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ABSTRACT

The purpose of this study is to contribute to an understanding of how errors could be employed in mathematics instruction so that the students use them constructively in support of their learning of mathematics. A teaching experiment was designed to create an ideal context in which the pedagogical approach to errors as springboards could be applied constantly. The teaching experiment was organized as 10 lessons on mathematical definitions for two 11th-grade female students. Twenty mathematical errors were recorded and analyzed. The error activities identified and described are different from learning goals and outcomes. A taxonomy of constructive uses of errors was developed based on level of mathematics discourse and stance of learning. Eight potential benefits associated with errors as springboards were generated from the analysis of error activities. (YP)

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STUDENTS' CONSTRUCTIVE USES OF MATHEMATICAL ERRORS: A TAXONOMY

by

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STUDENTS' CONSTRUCTIVE USES OF MATHEMATICAL ERRORS: A TAXONOMY

I. Introduction

The scope of this paper is to contribute to an undestanding of how errors could be employed in mathematics instruction so that *the students themselves* use them constructively in support of their learning of mathematics. The study reported here is a component of a more comprehensive research project on the educational potential and implications of alternative pedagogical uses of errors, and builds on prior results reached within that project.

A review of the literature on errors in various fields (Borasi, 1986a; 1988a) suggested that mathematics educators have not yet fully exploited the educational potential of errors. Mathematics teachers, as well as researchers in the field, have certainly been concerned with student errors for a long time, and have attached considerable importance and value to those errors as a means to diagnose learning difficulties and to suggest effective strategies to remediate them (Radatz 1979, 1980; Novak-Helm, 1983; CIEAEM, 1988). Though this approach to errors has provided valuable contributions to mathematics education, it also presents some important limitations. In an interpretation of errors as tools for diagnosis and remediation, only teachers or researchers, and not the students themselves, are engaged in the creative activity of analyzing errors. Furthermore, the scope of the error analysis is strictly focussed on eliminating the error.

An analysis of how errors are used in other contexts such as computer science and scientific research, on the contrary, suggests that there are other valuable ways to capitalize on errors, regardless of any tendency to repeat or eliminate them. Historian and philosophers such as Kuhn (1970), Lakatos (1976) and Kline (1980), for example, have shown us remarkable examples in the development of mathematics and the sciences when supposed errors stimulated inquiry which in turn led to new insights, unexpected discoveries, and sometimes even to the opening of entirely new areas of research. The common practice of *debugging* in computer programming has also shown that one does not need to be a genius or a professional scientist to engage productively in the identification and analysis of errors.

On the basis of this critical review of the literature, combined with conceptual analysis, I presented an argument for the creation of instructional activities where mathematics students themselves could similarly benefit from engaging in a constructive use of errors –

an instructional strategy I called *using errors as springboards for inquiry* (Borasi, 1986a; 1987). Various examples, dealing with a range of mathematical topics and levels, were also developed to show how an analysis of specific mathematical errors can lead to interesting explorations and insights, without necessarily requiring considerable mathematical background or ability (see, for example, Brown- Callahan, 1985; Borasi, 1986b).

To support and further elaborate the above claims with empirical data, a teaching experiment was designed to create an ideal context in which the pedagogical approach to *errors as springboards* could be applied consistently, and extensive data about such implementations could be collected. The teaching experiment was organized around the topic of "mathematical definitions", and consisted of ten lessons taught by the researcher to two sixteen-year-old eleventh-grade female students. Besides implementing pre-planned activities based on the analysis and study of pre-selected errors, the instructor also took advantage of other errors that the two students unexpectedly made in the course of the instructional experience. The constructive uses of errors made in the course of this teaching experiment were all identified and analyzed in-depth, and used generatively in the spirit of qualitative research to develop a taxonomy of constructive uses of errors available to mathematics students and to generate hypotheses about their educational value and implications for mathematics instruction.

While a complete report on the results of this empirical study are beyond the scope and space limits of this article, in what follows I have attempted to reconstruct the crucial steps in the creation of the taxonomy proposed, as well as the major insights about using *errors as springboards* in mathematics instruction gained as a result. [1]

The taxonomy presented in this paper should be considered as a working hypothesis, for which future experiences conducted in a variety of other instructional contexts may suggest further refinement and elaboration. However, as with most taxonomies, its value should be measured not so much in terms of its completeness and definitiveness, but rather in terms of its power to allow educators to become aware of new possibilities and to better interpret and evaluate their educational practice. In this respect, I believe that the taxonomy suggested here can contribute to mathematics education by identifying new ways in which students could be actively and productively engaged in mathematical inquiry motivated by specific errors, by suggesting the specific values and implications for mathematics instruction that each of these approaches may entail, and also by highlighting some important variables and

[1] A complete report on the results of the implementation of the teaching experiment, as well as of the articulated and in-depth analysis of the empirical data thus generated, can be found in the Preliminary Research Report on the Project "Using Errors as Springboards for Inquiry in Mathematics Instruction", sent to the National Science Foundation (Borasi, 1988b).

dimensions that teachers and researchers should take into account when planning learning activities making use of errors.

II. The Research Design

As mentioned in the introduction, previous components of the research project already suggested several directions in which errors could be used constructively in mathematics instruction, as well as the possibility and value of engaging mathematics students in such activities. The implementation and analysis of the strategy in a variety of instructional contexts seemed now necessary to gain a better understanding of how errors could actually be used by the students themselves to enhance their learning of mathematics, what benefits could de derived from each of these uses, and what would be the implications for school mathematics practice.

As a start, the teaching experiment methodology seemed to provide the most appropriate context to generate and collect empirical data for these research purposes. A teaching experiment set-up would in fact allow the researcher, in her role as both curriculum designer and teacher of the unit, extreme freedom and flexibility to take maximum advantage of the educational potential of errors whenever possible and appropriate. It would also make it possible to create a learning environment compatible with the pedagogical assumptions and goals of the strategy proposed. Finally, the small number of students involved in the instructional experience would allow for the careful monitoring and collection of the rich and varied set of data necessary to conduct a qualitative empirical study aiming at generating new hypotheses and insights.

The notion of *mathematical definition* was chosen as the topic to be addressed in this experience. Though other mathematical content could have well served the purpose, this specific topic presented some obvious advantages. First of all, it does not require specific technical prerequisites, and it could be easily adapted to suit students with different mathematical backgrounds and abilities. Secondly, inquirying into the nature of mathematical definitions naturally provides the opportunity to engage the students in a wide range of mathematical activities – going from the solution of specific problems to reflections about the nature of mathematics. Thus, within the context of learning about mathematical definitions one could explore the effects of *using errors as springboards* with respect to educational goals more varied and comprehensive than the mere mastery of mathematical facts and skills (to which, unfortunately, much of the current mathematics curriculum reduces to).

A two week unit was thus designed with the multiple goal of making students better recognize the various roles played by definitions in mathematics, appreciate the value and limitations of the criteria traditionally imposed by mathematicians on mathematical definitions, and become aware of the limitations as well as the power of mathematical definitions. A sequence of concrete and thought-provoking activities around the definitions of *circle, polygon, exponent* and *variable*, was created to guide the students' inquiry into this complex and abstract topic. These activities often involved the constructive use of some error previously selected by the instructor – for example, the analysis of a list of incorrect definitions of *circle* was proposed to help the students identify some of the requirements usually imposed on mathematical definitions and become aware of the rationale behind them; contradictions encountered in the extension of the definition of exponentiation beyond the whole numbers were introduced to illustrate the existence of unavoidable limitations or exeptions even within mathematical definitions. Besides this *pre-planned* use of *errors as springboards*, the instructor was also open and ready to take advantage *impromptu* of interesting errors unexpectedly made by the students in the course of a lesson or a homework assignment.

This unit on *mathematical definitions* was implemented with two sixteen-year-old female students, Kim and May.[2] The two girls, though quite bright and talented in other areas, did not like mathematics and agreed to participate in the experience only as a way to "make-up" for the many absences they had accumulated in another mathematics course taught by the same instructor. The instructional experience consisted of ten instructional sessions, followed by a take-home individual project.

In the spirit of ethnographic research (Eisenhart, 1988), extensive data was collected on both what happened in the experience and the participants' perceptions of it. More specifically, all ten lessons were audio-taped and transcribed, and the students' written work collected; both the researcher/instructor and a non-participant observer kept field notes; and a series of interviews (audio-taped and later transcribed) were conducted at the end of the experience with each student.

