableness. She was thrown by the numbers, but felt sure.

There are 16 football teams in the National Football League, each in a different city. To conduct their annual draft, each team has a direct telephone line to each of the other teams. How many direct telephone lines must be installed by the telephone company to accomplish this? Suppose the league expands to 24 teams?



||| ECH. = 15 - 119
= 14 - 104
= 13 - 90
= 12 - 77
= 11 - 65
= 10 - 54
= 9 - 4
= 8 - 35
= 7 - 27
= 6 - 20
= 5 - 14
= 4 - 9
= 3 - 5
+ = 2
179

16 teams
179 lines

21 →
20
19
18
17
16
140
+179
319

23
22

23 - 30
22 27
21 25
20 - 24
19 - 24
18 - 15
+17
140

24 teams
319 lines

Wendy reread it, was a little unsure, drew circles to represent teams because she "saw problems like this before...when you don't know what to do, draw". She reread it again, connected a few circles, started tallying, and "as I did it I noticed a pattern". She then began generating the pattern by listing numbers without counting connections, but in the course of doing this she did count some to verify the pattern. She added by subtotaling, using her pen to keep track (got it wrong - subtotaled and also added in 10's), checked it, then extended the method to 24. She took the result for 16 and added it to the extention (missed 16 teams), subtotaled and wrote out answer. She recognized the problem structure, and didn't check the extention implementation, nor the reasonableness of the result. She felt sure because of the problem type.

Kennedy collected 225 tape cassettes and 4 old shoe boxes to put them in. If he puts the same number of cassettes in each box, how many extra cassettes will there be?

$$\begin{array}{r} 56\,R1 \\ 4\overline{)225} \\ \underline{20} \\ 25 \end{array}$$

Anne

She reread the problem several times, subvocalizing because it made it "easier to find the meaning and figure out what to do". She spent a short amount of time thinking "trying to figure out what to do"(She seemed to recognize it as a / problem readily). She divided, vocalizing all the way through "4 goes into 225 how many times?", to help her keep her "mind on it". She then went back to the problem to "think about the question". Finally she said "1 cassette"

The sixth grade math teacher did an experiment with h[illegible] students. One at a time, students were to give her change for a 50 cent piece without using pennies. No student could use the same set of coins as someone else. How many students will be able to give her change?

50 ¢

~~25 25~~ Q Q

QDDN

DDDDD

NNNNNNNNNN

DDDDNN

DDDNNNN

DDNNNNNN

DNNNNNNNN

QDNNN

QNNNNN

10

Anne reread the problem three times because it was "unlike the math I've been working with". She spent some time thinking about it because at first she "didn't know what to do". She then planned to list "as many sets of coins that would go into 50". She started listing 25 25, but then crossed it out and listed Q Q because "numbers would take too long". Her listing of sets was somewhat but not completely systematic-a method she later described as "awkward". While writing down one set she was already thinking about the next. Several times she looked over her listed sets to see if she "could think of any more". Anne stopped at 10 sets, and quickly went through the sets with her pen to make sure they totalled 50.

Atlas Steel makes 4 different types of steel. From a shipment of 300 tons of raw steel, the factory produced 60 tons of type I, which sold for $60 a ton; 75 tons of type II, which sold for $65 a ton; 120 tons of type III, which sold for $72 a ton; and 45 tons of type IV, which sold for $85 a ton. Raw steel costs $40 a ton. It costs the factory $2,500 to convert every 300 tons of raw steel into the 4 types. How much profit did Atlas make on this shipment?

60
×60
00
360
3600

75
65
375
450
$4875

120
72
240
840
$8640

45
85
225
360
$3825

3600
4875
8640
3825
$20940
14500
$6440

300
40
000
1200
12000
2500
14500

$6440

Anne reread it because "it looked complicated", but she then "knew what to do". She had a global plan to calculate income then subtract cost, but she figured out the latter details while working on it. She multiplied the necessary quantities to figure income then quickly looked at the problem to see if there was "anything to do before adding" them. She vocalized through her calculations. Knowing in mind that she "had to subtract", she went back to the problem to figure out what to do. After rereading and thinking for a short while, she multiplied, then subtracted and came up with the correct answer. She was confident in her answer and didn't check her calculations.

There are 10 people at a party. If everyone shakes hands with everyone else, how many handshakes will there be?

9
9
9
9
9
9
9
9
9
9

10
9
90

9 18

9
9
7
6
5
4
3
2
1
45

Anne reread the problem several times. Unsure what to do, she tested out 10 times 10 in the air with her pencil while talking to herself, then said "no". She "realized people shake 9 times". She began listing 9's, but then wrote 9x10=90. She was confident in her answer and did not check anything. After I told her that her answer was not correct, she reread the problem and looked over her work. She asked me if she was really wrong and when I told her she was, she reread the problem again then listed 9,8,7... She counted to make sure all 10 people were accounted for, then added up the numbers while vocalizing. She was confident and didn't check anything.

Felipe's typewriter sticks when the 7,8, or 9 key is typed. If he types each number from 100 to 200, how many times will his typewriter keys stick?

1^{st} 10 = 3
2^{nd} " = 3
↓
10^{th} = 3

70's = 10
80's = 10
90's = 10

$$\begin{array}{r} 10 \\ \times\ 3 \\ \hline 30 \\ 30 \\ \hline 60 \end{array}$$

Anne claimed she knew how to solve it "before I came in here" because she "had a couple like that in class", but she reread and thought about it a little anyway. She then wrote 1st 10=3, 2nd 10+3, saw the pattern, and wrote --- 10th=3, then she added the 10's digits, and calculated "60". She carried it out very quickly, felt sure, and didn't check anything.

There were 347 people at a $150-a-plate luncheon to raise money for charity. Expenses were $5000. How much actually went to the charity?

```
  2 3
  347
  150
  000
 1735
 347
4 ~~5~~2050
  5000
 47050
```

$47,050

Anne claimed that she knew what to do after the first reading, but she reread it again anyway. Her plan was to calculate how much they received, then subtract 5000. She then thought it over "to see if it makes sense". She calculated, using some finger motions. She didn't check anything.

Mr Shuttlemeier's English class is printing a newspaper for the school and giving the proceeds to student government. The local newspaper is charging them $0.15 a paper for the first 100 papers printed, $0.10 each for the next 200, and $0.06 for each paper thereafter. The class has orders for 625 papers. What will they have to charge for each paper in order to give student government at least $100.00?

100
.15
500
100
1500
2000
19.50
34.50

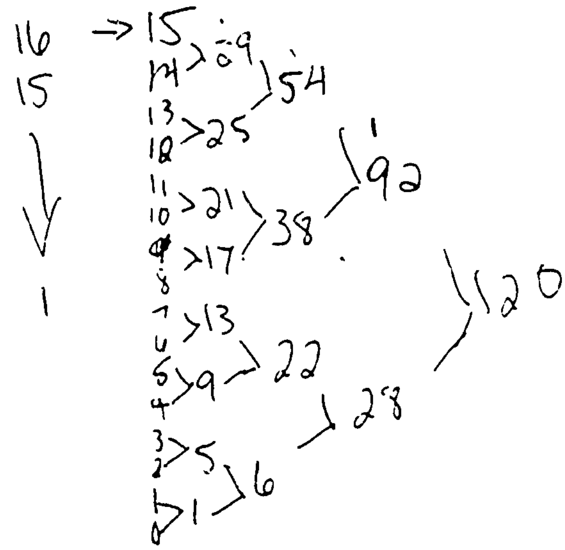
525
3
325
6
1950

22
625 | 134.50
1250
9500

625
.22
1250
1250
13750
34.50
103.00

Anne reread it, subvocalizing "to help it sink in, to get it fully". She was somewhat unsure how to solve it , but knew and planned to first calculate costs, then do something else. She began to carry out the first part of the plan, doing "what I know I can do first"-calculating the cost, vocalizing "now I have 325, now times 6...", doing some tallying of amount left. After this she thought a long while, thinking "how am I going to do this". Still a little unsure "how to figure out how much to charge for each", she thought she might need to add 100, and assessed this plan by carrying it out, because she "didn't want to sit a long time doing nothing". After she added (made a mistake and didn't catch it), she divided, then checked by multiplying to see if the found result did check out with the problem conditions. This was the first time she ever checked anything - problem type?, unsure of full plan at first?, training?

There are 16 football teams in the National Football League, each in a different city. To conduct their annual draft, each team has a direct telephone line to each of the other teams. How many direct telephone lines must be installed by the telephone company to accomplish this? Suppose the league expands to 24 teams?



