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ABSTRACT

This yearbook discusses instructional approaches that are consistent with the reformulation of the school mathematics curriculum by the National Council of Teachers of Mathematics (NCTM). Articles included cover: (1) Pennsylvania standards for mathematics programs (including goals, curriculum, instruction, evaluation, teachers, and administration); (2) metacognitive skills; (3) cooperative learning in calculus; (4) teaching the language of mathematics; (5) problem solving with number operations; (6) teaching geometry; (7) problem solving strategies; (8) learning from real life; (9) using realistic applications; (10) applied geometry; (11) mathematical modeling; (12) use of estimation; (13) geometrical probability; (14) computer based approaches in calculus; and (15) changes in teacher education. (YP)

NEW DIRECTIONS FOR MATHEMATICS INSTRUCTION

1989 Yearbook

PENNSYLVANIA COUNCIL OF TEACHERS OF MATHEMATICS

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PREFACE

The 1989 PCTM Yearbook — NEW DIRECTIONS FOR MATHEMATICS INSTRUCTION — is the fifth yearbook to be developed and distributed to members of the Pennsylvania Council of Teachers of Mathematics. We chose this theme because of recent calls for reformulation of the school mathematics curriculum, one of which has been elaborated in the recently-published *Curriculum and Evaluation Standards for School Mathematics* from the National Council of Teachers of Mathematics. This publication and others cited in articles in this *Yearbook* have called for new mathematics content in grades K-12 as well as new approaches to teaching that content.

The articles that follow provide suggestions for changes in emphasis in the K-12 mathematics curriculum and discuss instructional approaches that are consistent with those changes. *Standards for K-12 Mathematics Programs in Pennsylvania*, a cooperative effort by PCTM with the Pennsylvania Association for Supervision and Curriculum Development and the Pennsylvania Council of Supervisors of Mathematics, provides a guide for school districts to use in determining whether their mathematics programs are consistent with recent suggestions for curricular reform.

Several articles in this *Yearbook* address mathematics as communication, one of the standards that appears at each of the K-4, 5-8, and 9-12 grade levels in the NCTM *Standards*. The article by Zbiek suggests that self-communication can be an important part of mathematics learning, since it can enhance students' metacognitive capabilities. New approaches using small group cooperative learning to enhance student-student and teacher-student communication is addressed in the article by Saunders. The Heinrichs and Larrabee article provides techniques for helping students to deal with the interaction between the language of mathematics and English.

Problem solving is another of the K-12 NCTM standards. The article by Cohen illustrates how teachers can address problem solving in the elementary classroom by using activities that grow out of work with the four basic operations. The article by Camerlengo introduces a dynamic approach to problem solving in secondary-level geometry. Larson's article examines two general approaches, top-down and bottom-up, that teachers might use when teaching problem solving.

Five articles address the uses of applications of mathematics and the development of mathematical models for realistic situations. Dobransky, Kerrigan, Kerrigan, and Stopper explain why elementary teachers might include a focus on realistic problems in their curriculum. Matras describes a variety of uses for applications problems at the secondary level, and the article by Cacka presents an entire Geometry course that focuses on applications of mathematics. The articles by Blume and Swetz provide examples of situations one might use to engage students in the process of mathematical modeling.

Several of the articles address particular content areas or suggest specific techniques that are appropriate for new curricular emphases. The Jorgensen article argues that estimation should be an integral part of mathematics instruction at all levels. The article by Walton introduces the reader to a new approach that incorporates the use of geometric models for probability problems. The articles by Heid and Leinbach illustrate the impact of new graphing and symbolic-manipulation tools in calculus. These articles suggest that standard approaches to calculus are no longer appropriate for preparing students to be users of mathematics in the 21st century.

Changes in curriculum and instructional methods are not possible without teachers who are knowledgeable about and committed to such changes. The article by Trueblood suggests ways that teacher education programs can prepare teachers to be fluent with the content of a revised and reformulated mathematics curriculum and to acquire the background in teaching methods necessary to provide the instruction for such a curriculum.

The school mathematics curriculum recently has received unprecedented attention and will face numerous challenges in the years ahead. It is hoped that the suggestions offered in the aforementioned articles will stimulate teachers and researchers to continue to address the question of how mathematics instruction can help students to create, apply, reason with, and communicate about mathematics

Many people contributed in important ways to the 1989 Yearbook. The authors of the manuscripts deserve thanks for sharing their ideas with their peers. Eighteen referees selected the articles from among the many quality articles that were submitted. Their insightful suggestions also contributed greatly to the editorial process. The commercial and institutional advertisers deserve thanks for their investment in the yearbook. The PCTM Executive Board's support for the efforts of the Publications Committee continues to be appreciated. Bob Nicely, the current President of PCTM and Co-Editor of the previous PCTM Yearbooks, provided valuable assistance in all stages of preparation of the Yearbook. Linda Haffly and Suzanne Harpster of Penn State provided excellent support to the co-editors. We are grateful for being given the opportunity to work with so many helpful people. We hope the Yearbook proves to be useful and invite response from readers, authors, and advertisers.

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6

TABLE OF CONTENTS

8

9

STANDARDS FOR K-12 SCHOOL MATHEMATICS PROGRAMS IN PENNSYLVANIA: A GUIDE FOR A MATHEMATICS PROGRAM REVIEW

Pennsylvania Association for Supervision and Curriculum Development
in cooperation with the
Pennsylvania Council of Teachers of Mathematics
and the
Pennsylvania Council of Supervisors of Mathematics

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Introduction[1]

Now, perhaps more than at any time in this century, the K-12 mathematics curriculum is undergoing major changes. The National Council of Teachers of Mathematics (NCTM) has issued "Curriculum and Evaluation Standards for School Mathematics" and position statements on leadership, staff development, and the effective use of technology. The Mathematical Sciences Education Board (MSEB) has been developing a philosophy and framework for the K-12 mathematics curriculum, the Mathematical Association of America (MAA) has issued "Guidelines for the Continuing Mathematical Education of Teachers" and NCTM has issued "Guidelines for the Post-Baccalaureate Education of Teachers of Mathematics."

In an effort to help Pennsylvania school districts capitalize on the current opportunity for curriculum redesign and enhancement, the Pennsylvania Association for Supervision and Curriculum Development, in cooperation with the Pennsylvania Council of Teachers of Mathematics and the Pennsylvania Council of Supervisors of Mathematics, has developed a guide for conducting a mathematics program review. The authors of the guide — the aforementioned mathematics program review committee — carefully analyzed the NCTM and MAA publications, plus others listed in the references, and then identified and listed critical elements in the areas of goals, curriculum, instruction, evaluation, and teacher and administrator responsibility. The committee then designed the program review guide to enable school district personnel to analyze their mathematics programs with respect to these critical elements, both in terms of their current level of implementation and their levels of importance — both nationally and locally.

This guide lists elements that have been identified as important by national standards. It is intended to (1) stimulate critical analysis of content, methodology, assessment and management issues related to the K-12 mathematics program, (2) identify *some* of the major desired directions for the K-12 mathematics curriculum, and (3) help districts identify discrepancies between "what is" and "what could be."

Because of the many differences (staff needs, resources, goals, organizational patterns, etc.) among school districts, each district must determine the most meaningful application for the guide. School district personnel should develop an application plan that enables them to use the guide in a way that recognizes these idiosyncratic differences.

It is inappropriate to simply distribute the guide to teachers, have them "fill it out," and then tally the results! This document is only one piece of a comprehensive mathematics program review. It is a *guide* which will help districts gather data which teachers and administrators can discuss. They can then work toward consensus about future directions for the mathematics program. *The process may be more important than the product!*

The sample completed guide in this section contains six items that were selected from the program review guide (one item from each of the six sections) so that district personnel can see how they might complete the forms. When answering the question "How important is this to you?", respondents should encircle N for "not important," M for "moderately important," and V for "very important." When answering the question "To what extent does this happen in your setting?", respondents should encircle No for "not implemented," Lo for "low level of implementation or occurrence, " Mod for "moderate level of implementation or occurrence," Hi for "high level of implementation or occurrence," and NA for "not applicable."

SAMPLE

The following six items were selected from the program review guide (one item from each of the six sections) so that district personnel can see how they might complete the forms

	How important is this to you?	To what extent does this happen in your setting?	Comments
GOALS			
An exemplary mathematics program should			
1. systematically develop mathematical concepts and skills including measurement, estimation, computation, and geometry.	N M V	No Lo Mod Hi NA	
CURRICULUM, K-4			
5. Estimation is used to determine reasonableness of answers.	N M V	No Lo Mod Hi NA	
INSTRUCTION			
6. Teaching strategies that foster the development of higher order thinking, reasoning, problem solving, and communicating mathematical ideas are used in all courses.	N M V	No Lo Mod Hi NA	
EVALUATION			
3. Teachers match teaching strategies to the needs of students and the nature of the objective	N M V	No Lo Mod Hi NA	
TEACHERS, K-12			
Teachers of mathematics are actively involved in the			
1. design and development of their staff development activities	N M V	No Lo Mod Hi NA	

The authors of the guide suggest that districts:

1. form committees at grade levels K-4, 5-8, and 9-12;
2. give each committee member a copy of the mathematics program review document at least one week before the committee meeting is scheduled;
3. allot at least one full day for the committee members to discuss and analyze their respective (K-4, 5-8, 9-12) portions of the K-12 mathematics program as well as the other (K-12) sections of this document; and
4. use the results of the analysis to develop and implement realistic long-range curriculum and staff development plans for the district.

One way a district might want to organize the day (mentioned in item 3 above) is as follows.

1. Form committees of 5-10 members at grades K-4, 5-8, and 9-12. (There are a number of ways to organize these committees, depending on district organization patterns and goals.)
2. Have each committee member read the mathematics program review guide prior to the team meeting.
3. Have the committee members, in the meeting, come to consensus on the "level of importance" for each item in the guide.

4. Have the committee members, in the meeting, analyze their program to determine the "level of implementation" that currently exists in the district.
5. Have the chairs of the committees compile a "consensus" document that reflects the K-12 program. Concurrent or follow-up assistance from outside consultant(s) might be needed as part of this analysis and planning process.

The following sections — Goals, Curriculum: K-4, Curriculum: 5-8, Curriculum: 9-12, Instruction, Evaluation, Teachers: K-12, and Administration: K-12 — contain a listing of the elements that have been identified as important by national standards. The primary author(s) of each of the sections are listed with the respective sections. Copies of the complete (i.e. with columns for recording perceptions and comments) mathematics program review guide can be obtained from the Pennsylvania Association for Supervision and Curriculum Development. (Contact Robert F. Nicely, Jr., 277 Chambers Building, The Pennsylvania State University, University Park, PA 16802; telephone 814/865-2525 for details on how to get copies or to discuss strategies for implementation and data analysis.)

[1]The primary author of this section was Robert F. Nicely, Jr.

GOALS[2]

An exemplary mathematics program should:

1. systematically develop mathematical concepts and skills including measurement, estimation, computation, and geometry;
2. be a sequential, integrated, and articulated K-12 curriculum;
3. help all students develop proficiency in problem solving and higher-order thinking;
4. encourage all students to develop their full potential in mathematics;
5. promote a belief in the utility and value of mathematics;
6. provide a variety of experiences for students which will enable them to understand the relationship of mathematics to their world;
7. use technology in all forms to enhance mathematics instruction; and
8. be taught by knowledgeable, proficient, and active professionals.

[2]The primary authors of this section were James E. Renney, James S. Gibson, Joanna C. Good, and Carl A. Guerriero.

CURRICULUM: K-4[3]

A. Problem Solving (Critical Thinking Skills)

Problem solving provides a framework for the learning of most mathematics concepts and skills. Many of the best problem solving situations grow nat-

urally from the students' environment and are posed by both the student and the teacher. A variety of strategies and techniques can be used in the solution process.

1. Instructional activities regularly include problem solving with mathematical applications that are meaningful to students.
2. Instructional activities integrate other subject areas.
3. A variety of strategies (e.g., patterns, guess and check, working backwards, diagramming, simulation, reduction, logical thinking) are used to develop higher-level thinking skills.
4. Instructional activities include real problems with manipulative/laboratory/outdoor experiences and technology.
5. Estimation is used to determine reasonableness of answers.

B. Communicating Mathematical Ideas

Mathematics is a language which — when used early, often and appropriately — enhances and expedites learning. Communication of mathematical ideas should be an integral part of every district's program.

1. Understandings and relationships between and among mathematical concepts, procedures and symbols are communicated through writing and speaking. This is done at each stage of conceptual development — concrete, pictorial, abstract — and in every area of mathematics.
2. Mathematics information is presented in a variety of ways such as speaking, drawing, graphing, writing, concrete demonstrations and/or with projects.
3. Mathematics information is received by listening, visualizing, and reading.

C. Computation (Whole Numbers)

Computational skills should enable children to develop a complete understanding of the four operations and to apply these to real-world problems. When applying computational skills, students should have practice in choosing the appropriate computational method, i.e., mental arithmetic, paper-pencil algorithm, or calculating device. Careful attention must be given to underlying concepts when using manipulatives (concrete) to develop thinking patterns.

1. Practical experiences provide an understanding of the relationship of mathematics to the real world.
2. Computational skills in the basic operations — addition, subtraction, multiplication, and division — with whole numbers are sequentially developed.
3. Problem situations use informal language. Manipulative materials and activities precede the formulation of representative number operations.

4. Mathematical symbols, (e.g., $=$, $+$, $-$, $<$,$>$) are related to the expression of problem situations, models, number sentences and operations.
5. Operational sense (e.g., choice of operations, fact families, and inverse operations) is developed through the analysis of arithmetic operations.
6. A variety of paper-and-pencil algorithms and mental arithmetic techniques are used.
7. Computational algorithms are developed with an understanding of underlying principles (the "whys").
8. Estimation is used to determine whether a computed answer is reasonable.
9. Calculators are used as a computational tool.
10. Computers are used as an instructional tool.

D. Measurement

Measurement enables children to recognize mathematics in their own lives. The process of measuring requires active involvement and encourages participation in problem-solving. The study of measurement extends a child's concept of number and offers opportunities to practice estimation and fractions. Many ways exist to integrate measurement across the curriculum in real-life situations. Current recommendations suggest that one avoid arithmetic conversion between systems of measurement.

1. Instruction includes concrete experiences in both customary and metric systems as well as systems of measurement developed by the student.
2. Estimation of quantities is used to determine appropriateness of answers and in some cases, provides the answer.
3. Measurable attributes include length, mass, area, volume, angle, temperature and time.
4. Problems of measurement are made meaningful by using real-life situations.
5. Calculators and computers are used to solve measurement problems.
6. Fraction and geometric concepts are related to measurement.

E. Numeration

It is essential that children understand numbers so that they can establish order and make sense of number use in their everyday lives. Teachers should provide a variety of activities with physical materials before emphasizing work with symbols. These activities should enable students to value mathematics and become confident in their ability to do mathematics.

1. Number concepts are developed sequentially from concrete to semi-concrete to abstract.

2. Manipulative materials and activities are used to construct number meanings through real-world experiences.
3. Number concepts are developed through counting, writing, ordering, and comparing numbers, and through grouping and place-value.
4. Number sense (i.e., recognizing number relationships, relative magnitudes, and the effects of operating on numbers) is developed.

F. Geometry/Spatial Concepts

Geometry relates directly to several areas of the curriculum, particularly language, science, number and measurement concepts. Children's understanding of spatial terms and their ability to function in the classroom setting are enhanced by instructional activities involving spatial relationships and patterns. Geometric understandings enable students to visualize, thereby increasing their comprehension. Some children's spatial capabilities exceed their numerical skills. Developing spatial capabilities fosters success and interest in mathematics.

1. Geometric shapes are explored and attributes identified through concrete manipulatives.
2. Geometric models are constructed in a variety of ways including manipulating geometric shapes on a computer screen.
3. Concepts of space (e.g., the Boehm concepts) are explored through concrete experiences as well as congruence, symmetry, parallel and perpendicular lines. (The Boehm concepts are language terms which relate directly to concepts of space. The Boehm concepts include center, above, below, second from the left, third from the end, beside, between, in front of, middle, pair, top, and bottom.)
4. Geometry is used to solve problems.
5. Geometry is related to concepts of number and measurement.

G. Organizing and Interpreting Data

Construction and interpretation of charts, tables, and graphs are important skills that can be developed in kindergarten and the primary grades. Real-life situations can be translated into these mathematical representations and used to help students think analytically, categorize, compare, make conclusions, and predict outcomes.

1. Simple tables, maps, charts, graphs, and diagrams are constructed, read, and used.
2. Everyday real-life situations are translated into mathematical representations.
3. Questions about everyday living are answered by collecting, organizing, and using data.
4. Representational models and/or verbal arguments are used to

justify thinking and solution processes.

5. Patterns, models, experiences, and observations are used to make and validate conjectures and/or predict outcomes, (e.g., school election results, weather forecasts, and sporting event outcomes).

H. Fractions (Decimals to tenths in Grade 4)

Fractions and decimals are an integral part of the real world. Initial concepts in this area provide the basis for advanced work in later grades. A student's knowledge of number systems is greatly enhanced when fraction and decimal concepts are integrated with whole numbers.

1. Concepts are developed in a sequential manner using physical materials, diagrams and symbols.
2. Concrete models and pictorial representations using geometric shapes, congruent regions, diagrams and sets are employed to explore operations.
3. Real-world situations are used to show the relevance of fractions and decimals to daily life.
4. Fractional representations include regions, sets, symbols, and the numberline.
5. Skills such as comparing and ordering, finding equivalent fractions, and relating fractions to decimals are stressed.
6. Concepts and operations are applied to other areas of the curriculum such as problem solving, geometry, measurement and analysis of data (statistics).

I. Technology

The use of computers and calculators should be integrated throughout the mathematics curriculum rather than taught as a special topic. Students should be able to use these technological tools for unwieldly computations, to assess reasonableness of an answer, for problem solving, and to assist them in the mastery of mathematical topics.

1. Computers and calculators are used for mathematical applications, problem solving, and concept development.
2. Mathematics software is carefully selected for quality and appropriateness and used to supplement the mathematics curriculum and instruction.
3. Technological tools are used for complex calculations when the tedium of manual manipulation outweighs the educational benefits, i.e., multi-digit (more than two digits) divisors, multi-digit multipliers.
4. Current research is used to update the integration of computers and calculators into the mathematics program.
5. The use of computers is incorporated into all levels of the K-4 mathematics program.

6. Estimation and mental arithmetic skills are used to determine whether results obtained from computers or calculators are reasonable.

[3]The primary authors of this section were Lucy M. Young, Lois Barson, Linda Renney, and Brenda K. Wise.

CURRICULUM: 5-8[4]

A. Problem Solving (Critical Thinking Skills)

Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should solve non-routine problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. Students should see alternate solutions to problems, they should experience problems with more than a single solution.

1. Activities that require original thinking are routinely encountered in the presentation of the material.
2. Checks for reasonableness and completeness are included as an integral part of the problem solving process.
3. Topics are often applied to real-world situations that differ from those presented in the textbook.
4. Multi-step solutions and non-routine problems are posed on a regular basis.
5. Situations are provided that require the determination of the problem; the collection and use of missing data, formulas, and procedures; and the definition of an acceptable solution.
6. Activities require the collection, organization, and manipulation of data and the drawing of inferences from that data.
7. Computer simulations are used to model and/or analyze complex situations.
8. Mathematical information routinely appears in various forms (e.g. tables, graphs, formulas and functions).
9. Group problem solving is encouraged which requires students to share responsibility for the product of the activity and to discuss the results.
10. Activities are structured so that several strategies or techniques are available for use in the solution process.
11. Activities are sequenced to guide the development from identification of concrete instances to formal investigations.
12. Interdisciplinary projects are encouraged.
13. Strategies such as top-down analysis and stepwise refinement are used to analyze and solve complex problems.

14. Activities are used which require the generalization of results to other situations and subject matter areas.

B. Communication

The 5-8 program should provide opportunities for students to develop and use the language and notation of mathematics. Vocabulary that is unique to mathematics and terms which have a common as well as a mathematical connotation should be used throughout the curriculum. Mathematical ideas should be expressed by writing, speaking, making models, drawing diagrams and preparing graphs. Opportunities should be provided for mathematical discussions.

1. Mathematical situations are represented or described in a variety of ways (e.g. verbal, concrete, pictorial, graphical, algebraic).
2. Understanding of mathematics is developed through reflection and by organization and communication of ideas.
3. Positions on mathematical processes and/or solutions are defended through sound argument.
4. The need for mathematical symbolism is demonstrated.
5. The ability to read mathematics is emphasized.
6. The ability to write mathematics problems from real-world situations is emphasized.
7. Proper mathematical vocabulary and notation is stressed.

C. Computation

The 5-8 program provides students with a variety of opportunities to compute with whole numbers, decimals, and fractions. Calculators or computers should be used for long, tedious computations. Students should be permitted to study additional mathematics topics prior to their mastery of computational algorithms.

Students should be able to carry out rapid approximate calculations through the use of mental arithmetic and a variety of computational estimation techniques. When computation is needed, an estimate can be used to check reasonableness, examine a conjecture, or make a decision. Students should also learn simple techniques for estimating measurements such as length, area, volume, and mass (weights). They should be able to decide when a particular result is precise enough for the purpose at hand.

1. Pencil and paper is used to add, subtract, multiply and divide fractions with "reasonable" denominators.
2. Pencil and paper is used to add, subtract, multiply and divide decimals having a "reasonable" number of places.
3. A four-function calculator is used to add, subtract, multiply and divide "unreasonable" fractions and decimals.
4. Computational algorithms are developed with an understanding of underlying principles (the "whys").

5. Estimation is used to check for reasonableness of computation.

D. Measurement

Students must see the power, usefulness, and creative aspects of mathematics so that they will not view it as a static, bounded set of rules and procedures to be memorized but quickly forgotten. When measurement is explored through rich, investigative, purposeful activity, it provides such opportunity. Students should learn the fundamental concepts of measurement through concrete experiences.

1. Basic units of measurement in the metric system and the relationship between those units of measurement — both within the dimension (e.g. length or volume) and between dimensions (e.g. length and volume) are included.
2. Basic units of measurement in the customary system and the relationship between these units within a dimension (e.g. feet in a yard or pints in a quart) are included.
3. Appropriate instruments are selected and used to measure a dimension accurately.
4. Scale drawings are made and interpreted.
5. Procedures and formulas to determine area and volume are developed and used.
6. Measurements in both the metric and the customary system are estimated.
7. Student-developed systems of measurement are encouraged.
8. Concepts of perimeter, area, and volume are developed intuitively by counting units, covering surfaces, and filling containers.
9. Real-world measurements are used to generate student-collected data.

E. Number/Number Systems

A critical part of the middle school mathematics curriculum is a student's ability to generate, read, use, and appreciate multiple representations for the same quantity. A student's understanding of numerical relationships as expressed in ratios and proportions, equations, tables, graphs, and diagrams is of crucial importance in mathematics.

Additionally, students need to understand the underlying structure of arithmetic. Emphasis must be placed on the reasons why various kinds of numbers (fractions, decimals, and integers) occur, on what is common among various arithmetic processes (how the basic operations are similar and different across sets of numbers — whole numbers vs. fractions vs. decimals, etc.); and on how one system relates to another (integers as an extension of whole numbers).

1. The sets of numbers are developed starting with the counting numbers and ending with the irrationals.
2. Numbers are understood to have several representations (frac-

tions, decimals, etc.) and processes are available to convert from one to another.

3. Numbers are written as numerals, in words and in expanded notation.
4. The relationship between a number (or set of numbers) and its (their) graph(s) is emphasized.
5. The use of ratio and proportion is extended to cases different from the problems normally in the textbooks.
6. The most appropriate form of a number is used in computation. The concepts of precision and accuracy are introduced and included in the determination of answers.
7. Numbers with terminating, repeating or nonrepeating decimal forms are presented and used properly.
8. Computation performed on elements of a set of numbers is patterned after and developed from the application of binary operations to the whole numbers.
9. Feasible, reasonable and impossible solutions are explored and discussed.
10. Number theory concepts such as prime numbers, GCF, LCM and divisibility are introduced and developed.
11. Mathematics is viewed as a systematic development of a body of knowledge from a few accepted propositions by applying logical and procedural rules.
12. The concepts of relation and function are introduced and explored.

F. Geometry

Students should have knowledge of concepts such as parallelism, perpendicularity, congruence, similarity and symmetry. They should know properties of simple plane and solid geometric figures and should be able to visualize and verbalize how objects move in the world around them using terms such as slides, flips and turns. Geometric concepts should be explored in settings that involve problem solving ana measurement.

1. The identification and description of geometric figures are emphasized.
2. Opportunities to visualize, represent and manipulate one-, two-, and three- dimensional figures are provided.
3. The relationship between geometric properties and other mathematical concepts are explored.
4. Geometric relationships and their consequences are developed through nonclassroom experiences.
5. Appreciation of geometry and its relationship to the physical world is developed.
6. Construction, drawing and measuring are used to further under-

standing of geometric properties.
7. Technology is used to explore geometric properties.

G. Probability and Statistics

Understanding probability and the related area of statistics is essential to being an informed citizen and is important in the study of many other disciplines. Students in grades 5-8 have a keen interest in trends in music, movies, and fashion and in the notions of fairness and the chances of winning games. These interests can be excellent motivators for the study of probability and statistics.

1. Data are systematically collected and organized.
2. Collections of data are represented and described by developing and using charts, graphs and tables.
3. The likelihood of bias in a collection of data is recognized.
4. Predictions are made by interpolation or extrapolation from events or a given collection of data.
5. Basic statistical notions (e.g., measures of central tendency, variability, correlation and error) are developed.
6. The concept of probability is developed and applied both in a laboratory (classroom) and in the real world.
7. Simulations and experiments are devised and conducted to determine empirical probabilities.
8. The role of probability is emphasized in situations of chance, insurance, weather, and other activities.
9. When calculating from real data, the level of accuracy and precision needed are emphasized.

H. Algebra

One of the most important roles of the middle grades mathematics curriculum is to provide a transition from arithmetic to algebra. It is critical that students in grades 5-8 explore algebraic concepts in an informal way in order to build a foundation for subsequent formal study of algebra.

1. Informal explorations with algebraic ideas (e.g., variable, expression, equation) that can be abstracted from physical models, data, tables, graphs and other mathematical representations are used.
2. Concrete experiences with situations that allow students to investigate patterns in number sequences, make predictions and formulate verbal rules to describe patterns are emphasized.
3. Manipulative skills are de-emphasized and students' ability to use algebraic concepts to express mathematical relationships are emphasized.

[4]The primary authors of this section were James E. Renney, James S. Gibson, Joanna C. Good, and Carl A. Guerriero.

CURRICULUM: 9-12[5]

A. Problem Solving (Critical Thinking Skills)

Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-routine problems. Problem solving involves posing questions, drawing diagrams, analyzing situations, using guess and test, and illustrating and interpreting results. Students should see alternate solutions to problems, and they should solve problems with more than a single solution. Problems and applications should be used to motivate the study of mathematical concepts.

1. Non-routine problems that require application of previous knowledge to unfamiliar situations are assigned and discussed in all courses.
2. Applications involve building and using mathematical models of realistic situations.
3. A variety of problem-solving strategies are developed to solve a broad range of problems.
4. Formulation of original problems is stimulated by classroom discussions.
5. Problems and applications are used to introduce mathematical topics, to develop understanding of them, and to review them.
6. Questioning the reasonableness of a solution to a problem is emphasized as much as the method for obtaining the solution.
7. Opportunities are provided to compare several strategies for solving a problem and to solve problems that have more than one solution.
8. Incorrect solutions are analyzed to identify common errors in the problem solving process.

B. Communication

The 9-12 program should provide opportunities for students to develop, learn and use the language and notation of mathematics. Vocabulary that is unique to mathematics and terms which have a common as well as a mathematical connotation should be developed and used throughout the curriculum. Mathematical ideas should be expressed by writing, speaking, making models, drawing diagrams and preparing graphs. Opportunities should be provided for discussion of mathematical topics.

1. Communication skills are developed in small groups through listening, exploring, questioning, discussing, and summarizing.
2. Mathematical concepts are described and processes justified by speaking, writing, drawing diagrams and graphs, and demonstrating with concrete models.

3. Vocabulary used in the classroom is consistent with the level of the course and addresses precise terminology, paraphrased descriptions, and "everyday" use of mathematical language.
4. Appropriate symbolism and notation is used by students and in all material presented to students.

C. Computation

The 9-12 program provides students with a variety of opportunities to gain facility in computing with whole numbers, decimals, and fractions, and in using the four basic operations. Students should appropriately choose calculators or computers to perform computations that warrant their use. Lack of mastery of paper-pencil computation should not prohibit students from studying additional mathematics topics.

1. Choosing appropriate computational methods (mental arithmetic, paper-pencil algorithms, or calculating device) is emphasized in all courses.
2. Calculators and computers are used in daily work and on examinations.
3. Selecting the appropriate computation to be performed is stressed as much as performing the computation.
4. Students are provided the opportunity to study additional mathematics topics/courses even if they lack facility with paper-pencil computation.
5. Computation serves the purpose of strengthening conceptual and procedural understanding of number, numeration and operations in the context of applications and problem solving.
6. The approach to computation in 9-12 mathematics courses reflects the ways in which computation is and will be used outside of the school setting.

D. Estimation

The 9-12 program provides opportunities for students to develop and use estimation skills and concepts on a continuing basis throughout all courses.

1. Estimation is used to judge reasonableness of results.
2. Estimation is used frequently as part of the problem-solving process rather than being used only for computational exercises.
3. Calculators are used to develop estimation skills and reasonableness of results.
4. Situations are presented for which precision of results must be determined.

E. Reasoning

Provisions are made at all levels for introduction and use of simple valid arguments. The 9-12 program provides students with opportunities to learn *the basic tenets of logical argument and to validate arguments. Connections between various representations of mathematical ideas are used to develop arguments.*

1. Opportunities are provided to explore patterns and to make and test conjectures.
2. Construction of simple valid arguments using a variety of proof techniques (counterexample, proof by contradiction, etc.) is required in all courses.
3. Activities which necessitate following a logical argument and judging the validity of the argument are provided in all courses.
4. Higher-order thinking skills are taught and evaluated in all courses.
5. Different representations (e.g., diagrams, graphs, tables of values, equations) are used to draw conclusions about problem situations.