To prepare for the identification of alternative ways in which the students were engaged in a constructive use of mathematical errors, and for the analysis of the respective values of each use, all this extensive data was organized and analyzed as follows:

- To capture the wholistic nature of the teaching experience, a detailed narrative report

[2] The experience took place in Spring 1986, and took about two months to be completed. The students were eleventh graders attending the School Without Walls, an alternative school in the city of Rochester Public School System, and they had previously participated in an experimental 11th-grade mathematics course taught by the researcher in their school in Fall 1985.

on the ten lessons and the take-home project was created integrating field notes, selections of the transcripts from classes and of students' work, as well as the researcher's insights and observations.

- Each occasion in which the students were engaged in a constructive use of errors was identified and described, taking into account the following variables:
  (i) the nature of the error (i.e., what the error consisted of);
  (ii) the context in which the study of the error occurred;
  (iii) the origin of the error (i.e., who made the error and how was the error introduced to the students' attention);
  (iv) the students' level of participation in the study of the error;
  (v) the educational goals that the activity helped to achieve;
  (vi) the results of the activity, including the students' reactions to it.

This preliminary analysis helped to identify about 20 instructional episodes in which errors had been used constructively during the teaching experiment (in what follows, for brevity, I will refer to these instructional episodes as *error activities*). Several of these episodes presented considerable differences with respect to the various dimensions considered, and clearly suggested that the learning outcomes and benefits that the students might derive from a constructive use of mathematical errors highly depended on the *way* the study of the error was approached.

In the attempt to identify fundamental differences among possible uses of *errors as springboards*, and consequently better assess the educational potential of this instructional strategy, an attempt to categorize all the *error activities* thus identified was conducted. This generative process went through several cycles involving empirical "grouping" of episodes which presented interesting similarities, and a conceptual search for suitable logical categories which could characterize and make sense of the groups thus formed. In the end, a taxonomy of nine complementary ways in which students could be engaged in the constructive uses of errors emerged.

Once created, this taxonomy allowed the researcher to go back to each of the *error activities* previously identified, and from this analysis new insights were gained about the educational implications and values of variations within the strategy of using errors constructively in mathematics instruction.

The dimensions and variations highlighted by the taxonomy also helped in the analysis of other instructional experiences where a use of *errors as springboards* had been implemented by different instructors and in a variety of contexts. While this application of the taxonomy was not systematic, and thus will not be reported in this paper, it supported the contention that the dimensions identified within this teaching experiment are by no means specific to this context only. It also suggested there is value in employing the taxonomy proposed for the analysis of other instructional experiences implementing an *errors as springboards* approach.

Before presenting the taxonomy and the hypotheses about the potential values of using *errors as springboards* it suggested, in the next section I will attempt to provide a summary of the most crucial data – the *error activities* which occurred during the teaching experiment – for the readers interested in the process which brought to these results.

III. Summary of the Data: 'Error Activities' Developed In the Teaching Experiment

As mentioned before, about twenty episodes in which the students were engaged in a constructive and substantial use of mathematical errors were recorded in the teaching experiment. The analysis of both "what happened" in these episodes and the context in which each developed were crucial in generating the categories of the taxonomy and the hypotheses about the educational potential of each of the variations of the strategy. In the space constraints of this paper, it is unfortunately impossible to provide a satisfactory report of each of these *error activities*. However, I will attempt to give the readers an appreciation of the range of learning experiences which developed around specific errors, as well as a flavour of the richness of most of them, by the combination of two complementary types of report.

First, I will provide a brief chronological overview of how the ten instructional sessions developed, highlighting the *error activities* which occurred in each lesson. This narrative will allow me to present a brief description of <u>each</u> *error activity*, and at the same time provide information about the context in which it occurred – a crucial element for the analysis and categorization of the constructive uses of errors made in this study.

The disadvantage of this type of report, however, is that because of the brevity of each description, one cannot really appreciate the depth of the reflections and discussions which were stimulated by some errors, nor the quality of the students' actual responses and reactions to the activity. To complement the previous overview, then, I have selected a few *error activities* to be reported in more detail. These specific episodes have been chosen so as to highlight interesting differences in the approach and use made of errors, which played a key role in the construction of the taxonomy.

a) Overview of the instructional sessions and of all the error activities developed in them

Lesson 1. (20 minutes)

The unit started by asking the students to write tentative definitions for nine concepts, chosen from geometry, algebra and "real-life" (they were: *circle, square, polygon, variable, exponent, equation, cat, purple, crazy*). The intention was to gather some indirect data about what the students believed about mathematical definitions, as well as provide an initial stimulus for a reflection on mathematical definitions. Though a good number of the definitions produced by the students in response to this task were incorrect, no mention or use of these errors was made in this lesson.

Lesson 2. (40 minutes)

In this lesson we started our study of definitions of *circle*. Since *circle* is a very familiar figure, which any secondary student can identify without doubts, I expected that Kim and May would feel confident in evaluating tentative definitions of *circle*, and from this concrete task be brought to identify and appreciate the traditional requirements imposed on mathematical definitions – such as isolation of the concept, use of precise terminology, essentiality, non-circularity.

Error Activity 1: The lesson was totally driven by the analysis of a list of eight incorrect definitions of *circle* – proposed by the instructor, and including the two definitions of *circle* produced by Kim and May in the previous session. This analysis brought to the surface some of the criteria that the students were implicitly using to evaluate mathematical definitions, as well as new ones they had not considered before. In addition, this activity led the students to propose two acceptable definitions of *circle* (the usual metric definition, and a definition using the equation of a circle as derived in analytic geometry) and to engage in other mathematical activities which increased their understanding of circles. (See a more detailed report of part of this error activity in part **b** of this section)

Error Activity 2: In the process, Kim stated her confusion about the fact that the equation of a circle in analytic geometry ($x^2+y^2=r^2$) and the formula characterizing right triangles in the Pythagorean theorem ($a^2+b^2=c^2$) could look the same. The analysis of this "error" allowed the instructor to point out the fundamental difference between *variable* and *constant* in an equation – an issue which would also be further pursued in a later lesson.

Lesson 3. (30 minutes)

Most of this lesson was devoted to further explorations of the roles and nature of definitions in mathematics, by engaging the students in the solution of the problems requiring an essential use of the definition of *circle*: "Determining the circle passing through three given points", and "Finding the interior angle of a regular pentagon inscribed in a circle".

Error Activity 3: Since the second problem was particularly novel to the students, they naturally made some errors in the process of solving it, and with the instructor's help they were able to make use of the information provided by their errors and partial results to reach the solution.

Error Activity 4: A student's casual mention of an isosceles triangle as "A triangle with two equal sides and two equal angles" motivated an interesting digression. The instructor questioned whether this could be considered an acceptable definition of *isosceles triangle*. In pursuing this question, four alternative tentative definitions of *isosceles triangle* were considered ("A triangle with two equal sides", "A triangle with two equal angles", "A triangle with two equal sides and two equal angles", and "A triangle with two equal sides or two equal angles"). This analysis, especially in light of the need to apply a definition of *isosceles triangle* to solve the problem of "Finding the interior angle of a regular pentagon", brought us to conclude that the "AND" definition was not appropriate, and furthermore to realize the importance of the criterion of essentiality (or "non-redundancy") in mathematical definitions – a requirement the students had not previously appreciated.

(continuation of **Error Activity 1**) After the two problems were successfully solved, the students were asked to go back to the list of incorrect definitions analyzed in lesson 2, and to discuss why those definitions would not have been good enough to solve these problems. At the same time, the students were asked also to look at the list from a different perspective – that is, to use it to help them identify important properties of *circle* – and thus realize the value of these definitions, however incorrect, to provide insights into the notion of *circle*.

Lesson 4. (40 minutes)

Error Activity 5: When trying to solve the problem of "Finding the circle passing through three given points" at home by using the analytic approach, Kim had made some errors, which she asked the instructor to look at. She was concerned to know where she went wrong and whether she had done anything worthwhile at all. Besides helping the student understand how to approach and solve the original problem, the analysis of her errors also led to an exploration which had not been planned by the instructor and which produced new insights about circles. (See a more detailed report of part of this activity in part **b** of this section).

Error Activity 6: Errors were used explicitly once again when the instructor presented a "wrong" proof for the theorem "All angles inscribed in a semicircle are right angles" and asked the students to find the error in it. This "proof" tried to show that the triangle, obtained by considering the diameter of the circle and the sides of the inscribed angle, is a right triangle because its sides satisfy the "Pythagorean theorem" equation $a^2 + b^2 = c^2$. In other words, the proposed proof tried to use just one of the properties, instead of a definition, to verify whether a figure was a right triangle or not. This exercise helped the students further clarify the distinction between *definition* and *properties* of a mathematical object (though some doubts remained still unresolved). It also made the students curious about the truth of the theorem, so that they then eagerly engaged in the derivation of the an alternative "correct" proof of this theorem, using the usual definition of right triangle as "A triangle with a 90^o angle".

