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ABSTRACT

Latent class models for mastery testing differ from continuum models in that they do not postulate a latent mastery continuum but conceive mastery and non-mastery as two latent classes, each characterized by different probabilities of success. Several researchers use a simple latent class model that is basically a simultaneous application of the binomial error model to both mastery classes. W. A. Reulecke (1977) presents a version of this model that assumes that non-masters guess blindly, with a probability of success equal to the reciprocal of the number of alternatives. Assuming a loss ratio, these models enable the derivation of an optimal cutting score for separating masters from non-masters. To compute this cutting score, the model parameters must be estimated. J. A. Emrick and F. N. Adams (1969) suggest a method that is based on the average inter-item correlation but which, due to its assumptions, is only of restricted applicability. The sample applies to the maximum likelihood method in as much as this involves estimation equations that can be solved iteratively. In this paper, the method of moments is used to obtain "quick and easy" estimates. An endpoint that assumes that the parameters can simply be estimated from the tails of the sample distribution is discussed. A Monte Carlo experiment demonstrates t'at the method of moments yields excellent estimators and beats the endpoint method uniformly. Five data tables are included. (Author/TJH)

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SIMPLE ESTIMATORS FOR THE SIMPLE LATENT CLASS MASTERY TESTING MODEL

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Latent Class Mastery Testing Model

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Abstract

Latent class models for mastery testing differ from continuum models in that they do not postulate a latent mastery continuum but conceive mastery and nonmastery as two latent classes each characterized by different probabilities of success for the test items. Several authors, including Emrick and Adams (1969), Besel (1973), Davis, Hickman, and Novick (1973), and Macready and Dayton (1977), give a simple latent class model which is basically a simultaneous application of the binomial error model to both mastery classes. Reulecke (1977) presents a version of this model in which it is assumed that the nonmasters guess blindly, with a probability of success equal to the reciprocal of the number of alternatives . Assuming a loss ratio, these models enable us to derive an optimal cutting score for separating masters from nonmasters. In order to compute this cutting score the model parameters must be estimated. Emrick and Adams suggest a method which is based on the average inter-item correlation but which, because of its assumptions, is only of restricted applicability. The sample applies to the maximum likelihood method inasmuch this involves estimation equations which can only be solved iteratively. In this paper it is shown how the method of moments can be used to obtain "quick and easy" estimates. In addition, an endpoint is discussed which assumes that the parameters can simply be estimated from the tails of the sample distribution. A Monte Carlo experiment demonstrated that the method of moments yields excellent estimators and beats the endpoint method uniformly.

Key-words: Mastery Testing, State Models, Random Guessing, Latent Class Analysis

Simple Estimators for the Simple Latent Class Mastery Testing Model

All existing models in mastery testing can be classified as either continuum or state models (Meskauskas, 1976). Models of the former type postulate a latent or true score continuum underlying the observed test score and assume that a point can be denoted dividing the continuum into a mastery and a nonmastery region. State models differ from continuum models in that they do not postulate a latent continuum but conceive mastery and nonmastery as two latent classes each characterized by different probabilities of a correct response to the test items. Ideally, these probabilities would be equal to one for a master and to zero for a nonmaster, but the influence of extraneous factors can be expected to introduce a bias making these probabilities differ from their ideal values.

Besel (1973) and Macready and Dayton (1977) have given a latent class model for mastery testing which can be regarded as the most general model known so far. For each of the two latent classes, it has a different parameter for each of the items representing the probability of a successful reply to the item. Denoting the probability of a successful reply to item i for a nonmaster and a master by α_i and β_i, respectively, this model describes the probability of the jth response vector on an n-item test as

$$(1) \qquad P(j) = (1-\mu)\left[\prod_{i=1}^{n} \alpha_i^{\,a_{ij}}(1-\alpha_i)^{1-a_{ij}}\right] + \mu\left[\prod_{i=1}^{n} \beta_i^{\,a_{ij}}(1-\beta_i)^{i-a_{ij}}\right],$$

where μ represents the probability of a master, and $a_{ij} = 1$ if the response to the ith item for the jth vector is right and $a_{ij} = 0$ otherwise. The observed score distribution generated by (1) is a mixture of two compound binomial distributions (Walsh, 1953, 1959, 1963; Lord & Novick, 1968, sect. 23.10) with μ as mixing parameter.