F. Curriculum Emphasis

There should be a change in the content emphasis in the secondary curriculum. The strong emphasis traditionally placed on computational algorithms in the noncollege-bound curriculum should give way to the inclusion of a broad range of studies, including problem solving, estimation, geometric concepts, applications and mathematical reasoning. The program for college-bound students should integrate the same concepts and reduce the emphasis on algebraic manipulation skills. Throughout all courses calculators and computers should be used as tools for graphing, problem solving, performing tedious calculations, data generation, and concept development.

1. The systematic use of calculator and/or computer graphing to develop and support algebraic concepts reduces the need for paper-pencil graphing.
2. Function, as introduced in algebra, serves as a unifying concept across all mathematics courses (e.g., geometric transformations, trigonometric functions, and sequences).
3. Emphasis is on algebraic concepts such as linearity, function, equivalent expressions, solution, and the like, rather than on algebraic manipulation skills.
4. The concepts of limit, maximum, and minimum are developed informally throughout the algebra strand.
5. Investigations and comparisons of various geometries are used to enhance the study of geometric concepts.
6. Opportunities to explore patterns and to make and test conjectures precede the development of deductive arguments.
7. The writing of deductive arguments in paragraph form is encouraged.
8. The study of geometric properties is not restricted to formal geometry courses.

9. Opportunities to visualize and work with three-dimensional figures are provided to develop spatial skills.
10. Problems are chosen in such a way as to integrate various strands of mathematics with applications from other curricular areas.
11. The use of the scientific calculator is emphasized rather than table reading skills and interpolation.
12. The development of trigonometric function concepts progresses from the informal to the formal.
13. Data from real-world situations are needed to illustrate the properties of trigonometric functions.
14. Discrete mathematics topics such as representing problem situations using finite graphs, matrices, sequence and series, and combinatorics (combinations, permutations, probability) are included in both college-bound and noncollege-bound curricula.
15. Experimental probability or simulation methods are used when appropriate to represent and solve problem situations.
16. Applications of probability in related fields such as business and sports are integrated into the mathematics curriculum.
17. Charts, tables, and graphs are used to draw inferences from real-world situations.
18. Opportunities are provided to collect, organize and display data in all courses.
19. Statistical techniques developed in mathematics classes are applied to other areas.
20. Applications of measures of central tendency, variability and correlation are used at appropriate levels.
21. Information is provided to give the opportunity to analyze the validity of statistical conclusions and the uses and abuses of data.
22. Calculators and computers are used as tools in statistical investigations.

[5]The primary authors of this section were Glendon W. Blume, Bethlynne Cacka, Mary Moran, and Fred R. Stewart.

INSTRUCTION[6]

To learn the essential mathematics needed for the 21st century, students need a non-threatening environment in which they are encouraged to ask questions and take risks. The learning climate should incorporate high expectations for all students, regardless of sex, race, handicapping condition, or socioeconomic status. Students need to explore mathematics using manipulatives, measuring devices, models, calculators, and computers. They need to have opportunities to talk to each other and write about mathematics. Students need modes of instruction that are suitable for the increased emphasis on problem solving, applications, and higher-order thinking skills. For example, cooperative *learning allows students to work together in problem-solving situations to pose questions, analyze solutions, try alternative strategies, and check for reasonableness of results.*

A. Teaching Strategies and Instructional Activities
   1. Teaching practices include large group, small group, and individualized instruction when appropriate.
   2. A variety of instructional strategies that are based on research findings is used.
   3. The classroom environment encourages students to interact with peers and the teacher, take risks, explore, and seek their own solutions to problems.
   4. Calculators are used by all students as an integral part of the program at all levels.
   5. Computers and appropriate software are used by all students.
   6. Teaching strategies that foster the development of higher-order thinking, reasoning, problem solving, and communicating mathematical ideas are used in all courses.
   7. Activities for developing mathematical concepts are appropriate to students' levels of development and progress from the use of manipulatives to the pictorial to the abstract or symbolic.
   8. Teachers employ teaching styles compatible with students' learning styles.
   9. Problems using realistic application situations are used to introduce and develop mathematical concepts as well as to reinforce them.
   10. Instructional activities are designed to build on students' previous mathematical experiences.

B. Human and Material Resources and Facilities
   1. Appropriate curriculum materials in sufficient quantities are provided for all students, including those who are mathematically gifted or in need of remediation.
   2. Human resources such as department heads, teacher aides, technology specialists, volunteers, librarians, support staff, referral services, resource teachers, and consultants are available and used to support the mathematics program.
   3. Sufficient time is allotted for effective and efficient mathematics instruction.
   4. Instructional and resource areas — large group, small group, individual, math laboratories, computer laboratories, math centers, library media centers — are available to support activities that require flexible grouping.

5. A staff resource and office area is provided for individual and group planning.
6. Teachers have ready access to equipment — manipulatives, chalkboards, bulletin boards, calculators, computers, computer projection devices, overhead projectors, duplicating equipment—to support the instructional program.
7. Audiovisual materials are readily available and integrated into the instructional program.
8. Adequate and appropriate computer software is used and integrated into the instructional program at all levels.
9. Contemporary issues and careers in mathematics are explored through community resource persons, current articles, news items, and other resources.

[6]The primary authors of this section were Glendon W. Blume, Bethlynne Cacka, Mary Moran, and Fred R. Stewart.

EVALUATION[7]

Evaluation is perhaps the most discussed but most poorly conducted element of any curriculum or instructional program in the schools. In order to obtain a fair and comprehensive assessment of any K-12 program, evaluation strategies must focus on the curriculum, the personnel, the materials, and the facilities.

There is a certain amount of redundancy, but the evaluation section focuses more on the process of evaluation. One might think of this section as "meta evaluation" — an evaluation of the evaluation.

A. Program (Curriculum)
1. Appropriate planned course documents are available to all staff who teach mathematics.
2. Planned courses are up-to-date and reflect both the needs of the district's students and the current topics in mathematics articulated by PDE and NCTM.
3. The mathematics curriculum can be approached through a variety of instructional methods and is appropriate to students with different learning styles.
4. The mathematics curriculum articulated in the planned courses is the curriculum actually taught.
5. The articulated mathematics curriculum is periodically evaluated by:
   a. teacher-made tests
   b. criterion-referenced tests
   c. standardized tests
   d. classroom observations
   e. teacher feedback
   f. student feedback

6. Student evaluation data is periodically used to:
   a. assess the quality of the mathematics program.
   b. provide information about the effectiveness of the instruction.
   c. adjust curriculum content.
   d. inform the Board of Education, parents, and the professional educational community of both progress and needs.
   e. determine the focus of staff development programs.
   f. determine a part of student report card marks.
7. The revisionary process for the mathematics curriculum involves:
   a. school-based student evaluation data.
   b. current research findings.
   c. NCTM recommendations.
   d. PDE recommendations.
   e. state test data (TELS/EQA)
   f. school district instructional support staff.
   g. teacher feedback.
   h. outside consultants.
8. Evaluation instruments are changed and updated in consonance with curricular changes.

B. Student
1. Student mathematical progress, based on clearly defined district objectives, is monitored and reported.
2. The district testing program (standardized, criterion referenced, competency tests, etc.) influences the placement and instruction of students.
3. Teachers are involved in the development and utilization of procedures for tracking student mathematical progress and for improving student monitoring procedures.
4. Student mathematical progress within and across grade levels is communicated among teachers to improve student learing.
5. Student differences (gifted and talented, remedial, average) are considered in the development of evaluation devices.
6. Student evaluation is periodically analyzed to accommodate changes in goals.
7. Student developmental levels are considered in the development of evaluation devices.
8. Evaluation is a motivation to student achievement.

C. Teacher Evaluation
1. Assessment devices are administered by the teacher and data obtained are utilized in planning instructional activities for individual students.

2. Teachers base instructional activities for students on student needs, interests, and abilities.
3. Teachers match teaching strategies to the needs of students and the nature of the objective.
4. Teachers provide a classroom environment that is conducive to the learning of mathematics.
5. Teachers maintain and interpret appropriate records of student progress.
6. Teachers follow the district's planned courses for teaching mathematics.
7. Formal teacher evaluation is based on criteria for effective teaching adopted by the district and described in a written document.
8. Teacher evaluation is based on current research.
9. Teacher evaluation includes data from classroom observations.
10. Teacher evaluation includes feedback from students which focuses upon such aspects as interest of content, effectiveness of instruction, challenge of work, and individual success.

D. Staff Development

1. Provisions are made for sharing educational ideas among teachers.
2. Meetings are held to discuss teacher concerns and techniques.
3. Teachers and administrators meet regularly to plan and revise the mathematics curriculum.
4. Inservice is provided for each teacher prior to implementing a mathematics curriculum.
5. Teachers of mathematics participate in state and national mathematics workshops, meetings, institutes, and conferences.
6. Teachers of mathematics enroll in college-level courses.
7. Teachers of mathematics enroll in practical educational courses available through the intermediate units.
8. Teachers of mathematics observe one another in mathematics courses.
9. Peer coaching exists among mathematics teachers.
10. Summaries of relevant current research, and professional publications and materials are made available to teachers of mathematics.
11. Mathematics teachers are surveyed to determine their needs/desires for staff development.
12. Mathematics teachers are involved in the planning of staff development programs.
13. Inservice activities are planned to prepare teachers to teach current topics in mathematics.

14. Inservice activities show teachers how to utilize available instructional materials, facilities, and equipment.
15. Teachers of mathematics participate in the governance activities of professional mathematics organizations.
16. Training is provided for teachers of mathematics in the effective use of computer hardware/software.
17. Staff development addresses course content, methods of presentation, and learning strategies.
18. Assistance is provided to teachers for interpreting the results of all mathematics assessment.
19. Materials used and suggested in staff development sessions are purchased by the district.
20. Teachers are made aware of published diagnostic tests as a means for assessing student attainment.

E. Materials

1. Printed materials (including textbooks, teachers manuals, instructional aids)
   a. Procedures have been established to evaluate and select textbooks and other instructional materials.
   b. Teachers participate in the evaluation and selection of textbooks and other instructional materials.
   c. Provisions are made for teachers to become familiar with selected textbooks before using them in the classroom.
   d. Textbooks are periodically reviewed and updated.
   e. Teachers work with the media specialist on evaluating and selecting mathematics materials for the media center.
   f. Printed material is appropirate to the mathematical and reading level of the targeted group.
2. Instructional aids
   a. An inventory of available instructional aids is provided for the teaching staff.
   b. Teachers have easy access to instructional aids.
   c. Procedures have been established to evaluate and select instructional aids to ensure that selected aids support the mathematics curriculum.
   d. Teachers participate in the evaluation and selection of instructional aids.
   e. Provisions are made for teachers to become familiar with selected instructional aids before using them with students.
   f. Instructional aids are periodically reviewed and updated.
3. Student manipulatives
   a. Procedures have been established for the evaluation and selection of student manipulatives.

b Teachers participate in the evaluation and selection of student manipulatives.
c. Provisions are made for teachers to become familiar with the uses of manipulatives before using them with students.
d. An inventory of available manipulatives is provided for the teaching staff.
e. An adequate supply of manipulatives is available per child in the targeted group.
f. The supply of manipulatives is periodically reviewed and updated.
g. Manipulatives are appropriate for the mathematical and developmental levels of the students.

4. Technology
   a. The district has a plan for integrating technology into the mathematics curriculum.
   b. A systematic process exists for evaluating and selecting hardware and software.
   c. Provisions are made for teachers to learn the use of hardware.
   d. Teachers work with the media specialist to evaluate and select math software for the media center.
   e. An inventory of available hardware and software is provided for the teaching staff.
   f. Hardware/software is readily available for every classroom.
   g. Incentives are offered to teachers to utilize technology in their classrooms.
   h. Hardware/software is appropriate to the mathematical and developmental levels of the students.
5. Audio-visual materials
   a. Procedures have been established to evaluate and select audio-visual material and hardware which support the mathematics curriculum.
   b. Teachers work with the media specialist to evaluate and select audio-visual mathematics software/hardware.
   c. The district provides funds to purchase audio-visual software/hardware.
   d. Provisions are made for teachers to become familiar with audio-visual mathematics materials and equipment before using them with students.
   e. An inventory of available audio-visual mathematics materials is provided for math teachers.
   f. Audio-visual equipment and materials are periodically reviewed and updated.

F. Facilities
1. Instructional areas are available for large group, small group and individualized instruction.
2. Books and resource materials (such as instructional aids, audio-visual equipment, hardware, software, manipulatives) are safely stored and easily accessible.
3. The resource facility and its contents are accessible to physically challenged students.
4. Resource materials are organized in containers designed to be manageable by students and to facilitate easy use.

[7]The primary authors of this section were Lucy M. Young, Lois Barson, Linda Renney, and Brenda K. Wise.

TEACHERS: K-12[8]

In order to have a high-quality mathematics program, teachers of mathematics have the responsibility to be well-prepared, to possess and demonstrate positive attitudes, to continually grow professionally, and to be actively involved in educational issues that effect the quality of their students' learning.

A. Professional Preparation

Teachers of mathematics:
1. have a strong background in mathematics;
2. have a strong background in mathematics education;
3. have a strong background in child or adolescent development and behavior; and
4. meet state certification standards for their assigned teaching responsibilities.

B. Professional Development

Teachers of mathematics:
1. have and use personal "professional self-development" plans;
2. belong to the Pennsylvania Council of Teachers of Mathematics;
3. belong to the National Council of Teachers of Mathematics;
4. read the publications of the Pennsylvania Council of Teachers of Mathematics and the National Council of Teachers of Mathematics; and
5. attend, and use the knowledge gained from, mathematics workshops, meetings, conferences, institutes, and inservice programs.

C. Involvement

Teachers of mathematics are actively involved in the:
1. design and development of their staff development activities;
2. critical analysis of their own teaching behavior;

3. selection of textbooks and other instructional resources; and
4. critical examination and revision of the mathematics curriculum.

D. Attitude
Teachers of mathematics:
1. exhibit positive attitudes toward, and have a positive rapport with, their students;
2. exhibit positive attitudes toward mathematics and its use;
3. set high, but reasonably achievable, expectations for their students' mathematics performance and classroom behavior; and
4. develop and reinforce the notion of lifelong learning as a desirable behavior.

[8]The primary author of this section was Robert F. Nicely, Jr.

ADMINISTRATION: K-12[9]

In order for a school district to ensure that it has a high-quality mathematics program, the school board and administration have the responsibility to provide leadership and support for personnel development, curriculum and instructional improvement, communications, and compliance.

A. Professional Development (outside of the school district)
School board policy and administrative practice:
1. Encourage and support teachers of r. ˀthematics to attend local, state and national professional ., .athematics education conferences.
2. Encourage and support teachers of mathematics to attend mathematics workshops, meetings, institutes, conferences, and in-service programs.
3. Encourage and support teachers of mathematics to enroll in mathematics, mathematics education, and related courses in colleges and universities.
4. Provide a professional library — including publications and materials from professional organizations — in the school for use by teachers and mathematics.

B. Staff Development (within the school district)
School board policy and administrative practice:
1. Provide for the systematic orientation of newly-hired teachers of mathematics to the district's curriculum and instructional resources.
2. Provide for the systematic and regular updating of all teachers of mathematics regarding current research and recommendations from professional organizations.
3. Enable teachers of mathematics to be systematically involved in the

design and development of their staff development activities.
4. Provide adequate financial support for "expert help" for staff development programs for teachers of mathematics.
5. Encourage and enable teachers of mathematics to observe other teachers of mathematics.

C. Curriculum and Instructional Leadership
School board policy and administrative practice:
1. Provide for instructional leadership and support for the mathematics program *at the building level.*
2. Provide for coordination of the K-12 mathematics program among all schools in the district.
3. Provide for teachers at different grade levels and in different buildings to meet on a regular basis to review K-12 mathematics program issues.
4. Provide for the development and/or revision of the K-12 mathematics curriculum on a regular basis.
5. Ensure that each teacher of mathematics has a copy of the planned course for his/her assigned grade(s) or course(s).
6. Use a curriculum and instruction management model that enables district personnel to determine if the planned curriculum is being implemented.
7. Provide for adequate and appropriate physical facilities (laboratories, furniture, space) for the K-12 mathematics program.
8. Provide for adequate and appropriate instructional materials (calculators, computers, concrete materials, books, measurement devices) for the K-12 mathematics program.
9. Ensure that adequate time is allotted to plan and teach the K-12 mathematics program.
10. Ensure that the instructional organization pattern accommodates student differences.
11. Enables teachers of mathematics to be systematically involved in the selection of textbooks and other instructional resources.

D. Compliance
School board policy and administrative practice:
1. ensure that the K-12 mathematics program is in compliance with state (Chapter 5) requirements.
2. ensure that all teachers meet state certification requirements.

[9]The primary author of this section was Robert F. Nicely, Jr.

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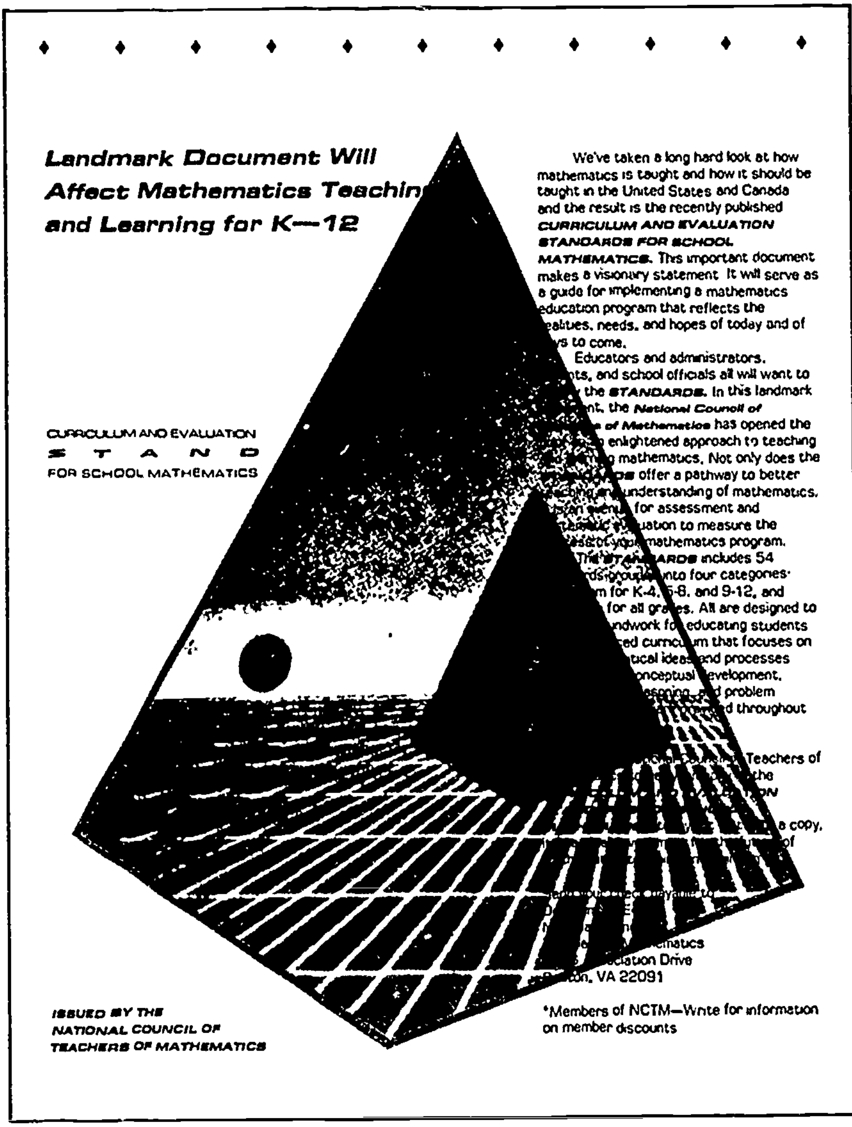
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METACOGNITIVE SKILLS, PROBLEM SOLVING, AND THE MATHEMATICS CLASSROOM

Rose M. Zbiek
Lake-Lehman High School

"Stop and think about what you're doing." How many times a day does a teacher wish that he or she could encourage students to reflect upon what they are doing and why they are doing it? Within the past two decades, metacognition—"thinking about thought"—has become a major issue in psychological and educational research. Its influence upon increasing understanding and improving learning has become the focus of many studies. The intent of this article is first to explore the role of metacognition in research and particularly in mathematical problem solving, and then to propose techniques through which metacognitive skills may be developed during mathematics instruction.

Nature of Metacognition

Metacognition may be viewed as the integration of two separate although related ideas: a learner's knowledge and beliefs about cognitive phenomena, and the self-regulation and control of the learner's cognitive action. The former includes beliefs about oneself and others as cognitive beings, knowledge of factors and conditions that make tasks more difficult, knowledge of the scope and requirements of tasks, knowledge of cognitive strategies, and awareness of the potential usefulness of these strategies. As an example, a student may know that solving an equation is more difficult when there are trigonometric expressions or higher powers involved. The second aspect of metacognition involves the influence of knowledge and beliefs on decisions to employ a certain strategy, on understanding a given task, and on monitoring the use of strategies (Garofalo & Lester, 1985). That same student may use his or her knowledge of the ease of solving quadratic equations versus the difficulty of solving trigonometric and quartic equations to begin to solve

$$\sin^4 x - 3\sin^2 x = 4$$

by writing it as

$$A^2 - 3A = 4.$$

Metacognition in Literature

Metacognition in Mathematics Education

Metacognition has only recently drawn the attention of mathematics educators. Many factors contribute to this delay. First, metacognition is poorly defined and is often hard to distinguish from other cognitive activity (Garofalo and Lester, 1985). As an example, when dividing 4 into 712, consider the idea of "checking your answer" by multiplying 4 by 178. Is this a metacognitive effort to verify the solution process, or is it part of the cognitive strategy for doing long division? In addition, metacognitive activity is rarely encountered in a classroom, where the focus most often is on getting "the answer" rather than on the solution process. Students seem to react to their environment (Schoenfeld, 1985). After all, why think about how exercises 1 and 2 differ as long as you get the answers in the back of the book? Metacogitive skills are also overlooked because they are covert. To study them requires relying on self-reports. It certainly is not hard to imagine the difficulty a young child would have in explaining why he or she chose to add $4+8$ by counting on from 8 rather than by regrouping to have $2+(2+8)$. Lastly, as long as students do get the right answers, metacognitive skills are deemed unimportant as well as unresearchable (Lester, 1982).

Metacognition and Problem Solving

The role of metacognition in problem solving has recently received increased attention (Garofalo & Lester, 1985; Kilpatrick, 1985; Silver, 1982). Many mathematics educators believe metacognitive activity is the "driving force" behind success (Schoenfeld, 1983). In fact, being able to use these skills to approach and evaluate an unusual situation may explain why expert problem solvers are successful even when faced with problems about ideas with which they have little experience. This could explain why one student can solve a puzzle in minutes while it takes another student (or the teacher) hours.

Occurrence of Metacognitive Behavior

Despite the potential need for metacognitive skills in successful problem solving, researchers have found an absence of metacognitive activity during classroom and laboratory observations. This lack was found at all age levels. Such findings tend to substantiate Schoenfeld's (1985) observation: although students may be spending a large number of hours seeing and learning about *performing* mathematically, they are not necessarily spending any time seeing and learning about *thinking* mathematically. (For examples see Garofalo & Lester, 1985; Heid et al., 1988; Lester, 1980; Schoen & Oehmke, 1980; Suydam, 1987.)

Feasibility of Successful Metacognitive Training

The lack of metacognitive skills among students suggests the need to consider how we can train students in these skills. Several researchers have voiced this thought (See Flavell, 1979; Rigney, 1980; Garofalo & Lester, 1985).

This concern about whether students can be taught to use metacognitive skills was considered by Lester (1980). He noted that Schoenfeld had addressed this idea with some success. Schoenfeld actually developed a schematic outline of a "managerial" strategy for problem solving and discussed how to train students to use it. He stressed specific managerial strategies that apply to mathematical problem solving. These strategies were efficient ways to choose an approach, to avoid approaches that go nowhere, and to allocate resources such as time. Thus, students were not only taught the problem-solving techniques, but also ways to choose among them. The weakness of his results, however, is that he worked with a very limited population: college students with substantial mathematical backgrounds who voluntarily enrolled in his course on mathematical problem solving.

Classroom Implementation of Metacognitive Training

Techniques Implied in Literature

Many students undoubtedly develop some awareness of metacognitive activity on their own. Most teachers, with comments such as "check your work," do give some indication that regulation is important. However, mathematical instruction, classwork, and homework still focus primarily on developing knowledge of concepts and procedures, but not on acquiring and controlling metacognitive behaviors. Several ways of improving metacognitive skills are suggested in the literature. These ideas fall under three main categories: classroom structures, communication, and technology.

Teachers could include the modeling of metacognitive skills, small group interaction while solving problems, large group work during which the students direct the solution, and reciprocal teaching episodes wherein individual students play the teacher. Metacognitive skills also may be enhanced when teachers encourage communication, including questioning to induce thought, self-communication, and student writing. Technology may be used by teachers to increase metacognitive activity through student programming assignments and software designed to model and cue the regulatory process. The common element of all these techniques is the transfer of responsibility for gaining and evaluating understanding from the teacher to the student.

Examples of Activities Using the Techniques

The material chosen for examples might involve secondary school geometry or algebra, perhaps trigonometry or even calculus. However, the first two examples given here are appropriate for a first-year course in general mathematics. This choice reflects three considerations. First, the emphasis in the examples is on the techniques, not on the mathematical content. More importantly, problem solving and metacognitive skills are relevant for *all* students. Lastly, training in metacognitive skills may particularly aid academically marginal students.

The first example uses writing with pretest questions as a self-communication exercise. Figure 1 contains pretest questions from a commercially produced textbook (Bolster & Woodburn, 1985). Figure 2 provides an example of the types of responses students may give when directed to write "what they are thinking" while doing the problems. This writing process focuses the students' attention on their own thinking. The need to put something on paper makes this task unavoidable. Having these metacognitive remarks in writing may provide direction for exploration, remediation, and discussion. The teacher can aid the student to use these comments to develop metacognitively.

C. Reduce $\frac{8}{20}$ to lowest terms.

F. Write $\frac{2}{5}$ as a decimal.

Figure 1. Skills Pretest Exercises for Multiplying and Dividing Fractions and Mixed Numbers, taken from the Teacher's Edition of *Mathematics in Life* (Bolster & Woodburn, 1985, p. 149).

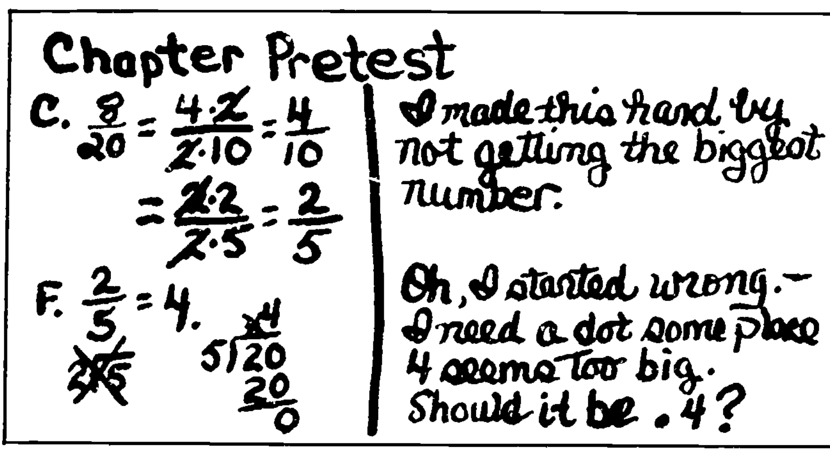


Figure 2. Potential Responses Evoked by Writing Activity During Skills Pretest.

The second example is a small-group activity. Exercises, as found in the textbook (Bolster & Woodburn, 1985) are given in Figure 3. The supplemental questions given in Figure 4 are designed to address the metacognitive skills of "varying conditions" and "evaluating results." The consideration of the data set raises questions about the use and acceptance of the mathematical models. The specific questions requiring students to evaluate their results encourages them to verify their own answers rather than to remain dependent on the teacher.

Eva and Luis Ortiz exercise regularly at a nearby track. Luis rides a bicycle and Eva jogs. There are small posts in the ground to mark the fractional parts of one lap of the track. Luis rides at a speed of one lap every 1½ minutes. How long will it take him to complete

a. 3 laps? d. 1½ laps?
b. 7 laps? e. 5¾ laps?
c. 12 laps?

Figure 3. Consumer Applications Exercises, taken from the Teacher's Edition of *Mathematics in Life* (Boister & Woodburn, 1985, p. 161).

DIRECTIONS: Answer these questions after you do page 161.

21. Luis' friend Dave actually timed Luis as he did his lap yesterday. The data Dave collected is shown below.

Lap number	1	2	3	4	5	6	7
Time to complete this lap	1½	1¼	1¼	1¼	1⅜	1½	1¾

a. According to this data, how long did it take Luis to complete the first three laps?
b. Compare this answer to the answer you had for exercise 1 on page 161.
c. Explain why the answer to part a is different from what you calculated in exercise 1.

22. a. Use your numbers from exercises 1 through 5 on page 161 to complete the following chart.

Number of laps	1½	3	5¾	7	12
Time to complete this many laps					

b. Assume that Janice, Frank and Ellen were asked to calculate how long it would take Luis to complete exactly 4 laps. They gave these answers:

Janice	5 minutes
Frank	9 minutes
Ellen	6½ minutes

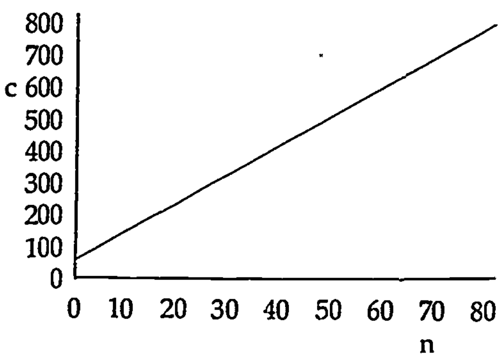
Without doing any calculations, whose answer would you accept? Why?

Figure 4. Supplemental Questions for Applications Exercise Done in Small Groups.

The last example is appropriate for developing the metacognitive skills of "choosing strategies" and "comparing alternative approaches." As well as obtaining answers to realistic questions, the students must verbalize the reasoning behind their choices. They must also consider using other approaches—ideas which may not have otherwise occurred to them without this model.

Consider the following situation:

The cost c of having a meal catered by E-Z Serve depends upon the number n of people served. The graph, equation, and table below represent this relationship.