Toward the end of this lesson we moved to consider the definition of a different geometric concept: *polygon*. Since the students were relatively unfamiliar with this concept, I expected that working towards the creation of a good definition of *polygon* would provide them with an experience similar to that of a research mathematician trying to define a new concept – an activity that characterizes much of a mathematician's research when exploring a new mathematical area, and which challenges some of the traditional requirements of mathematical definitions as identified in previous lessons.

Error Activity 7: Our search for an appropriate definition of *polygon* started again, as previously done in the case of *circle*, with an analysis of a few incorrect definitions of this concept (the ones produced by the students themselves in lesson 1). Several figures were analyzed in the attempt to decide whether they could be considered examples of polygons, and the original tentative definitions modified accordingly. The students had to realize, however, that this time this strategy could not lead them very far, since they were not always sure in the first place whether certain figures (such as a "bow-tie") should be considered *polygons*. As expected, this impossibility alerted the students to limitations in the strategy so successfully employed in the case of the familiar definition of *circle*. They were at a loss, however, about what alternative procedure to follow in order to come up with a good definition of *polygon*.

Lesson 5. (40 minutes)

In this lesson, we continued to work towards the creation of an appropriate definition of *polygon*, but this time followed an approach inspired by Lakatos' *Proofs and Refutations* (1976).

Error Activity 8: To get out of the impasse experienced at the end of the previous lesson, the instructor suggested to discuss some results about polygons which mathematicians could be interested in verifying and using. As an example, the tentative theorem "In an n-sided polygon the sum of the interior angles is $180^\circ \times n$" was suggested.[3] The activity of verifying the validity of this theorem in the case of a few "typical" examples of polygon involved the students in genuine problem solving, during which errors and steps in the wrong direction were naturally made and used constructively towards the solution of the problem (see a more detailed report of this part of the activity in part **b** of this section). As a result of this preliminary activity, the students proposed a modified version of the theorem ("In an n-sided polygon the sum of the interior angles is $180^\circ \times (n-2)$"), which they were convinced would work in the case of all the polygons they were familiar with, and that they would have liked to assume as characteristic of all polygons. This theorem thus provided them with some concrete criteria to examine whether some of the "borderline figures" considered at the end of the previous lesson should be considered as polygons, and in turn suggested some additional refinements in the tentative definition of *polygon* reached thus far.

Lesson 6. (30 minutes)

(Continuation of **Error Activity 2**) Starting from the confusion previously expressed in Lesson 2 by one of the students about the equations $x^2 + y^2 = r^2$ and $a^2 + b^2 = c^2$, we came back to the issue of the difference between variables and constants in equations, and touched upon the topic of ambiguous notation in mathematics.

The problem of writing a precise definition of *variable* was then addressed. The instructor's goal in this case was to make the students realize that there are mathematical concepts which we may intuitively understand and use, but which defy a precise definition.

Error Activity 9: The incorrect definitions of *variable* produced by the students in the first lesson were analyzed with this intent, though the activity this time seemed a bit too vague and abstract, and did not turn out very successful.

Lesson 7. (40 minutes)

The consideration of the operation of *exponentiation* was now suggested, since it seemed worthwhile to examine a case when a definition needs to be modified as the concept it characterises is extended to new domains. In this case, while exponentiation is originally defined as "repeated multiplication" when first introduced within the set of whole numbers, this meaning and definition have to be soon relinquished if we want to consider negative or fractional exponents as well. By engaging in the exploration of what happens to exponentiation in new domains, the students could then experience an activity very typical of

[3] Note that the text of the theorem proposed is incorrect, since the formula should read "$180^\circ \times (n-2)$".

a mathematician's research – that is, the extension of known concepts and results to new domains – and thus become aware of the power as well as the limitations of this process.

Error Activity 10: Since the students showed less familiarity with exponents than expected, however, the extension of multiplication to negative numbers and fractions was first briefly revisited. In particular, the students were brought to realize that, even in this more familiar case, the definition of *multiplication* as "repeated addition" fails as soon as we move outside of the counting numbers. In turn, this brought them to the realization that in mathematics the "correctedness" of a definition may depend on the mathematical context in which we are operating at the time.

In analogy to what had been done to extend the notion of multiplication, the students were now led to do the same with the notion of exponent. Through the creation of appropriate patterns, they developed a tentative definition for negative exponents ($a^{-n} = 1/a^n$), and checked whether known properties of exponents (such as $a^b \times a^c = a^{b+c}$) would still hold in this case. In the process, twice May spontaneously engaged in initially erroneous computations, which eventually brought us all to realize some properties of the extended operation of exponentiation not considered before.

Error Activity 11: In the first case, May's mistaken result $2^{-6} = 1/2^6 = 1/128$ brought the instructor to remind the students of the value of "breaking down" large exponents – in this case, $2^6 = 2^3 \times 2^3$. This observation made the students immediately identify and correct the mistake, and at the same time obviously contributed a new dimension to their understanding of the known property $a^b \times a^c = a^{b+c}$.

Error Activity 12: The second mistake occurred in the following computation:
$3^2 \times 12^{-1} = 3^2 \times 4 \times 3^{-1} = 3^{2-1} \times 4 = 12$. Rather than pointing out the mistake, in this case the instructor suggested that the students verify the result of this computation by using the definition of negative exponent ($a^{-n} = 1/a^n$) previously produced. This alternative procedure yielded a different result, and thus brought the students to reexamine May's original calculation. Without the instructor's intervention, May herself was then able to observe that the distributive property of exponentiation over multiplication should hold also in the extended domain, and to correct her first procedure.

Lesson 8. (40 minutes)

In the first part of this lesson, the extension of the notion of exponent was continued to include fractional exponents, and contributed the definition $a^{1/n} = \sqrt[n]{a}$.

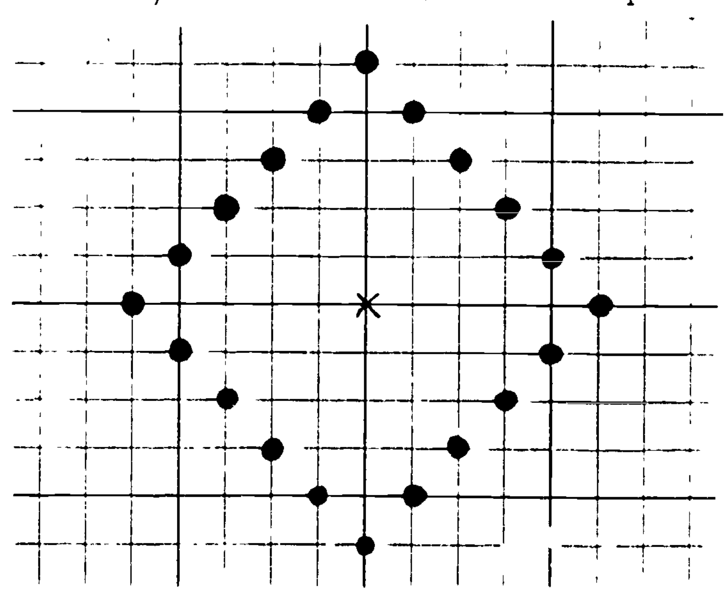
Error Activity 13: At home, Kim had attempted to use the following pattern to motivate the definition $2^{1/2} = \sqrt{2}$: $12^{1/2} = \sqrt{12}; 8^{1/2} = \sqrt{8}; 4^{1/2} = \sqrt{4}; 2^{1/2} = \sqrt{2}$. When she suggested this pattern in class, the instructor took the opportunity to clarify some fundamental points in the use of patterns as a heuristic to extend a definition – most essentially, the fact that we need to construct a pattern which will bring us from something we already know, to something we do not know yet.

The rest of the lesson was devoted to the exploration of what happens to *circle's* (as described by the usual metric definition) when we move to a context different from Euclidean geometry. The students were thus presented with *taxigeometry*, i.e., the geometrical idealization of a city with a regular grid of streets, where distance can no more be measured "as the crow flies", since we cannot go across buildings (Krause, 1986). In this context, *distance* is thus redefined as "the length of a minimal path to go from one point of the city to another along the streets". The activities and discussions which were generated around the definition of circle in *taxigeometry*, brought the students to realize the key role played by the context in interpreting mathematical definitions and also contributed in shaking some of their deterministic expectations about mathematics.