Anne reread it and "thought it would be hard", so she read it over a couple of times. She thought of a map, with dots and connecting lines, and then "realized it was just like the others". She proceeded to write 16 - 15, 15 - 14,..., and continued listing, then "justed added them up", using a form of subtotalling to keep track. She talked to herself when calculating, seemed confident of plan, and later when referring to this type of problem remarked "once you figure it out you can do all like them". She quickly looked over her work after adding and said "OK".

Kennedy collected 225 tape cassettes and 4 old shoe boxes to put them in. If he puts the same number of cassettes in each box, how many extra cassettes will there be?

$$\begin{array}{r} 56 \\ 4\overline{)225} \\ \underline{20} \\ 25 \\ \underline{24} \\ 1 \end{array} \qquad \begin{array}{r} {}^{2} \\ 56 \\ \underline{\times\ 4} \\ 224 \\ \underline{+\ 1} \\ 225 \end{array}$$

Christy read, then reread silently, subvocalizing "to make sure I understand it...when I read it outloud I just read the words". She quickly divided 4 into 225, "I really don't know how I decided it...I didn't really think about it, I just started writing it out on paper". She seemed to be fairly confident. She then responded "there will be 1 cassette". DISCUSSION. Only later she checked by multiplying because "I wasn't sure if I was on the right track or not, so I went back and checked it". She seemed to check her work merely by checking the calculation, not the sensibleness. Not much to analyze or monitor.

The sixth grade math teacher did an experiment with her students. One at a time, students were to give her change for a 50 cent piece without using pennies. No student could use the same set of coins as someone else. How many students will be able to give her change?

1 25 25   2 10¢×5   3 5×10

4 1 25¢ 2 10¢ 1 5¢

5 3×10¢ 4×5¢

6. 1 25¢ ~~3¢ 4¢~~ 5¢

Christy read using her pen, then reread moving her lips, using her pen "so I can find the key words...the numbers and the problem". She didn't seem to do much analysis. Listed the 3 unmixed sets, plan was "I just started with anything I could think of that would equal 50... I started with the 25 and then I got less and less and I couldn't find anymore". She labelled the sets "just to see how many students would be able to give her change". Some plan and monitoring. Some analysis - "Can they mix up the change or do they have to be the same..I think they can as long as they don't use pennies". She listed a 4th (mixed) set. She labelled amount of each coin "these little numbers we just to tell me how much of each", put in c sign to "help me from getting 25 and 2 mixed up, all of the numbers were jammed together...I could tell if I started over" - some monitoring. I'm confused". She then asked "does it mean they couldn't use 10's if some else did?" More analysis, then put down 5th set, latter plan being "I made sure I had one of each...just 10 and 5, and this one 25 and 5 (6th which came later after discussion). Didn't follow thru on all mixed possibilities after she cleared up exact condition. Analysis piece meal, plans followed level of understanding, no real effort for completeness.

Atlas Steel makes 4 different types of steel. From a shipment of 300 tons of raw steel, the factory produced 60 tons of type I, which sold for $60 a ton; 75 tons of type II, which sold for $65 a ton; 120 tons of type III, which sold for $72 a ton; and 45 tons of type IV, which sold for $85 a ton. Raw steel costs $40 a ton. It costs the factory $2,500 to convert every 300 tons of raw steel into the 4 types. How much profit did Atlas make on this shipment?

60
× 60
120
1200
1320

75
× 65
375
4500
4875

120
× 72
240
8400
8640

45
× 85
225
2800
3025

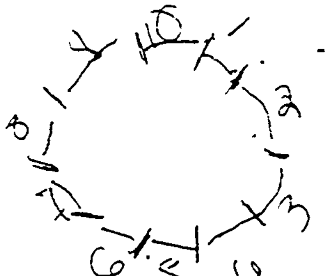
1320
4875
8640
3025
17860

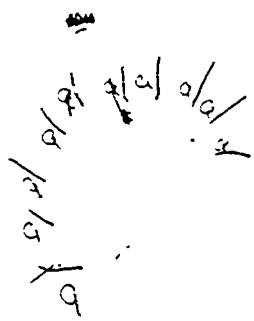
Christy read and reread "because if I don't understand a problem I keep reading and rereading until I understand it a little bit", with pen "helps me follow the words better, pick out the information I need". She reread it again, "thinking how to solve the problem...it had so many numbers, I didn't know what to do". She was thinking "if I should go ahead and do that ..if it's the right thing to do.. to multiply it by that number (60x60). She "knew it wasn't division, I decided to multiply...I don't know" Not enough analysis, understanding. She planned to multiply 60x60 to get "how much money they made off of the one type of steel...then I did the rest of them". Multiplied incorrectly, not close, didn't catch. She reread again, then listed the products to add, added them up (some pen movements) "I should add them up because it said how much profit on each shipment", got 17,860. Not enough analsis of conditions, didn't use all needed information, no check of calculations, maybe didn't understand profit. No final assessment of any kind.

There are 10 peop'e at a party. If everyone shakes hands with everyone else, how many handshakes will there be?

$$\begin{array}{r} 10 \\ \times 10 \\ \hline \cancel{100} \end{array} \qquad \begin{array}{r} 9 \\ \times 9 \\ \hline 81 \end{array}$$

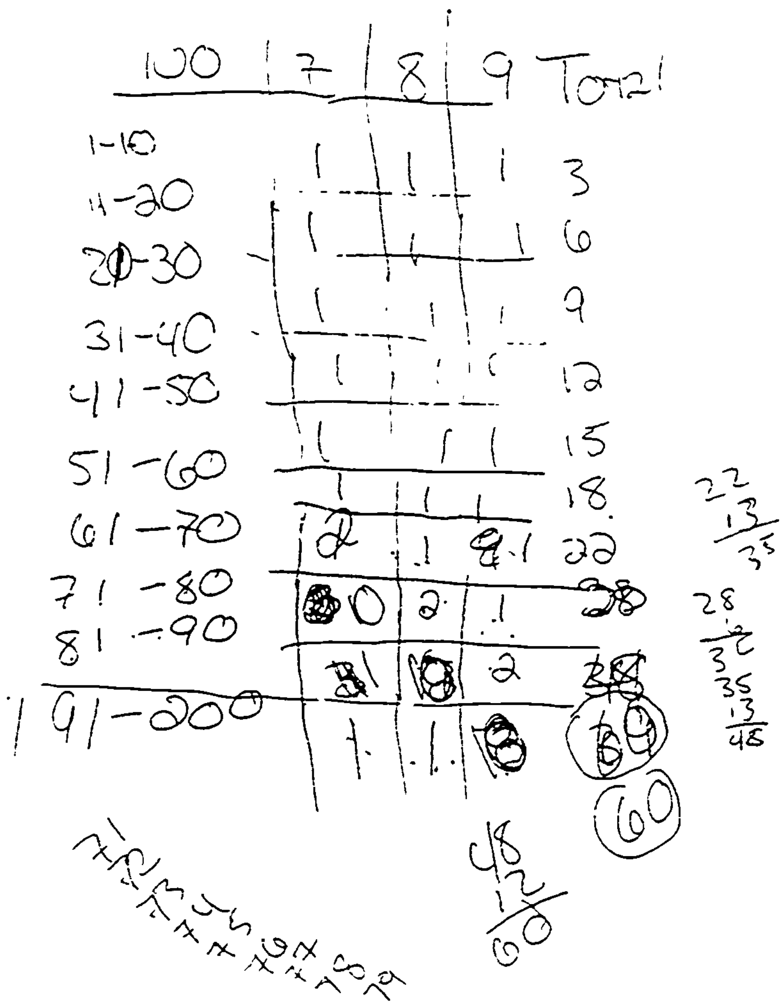
5





Christy read and reread, paused, then multiplied 10x10, saying "there will be 100 handshakes".." because "there's 10 people, one person shakes hands with 10 people, another shakes hands with 10 people, just keep going to 10x10". No analysis of conditions, it seemed reasonable to her, no final assessment, check. DISCUSSION - shake with themselves? "No", multiplied 9x9, "I guess it will be 9x9" DISCUSSION of simpler cases. She later remarked "I read the problem too fast..I should have thought about it more..they weren't gonna shake with themselves".