$c = 8.84\,n + 53$

n	c
0	53
10	141.4
20	229.8
30	318.5
40	406.6
50	495
60	583.4
70	671.8
80	760.2

1. A customer calls and asks, "What it would cost him to have a party catered for 32 people?"
   a. Which form of the relationship would you use to answer his question?

b. Why did you choose this form?
c. What answer would you give the customer?
d. With which other forms could you have answered this question?

2. A potential customer call and asks. "Could I have a party for 18 people catered for under $250?"
   a. Which form of the relationship would you use to answer his question?
   b. Why did you choose this form?
   c. What answer would you give the customer?
   d. With which other forms could you have answered this question?

Figure 5. Algebra I Problem Situation and Corresponding Questions. Questions like these appear in the materials used in the "Algebra With Computers" curriculum (Fey, et al., 1986).

Implications for Further Study

Additional research must determine the possible roles of classroom structure, communication and technology in developing metacognitive skills. Researchers must ascertain which combinations of these techniques are most effective. Future investigations must consider which particular aspects of metacognition are most directly influenced by each technique. Longitudinal studies done in naturalistic classroom settings are needed.

A model which explains metacognition must be developed. Similarly, new or revised problem-solving models should include the role of metacognition (Silver, 1985). These models should reflect its potentially developmental nature.

The development of metacognitive skills helps students to learn, to understand, to use, and to appreciate problem-solving processes. Mathematics instruction encouraging metacognitive as well as mathematical concepts enables students to apply, reason with, and communciate about mathematics.

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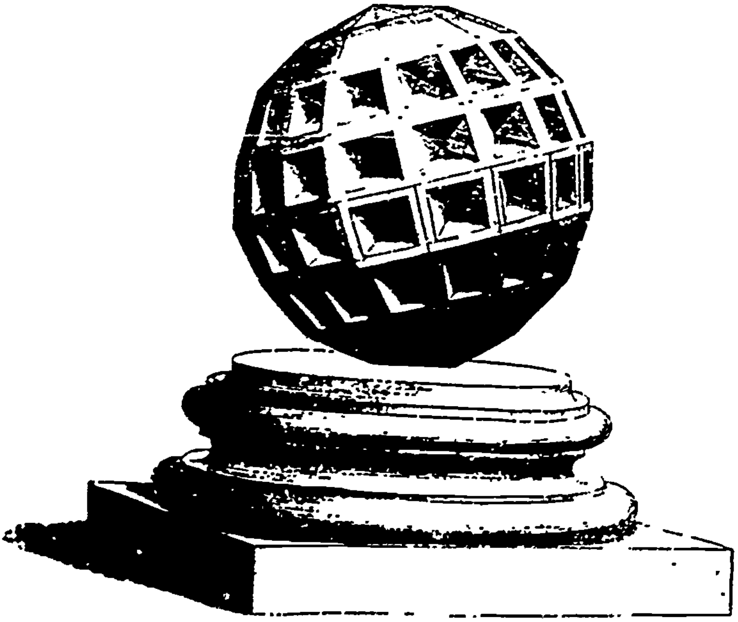

HIGH SCHOOL CALCULUS STUDENTS TRY COOPERATIVE LEARNING: WHAT CAN THEY TELL US?

Dr. James Saunders
Upper St. Clair High School

One of the main goals of cooperative learning is to prepare students for a team-oriented world. There is reason to believe that formal education is one of the few experiences left where the individual has to learn and be evaluated without the benefit of consultation and assistance from others. Individual learning is so much a part of traditional education that students often resent and ridicule the high-achievers because they make the others look bad (Brandt, 1987). Advocates of cooperative learning think that education can be structured in such a way so that all students will be encouraged by and feel a part of the accomplishments of each learner in the group. The best analogy is a baseball or basketball team where the members appreciate the contributions of the exceptional player, but realize that all have to do well in order to help the team to succeed (Slavin, 1987). How can educators unravel all of the many years of concentration on individual, competitive-type learning that students have come to accept as a way of life? One suggestion, which is the subject of this paper, is to introduce students to cooperative learning in a four or five week unit. During and after the experimental unit listen to what the students have to say. What are their fears? What do they like about cooperative learning? What suggestions do they have? Students and teachers, working together, can implement a more team-oriented approach to learning by gradually working toward greater emphasis on the achievement of the group rather than the individual.

There are approximately one hundred students who take calculus each year at Upper St. Clair. Calculus students are among the most highly motivated individuals in the school. These students work hard to achieve good grades which they naturally consider as a result of individual achievement. Up to the point of the experiment discussed in this report, all students involved would probably define cooperative learning as simply a method of working in small groups to help learn a task. The concept of determining one's grade based on the achievement of all members of a group was an unusual, if not foreign, teaching-learning method for most of these students.

The intent of the author in this report is to explain how a cooperative learning experiment was carried out, to present some of the statistical results, and most importantly to relate what the students had to say about their reactions to cooperative learning. As a teacher of mathematics for many years, I am continually impressed by how students give fresh insights into solving problems. Similarly, they have much to tell us about what methods work in the classroom. All we have to do is listen.

Students were informed ot the cooperative learning experiment at the beginning of the third quarter of the 1987-88 school year. Four classes of students were divided into "equivalent" groups of three or four students each. Equivalence was based on the teacher's judgement of previous calculus achievement. Calculus at Upper St. Clair is presented in a class, large-group format. Students meet as a class for two mods (25 minutes each) on Monday, Wednesday and Friday. Two classes are joined (50-60 students) to form a large group on Tuesdays and Thursdays. Most of the cooperative learning activities were carried out on Mondays and Wednesdays. The Tuesday large group usually consists of a lecture on new material, but there is ample opportunity to "check for understanding" by breaking up into cooperative learning groups. Thursday is reserved for quiz and test evaluations. The unit includes three quizzes and a test.

Before the students were told about the details of the cooperative learning experiment, a cooperative learning attitude survey (see Appendix) was administered. The survey was constructed from ideas about cooperative learning discussed by Ron Brandt (1987) and Robert E. Slavin (1987). The same survey was also administered at the end of the four-week experiment. Results of the survey showing class averages for each item are displayed in Table 1 and Figure 1.

Student Attitudes Toward Cooperative Learning

		PRE-TEST					POST-TEST				
		Mods No. 1 2 20	Mods No. 3 4 17	Mods No. 5 6 27	Mods No. 9 10 31	Average	Mods No. 1 2 21	Mods No. 3 4 18	Mods No. 5 6 28	Mods No. 9 10 31	Average
Item	Bias*										
1	-	3.70	3.53	3.50	3.53	3.57	3.56	3.55	3.29	3.52	3.48
2	+	3.15	3.29	2.89	3.00	3.08	3.00	3.01	2.21	2.87	2.77
3	-	3.35	3.06	3.14	3.08	3.16	3.01	3.01	2.54	2.97	2.88
4	+	4.00	4.18	4.11	3.98	4.07	3.91	3.93	3.57	3.71	3.78
5	+	4.50	4.47	4.43	4.36	4.44	4.33	4.31	4.18	4.10	4.23
6	+	3.50	3.29	3.29	3.36	3.36	3.34	3.32	2.79	2.94	3.10
7	+	4.00	4.24	3.93	3.95	4.03	3.96	4.04	4.07	3.94	4.00
8	-	3.15	2.82	2.71	2.78	2.87	2.78	2.72	2.14	2.45	2.52
9	+	4.10	4.41	4.14	4.12	4.19	4.13	4.16	4.25	4.16	4.18
10	+	3.15	2.82	2.96	3.07	3.00	3.08	3.04	2.50	2.74	2.84
11	+	3.75	3.41	3.25	3.39	3.45	3.43	3.39	3.29	3.48	3.40
12	-	4.35	4.76	4.79	4.58	4.62	4.51	4.54	4.86	4.35	4.57
13	-	3.40	2.94	2.96	2.93	3.06	2.96	3.00	2.64	3.06	2.92
14	-	3.90	3.94	3.71	3.64	3.80	3.67	3.71	3.57	3.71	3.67
15	+	4.05	4.00	4.04	3.98	4.02	3.92	3.97	4.11	4.06	4.02
16	+	4.35	4.53	4.54	4.42	4.46	4.44	4.43	4.36	4.19	4.36
Class average		3.78	3.73	3.65	3.64		3.63	3.63	3.40	3.52	

*Bias refers to positive or negative statement toward cooperative learning. Responses were scored as A=5, B=4, C=3, D=2 and E=1 on positive items. A=1, B=2, C=3, D=4 and E=5 on negative items.

Table 1

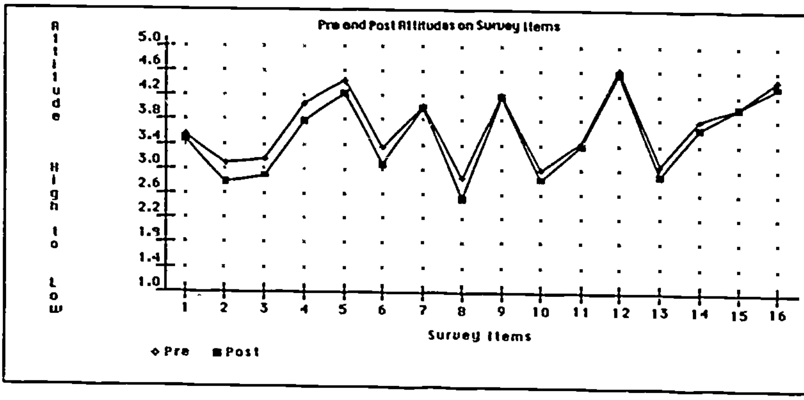

Figure 1

One of the key features of this version of cooperative learning is the emphasis on students teaching each other. Concepts and terminology are discussed by the students. When new material was to be presented, a representative "teacher" was asked to come to the front of the room from each group. While the instructor discussed the lesson with the representative "teachers," the groups of students worked on completing assigned problems. The "teachers" would then go back and teach the other members of the group. Quizzes and the unit test continued to be administered on an individual basis. However, group members' scores were compared and the individual's score recorded in the gradebook was the average of the highest and lowest scores for the group. Thus, all can benefit from the superior work of the high achiever and still have an incentive to encourage and help each other to understand the material presented. There were many opportunities to remind students that the crucial component of cooperative learning is that it is not good enough just for *you* to understand; *everyone* in your group must feel confident about the task at hand. Student reaction to grades based on group achievement was relatively passive. There seemed to be more a sense of adventure as reflected in a comment like "Let's see how this works." rather than "No way!".

Now, let's look at what the students thought about cooperative learning. Student reactions were compiled from the free response questions listed on the back of the post attitude survey. Sample responses are listed below as positive, negative and suggestions.

What did you like about the cooperative learning experiment? It was a chance to get help and sometimes a better understanding of the material. When you are unsure about something, someone in the group will always volunteer to help you. . . . I like teaching others how to do the problems. Also, I liked being able to ask others for help and have them help me because they know that it is a group effort. . . . It was easier to ask for help because you knew that other people's grades depended on you. Other people in the group were more willing to take the time to help for the same reason. . . . Talking about the problems helped because one of us might be able to explain a problem that the others didn't get. Thus if you had to do a problem on the board that you didn't get, there were other people to help you figure it out. . . . It brought my grades up and I also think I learned more. My group always explained everything to me, so I understood. I was more interested to learn. I also like working out the problems with them because every time I didn't understand I would just ask them. . . . I also liked how a group can solve problems together faster, because each person has input, each knows different parts of the problem — so with a group effort it is done faster. . . . The group explained many questions and we talked about the problems so we had a better understanding of what we were doing. . . . It gave you many "study partners" and a reason to work harder when many students want to ease up. . . . By talking with other students about the assignments in class, I felt I received more individual attention and instruction than before. . . . When I did really bad on a quiz my teammates were there to help me. I didn't want to put my team down so I had to study more and try harder. . . . I thought that I learned more, or understood it better coming from someone whose comprehension of the subject would be about the same as mine. There was a sense of caring about how the others did in the group. . . . It enables you to get a higher test score. It also encourages those who normally don't do well to try a lot harder because they feel guilty about pulling down the grades of the other students.

What did you not like? Having to be taught the material by someone else and not getting to hear it myself first hand. . . . I felt a lot of pressure from people who did better than me, like you better do well or else! I also did not like my group. . . . Some groups had more potential to do well because they got all smart people. I didn't like depending on others for my grade and having others depend on me for theirs. . . . Some people rely too much on the other group members. . . . Some people didn't care that they were hurting the group and didn't study. . . . I wouldn't like it if I was getting A's, and someone else was pulling me down. . . . How some of the "smarter" people would get really mad at others in their group because they didn't catch on as fast and would bring their score down. If they really cared they would help them, not criticize, whether in cooperative learning or a regular program. . . . My group lacked motivation. When they did not understand something, they didn't ask for help. When I tried to help, I don't think they were listening. . . . I don't feel that it is fair for an individual's grade to suffer because other people do not score as well on quizzes. It should not be other students' responsibility to teach, that is why we have a teacher. . . . Although the group is supposed to be a team, it isn't. It is still mostly individual. . . . Earning the lowest score and have others know was embarrassing. . . . That the people who tried the hardest and worked most efficiently were penalized. . . . I feel time was wasted when every member of the group was confused. . . . I felt cheated out of my own individual scores. . . . Coop learning made the high achievers work hard and the low achievers feel guilty. . . . An individual can not work to his/her own pace. They must keep up with or slow down to help the group rather than simply spending the most time on what he or she individually finds most difficult. I sort of was planning to take it easy and not study so much this semester, but I felt like I had a responsibility to do my best, because it's not fair to the others.

Do you have any suggestions about how to introduce and/or implement cooperative learning? Make it a gradual introduction in the first semester by having a group of 3 or 4 people take a quiz together and try to get used to the idea of an average grade. . . . It would be tough, because I'd bet most of the community would be against it. . . . Explain teamwork will be a part of our lives, in jobs, marriages, etc. — and we should learn how to cooperate. . . . I think grades should be the average of the highest, lowest, and the individual's score, so that the group score is important, but the individual's score can also help or hurt the individual. This way the student would benefit from both the cooperative and independent learning. . . . Have one group test (not all) which is really tough and have group members pool their abilities into taking the test together. . . . Let those who prefer co-op to do it and those who don't continue individual work . . . Still keep the groups — but make it some way that the lowest person doesn't over benefit and the highest person isn't the one who is hurt the most. But I don't know how. Sorry. . . . I think more interest might be generated by greater competition between the teams. . . . There should be an emphasis placed on the idea of being a team — working toward the common (all in the group) goal of understanding.

In summary, there is a clear message from the students. They appreciate the opportunity to try cooperative learning, they understand the need for developing cooperative skills and, in many instances, they even work harder; but, it is difficult to give up individual grades. Several students even expressed feelings of guilt for "pulling down" the grades of others. It takes more than one experiment in cooperative learning to realize that the goal is not for the individual to get a good grade, but for the group to be successful. The suggestion for including the individual score with the group average may be helpful. Also, several comments were made to suggest more group competition to provide incentives to assist each other. As one student suggested, "We have to think up ways so that the lowest person doesn't over benefit and the highest person isn't hurt."

Attitude survey results (Table 1) show that attitudes toward cooperative learning decreased on all but one of the items from pre-test to post-test. Also class averages for the whole survey decreased for each class. Significance tests were not applied to this data, but we can be reasonably sure that attitudes toward cooperative learning did not improve as a result of this study. This result is consistent with the free-response comments when you consider the student feeling toward group grading. Whether or not the students achieved more as a result of cooperative learning was not addressed. This author, from the student comments and observation of the experiment, tends to feel that the students did work harder than normal for second semester seniors. Students were aware that cooperative learning "moved" them to change some of their traditional patterns of learning. One student noted, "I feel cooperative learning would work but only for a more enthusiastic, less *senior* class!".

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APPENDIX

Cooperative Learning Attitude Survey

A If you strongly agree
B If you agree
C If your feeling is neutral
D If you disagree
E If you strongly disagree

____ 1. Cooperative learning is contrary to basic American values.
____ 2. Individual learning patterns do not prepare students very well for today's team oriented world.
____ 3. Students joining forces to "sink or swim" together in academic classes is not practical.
____ 4. Group work is a powerful tool for learning.
____ 5. People understand and remember things much better if they talk about them with others.
____ 6. When students compete individually the student who gets the "best grades" is sometimes ridiculed because he/she makes the others look bad.
____ 7. In cooperative classrooms, students encourage their teammates to do well, because they also benefit.
____ 8. It is unfair that students should benefit from each other's efforts and share responsibility for what others do or don't do.
____ 9. Americans have always prized individuality but in the modern world we also need teamwork.
____ 10. Typical citizens will support the use of cooperative learning in schools.

_____ 11. Individual scores should be based on the *average* score of the group.

_____ 12. Individual scores should be based on the *lowest* score of the group.

_____ 13. Cooperative learning causes the smartest or highest achieving persons to do most of the work.

_____ 14. In cooperative learning the lowest achieving persons will have little to do.

_____ 15. The group's task is to ensure that all will be successful on individual learning assessments.

_____ 16. If students care about the success of the team, it becomes legitimate for them to ask one another for help and to provide help to each other.


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TEACHING THE LANGUAGE OF MATHEMATICS IN THE UPPER ELEMENTARY GRADES

Audrey S. Heinrichs
Widener University

Virginia Larrabee
Castleton State College

As one of the Curriculum Standards adopted by the National Council of Teachers of Mathematics (1989), problem solving is of great importance in school mathematics. Since many mathematical problems are verbally stated, teachers of mathematics must deal with 1) mathematics as a language, 2) everyday language, and 3) the interaction between the two.

In addition, the importance of familiarity with mathematical words is underscored by the high correlation that exists between vocabulary knowledge and comprehension (Anderson & Freebody 1981; Carr & Wixson 1986). Unquestionably children must understand the words they hear and read in order to comprehend the verbally expressed problems that they meet (Vacca & Vacca 1986). In real life as in school, problems are most often met in verbal form, and as often must be analyzed by processes that involve words.

This article will present instructional methods which emphasize the interaction of the languages of mathematics and English in order to provide a necessary route into the language of mathematics. The methods will also increase comprehension for all children, as concepts, relationships, and vocabulary pertinent to the field of mathematics are developed.

Semantic Mapping or Webbing

Making a two-dimensional visual image of the relationships of mathematical words has been found to be useful (Johnson & Pearson 1984, in Stahl & Clark 1987). In making a semantic map or web, students analyze the commonalities between certain words, including their hierarchy or equivalence of rank. This causes the children to become involved in sorting out the meanings of such words, creating greater familiarity with them. An example of a semantic map of terms related to the concept "Multiplication" appears in Figure 1. The teacher directs students to the glossary of their math textbook and assigns them, in small groups to facilitate discussion, the task of locating related terms, then defining their relationship by drawing a semantic map of them.

A number of alternate terms and different paths of relationship might be chosen, and this flexibility in itself creates discussion and specific definition by the students drawing the maps. When the semantic maps are drawn on transparencies and shown to the whole class, the originating groups can describe how and why they selected and related their terms as they did on their map. One result is a gain in their sense of power over the concepts, and another result is a sense of flexibility in approaching mathematics. They see, for instance, that a word like "addend" may be placed in different categories and at differing levels within a category, depending upon how the categories are defined by the student who is making the semantic map. (See Figures 1 and 2.)

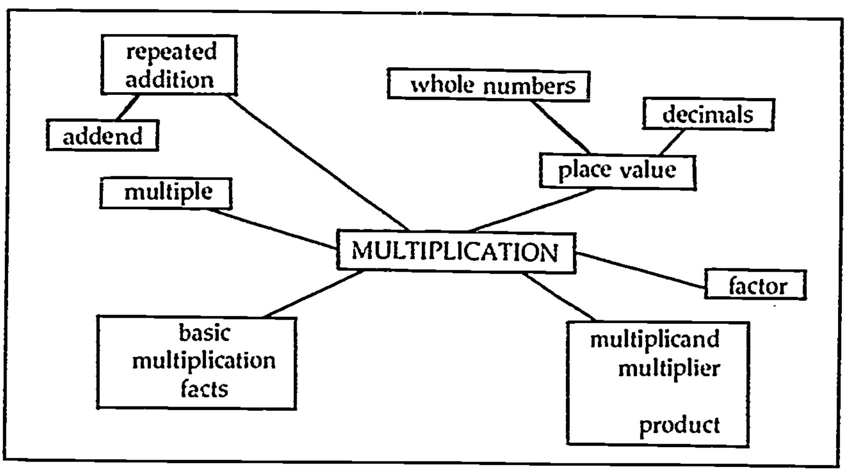

Figure 1
SEMANTIC MAP OF MULTIPLICATION-RELATED TERMS.

Multiple Meanings of Technical Words

A number of mathematical words have additional meanings that are in common use. Deliberate study of these words expands the student's sense of 1) flexibility, again, 2) familiarity with the words, 3) the importance of specific definition, and 4) the value of having a wide knowledge of word meanings (Heinrichs 1987). In either of the following activities, the teacher may provide the students with the list of words in Figure 3, which contains words from math textbooks for fourth-, fifth-, and sixth-grade levels, or direct students to the glossary of their math textbook where they can locate words with multiple meanings.

For the first activity, with the list before them on paper, on overhead transparency, or on the chalkboard, the children create a story which uses

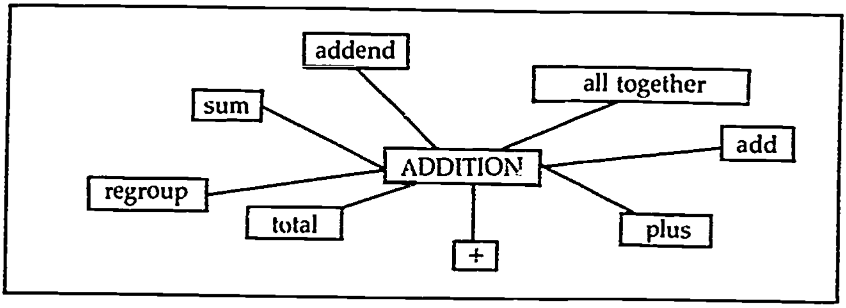

Figure 2
"ADDEND" AS A MAJOR SUB-CATEGORY.

Directions: Select the option which uses the same meaning of the underlined word that appears in the numbered sentence.

1. After Sally typed *RUN*, she was able to see the results of her BASIC program.
   a. The rabbit run was full of holes.
   b. She found a run in her stocking.
   c. Let's take a run around the block.
   d. George likes to run for exercise.
   e. One important computer program command is "run."

2. A *right* triangle contains an angle of 90 degrees.
   a. A rectangle has four sides and four right angles.
   b. Did Anna really have the right to tell John?
   c. Larry, your answer is right.
   d. The Bill of Rights is a famous and important document.

Figure 3
MULTIPLE MEANINGS — SELECTING THE CORRECT ONE.

the mathematical words in their alternate meanings. Each word of the story that comes from the list should be underlined in order to highlight its multiple-meaning potentialities. The students remember the words as friends when they next meet them in mathematical context.

A second activity that engages the students with multiple-meaning words is that of copying the textbook sentence in which each mathematical word appears, and creating several sentences which use either the correct meaning or alternate meanings in separate sentences. These can be typed and copied or written on transparencies for discussion by the whole class, as they select the meaning which is mathematically correct. The selection, the use of a dictionary to locate additional meanings, and the processing of definitions involve the students intensely in all of the meanings of the mathematical words, and, once again, underline the importance of exact definition in the use of technical vocabulary. An example follows in Figure 4.

area	list	reduce
average	loop	reflection
base	mean	remainder
basic	measure	right
bit	meter	round(ing)
capacity	mode	run
carry	multiple	sample
circle	ordering	scale
deduct	part	segment
degree	perimeter	space
diameter	place value	square
difference	plane	store
digit	point	straight
discount	power	sum
expanded	prime	time
factor	print	translation
figure	prism	triangle
flip	product	unit
foot	property	variable
fraction	proportion	volume
graph	protractor	yard
	range	zero

Figure 4
WORDS WITH MULTIPLE MEANINGS. GRADES 4, 5, AND 6.

Both of these activities are fun and effective in involving children in processing mathematical words, releasing the terms from their sometimes threatening aura. The teacher must be prepared for some noise and discussion as the students work on these activities (Guerriero 1988).

It is important to recognize that efforts to increase the richness of language possessed by students has an incalculable payoff. The more words they know, the more words they can learn and the more information they can comprehend, whether orally or in written form. Children can be confused — unnecessarily — by the multiple meanings of words. Using an incorrect definition, from a different language, will not solve mathematical problems; such problems require proper technical definition of terms. This confusion is unfortunate and can be avoided by teachers who see mathematical terms in their own and other contexts and deliberately pass on that knowledge to their pupils.

Summary

Deliberate language development by teachers of upper elementary grade mathematics can produce gains in students' knowlede of, ease and familiarity with, and use of technical terms in that subject. Ability to solve word problems, the capacity to analyze, and an understanding of the need for exact definition can be enhanced by the methods described here. Benefits in general richness and awareness of language use can result, as mathematical concepts and terms are processed and remembered.

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PROBLEM SOLVING WITH ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION

Martin P. Cohen
University of Pittsburgh

Basic number operations can provide many rich opportunities for problem solving. A variety of activities that involve addition, subtraction, multiplication, and division can be used for problem solving. Students can be presented with problems that require them not only to know the basic number operations but also to use good reasoning skills. When necessary, a calculator can be used to facilitate computation and focus students' attention on problem solving.

Activity 1

This activity is called *digit detectives* (Cohen, 1985, p. 44). Replace each blank with a digit so that the solution to the problem will be correct.

(1)
$$\begin{array}{r} \_17 \\ +\,2\_\_ \\ \hline 975 \end{array}$$

(2)
$$\begin{array}{r} 43\_ \\ -\,2\_9 \\ \hline \_86 \end{array}$$

(3)
$$\begin{array}{r} \_6 \\ \times\ 1\_ \\ \hline \_2 \\ \_6 \\ \hline \_3\_ \end{array}$$

(4)
$$\begin{array}{r} 7\overline{)5\_\_} \\ 5\_\_ \\ \hline \_ \\ 7 \\ \hline 0 \end{array}$$

Some students may use guess-and-test, especially in problems 3 and 4. In problem 3, students may realize the second factor is either 12 or 17. If 17 is chosen, then the first partial product would not be a two-digit number. Hence, the second factor must be 12.

Activity 2

Write +, −, ×, or ÷ in the appropriate boxes to make the statement correct. Each operation must be used exactly one time.

(1) $9 \square 2 \square 5 \square 80 \square 20 = 19$

(2) $7 \square 6 \square 2 \square 8 \square 1 = 12$

What happens when parentheses are used?

(3) $4 \square 9 \square (8 \square 4) \square 6 = 34$

Strategies might include not subtracting a larger number from a smaller number, not dividing unless the quotient is a whole number, and considering the size of the numbers in relation to the answer. For example, in problem 1, since 80 is so large in relation to 19, it is probably a good idea to first divide 80 by 20. Students may then proceed to try 9×2, since it is close to 19.

Students can explore whether there is more than one way to assign the operations, and if more than one way exists, they can determine the number of different ways the operations can be assigned to produce the correct result.

Activity 3

The problems in this activity are called *number search* problems (Cohen, 1985, pp. 45-46). You are given an initial, or starting number, and your goal is to reach an ending, or final, number by applying two basic operations as often as needed. Try to be as efficient as you can. Consider this sample problem.

Given the number 9, try to reach the number 11 using the operations of Adding 4 and Subtracting 5.

Solution:

$$9 \xrightarrow{+4} 13 \xrightarrow{-5} 8 \xrightarrow{+4} 12 \xrightarrow{-5} 7 \xrightarrow{+4} 11$$

Students can first be asked whether it is possible to reach 11. Given the solution above, they can be asked whether there are other solutions. Another interesting queston is whether one can find a solution for *any* two operations (e.g., Adding 4 and Subtracting 9). Now try these.

(1) Given the number 39, try to reach the number 40 using the operations of Adding 11 or Subtracting 7.
(2) Given the number 107, try to reach the number 17 using the operations of Adding 11 or Dividing by 3.
(3) Given the number 90, try to reach the number 80 using the operations of Multiplying by 2 or Dividing by 3.

Students can use specific strategies as well as more general strategies on these problems. A very powerful general strategy involves comparing one's number to the final or goal numbers. In problem 1, 39 is below the goal of 40, therefore, add 11. Now, 50 is above the goal of 40, therefore, subtract 7. For the same reason, subtract 7 from 43 and continue this process. Some students may realize that problem 1 can be solved by representing the difference, one, as two groups of 11 minus three groups of 7.

In problem 2, one can use the general strategy described for problem 1, but one must also avoid dividing unless the quotient is a whole number.

Again, questions can be raised about whether a solution exists and how many solutions exist. (In problem 1, what happens if one adds 11 seven times and subtracts 7 eleven times?)

Activity 4

In this activity, one is to use each of the digits 2, 3, 4, 5, and 8, to yield a sum of 662. One might write:

$$\begin{array}{r} 524 \\ +\underline{\ 38} \\ 662 \end{array}$$

(1) Use all five digits to yield a sum of 490.
(2) Use all five digits to yield a difference of 778.
(3) Use all five digits to yield a product of 28,290.
(4) Use all five digits to yield a quotient of 882.

These problems can lead to consideration of problems such as the following.

Using the digits 2, 3, 4, 5, and 8, arrange two factors so that multiplication will yield (1) the greatest possible product, (2) the smallest possible product, and (3) the product closest to 15,000.

(1) $832 \times 54 = 44{,}928$
(2) $2 \times 3458 = 6916$
(3) $3825 \times 4 = 15{,}300$

Each of the preceding activities and problems associated with them have many variants. For example, one might begin Activity 2 with only two boxes. Also, the whole numbers in that activity may be replaced by fractions or decimals. Additional problems in Activity 3 can require more than two operations. Variants or problems similar to those presented are easy to construct. Students can be asked to construct their own problems. Usually, they find this assignment very enjoyable.

Activities 1-4 are simple, in that they only involve the four basic operations. However, problems posed in conjunction with these activities can raise challenging questions and provide students with an opportunity to learn important problem-solving strategies.