Error Activity 14: At the very beginning, the students jokingly challenged the instructor's definition of *distance* in taxigeometry, thus disregarding some of the constraints that had been previously set-up. At the same time, their "error" started a valuable discussion about alternative types of distances which could be considered in a city, depending on the constraints which one wants to considered.

Error Activity 15: Once we had finally agreed on how to compute distances in taxigeometry, the problem of "finding all the points in the city at distance 5 from a given point" was posed by the instructor. Kim's initial suggestion that the solution would be a usual circle of radius 5 contrasted with May's drawing of the correspondent of this locus in taxigeometry (i.e. the *taxicircle*, to which the students refered to spontaneously as *diamond*):

(INSERT FIGURE)



The students debated the issue of what was the correct answer for a while and finally concluded for May's taxicircle. In the process, a better understanding of the context of taxigeometry was achieved by both students.

Error Activity 16: At this point the instructor presented the students' with the apparent contradiction that the same definition ("All points equidistant from a given point in the plane") could characterize such different figures as the usual circle and the *diamond*, and asked them to re-evaluate the appropriateness of the metric definition accepted so far as a definition of *circle*. Realizing that the usual circle and the taxicircle did not share the same properties, brought May to conclude that the metric definition we had previously accepted needed some refinement. She set up to discover the "error" and fix it, and concluded very creatively that indeed the definition itself was correct, but <u>we</u> had misinterpreted it when applying it to the taxicircle, because in the case of taxigeometry we were not really considering the *Euclidean plane* (as we were keeping into consideration "buildings that are coming up in 3D") – a very interesting insight on what is the essence of the Euclidean plane! At the same time, this activity spontaneously generated in May the desire to define precisely *diamond*.

Error Activity 17: May and Kim engaged in the activity of creating a good defintion of *diamond* with very minimal intervention from the instructor. The process they used was based on the successive refinements of tentative definitions, so as to achieve the inclusion or exclusion of specific examples as appropriate – a constructive use of errors very similar to that previously made in **Error Activity 1** under the instructor's guidance, which now the students were able to employ on their own.

Lesson 9. (30 minutes)

In this lesson, we returned to the definition of exponent, in the attempt to challenge the students' expectation that mathematical definitions should eventually be "perfectly satisfactory and universal".

Error Activity 18: First, the new problems created when working with negative bases in conjunction with fractional exponents was proposed by the instructor and debated with the students. Possible solutions to this problem were discussed, which involved both imposing restrictions on the extended definition (by identifying "exceptions" to which the definition could not be applied) and other creative alternatives suggested by the students themselves.

Error Activity 19: The consideration of two different yet equally reasonable patterns brought the contradictory definitions of $0^0 = 1$ and $0^0 = 0$. The discussion of this unexpected result and the realization of the impossibility of resolving this contradiction had a profound impact on the students, and motivated interesting reflections on the nature of mathematical activity and mathematics itself (see a more detailed report of this activity in part b of this section).

Lesson 10. (30 minutes) – conducted with May only

We had planned to conclude our instructional unit with a discussion on similarities and differences between mathematical definitions and definitions in other fields, while revisiting the task of writing an appropriate definition for the nine concepts of *circle, square, polygon, variable, exponent, equation, cat, purple,* and *crazy* – with which the unit itself had started. Unfortunately, only one of the students (May) participated to this session, due to logistic reasons.

Error Activity 20: May experienced considerable difficulty in writing the definition of *cat* and was not satisfied with any of the alternatives she had been able to produce. Invited to comment aloud on her problems, she engaged in a very interesting discussion about what should be a "good" definition for *cat*. Though this discussion did not finally resolve her doubts, it certainly revealed a high degree of sophistication in her notion of definition and her ability to apply what she learned in the unit. In the course of this discussion, the instructor invited May to compare and contrast this situation with some of the mathematical definitions previously discussed. This enabled her to better realize some of the limitations existing when operating with mathematical definitions, and to appreciate the considerable variety existing within mathematical definitions.

b) In-depth report of selected error activities

Amongst the twenty *error activities* just described, parts of Error Activity 1, 5, 8 and 19 will now be reported in more detail.

Error Activity 1: *Analyzing a list of incorrect definitions of circle*

This first *error activity* illustrates well the variety of learning objectives which the analysis of the same error – in this case, a list of incorrect definitions of *circle* – can lead to, if approached from a variety of perspectives.

The students were presented with the task of analyzing the following list of definitions of *circle*:

A. *All the possible series of points equidistant from a single point (A)* (May)

B. *πr^2 circumference formula, = radius, an exact center,* 360^0. (Kim)

C. *Round - 3.14 - shape of an orange, coin, earth - Pi.*

D. *Circle = something whose area is = to* πr^2.

E. *Circle:* $(x)^2 + (y)^2 = r^2$. *Round.*

F. *A circle is a geometric figure that lies in a two dimensional plane. It contains 360 and there is a point called the center that lies precisely in the middle. A line passing through the center is called the diameter. 1/2 of the diameter is the radius. I don't like circles too much any more because they look like big fat zeros but they can be fun because you can make cute little smiley faces with mohawks out of them.*

G. *A closed, continuous, rounded line.*

H. *I sometimes find myself going around in them...*

The exercise was designed with the hope that by trying to understand why a specific definition was not "good," the students would become explicitly aware of the role of criteria such as *isolation of the concept*, *precision* or *essentiality* in mathematical definitions. Students are usually expected to know these criteria implicitly, though it may not really be the case. The students' familiarity with the notion of circle allowed them to quickly realize that all the definitions in this list were somehow incorrect. Trying to make explicit the rationale behind their evaluation, however, was not always immediate, and forced the students to make explicit some of the requirements they had subconsciously used. This process is well illustrated with respect to the requirement of "isolation of the concept" in the following excerpts[1] :

M.: "Closed continous rounded line": that could be just a spiral ... a closed one (she draws it)

...

M.: "[Circle = something] whose area is πr^2" ! ... improper English ...

R.: *Would that be a definition of circle?*

M.: No, it's an element of the circle ...

R.: *Why would you exclude [this definition] then? Can you show a figure which satisfies this definition and it is not a circle?*

(May suggested her "closed" spiral once again, and observes that it has the same area as the outer circle.)

The analysis of some of the incorrect definitions of *circle* in the list also brought the students to realize the possibility of requirements that are not easily appreciated by mathematics students, as shown in this exchange:

R.: *All the things you wrote down are correct, right? (referring to Kim's corrected definition of circle "$2\pi R$ circumference formula, = radius, an exact center, 360 degrees ")*

K.: *Yes. But I was not able to put down a round answer, I just put what came to my mind ...*

R.: *That's also what this definition F does. Why do you think we may not want to have a long list of*

[1] The dialogues reported in this paper have all been taken verbatim from the transcript of the instructional sessions; the following abbreviations have been employed: M.= May; K.=Kim; R.= researcher/instructor.

properties?

K.: *I am not saying that it would not be good, but ...*

R.: *Oh, you would like to have put even more?*

K.: *I just did not remember ...*

M.: But for a definition ... it should be stated as simply as possible ...

R.: *So, we want a definition to be able to identify only circles. And a long list of properties would probably do that even better. Then why you would not like it?*

M.: Why? Because a definition is something you have to remember ... you don't want to remember all the little things ... the whole list ...

This brief discussion opened the stduents to the consideration of the requirement of "essentiality" – an issue which would be further pursued other times along the unit.

The analysis of some definitions in the list also raised new questions which the instructor had not originally planned. For example, in the case of def.E, the students did not immediately see that the equation proposed ($x^2 + y^2 = r^2$) would describe only circles with center in the origin, and was thus too restrictive – which was the primary reason why the instructor had included this definition in the list! Rather, the students' analysis of def.E developed in other, even more interesting, directions, as shown in the following dialogue:

K.: *I don't know what x and y are ...*

R.: *You are right, we have to say what x and y are, or it does not make any sense ...*

K.: *Like, in mine, if I should do it over, I should say what r means ...*

R.: *Let's say we were using graph paper ...*

K.: *Oh, that makes sense!*

M.: But this is not the full sense of what a circle is ... because you do not always have graph paper ...

R.: *That's a good point. But we can say ... if you have a circle, you can put on it graph paper ...*

M.: With some work ... This is a good definition, though, because it will only give circles ...