The simple latent class model for mastery testing follows from (1) on assuming $\alpha_i = \alpha$ and $\beta_i = \beta$ for all items. Written not in response vector form but as a model for the observed score distribution, it is equal to

$$(2) \qquad p(x) = \binom{n}{x}\left[(1-\mu)\alpha^x(1-\alpha)^{n-x} + \mu\beta^x(1-\beta)^{n-x}\right].$$

From this expression it is clear that this model involves an observed score distribution which is a mixture of two simple binomial distribution with success parameters α and β and mixing parameter μ. The simple latent class model was introduced in the area of mastery testing by Emrick and Adams (1969), and also given by Davis, Hickman, and Novick (1973) and Macready and Dayton (1977).

Dayton and Macready (1976) consider a further simplification of (2) which is obtained by assuming $\alpha = 1 - \beta$. This amounts to the notion that the deviations of α and β from their ideal values of one and zero are equal to each other. They also show the possibility of incorporating hierarchical relations into the aforementioned models and using them for validating behavioral hierarchies.

An important property of the simple latent class model is that it allows the derivation of a simple rule for separating masters from nonmasters. This rule, which in statistical terminology is a monotone, nonrandomized Bayes rule, has the form of a cutting score on the test, c say, so that students with

$X \geq c$ are declared to be a master and those with $X < c$ a nonmaster. This cutting score is equal to

$$(3) \qquad c^* = \frac{\left[\ln \frac{1-\beta}{1-\alpha} + \frac{1}{n}\ln \frac{\lambda\mu}{1-\mu}\right]n}{\ln \frac{\alpha(1-\beta)}{(1-\alpha)\beta}}$$

where λ is the loss ratio ℓ_1/ℓ_0, ℓ_0 being the loss associated with misclassifying a nonmaster and ℓ_1 with misclassifying a master (Emrick & Adams, 1969; for a derivation, see also van der Linden, 1980a).

Estimating the Model Parameters

In order to be able to use the above cutting score, the model parameters α, β, and μ must be estimated from a sample of test scores. In this paper we will review the use of some of the available estimation methods for the simple latent class model and report results from a Monte Carlo investigation into the statistical properties of two classes of estimators -- one known as moment estimators and another as endpoint estimators. But before proceeding, we observe that the optimal cutting score given in (3) can also be used when not a latent class model but a continuum model with indifference zone is adopted. In that case no parameter estimation is involved. This possibility has been worked out elsewhere (van der Linden, 1980b), and will not be considered here.

Emrick and Adams (1969) and Emrick (1971) show how estimators for α and β can be obtained via the square root of the interitem correlation. Their method does not yield an estimator for μ but, on the contrary, assumes that μ is a priori known and has a value equal to .50. It also assumes an a priori known ratio of the two success parameters, α and β. Since there will hardly be situations for which these two assumptions hold, Emrick and Adams' method is of restricted applicability and will be disregarded further in this paper.

From a statistical point of view, maximum likelihood estimation is an attractive method. The use of maximum likelihood estimation for the parameters of the simple latent class model for mastery testing has become available through the computer programs by Dayton and Macready (1978; see also, Macready & Dayton, 1979). Maximum likelihood estimates for the parameters of this model are by no means simple, however. The estimation equations are intractable and can only be solved using iterative procedures as implemented in Dayton and Macready's programs. Since in this paper the emphasis is on simple estimators, to be computed by hand or with the aid of a pocket calculator, the possibility of maximum likelihood estimation will, in spite of its favorable properties, be disregarded further in this paper as well.

As noted earlier, the model given in (2) is a mixture of two simple binomial distributions. Mixtures of binomials have been extensively studied in the statistical literature (e.g., Blischke, 1962, 1963, 1964; Pearson, 1915; Rider, 1961), and on several independent occasions the method of moments has been used to derive estimators for mixtures of two binomials with results applying to the mastery testing model dealt with in this paper.