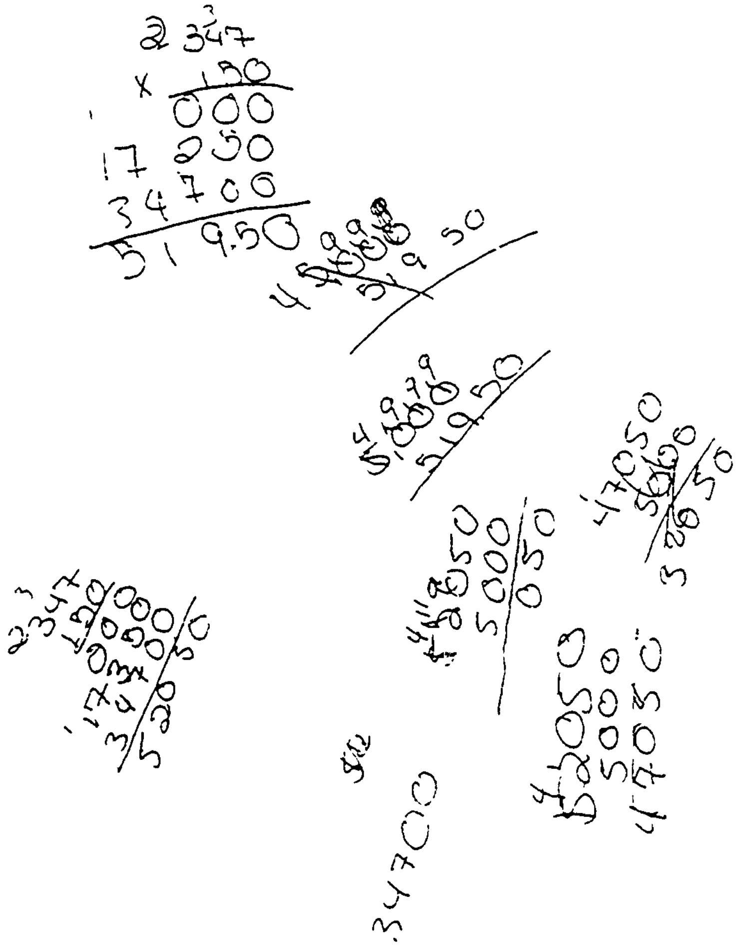
Felipe's typewriter sticks when the 7,8, or 9 key is typed. If he types each number from 100 to 200, how many times will his typewriter keys stick?



Christy read, did not reread (for the 1st time) - she saw "like one done in class - handwriting numbers on pages". She wrote 100 and started making a chart , left out 1-- digits. Seemed organized and confident. She went though filling in the chart going across then down, talking to herslf all the way, adding spaces as needed. At 61-70 made some on-line corrections, moved slower though 71-200, made some errors , didn't check. Used pen to tally across, subtotalled along the way, got "38". Not enough analysis of numbers, didn't catch 10's digits all the way. Plan was good, organized. No check for completion. DIANA told her she missed some. Christy went through her work carefully, checking totals, "I can't find my mistake" DIANA asked her to explain 61-70 and 71-80, "71 there's one 7, 77 there's 2, that makes 3...I see". She put 8 7's, started working on last 3 classes (71-200), then wrote out all the 70's, then changed total to 10, redid later entries, retallied, subtotled, and got 60. No rereading or further analysis or evaluation on her own, only after she had to explain. Some analysis and plan/organization - recognized type.

Name ______________________

There were 347 people at a $150-a-plate luncheon to raise money for charity. Expenses were $5000. How much actually went to the charity?



Christy read, reread, no pen, no lips. She multiplied 347x150=51950, didn't catch mistake. She reread and asked about expenses. DISCUSSION with Diana. Christy put in decimal point 519.50, then subtracted from 5000 twice, but didn't finish. She was "confused thinking there was a decimal point in there". She reread it and asked "..150 a plate for one person, do they pay for expenses too?". She seemed confused by the size of the numbers, some analysis, but not enough. Remultilplied at Diana's hint, got 52,050, didn't catch discrepancy, still confused "520.50 doesn't sound right, maybe 5205 or 52050 - but that seems too much". She subtracted 52050-5000=47050 "sounds too much". Confused, she tries something else - adding 47050 and 5000. Assessed earlier results by number size, not sensibleness of plan. Didn't check calculations, not enough analysis, had some idea, plan, "sound" most important, didn't reason.

Mr Shuttlemeier's English class is printing a newspaper for the school and giving the proceeds to student government. The local newspaper is charging them $0.15 a paper for the first 100 papers printed, $0.10 each for the next 200, and $0.06 for each paper thereafter. The class has orders for 625 papers. What will they have to charge for each paper in order to give student government at least $100.00?

15¢ 100
10¢ 200
6¢ Thereafter

625P

100 15¢ a paper
200 10 " "

19.50 6¢
15.00 15¢
20.00 10¢

525
325
.6
19.50

54.50
109.00

19.50
15.00
20.00
54.50

54.50.

2 54.50
1.50
27250
5450
81.75

Christy read aloud, wrote out information, then reread with pen. She calculated the total cost, rewriting info, doing the 100 and 200 mentally, writing down total left. She listed costs with each price, then added them up and got 54.50. Understood total cost idea up to this point, plan good, not full. She then reread with pen, and seemed confused "what will they have to charge..a east 100...are they talking about students here? Some analys she had trouble with who's paying whom, relationships in problem. She was lost, then tried to hit 100, by 2x54.50, 1.50 "less than $2" x54.50. Confused charge for each and total cost. No assess for sensibleness. No check of calculations.

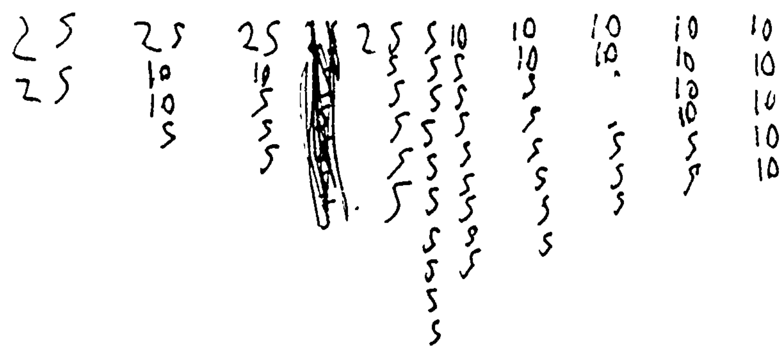
Kennedy collected 225 tape cassettes and 4 old shoe boxes to put them in. If he puts the same number of cassettes in each box, how many extra cassettes will there be?

$$
\begin{array}{r}
56 \\
4\overline{)225} \\
-20 \\
\hline
25 \\
-24 \\
\hline
1
\end{array}
$$

Jason read, but didn't reread right away. He "divided 4 into 225" because "you got 4 boxes and 225 tapes...you want to find out how many tapes in each box, and that would obviously be division"(understood division). He started dividing then reread because "I didn't read the last part really well...didn't understand why I should get the remainder" (confused by what was asked for?). He finished dividing, looked over the division, then said "56 remainder 1, the number of extra cassettes is 1".

The sixth grade math teacher did an experiment with her students. One at a time, students were to give her change for a 50 cent piece without using pennies. No student could use the same set of coins as someone else. How many students will be able to give her change?

10



Jason read then reread "I did reread it a few times". Thought awhile and seemed to understand. "I was trying to figure out an easier way to do it than listing, but I couldn't think of one..some clue to figure out a formula to make it easier, I thought it would probably be more than 10". He began listing "I just tried to put down all the combinations I could think of". He listed in columns, starting with 25,25 "I started with 2 quarters and went to q,d,d,n...2 dimes and I subtract one and add 2 nickels..the shortest way to do it". He checked each with his pen "to see if done well..make sure I put the right amount of nickels..I was counting 10,20,..50..I needed to figure it out pretty carefully". He inserted a combination "I started with a dime and ended with all dimes..I forgot to begin it with all nickels, I needed it for the right pattern, if it ended with a bunch of dimes, it should begin with a bunch of nickels". He looked back at the problem, looked over his work, counted his sets with pen, wrote 10. Then he added another combination "I thought I didn't have that one..I had it right there..I thought since I didn't have all 5's, I might have forgotten to write 25 with the rest 5's". He crossed it out and said "I come up with 10".

Atlas Steel makes 4 different types of steel. From a shipment of 300 tons of raw steel, the factory produced 60 tons of type I, which sold for $60 a ton; 75 tons of type II, which sold for $65 a ton; 120 tons of type III, which sold for $72 a ton; and 45 tons of type IV, which sold for $85 a ton. Raw steel costs $40 a ton. It costs the factory $2,500 to convert every 300 tons of raw steel into the 4 types. How much profit did Atlas make on this shipment?