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CLIMBING TO NEW HEIGHTS IN MATHEMATICS

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MATHEMATICS, MOTION, & GENERALIZATIONS

Vivian Camerlengo
The Pennsylvania State University

Every mathematics teacher is aware of students who respond accurately to items of factual knowledge on a unit test but later are unable to recall them for use in a problem solution. The fact that memory can serve to produce what is needed at the time of evaluation with little or no permanency of conceptual meaning is not surprising to the experienced professional, although it may seem inconsistent with the teaching strategies and activities that have been implemented in the classroom. Teachers ponder this fact and often decide that what is needed is more review of the "basics." This article suggests another possibility to the "more of the same" strategy — a way of linking conceptual knowledge through the use of dynamic motion and generalizations. The approach that will be discussed addresses the structure of the mathematics involved and attempts to minimize the amount of rote recall required from the student. Examples will include an excerpt from a geometry unit on the measurement of angles involving dynamic motion, the use of spatial relationships in a proof, and an application of visualization techniques in the solution of a non-routine problem.

Teaching "Today's" Geometry

The mathematics education community today is taking a second look at the teaching and learning of high-school geometry. Mathematics educators have reviewed the role of rigor and proof, as seen in the van Hiele levels; they have recommended the incorporation of technology through the use of computer packages such as Logo and the Geometric Supposer and they have suggested that we teach geometry on the secondary level from both a synthetic and algebraic perspective. This last suggestion includes objectives that address the student's ability to:

— represent problem situations with geometric models,
— interpret and draw spatial phenomena,
— deduce properties of figures from given assumptions, and
— apply and analyze Euclidean transformations (NCTM, 1987).

Geometry teachers would surely agree with the appropriateness, albeit incompleteness, of this set of objectives. It is difficult, however, to teach the concepts necessary for achievement of these objectives. Students who have completed a high school geometry course often remember it with comments like,

— "Sure glad it is over!"

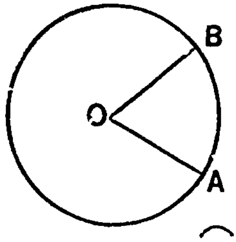
— "Theorems and more theorems to be memorized."
— "Proofs, proofs, proofs!"
— "Don't remember much." ((Martell) Camerlengo, 1982).

An expanded application of the notions of dynamic motion[1] and generalization addresses these student responses. This idea has been applied to high school geometry by the Geometry Problem Solving Project (NIE-G780225) and results in a generalization that can unify several, apparently disparate, theorems and proofs.[2]

One Equals Six!

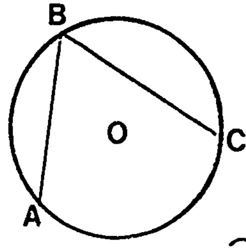
Consider the six "different" theorems presented in a conventional high-school geometry text involving the measurement of angles formed inside and outside of circles.

CASE 1 — CENTRAL ANGLE THEOREM:
A central angle is measured by its intercepted arc.



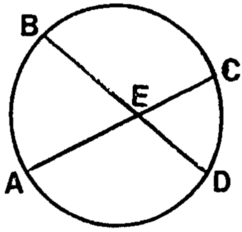
$m\angle BOA = m(\overset{\frown}{AB})$

CASE 2 — INSCRIBED ANGLE THEOREM:
The measure of an inscribed angle is one-half the measure of its intercepted arc.



$m\angle ABC = \frac{1}{2}\, m(\overset{\frown}{AC})$

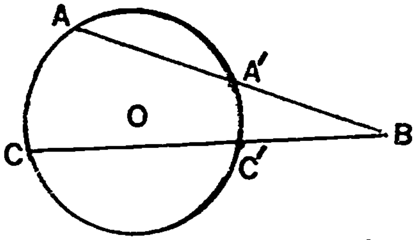
CASE 3 — ANGLE FORMED BY CHORDS:
The measure of an angle formed by two chords intersecting within a circle is one-half the sum of the measures of its intercepted arcs.



$m\angle AEB = \frac{1}{2}(m\,\overset{\frown}{AB} + m\,\overset{\frown}{CD})$

CASE 4 — ANGLE FORMED BY SECANTS INTERSECTING OUTSIDE THE CIRCLE:

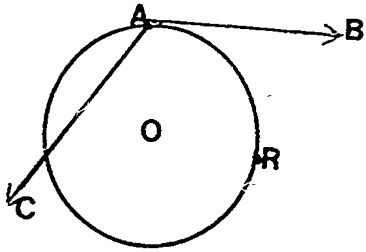
The measure of an angle formed by two secants intersecting in the exterior of a circle is one-half the difference of the measures of its intercepted arcs.



$$m\measuredangle ABC = \tfrac{1}{2}\,(m\overset{\frown}{AC} - m\,\overset{\frown}{A'C'})$$

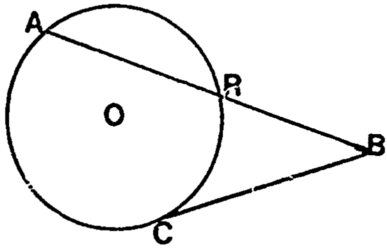
CASE 5 — ANGLE FORMED BY A TANGENT AND A SECANT:

a) The measure of an angle formed by a tangent and a secant intersecting at the point of tangency is one-half the measure of its intercepted arc.



$$m\measuredangle BAC = \tfrac{1}{2}\,m(\overset{\frown}{ARC})$$

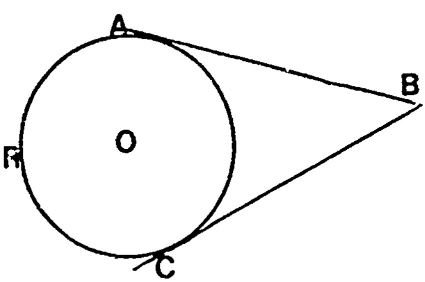
b) The measure of an angle formed by a tangent and a secant intersecting in the exterior of a circle is one-half the difference of the measures of its intercepted arcs.



$$m\measuredangle ABC = \tfrac{1}{2}\,(m\overset{\frown}{AC} - m\overset{\frown}{RC})$$

CASE 6 — ANGLE FORMED BY TWO TANGENTS:

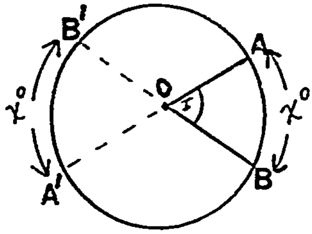
The measure of an angle formed by two intersecting tangents is one-half the difference of the measures of its intercepted arcs.



$$m\measuredangle ABC = \tfrac{1}{2}\,(m\overset{\frown}{ARC} - m\,\overset{\frown}{AC})$$

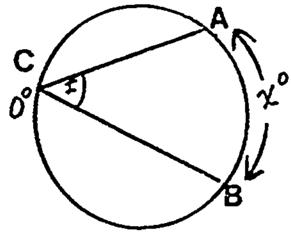
All of the above cases can be reduced to one, Case 3, if we adopt an appropriate convention for the construction of the rays forming the angle and note the location (in motion) of the vertex of the angle. In the diagrams below consider the vertex of angle 1 "moving" from the *center* of the circle, O, (Case 1) to another location (off of O, the center of the circle, but still *within* the circle's interior (Case 3) to a position *on* the circle (Cases 2 and 5a) and eventually to a position *outside* the circle (in the exterior — Cases 4, 5b, and 6). The vertex of the angle is either interior to the circle, on the circle, or exterior to the circle.

In the central angle case we can extend the two radii to intercept an arc of measure equal to the one subtending the central angle; the generalization of angle measure being determined by one-half the sum of the measures of the intercepted arcs holds.



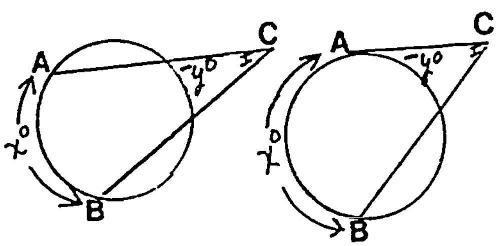
$$m\angle AOB = \tfrac{1}{2}\,(m\widehat{AB} + m\widehat{A'B'})$$
$$= \tfrac{1}{2}\,(2x) = x$$

In the inscribed angle case we can consider the minor arc at the vertex to have the measure of zero degrees. The generalization that the measure of an angle is determined by one-half the sum of the measures of its intercepted arcs applies here.



$$m\angle ACB = \tfrac{1}{2}\,(x + 0)$$
$$= \tfrac{1}{2}\,x$$

As the vertex of the angle moves from the interior to the exterior of the circle, consider the minor arc to be of negative measure; the generalization holds.



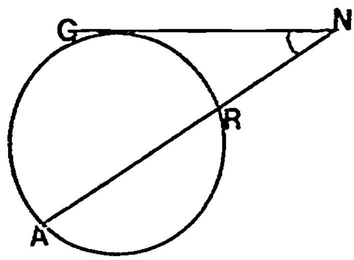
$$m\angle ACB = \tfrac{1}{2}\,(x + -y)$$
$$= \tfrac{1}{2}\,(x - y)$$

Thus, all angles formed by radii, chords, secants, and/or tangents intersecting in the *interior* of the circle, *on* the circle, or in the *exterior* of a circle are determined by one-half of the sum of the measures of the intercepted arcs, noting that the arc may be zero or negative measure depending on the movement and location of the angle's vertex.[3] Not only does one generalization replace six but in addition, the process of visualizing some parts of figures in motion and other parts as stationary allows the relationship of specific components to "completed whole" to be seen from a different perspective.

Visualizing a Proof

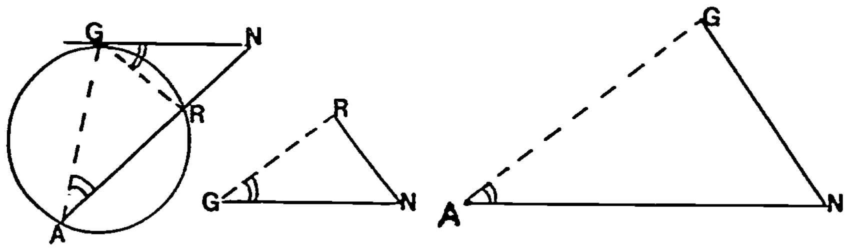
Another application of dynamic motion results in an overview of a proof that can be readily "seen" and provides a plan for generating a logico-deductive corroboration in T-proof format. Consider the theorem:

If a tangent and a secant are drawn to a circle from an exterior point, the square of the length of the tangent segment is equal to the product of the lengths of the secant segment and its external secant segment.



$$GN^2 = (AN)(RN)$$

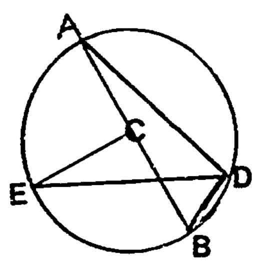
If we use the heuristic of selecting an appropriate auxiliary line and construct $\overline{GA}$ and $\overline{GR}$, we note that both triangle GRN and triangle GAN contain angle N (see figure at top of next page). If we can locate one more pair of congruent angles, we will have similar triangles and can utilize its consequence, the proportionality of corresponding parts. Applying the generalization discussed in the first part of this paper, we note that both angles, NGR and NAG intercept the same arc and therefore, have the same measure; similarity is established! If we "pick-up," reflect, and rotate the two triangles RGN and GAN and arrange them as illustrated, it is clear that the proportion GN/AN = RN/GN can be generated. One more algebraic manipulation produces the desired result, $GN^2 = (AN)(RN)$. Similar "movement" of triangles will yield the other theorems concerning segments formed by chords, secants, and tangents. The reader is encouraged to try these.



Incorporating Problem Solving

In keeping with a problem solving model that posits an interaction between conceptual knowledge and successful heuristic implementation (Kantowski, 1977, 1981), a problem like the following could be introduced during the unit.

A diameter $\overline{AB}$ is drawn through C, the center of a circle. Two points, D and E are taken on the circumference of the circle such that D and E are on opposite sides of AB, and $\overline{CE} \perp \overline{AB}$. Chords $\overline{AD}$, $\overline{DB}$ and $\overline{DE}$ are drawn. Find the measure of the angle formed by chords $\overline{DE}$ and $\overline{DB}$ (GPSP-NIE-G78-0225).



Not only do students need to employ several concepts concerning angle measure but they also need to utilize them in a nonroutine fashion. An insight that could lead a student to solution depends on visualizing D as "moving" and E as fixed by the perpendicularity constraint. Wherever D is fixed, however, the specific heuristic that directs the students to focus on angles that intercept the same arc applies. If the student can employ the notion of reversibility and "see" arc BE along with angle EDB, (s)he will notice the arc subtends *two* angles, ECB and EDB. Knowledge of *one* fact concerning a central angle and its measure (or the one generalization presented earlier) leads to solution. When high-school students used this heuristic approach in conjunction with the generalization relating angles and their measure, 72% were successful in solving the problem ((Martell) Camerlengo, 1982).

Time for Problem Solving

In this article, we employed the notion of dynamic motion of an angle's vertex to collapse several theorems involving the measures of angles in circles into one generalization. In addition, we used the rigid motion transformations of reflection, rotation, and translation in conjunction with appropriate geometric heuristics to visualize an overview and a plan for a geometric proof. Finally, we integrated the movement of one point and the fixity of another in a problem solving situation through the specific heuristic that directs one to focus on angles that intercept the same arc.

Employing dynamic motion and visualization techniques in teaching mathematics, as well as using generalizations to establish meaningful links among properties, minimizes what students have to produce from rote memory. With less time needed for drill on factual and conceptual knowledge, mathematics teachers may increase the time spent on solving nonroutine problems in geometry. This goal is consistent with the first standard of the National Council of Teachers of Mathematics (1987). The concluding comments of the *Standards* document suggest that the quality of instruction in the classroom will improve only with a different perception *of* mathematics along with appropriate student activities (*National Council of Teachers of Mathematics*, 1987). This paper argues for the need of perception *in* mathematics as well — perceptions that are influenced by the use of visualization techniques, dynamic motion, and unifying generalizations. Mathematics in motion may well be mathematics internalized, conceptualized, and available for use in problem solving — a direction we all want our students to follow.

FOOTNOTES

[1]The concept of dynamic motion in geometry encompasses all rigid transformations on the figure as a whole: rotation, reflection, and translation, as well as movement of component parts of the figure within the constraints of the whole. Guilford (1959) referred to these abilities, in his model of the structure of the intellect, as Vz (visualization) and S-R (spatial relations).

[2]The author is currently developing applications of visualization and motion techniques in other domains of mathematics, including algebra and trigonometry.

[3]The ideas presented in this section were developed by the project staff of the Geometry Problem Solving Project at the University of Florida, 1978-82, research supported by the National Institute of Education, grant #G78-0225.

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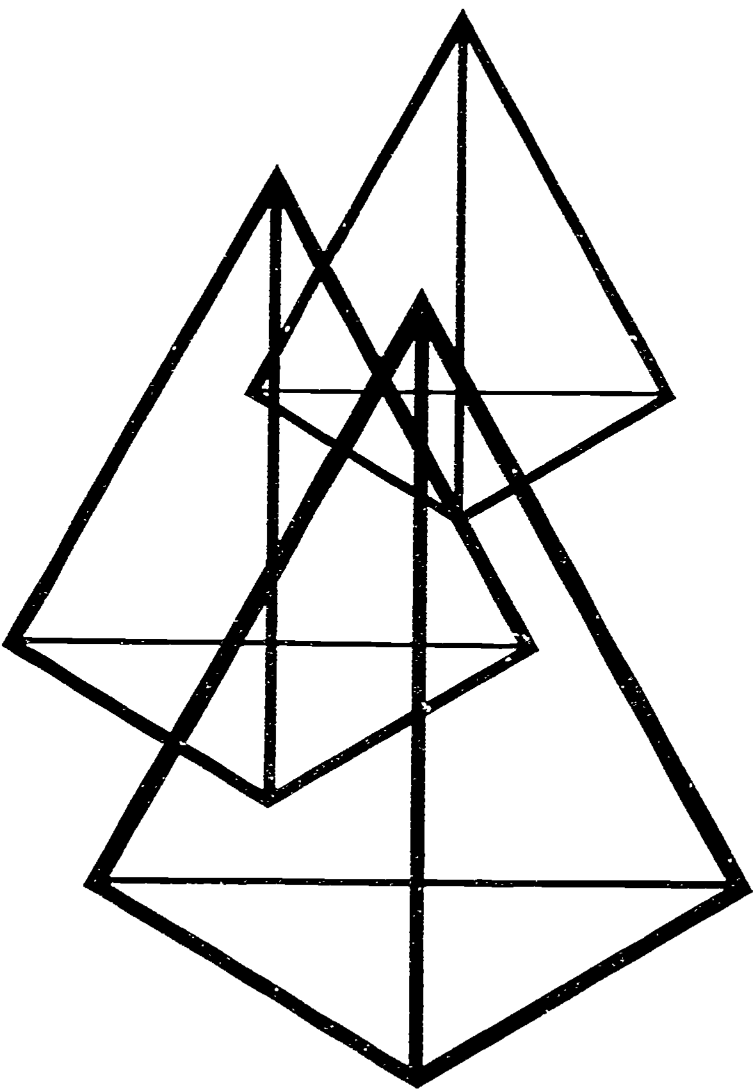

MATHEMATICAL PROBLEM SOLVING: SHOULD WE BE TOP-DOWN, BOTTOM-UP, OR MIDDLE-OUT?

Philip D. Larson
Upper Bucks Christian School

Not long ago an ad appeared in a nationally-distributed newspaper featuring a reprinted version of Ray's *Arithmetics*, a set of volumes used in the nineteenth century. The ad's headline stated, "Let the experts scoff, my child knows math!" At the end of one of Ray's texts appears the following:

> Youth in that generation finished their schoolbooks and then read the Bible, sang from the hymnbooks of Lowell Mason, and read Roman and Greek classics in the original languages. It was not unusual for a blacksmith to carry a Greek New Testament under his cap for reading during his lunch break. The literacy rate, even on the frontier, was higher than today's rate. (Ray, 1985, opposite p. 408)

Some people would assume that students' knowledge of arithmetic and geometry also must have been greater then than it is now. In response to this, proponents of back-to-the-basics might suggest that purchasing texts similar to these nineteenth-century texts would improve students' performance.

Others agree that we must return to the basics — but contend that we also must redefine those basics. One of the new "basics" is problem solving (Coleman, et al., 1983, p. v). "But that's not new," someone says. "Kids have *always* had to solve problems!" However, if problem solving is defined as follows, perhaps one should say that kids have *not* always had to solve problems. Problem solving embraces:

> a situation that involves a goal to be achieved, has obstacles to reaching that goal, and requires deliberation, since no known algorithm is available to solve it. (House, Wallace, & Johnson, 1983, p. 10)

How should schools teach this new "basic?" Should one approach problem solving by refining and sharpening old, tried-and-true methods? One textbook author, John Saxon, seems to believe that the idea of emphasizing procedures is basically sound, but that teachers just need to work the system better. Saxon's textbooks are similar to many traditional textbooks in that both aim to teach students algorithmic skills. Saxon has sharply modified the traditional modular approach, however, by exposing students to algorithms in increments rather than units. In a typical Saxon text, a lesson may begin a new topic, such as negative exponents, however, the accompanying assignment would have only a few exercises on this new topic. Each subsequent assignment would continue to have a few exercises on negative exponents. In these books:

> the learning process is spread out and comprehension will come in time. The emphasis is on review and not on an all out [sic] attack on the new concept. This development spreads out the learning process, increases the depth of understanding and improves long term [sic] retention. (Saxon, 1981, p. vi)

Saxon also has said that "understanding comes downstream from doing;" he assures teachers that "after [students] do a problem like this every night for a year or a year-and-a-half, . . . most of the kids can do the problem and some of them even understand" (Bloom & Saxon, 1988).

Back-to-basics advocates would claim that students need only to practice-practice-practice the basic skills (or algorithms). Becoming well-acquainted with textbook exercises and textbook "problems" (such as coin or age problems) will give students positive experiences from which they will be able to synthesize solutions to many real problems. As students repeatedly practice these skills, they will become problem solvers. In other words, since becoming a successful problem solver requires a foundation of basic procedural skills, teachers should start from the bottom (skills) and move up to problem solving (insight).

Others argue that the way to produce good problem solvers is not to appeal to lower-level thinking by stressing repeated, rote exercises. Rather, one must require higher-level thinking. Following the inspiration of Polya and Wickelgren, children should be taught to do such things as draw diagrams, work backwards, or use some other generalized plan-of-attack (Polya, 1945; Wickelgren, 1974). Since children would be much better prepared to solve the world's problems if they knew a set of general approaches to problem solving, some say we must start at the top and work down to the actual problems!

Two Approaches

Most people likely would agree that students need to be problem solvers. Unfortunately, there is little agreement about how to teach them to become problem solvers. Many opinions seem to be split between two ends of an apparent dichotomy, the bottom-up approach and the top-down approach.

Consider the following illustrations of how these two approaches might influence an instructor teaching the division of fractions. Traditionally the bottom-up approach has been used and students have simply been given a technique. they have been told that they should always invert the second fraction and multiply. After students are skilled in the procedure, they would be given problems to which their procedure would apply. A teacher from this school of thought would place great emphasis on *how* to divide the fractions, the reason *why* would be of secondary importance. If students would ask why, they might even be told, "It's just a rule. That's the way you're supposed to do it."

Top-down instruction would be very different. While a pure top-down approach may not have a sequence of content to cover (as in a Piagetian classroom-without-curriculum), nevertheless the problem of ratio apportionment might arise in an applied situation, and the class might wish to devise a general algorithm for dividing fractions. As indicated in the example below, the result from their exploration might ostensibly be the same as the rule taught from the bottom-up, but, in a top-down approach, the reasons would be stressed much more than the procedural steps.

$$\frac{4}{5} \div \frac{6}{7} = \frac{\frac{4}{5}}{\frac{6}{7}} = \frac{\frac{4}{5}}{\frac{6}{7}} \cdot \frac{\frac{7}{6}}{\frac{7}{6}} = \frac{\frac{4}{5} \cdot \frac{7}{6}}{1} = \frac{4}{5} \cdot \frac{7}{6}$$

If the quadratic formula were being considered in an algebra class, bottom-up teachers would stress the procedures; top-down teachers might not emphasize the procedures, but rather would try to develop each student's intuition by exploiting the geometry of the quadratic formula (such as the symmetry inherent in the plus-or-minus sign). If the simplex method of solving linear programming problems were being taught, bottom-up teachers would emphasize *how* to solve the problem and apply the results; a top-down teacher would lead students to re-invent the simplex method by thinking what should be done graphically in a two-dimensional situation, how corner points could be found via Gauss-Jordan reduction, extrapolating to hyper-space, etc.

How does a teacher choose from these two approaches? Certainly the goal is clear: teachers want to develop problem solvers. But is a bottom-up approach most productive or is top-down a preferable approach? Recent research can cast some light on this question.

Problem-solving Literature

A major premise of top-down problem solving is that traditional procedures should be moved to the side to make way for heuristics, or content-independent strategies such as working backwards, drawing a diagram, etc. Does this policy produce students with better problem-solving skills? Many research studies have found that teaching heuristics or content-general strategies was effective. The results of the following projects are significant.

Since 1950 numerous investigations of mathematical problem solving have been conducted. Marcucci (1980) examined thirty-three studies in which researchers compared instruction between control groups and groups using approaches such as modeling, guided discovery, and heuristics. (In each of these thirty-three studies, problem solving meant solving traditional textbook-type problems.) Instruction in heuristics proved to be the most successful technique at the elementary level; however, these approaches proved to be less successful than even the control treatment in the secondary studies.

Swoope (1983) prepared a ten-week course for second-year algebra students in which they were taught problem-solving strategies. At the end of the course, experimental and control groups were compared for (1) their performance on the mathematics section of the SAT, and (2) their use of problem-solving strategies. While the experimental group means on the SAT were not significantly higher than the control group means, the experimental group *did* score significantly higher on use of problem-solving techniques.

Many students seem to be able to see a problem only one way, but Vissa (1985) discovered that students who had been taught top-down strategies had greater "flexibility in incorporating other heuristic strategies when one alone seemed unproductive" (p. 2431A). Surprisingly, however, Vissa found that in numerous cases "students who had no instruction in the targeted heuristics seemed to develop problem-solving skills comparable to students who experienced such instruction" (p. 2431A). No report was made on whether or not the students in the experimental group were more successful resolving the problems.

In a similar study, Brewer (1981) taught an experimental group of average fifth-graders the four-part heuristic of George Polya: (1) understanding the problem, (2) devising a plan, (3) carrying out the plan, and (4) looking back (Polya, 1945). The control group had the same opportunity as the experimental group to solve problems (and compare results), but they received no instruction in problem-solving methods. Brewer concluded that the experimental group "scored significantly higher on only one criterion, Devising a Plan." She also noted that the "treatment did not produce a significant difference on the total written test scores . . ." (p. 1944A).

Proudfit (1981) also compared fifth-grade students who had been instructed in Polya's four-pronged approach with those who had not. Students in the experimental group were found to be more adept in steps two (devising a plan) and four (looking back). No difference was found in steps one (understanding the problem) and three (carrying out the plan).

In an important investigation by Charles and Lester (1984), an experimental group was taught Polya's model of problem-solving and attention was especially given to providing time for process problems and helping the students to choose and use strategies. They reported that "the experimental classes scored significantly higher than the control classes on measures of ability to understand problems, plan solution strategies, and get correct results" (pp. 15-16).

One may notice that all of the above studies found that students developed superior use of heuristics, but only the last of the studies found that the students were any better at actually solving problems.

Lesh (1985) found that "students more often construct solutions by gradually organizing, integrating, and differentiating unstable *conceptual* structures . . . than by linking together stable procedural systems . . ." (p 320). That is, the students in this study did not usually use automated skills to solve the problems in the research. Lesh also wrote that for researchers to discover the heuristics of experts in order to teach them to novices would probably be of little value.

> It seems plausible that childrens' [sic] conceptions of problem-solving processes, strategies, and heuristics must develop in a manner similar to the way other mathematical ideas . . . are known to develop. Yet, research has seldom viewed heuristics developmentally. (p. 325)

Very often problem solving means simply to teach students Polya's heuristics. But, as Silver (1985) says, "Polya does not present a theory of instruction" (p. 249).

More Problem-solving Literature

What about other literature? Do top-down strategies always come out on top? Not at all. Larkin and others (1980), in reviewing the research, found that poor problem solvers often were prone to use general heuristics while good problem solvers would usually choose content-specific approaches. In examining the differences between an expert and a novice in physics (kinematics), they noted the following:

> The novice solved most of the problems by working backward from the unknown problem solution to the given quantities, while the expert usually worked forward from the givens to the desired quantities. This was surprising, since working backward is usually thought to be a more sophisticated strategy than working forward. . . . Novices having little experience . . . seem to require goals and subgoals to direct their search. The management of goals and subgoals — deciding periodically what to do next — may occupy considerable time and place a substantial burden on limited short-term memory. (p. 1338)

Lesh (1985) found similar results:

> Students who do not have relevant ideas in a particular domain are, in general, poor problem solvers in that domain, even if they have had extensive training in the use of general, content-independent heuristics and strategies. (p. 313)

One may infer from these studies that one well-trained in general problem solving is not likely to be able to compete in a content domain with someone skilled in that domain. Bloom and Broder agree. They found that "methods of problem solving, by themselves, could not serve as a substitute for basic knowledge of the subject matter" (Bloom & Broder, in Mayer, 1985, p. 131).

Some advocates of bottom-up have argued that rotely-learned knowledge can evolve into general concepts. So, if one rotely acquires some declarative or data knowledge, that knowledge can be abstracted into more general procedural knowledge (or concepts). And so, the argument goes, bottom-up learning perhaps can evolve into top-down thinking. Winograd (1975), in studying artificial intelligence, found that the line between declarative and procedural knowledge is not absolute; for example, one can characterize declarative knowledge via a computer program or procedural knowledge, and one can easily characterize procedural knowledge with linear data or declarative knowledge.

Perhaps some of these strange loops between bottom-up and top-down are related to memory abilities. Hunt (1980) suggested that students with high-verbal skills (and apparently greater short-term memory) found it easy to use more strategies in problem-solving tasks. He contended that, "The less effort required [by the memory,] the more capacity there is available for other tasks" (p. 94). So the fact that some students could spend more time in metacognitive activities was related to memory capacity. Those who did not become involved in metacognition were too busy with more basic details.

Needed Research

1. Modern problem-solving methods are often taught in a bottom-up way, even though they are supposed to be the beginning of top-down. That is, students are sometimes given a list of superprocedures (or heuristics) which they are to memorize before trying to solve a problem. Should this be changed?
2. Is the apparent dichotomy between bottom-up and top-down an example of idiosyncratic cognitive styles? Silver has indicated that "individual cognitive styles are modifiable" (Silver, p. 259, 1985). If it is true, then, that cognitive styles can be changed, should one of these approaches be changed to the other?
3. Several of the studies cited took place over a period of less than three months. Perhaps more time is needed to find positive results. Therefore, one might design a longitudinal study to identify students receiving bottom-up instruction and students in a top-down environment. How would these groups of students compare on a general problem-solving test, perhaps something like problems from the MATHCOUNTS competition?
4. Is the best instructional approach to provide rich experiences for our students that allow them to build many content-*specific* schemata? Is this more efficient than teaching content-*independent* heuristics?
5. Do poorer problem-solving students need greater facility with skills than experts? If so, why?

Final Comments

In summary, much research posited in favor of traditional problem-solving showed that students are better able to use heuristics, but it rarely showed that they would be better able to solve problems. Other research showed that students who had only learned content-independent strategies (heuristics) could not compete with those who followed content-specific tactics.

A back-to-basics approach does not seem appropriate, although perhaps some algorithmic skill is needed. Can any insight be gained into division or square roots by learning the traditional algorithms? Most students who learn these procedures (and most teachers who teach them) haven't the slightest notion why these algorithms work. Can one get a better understanding of division or root extraction by learning these traditional methods? Yet it appears that expert problem solvers are experts partly because of their algorithmic proficiency.

One potential problem with the back-to-basics movement appeared in a study (Reed, 1983) comparing students using the texts by John Saxon to students using conventional texts. While students using Saxon's books performed better on algorithmic skills (as one would expect), attitudes toward mathematics degenerated more among students using Saxon's texts than among students using the traditional texts. One might conclude that students find bottom-up approaches boring.

Consider an analogy between learning math and learning automotive repair. Do we expect one who is highly skilled in using wrenches and other tools necessarily to be qualified to fix our own car? Of course not. Likewise, can a back-to-basics emphasis on skills alone prepare a student to be able to solve real problems? Hardly. One would certainly think that emphasizing skills alone could not prepare a student for problem solving.