R.: *But, how can you check if it does?*

On the instructor's suggestion, a specific value (of 5) is given to the radius, and the students start to complete a x-y chart, in order to plot some points on the graph paper. To check whether the points we plotted really belong to a circle, the instructor uses a compass to draw a circle with center in the origin and radius 5 - our points are right on the circle! This demonstration convinces the students, but also raises a new question:

M.: How would you figure out if something is a circle, if there is no measurement for the radius?

R.: *Ah! This is a good point!*

M.: What if they just say "circle", "draw a circle" and you are ... what's its r?

M.: So it doesn't work, because you don't know a circle if you don't know its radius.

R.: *Do you think you need also to know where it is placed? Where is the center of the circle?*

M.: No ...

Thus, the analysis of def.E unexpectedly generated valuable questions and reflections regarding analytic geometry, the representation of circles in that context, and what we should expect from mathematical definitions, which in turn resulted in new insights for the students about these important mathematical concepts. In addition, in the effort to pursue some of these questions, the students became engaged in the meaningful performance of a concrete mathematical task – the derivation of the equation of circle in Cartesian coordinates.

In the process of examining the incorrect definitions provided by the instructor, the students also spontaneously engaged in an attempt to create a "correct" definition of *circle*. One such definition was produced by a continuation of the previously described activity, and led to characterization of circles in the context of analytic geometry by means of the more general equation $(x-h)^2+(y-k)^2=r^2$. Another "correct" definition of *circle* resulted from an improvement of def.A (*All the possible series of points equidistant from a single point*). This activity was initiated by May herself (the author of this definition), in response to the instructor's comment that spheres were also described by this definition. By adding the condition that "all the points should be on a two-dimensional plane", May thus produced a version of the metric definition of *circle* reported in most geometry textbooks, a definition which would be used often in later activities.

Notice how the various activities developed around incorrect definitions of *circle* in this instructional episode engaged the students in valuable problem solving and genuine mathematical thinking, provided new insights into the notion of *circle* as well as into the principles of analytic geometry, and finally provided a concrete starting point and means to examine a rather abstract issue such as the characteristics of good mathematical definitions.

Error Activity 5: *Expected and unexpected insights are gained from debugging an unsuccessful homework assignment*

The analysis of one of the errors made by Kim at home, when trying to solve the problem of "finding the circle passing through the points (7,0), (5,4) and (6,-3)", provides a good example of how an analysis of errors can sometimes bring rewards that go beyond understanding what went wrong and how the problem should have been solved.

The activity was actually initiated by Kim's overwhelming concern with understanding and remediating her error, implicit in her request to the instructor:

K.: *It didn't seem right to me ... was any of this right?*

The student had actually set up the system of equations:

$$\begin{cases} (7-h)^2 + (0-k)^2 = r^2 \\ (5-h)^2 + (4-k)^2 = r^2 \end{cases}$$

and proceeded to solve it correctly (except for a minor computational mistake which was soon discovered and corrected). However, having considered only two equations, this procedure brought her still to an equation in two variables (2k=h–2), which she could not "solve".

By comparing Kim's work with the procedure presented by the instructor in the previous class to solve the same problem, it was possible for the student to realize that she had not used all the information provided in the problem, and also to see how her partial result could be used to reach the solution.

At this point, however, the instructor also realized that Kim's partial solution could provide new insights into the problem under study. This new awareness was communicated to and discussed with the students in the following conversation:

R.: *If you use only [the information from] two points, what this tells you is that there are really infinitely many circles that pass through those two points. Do you understand what I am saying?*

M.: Many circles are passing through those two points just because they can change all the time?

R.: *Right, many values of h and k satisfy this equation ...*

M.: k and h are the [coordinates of the] center. right?

R.: *Yes!*

M.: Wait, wait! No!

R.: *I might be wrong!*

M.: It can't be a bigger circle ... because just as you drew it goes like that (she shows on the figure that this circle would not pass through the third point) ... if you proved, the way she got this ... did she use two of these points?

R.: *Yes, she used only two of them ... these two ...*

M.: If she pulled in the third point, then it would make it definite, because ... there is no other circle than that one that would pass through all three.

R.: *So if we have three points, only one circle passes, If we have two – which is what Kim started to do – then we find many circles that pass through those points. But only one of those will pass also through C (the third point).*

(pause)

R.: *Can you notice anything special about [the circles passing through the two given points]?*

(a few values for the coordinates of the center of those circles are computed once again, and plotted on the graph)

M.: It's going to be a straight line! Because I remember when we were doing the line equation [in class, the previous semester] it always ended up like that (pointing to the equation 2k=h-2).

Thus, Kim's error led us all to learn something new about the circles passing through two points, as well as the relation between this new problem and the original one of "finding the circle passing through three given points". Paradoxically, all this would have been missed had Kim not made an error!

Error Activity 8: *Encountering errors in the process of solving a novel problem about polygons.*

Kim and May often engaged in the solution of original problems throughout the teaching experiment. Inevitably, in those cases they did not immediately hit on a successful approach, and consequently had to deal somehow with their false steps or errors. I chose to focus here on one such occasion, which occurred at the beginning of Error Activity 8, when the students were trying to prove informally the following (incorrect) theorem about polygons, suggested by the instructor:

"In any polygon the sum of the interior angles is 180° times the number of sides. "

Here is how the conversation started:

R.: *How do you think we can prove something like this?*

M.: I don't know. Take a polygon as an example. (The instructor immediately draws one, an "almost regular" pentagon.)

M.: We never really got to the definition of a polygon. We think this is a polygon.

R.: *Right, so for the moment we think it's a polygon. How to figure out the sum of the interior angles?*

M.: Put a circle around it that meets all the points.

R.: *(draws a very "skinny" polygon) What if the polygon is like this?*

K.: *(has a great insight)* Make it into triangles.

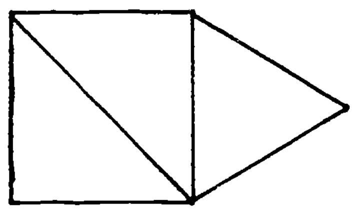
M.: Take a center point. But if it's really weird shaped you can't do it... Oh yes, you could do it.

R.: *It seems that whenever I have a polygon I can pick a point in the center, more or less. Does it matter which point I pick? Maybe we should ask Kim why it is that you wanted to break it down into triangles like this?*

Notice how, in this beginning stage of the process, various suggestions and ideas were raised, some of which could be considered as "errors" since they led nowhere or could have engaged the students in an unnecessarily complex solution process (as May's idea of drawing a circle around the polygon in this case). In the dialogue reported above, the instructor played the role of moderator, selecting which ideas to follow among the many suggested.

Later on, however, the students themselves showed their ability to evaluate alternative options, and to identify and correct potential errors. An example of this can be found in the conversation which developed a little later during the same activity, after Kim had pursued her idea of breaking up the original polygon in triangles and using the property that the sum of the angles in a triangle is 180°, and produced the following figure:

INSERT FIGURE



R.: *Now that we broke [the pentagon] into triangles, what can we say about the angles of these triangles?*

M.: That is 3×180^o altogether.

R.: *Okay. Correct? Then is my theorem correct?*

K.: *(not having seen it yet) Why do you say this is 3×180^o?*

M.: (pointing to the 2 triangles in the "square" part of the pentagon) there is two triangles in there.

K.: *(still doubting) But you could make this* (and she draws another diagonal in the square, thus producing *5 triangles in all).*

R.: *Okay. interesting. Then you would have 5 triangles.*

M.: Good point... (almost believing she has been proven wrong).

R.: *Well, the sum of all the angles of these triangles is 5×180^o. We are sure of that. But is it the sum of the angles of the original polygon? Or are we having something extra here?*

(Kim immediately points to the four central angles).

R.: *So if you want to use this idea, you should take away these extras. How much are these extra angles?*

M.: All together?

K.: *...okay, these* are *all* 90^o

M.: Wow! 90^o, these are $180^o \times 4$... (she has the right idea, but misspeaks. The instructor quickly corrects her.)

M.: (doubting that such a conclusion can be drawn) But you haven't proven that's a square yet.

R.: *No, that's correct. Can we say this is 360^o even without knowing if this is a square?*

M.: (with another insight) Yes, you can, because it's a circle! All the angles put together make one big angle all the way around! Wow! We are the "discovery channel" today!!

R.: *So we have 360^o, which is like 2×180^o. So this is like $5 \times 180^o - 2 \times 180^o$, which is like 3×180^o. So now the two [methods] have given us the same result. Does it fit with the theorem I stated? How many sides did the polygon have here?*

M.: Five.

R.: *So, instead here we have 3 (pointing to the 3×180^o) so do you trust more my theorem, or do you trust this?*

M.: (without hesitation) Trust ourselves.