60 × 60 = 3600

75 × 65: 375 + 4500 = 4875

120 × 72: 240 + 8400 = 8640

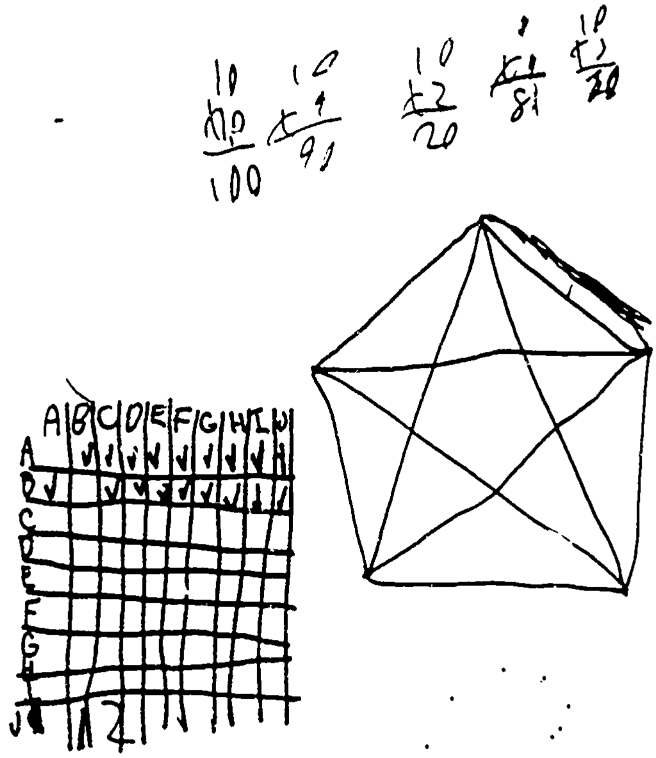
45 × 85: 225 + 3600 = 3825

300 × 40 = 12000
+2500
$14,500

3600
4875
8640
3825
$20,940
−14,500
$6,440.00

Jason read aloud then silently "read it twice". He used his pen a little near the end of rereading (seemed to have had a good idea of what to do) He got the plan "after I reread it". His plan was "gotta multiply number of tons by how much a ton costs..now I got to add the answers together..now I have to find out how much it costs to make it"(verbalized plan as he worked it). He realized the quantities could be found in reverse order. Feeling confident, he multiplied slowly and carefully and looked it over using pen to get out numbers, added products "they sold it all for $20940", multiplied and added to get costs "it costs $14500 to make the steel", subtracted and got a "profit of $5440". No check of final answer. Confident of plan, not calculations. At Frank's urging, he checked it over with his pen.

There are 10 people at a party. If everyone shakes hands with everyone else, how many handshakes will there be?



#people	# h s
1	0
2	1
3	3
4	6
5	10
6	15

Jason read, then reread. Thinking, some looking around. Multiplied 10x10 "I think multiplying 10x10 to get 100...I think that'd be the answer..I think I'm pretty sure...I think I see how it is done". No check or evaluation. DISCUSSION He mentioned "10 dots, 5 on each side, 10 dots in a circle and draw a line from each dot to every dot and count them." FRANK GAVE 5 VERSION. No reanalysis. He multiplied 5x5=25 "wait a minute let me think about this...it would be 20...5x5 means 1 guy shakes hands with 5 people but he doesn't shake hands with self..I think that would be 10x9 to get 90". No check. FRANK ASKED HIM TO DRAW. He drew 5 dots, connected them, counted with pen, "I think there would be 9..they'd only have to shake hands once". For 10 "I think there'd be 20..just double the answer".

Felipe's typewriter sticks when the 7,8, or 9 key is typed. If he types each number from 100 to 200, how many times will his typewriter keys stick?

9

$$\begin{array}{r} 20 \\ \times 3 \\ \hline 60 \end{array}$$

Jason read and reread because "I had trouble with this kind of problem in class..I didn't read it through carefully". He read DIANA INTERRUPTED HIM. His plan was to count the 7's not in 70s and add those 7s in 70s. Seeming to visualize, "9 7s from 1-100 plus 1 and all the 70s and 80s and 90s..there's 9 not in the 70s, plus 11 that's 20..so it'd be the same with all of them, so if you multiply by 3 you'd get 60". He wrote 9, multiplied 20x3, looked over, some thinking, some finger counting (evaluation-check), looked over. DIANA DIDN'T ASK META QUESTIONS.

5\2

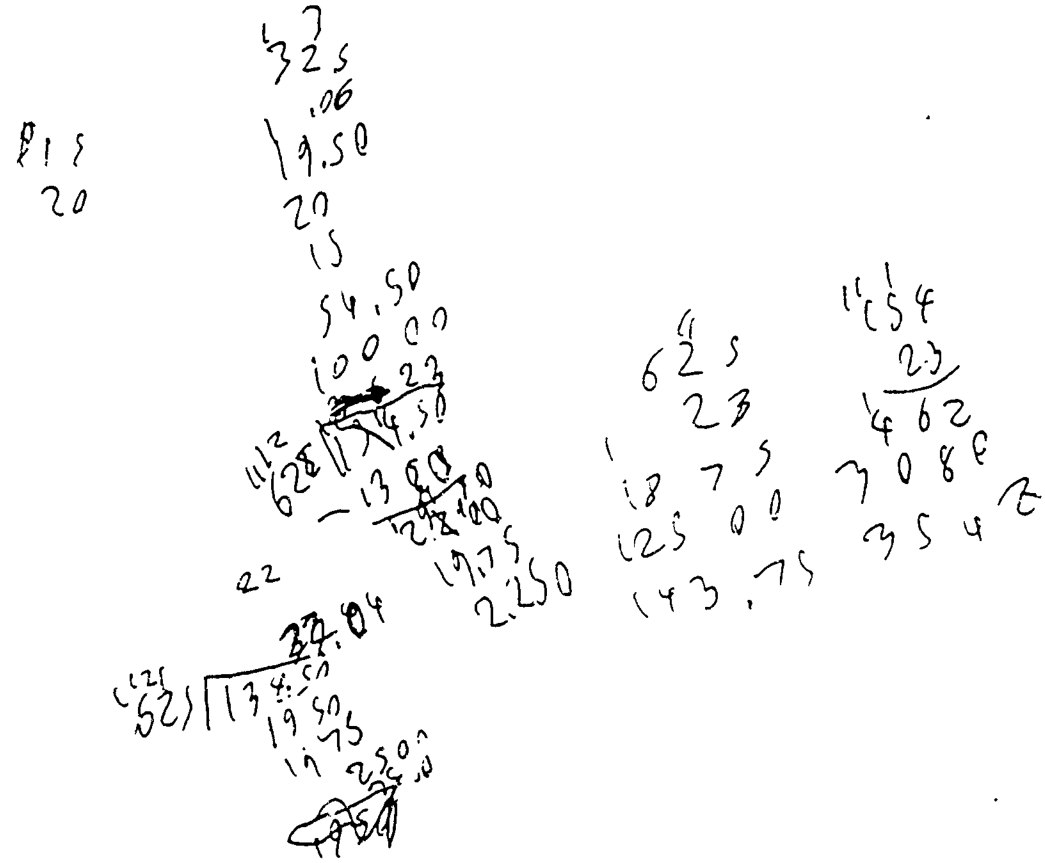
There were 347 people at a $150-a-plate luncheon to raise money for charity. Expenses were $5000. How much actually went to the charity?

```
   23
   347
  x150
 17350
+34700
 52,050
- 5000
 47,050
```

Jason read aloud with pen, then reread. He seemed to understand. Plan was "1st thing I need to find out the total money they made, can do that by mult". He mult 347x150 slowly and deliberately and got "$52050", then went through it again (he was "not really sure about the mult..doesn't make sense if they sold them for 150, and 347 is $52050 and there's only 5000..there's a bid difference).. "I nee o subtract expenses which is $5000". He subtracted and got 50 for charity". "I think I got it right". No check or review.

5/12

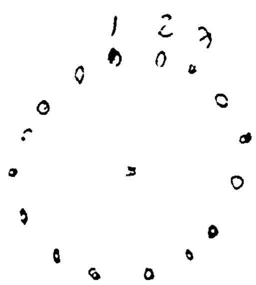
Mr Shuttlemeier's English class is printing a newspaper for the school and giving the proceeds to student government. The local newspaper is charging them $0.15 a paper for the first 100 papers printed, $0.10 each for the next 200, and $0.06 for each paper thereafter. The class has orders for 625 papers. What will they have to charge for each paper in order to give student government at least $100.00?



Jason read slowly, reread. Thinking, seemed to understand. Plan was to figure costs, need 100, divide by 625. "1st I need to figure out what the 100 papers cost..100x15 is $15 (in head), then the next 200 are 10 each (in head), recorded product ..so that's $20, that's 300 papers bought and 325 left...325 times 6 (out)..19.50"..listed costs.. "need to get $1 (added 100 in head) so its 154.50", divided "625 into that number to find out how much to charge". He divided slowly, "it would be roughly 23 cents..there's a remainder". MISTAKE not found, no check or evaluation (not interested in accuracy) "I might have messed up the division..I usually check along the way..if the answer doesn't seem right I sometimes do it over again..I don't like checking division..I'd probably do 625x23". CHECKED by multiplying, mistakes, confused "you have to multiply the right answer", divided again - wrong.