On the other side, those who would advocate heuristics without skills have too often been like the emperor who had no clothes: the clothing of research simply does not conclusively show any practical value in generalized approaches. The apparent face validity of teaching heuristics has not held up very well under the scrutiny of research.

Those who advocate primarily either a top-down *or* a bottom-up approach to instruction in problem-solving could be considered extremists in their opposition to the other approach. Students who would be problem solvers certainly must have some algorithmic skill, although the needed algorithmic skills may be substantially different from those stressed at present. Yet students also must have developed creativity and the ability to handle unfamiliar problems, they need many things offered by a top-down approach.

The apparent dichotomy between these approaches and the chasm between skill and intuition disappears when one considers that mathematics is partly a facet of our culture. many marvelous ideas have been synthesized in the past, and we will continue to use our creativity to add to the body of knowledge about mathematics. In this cultural model, bottom-up teachers would emphasize looking to the *past*, following ideas that have already been synthesized, and perhaps even encouraging students to understand them. Likewise, top-down teachers would emphasize looking to the *future*, and would be intently preparing students to create new mathematics by various means including reinventing already-existing mathematics. Fr..m this cultural model, it appears that top-down/bottom-up is not an either-or, but is rather a both-and.

Our students need to be well-aware of the past, and they certainly need to be prepared for the future. The task facing us, then, as mathematics educators, is to find the optimum symbiosis of top-down and bottom-up, a "middle-out" approach (Zbiek, 1988), in which both top and bottom levels of thinking are emphasized. Such an approach will eye both the marvels of the past and the excitement of the future. Mathematics students of tomorrow will, I believe, find us using a synthesis of top-down and bottom-up; perhaps we will be using a "middle-out" approach.

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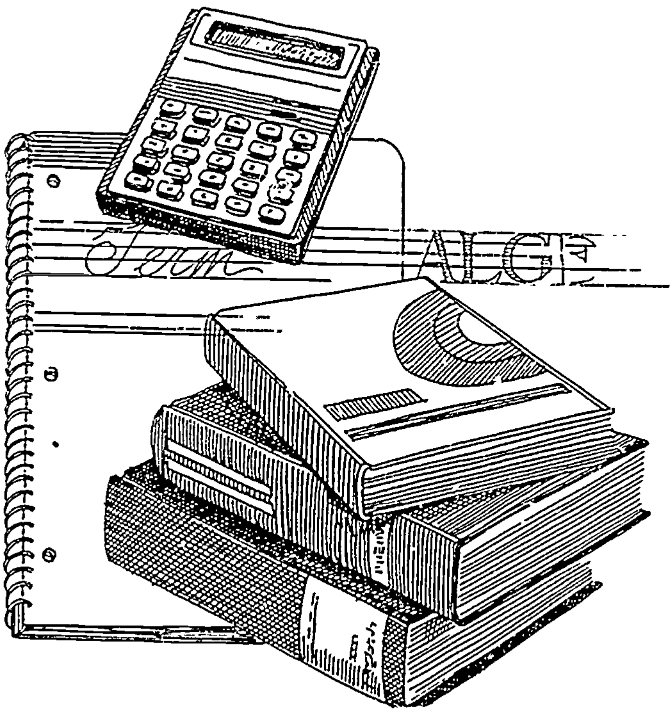

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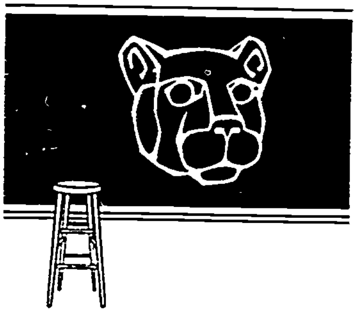

DOCTORAL PROGRAMS IN MATHEMATICS EDUCATION

The Division of Curriculum and Instruction at The Pennsylvania State University has strong and growing masters and doctoral programs in mathematics education.

Graduate teaching and research assistantships are available for the 1989-1990 academic year. In addition there are several other sources of support for minorities interested in mathematics education at the higher and basic education levels. Students typically gain college-level teaching and/or research experience as they pursue graduate work in mathematics education.

Currently, ongoing research projects include a three-year NSF project focused on writing and evaluating an innovative computer-intensive beginning algebra curriculum and a national evaluation of the use of calculators in competitions. Other current specialty areas of the mathematics education faculty include the impact of computing and computer programming on the teaching and learning mathematics, constructivist approaches to the teaching and learning of mathematics, the assessment of attitudes toward mathematics, and the evaluation of the cognitive levels of mathematics textbooks. If you have been considering pursuing a masters of doctorate in mathematics education find out about the Penn State program.

For more information, please contact:

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EVERYDAY MATHEMATICS: LEARNING FROM REAL-LIFE EXPERIENCES

Paul Dobransky
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Jean Kerrigan
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Judy Stopper
Marple Newtown School Distict

One of the most powerful ways for children to achieve in mathematics is to experience constructive real-life mathematical situations in the classroom (Commission on Standards for School Mathematics, NCTM, 1987). Examples involve problems drawn from the child's local environment, for instance, actual and simulated excursions such as estimating gas mileage while riding in a car or estimating the amount of change to be received while shopping at a local store. Primary students might learn to identify coins by using real coins to purchase ice cream from the school cafeteria. Intermediate students can learn about percent by computing sales tax on actual purchases at a local department store or discounts on records or tapes from a music store. Situations like these, which will be referred to as "everyday mathematics," fill the child's world with activities involving thinking skills and estimation. Beginning in kindergarten, students should form the habit of estimating before calculating answers (Mathematics Framework for California Public Schools, K-12, 1985).

> *WHAT IF* . . . I have $10.00 to spend on a music tape, how much tax will I pay? What will be the total cost? How much cheaper is the price in one store than another? What might be a good price? If I can get the tape for 25% off, what will I have to pay?

Mathematics involving such "what if" questions is the mathematics encountered in everyday situations. It is the mathematics one does in coping with the real world. Pencils and calculators are not always used, but thinking skills, approximation, estimation, and common sense are required.

Motivation for Learning

Students who apply skills in their daily lives are much more motivated to internalize the learning of these skills. For instance, students who count money to see if they have enough to purchase cookies are much more motivated than students who count plastic money for no real reason; students counting M & M's so that each student receives an equal amount are more highly motivated than students using plastic counters to learn counting. Representing these experiences on paper or on the chalkboard is more meaningful because students are involved in the action. Students using everyday mathematics are motivated to achieve beyond their grade-level objectives because their goal is to solve real problems.

A challenging activity might involve developing a booklet of everyday problems to be solved by the class. Such a publication can be extremely motivational when students share problems with family and friends. Ownership of problems motivates students, and this increased motivation leads to increased learning.

Emphasis on Interaction

Everyday mathematics can utilize common experiences which stimulate interaction between students and the teacher as well as among students. Since most classes have students who shop at the same stores and attend the same recreational activities, students can work in groups to formulate and solve everyday problems. "Interaction among children is one of the chief motivators for the learning process. Children's thinking and insight are stimulated by explaining their solution processes to others, as well as by the ideas and questions offered within the group. They come to examine a problem more objectively and begin to use new perspectives when they are exposed to different points of view" (Gilbert-MacMillan and Leitz, 1986). This interaction promotes problem solving strategies and develops thinkers The teacher acts as a catalyst to stimulate thinking by responding to the students' inquiries as they think through the development and solution of relevant local problems. Children are encouraged to offer varied approaches to a solution. An example of interaction that might occur follows:

Teacher: All 18 of you brought in your $1.00 for the pizza party. What shall we do?

Student 1: We can get pizza from the cafeteria for $3.00 for each pie.

Student 2: If each pizza can be cut into 8 pieces, we will need 4½ pizzas for each student to get 2 slices.

Teacher: We cannot buy ½ of a pizza.

Student 2: We will have to buy five pizzas. That will cost $15.00.

Student 3: Can we have soda?

Teacher: Six packs of Pepsi will cost $1.89.

Student 3: Then we will need about $6.00 for soda, but we only have $3.00 left.

Student 4: We can get two-liter bottles for $0.79 each. Six people could easily share a bottle. Three bottles times about $0.80 each is $2.40.

Student 5: Mrs. Jones, do you like Pizza?

Teacher: Why do you ask?

Student 5: Well if you don't like pizza, then it would be easier to buy 6 pies, cut them into 6 slices each, and forget the soda. Then we won't have to fight over the extra slices and worry about giving money back.

This type of interaction contrasts sharply with that which occurs when a teacher stands in front of the room with answer book in hand, while students recite their answers to the odd-numbered exercises.

Everyday Mathematics and Textbooks

Because many word problems in textbooks are contrived, it is critical to include locally-developed problems in mathematics lessons. Textbook problems often are so predictable in their makeup that students do not even bother to read them, but just extract the numerical data and compute (Paige, 1988). For example, adding prices from a menu in a local fast-food restaurant may be more meaningful than doing so from a menu in a textbook. Measuring distances that involve local landmarks can be more effective than computing distances in textbook problems. Students improve their understanding of measurement by measuring things meaningful to them. Recording and comparing growth patterns or performances in local "olympic games" are suggested activities. Problems involving personal statistics of students (e.g., height or number of siblings) take on added significance to them.

When students use only word problems from textbooks, there is little transfer to the everyday world. Since the students are consistently working toward the type of solution usually found in answer keys, they may not use higher-level thinking skills nor have the opportunity to answer the "what if" questions commonly encountered in everyday problems.

Everyday Mathematics and Its Integration with Language Arts

Everyday mathematics activities provide opportunities for students to develop oral and written communication skills. Students can discuss their daily experiences and mentally "hold conversations with themselves" to organize language that can be understood by others. Reading and writing skills improve when students are writing about topics they know well and have experienced. Furthermore, an interest in word problems increases when there is a positive response from peers and a sense of ownership. As an example, when students' mathematics problems are duplicated for class members, they are motivated not only to solve the problems but also to improve their language so that peers are able to read and understand their written work.

Conclusions

In today's world, estimation, mental arithmetic, and knowing what operation to use in solving problems are more important than the actual algorithmic method of calculation. However, in many classrooms, students spend most of their time on repeated manipulations. What is even more tragic is that, all too often, worksheets are simply marked as right or wrong, with the assumption that errors are mainly due to carelessness. Everyday mathematics can be highly diagnostic and offers teachers opportunities to detect incorrect methods employed by students and to improve students' problem-solving ability.

If one believes that understanding solutions and developing thinking skills are as important as getting correct answers, then everyday mathematics is a method to consider. Furthermore, if one truly believes that the primary function of school is to help students survive in the real world, then the real world should be the primary source of learning experiences. In this context, everyday mathematics becomes a vital component of the mathematics program in every school.

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FINDING AND USING REALISTIC APPLICATIONS PROBLEMS IN THE MATHEMATICS CLASSROOM

Mary Ann Matras
East Stroudsburg University

Rapidly advancing technology has precipitated recent calls for changes in school mathematics. The availability of powerful computers and calculators in both classrooms and homes and subsequent changes in the content and uses of mathematics have forced a substantial reexamination of the entire school mathematics curriculum.

Recommendations made by professional organizations like the National Council of Teachers of Mathematics (NCTM, 1980, 1987) emphasize using the power of calculators and computers to allow instruction to focus on problem solving, concept learning and realistic applications problems. The search has begun for ways to implement these recommendations in mathematics classrooms at all levels of instruction, and mathematics teachers have the opportunity to explore these ideas in their own classrooms. The purpose of this article is to look at ways that new technology might change the use of applications problems in the mathematics classroom and to suggest ways that teachers might write their own applications problems for classroom use.

Classroom Applications Enhanced Through Technology

Applications problems have always been a part of the mathematics curriculum. Mathematics educators have felt that a central purpose of mathematics education was instruction in solving problems. In the past, however, because of the difficult level of computation or symbolic manipulation involved in realistic application problems, these problems have often been written in such a way that these computations or manipulations could be readily done by hand. As a consequence, many realistic problems were not included in the curriculum or instruction. With computers and calculators available to handle difficult computations or symbol manipulations, mathematics materials and lessons are now free to include realistic applications problems at all grade levels and for a wider variety of purposes than were previously available. Classroom teachers can begin to explore new uses of application problems in instruction.

Using Applications to Introduce a New Skill or Unit

Students at all levels of mathematics learning are apt to question what use the mathematics that they are learning will have to them. Using a realistic application problem as an introduction to a new skill or unit is one way in which teachers may answer those "What use is it to me?" questions and may motivate students to understand and to use new skills. For example, an algebra teacher who is introducing students to quadratics might begin instruction using a projectile motion problem such as one that discusses the height of a pop-up hit in baseball (Fey, 1986). For the teacher who has a classroom computer with a large monitor (or a computer-screen projection device) or whose students are good users of calculators, questions about the height of the pop-up and when it hits the ground are readily answered even on the first day of instruction in quadratics. This situation can then lead to a discussion of the variety of situations where quadratics are used, in preparation for later development of manipulative skills dealing with quadratics. Middle school teachers who are introducing units on decimals can begin with applications involving money. Calculators can be used to handle the initial computations as students and teacher discuss the situation and develop the decimal concepts. The use of technology allows teachers to choose a wider variety of application problems to help with the introduction of units or skills.

Using Applications for Practice of a Newly Learned Skill or Concept

Application problems are also useful for helping students practice a newly learned skill or concept. For example, students learning slope can be asked to look at a group of linear situations, to compute the slope for each situation and to explain what it means both in terms of the situation and in terms of the computational definition of slope. Students who are practicing almost any computational skills may be asked to answer a wide variety of questions based on a table or graph relating to an application problem.

Using Applications to Review or Consolidate Skills

Applications problems are especially good for providing an opportunity for students to review and consolidate skills. Teachers who are planning a review prior to a test or quiz or who are reviewing prerequisite skills prior to introducing more difficult skills may find that a carefully selected application problem will provide a new and exciting basis for classroom discussion. This end-of-the-chapter time might be a good opportunity for students and teacher to discuss longer and more complicated applications problems. For example, the projectile motion problem about the baseball pop-up or one concerning flight in space (Fey, 1986) might be used at the end of the unit to review and consolidate quadratic skills. For review of decimals, a computer-spreadsheet table with information expressed in decimal form may serve as a basis. For review of percents, election year polls and other poll data may provide a beginning point.

Using Applications to Enhance Motivation

Applications can be infused into instruction at times when students particularly need motivation. This motivation may be particularly helpful for students trying to review concepts that have been studied but not yet mastered. Students in these situations often feel that they have already seen all of the materials, and as a result the may turn off instruction. The use of different and unusual applications may make it easier for students to participate in class or understand concepts that they have seen before.

Convincing Students of the Usefulness of Mathematics

Interdisciplinary units filled with applications problems may help students see the usefulness of mathematics in other areas, may help them see mathematics skills in new ways and may, by sheer repetition, help them master difficult skills. Mathematics teachers can construct these interdisciplinary units to coincide with relevant units in other subject-matter areas. For example, when the science teacher is teaching an ecology unit, the mathematics teacher can use information from the science text and from the nearest EPA reporting station to do a graphing unit. This approach allows the science teacher to expect a higher level of mathematical skills from students in the science classroom and allows the mathematics teacher to help students develop a higher level of graphical interpretation skills. Mathematics teachers may team with social studies teachers to present ratio and proportion in the mathematics classroom when map skills are being taught in social studies. High school mathematics teachers may ask science and social studies teachers for copies of their textbooks to use as resources in developing classroom application problems. Through "double teaming" by mathematics teachers and teachers of other subjects, students can often learn difficult skills in one class and then have them reinforced or used in another class.

Summary of Uses

A wide variety of classroom uses of applications problems are available to teachers. Using applications to motivate the need for a skill, to practice a newly learned skill, to practice a number of skills together, to review either prior to a test or before the introduction of subsequent material can help students learn mathematics. The appropriate use of calculators and computers may make it possible to use problems with realistic data. Interdisciplinary efforts by mathematics teachers and teachers of other subjects can help students see the usefulness of mathematics while they learn well in both subject areas.

Finding Realistic Applications Problems

Teachers who seek to use a wide variety of applications in their classrooms must be able to find suitable problems for classroom use. Older textbooks may not have been written with an applications approach; some newer textbooks that have applications problems in them have very simple, unrealistic data. Teachers with excellent texts available will still seek problems for use in classroom instruction, supplemental worksheets and tests and quizzes. While there are a number of excellent supplementary materials (Sharron & Reys, 1979; NCTM, 1980) available for teachers to use in preparing applications problems, teachers who seek realistic data may have to research their own problems.

The most difficult part of writing applications problems involves locating new and interesting situations to meet the specific instructional needs of the students. The location of situations requires detective effort on the part of the teacher. Teachers need to be aware of places where application data may be found and they need to be in constant search of applications.

The newspaper has historically been a major resource in finding situations for applications problems. The sports page has many interesting statistics, the business page has tables and graphs, and the news pages frequently have stories with a mathematical slant. Newspaper graphs are often a good source of ideas and questions. Opinion polls and headlines may be sources for statistics and percent problems. Teachers may provide students with a number of newspapers, and ask them to write their own application problems using recently acquired skills.

Though newspapers remain a good source of applications, a variety of other sources are available for problem situations. One major source is the government. As a collector and user of statistics, all forms of government (federal, state, and local) produce a wide variety of reports filled with applications of mathematics. For example, the EPA publishes reports which give pollution and air quality readings for local reporting stations; these reports are rich with ideas and situations. The Superintendent of Public Documents (Pueblo, CO) prints a great number of free or inexpensive materials full of applications for the mathematics classroom. Regional governments often print pamphlets full of statistics on the local area, while state governments print their own statistical reports.

Textbooks from other subjects and from advanced mathematics can also be sources of problems. A college calculus, physics or business text may contain situations that can be modified for use in particular mathematics classrooms. Copies of current textbooks in other subject areas can also provide a source of problems. School libraries are rich resources of possible situations; even the fiction section can give clues concerning students' interests.

But perhaps the most interesting problems are developed by teacher detectives looking around at the uses of mathematics in everyday life. The placemat on the tray of a fast-food restaurant may contain a table telling the calories and nutrients in their various salads and salad dressings. Such a table can be used to generate addition and subtraction problems for upper elementary students, decimal problems for junior high students or general mathematics students in high school, or a sampling distribution problem for statistics students. The table on the back of a radio station's advertising rate flier can be used to write a linear function problem for an Algebra I test. Doctors' offices offer statistics in brochures on different drugs or diseases, and in children's growth charts and adults' weight charts. Teachers who use applications problems are continually looking for unusual sources of problems. They find that when approached for information, most people are more than willing to help. The cashier at the fast food restaurant will find them a clean placemat to take home, the doctor a copy of a chart, the friend a sales brochure.

Applications problems provide exciting and different ways for teachers to help students become knowledgeable adult users of mathematics. By using new technology available to them, teachers can use more realistic applications problems in their classrooms. And by becoming detectives of sorts, teachers can search for and find new realistic problems for use in instruction.

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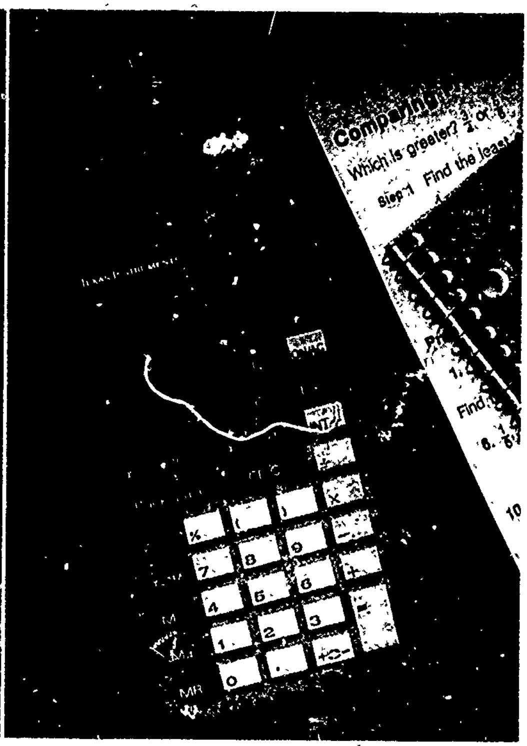

APPLIED GEOMETRY — AN INTEGRATED APPROACH

Bethlynne Cacka
Garden Spot High School

"But when will I ever use this?" How many times have you heard that question? How many times have you attempted to convince students that they may, indeed, need to apply the math that they are learning? Garden Spot High School's Mathematics and Industrial Arts/Technology Education (IA/TE) Departments have collaborated in an attempt to convince at least some of their Geometry students that mathematics is useful.

In a course being piloted in the 1988-1989 school year, low ability math students are participating in an Applied Geometry course that is being team taught by a math teacher and an IA/TE teacher. The course integrates the student of geometry, mechanical drawing, constructions and model making.

The first "seeds" of the course resulted from a faculty room conversation. A Geometry teacher, who had difficulty grasping spatial relations and concepts related to drawing, asked a drafting teacher for some insight into a particular problem. The drafting teacher was quite helpful, but realized that he knew little of the geometric reasoning behind his suggestions. Further discussions lead to the "Wouldn't it be interesting if . . ." stage of planning. From there the ideas were passed on to the mathematics department, where they were received with interest and enthusiasm. Work was then begun on developing the integrated curriculum for the Applied Geometry course.

Todd Reitnouer, an IA/TE teacher at Garden Spot High School, explains the willingness of his department to support the integrated approach:

> Industrial arts has traditionally used a hands-on approach to teaching its content. In the switch in philosophy to Technology Education, we are interested in working some of our learning activities into the courses of other departments. These activities allow the students to see how the mathematics that they study has everyday applications and it also introduces them to new technological areas.
>
> Some of the projects being used in the Applied Geometry course were taken from the basic Mechanical Drawing course. Where we once taught shapes and how to draw them with instruments, we will now include the geometric reasoning behind the constructions. Other projects were newly designed to fit into the units being taught. (Reitnouer, 1988)

The Applied Geometry course is based on three major premises.

1. STUDENTS LEARN GEOMETRY THROUGH A PROBLEM-SOLVING APPROACH.
   The course consists of a series of projects or problems that students are to solve. One of the first projects is the design, layout and construction of a storage box with a lid and compartments to hold the student's drawing instruments. Each student develops a design that is submitted to the class as a whole. The class then studies and compares the designs to decide which one will best suit the purpose.
2. STUDENTS LEARN ON A "NEED TO KNOW" BASIS.
   As a project progresses, students are introduced to the geometric and Industrial Arts concepts that they will need to complete their project. For the storage box project, students are introduced to measurement, perpendicular lines, parallel lines, and board feet.
3. STUDENTS FOLLOW A PROBLEM FROM DESIGN, TO LAYOUT, TO CONSTRUCTION.
   Each project is designed to have students meet three goals:
   (1) to design a solution to the problem,
   (2) to draw a layout for the solution, and
   (3) to construct or build the final project.
   In the storage box project. the student design chosen by the class is blueprinted. Students then plan a list of the necessary materials (board feet of wood, hinges, stain, etc.). In the final stage, students construct their instrument boxes in the school's woodshop.

A course outline for Applied Geometry is given in the Appendix. Also included is an example of one of the course projects.

The planning for the Applied Geometry course took approximately three years. First a presentation was made to the Eastern Lancaster County School Board. The Board received the course with interest and agreed to provide more than $2000 to support the endeavor.

Dr. John Gould, Assistant District Superintendent, explains the district's philosophy:

> In 1984, as part of a curriculum redesign process, secondary teachers were encouraged to analyze their courses for possible integration with courses outside their content areas. This process utilizes the developing research in teaching and learning styles and a need to offset the intuitive understanding that high school curriculums oftentimes fragment a student's perception of learning. Current educational literature calls for schools to develop programs that will help all students to understand the interrelationships between information . . . with the development of Applied Geometry, the district believes that we are beginning to address these relationships and that the course will serve as a prototype for other courses. (Gould, 1988)

Following the Board's approval, the second step required the IA/TE teacher to select the drawing equipment for each student. Because the regular drafting room was not available, portable drawing boards, and quality compasses, architect's scales, triangles, protractors and lead holders were ordered. The geometry teacher reviewed textbooks in search of a geometry book that would support the ideas of the course. Addison-Wesley's *Informal Geometry* was selected, but intended for reference only.

Together the teachers prepared the classroom, replacing desks with cafeteria tables and chairs. A drafting machine was installed and a cabinet was "borrowed" to store the equipment.

A last necessary step in the process of piloting a course is evaluation. A second geometry course was offered to other low ability students using the Addison-Wesley textbook. The course will be taught "traditionally" by the same Geometry teacher that is teachng the pilot. As the year progresses, the Mathematics Department will compare the two classes for ability and knowledge. Based on the final June evaluation, the Applied Geometry will or will not be offered to all low ability students.

Students have been receptive to the Applied Geometry concept and patient with their sometimes faltering teachers. Is the course succeeding? It is too soon to tell—but not one student has asked the dreaded question, "But when will I ever use this?"!

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APPENDIX

APPLIED GEOMETRY COURSE OUTLINE, FALL 1988

I. INTRODUCTION
   - A. Spatial Visualization
   - B. Vocabulary

II. MEASUREMENT

III. GEOMETRIC CONSTRUCTIONS
   - A. Drafting Instruments
   - B. Non-intersecting lines
   - C. Intersecting lines
   - D. Planes
   - E. Circles
   - F. Regular polygons

   PROJECT — Supply Box (detailed in the article)

IV. TRIANGLES

   PROJECT — Use drafting instruments to duplicate a given triangle. Lead to the discovery of the triangle congruence postulates.

   PROJECT — Design and build a geodesic dome from toothpicks.

V. POLYGONS AND POLYHEDRA
PROJECT — Design, layout, and construct a box with a lid (metal shop).
PROJECT — Design, layout and construct 5 different polyhedra and create a mobile. (construction paper)
PROJECT — Design and build a sphere with toothpicks using a combination of geometric shapes.
PROJECT — Draw various crystalline structures. (this project is team taught with a science teacher)

VI. AREA AND PERIMETER
PROJECT — Plot plan
PROJECT — Design and layout a cover for a machine; calculate the surface area.

VII. VOLUME
PROJECT — Design, layout and construct a sugar scoop (metal shop).

VIII. ADDITIONAL IDEAS
A. Discovering geometry in nature through photography
B. Programming in LOGO
C. Introduction to trigonometry through surveying

SAMPLE APPLIED GEOMETRY PROJECT

Construction of a Scoop

I. Geometric Concept — Cylinder
A. (1) Define circumference — example roll a coin along a segment to measure its circumference, measure the coin's diameter and then divide circumference by diameter (c/d). Do for 3 coins. Show answer is the constant pi.
(2) Define volume of a cylinder. Determine the volume of the cylinder.
(3) Define the surface area of a cylinder. Find the surface area of the cylinder.
B. Extend figure to show various types of cylinders.
(1) right cylinders
(2) oblique cylinders
C. Show a scoop and discuss the relationship between the scoop and a cylinder. Allow class to discover correlations.
D. Explain, discuss and demonstrate a truncated right cylinder. Develop and sketch the three basic views: top, side, and auxiliary.
E. Discuss what information would be needed to make a scoop.

II. Mechanical Drawing Section
To make a scoop:
(1) Draw the front and top views of a truncated cylinder.

(2) Divide the top view into convenient number of equal parts.
(3) Project divisions down to front view.
(4) Draw stretchout line (circumference)
a. include tabs
b. divide into the same number of equal parts as the top view (12).
(5) Project lines from front view to stretchout.
(6) Plot curves.
(7) Draw the circular base with tabs.
(8) Design and draw the handle for the scoop.

III. Construct a pattern and transfer the layout onto paper to make the model.

IV. Discussion of the project
A. The class will discuss the various models and decide on the best drawing.
B. The selected drawing will be sent to the metal shop to be made (if possible) as a special project by a metal shop student.
C. Finished product will be returned and discussed by the class.

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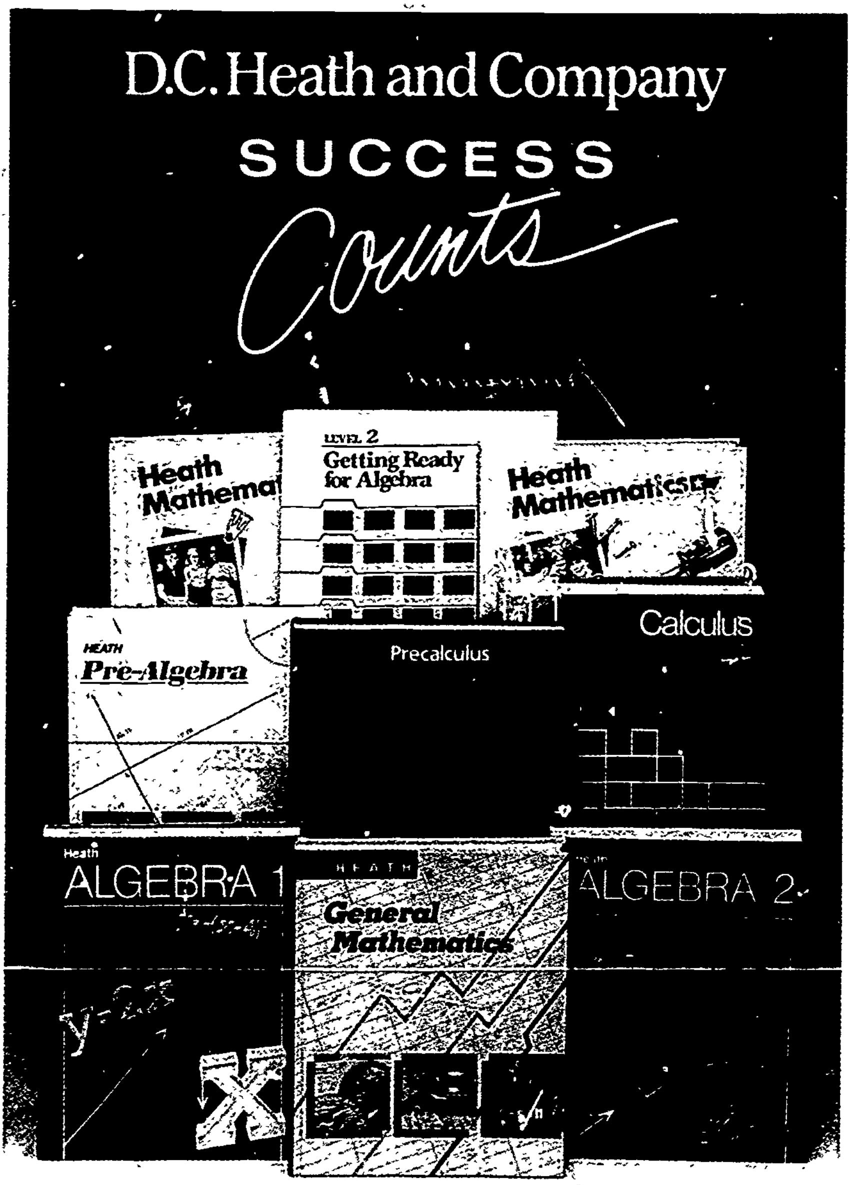
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General Mathematics
ALGEBRA 2

MATHEMATICAL MODELING: A FOCUS FOR TEACHING STUDENTS TO APPLY MATHEMATICS

Glendon W. Blume
The Pennsylvania State University

The existence of computer technology has changed the nature and use of mathematics in many ways. Prior to the accessibility of computers, many applications of mathematics centered on analysis of well-established mathematical representations of physical systems. Since the mid-1970's mathematicians and users of mathematics have given increased attention to formulating mathematical representations that model a process or system and less attention to carrying out the computations associated with such a system (Cross & Moscardini, 1985). In short, the process of building and testing mathematical models has become an increasingly important capability for those who create and use mathematics.