Generally, the method of moments expresses the parameter to be estimated as explicit functions of population moments and next substitutes the corresponding sample moments to obtain estimators for these parameters. The method of moments ordinarily yields simple, closed-form estimators, which are consistent under mild conditions (Rao, 1972, p. 351).

As can be verified from, for example, Blischke (1962, 1964), the method of moments yields the following estimators for mixtures of two binomials

$$(4) \qquad \hat{\alpha} = \frac{1}{2}A - \frac{1}{2}(A^2 - 4AF_1 + 4F_2)^{1/2};$$

$$(5) \qquad \hat{\beta} = \frac{1}{2}A + \frac{1}{2}(A^2 - 4AF + 4F_2)^{1/2};$$

$$(6) \qquad \hat{\mu} = \frac{F_1 - \hat{\alpha}}{\hat{\beta} - \hat{\alpha}},$$

with

$$(7) \qquad F_k = \frac{1}{m} \sum_{x=k}^{n} \frac{x(x-1)\ldots(x-k+1)}{n(n-1)\ldots(n-k+1)} O(x);$$

$$(8) \qquad A = \frac{F_3 - F_1F_2}{F_2 - F_1^2},$$

and where m is the sample size, $O(x)$ the observed frequency of $X = x$ and F_k is an expression which is, up to the factor $(n - 1)\ldots(n - k + 1)$ in the denominator, equal to the kth factorial moment (see also, Johnson & Kotz, 1969, sect. 3.11). For data sets normally encountered in mastery testing (7) can be computed easily by hand for $k = 1, 2, 3$. Once A has been calculated from (8), (4) - (6) give the desired estimates for α, β, and μ.

In some applications of the model in this paper, it may be meaningful to take a "mastery or random guessing" point of view and to assume that the students either master the items or not master them and guess blindly. In that case the success parameter α can be treated as a known parameter and set equal to reciprocal of the number of item alternatives. This point of view has been taken by Reulecke (1977a, 1977b) and was alluded to in an example by Emrick (1971). If α may be supposed to be a known parameter, the method of moments can still be used but now leads to a different set of estimators since only the first two sample moments are needed for estimating the remaining success parameter, β, and the mixing parameter μ.

As can be verified from van der Linden (1980c), the method of moments yields the following estimators for mixtures of two binomials with one known success parameter

$$(9) \qquad \overset{\circ}{\beta} = \frac{(F_2 - \alpha^2)}{F_1 - \alpha} - \alpha;$$

$$(10) \qquad \overset{\circ}{\mu} = \frac{F_1 - \alpha}{\overset{\circ}{\beta} - \alpha},$$

where F_1 and F_2 are again obtained from (7) by substituting $k = 1, 2$.

Reulecke (1977a) has proposed a method of estimation in which α is treated as a known parameter and β and μ are estimated from the frequencies in the right-hand tail of the sample distribution. This method can also be used when α is to be considered an unknown parameter, however. Then α is estimated similarly from the left-hand tail of the sample distribution (van der Linden, 1980b).

More in particular, this "endpoint" method of estimation assumes that the tails of the sample distribution are virtually unmixed and that the parameters can be estimated from the ratio of the observed frequencies in the two outmost categories in each tail as

$$(11) \qquad \tilde{\alpha} = \frac{0(1)}{n0(0) + 0(1)} ;$$

$$(12) \qquad \tilde{\beta} = \frac{n0(n)}{0(n-1) + n0(n)} ;$$

$$(13) \qquad \tilde{\mu} = 0(n) \left[\frac{0(n-1) + n0(n)}{n0(n)} \right]^{n} m^{-1},$$

where m still denotes the sample size, n the test lenght, and 0(.) the observed frequences.

The estimator for μ in (13) is based on the observed frequences of $X = n - 1$ and $X = n$. It is also possible to derive an estimator for μ from the frequences of $X = 0$ and $X = 1$. However, the problem this creates can be circumvented by using neither of the two and instead substituting $\tilde{\alpha}$ and $\tilde{\beta}$ in the first moment equation. This results in

$$(14) \qquad \dot{\mu} = \frac{F_1 - \tilde{\alpha}}{\tilde{\beta} - \tilde{\alpha}}$$

(van der Linden, 1980a).