51.2

There are 16 football teams in the National Football League, each in a different city. To conduct their annual draft, each team has a direct telephone line to each of the other teams. How many direct telephone lines must be installed by the telephone company to accomplish this? Suppose the league expands to 24 teams?



6
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
120

3
23
22
21
20
19
18
17
16
156
120
276

Jason read aloud, no reread (he claimed he did problems like this before), seemed to understand right away, much analysis not necessary, knew what to do. He drew 16 dots in a circle (not in order), listed 15, 14, ...1,0 (while saying 15,14 as he wrote) because "this one called team 1 would connect to 15 other teams..so 15, this team here..they would only have to connect 14 teams..keep on going down..13, 12,..1". He counted them up with pen "making sure I had the right number of teams..I had 15 teams, I just added the 0". Then "what I need to do now is add them up". He added them up in head, subtotaling aloud with finger, then counted 10s digits, got "120". FOR SECOND QUESTION, he said "24 is 8 more than 16, so I need to do is go up..16, 17,,,23". He wrote up 16,...23, pointing "I counted until I had 8", added and got "276". Confident, no check.

Kennedy collected 225 tape cassettes and 4 old shoe boxes to put them in. If he puts the same number of cassettes in each box, how many extra cassettes will there be?

551.25
4) 1225.00
20
25
20
5
4
10
8
20

Melissa read and reread because "I wasn't quite sure of what to do then I read it again". She began to divide "I'm not sure..I think it's right..I figured if you divided you would figure it out, but I don't think it's right". She seemed to have an understanding, but was thrown off by the quotient she got, and didn't check. Before finishing the division, she went back and reread with pen, because "after I got the 1, I knew it was wrong so I went back to the problem to make sure I did it right..to make sure I had the numbers copied right". She continued the division because she "didn't know what else to do", and got 551.25. She caught one on-line mistake, but not her wrong division, didn't check it even though the answer wasn't reasonable to her, "there's only 225 tapes".

The sixth grade math teacher did an experiment with her students. One at a time, students were to give her change for a 50 cent piece without using pennies. No student could use the same set of coins as someone else. How many students will be able to give her change?

50¢ 25-25 5

25-10-5-10

10-10 10-10-10

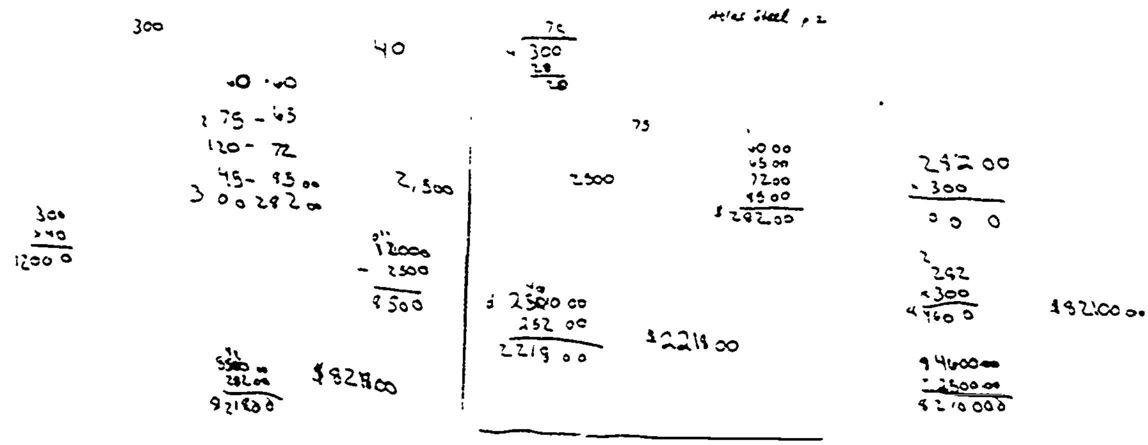
5-5-5-5-5-5-5-5-5-5

10-5-10-5-10-10

25 5-5-5-55

Melissa read and reread because "sometimes when I read out loud I don't exactly understand". She wrote down '50" cents "so I wouldn't have to go through that again". Her plan was to list sets, she started with 25-25 and then "broke up the 25 into 2 dimes and a nickel...I broke up the 25 as much as I could". She didn't consider breaking up the 10s in the same way, not enough analysis of the coin condition. She was somewhat systematic "the 10s went in sequece so I knew the 5s would go in sequence...I broke up the other 25". After each set was written down she went back and pointed with her pen "I was going 5, 10, 15, 20, 25". Afterward, she looked back "I was checking over..make sure I had gotten all.. making sure I hadn't missed something like a 10 ther or a 5 there". Her only problem was that her analysis and plan failed to consider the 10s, focussed on breaking 25s. DISCUSSION, she realized "I just figured it out - 25 and 5 5 5 5 . I had forgotten you can take 10 down to 5" - broke all 10s, (but not one at a time), didn't approach this systematically, or analyze or re-evaluate her work.

Atlas Steel makes 4 different types of steel. From a shipment of 300 tons of raw steel, the factory produced 60 tons of type I, which sold for $60 a ton; 75 tons of type II, which sold for $65 a ton; 120 tons of type III, which sold for $72 a ton; and 45 tons of type IV, which sold for $85 a ton. Raw steel costs $40 a ton. It costs the factory $2,500 to convert every 300 tons of raw steel into the 4 types. How much profit did Atlas make on this shipment?



Melissa read aloud using her pen to "help to keep where I am", but did not reread (claimed she usually did) - she just listed the data on the side "so I can sort it out in my head and don't have to read thru the whole paragraph". No analysis of information. She multiplied "300x40 to get the total cost", then "subtracted 2500 from the total it cost the factory to convert, I added this up (prices per) and subtracted it from that". Some pen movements "I lose my place when I am counting in my head". No check or evaluation. DISCUSSION She realized "this is definitely wrong". No reread or analysis, even though she was somewhat unsure. She started dividing 4 into 300, but "it was useless", she then added prices and subtracted total from 2500. Her new plan was to "add the totals" ..and subtract the 2500, but she subtracted the total from 2500 because "the number (282) wasn't big enough..so I turned it around". She decided this was not complete, not using the 300. (but there were a lot of unused numbers). Her overall plan was "I knew I had to add (prices)..and the 300 and the 40, I wasn't sure what to do so I just kind of multiplied and divided both times.. it was useless.. I knew I had to subtract the 2500 because you had to take away how much to convert, but I wasn't sure what to take it away from". She then multiplied 282x300, and subtracted 2500==82,100.00. While multiplying she took off 0's because "this is confusing..so I left off the 0's". Unsure, no analysis, some pen monitoring and other. No check of calculations, no final evaluation, no assessment of reasonableness of plan (had some idea of income-expense), no asses of reasonableness of result.

There are 10 people at a party. If everyone shakes hands with everyone else, how many handshakes will there be?

$$\begin{array}{r} 10 \\ \times 10 \\ \hline 100 \end{array}$$

$$2\overline{)100} \quad 50 \quad 50$$

$$\begin{array}{r} 12.25 \\ 2\overline{)25.0} \\ 2 \\ \hline 05 \\ 4 \\ \hline 10 \end{array}$$

$$\begin{array}{r} 50 \\ \times 50 \\ \hline 2500 \end{array}$$

1250

$$\begin{array}{r} 1250 \\ 2\overline{)2500} \\ 2 \\ \hline 05 \\ 4 \\ \hline 10 \end{array}$$

Melissa read aloud, looked like a brief reread. She multiplied 10x10 because "if there are 10 people..if you shake with everybody, and the next person shakes hands with 10 people, you go on and on..there would be 100". Not analysis of conditions, no final evaluation - it seemed reasonable to her. DISCUSSION (if I shake with you how many handshakes, people?). Melissa thought briefly, then divided 100 by 2 because "I divided because there's 2 people shaking each time". Seemed reasonable to her, but when she extended her method to 50 she remarked "I don't think it's right.. there'd be 1250 handshakes!". Seemed too high, but no reconsideration.

Felipe's typewriter sticks when the 7, 8, or 9 key is typed. If he types each number from 100 to 200, how many times will his typewriter keys stick?