What Is Mathematical Modeling?

To understand the meaning of the term "mathematical modeling," it is helpful to note similarities in several definitions of the mathematical modeling process. James and McDonald (1981) describe mathematical modeling as "the process of translating a problem from its real environment to a mathematical environment, in which it is more conveniently studied, and then back again" (p. 5). Kerr and Maki (1979) describe such translation in more detail. They define the process as one in which the first step is usually the identification of a *real-world problem*. That problem is often modified and simplified so it can be described in a reasonably precise, but succinct, manner. At this point one has formed a *real model*, a written description of the problem. The words and concepts of the real model are replaced with mathematical symbols and expressions. That produces the *mathematical model*. Then one uses mathematical techniques to arrive at conclusions based on the model. These conclusions are then tested and compared with the original real-world problem to determine the appropriateness and usefulness of the model. Similarly, Andrews and McLone's (1976) description of mathematical modeling includes simplification or idealization, translation into mathematical entities, validation of the model, and interpretation of the results of the model.

The diagram in Figure 1 (NCTM, 1987) summarizes the preceding descriptions by illustrating the process of mathematical modeling. Notice that the process begins in the upper left with the identification of a real-world problem situation. That situation often is simplified in some way so an accessible problem can be formulated. A mathematical representation of the simplified situation is constructed and constitutes the "mathematical model." Mathematical transformations or operations are used to generate a (mathematical) solution, which is then interpreted in light of the problem that was formulated. This interpreted solution is then verified to determine whether it offers an adequate (real-world) solution to the real-world problem. At this point, the modeling process may begin again, using a refined version of the model.

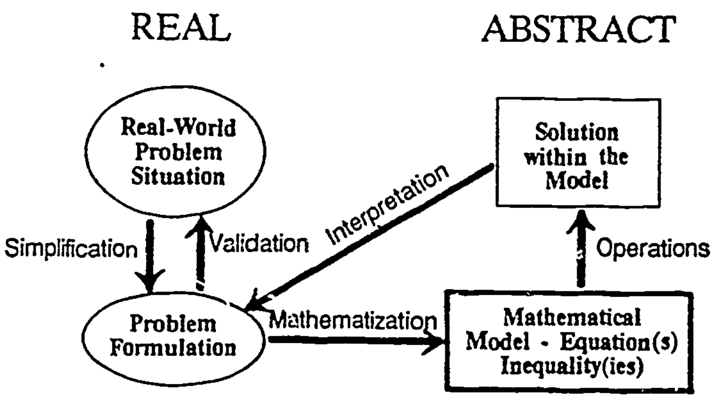

Figure 1. The Mathematical Modeling Process.

An Example of Mathematical Modeling

The following example of an application situation was developed while working with two teachers who were participating in an algebra curriculum development project that emphasizes mathematical modeling.[1] It is intended for use with first-year algebra classes.

The Real-World Problem Situation

In certain states owners of automobiles are required to have a safety inspection done on their vehicles each year. These inspections usually check, among other things, whether headlights work properly, whether brake linings are sufficient, and whether the tires have enough tread on them to provide a safe level of traction. If the tread depth of a tire is less than some minimum amount, the owner is required to replace the tire.

Formulation of the Problem

Suppose an automobile owner wanted to estimate the distance he or she could drive before needing to replace a tire. The owner might base such an estimate on the current tread depth of the tire. Alternately, an owner might want to know how much tread would be needed to be able to drive another 10,000 miles, about one year's driving distance for many people. Hence, the problem consists of determining a relationship between a tire's current tread depth and the number of miles of driving distance remaining for that tire.

There are a number of simplifications and assumptions that arise in the formulation of this problem. For example, one might assume that the tire is of "average" quality, that driving conditions are "normal," and that when the tire is new it wears similarly to what it does when little tread remains. Also, one might exclude indicators of tire quality other than tread depth, for example, uneven wear or sidewall cracking. One also might assume that tread depth is uniform over the entire circumference of the tire and accept a single tread-depth measurement as representative of the tire's tread depth.

The Mathematical Model

To describe remaining driving distance, D, as a function of current tread depth, t, one needs to develop a function rule of the form,

$$D(t) = \text{an expression involving } t.$$

An acceptable function rule can be found by following a chain of reasoning similar to the following. An "average" tire might be sold with a 40,000 mile tread wear-out guarantee. If the minimum tread depth for passing inspection is 2/32 in., and a new tire has approximately 11/32 in. of tread, then the "usable" tread is 11/32 − 2/32, or 9/32 in. Since about 40,000 miles of driving wears away this 9/32 in. of tread, $40{,}000 \div (9/32) = 142{,}222$ provides an estimate of the number of miles of driving distance remaining per inch of tread. Once this is known, the number of miles a tire can be driven can be determined by multiplying the number of usable inches of tread remaining by 142,222 miles per inch of tread. This can be represented symbolically by

$$D(t) = 142222\,(t - 2/32),$$

where t is the current tread depth in inches, $(t - 2/32)$ is the "usable" tread depth, and $D(t)$ is the distance, in miles that the tire can be driven. It should be noted that this model assumes a constant rate of change for remaining driving distance as a function of tread depth, thus giving rise to the linear model described by the function rule above.

Solution Within the Model

Given the preceding model, the driver who measures ⅛ inch of tread on a tire can expect to be able to drive 142222 (1/8 − 2/32), or slightly under 9000 miles on that tire. Similarly, if one is interested in determining whether a tire has 13,000 miles of driving distance remaining on it, one needs to solve the equation

$$D(t) = 13000 \quad \text{or} \quad 142224\,(t - 2/32) = 13000.$$

This model predicts that one needs a minimum tread depth of approximately 5/32 inch to get an additional 13,000 miles of driving distance out of a tire. Note that students need not possess the skills to be able to solve the preceding equation in order to develop and discuss the mathematical model. A calculator could be used with a guess-and-test strategy or a computer could be used to solve directly for t or to generate a table of values or graph from which an acceptable solution could be determined.

Interpretation of Results From the Model

Once the preceding mathematical solution has been generated, it must be interpreted in light of the problem formulated from the real-world situation. At this point one must determine whether the solution is reasonable, the extent to which assumptions that have been made might affect the accuracy of the solution obtained, and whether further refinement of the model is warranted. Discussion likely would center on a tire's uniformity of wear over time, the need to take mutiple measurements of tread depth around the tire, and other factors that might be taken into account in interpreting the solution generated from the model.

The Role of Applications in School Mathematics

If one examines the role that applications play in secondary (and elementary) mathematics curricula, one often finds that applications are "afterthoughts," serving primarily to provide practice on recently-taught skills and concepts. Typical approaches to applications require skills to be developed prior to consideration of applications, and, often, little emphasis is placed on any of the aspects of modeling other than working with the symbolic representation to find a solution within the model. Because most students do not encounter the entire mathematical modeling process, they cannot take advantage of the rich set of processes associated with applying mathematics: making reasonable assumptions, formulating problems from ill-defined situations, translating quantitative, spatial or logical information into a mathematical representation, verifying the validity of a model, and interpreting a solution in light of real-world information. Such *processes* deserve emphasis that often is given exclusively to the *products* (methods of solution and solutions) associated with the teaching of applications.

A focus on mathematical modeling allows teachers to use applications (realistic situations) to introduce concepts rather than to provide practice on skills that have been taught. The existence of technology allows teachers and students to focus on building and testing mathematical models, not just on developing solutions within a given model. Using a computer and/or calculator to aid the process, students can easily build, test, discuss, and then modify a mathematical model for a situation.

The use of mathematical modeling in the school mathematics curriculum can provide interesting, realistic situations (see Swetz, 1987) that can motivate students and develop their ability to become users of mathematics outside of the classroom. Appropriate use of applications should include a focus on:

1. recognizing mathematics in a situation and posing a problem,
2. expressing relationships in mathematical form,
3. generating a mathematical solution, and
4. interpreting the solution and reconsidering the model.

When students encounter a situation that involves mathematical relationships, whether they be numerical, spatial, or logical ones, students should be able to determine how mathematics applies to that situation, pose a problem in mathematical terms, create some useful mathematical representation(s) of the situation, generate a solution, and interpret the appropriateness of that result for the situation at hand. If students can demonstrate such abilities, then their mathematics instruction will have been far more successful than instruction that focuses primarily on practicing the setting up and solving of equations that arise from fairly well-defined, and not necessarily realistic, application exercises.

FOOTNOTE

[1]"Computer-Intensive Algebra" is a project funded by the National Science Foundation (MDR-8751499). This research was supported in part by the National Science Foundation. Any opinions, findings, conclusions or recommendations expressed herein are those of the author and do not necessarily reflect the views of NSF. Donna Bettcher and Mike McRae of Northwestern High School, Adelphi, MD, were responsible for developing and clarifying this mathematical model.

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MATHEMATICAL MODELING

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Introduction

What is mathematical modeling?

Intuitively, most people understand what a "model" is in the physical sense. It is a replication, usually scaled down, of an object. Children may make model boats or airplanes. The model shares many of the properties of the original object: it may have the same features, be the same color and may even function similarly to the object it represents. For example, a model sailboat can float and is propelled by the wind. The model is convenient to work or play with precisely because it does not share all of the properties of the parent object. Properties such as size and weight can prevent us from working with a real object, whereas its model can be easily handled. A model can be readily manipulated and studied, and in the process, information on the parent object can be obtained. The aeronautical features of a supersonic passenger plane often are determined by the use of models in a windtunnel, since the alternative strategy of building a full-scale plane and testing it in a windtunnel would be prohibitive in cost. Physical models supply a valuable tool in many technological and industrial research fields.

Theoretical models also can be constructed. A theoretical model of an object or phenomenon is a set of rules or laws that accurately represents that object or phenomenon in the mind of an observer. When those rules or laws are mathematical in nature, a mathematical model has been developed. Thus, *a mathematical model is a mathematical structure which approximates the features of a phenomenon of concern. The process of devising a mathematical model is called mathematical modeling*. Some basic mathematical structures that frequently are used in modeling situations are. graphs, equations or systems of equations or inequalities, digraphs, index numbers, numerical tables and algorithms. For a civil engineer, the amount of deflection (bending) of a beam under load is important. One could always set up a beam, subject it to a load and measure its deflection, however, this process would be time consuming and expensive. It would be more convenient if a theoretical model for a beam under load existed. Through experimentation, observation and calculation, such a model was found:

$$\text{Deflection} = \frac{PL^3}{48EI}$$

where L = the length of the beam under concern
P = the load
E = the modulus of elasticity, which depends on the material from which the beam is made.
I = the moment of inertia, which depends on the cross-sectional area of the beam.

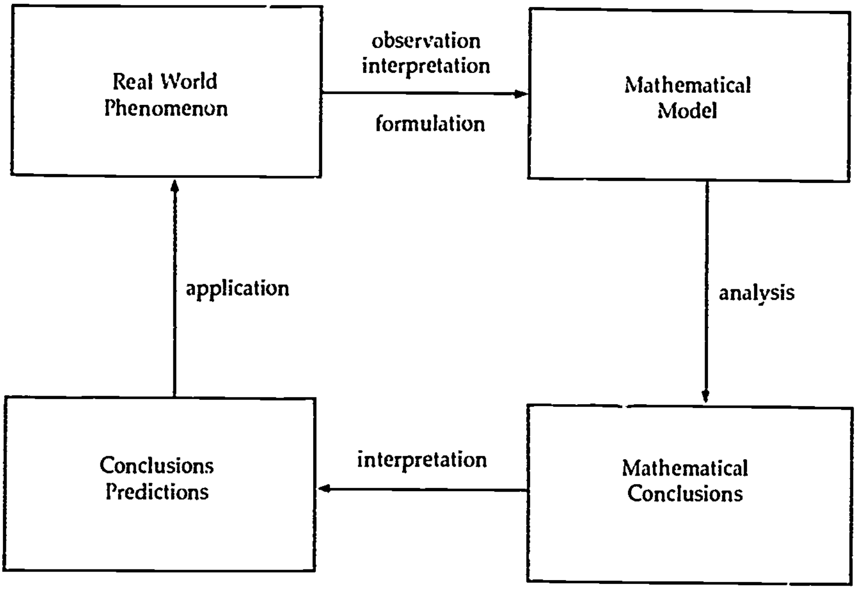
In this instance, the model for deflection is a single equation. Most scientific formulas are really mathematical models of the phenomena they describe.

What is the difference between problem solving and mathematical modeling?

Mathematical modeling is a type of problem solving. While mathematical modeling shares characteristics with many problem-solving situations, it is distinctly different. Frequently, in a mathematical modeling situation, a phenomenon is to be modeled that is seemingly unmathematical in context. It may be an event in the realm of politics, such as the prediction of election results in a particular county, or economics, concerning the long-term behavior of oil prices, or even ecology, when future growth patterns of a forest are to be predicted. These events must be interpreted as problems, their important factors discerned, relationships determined and then interpreted mathematically. The mathematical interpretations of relationships allows for an analysis of the phenomenon, and conclusions (solutions) can be found. Thus, mathematical modeling is a systematic process drawing on many skills and employing the higher cognitive activities of interpretation, analysis and synthesis. The modeling process is comprised of four main stages:

1. Observing a phenomenon, delineating the "problem situation" inherent in the phenomenon and discerning the important factors, (variables/parameters), affecting the problem;
2. Conjecturing about the relationships between factors and representing them mathematically, thus obtaining a model for the phenomenon,
3. Applying appropriate mathematical analysis to the model; and
4. Obtaining results, reinterpreting them in the context of the phenomenon under study and drawing conclusions.

These stages can be schematically represented in the form of a closed cycle:



A fifth stage also could be added to this process, namely, the testing and refinement of the model. Are the conclusions usable? Do they "make sense?" If not, a reexamination of the models, factors and structure is called for, and a possible reformulation of the model may result.

Sample Modeling Exercises

The following modeling exercises are examples of ones that were developed as part of a National Science Foundation workshop for teachers directed by the author during 1987 and 1988. Information about the availability of the complete set of modeling exercises, entitled *Mathematical Modeling in the School Curriculum. A Resource Guide of Classroom Exercises*, can be obtained by writing to the author at the address given at the end of this article.[1]

A Trash Collector's Dream

This modeling exercise is appropriate for students in a General Mathematics course. The mathematics topics involved are basic operations and an introduction to the use of variables and algebraic expressions. Georgia Voegler of the Mechanicsburg Area School District developed this example.

Description of the Problem

A trash collection contractor enters a new development of 200 homes with this proposal: the annual service charge of $300 per household will be reduced by $1.25 for every enrollment over 100 homes. How many homes within the development must the contractor provide services to in order to maximize his revenue?

The Model

The trash collector's revenue is the money that is paid by the enrolled customers. If 100 or less than 100 customers enroll, the collector will get $300 from each one. In this case, the maximum revenue will be 100 x $300, or $30,000.

If more than 100 customers enroll, there will be more customers but each will pay less. The bill will go DOWN $1.25 per household each time the number of customers goes UP.

A table can help to organize this information.

Customers Extra	Total	Bill for Each Customer	Revenue
0	100	$300	$30,000.00
1	101	$300 − $1.25 = $298.75	$30,173.75
2	102	$300 − $1.25 − $1.25 = $297.50	$30,345.00
3	103	$300 − $1.25 − $1.25 − $1.25 = $296.25	$30,513.75

Do you see that if there are three extra customers, $1.25 is subtracted three times? This fact can be used to speed up the calculations. Let n represent the number of customers over 100. Then $1.25(n) must represent the amount to be subtracted from $300 for each bill. Remember, we are looking for the maximum revenue, that means the largest amount of money.

Continue the table, but this time use some larger numbers.

Customers n	100 + n	Bill for Each Customer 300 − 1.25(n)	Revenue (100 + n) (300 − 1.25n)
20	120	$300 − $1.25(20) = $275	$33,000.00
40	140	$300 − $1.25(40) = $250	$35,000.00
60	160	$300 − $1.25(60) = $225	$36,000.00
80	180	$300 − $1.25(80) = $200	$36,000.00
100	200	$300 − $1.25(100) = $175	$35,000.00

From the table it appears that the maximum revenue should occur with the enrollment of between 160 and 180 customers. Continue the table to show these values. Complete this table.

Customers n	100 + n	Bill for Each Customer 300 − 1.25(n)		Revenue (100 + n) (300 − 1.25n)
60	160	\$300 − \$1.25(60)	= \$225	\$36,000.00
65	165	\$300 − \$1.25(65)	=	
70	170	\$300 − \$1.25(70)	=	
80	180	\$300 − \$1.25(80)	= \$200	\$36,000.00

(1) What appears to be the number of customers that will provide the maximum revenue?

(2) Check your result by trying one customer more and one customer less than your answer above.

(3) Does this prove your answer?

(4) Does this type of enrollment plan seem to be reasonable for a business to try?

(5) Find the maximum amount of revenue if the reduction is \$1.50 for each enrollment over 100.

Students also may be encouraged to graph their results as revenue vs. number of customers. They should be able to see the growth of revenue to 170 customers and a decline thereafter.

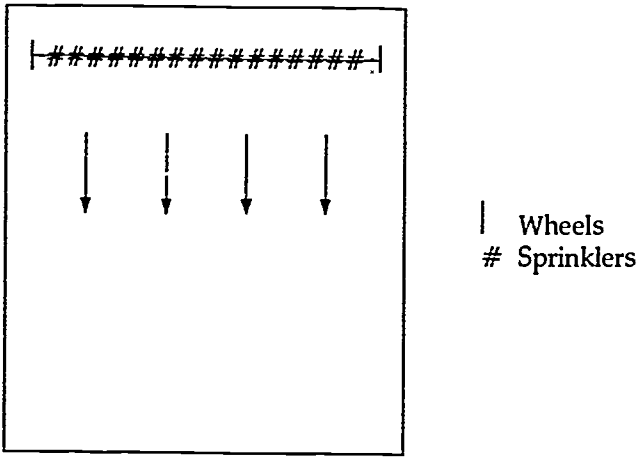
An Irrigation Problem

This modeling exercise is appropriate for students in second-year algebra or precalculus classes. Geometry, analytic geometry and advanced algebra topics are required, and, although calculus could be used, it is not required. This example was developed by Jefferson Hartzler of the Department of Mathematics, The Pennsylvania State University at Harrisburg.

Description of Problem

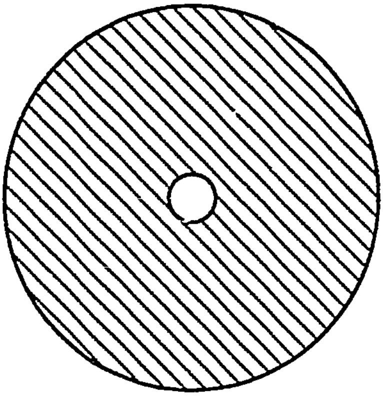
A linear irrigation system consists of a long water pipe on wheels with sprinklers mounted on regular intervals along the pipe. The system moves slowly across a rectangular field to give all parts of the field the same designated amount of water.

The manufacturer wants you to decide how to space the sprinklers on the pipe to give the *most uniform coverage* of water possible. After you have established the best spacing you should also decide how fast the system should move across the field to drop one inch of water in one pass.



Here are the specifications:

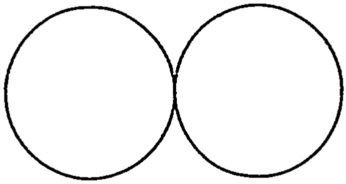
1. Each sprinkler head produces the spray pattern shown below. The flow rate is 5 gallons per minute for each sprinkler. That is, water falls uniformly on the area between two concentric circles with radii 1 foot and 20 feet.



2. The field is 1000 feet wide and 2000 feet long.
3. There should be no more than double overlap of spray patterns to avoid runoff.

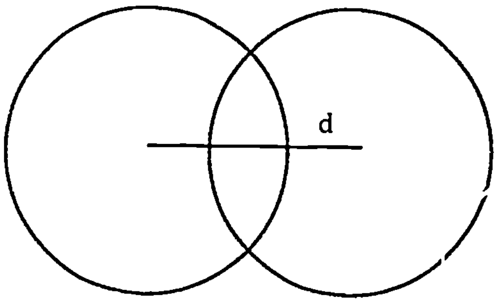
The Model

To have any hope of getting uniform coverage it is necessary for the spray patterns to overlap. What would happen if the outer circles of the spray patterns were tangent to each other?

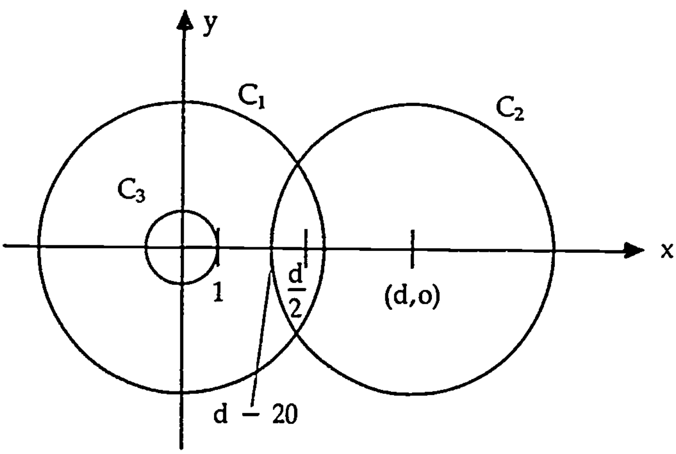


If the distance between the sprinklers is called d, you need to find the "best" value for d so that $20 \leq d < 40$. ($d < 20$ results in at least triple overlap of spray patterns! Why?)

Here is a diagram for $20 < d < 40$.



Now put the circles on a rectangular coordinate system in a convenient way. (It is good enough to look at only two spray patterns, since the same results will occur all along the pipe.)



Now we can write the equations for the three circles:

$C_1 : x^2 + y^2 = 20^2$
$C_2 : (x - d)^2 + y^2 = 20^2$
$C_3 : x^2 + y^2 = 1^2$

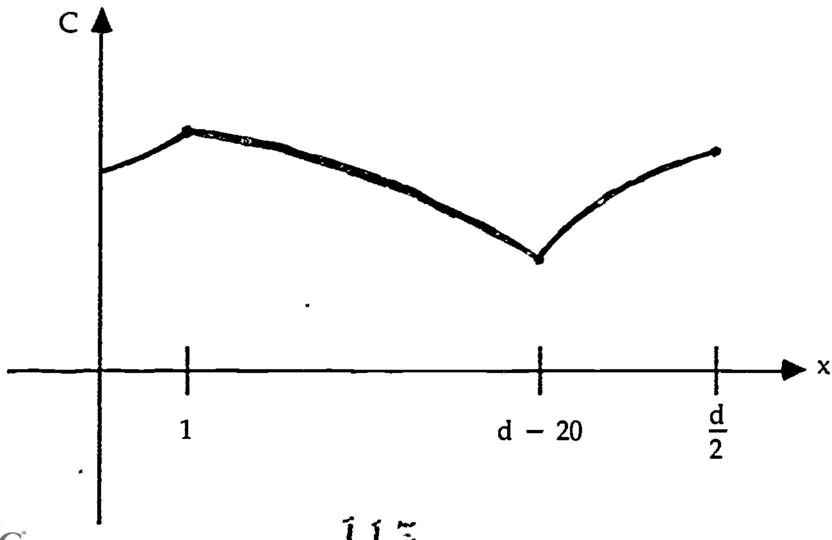
Recall that your objective is to provide uniform coverage of water. This must be translated into a mathematical statement. The amount of water placed on a square inch of land x units away from the y-axis depends on how long that area is under one or more spray patterns. Since the system is moving across the field at a constant rate, the time under spray will depend on the vertical chord lengths through the square inch of land. For some x values there will be one chord and for others there will be two.

Complete uniformity would require that the sum of the vertical chord lengths through x be the same for all x. You can see that this is not possible. The most uniform coverage will then be obtained by selecting d so that the difference between the maximum and minimum chord sums is as small as possible.

Call $C(x)$ the "sum of vertical chords" function.

$$C(x) = \begin{cases} 2\left(\sqrt{400 - x^2} - \sqrt{1 - x^2}\right) & \text{for } 0 \leqslant x \leqslant 1 \\ 2\ \sqrt{400 - x^2} & \text{for } 1 < x \leqslant d - 20 \\ 2\left[\sqrt{400 - x^2} + \sqrt{400 - (x - d)^2}\right] & \text{for } d - 20 < x \leqslant \frac{d}{2} \end{cases}$$

The graph of $C(x)$ will look like this:



You should graph $C(x)$ for at least one value of d by plotting points, using a computer, or using calculus.

From the graph it is clear that the largest value of $C(x)$ occurs at 1 or $d/2$ and the smallest at $d - 20$.

The problem now can be solved by completing the table below and selecting the d value which yields the smallest entry in the last column.

d	C(1)	$C(\frac{d}{2})$	C(d − 20)	$C_{Max} - C_{Min}$
21				
22				
23				
24				
25				
26				
27				
28				
29				
30				
31				
32				
33				
34				
35				
36				
37				
38				
39				

Now you know how far apart the sprinklers should be. How many sprinklers are needed? Where should the end one be placed? How fast, in inches per hour, should the device move across the field so that the *average* waterfall is one inch? What is the most water any portion of land will receive? What is the least?

Suggestions for Further Study

Resolve the problem if triple overlap of spray patterns is permitted. Does this solution give more uniform coverage?

Conclusion

Why incorporate mathematical modeling into the secondary school curriculum?

One of our ultimate goals as teachers is to prepare young people to function confidently and knowledgeably in "real-world" situations. Mathematical modeling is a form of real-world problem solving. The techniques discussed above are exactly those employed by mathematicians to solve the problems they encounter in their workplaces. A modeling approach to problem solving focuses a variety of mathematical skills on the task of obtaining a solution and helps a student see mathematics in a broad spectrum of applications. The strategies and skills learned in modeling exercises are readily transferable to new situations. Students who have been involved in modeling experiences obtain a greater appreciation of the power of mathematics. As one student in an Algebra I class commented after doing a modeling exercise, "Now that's real mathematics!"

How can mathematical modeling be incorporated into secondary mathematics teaching?

Modeling can be incorporated in a variety of ways. There is no need for separate courses or sections of a course devoted exclusively to mathematical modeling. The separation or isolation of mathematical modeling from the rest of the mathematics curriculum tends to raise suspicion in the minds of students — that mathematical modeling is something exotic or difficult. A modeling approach to problem solving and modeling theory should be incorporated gradually and in a low-key manner into the existing curriculum. The relevant mathematics and many of the problem situations are already in place, they merely need a slightly different orientation to become modeling situations.

How can a teacher prepare to undertake modeling exercises with his or her students?

First, the teacher should learn more about mathematical modeling by doing some additional reading on the subject. *The Mathematics Teacher* often publishes excellent articles on mathematical modeling. Second, the teacher should examine existing modeling exercises, such as the examples presented above, to determine their appropriateness for use in a given class. After working through the mathematics, students' potential reactions to the exercises should be assessed. If the anticipated reactions are positive, some of the prepared exercises should be given to the students. Finally, as the teacher gains confidence with the ideas and techniques of mathematical modeling, he or she can seek out other appropriate modeling situations, or better yet, devise his or her own.

FOOTNOTE

[1] *Mathematical Modeling in the School Curriculum* was developed under a project funded by the National Science Foundation (Award No. TEI—8550425). Any opinions, findings, conclusions, or recommendations expressed herein are those of the author and do not necessarily reflect the views of NSF.

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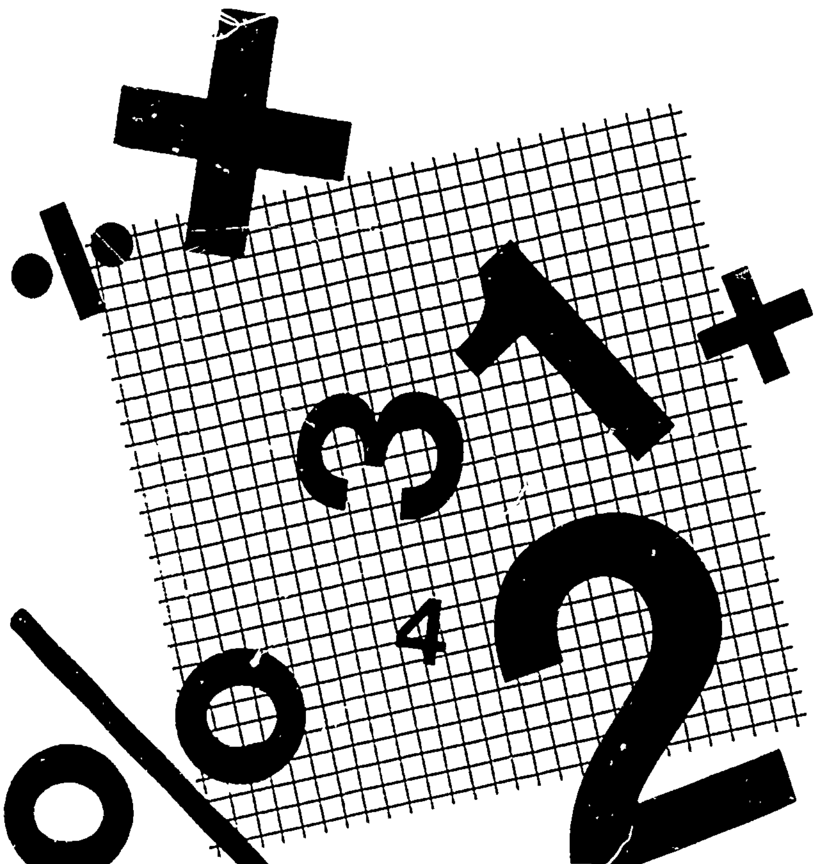

KEEP 'EM GUESSING

Fred H. Jorgensen
Middle Bucks AVTS

From the corner of the room comes this comment: "I know what you are going to say."