We finally observe that each of the above sets of estimators can be used to obtain an estimate of the cutting score c^* given in (3).

Results from Monte Carlo Experiments

Using random procedures from the NAG Fortran Library (1977), Monte Carlo experiments were run in which we varied the two success parameters, the mixing parameter, test length, and sample size and determined the consequences for the expected error of estimation and the risk function using squared error loss. This was done for both sets of moment estimators, the endpoint estimators and the corresponding estimators for the optimal cutting score c^*.

The results can be found in Tables 1 - 5 at the end of this paper. In these tables ε is a generic symbol for an error of estimation and $E\varepsilon$ and $E\varepsilon^2$ denote the estimated expected error of estimation and the estimated risk function, respectively. As indicated by their subscripts, the results for the estimators of the optimal cutting score c^* are reported for loss ratio values equal to .25, 1, and 4. Each figure in Tables 1 - 5 is based on 1,000 replications.

From the results presented in Tables 1 - 5 the following conclusions can be drawn:

(1) On the whole, the results for the moment estimators with α unknown (equations 4 - 6) are excellent and suggest estimators which can safely be used in most practical situations. The results are better the larger the difference between α and β, the closer μ to .50, the longer the test, and

the larger the sample size. To see a result typical of what was encountered in our experiments, for example, the center column of Table 3 can be used. The values of $E\varepsilon$ and $E\varepsilon^2$ for $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\mu}$ differ only in their third decimal from zero, while the values for $\hat{c}^*$, although somewhat larger, are almost equally good. The only exception to this favorable conclusion has to be made for the results for $\hat{\mu}$ and $\hat{c}^*$ for $(\alpha, \beta) = (.40, .60)$ in Table 1. This set of parameter values represents a situation in which our model comes close to a single binomial, and in that case it is clearly difficult to estimate the mixing parameter and the optimal cutting score reliably.

(2) The results for the moment estimators with α known (equations 9 - 10) are even better than those with α unknown. Especially the results for $\overset{\circ}{c}{}^*$ show a considerable improvement in their $E\varepsilon$ and $E\varepsilon^2$ values compared with the results for $\hat{c}^*$. The same trends as were observed for the case with α unknown can be seen in Tables 1-4: The values of $E\varepsilon$ and $E\varepsilon^2$ tend to zero when the difference between α and β, the test length, and the sample size increase and when μ goes to .50. Inspecting Table 1, it appears that the moment estimators with α known get less upset when α and β approach each other in value and the situation resembles that of a single binomial. Only the values of the risk function for $\overset{\circ}{c}{}^*$ are still too large, indicating that in this situation inaccurate estimates of c^* can be expected.

(3) The Monte Carlo results for the endpoint estimators show that these are completely unreliable. From the great variety of parameter sets we used, three are picked out and given in Table 5. Parameter set I ($\alpha = .10$, $\beta = .90$, $\mu = .70$, $n = 10$, $m = 100$) is the set with the best results we have seen, parameter set II ($\alpha = .25$, $\beta = .75$, $\mu = .70$, $n = 10$, $m = 25$) was one of the worst sets, while results typical of what we normally

encountered were obtained for parameter set III (α = .25, β = .75, μ = .50, n = 10, m = 100). Even for parameter set I the results are still too poor for practical purposes.

(4) From Table 5, it can be concluded that the use of the endpoint/moment estimator for μ defined in (14) is in all cases a considerable improvement on the use of $\tilde{\mu}$. As can be seen in the last three rows of this table, substituting $\dot{\mu}$ instead of $\tilde{\mu}$ into the optimal cutting score results in a remarkable gain in efficiency but not enough to yield an estimator with errors of estimation that can be tolerated when used in practice.

(5) During the Monte Carlo experiments it was counted how often inadmissible estimates, i.e., estimates lying outside the interval the parameter is defined on, were met. For the moment estimators with α unknown only one percent of the replications resulted in such estimates, while such estimates were not at all encountered for the moment estimators with α known. The situation was completely different for the endpoint estimators $\tilde{\mu}$ and $\dot{\mu}$, however. Percentages of inadmissible estimates exceeding 20 or 30 per cent were no exception, and in one case no less than 47 per cent of the replications yielded values for $\tilde{\mu}$ larger than one.