R.: *So the theorem was wrong, so I suggest a correction. (She adds -2 to the original theorem, which now reads: "In any polygon, the sum of the interior angles is* $180°$ *times the number of sides -2".)*

Notice how the students showed considerable creativity and mathematical intuition in this process, and used their own errors constructively, though with some help from the instructor. It seems also important to remark how the students' attitude in this episode was quite alert and critical, and how they showed more trust in their own logical reasoning than in a statement initially proposed by authority. Starting with something that later on proved to be incorrect made the students even more cautious in their activity, and brought May to spontaneously suggest trying the "correct" theorem out with other, stranger examples of polygon.

Error Activity 19: *Facing an inevitable contradiction in trying to define* 0^0

In the pre-college mathematics curriculum, students rarely experience "errors" that come as a result of limitations of mathematics itself, rather than being the product of their own shortcomings or ignorance. The unresolvable contradictions encountered when trying to define 0^0 by using patterns presented an interesting example in this sense, accessible to the two students.

The instructor first of all set up a situation which would lead the students to consider the plausibility of either 1 or 0 as possible solutions, and then faced them with the contradiction this led to (which in turn could make one consider either solution as an "error"):

R.: *I'll show you another thing that gave me a problem. What do you think is* 0^o*?*

K.: *Zero?*

R.: *And how do you think...why do you think that?*

M.: It's not...it's undefined.

R.: *How would you justify that it is undefined?*

M.: Because....well, I can see how she's going to say zero times zero equal zero; but I would say undefined because you can't raise something to the power of zero; that makes it totally imaginary.

K.: *(emphatically) You can!*

R.: *You can, right? Do you remember [how we computed]* 2^o*?*

(The instructor briefly reminds them how we had created a pattern that justified the definition $2^o = 1$, and invites the students to repeat the procedure to evaluate 3^o and 4^o. As a result, the following pattern is produced: $5^o = 1, 4^o = 1, 3^o = 1, 2^o = 1, 1^o = 1$.)

R.: *So a pattern would give you that. All these numbers will give you one. So it seems to be reasonable that [*0^0*] gives you one too (writes* $0^o = 1$*).*

(The students seem convinced, but at this point the instructor shows the possibility of an alternative pattern)

R.: *What about this pattern?* $0^5 = 0, 0^4 = 0, 0^3 = 0, 0^2 = 0, 0^1 = 0$.

K.: *Equal zero.*

R.: *So in this case we might even argue that this [0^0] should be zero. (writing $0^\circ = 0$)*

K.: *Makes more sense to me that that would be zero.*

R.: *But this is really the kind of situation that I think makes mathematicians mad. I mean, you have two patterns. [Patterns] seemed to work so nicely before. We got all this nice extension of the definition. And this time one pattern gives you one result, and the other pattern gives you the other result. So, how do you think we can deal in situations like this? (long pause)*

K.: *I don't know...I don't know with this one...*

M.: I... that zero... negative zero, or something like that. I don't like it.

R.: *(agreeing) That's probably what the mathematicians would have said too: I don't like it!*

M.: Pretend we never saw it...

R.: *Well, pretending that you haven't seen it, that doesn't help too much, because you might find yourself in a situation in which this occurs.*

The importance and educational value of presenting students with a situation like the impossibility of defining 0° is further made evident by the following exchange:

R.: *Was there any other time in which you had to deal with a situation more tough than you really wanted, in mathematics. A definition didn't fit.*

M.: Just about every test I took for the last three years actually. (more seriously now) In which a definition didn't fit? (thinking)

R.: *No? It never happened?*

M.: Not off hand, no, but I do know it did happen....but I just don't remember it... I figure it was... but it might have been my fault... (inaudible)

R.: *But sometimes it may not be your fault. It may be more the peculiarity of the situation we are in here.*

The students were clearly surprised to find limitations inherent in mathematics itself, since they never encountered any before! It is interesting to see how May, especially, struggled with this realization both in this very lesson and later on when asked to reflect on it in various interviews. The first reaction was (quite reasonably) one of dismay and frustration:

M.: The thing about this confusion is, I sort of doubt, like, I don't know, if we can't figure that out, who's to say the rest of these are right? Then we are going into say, what mathematicians that were there said, was the pattern, and that's the way it's supposed to be, if like, we're already confused about that. And that's relatively simple, and that's just an exponent. I'm wondering about all this stuff that we are learning, ten years from now well find out, you're wrong!

R.: *I see your point. How can we be sure, say, that this setup works. The pattern is only kind of a way to start guessing it. The second step, if you remember, was I was trying to see if rules and operations of exponents seem to be maintained with this definition. And it seems to work, but sometimes it* *breaks down, like here you'll notice it breaks down with zero, it breaks down with negative numbers And the reason is, that first definition is....*

M.: (interrupting) those are pretty large groups.

R.: *They are. Okay, what do you think then mathematicians have to do though. It doesn't always work the way they want.*

M.: What I think is that they would be working towards one that would be more universal, that included all of the exceptions. I don't know. What seems to be wrong with that, if this seems to be so completely right, if the entire negative part of whole number is not included?

R.: *Yes.*

M.: It's....it's not right! (emphatically) It's "prejudiced".

May, however, tried to overcome her first negative feelings and deal more constructively with this immediate discomfort. Her first attempts aimed at "fixing" the problem somehow, by finding a different "pattern" which would work better:

M.: I don't think there is an error in our thought process, but I think, Aaah, there might have been an error in the patterns we were looking for. Right? I can believe that there are patterns that would show up and be correct but maybe we were looking for the wrong patterns. And it might have seemed like a pattern all the way down and then it wasn't a pattern. So it is not a pattern at all and we should have tried a new one. But I couldn't think of a new one that was the most practical pattern we could find. Aaah, with things like that I really with some would consider them errors because even the ... I might consider them errors in the whole creation of mathematics and 0-10, but that's pretty broad. So, I'd rather not see it as an error just that you should have been looked at a different way. And if we had looked at it every way possible, all the way around it, then it's not really an error it's just a complication. That we will avoid.

Since this approach did not lead her very far, May then suggested the idea of trying to "change systems," so as to overcome the problem – a quite creative approach and one of which any mathematician could be proud! Though this idea was unfortunately not pursued, it is interesting to see in the dialogue how even just conceiving this possibility had an impact in this students' conception of mathematics:

M.: I like concentrating on things that can't be solved in math. I know we came to that point with the zero [to the power zero].

[...]

M.: I was thinking about, you know, when we come to an equation like that, when it just cannot be figured out by me, or by the next person, and then it just reminds me that this was all invented by people. It's not something like we are born and there is a tree and it has been there forever. It's like we invented this, out of our minds. And we invented zero to ten, and the whole number system and all the other number systems, and, and, so...

R.: *And it could have been invented in a slightly different way.*

M.: Yeah, that too. And then I was thinking about, what if we had decided that it was one through ten, and there is no zero (laughter), then our problems would be solved! Then we wouldn't have those essays to write!

R.: *Maybe some problems would be solved, but others would remain. And in fact some other would be so big that that's why they invented zero.*

M.: It would be neat though if some people just decided that that's the way their numbers system is going to be so the whole world uses it, umm, I don't know.

This episode suggests that some errors have the potential to stimulate students' reflections on the nature of mathematics, make them realize that mathematics has been created by human endeavor, and thus make it feel more accessible to them.

IV. A Taxonomy of Constructive Uses of Errors

The variety within the *error activities* developed in the teaching experiment is indeed remarkable, and reveals that using *errors as springboards* cannot be perceived nor evaluated as a monolithic instructional strategy. In particular, the *error activities* identified and described in the previous section presented important differences in learning goals and outcomes. If mathematics teachers are interested in engaging their students in the constructive use of some mathematical error, it will be important that they are aware of what could influence the scope and results of such an activity.

A first important dimension comes to mind when we look at an articulated *error activity* such as **Error Activity 1**. In that case, the analysis of a list of incorrect definitions of *circle* from different perspectives brought the students to different kinds of mathematical learning, all valuable and complementary to each other. On one hand, trying to identify what was wrong in a few specific definitions and "fix them up" was instrumental in allowing the students to successfully complete the task of producing a correct definition of *circle*. An analysis of the mathematical properties mentioned in the list of definitions proposed (regardless of their appropriateness as definitions of circle), brought the students to appreciate the relationships and relative importance of characterizing properties of circle, and ultimately contributed to a more sophisticated understanding of the concept of *circle*. Finally, operating at yet a higher level of abstraction, the attempt to identify and characterize what made some of the definitions in the list obviously unacceptable, helped the students become aware of the rationale behind some of the requirements usually imposed on mathematical definitions, and thus led them to a deeper understanding of a meta-mathematical notion such as *definition* and to a better appreciation of how criteria and rules come about in mathematics.