What's that?", I ask.

The response: "Give me an approximation."

Hearing that made me feel good—another student had evidently caught on. Seldom does a period go by in which I don't plead (often many times) for students to approximate, estimate, or just plain guess what an answer to a particular problem might be. In this case the student no longer needed prodding, he knew what to do.

Estimation is considered high on just about anyone's list of mathematics priorities today. State, local, and federal recommendations invariably, often quite vigorously, advocate the teaching of estimation in school mathematics classes. The National Council of Teachers of Mathematics has dedicated its entire 1986 Yearbook to the subject.

My aim here, however, is to propose that estimation is not just another topic to be covered if and when time permits. Rather, estimation is a way of teaching regardless of the subject matter, regardless of the grade level at which mathematics is taught and regardless of the "category" of students. For one thing, estimation can be thought of as a tool which one uses to help solve problems of all types and at all degrees of difficulty. Making good estimates, or educated guesses, forces one to at least know what the problem is asking for. Often it leads to a method of solution.

For example, a lesson on addition of fractions with unlike denominators begins with the teacher putting this problem on the board:

$$\frac{3}{4} + \frac{3}{8} = ?$$

Consider all the discussion and thinking that could take place by simply asking the class "What do you think the answer should be?" No doubt, someone will be quick to volunteer 6/12 as an answer, and you might write this down as a "possibility." Patience is the rule here while other students have time to speculate, respond, and possibly argue in favor of additional "possibilities." Someone may come up with the correct answer—possibly even a brilliant observation unpredicted by the teacher. How wonderful! If not, nothing is lost and the teacher can begin instruction with an alerted class.

Estimates can also be used to verify an answer found by other means. That is, if the answer is close to the estimate, it is probably correct.

Suppose, for example, that a student arrives at an answer of 20 inches (Ah! The English system still remains) for the hypotenuse of a right triangle whose legs are 15 inches and 5 inches. Which is the better teacher comment?: "Your answer is wrong," or "Look at the picture and tell me if your answer is reasonable." The second comment would probable stimulate additional thought by the student, whereas the former might only serve to discourage.

This use of estimation (to verify an answer) is particularly helpful when using calculators or computers, since these machines, as useful as they are, do not generally "know" if an answer is reasonable or not. All too frequently, students blindly accept the most absurd electronic output when a simple estimate could have caught the obvious error.

So, by constantly encouraging students to estimate, we are giving them a tool for both analyzing and checking problems. Moreover, in some circumstances estimating is all that is required. At other times, when data is either limited or inexact, estimation is all that is possible. These are additional reasons for stressing estimation in mathematics classes. When will you finish your homework? How many points will our team score? How long before we reach our destination? These are examples of questions which cannot, or maybe need not, be answered with precision or certainty, but may never-the-less require answering—albeit with an estimate.

Teachers might consider giving whole assignments that are focused on estimating or mental computation. This can be particularly appropriate when introducing a new topic. Ask your students to "try by any means" problems on the not-yet-learned material. Suppose your next topic in Trigonometry is conditional equations. Why not assign a list of such problems without any instruction other than to say "Find as many values of the variables in these equations which make the equations true." Some students may not be able to do this on their own, but others may surprise you with their ingenuity. In any case, as with the earlier problem on addition of fractions, they are likely to be ready or even anxious for your upcoming instruction.

Teachers can use student estimates as a teaching aid. If a student's estimate is "off target," it may reveal reasons for his or her misthinking. An answer of 30 for the approximate square root of 93, for example, may mean that the student is confusing the square root of a whole number with an even number of digits for one with an odd number of digits.

Finally, estimation is timeless. While many other tools and methods we use or topics we choose may be here today and gone tomorrow, estimation is here to stay. That is, as long as teachers continue to teach students to think instead of just producing answers which have little or no meaning to them, estimation will be an important part of mathematics instruction.

As we search for *New Directions for Mathematics Instruction* in the 1990's, indeed in any decade, let us include a strong does of estimation as an essential ingredient. Day, after day, after day, let's keep our students guessing, guessing, guessing. They will enjoy it!

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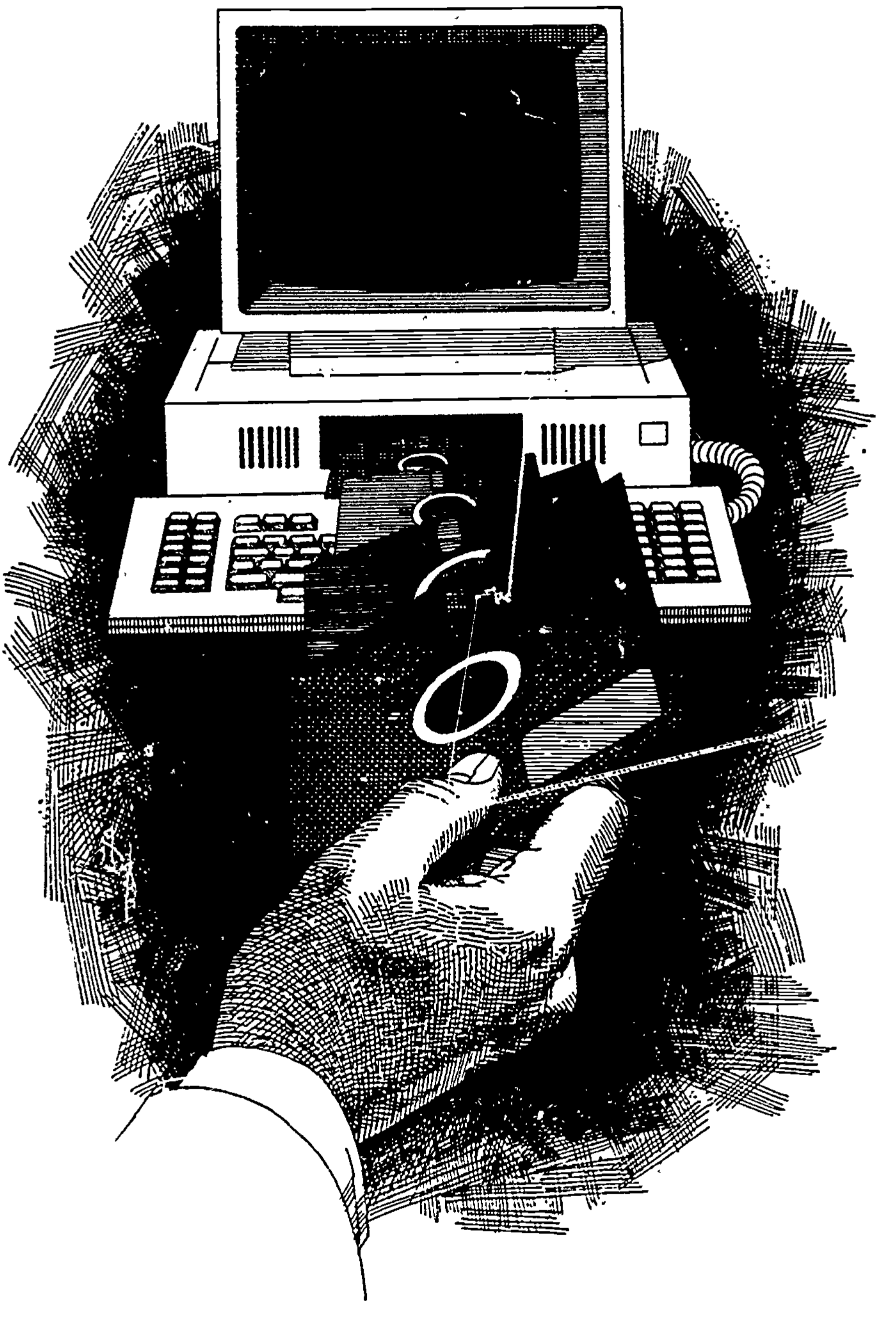


GEOMETRICAL PROBABILITY

Karen Doyle Walton
Allentown College of St. Francis de Sales

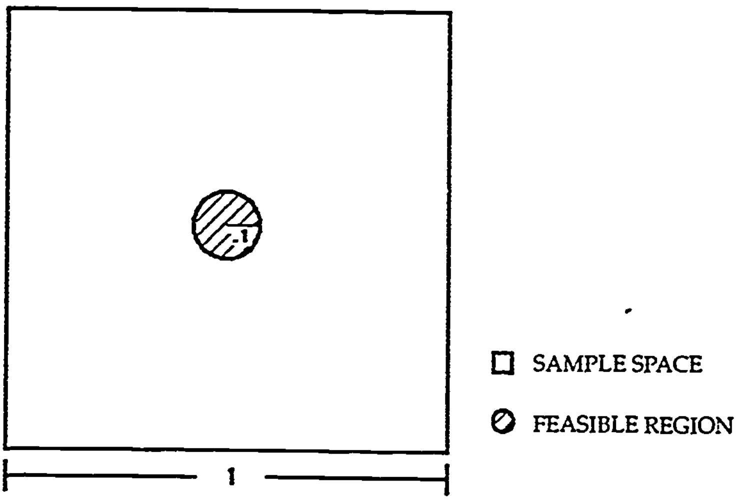
Probability is an area of mathematics that has been given little attention in the traditional mathematics curricula of most elementary and secondary schools in the United States. Recognizing this deficiency, the National Council of Teachers of Mathematics (NCTM) has drafted *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1987), which prescribes the inclusion of probability and statistics at each level of a child's education.

Standard 10 states that students in grades K-4 should be introduced to probability in terms of simple experiments through investigation and actual involvement, generating results and exploring the ideas of chance and likely outcomes. Students in grades 5-8 are to be involved in modeling and simulation as well as the study of both empirical and mathematical probability. In grades 9-12, while continuing to study experimental and theoretical probability, students should apply the concept of randomness to simulations as well as actual probabilistic experiments (Burrill, 1988).

Geometrical probability is an approach to the subject which represents a new direction for mathematics instruction in the traditional K-12 curriculum. The NCTM is not alone in recognizing the importance of geometrical probability. The Consortium for Mathematics and its Applications (COMAP) received funding from the National Science Foundation which enabled Fred Djang (1988) to author "Applications of Geometrical Probability," a teaching module which introduces probability by using experimentation, estimation, and calculation to model real-world problems. Richard Dahlke and Robert Fakler's (1986) "Applications of High School Mathematics in Geometrical Probability" was published by COMAP in cooperation with the Society for Industrial and Applied Mathematics, the Mathematical Association of America, the American Mathematics Association of Two-Year Colleges, and the NCTM—organizations spanning mathematics education from kindergarten through the postdoctoral level, and attesting to the consensus on the importance of geometrical probability.

What is Geometrical Probability?

Geometric probability is a means of discovering the likelihood of a real-world experiment by translating the problem into the random choice of points in a geometric region. The model can be illustrated by throwing darts at a one-unit square dartboard on which a circle of radius 1/10 unit has been drawn. Assuming that the dart throws will be distributed randomly on the dartboard, the question is asked "What is the probability of a dart landing inside the circle?" The dartboard is called the *sample space* and the region inside the circle is called the *event* or *feasible region*. The probability of a dart landing inside the circle is equal to the area of the circle divided by the area of the square, as illustrated in Figure 1.



$$\text{PROBABILITY} = \frac{\text{MEASURE OF THE FEASIBLE REGION}}{\text{MEASURE OF THE SAMPLE REGION}}$$

$$= \frac{\text{AREA OF THE CIRCLE}}{\text{AREA OF THE SQUARE}}$$

$$= \frac{\pi\ (0.1)^2}{1}$$

$$= (0.01)\,\pi$$

$$= \pi\,\%$$

Figure 1.

Why Study Geometrical Probability?

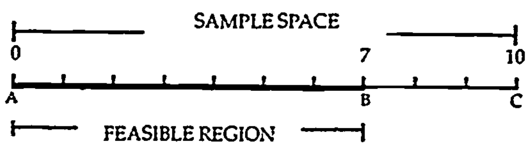
Geometrical probability is an intuitive topic replete with interesting problems appropriate for all levels of mathematics instruction. Each problem is translated into a mathematical model dealing with continuous (as opposed to discrete) sample spaces and events. The process is the foundation of problem solving, ranging from calculating simple areas, graphing inequalities, and applying theorems, and leading to calculus-based probability and statistics on the college level (Dahlke and Fakler, 1986).

The following problems illustrate the potential of geometrical probability to provide fertile opportunities for creative problem solving, integrating various areas of mathematics and computer learning.

Examples of Geometrical Probability

Problem 1: (Djang, 1988) Lisa rides to work on a subway which runs every ten minutes. If she arrives at the subway stop at a random time in the morning, what is the probability that her wait for the subway will be more than three minutes?

Mathematical Solution: Since Lisa arrives at a random time during a ten-minute period, the ten-unit line segment below can be used as a mathematical model to represent the *sample space*. The segment from 0 to 7 represents the *feasible region* for waiting more than three minutes. Hence the probability of waiting more than three minutes is the length of the feasible region divided by the length of the sample space.



$$\text{PROBABILITY} = \frac{\text{MEASURE OF THE FEASIBLE REGION}}{\text{MEASURE OF THE SAMPLE SPACE}}$$

$$= \frac{\text{LENGTH OF SEGMENT AB}}{\text{LENGTH OF SEGMENT AC}}$$

$$= \frac{7}{10}$$

Figure 2.

Problems such as this one lend themselves well to the introduction of the concepts of chance, randomness, and likely outcomes. A successful motivational device that is consistent with Standard 10 of the draft of *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1987) is to have students experience an empirical approach, perhaps followed by a computer simulation, before the mathematical solution is discussed.

Actual Trials: Students could construct a spinner with 10 units on its face. After each spin, the outcome is recorded, ignoring spins which land on 10 or 7. This is permissible since the point on the spinner corresponding to a number such as 7 or 10 will, theoretically, be infinitely small, making the probability of landing on that point zero. After recording the data from 100 spins (excluding those landing on 10 or 7) as in Figure 3, the empirical probability can be calculated. Students can perform additional trials (e.g., 200 and 300) and observe changes in the relative frequencies. The teacher can introduce such terms as sample space, feasible region, experiment, trials, events, relative frequencies, complementary events, successful trials, and randomness in a natural, unforced setting.

Figure 3.

			Relative Frequencies	
Number of Trials	Outcomes 0-7	7-10	Waiting More Than 3 Minutes	Waiting Less Than 3 Minutes
100	62	38	62/100 = 0.620	38/100 = 0.380
200	135	65	135/200 = 0.675	65/200 = 0.325
300	212	88	212/300 = 0.717	88/300 = 0.293

Computer Simulations: A computer simulation of the problem provides additional motivation and reinforcement. A class with minimal computer programming experience can write a simple BASIC program such as the one below, or the teacher can provide the program for a class with no programming experience.

```
10 HOME
20 FOR I = 1 TO 10
30 X = RND (1) * 10
40 PRINT X
50 NEXT I
60 END
```

Note: In some versions of BASIC, RND(1) generates random numbers in the interval [0,1).

Since each run of the program generates 10 outcomes ranging between 0 and 10, results can be tallied for 100 trials, 200 trials, etc., as illustrated in Figure 4. Relative frequencies can be calculated.

Figure 4.

Number of Trials	Outcomes 0-7	7-10	Relative Frequencies: Waiting More Than 3 Minutes	Waiting Less Than 3 Minutes
100	65	35	65/100 = 0.650	35/100 = 0.350
200	139	61	139/200 = 0.695	61/200 = 0.305
300	204	96	204/300 = 0.680	96/300 = 0.320

Comparisons of the results of the actual spinner trials, computer simulation, and mathematical solution can generate thoughtful discoveries and questions.

The following geometrical probability problem also lends itself well to solution by these three methods.

Problem 2: (Djang, 1988) At a county fair a game is played by tossing coins on a large table ruled into congruent squares with sides of 5 centimeters. If a coin lands entirely within a square, the player wins a prize. (It is assumed that the markings on the table have no thickness.)
Actual Trials: Using a thin pen, draw a grid with lines 5 centimeters apart on a large piece of posterboard. Place the grid on the floor and allow students to throw pennies onto it, tallying the number that fall entirely within a square. Repeat the trials 100 (or 250, or 500) times, as in Figure 5.

Figure 5
Sample Trials

Successful Throws	30	81	160
Total Throws	100	250	500
Relative Frequency of Successful Throws	30/100 = 0.3	81/250 = 0.32	160/500 = 0.32

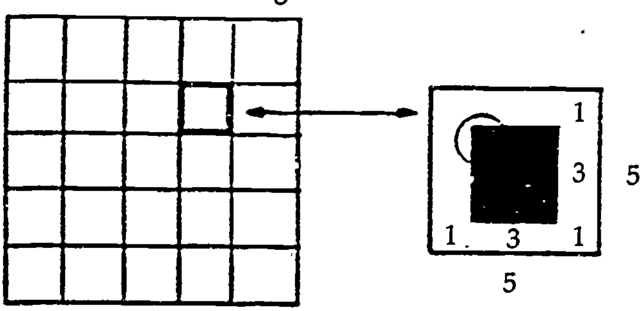
Computer Simulation: Run the program PENNY TOSS listed in Figure 6, letting T = 100. Record the results and repeat for T = 250 and 500. Observe that line 80 randomly generates x- and y-coordinates for the center of the tossed coin. Line 100 determines whether the coin falls within the square and counts the number of successful trials. Line 120 computes the relative frequency of successful trials.
Sample Runs: When 100 pennies are "tossed" the relative frequency of success is 0.34; when T = 250, the relative frequency is 0.364; when T = 500, the relative frequency is 0.378.
Mathematical Solution: Using geometrical probability, the experiment can be considered as randomly choosing a point of a region. Each outcome of the experiment becomes the position of the coin's center on or within whichever square it falls. In the diagram below, a single square is isolated

from the grid since a single square and its interior can be considered the sample space.

Figure 6.

```
10   REM PENNY TOSS
20   HOME
30   S = 0
40   PRINT: PRINT "HOW MANY TOSSES DO YOU WANT?"
50   PRINT: PRINT "T SHOULD BE LESS THAN OR EQUAL TO 1000"
60   INPUT T
70   FOR I = 1 TO T
80   X = 7 * RND (1):Y = 7 * RND (1)
90   IF (X < 1) OR (X > 6) OR (Y < 1) OR (Y > 6) THEN GOTO 80
100  IF (X > 2) AND (X < 5) AND (Y > 2) AND (Y < 5) THEN S = S + 1
110  NEXT I
120  PRINT: PRINT "THE RELATIVE FREQUENCY OF SUCCESS IS
     ";S / T
130  PRINT: PRINT "DO YOU WANT TO TOSS PENNIES AGAIN?"
140  INPUT C$
150  IF LEFT$ (C$,1) = "N" GOTO 170
160  GOTO 20
170  END
```

Figure 7.



A successful outcome occurs if the coin is interior to the square. Since the diameter of the coin is 2 centimeters, its center must lie at least the radius length (1 centimeter) from the boundary of the square region. The feasibility region is the shaded square region of side $5 - (1 + 1) = 3$. Therefore the probability of success, P, equals the area of the feasibility region divided by the area of the sample space: $P = 3^2/5^2 = 9/25 = .36$. Compare results from the three methods.

Conclusion

Numerous additional problems are presented in "Applications of High School Mathematics in Geometrical Probability" (Dahlke & Fakler, 1986) and "Applications of Geometrical Probability" (Djang, 1988), which includes examples, black-line masters for transparencies, and supplemental exercises with solutions. Additional problems for grade seven through the beginning college level can be found in the Additional Readings on Geometrical Probability that follow the references. Approaching problems in more than one way encourages students to create, apply, reason, and communicate about mathematics. Success at problem solving, modeling, and simulation empowers students to become articulate, confident, and able users of mathematical reasoning.

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NEW DIRECTIONS FOR CALCULUS: THE POTENTIAL FOR A COMPUTER-BASED APPROACH

M. Kathleen Heid
The Pennsylvania State University

The problem is that everybody talks about it and *everybody* has suggestions on what to do about it. The topic is not "the weather" but the content of introductory calculus courses. The purpose of this paper is to outline some of the issues and recommendations surrounding the calculus debate, to discuss one major force for change, and to illustrate one specific curriculum that provided a new direction for calculus responsive to those recommendations for change.

The Present Status of High School Calculus

Many students arrive in introductory college calculus courses with some high school exposure to the subject. In a survey of Pennsylvania high school calculus teachers, for example, Wilson (1987) found that eighty-four percent of the teachers surveyed believed that their students were taking calculus in high school for the express purpose of preparing themselves to succeed in the college course. They, and their teachers, presume that a high school introduction to calculus will give them a "leg up" on the calculus course they will take in college. As early as 1973, Mullenex and Neatrour (1973) reported that 56 percent of the Virginia high school calculus students actually took the course over in college.

In spite of this rather widely accepted practice of taking high school calculus as preparation for its repetition in college, nevertheless, there is ample evidence that this practice has not been successful. In research studies that span almost twenty years, McKillip (1965) and O'Dell (1983) have found that students who follow one or two semesters of high school calculus with a college calculus course do not perform better in the college course than students with no high school calculus.

At present, the only clearly discernible advantage attributable to high school calculus occurs for students who take the AP calculus course. These students regularly outperform their college counterparts on measures of calculus performance (Fry, 1973; Haag, 1977; Dickey, 1982). This advantage was recognized in a February 1986 statement from the National Council of Teachers of Mathematics and the Mathematical Association of America: "MAA and NCTM recommend that all students taking calculus in secondary school who are performing satisfactorily in the course should expect to place out of the comparable college calculus course." One might think that, with such a consistent message from the research and with such an unequivocal position taken by the two largest bodies of mathematics educators in the country, the issue is settled. Au contraire — the debate has just begun.

New Directions for College Calculus

Since the time of the joint NCTM/MAA statement on high school calculus, there have been significant efforts to rethink and reformulate the beginning calculus curriculum. During the past few years, the Tulane Conference (Douglas, 1987) suggested that the new introductory calculus courses be "lean and lively" and the follow-up conference in October 1987, "Calculus for a New Century" (Steen, 1988), pointed out the need to give a greater number of students a deep and practical understanding of calculus. Following the lead of those conferences, colleges and universities across the nation are now working on radically reformed versions of college calculus. These versions, as they emerge, are likely to be characterized by a deemphasis on symbolic manipulation skill, an increased emphasis on applications, and a considerable role for the graphics and symbolic-algebra capabilities of calculators and computers. While the carefully designed AP calculus curriculum assured success in learning the concepts and skills of the "old" calculus, its success with the new calculus courses is yet to be tested.

New Directions for Calculus in High School

Forces for change in the treatment of calculus are coming from the secondary school level as well. The National Council of Teachers of Mathematics, in its *Curriculum and Evaluation Standards for School Mathematics*, has proposed that the core curriculum for school mathematics in grades 9 through 12 include the "conceptual underpinnings of calculus." According to the 1987 draft of the *Standards* (NCTM, 1987) the secondary mathematics curriculum should include:

"the informal exploration of calculus concepts from both a graphical and numerical perspective so that all students can:
— determine maximum and minimum points of a graph and interpret the results in problem situations; and
— investigate the concepts of limit and area under a curve by examining sequences and series."

In addition, the *Standards* recommend that all college-intending students have an understanding and an ability to apply the concepts of limits, area under a curve, rate of change, and slope of a tangent line in addition to the ability to analyze a variety of function graphs. These goals are probably consistent with those in the college calculus reform movement. Moreover, when the Standards are implemented, students will enter college with many of the basic understandings on which the newly-designed college course could profitably build.

The Computer: A Preeminent Force for Change

One of the most powerful forces for change in the introductory calculus curriculum has been the growing availability and sophistication of computing technology. Calculators and computers provide students with finger-tip access to function graphs, limits, derivatives, and definite and indefinite integrals. Programs like *Mathematica, Derive* (the successor to *muMath*), and *Maple* now provide the personal computer user with automatic graphical and symbolic manipulation ability within a single package. Calculator companies are making similar strides.

The Hewlett-Packard HP-28S, for example, is a recently available calculator that can perform differentiation either directly or in a step-by-step mode. For example, to compute the derivative with respect to x of $4x^3$, the user merely types in $4x^3$ on one command line and x on the next command line, and then depresses the differentiation key. In a step-by-step mode the conversation between the user and the calculator looks like the following:

User Command	Calculator Display	Interpretation
δ X (4*X^3)	δ X (4*X^3)	$d/dx\ (4x^3)$
Differentiate	4 * δ X (X^3)	$4\ d/dx\ (x^3)$
Differentiate	4*(δ (X)*3*X^(3-1))	$4\ (d/dx(x))\ (3x^{3-1})$
Differentiate	4 * (3 * X ^ 2)	$4\ (3x^2)$

The HP-28S also has a limited capacity to perform symbolic integration and the capability of producing function graphs. Other calculators now available feature function graphers and incorporate some of the numerical calculus capabilities (numerical integration, for example).

The existence and widespread availability of this kind of computing technology now makes it feasible to construct calculus curricula that focus on major ideas, concepts, and applications, instead of on by-hand symbolic manipulation.

A Computer-based Conceptually Oriented Calculus Curriculum

I constructed and field-tested one such curriculum with an introductory applied calculus course at a large university (Heid, 1988). In the context of a fifteen-week course, the experimental classes devoted the first twelve weeks to the concepts and applications of calculus and used the computer to take limits, to compute derivatives and integrals, to solve equations, and to graph functions. Instruction on by-hand symbolic-manipulation skills was reserved for the last three weeks of the course. As reported in Heid (1988), students in the experimental classes seemed to understand calculus concepts of derivatives and integrals more broadly and deeply than students in the traditional course, and performed comparably on a final examination of by-hand symbolic-manipulation skills.

The most important feature of the experimental curriculum was the singular priority given to conceptual understanding. The examples below give the flavor of test questions and assignments that can be given to introductory calculus students who have access to graphing and symbolic manipulation programs and who have not yet learned the manual procedures for computing derivatives and integrals. They serve to illustrate some ways that computer-based calculus courses could shift their emphases to conceptual understanding.

Sample test questions:

Instead of asking students to compute a derivative or an integral, the following questions aimed at assessing student understanding of the concepts.

1. Thus far in the course you've learned no rule for finding the derivative of a function like
$$f(x) = 3^x$$
Explain how you could find $f'(4)$ to any desired accuracy.
2. A graph of the acceleration of a vehicle in miles per hour is shown below where $D(t)$ is the distance (in miles) that a vehicle travels in t hours. At time $t = 0$, the velocity of the vehicle is 25 miles per hour. Estimate the velocity, $D'(t)$, of the vehicle (in miles per hour) at time $t = 5$.

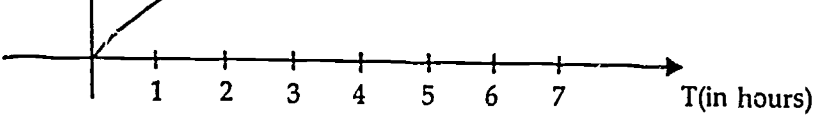


Figure 1

3. Suppose transportation specialists have determined that $G(v)$, the number of miles per gallon that a vehicle gets, is a function of the vehicle's speed, v, in miles per hour. Interpret, in terms of mileage and speed, the fact that $G'(55) = .4$.

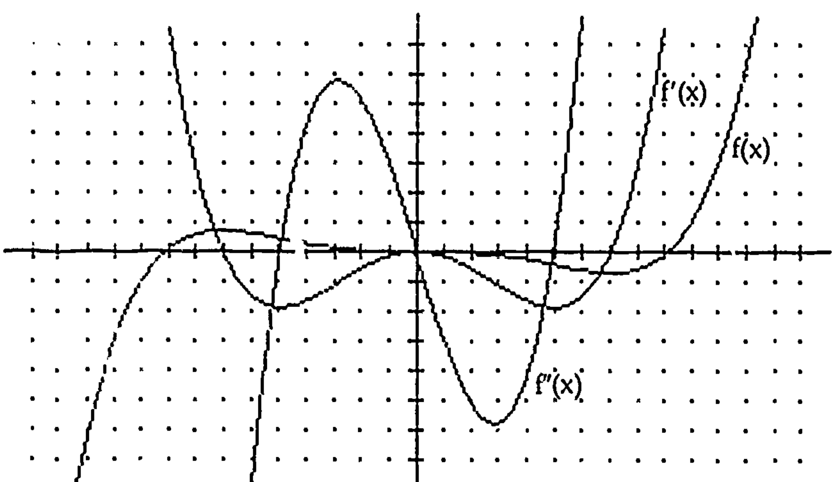
Within the context of a course that allocated most routine symbol manipulation to the computer, students in the experimental course outperformed their counterparts in the traditional course on test and quiz items like those above. For example, they were more able: to interpret information about graphs, slopes, and derivatives; to translate between symbolic representations and their graphs; and to explain a theoretical basis for a derivative.

Sample assignments:

An important feature of the assignments was the fact that many of the assignments engaged students in the active consideration of multiple representations (graphic, numeric, and symbolic) of derivatives and integrals.

1. Given: $f(x) = 6x^5 - 5x^3$
   Use the computer to sketch graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$. Use muMath to verify the maximum and minimum points, inflection points, x- and y-intercepts, and the intervals of upward and downward concavity. Record the graphs as well as the muMath results, and comment on how the characteristics of $y = f(x)$ are reflected in the graphs of $y = f'(x)$ and $y = f''(x)$.

   [As part of the assignment, students created superimposed graphs of f, f' and f''. A sample is shown in Figure 2.]



F(X) = 6*X^5−5*X^3
X−UNIT = .1
F(X)−UNIT = 1

Figure 2

2. Create your own original and realistic optimization problem. Problems modelled on textbook problems (yours or others) are not fair game! Once your problem is written, use muMath to find

an exact solution. Experiment with how the answer to your problem would change if you change some of your parameter values. Sketch graphs comparing these changes.