107, 108, 109, 117, 118, 119, 127, 128, 129, 137, 138, 139, 149, 149, 147, 157, 158, 159, 167, 168, 169
177, 178, 179, 188, 187, 189, 197, 198, 199

Melissa read it, but didn't reread because "I understood it the first time". She had an understanding of what was being asked, and some conditions, but no analysis of the conditions, so she didn't catch the 10's digits. She knew what to do because "we did one just like it in class so I knew what we were doing...we were doing 1, 12, 22..". She listed "you have to take 107, 108,...and so forth. After she listed to 199, she counted them up with her pen "I counted them up afterward", using her pen to help count. No final check or evaluation for completeness, even though she remembered missing some on the one she did before. She claimed she checked "as I went along".

Answer. 39 times

21.

There were 347 peop e at a $150-a-plate luncheon to raise money for charity. Expenses were $5000. How much actually went to the charity?

150
× 347
1050
6000
45000
52050

52050
5000

512050
5000
47050

47050

Melissa read but didn't reread because "I knew some of it..it stayed in my mind..I didn't really need to know what it was for". She seemed to understand the structure. She "multiplied 150x347 and got 52,050", she used her pen to make guidelines because "I write crooked..if I don't put the lines, I screw up the calculation". She then cheched the multiplication "to make sure I had 3 things going down". She looked back "for the 5000", began to subtract the 5000 "and then the expenses were $5000...they'd probably take that back, I'm guessing here, it doesn't really say". She started subtracti but didn't finish, she got confused "I was trying to subtract 10 from 0, I thought it was a 1 I guess". She started over, subtracted and got $47,050, and wrote it out. Had the right idea all along, no final evaluation.

Answer. $47050.00

Name ____________________

Mr Shuttlemeier's English class is printing a newspaper for the school and giving the proceeds to student government. The local ne spaper 's chargi g them $0.15 a paper for the fir t 100 papers printed, $0.10 each for the next 200, and $0.06 for each paper thereafter. The class has ord rs for 625 papers. What will they have to charge for each paper in order to give student government at least $100.00?

$15.00

100 = .15
200 = .10
$20.00

100 × 15 = 500, 1000, 1500

25¢.

200 × 10 = 2000

$35.00 for 300 pp.

625 − 300 = 325

325 × .06 = 19.50

625) 154.500 = .247

3500 + 19.50 = 54.50

5450 × 100 = 545000

100.00

100.00 − 54.50 = 45.50

54.50 × .10 = 5.4500

54.50 × .50 = 272500

Answer 100

Melissa read, but didn't reread, and didn't fully understand "No, .I still don't understand". She did grasp the first part - figuring costs, but didn't know what to do after that. She listed some cost information on the side "so I wouldn't forget..I knew that those were important". She multiplied amounts by costs (100, 200), and put results near listing "because I forget", subtotal. $35 for 300, subtracted 300 from 625 with pen, multiplied, 625x6 with pen, added 35+19.50 with pen, "that's how much it cost". looked back "to make sure its 100, not 1000". She wrote dow the $100, and started multiplying 54.50 times .10., .50 "I didn't really know..I was just trying..if it looked good I would use it..compare to 100..about half..I don't know...I tried .50 and it wasn't enough". She subtracted 54.50 from 100, left that and multiplied 54.50 by 100 "I think that's right". Calculated costs correctly, but lost after that. No rereading or analysis, check of calculations, no final evaluation. Didn't think rereading would help. DISCUSSION, "it would have to be at least 154.50". She divided "I guess it sort of popped in (because of how I said it), using side multiplications to help "to multiply out..see what it equals". She rounded .247 to 25 "because 7 is more t half..more than 5", not because of the sense of the problem. No check of division.

There are 16 football teams in the National Football League, each in a different city. To conduct their annual draft, each team has a direct telephone line to each of the other teams. How many direct telephone lines must be installed by the telephone company to accomplish this? Suppose the league expands to 24 teams?

```
  16
 ×15
 ---
  80
 160
 ---
 240
```

240 lines

```
  24
 ×23
 ---
  72
 480
 ---
 552
```

552 lines

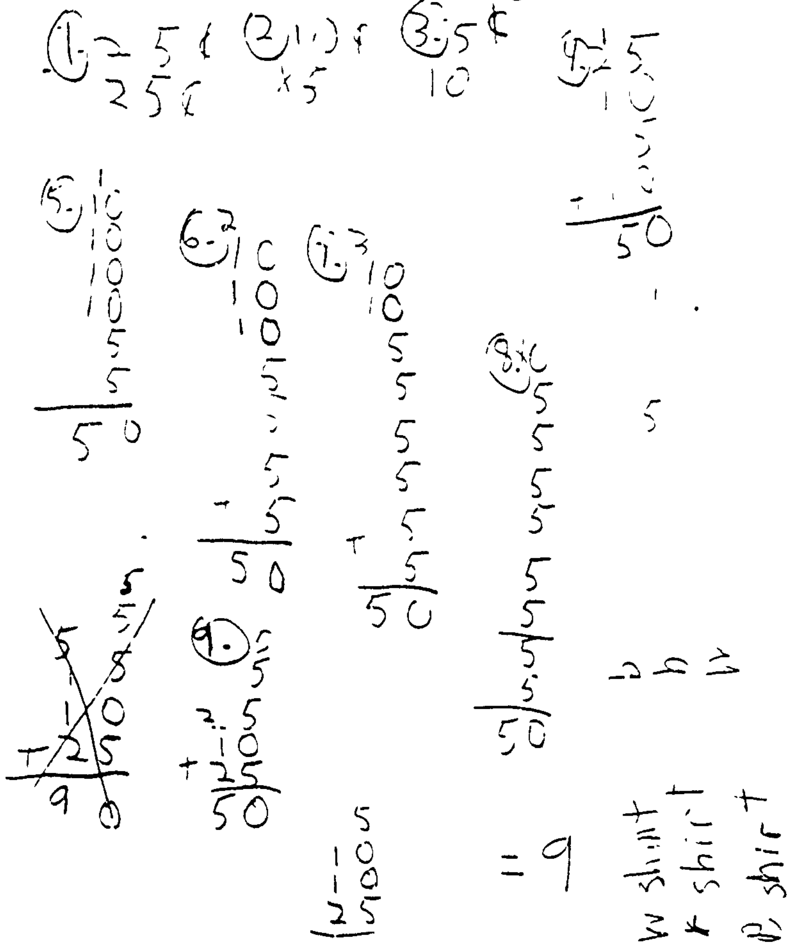
Melissa read but no reread. "No..I used to when I first came in here, I got better. I can pick out information and I can keep it until I finish reading and when I'm done I sort of write it out..I automatically picked out the 16". Recognized problem, didn't analyze, knew what to do "I've done a problem like this before..it was a streamer and pole problem". She multiplied "16x15 because you don't want a line to yourself", but she didn't consider the double aspect. She then multiplied 24x23, some pen movements during the multiplication. No final check or evaluation.

Kennedy collected 225 tape cassettes and 4 old shoe boxes to put them in. If he puts the same number of cassettes in each box, how many extra cassettes will there be?

$$
\begin{array}{r}
56 \text{ r}1 \\
4\overline{)225} \\
\underline{20} \\
25 \\
\underline{24} \\
1
\end{array}
$$

Tommy reread the problem "to understand better". He pointed at the words "maybe to make it clearer to me which words I was reading". He knew to divide "kinda by instinct", but he "thought I should still think about it more". He divided, then looked over the problem and his work "to make sure I did the right thing, and didn't leave anything out". No check of calculation. Correct.

The sixth grade math teacher did an experiment with her students. One at a time, students were to give her change for a 50 cent piece without using pennies. No student could use the same set of coins as someone else. How many students will be able to give her change?



Tommy reread "to get all the details". He thought about the conditions, "thought it might have to do with if one person used a quarter, another couldn't", but figured "if they wanted me to do that they would have said it in a better way". He also thought about the question because he was "not all that sure what the question asked". He was "confused how to do the problem", but started adding up sets, begining with 25 25 to "take care of these possibilities, then the possibilities with coins mixed up". He labelled to keep track of coins and amounts, had some trouble with set 4, erased, fixed. He went back to check for "anything I missed before, I didn't understand well". He then began adding other sets with 5 10, somewhat systematically, using his pen to help "count over my numbers...instead of just thinking in my head, I'm doing something physical...it helps...but slows me down". He went back to finish 25 sets (but missed 1 of 2), had some trouble with one, crossed it out, started over. He looked for more possibilities "I wanted to make sure I had all of them", then numbered them "to make them definite, separate them, so I know they're different". He added another set, but realized he already had it. He again looked to "make sure they're different, to recount again, make sure if there are any more poss" He reread to "make sure I didn't skip anything". He got 9, and thought his method "isn't very good...there's a better way..like a chart, so after your'e done you can check over..more self-contained".