More generally, it seems important to appreciate that the mathematical inquiry and insights which could be motivated by an error could take on different forms depending on the mathematical context in which one is operating. Even more specifically, it seems important to distinguish between three different **levels of mathematical discourse**, which are respectively:

- **performing a specific mathematical task** – that is, solving a problem, performing a computation, attempting to prove a result, producing an acceptable definition for a given concept, etc.
- **understanding some technical mathematical content** – be it a concept, rule, or topic such as *circle*, *exponentiation*, or *Cartesian coordinates*.
- **understanding about mathematics** – this could involve understanding meta-mathematical notions such as *definition, proof* or *algorithm*; becoming aware of helpful heuristics as well as of their domain of application and limitations; appreciating what characterizes mathematical thinking and mathematics as a discipline; and so on.

Another factor, independent of the mathematical context or content of the *error activity*, seems also to have affected considerably the students' attitudes and goals in the analysis of specific errors. For example, when the students analyzed obviously incorrect definitions of *circle* in the effort to come up with a correct one (*Error Activity 1*), or when one of the students asked the instructor to go over her homework to see "what had gone wrong" (first part of *Error Activity 5*), the activity was essentially one of *debugging*, where a result was clearly recognized as incorrect and analyzed in the belief that it could contain valuable information to reach the original objective. The situation and use of errors was quite different instead when the students were engaged in a genuine problem solving or discovery activity. Since in this case they did not know a-priori what results to expect, it was difficult for them even to recognize when something was an error or not, and yet it was essential to take advantage of steps in the wrong as well as in the right direction in order to come up with a solution to the set task (see for example *Error Activity 8*). Finally, throughout the teaching experiment we can also notice many instances when an error generated new questions which may not have directly contributed to reaching the goal originally set for the activity, and yet stimulated inquiry and learning in valuable directions. Think for example of how, in the second part of *Error Activity 5*, the partial and insufficient result reached by the student with respect to the task of *finding the circle passing through the 3 given points* provided valuable information for yet a different problem, *finding the circle passing through 2 given points*; or consider the remarkable surprises and new insights about exponentiation, definition, and mathematics more generally, that were derived from the impossibility of avoiding contradictions in the definitions of 0^0 attempted (*Error Activity 19*).

The differences highlighted in the above paragraph seem to depend not so much on the mathematical content of the *error activity* but rather upon the nature of the learning context and the degree of "open-endedness" of the lesson. It seems thus important to identify and consider the following distinct **stances of learning** as another element which may considerably affect the constructive use made of specific errors:

- **Remediation:**
  The spirit of this learning stance, and its implications in terms of the constructive uses of errors it may invite, are well captured by the following quote, produced by one of the two students participating in the teaching experiment:

  K.: *I tried to look at my error and tried to figure out why I made that error and to see what that error led to. And to see if I could learn something from that, instead of just saying 'Oh, I made this error, I did it wrong, and this is the right way.' To look at the reasons behind it*

  Notice how this stance of learning assumes that the student is aware that his/her result is not correct (though he/she may or may not know what the correct result is), and that both the question and the answer are predetermined and known to authority. There is also a sense that the student should have known how to produce the correct answer in the first place – though something obviously went wrong, and now it needs to be identified and corrected.

- **Genuine learning/discovery:**
  Whenever we attempt to learn something new, or to solve a genuinely novel and original problem, error-making will take on a different connotation. Obviously, we would not expect to be able to get to the correct solution immediately, but rather that in the process we will make some moves in the wrong direction, which must be eventually identified and corrected. Once again, this approach was implicitly identified by one of the students, who described it in the following words:

  M.: Before I said something ... about how we were presented with errors. But we weren't really, we were presented with a question like: "How?". And then we would try to figure out and we would come to errors first. ... you make an error, and you try to correct it and you just work through, and ... I guess that errors were a big part of it. Just because they came with our learning process.

  Note how, when this stance of learning is assumed, the student is not expected to know the answer in advance, and the question or task set to him/her, as well as the way to approach it successfully, is more open-ended than in the previous case. However, the task/question itself is still set for the student and unquestioned, and its solution is perceived as predetermined and known to authority.

- **Openness to challenging the given:**
  The most radical uses of *errors as springboards* realized in the teaching experiment occurred when the students were allowed to change the direction of the inquiry or the definition of the task originally set in the lesson. Once flexibility in redefining

the learning objectives is accepted, it is possible to recognize and follow up questions and inquiry which could have been motivated by an error within a specific situation, but which may have little to do with the task originally set. Or, the analysis of the errors encountered may reveal the existence of inherent limitations (in the concept, problem or mathematical issue studied) which had not been initially envisaged but may now suggest the appropriateness of redefining the original task as well as of our expectations.
In other words, within this learning stance, one assumes that neither the answers nor the questions are necessarily predetermined, and that detours as well as redefinitions of the original task may be appropriate and desirable.

By combining the categories identified above, a 3x3 matrix of constructive uses of errors, accessible to mathematics students and appropriate to enhance learning in mathematics instruction, can be generated, as illustrated in a matrix form in Table I.

Table I

A Taxonomy of Constructive Uses of Errors

Level of Math Discourse / **Stance of Learning**	*Performing a Math Task*	*Learning Technical Content*	*Learning ABOUT Nature of Mathematics*
Remediation	R_p: Analysis of recognized errors to understand what went wrong and fix it, so as to perform the set task successfully	R_c: Analysis of recognized errors so as to clarify misunderstanding of technical mathematical content	R_m: Analysis of recognized errors so as to clarify misunderstandings regarding the nature of mathematics or general mathematical issues
Genuine Learning – Discovery	L_p: Errors are used constructively in in the process of solving a novel problem or task	L_c: Errors are used constructively for for the learning of a new concept, rule, topic, etc.	L_m: Errors are used constructively for for learning about the nature of mathematics or some general mathematical issues
Openness to Challenging the Given – New Directions	O_p: Errors motivate questions which may generate inquiry in new directions and new mathematical tasks to be performed	O_c: Errors motivate questions which may provide new perspectives and insights on a concept, rule, topic, etc.	O_m: Errors motivate questions which may provide new perspectives and insights on the nature of mathematics or some general mathematics issues

Illustrations of the application of *each* of the 9 uses of errors thus identified can be found in the teaching experiment, as summarized in table II – where in the box corresponding to each use of *errors as springboards*, I have reported the numbers corresponding to the *error activities* where such a use of error was employed. It is important to note that in most cases a combination of constructive uses of error was employed in the same *error activity* (as it is evident in the episodes reported in more detail in section IIIb), and thus the same number will often appear in more than one box in table II.

Table II
Examples of Variations of a Use of Errors as Springboards Observed in the Teaching Experiment

Level of Math Discourse / Stance of Learning	*Performing a Math Task*	*Learning Technical Content*	*Learning ABOUT Nature of Mathematics*
Remediation	R_p: 1,5,6,9,11,13	R_c: 1,2,9,11,13,14	R_m: 1,6,13
Genuine Learning - Discovery	L_p: 3,7,8,12,15,16,17	L_c: 7,8,12,15,16	L_m: 8,16
Openness to Challenging the Given -	O_p: 5,6,8,10,16,18,19	O_c: 1,2,4,5,10,14,18,19	O_m: 2,4,7,9,10,18,19,20

Results reported in Table II show that all the uses of errors identified by the taxonomy are feasible for secondary school students and that they are obviously complementary to each other. At the same time, the dimensions and differences highlighted by the taxonomy suggest that using *errors as springboards* is a more articulated strategy than what it might have appeared at first, and that it may thus be worth assessing the educational values and implications of variations within the strategy, rather than of the strategy as a whole.

In the next session, the empirical data collected in the teaching experiment will be used once again to generate some working hypotheses to this regard.

