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ABSTRACT

Teacher preparation curriculum at Technion--Israel Institute of Technology (Israel) includes courses intended to bridge the gap between the pure mathematics courses and those in psychology. The focus of this paper is an experimental program for one of these courses and data collected while implementing it. This is a second report on a naturalistic study in which mathematical paradoxes were used in the preservice education of high school mathematics teachers. The potential of paradoxes was tested for improving student-teachers' mathematical concepts and raising students pedagogical awareness of the role of falacious reasoning in the development of mathematical knowledge. Discussions include the psychological and mathematical background, the experimental courses and data collection procedures, the students, and findings. Examples of paradoxes are given. Included are 36 references. (DC)

Preservice Education of Math Teachers Using Paradoxes

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Preservice Education of Math Teachers Using Paradoxes

INTRODUCTION

This is a second report on a naturalistic study of the role mathematical paradoxes can play in the preservice education of high school mathematics teachers. We wanted to test the potential of paradoxes as a vehicle for:

(a) sharpening student-teachers' mathematical concepts;

(b) raising their pedagogical awareness of the constructive role of fallacious reasoning in the development of mathematical knowledge.

The first report (Movshovitz-Hadar 1988) was a brief one, and focused on the problem, the procedures and the findings in a general way. The present report is more detailed. Results obtained in parts of the study are added, analyzed and discussed, in order to substantiate the general findings.

THE PROBLEM

Mathematics teacher-educators all over the world look for means for integrating the teaching of contents with the teaching of psychological and pedagogical issues (Dorfler 1988 pp. 181-182). Teacher preparation curriculum at Technion, Israel Institute of Technology, includes courses intended to bridge between the pure mathematics courses, taken towards a B.Sc. degree in mathematics, and the psychology and methods courses taken towards a high school mathematics teaching certificate. At the focus of this paper are an experimental program the first author has been developing for one of these courses, and the qualitative data systematically accumulated in recent years, while implementing it.

PSYCHOLOGICAL and MATHEMATICAL BACKGROUND

Cognitive conflicts have long been a part of psychological theories of cognitive change (Cantor 1983). According to Piaget's theory, adaptation-accommodation on the one hand, and assimilation on the other, are usually in balance, creating a state of cognitive equilibrium. There are critical times, however, when cognitive conflict produces a state of disequilibrium, which results in the creation of a more advanced mental structure (Flavell 1963, Piaget 1970, 1977). Fujii (1987) explored the role of cognitive conflicts in enhancing a change from instrumental to relational understanding of mathematics as proposed by Skemp (1976).

A cognitive conflict is strongly related to paradoxes. A paradox is created when two (or more) contradicting statements seem as if both are logically provable. Clearly, at least one of the statements must be false, and its proof must have a flaw. However, as long as both statements seem convincing enough to forbid a resolution on the part of a human being facing them, this person is in a state of a conflict. Such a conflict, between two competing ideas, is one of three types of cognitive conflicts defined by Sigel (1979).

Cognitive conflict is usually a tense state. According to Berlyne (1960) it plays a major role in arousing, what he called, epistemic curiousity. An ordinary person has a strong incentive to relieve the conflict as soon as he or she can. Consequently, it drives one to an intensive, and hopefully fruitful, activity of thinking, and of critical examination of existing knowledge.

Resolving the cognitive conflict a paradox presents, involves turning at least one of the proofs into a fallacy, by pinpointing the invalid reasoning, and 'debugging' it. Debugging, as Papert (1980) indicated, is a powerful mechanism in the process of learning.

Paradoxes played a very important role throughout the history of mathematics. Some of the greatest mathematicians were astound when their theory yielded a paradox. Untangling it, which in some cases took quite a long time, and required a major intellectual effort, contributed significantly to further developments of the mathematical theory. Some nice examples can be found in Kleiner (1988), in Mieddleton (1986), and in Eves (1983).

The educational lesson is twofold:

a. In mathematics, as in other areas, to err is human. The freedom to err is at the heart of developing mathematical knowledge.
b. It is insightful to find the invalid logic underlying a mistaken proof. The outcome of this process is further development, refinement and purification of mathematical concepts and theorems.

We assumed these two to be very basic ideas, which every mathematics teacher should acquire. Providing for their acquisition was the ultimate goal of, and the leading thought in the development of the experimental course program.