Atlas Steel makes 4 different types of steel. From a shipment of 300 tons of raw steel, the factory produced 60 tons of type I, which sold for $60 a ton; 75 tons of type II, which sold for $65 a ton; 120 tons of type III, which sold for $72 a ton; and 45 tons of type IV, which sold for $85 a ton. Raw steel costs $40 a ton. It costs the factory $2,500 to convert every 300 tons of raw steel into the 4 types. How much profit did Atlas make on this shipment?

300
× 40
$12,000

12,000
+ 2,500
$14,500

60
× 60
$3600

65
× 75
325
4550
$4,875

$20,940
− 14,500
$6,440

3,600
4,875
8,640
+ 3,825
20,940

~~120~~
~~× 72~~
~~240~~
~~840~~
~~8640~~

85
× 45
425
340
3825

120
× 72
240
840
8640

Tommy reread "so I could comprehend it better". He thought a long time. His plan was to figure "how much it would cost them to process and buy the steel...then see how much they would make altogher, then subtract". He calculated products, sensed error in one - redid it in reverse, then again, crossed out 1st two - "it didn't make sense to me because these numbers should have equaled something higher then thes 2' He reread "just to make sure that's what I should have done". He then listed the products, added them, then subtracted. He then looked over it or reread the problem. "6440". No obvious check of calculations.

There are 10 people at a party. If everyone shakes hands with everyone else, how many handshakes will there be?

9 —   —9
9 —   —9
9 —   —9
9 —   —9
9 —   —9

$$\begin{array}{r} 10 \\ \times 9 \\ \hline 90 \end{array}$$

N

$1 \ldots N$

$$\frac{N \times (N+1)}{2}$$

$$\frac{10 \times 11}{2}$$

10-1=1&2, 1&3, 1&4, 1&5, 1&6, 1&7, 1&8, 1&9, 1&10
9-2 2&3
8-3
7-4
6-5
5-6
4-7
3-8
2-9
1-10
45
55

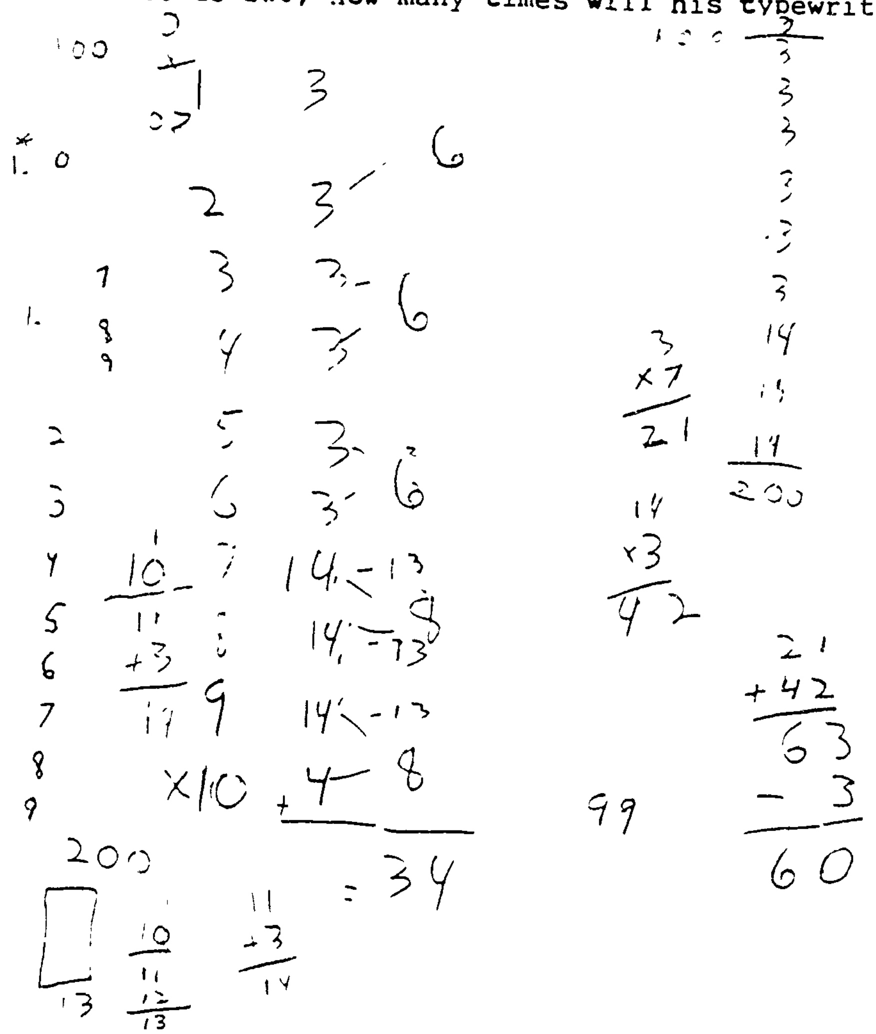
Ted - Ted & Jim - Ted & Lisa
~~Jim~~ Jim - ~~Jim & Ted~~ - Jim & Lisa
~~Lisa~~ - Lisa & Ted - Lisa & Jim

10 1
9 2
8 3
7 4
5 5
5 6
4 7
3 8
2 9
1 10

Tommy reread it, thought about it, but didn't realize they wouldn't shake twice. He listed 2 columns of -'s, connected some across and diagonal, reread problem, thought about it, moved pencil through -'s, counted, paused, calculated 10x9=90, wrote 9's next to -'s (verify 90), looked at his work and calculations, got 90. AFTER DISCUSSION WITH 3, he reread again, thought, wrote out handshakes (like his 3 method) 1&2, 1&3,...1&10, 2&2, then generalized, listed/numbered persons 3...10, then put handshake counts 10 9 8 ...(from generalizing) next to each person (errors: started with count of 10, missed 7). He added up counts, using pen, got 48. Checked calculation, but not listing of counts. Sloppy implementation. FRANK DIDN'T ASK ABOUT HIS WORK.

Name ____________________

Felipe's typewriter sticks when the 7,8, or 9 key is typed. If he types each number from 100 to 200, how many times will his typewriter keys stick?



Tommy read and reread "just to make sure I really understood..I can't do something hard without knowing what I'm doing, that's why I think ahead". He planned to "break the span into groups of 10, put the number of keys in each, then total up" He started listing, but only put 4 in 70, 80, 90's, realized grouping ones and teens could throw him so he noted it, used number (decade) guide to help "check that I had all possibilities (groups), subtotaled to help total, checked addition, tapped it out, "kind of checking". He did a lot of monitoring related to totalling, but miscounted in 7,8,9 groups, got "34". DURING DISCUSSION REALIZED "I forgot..wish I would have thought about it more", erased 4's, put 14s, used some multiplication to help total, added up to check 14, moved work to clean spot "not to get messed up", counted groups (check), got "63". DURING DISCUSSION realized 13 not 14, rechecked correctness of 13, subtracted, got "60". "I checked.. but I wasn't thinking ,I know I have a procedure to follow..I block out everything..instead of analyzing every aspect...to get the answer.. right calculations, I'm not so much interested in that..more interested in knowing how to do it". But he did a lot of monitoring for calculations but not checking for details of implementation. Sloppy.

There were 347 people at a $150-a-plate luncheon to r. se money for charity. Expenses were $5000. How much actually went to the charity?

$$\begin{array}{r} 347 \\ \times 150 \\ \hline 17350 \\ 34700 \\ \hline 52,050 \end{array} \qquad \begin{array}{r} 52050 \\ -\ 5000 \\ \hline \$47,050 \end{array}$$

Tommy read and reread "3 times", using his pen to help - "at 1st I didn't understand what expenses meant..probably costs". He seemed to have an understanding and planned to "multiply number of people times amountt...then subtract the expenses". He carried it out, caught mistake, then reread to "check I didn't leave anything out and that my method was correct". Quickly used pen to go through calculations.

Mr Shuttlemeier's English class is printing a newspaper for the school and giving the proceeds to student government. The local newspaper is charging them $0.15 a paper for the first 100 papers printed, $0.10 each for the next 200, and $0.06 for each paper thereafter. The class has orders for 625 papers. What will they have to charge for each paper in order to give student government at least $100.00?

100
x.15
$15.00

200
.10
$20.00

325
x0.6
$19.50

625) 154.50

15
20
+19.5
54.5

100
+54.5
$154.50

~~.35~~
~~.625~~
625
.344
2500
5000
18750
215.000

625) 154.5000
1275
2700
2500
2080
1275
725
1000

.3441

Tommy read and reread with some pointing and vocalizing. He had a plan to "multiply 1st, then do something else". He multipliplied out costs correctly but had trouble with one decimal pt.-did it 3x, added up costs, added 100 "because they needed 100 to give to students", when addidng "knew to divide", but looked back at problem "to make sure I was supposed to divide". He divided wrong but didn't catch it. Checked answer by multiplication, realized he had some error.."probably made a mistake in division or multiplication."