(6) A different problem was met for the endpoint estimators $\tilde{\alpha}$ and $\tilde{\beta}$. As is clear from (11) and (12), these expressions are indeterminate whenever the tails of the sample distribution are empty. This happened very often and especially for α and β values close to each other, μ values far away from .50, small samples, and long tests. The largest percentage of indeterminate estimators found in one experiment was equal to 48 per cent.

Conclusion from the Results

The foregoing suggests that the moment estimators can safely be used in most practical situations. The only exception which has to be made is for the case parameter values can be expected for which the model comes close to a single binomial. In that case less accurate estimates of the mixing parameter and the optimal cutting score can be expected. When α can be set equal to the reciprocal of the number of response alternatives, even better results can be expected than when α is unknown, especially when the model comes close to a single binomial. It must be taken into account, however, that this only applies to the extent the model of blind guessing holds for the nonmasters. If this is not the case, then the value of α is specified incorrectly and the moment estimators given in (9) - (10) may, in fact, be worse than those for α unknown.

It is not recommended to use the endpoint estimators in any situation. Not only were they beaten by the moment estimators for all parameter sets we used, they are also likely to lead to situations in which the estimates are indeterminate or, if not so, take inadmissible values. The endpoint/moment estimator for μ given in (14) offers an improvement, but is still too poor for applications in practice.

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TABLE 1

Results for Moment Estimators with Varying Success Parameters and μ = .70, n = 10, and m = 100

(α, β)	(.10, .90)		(.25, .75)		(.40, .60)	
	$E\varepsilon$	$E\varepsilon^2$	$E\varepsilon$	$E\varepsilon^2$	$E\varepsilon$	$E\varepsilon^2$
$\hat{\alpha}$	.000	.000	-.002	.002	-.012	.016
$\hat{\beta}$	.000	.000	-.001	.001	.021	.007
$\hat{\mu}$	.001	.002	.001	.004	-.149	.133
$\mathring{\beta}$	.000	.000	-.001	.000	-.002	.001
$\mathring{\mu}$	.002	.002	.001	.003	.024	.020
$\hat{c}^*_{.25}$	-.013	.067	.029	.182	.104	10.468
$\hat{c}^*_1$	-.013	.064	-.029	.161	.433	10.734
$\hat{c}^*_4$	-.012	.061	-.028	.145	.760	11.843
$\mathring{c}^*_{.25}$	-.003	.024	-.013	.039	-.044	2.010
$\mathring{c}^*_1$	-.003	.027	-.015	.051	-.510	2.925
$\mathring{c}^*_4$	-.003	.031	-.017	.064	-.584	4.102

^ moment estimator with α unknown

° moment estimator with α known

TABLE 2

Results for Moment Estimators with Varying Mixing
Parameter and α = .25, β = .75, n = 10, and m = 100

μ	.50		.70		.90	
	$E\varepsilon$	$E\varepsilon^2$	$E\varepsilon$	$E\varepsilon^2$	$E\varepsilon$	$E\varepsilon^2$
$\hat{\alpha}$	.001	.001	-.002	.002	.009	.011
$\hat{\beta}$	-.001	.001	-.001	.001	.002	.000
$\hat{\mu}$	.002	.004	.001	.004	-.020	.012
$\mathring{\beta}$	.000	.001	-.001	.000	.000	.000
$\mathring{\mu}$	.000	.004	.001	.003	.000	.001
$\hat{c}^*_{.25}$	.003	.115	-.029	.182	.006	.570
$\hat{c}^*_1$	.000	.113	-.029	.161	-.004	.411
$\hat{c}^*_4$	-.004	.116	-.028	.145	-.015	.291
$\mathring{c}^*_{.25}$	.002	.048	-.013	.039	-.004	.079
$\mathring{c}^*_1$	.001	.067	-.015	.051	-.037	.088
$\mathring{c}^*_4$	.000	.090	-.017	.064	-.038	.099