THE EXPERIMENTAL COURSE and DATA COLLECTION PROCEDURES

Course material development was based upon two major sources: the literat.re on mathematical paradoxes (Bunch 1982; Gardner 1982, 1983; Huck and Sandler 1984; Northro 1975; Smullyan R. 1978) and an inventory of mathematics errors commited by high school students (Movshovitz-Hadar N. et al. 1986, 1987).

In each class period, following a brief expository introduction, students were handed a worksheet, designed to make them struggle with a paradox on an individual basis (see a sample task, below). After about 15-20 minutes of work in a complete non-interactive manner, they had to turn in their worksheets. Then the rest of the time was devoted to interactive group work on the following issues:

(a) Resolution of the paradox.

(b) Discussion of the problem solving strategies applied by individual students to resolve the paradox.

(c) Discussion of the educational merits of the particular paradox.

(d) Reflection on the cognitive roots of the paradox and on the psychological aspects of being in a state of a cognotive conflict.

(For more details, see Class Session Description, below)

For a midterm and final (take home) exams, students were assigned paradox development tasks, coupled with a field trial.

In the spring semester of 1987 and of 1988 all class periods were videotaped. Qualitative analyses were carried out of the videotaped discussions, and of students' written resolutions returned during each session. These constitute the core of the data for this study. Additional information was available from students' written response to the question: 'What is a paradox?', asked at the first and the last class sessions, and from students' course evaluation, submitted at the end of the course.

A word about naturalistic studies is appropriate here. As argued by Woods (1985), the increasing use of qualitative techniques, especially ethnography, in the study of education, offers strong possibilities for bridging the traditional gaps between theory and research, on the one hand, and teacher practice on the other. In particular, Woods refers to preservice teacher-students saying: "They can be encouraged to draw on their own experience as pupils to appreciate some basic elements of the sociological perspective:... how one's own actions are constrained by wider forces; how failures they experienced or wittnessed may be seen as not necessarily pathological..."(Ibid p. 57). The experimental course provided such experiences and an opportunity to reflect on them, thus a theoretical base could be laid in a context of relevance for the future teachers.

A SAMPLE TASK

We confine this report to findings obtained from the data collected during the sessions devoted to one of the paradoxes. This paradox is presented below. Another sample paradox appears in the appendix.

Historical Background

To the early Greek mathematicians, it seemed evident, as indeed it seems to anyone today, who has not been initiated into the deeper mysteries of the number line, that the length measure of any line segment can be expressed by a rational number (fraction); ordinary common sense indicates this to us. It must have been a genuine mental shock for the Pythagorean foundations of mathematics to face the fact that there is no rational number corresponding to the measure of the diagonal of a square having a unit side. (Eves 1983 p. 43-44).

By the well known Pythagorean theorem, which states that the square of the longest side of a right angle triangle is equal to the sum of the squares of the other two sides, the Pythagoreans found that the diagonal of a unit square measures SQRT 2 units ('SQRT' stands for 'square root'). So, following their theory, the Pythagoreans took for granted that there must be two natural numbers, _a_ and _b_ , such that _a/b_ = SQRT 2. Moreover, many pairs having this property exist. One such pair is the fully reduced fraction, call it _p/q_ . As _p_ and _q_ cannot have a common factor greater than 1, they must be both odd, or one even and one odd. From here the Pythagoreans followed a few very simple and valid steps:

Square both sides of _p/q_ = SQRT 2 to get

$p^2/q^2 = 2 \implies p^2 = 2q^2 \implies 2 \mid p^2 \implies 2 \mid p$

Therefore there exists an integer _n_ such that _p_ = _2n_ , hence

$2q^2 = 4n^2 \implies q^2 = 2n^2 \implies 2 \mid q^2 \implies 2 \mid q$.

Therefore _p_ and _q_ are both even. This is contrary to the fact established earlier, that they have no common factor greater than 1.

This contradiction, having occured among the ranks of the Pythagorean brotherhood, "was not only a surprise, but, given what the Pythagoreans expected of mathematics, an unresolvable paradox" (Bunch p.85). Later on, accepting the fact that the Pythagoreans' mathematical foundations must be changed, mathematicians resolved this paradox by introducing a new kind of numbers, the irrationals (like SQRT 2, SQRT 3, etc.), and incorporating them into arithmetic.