Appendix E

PROBLEM-SOLVING DATA BANK

APPENDIX E

PROBLEM-SOLVING DATA BANK

Notes on the Data File of Problems

The disk of problems was created using Appleworks (an integrated word-processor, data-base, and spread-sheet program).

The disk contains six files. Four are data-base files (B Problems, D Problems, J Problems, and FileFormat). The fifth (KeyToCodes) and sixth (ProblemNotes) are word processing files. This note is a hard copy of the file ProblemNotes. A hard copy of the file KeyToCodes is also attached. That file contains an explanatory list of the codings that were applied to all the problems collected in the data-base.

The Data-Base Files

There are three data-base files which actually contain problems and their codings. These files are B Problems, D Problems, and J Problems (where the letter B, D, or J simply represents the first initial of the graduate assistant who compiled the file).

To get an overview of the problems in any one of these files, scroll through by using the up and down arrow keys. To view the individual problem entries (records) one by one, press OPEN-APPLE and Z simultaneously to move to single record mode. Then use OPEN-APPLE and the up or down arrow keys simultaneously to move from record to record.

The file named FileFormat is a blank template for problems and their various codings. Its form is the same as each of the actual problem files, but it contains no data.

Printing from the Data Base

To print out information from one of the data base files, press OPEN-APPLE and P simultaneously. Next press RETURN to indicate that you want to use one of the two Report Formats which have already been set up. Highlight the Report Format of your choice (see below) and press RETURN. An example of a record will appear on the screen again. Press OPEN-APPLE and P again to indicate that you want to print. Press RETURN to affirm printing using whatever printer your Appleworks disk is set up to use. (To set up your Appleworks disk for use with a particular printer return to the Main Menu of the Appleworks disk--by pressing ESC repeatedly--and choose OTHER ACTIVITIES.) Indicate the number of copies and press RETURN. Note: at any stage in this process, if you want to back out, press ESC (repeatedly, if necessary).

There are two Report Formats. The one called AllInfo prints out all the information for each record. If you choose this, you will get a hard copy of <u>all</u> the information for <u>all</u> the records (i.e. exactly what appears on the screen). The

Report Format called Cards will also give information from all the records in a file, but only the ID# assigned to the problem, and the problem statement. Furthermore, Cards is formatted to be printed on 4 x 6 inch continuous feed cards.

The data base is presently set up to print all the records (you can see this at the top left of the screen, where it says "All Records"). But, of course, the beauty of using a data base is that the records can be sorted by any of the variables listed and any subset can be selected, manipulated, and/or printed. To select a different subset of the records, press OPEN-APPLE R, and follow menu directions from there.

Key to Problem Codes

ID#. Three lists of problems, D (Diana), J (Jackie), B (Bea), each numbered consecutively

Source: Abbreviations for the books from which the problems were taken

Billstein
Billstein, R., Libeskind, S., & Lott, J. W. (1984). A problem solving approach to mathematics for elementary school teachers (2nd ed.). Menlo Park, CA: Benjamin Cummings.

Charles
Charles, R. I. (1985). Problem solving experiences in mathematics, Grade 6. Menlo Park, CA: Addison Wesley.

Dev e
evine, D., & Kaufmann, J. (1983). Elementary mathematics for teachers. New York: John Wiley.

Heath6
Rucker, W., & Dilley, C. (1979). Heath mathematics, Grade 6. Lexington: D. C. Heath.

Heath7
Rucker, W., & Dilley, C. (1979). Heath mathematics, Grade 7. Lexington: D. C. Heath.

Holden/Garofalo
Holden, L., & Garofalo, J. In-house handout. Indiana University, Bloomington, IN.

Krulik
Krulik, S., & Rudnick, J. A. (1980). Problem solving: A handbook for teachers. Boston: Allyn & Bacon.

Kennedy

Kennedy, L. M. (1980). Guiding children to mathematical discovery. Belmont, CA: Wadsworth.

LeBlanc

LeBlanc, J. In-house handout. Indiana University, Bloomington, IN.

PS Using Calc

Problem solving using the calculator. (1984). Project Impact, Price Laboratory School, University of Northern Iowa, Cedar Falls, Iowa.

Problem: Statement of the problem

Answer: Answer (s)

Type:
CC = complex computation (not word) problem
MS = multi-step word problem
P = process problem
SS = single-step word problem
Z = puzzle

Info:
IS = insufficient
IC = inconsistent
IR = irrelevant
S = sufficient

Diff: Difficulty level (for 6th grade)
E = easy
M = medium
H = hard
C = intended for use with calculator

Strat: (* indicates "best strategy")
G = guess & test
B = work backwards
P = look for a pattern
E = use equations
L = use logic
D = draw a picture
O = make an organized list
T = make a table
A = act it out
M = make a model
S = simplify
K = look for key words
R = use resources (books, calculator, teacher)

Meta: Which metacognitive phase is likely to be tapped
OR = orientation
OG = organization
E = execution
V1 = verification throughout the solution
V2 = verification at the end of the solution

OPER/#S: Operations involved (+, -, x, /)
Numbers involved: W = wholes only
D = decimals
F = fractions

Content: Numerous content areas are possible.
Among these are: ratio, money, measurement, logic, spatial, geometry, etc.

#Sols: Number of solutions
0 = none
1 = unique
+ = multiple

IsoTo: Problem is isomorphic to _____.
(Example: chickens & pigs, handshake, fox & goose)

Appendix F

PUBLICATIONS AND PAPER.

RELATED TO THE PROJECT

APPENDIX F

PUBLICATIONS AND PAPERS RELATED TO THE PROJECT

Publications

Garofalo, J. (1987). Metacognition and school mathematics. *Arithmetic Teacher, 34* (9), 22 - 23.

Lester, F. K (1988). Reflections about mathematical problem-solving research. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 115 - 124). Reston, VA: National Council of Teachers of Mathematics.

Lester, F. K., Garofalo, J., & Kroll, D. L. (1989). Self-confidence, interest, beliefs, and metacognition: Key influences on problem-solving behavior. In D. McLeod & V. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 75 - 88). New York: Springer-Verlag.

Goodwin, V., Mitchell, R., Mtetwa, D. & Garofalo, J. (in press). Some observations of seventh graders solving problems. *Arithmetic Teacher.*

Mtetwa D. & Garofalo, J. (in press). Beliefs about mathematics: An overlooked aspect students' difficulties. *Academy Therapy.*

Garofalo, J., Lester, F. K. & Kroll, D. L. (in preparation). Teaching mathematical problem solving: A discussion of a research project. In J. M. Pressley (Ed.), *Cognitive strategies instruction research* (tentative title). New York: Springer-Verlag.

Papers

Lester, F. K. (1986, April). *Difficulties in studying metacognition.* Paper presented at the research presession of the annual meeting of the National Council of Teachers of Mathematics, Washington, D. C.

Lester, F. K. & Garofalo, J. (1986, April). *An emerging study of children's metacognition during mathematical problem solving.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Lester, F. K. & Garofalo, J. (1987, April). *The influence of affects, beliefs and metacognition on problem-solving behavior.* Paper at the annual meeting of the American Educational Research Association, Washington, D. C.

Garofalo, J., Kroll, D. L. & Lester, F. K. (1987, July). *Metacognition and mathematical problem solving: Preliminary research findings.* Paper presented at the 11th annual meeting of the International Group for the Psychology of Mathematics Education, Montreal, Quebec.

Lester, F. K. (1987, July). *Why is problem such a problem?* Paper presented at the 11th annual meeting of the International Group for the Psychology of Mathematics E cation, Montreal, Quebec.

Garofalo, J. (1988, February). *Metacognition and mathematical problem solving.* Paper presented at the annual meeting of the Research Council for Diagnostic and Prescriptive Mathematics, New Orleans.

Garofalo, J. (organizer/presenter), D. L. Kroll (presenter), F. K. Lester (presider) (1988, April). *Methodological issues in problem-solving research.* Research workshop at the research presession of the annual meeting of the National Council of Teachers of Mathematics, Chicago.

Garofalo, J. (1989, February). *Some problem-solving strategies of seventh graders.* Paper presented at the annual meeting of the Research Council on Diagnostic and Prescriptive Mathematics, College Station, Texas.

Garofalo, J. (1989, April). *Effects of instruction on seventh graders' mathematical problem-solving performance.* Paper presented at the annual meeting of the National Council of Teachers of Mathematics, Orlando, Florida.