3. a. Solve the following problem using muMath.
   b. Graph both the original function and a possible appropriate antiderivative function, illustrating on your graphs the problem data and the solution.
   c. Create another mathematical question related to your graph and show the answer to your question both on your graph and using muMath.
   d. Estimate the areas under one of your graphs. Explain how your estimate corroborates the Fundamental Theorem of Calculus.

   People in manufacturing industries have observed in many instances that employees assigned to a new job or task become more efficient with experience. Some companies have enough experience with job training that they can project how quickly a new employee will learn a job. Very often a learning curve can be constructed which estimates the rate at which a job is performed as a function of the number of times the job has been performed by an employee. The learning curve for a particular job is

   $$h(x) = 10/x + 5 \text{ for } x > 0$$

   where h(x) equals the production rate measured in hours per unit and x is the unit produced. For example, when the 7th unit is produced $x = 7$ and h(7) is the rate (in hours per unit) at which the employee is working. Determine the total number of hours expected for producing the first 10 units. . . . the first 1000 units. Is there any limit suggested as to how efficient an employee can become at this job?

As the semester progressed, students became adept at interpreting derivatives and integrals symbolically, graphically, and numerically. With the computer as a tool, students found that they concentrated study time on the ideas instead of on the procedures of calculus.

Conclusions

At no time in recent history has the teaching of calculus received so much attention. Decisions made now about new directions for calculus courses will influence the pre-calculus secondary mathematics of the future and impact on the nature of mathematical understanding of tomorrow's mathematicians, natural and social scientists and engineers. Those new directions must result in a calculus with broader appeal and with greater understanding for a larger pool of students. Computers and calculators have the potential for helping calculus courses to fulfill that promise.

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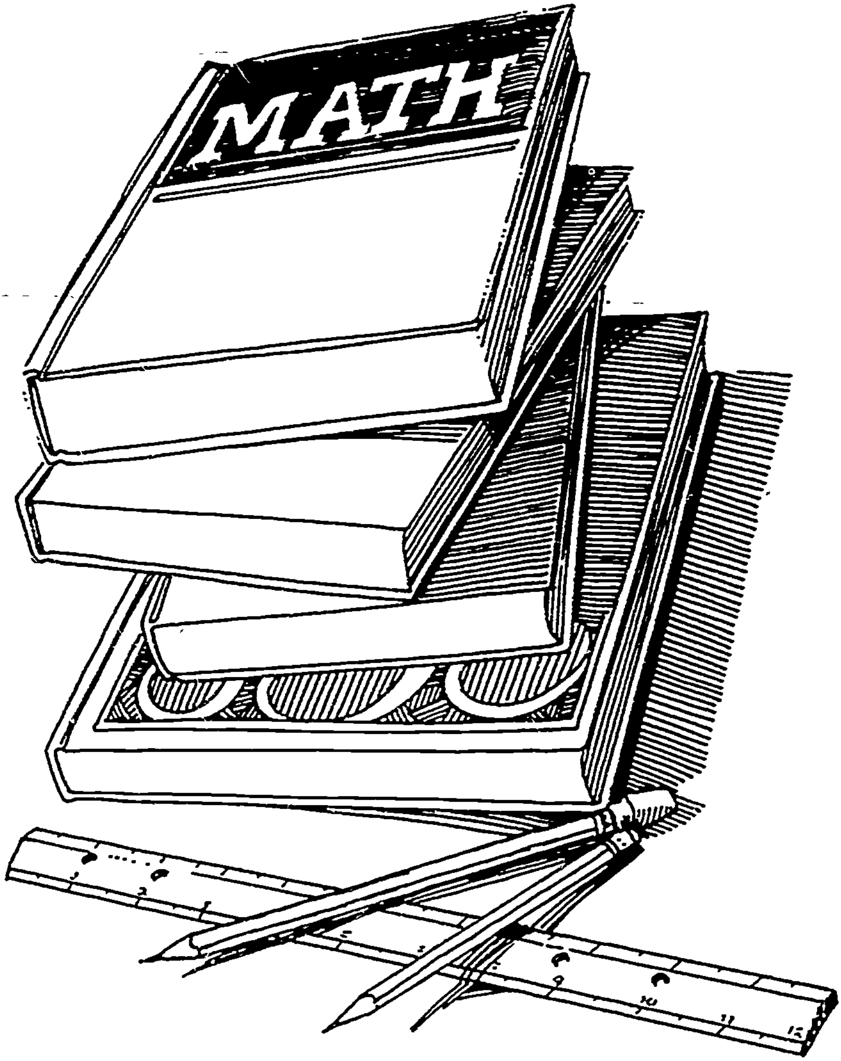
MATH

REDISCOVERING THE FUNDAMENTAL THEOREM OF CALCULUS

Carl Leinbach
Gettysburg College

Introduction

Very few first-year Calculus students appreciate the fact that the Fundamental Theorem of Calculus is making a statement about a function that is created from a given, bounded, piecewise continuous function defined on some interval, [a, b]. They view the Theorem as a tool for calculating and give very little thought to the nature of an integral. As a result, they have little appreciation for the concept of integrating a function and the applications of the definite integral become mechanical processes that they barely understand and will quickly forget. If at the end of the term, we ask our students (without warning them to prepare for the question) to state the Fundamental Theorem, we will get an answer that is equivalent to "$F(b) - F(a)$." More insightful students may mention the fact that $F'(x) = f(x)$.

The students in our first-year calculus course complete laboratory projects using the computer algebra system, *muMath*, enhanced with the graphics capabilities that are provided in the CALC-87 package marketed by Professors Ralph Freese and David Stegenga of the University of Hawaii. In muSimp, the language in which muMath is written, users can easily enter function definitions into the workspace. To define the function $g(x) = x^2$, for example, the user would enter:

```
FUNCTION G(X), X^2, ENDFUN;
```

To define a function, MEAN (A,B), that computes the mean of two numbers, A and B, the user could enter:

```
FUNCTION MEAN(A, B), (A + B)/2, ENDFUN;
```

It is crucial that the system have the muSimp function definition capabilities and that the system be able to display graphs.

The laboratory project we will describe is positioned in the first-year calculus course after the students have had some experience with approximating sums for the definite integral and prior to the presentation of the Fundamental Theorem of Calculus. The lab proceeds from the definition of a new function based on the Midpoint Rule, to the calculation of several approximations, to the tabulating of several approximations for definite integrals of the same function over an interval, to the graphing of these approximations. The student is then asked to compare the graph of the approximations to the graph of the original function. In several prior labs the students have looked at and analyzed the relationship exhibited by the graph of a function and the graph of its derivative. They are now asked to compare the graphs of the approximations and the graph of the function for any relationships.

The environment in which the students work contains a microcomputer lab having several MS-DOS computers and a central workstation with an overhead display for the instructor. The students work in pairs, and gather data in the form of strings of symbols, graphs, or numerical results. Each pair of students completes a worksheet to be handed in at the end of the lab period. They then use their data to make conjectures that are presented in their written laboratory report. Although the students may discuss the results they obtained in the lab, the two- to three-page written lab report is an individual effort. The lab report contains three main sections: a discussion of the purpose of the lab; a brief description of the lab procedure; and a discussion on the student's observation and conjectures. Obviously, the last section is the most important section of the report and requires the most thought. It is this section of the lab reports that the instructor can use as a basis for lectures on a particular topic. In general, the students are much more interested in seeing how their conjectures fared than in having a presentation based solely on material from their text.

The Laboratory Project

Before beginning the laboratory project described here, students will have done some homework exercises computing the values of definite integrals using the definition. They have had experience using the function values at the left-hand endpoints, right-hand endpoints, and midpoints of the subintervals of the partition. The lab begins by having the students enter the following muSimp function definition into their workspace.

```
FUNCTION MIDPOINT(F,A,B,N), H:(B-A)/N,
      H*SIGMA (EVSUB(F,X,A+H/2+I*H),I,1,N-1), ENDFUN;
```

The colon (:) signals the computer to assign to H the value of (B – A)/N. Therefore, the MIDPOINT function gives the value

$$H\sum_{i=1}^{N-1} F(A + \frac{H}{2} + IH)$$

to MIDPOINT (F, A, B, N) where $H = \frac{B-A}{N}$

This is merely a function for evaluating the sum of the areas of N rectangles having bases of width h and height equal to the function value at the midpoint. (Rectangles lying below the x-axis have "negative" height.) Although the same general results can be obtained using either the right-hand or left-hand endpoints, the midpoint approximating sum was chosen because it tends to give a more accurate approximation for a given value of N.

After typing in this function the students test it using some known results from class. For example, MIDPOINT (X^2,0,1,10) yields a result of 133/400 or 0.3325. The students do three or four more familiar examples to test their homework and gain familiarity with the use of this new function.

Now that the students are familiar with the use of the MIDPOINT function, it is time to convince them that this is indeed a function. They are asked to make a table of the values of MIDPOINT (X^3,0,T,10) for T = 0,0.1,0.2,0.3,...,1. From this table the students can plot by hand the values of MIDPOINT (X^3,0,T,10). But what does the graph look like for a broader range of values and for some different functions? To answer this question, the students use the graphing routine supplied by the CALC-87 enhancement to the package. From the very first lab period they have used this graphics capability so this is not a new use of the system.

The command GRAPH(MIDPOINT(X^3,0,T,10),T); plots the function in the range from T = −4 to T = 4. This graph obtained by using a screen dump of the result is shown in Figure 1.

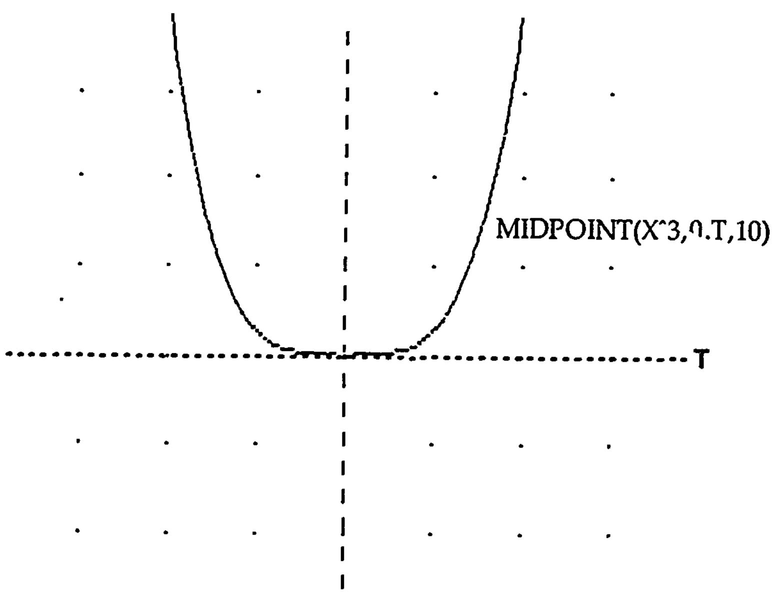

Figure 1

A byproduct of this investigation is that the students can see that it is possible to define a function by means of an algorithm as opposed to using a formula.

We have arrived at the crucial question for this lab period. What is the relationship between MIDPOINT (F,0,T,10) and F(T)? The students are instructed to graph both of these functions together by issuing the command: GRAPH(MIDPOINT(F,0,T,10),T,F(T)). The students display similar pairs of graphs for five different functions and draw the results on their lab sheet. Figure 2 shows the graph of MIDPOINT (SIN(X^2),0,T,10) and SIN(T^2) in a range from −2 to 2. Although the accuracy of the estimate deteriorates as T gets further away from 0, it is clear to those who can recognize the relationship between a function and its derivative, that SIN(T^2) is the derived function for MIDPOINT(SIN(X^2),0,T,10), or at least a close approximation of the derived function. Other functions graphed together with MIDPOINT(F,0,T,10) in this lab are: X^3−X+1, X^4−X^2+2X,EXP(−X^2), and SIN(X).

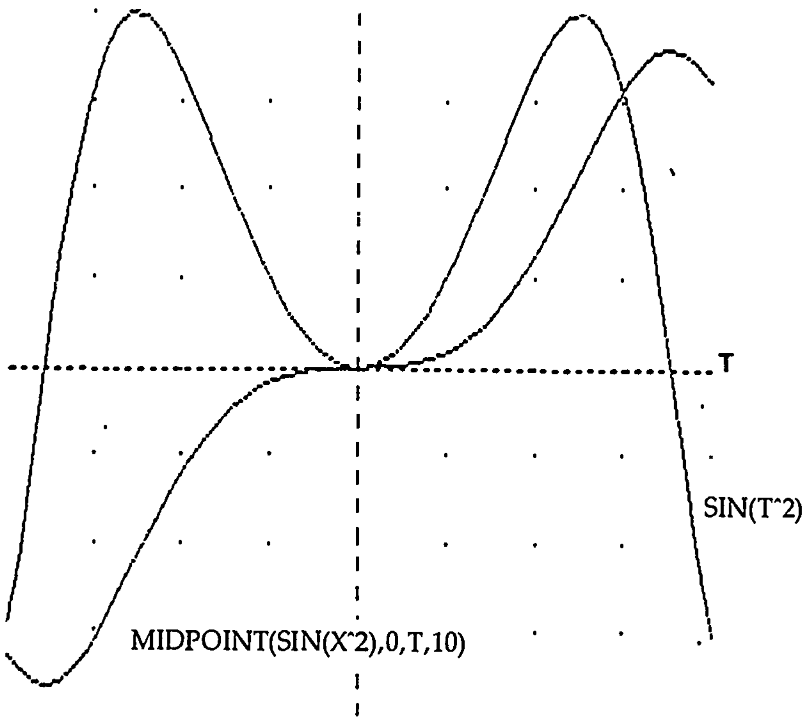

Figure 2

Summary

No one approach or technique is going to work for all students. This lab does, however, allow the instructor to make some very important points. The first point is that the computer algebra system provides a way of implementing the Riemann Sum definition as an algorithm. The second is that this algorithm defines a function that depends on a given function and an interval. The last point is the object of the lab. The function defined by this algorithm is an approximation of an antiderivative for the original function. Not all students recognize this relationship. However, the lecture on the Fundamental Theorem of Calculus that was given on the day the lab reports were due had much more class participation and apparent appreciation by the students than was evident in presentations given prior to the inclusion of a lab in the course. There was also a much more lively discussion of the theorem and the proof was followed much more closely than had been the case in previous presentations. This was especially the case among those students who did not notice the correct relationship between the functions.

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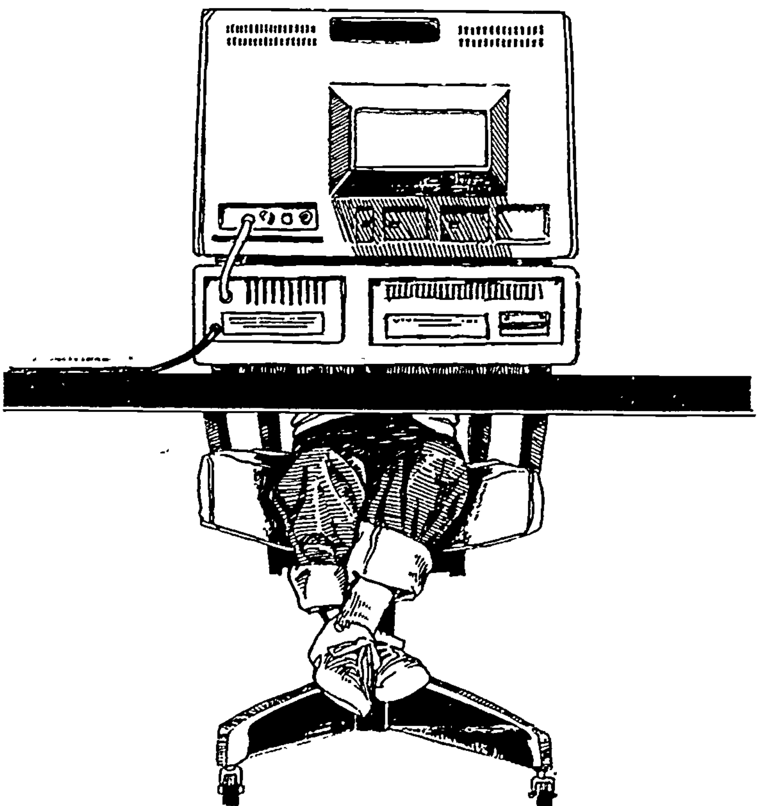

CHANGES IN SECONDARY MATHEMATICS TEACHER EDUCATION FOR THE 90'S

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Major changes are underway in the preparation of mathematics teachers. The changes are driven by reams of critical reports, ominous sounding data and the general public's confusion about what these data mean. The major changes are included in reports from The Holmes Group (1986), The National Commission for Excellence in Teacher Education (1985) and the Mathematical Association of America (1983 & 1988). What effect changes recommended by these reports might have on mathematics teacher education during the 1990's is not well documented. In fact, what specific effects some of these changes might have has not been delineated much beyond their cosmetic and political value. On the other hand, changes recommended by the national reports are based upon an on-going stream of research showing what in fact will help improve the quality of teacher preparation programs and performance of mathematics teachers in the classroom. Because of the field and clinical experience focus of these changes, mathematics teachers in schools, in collaboration with colleges and universities will be asked to play a significant role in their implementation. If the recommendations are implemented, the participation of high school mathematics teachers will extend from recruiting prospective teachers to field experience supervision.

Recruiting and Retaining Quality Mathematics Teachers

Research data (Feistriter, 1986) indicate that the current rate of supply of certified mathematics teachers will not be sufficient to meet the demand over the next ten years. Additionally, Schlechty and Vance (1983) report that 40-50 per cent of those entering teaching leave within the first five years. This teacher shortage and high turnover rate is occurring at a time when school boards have increased the number of mathematics credits required for high school graduation. Further, current demographics (Commission on Minority Participation, 1988) indicated that during the 1990's one-third of our school population will be minorities. Therefore, the critical issue now facing mathematics education is how to recruit and retain quality mathematics teachers in general, and minority mathematics teachers in particular.

To deal effectively and immediately with this issue requires that colleges, universities and schools work together to develop creative solutions that focus upon what Mary Futrell, the president of NEA, calls the "pipeline" factors. Because of the size of the teacher shortage these solutions must become a pump rather than a biased filter in the pipeline that places quality prospective teachers into America's teacher education programs.

Such solutions include increasing the academic support services and the collaboration among guidance counselors, college and high school mathematics teachers, and parents at the point where students select mathematics courses and plan their high school program. To become a mathematics teacher, a student must schedule and successfully complete a quality academic program in high school.

We know that many prospective teachers become interested in teaching mathematics because they are attracted to and come to respect one of their junior or senior high school mathematics teachers. Reinstating and supporting future mathematics teacher organizations in high schools will facilitate this connection. Effective leaders of such organizations should use current knowledge about what influences a person's decision to teach (Berry, 1984; Stratford and Berd, 1985). In addition to identifying with a teacher role model, research has identified two other factors that attract and encourage teachers to remain in the profession. These include the prospective teachers' knowledge about the lifestyle and benefits the profession has to offer and the quality of the work environment experienced during the first five years of classroom teaching. It is the combination of these two factors, rather than just salary, that attracts and convinces teachers to stay in the profession. In fact, research indicates the content of these two factors become the dominant influence on teachers as they decide to leave or remain in the profession.

Finally, new strategies that will immediately attract and retain minorities in mathematics education should be developed. Such a strategy consists of initiatives like developing partnership programs between school districts having high concentrations of minority populations and colleges or universities having mathematics teacher education programs geared to helping students complete their program requirements. An example of this strategy in action is the recently announced partnership between Penn State and the Reading (PA) School District (Colling, 1988; Abrams, 1988). The purposes are to attract qualified minorities to become involved in schools and to interest them in enrolling in Penn State's mathematics teacher education program. Interested students will be guaranteed admission to Penn State's teacher education program and will be assisted in obtaining financial help and the academic support needed to complete the mathematics teacher education program's requirements.

Improving the Quality of Mathematics Teacher Preparation Programs

All of the national reports maintain that the quality and rigor of teacher education programs must be improved. They also point to the need to increase the level of funding needed for colleges of education to carry out the recommended reforms. The reports' recommendations have become the springboard for the substantive program changes now being considered by most mathematics teacher education programs across the country. The recommendations can be placed into the three traditional teacher education program components: namely, general education, content specialty and professional education requirements. The recommendations that follow are aimed at restructuring programs to provide prospective mathematics teachers with a balance of high quality coursework in each of these components at the undergraduate level. They do not support the simplistic view that increasing the number of credits allocated to general education and the prospective teacher's content specialty at the expense of professional preparation will produce more effective classroom instruction. As most mathematics teachers know, the educational problems we now face are much more complex than this simplistic solution implies. In fact, if more credits in general education and mathematics is the answer to better classroom instruction, the critics must answer the question — Given the increase in general education and mathematics credits in mathematics teacher education programs since the 1930's, why have the mathematics achievement problems identified by the recent national reports arisen? The answer is that the current problems are more complex and are driven by many more pervasive societal forces than a one-dimensional type of solution suggests. The discussion that follows highlights the nature of this complexity and the changes that must be made by all those in colleges and universities responsible for preparing teachers. Psychological traps like scapegoating and fingerpointing should be avoided because they only serve to prevent the type of comprehensive changes required to redesign mathematics teacher education programs.

General Education

Recent transcript studies, such as the one conducted by the Southern Regional Education Board (1985), and the Association of American Colleges (Zemsky, 1987) show that there is a need to improve the distribution of general education courses between lower or introductory courses and in-depth or upper level courses. If implemented this recommendation means that the smorgasbord or free choice approach that has led to electing only lower-level survey courses would be replaced by specifying breadth and depth requirements for each of the traditional areas of study known as general education. For mathematics teachers, special attention needs to be given to increasing coursework in the area of science, technology and society and its impact on adults and children in an electronic-oriented world. Since mathematics teachers would have a strong background in mathematics, some of the quantification credits normally allocated to this general education education category could be devoted to building depth and breadth study in this area.

A panel established by the National Institute of Education and chaired by Kenneth P. Mortimer (1984) points out that reallocating credits alone does not address the major problems associated with improving the general education programs in colleges and universities. Rather, as the report points out, it is the ingredients that make up the quality of learning that should be given the most attention. The report singles out improving students' analytic, synthesizing and problem-solving capacities as essential to improving general education offerings. The report also stresses the need for the departments responsible for general education to base changes in general education requirements on data regarding students learning and growth rather than just to reshuffle the number of credits taken, course prerequisites, admission standards, etc.

Additionally, this comprehensive report maintains that prospective mathematics teachers need high quality college teacher models who demonstrate how to teach higher order thinking using microcomputer technology, group problem solving, collaborative learning and other teaching techniques and concepts which research shows maximize student involvement in learning. Such techniques and concepts involve using strategies to enhance students' achievement motivation, to increase engaged learning time, and to provide performance assessments and feedback on a regular basis. In short, the report maintains that all college teachers should enrich their instructional strategies beyond typical lecture to include more active modes of learning such as computer simulations, in-class debates, small group seminars and the like to encourage students to become creators as well as receivers of knowledge. Perhaps it is time to consider how all professors in colleges and universities can be better prepared as teachers during their graduate programs. For example, colleges of education could create education minors and teaching internships that could become part of doctoral programs for graduate students whose goal includes teaching.

When these general recommendations are also extended to college mathematics departments, these changes mean that prospective teachers should learn mathematics in an enriched learning environment where they are taught by professors prepared to teach and who demonstrate they care about the subject matter and about improving their students' critical thinking abilities. The report points out that these changes do not mean that course content should be watered down in favor of "exotic" teaching strategies. On the contrary, it means mathematics courses should be enriched to include the teaching of both mathematics content and mental processes required to learn this content at a deeper level of meaning. Additionally, use of such approaches will probably increase the contact between mathematics faculty and prospective mathematics teachers.

Mathematics Content Specialty

In addition to improved general education requirements, what type of undergraduate mathematics course content and what kind of instruction should be included in a prospective teacher's program?

The national reports indicate mathematics teachers need college level mathematics courses that provide them with the background and maturity needed to pursue additional coursework in mathematics and mathematics education at the masters level during their first five years of teaching and beyond. Additionally, they need to have an in-depth understanding of the mathematics content contained in the school curriculum. Currently the National Council of Teachers of Mathematics (Committee on Standards for School Mathematics, 1987) and the Mathematical Association of America (1983) suggest prospective teachers need courses in the following areas:

- Abstract & linear algebra
- Geometry
- Calculus
- Number theory
- Discrete mathematics
- Probability & statistics
- Introduction to computing
- History of Mathematics
- Computer science
- An educational computing language

Beyond the mathematics content included in these areas, The Mathematical Association of America (1983) indicates it is important that the emphasis in each course be on understanding, on foundations that model good teaching techniques, on class discussion that motivates thinking, on the mental processes that guide the search for meaning, and on sharing and communicating this understanding and meaning with others. The Association's recommendations point out that instruction in these areas should not be restricted to exposition that pushes students to reproduce information from the current frontiers of mathematics theory and related research. This observation is included here because one of the major problems in some colleges and major research universities is the lack of middle-range mathematics courses having the focus on teaching for meaning just described. Such middle-range courses should make extensive use of examples, applications, models to teach problem solving, and projects that encourage exploration and inductive thinking.

Such courses existed prior to the 1960's. During the 1960's, however, this type of course was eliminated and replaced by upper-level courses to accommodate the increased demand for mathematics majors whose goals were the Masters and Ph.D. in pure mathematics. Most mathematics departments agree that this preparation is not appropriate for mathematics teachers whose need is for a different level of understanding and a variety of examples that can be used as they teach junior and senior high school students. In the past the middle-range courses have attracted majors from other programs whose needs are relatively the same as prospective mathematics teachers. Therefore, it would not be necessary to advertise and restrict such courses to mathematics teachers.

Given the current financial problems faced by mathematics and mathematics education departments, the major problem that must be addressed by those responsible for funding these departments is how to increase the monies needed to recruit and retain professors with the dual interest of teaching such middle-range mathematics courses and of conducting research in mathematics teacher education. Perhaps the time has come to create new positions in both departments for which professors with interests in teacher education are hired and given tenure because of their specialized interests and productivity in research in mathematics teacher education. This approach suggests that developing collaborative doctoral programs in mathematics teacher education could then become a viable graduate program option.

Generic and Specialized Professional Knowledge and Practice

To date the mathematics education community has tended to focus on increasing the number of mathematics courses in a secondary teacher's program. This focus predominates despite the research by groups such as Pennsylvania's Association for Supervision and Curriculum Development's Professional Preparation Committee (1986) showing that beginning teachers' problems during the first five years of teaching are primarily pedagogical. One result of this narrow focus on mathematics content is that insufficient attention is now being given to improving the rigor and focus of the professional courses in a mathematics teacher's preparation. A broadening of the focus is supported by Pennsylvania's survey of beginning teachers (Professional Preparation Committee, 1986) that found that problems such as maintaining classroom discipline, motivating and stimulating student interest in mathematics problem solving rather than their lack of understanding of mathematics concepts were the major concerns of beginning teachers.

The professional preparation of teachers was also reviewed by a variety of national task forces such as the National Research Council (1989), the National Commission for Excellence in Teacher Education (1985) and the Mathematical Association of America (1983 & 1988). They concluded that the product of any teacher education program must be a graduate well grounded in *both* a subject specialty (e.g., mathematics) and the professional knowledge and skills needed to be an effective teacher. For mathematics teachers, this means preparing teachers so they have an in-depth understanding of the mathematics in the school's curriculum and a broad repertoire of instructional skills needed to facilitate the learning of mathematics of a variety of students with different ability levels and interests.

In addition to the state and national studies, the effective teaching research shows that mathematics teachers must learn to be analytical and reflective practitioners whose major concern is the cognitive and affective growth of their students. As one of my colleagues pointed out, this means effective mathematics teachers need to learn how to use action research in their classroom to evaluate their teaching.

The effective teaching research also suggests that mathematics teacher education programs must answer the question: What generic and specialized professional knowledge should be part of a mathematics teacher's background?

Briefly, effective teaching research and the national task forces agree that it is critical for mathematics teachers to have experience in classrooms with how a broad range of children, adolescents and adults develop and learn mathematics. This means prospective teachers need a broad-range course in the psychology and learning of mathematics where the connections among learning theory, patterns in students' mathematical development and the teaching of mathematics are the singular focus. Mathematics teachers also must learn how to use state-of-the-art technology and methodology in carefully supervised clinical experiences that include microteaching, computer simulations and school-based internships. In these clinical experiences, teachers need to learn the basics associated with teaching mathematics in a regular classroom as well as how to diagnose the typical learning difficulties and to conduct remediation activities. Mathematics teachers must also understand the structure of the school's mathematics curriculum and the methodology associated with specific instructional goals such as teaching concept development, problem solving and skill building. To monitor and evaluate their teaching effectiveness, teachers should be taught by master teachers to organize, manage and conduct self-analysis of both individualized and group-based instruction using current video technology. Additionally, because the professional knowledge base in methodology, learning theory and curriculum is continually changing, teachers must develop the habits of reading and integrating the knowledge reported by professional journals into their teaching repertoire.

The above description suggests mathematics teacher education programs should include courses in the following areas:

- Psychology of mathematics learning;
- School mathematics curriculum;
- Teaching methods in school mathematics;
- Current issues and research in school mathematics;
- Design of instruction and assessment of student performance,
- Systematic observation and analysis of classroom instruction, and
- Supervised internships with students in schools.

The rigor and the extensive nature of these requirements mean that colleges and universities must eliminate the cleavage between mathematics and mathematics education departments so that undergraduate programs can be designed to accommodate both the content and pedagogical needs of future mathematics teachers. Or, teacher education programs will become graduate programs as recommended by the Holmes Group Report (1986).

Finally, regardless of which option is chosen it is crucial to the advancement of the mathematics education profession to recognize that the professional education of mathematics teachers be viewed as an on-going life-long process and professional responsibility. The implication of this point of view is that schools, colleges and universities will collaborate and develop new extended graduate level mathematics education courses and experiences in partnership programs aimed at motivating teachers at all levels to want to continue their professional development beyond the Masters degree. The current career and staff development research demonstrates how important staff development and extended graduate programs are to the future quality of what happens in the classrooms with students.

Summary

Major changes are underway in the preparation of mathematics teachers. These changes are driven by recent national reports on teacher education in general and mathematics teacher education in particular. If the changes underway are implemented, participation of classroom teachers will extend from recruiting prospective mathematics teachers to field experience supervision. All of the national reports maintain the quality and rigor of teacher education programs must be improved. Suggestions for strengthening three program components, general education, content specialty and professional education, are presented and discussed.

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