V. Hypotheses about Pedagogical Implications Of Using Errors as Springboards

A further analysis of the twenty *error activities* developed in the teaching experiment, conducted with respect to the learning outcomes produced, brought to the identification of a number of potential benefits which could be associated to a use of *errors as springboards.*

These potential benefits have been briefly stated and listed below (in boldface), followed by the identification of the *error activities* where they were observed. To each potential benefit, the specific use(s) of errors made in the associated *error activities* were then examined, searching for emerging patterns which would suggest a relationship with specific variations within the strategy of *using errors as springboards*. As a result, in most cases it was possible to identify specific elements in the taxonomy as the most appropriate ways to use errors constructively which could bring along such an outcome. The results of this analysis have been reported below in schematic form:

a. **The study and analysis of errors can provide opportunities to engage the students actively in valuable and creative mathematical activities.**

Evidence supporting this claim can be found in the following error activities:

practically every one

Most appropriate ways of using errors to this end:

all nine

b. **Errors can motivate students' curiosity and attention, because an error shows a contrast with what is initially expected, or it may present the possibility of new alternatives.**

Evidence supporting this claim can be found in the following error activities:

practically every unit (though it is especially evident in 5, 8, 10, 16)

Most appropriate ways of using errors to this end:

all nine

c. **An analysis of errors can help students gain a better conceptual understanding of mathematical content, by identifying and clarifying misconceptions, highlighting new aspects and uncovering unexpected elements.**

Evidence supporting this claim can be found in the following error activities:

2, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18

Most appropriate ways of using errors to this end:

- R_c (Remediation w.r.t. learning technical Content) ,
- L_c (Use of errors within Learning/discovery w.r.t. learning technical Content) ,
- O_c (Errors Open new directions w.r.t. learning technical Content)

d. **A constructive use of errors can help students in the solution of problems or the performance of other mathematical tasks, by providing relevant information and concrete starting points.**

Evidence supporting this claim can be found in the following error activities:

1, 3, 4, 5, 6, 8, 16, 17, 20

Most appropriate ways of using errors to this end:

- R_p (Remediation w.r.t. Performing a task) ,

- L_p (Use of errors within Learning/discovery w.r.t. Performing a task) ,
- O_p (Errors Open new directions w.r.t. Performing a task)

e. **The analysis of errors can make more concrete, and thus accessible to the students, the discussion of more abstract issues (either regarding specific mathematical content or the nature of mathematics and general mathematical notions).**

Evidence supporting this claim can be found in the following error activities:

1, 2, 3, 4, 6, 7, 9, 13, 16, 19

Most appropriate ways of using errors to this end:

- R_m (Remediation w.r.t. learning about Mathematics) ,
- L_m (Use of errors within Learning/discovery w.r.t. learning about Mathematics) ,
- O_m (Errors Open new directions w.r.t. learning about Mathematics)

f. **Errors may invite the students to generate new questions and problems, and thus engage students in problem posing as well as problem solving activities.**

Evidence supporting this claim can be found in the following error activities:

1, 5, 6, 8 ,11, 12, 14, 16

Most appropriate ways of using errors to this end:

- O_c (Errors Open new directions w.r.t. learning technical Content) ,
- O_p (Errors Open new directions w.r.t. Performing a task) ,
- O_m (Errors Open new directions w.r.t. learning about Mathematics)

g. **The consideration of certain errors may help students realize some inherent limitations existing in mathematics, and could thus help them appreciate some of the more humanistics aspects of the discipline.**

Evidence supporting this claim can be found in the following error activities:

8, 9, 10, 16, 18, 19

Most appropriate ways of using errors to this end:

- O_m (Errors Open new directions w.r.t. learning about Mathematics) , and to a lesser extent:
- O_c (Errors Open new directions w.r.t. learning technical Content)

h. **The experience of paying attention to and working with errors may make the students more cautious in their mathematical activity, and more independent from authority for the verification of their work.**

Evidence supporting this claim can be found in the following error activities:

most error activities (8, 10 - 18 in particular)

Most appropriate ways of using errors to this end:

all nine

These results confirmed the expectation that, while all the nine variations identified in the taxonomy can provide valuable outcomes and contribute to students' learning of mathematics, they may do so in different ways. Specifically, it is worth remarking that variations in the level of mathematical discourse chosen for the analysis of errors may be especially important to foster a comprehensive learning and understanding of mathematics – integrating fundamental aspects such as the conceptual understanding of technical concepts and rules, the ability to apply these notions appropriately in a variety of circumstances, and the appreciation for their role in mathematics as a discipline. More indirectly, the learning stance assumed in the study of an error can also influence a student's approach to mathematical activity (and, consequently, his/her performance in this subject). As suggested by hypotheses f and g in particular, taking advantage of the potential of errors to open new directions for inquiry or to challenge the given may have an important role in challenging most students' view of school mathematics as an impersonal and uncreative domain, where their task is reduced to assimilate and reproduce rules provided by the teacher.

In sum, this analysis suggests that maximum benefits from engaging students in a constructive use of errors can be achieved when all the nine variations of this stategy are taken into consideration and employed, as appropriate, in mathematics instruction.

At the same time, it may be important to contrast this conclusion with the current reality of school mathematics practice. Within most of the current mathematics curricula, the educational objectives tested (and consequently stressed in instruction) are unfortunately often reduced to the performance of specific tasks and the development of technical skills. As a consequence, little attention is paid to developing conceptual understanding of the technical content addressed, and even less to the development of better appreciation and understanding of mathematics as a discipline. Similarly, the mandate from States or School Districts to cover considerable amounts of "material" in relative short periods of time may discourage teachers from allowing for "digressions" from their carefully planned lessons (regardless of the mathematical interest of the inquiry thus generated or the benefits the students could gain). Under severe time constraints, it often difficult to employ a problem-solving /discovery approach even for topics required by the curriculum – as this would obviously require considerable more time than other more "direct" instructional approaches. Thus, in an instructional climate where teachers' explanations followed by students' practice is the norm, one can expect considerable obstacles to engage students in a use of errors along the stances of learning characterized as "new learning" and "openness to challenging the given".

In sum, one would expect that an implementation of variations of the strategy of using *errors as springboard* in current mathematics instruction will encounter obstacles as one moves from the top-left corner to the bottom-right corner of the matrix representation of the taxonomy, as reported in table I. If educators really value the kind of learning outcomes and educational benefits associated with all the nine uses of errors identified by the taxonomy, it will be important for them to create compatible conditions in their classroom in order to implement successfully all variations of the strategy.

VI. Conclusions

At the beginning of the study reported in this paper, the value of engaging the students' themselves in the analysis and study of specific mathematical errors had already been suggested. This approach had been distinguished from other more common pedagogical uses of errors in mathematics instruction, which could be briefly described as:

- the teacher ignores the error (either voluntarily, or because s/he does not recognize it);
- the teacher identifies and corrects the error by providing the correct answer;
- the teacher uses the error as a tool to diagnose the student's specific learning difficulties, and to plan remediation accordingly.

The development and analysis of the teaching experiment discussed in this paper has now contributed new insights into the proposed strategy of using *errors as springboards*, which I will try to briefly summarize here.

First of all, the study has suggested that a pedagogical approach to mathematical *errors as springboards* is not a monolithic strategy, but rather one which can vary considerably in relation to both the **level of mathematical discourse** and the **stance towards learning** assumed in the lesson.

With respect to the mathematical dimension, it may be important to distinguish whether the analysis of the error engages the students in *performing a specific mathematical task*, *learning about some technical mathematical content*, or *learning about mathematics as a discipline*. While each of these mathematical activities are valuable and important for mathematics instruction, they may serve different and complementary goals in increasing students' learning of mathematics – such as increasing the students' ability in doing mathematics, the students' conceptual understanding of mathematical concepts and topics, and finally also the students' appreciation of the nature of mathematics and mathematical thinking.

Within each of these mathematical contexts, the **stance towards learning** assumed can also influence considerably the use made of errors. An analysis of errors may help in determining what went wrong and may suggest possible *remediation.* Or, errors might be approached as *steps in the wrong direction* which can help when learning or solving something genuinely new; here the goal is to see how the new information errors provide can shed light on the original problem/topic under study. Or, finally, the error might *stimulate new questions and explorations*, which may move the learner away from the original pursuit and towards new mathematical discoveries and learning.

By combining the categories identified above, a 3x3 matrix of constructive uses of errors accessible to mathematics students has been generated (see table I on page 27).

The empirical data collected so far suggest that the nine uses of errors identified by this taxonomy are all feasible to high school mathematics students and complementary to each other, yet they also present important differences – in terms of the types of mathematical results that they can help to achieve, the educational objectives they may facilitate, their compatibility with current instructional goals and practices, their demands in terms of creativity and improvisation from both teacher and students, and their potential impact on students' conceptions of mathematics and errors.

Mathematics teachers interested in implementing an approach to *errors as springboards* should be aware of these differences, if they want to take best take advantage of the educational potential of this strategy in relation to their own instructional goals and teaching styles.

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