^ moment estimator with α unknown

° moment estimator with α known

TABLE 3

Results for Moment Estimators with Varying Test

Length and α = .25, β = .75, μ = .70, and m = 100

n	5		10		20	
	$E\varepsilon$	$E\varepsilon^2$	$E\varepsilon$	$E\varepsilon^2$	$E\varepsilon$	$E\varepsilon^2$
$\hat{\alpha}$	.001	.007	-.002	.002	.001	.000
$\hat{\beta}$	.003	.002	-.001	.001	.000	.000
$\hat{\mu}$	-.013	.012	.001	.004	.002	.002
$\mathring{\beta}$	.001	.001	-.001	.000	.000	.000
$\mathring{\mu}$	.000	.004	.001	.003	-.001	003
$\hat{c}^*_{.25}$	.004	.301	-.029	.182	.009	.156
$\hat{c}^*_1$	.015	.246	-.029	.161	.007	.144
$\hat{c}^*_4$	.026	.209	-.028	.145	.005	.134
$\mathring{c}^*_{.25}$	-.008	.039	-.013	.039	.004	.050
$\mathring{c}^*_1$	-.009	.056	-.015	.051	.005	.059
$\mathring{c}^*_4$	-.010	.078	-.017	.064	.005	.069

^ moment estimator with α unknown

° moment estimator with α known

TABLE 4

Results for Moment Estimators with Varying Sample Size and α = .25, β = .75, μ = .70, and n = 10

m	25		50		500	
	$E\varepsilon$	$E\varepsilon^2$	$E\varepsilon$	$E\varepsilon^2$	$E\varepsilon$	$E\varepsilon^2$
$\hat{\alpha}$	.007	.010	-.002	.003	.000	.000
$\hat{\beta}$	.001	.002	.000	.001	.000	.000
$\hat{\mu}$	.021	.021	-.002	.007	-.001	.001
$\mathring{\beta}$	.002	.002	-.001	.001	.000	.000
$\mathring{\mu}$	.005	.012	.002	.007	.000	.001
$\hat{c}^*_{.25}$	-.002	.859	-.031	.353	-.002	.032
$\hat{c}^*_1$	-.002	.738	-.029	.303	-.002	.028
$\hat{c}^*_4$	-.004	.650	-.028	.266	-.002	.024
$\mathring{c}^*_{.25}$	-.031	.192	-.023	.087	.000	.007
$\mathring{c}^*_1$	-.031	.245	-.026	.112	-.001	.010
$\mathring{c}^*_1$	-.032	.308	-.029	.142	-.001	.012

^ moment estimator with α unknown

° moment estimator with α known

TABLE 5

Some Results for the Endpoint Estimators

Parameter Set	I		II		III	
	$E\varepsilon$	$E\varepsilon^2$	$E\varepsilon$	$E\varepsilon^2$	$E\varepsilon$	$E\varepsilon^2$
$\tilde{\alpha}$	.009	.002	.386	.346	.072	.053
$\tilde{\beta}$	-.004	.001	-.240	.219	-.062	.041
$\tilde{\mu}$	.025	.015	-.180	.140	.102	.124
$\dot{\mu}$	.001	.003	-.039	.119	.015	.085
$c^*_{.25}$	-.159	.842	-.920	6.512	-1.507	12.257
c^*_1	-.166	.853	-.617	5.614	-1.304	9.923
c^*_4	-.172	.868	-.287	4.726	-1.096	7.993
$\dot{c}^*_{.25}$	.017	.270	-.395	1.828	-.032	6.585
$\dot{c}^*_1$	.011	.254	-.168	1.633	-.009	6.352
$\dot{c}^*_4$	.004	.241	.058	1.582	.026	6.401

~ endpoint estimator

• endpoint/moment estimator

I $\alpha = .10,\ \beta = .90,\ \mu = .70,\ n = 10,\ m = 100$

II $\alpha = .25,\ \beta = .75,\ \mu = .70,\ n = 10,\ m = 100$

III $\alpha = .25,\ \beta = .75,\ \mu = .50,\ n = 10,\ m = 100$

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