Today mathematicians think of the statement "Sqaure root of 2 is irrational" as just another high school theorem. The Pythagorean's paradox became its proof by the method known as reductio ad absurdum. It is not considered a paradox today, because it does not contradict anything known to be true. For the Greeks, however, it did. For them it seemed impossible to find any line segment that could not be measured exactly as the ratio of two natural numbers. In fact, it still does today to the mathematically naive.

The Handout

The above paradox, which mathematics accomodated by defining a new kind of numbers, the irrationals, gave rise to our paradox, which appears in the frame below. It consists of a 'proof' that SQRT 4, known even to the Pythagoreans to be the rational number 2, is not a rational number...

<u>2 is ... irrational</u>

By definition, we call a number <u>r</u> irrational iff there are no two integers <u>a</u> , <u>b</u> for which <u>r</u> = <u>a/b</u> . To show that 2 is irrational, we apply an indirect proof and assume that 2 = SQRT 4 is rational. We'll show that this assumption leads logically to a contradiction. The proof is analogous to the one that shows that SQRT 2 and SQRT 3 are irrational:-
According to our assumption, there exist two integers <u>p</u>, <u>q</u> relatively prime (i.e. having no common factor other than 1) such that

$p/q = \text{SQRT } 4 \implies p^2/q^2 = 4 \implies$

$4q^2 = p^2 \implies$

$4 \mid p^2 \implies 4 \mid p$ (*)

$\implies$ there exists an integer <u>n</u> such that <u>p</u> = <u>4n</u>

hence $4q^2 = 16n^2 \implies q^2 = 4n^2 \implies$

$4 \mid q^2 \implies 4 \mid q$ (*)

and therefore <u>p</u> and <u>q</u> have a common divisor greater than 1, which contradicts the initial assumption that they are relatively prime.
It follows that our assumption was false and therefore SQRT 4, that is 2, is <u>not</u> a rational number.Q.E.D

... and you have always thought that 2 is an integer.

Where is the flaw?

(*) This is an invalid step. Needless to say, this mark was absent from the students' handout.

Class Sessions Description

THe study was conducted in two successive years. Each year, one class session (50 minutes), in the beginning of the semester, was devoted to this activity. Class session opened in a ten minutes of instructor-with-class 'Socratic dialogue', leading to the proof of the irrationality of SQRT 2. Intentionaly, none of the steps was ~~the critical steps (marked * above), like all others, was not~~ given an explicit argument. The historical background was presented at the end. Instructor's impression was that, for most of the students, the latter was new, and the earlier was a review of high school mathematics.

The full handout, assigned for individual work right afterwords, appeared in a slightly different version in each year: Both versions opened in a detailed proof (no arguments) that SQRT 2 is irrational.

- In the 1987 version, the proof for SQRT 3 was left to the student to work out independently. The SQRT 4 part was introduced by a comment that on the one hand it is known to be equal to the rational number 2, yet on the other hand, it can be shown to be irrational, similarly to SQRT 2 and SQRT 3.

- In the 1988 version, the proof for SQRT 3 was outlined, and blanks were provided for the student to fill in for its completion. For the 'proof' for SQRT 4, only the starting assumption and the beginning of the last statement were given, and the student had to fill in the rest in the space provided.

In both versions, the argument in the critical steps (marked above by *) was not provided. Both ended in asking the student: "Where is the flaw???"

After about 15 minutes of a quiet individual work, the students turned in their worksheets, and the rest of the time was devoted to reflections and discussions as described above (see Experimental Course and Data Collection Procedures). We shall come back to this part, in more details, in the Findings section.

Resolution and Pedagogical Remarks

The critical step (marked *, above) holds for primes only. Therefore, for the irrationality of square roots of prime numbers (e.g. 2, 3, 5, 7, 11, etc.), the same line of proof is valid. However, in the case of a composite number, such as 4, it is false. $4 \mid p^2 \not\Rightarrow 4 \mid p$. Even if 4 divides an integer squared, it does not necessarily divide the integer itself (e.g. 4 divides 36 but does not divide 6).

Students, who may have overlooked this critical issue in the proof of SQRT 2, are forced to give it a deep thought, through this activity. It also provides an opportunity to review the logic underlying the difference between primes and composites, with respect to their occurence in factorization of squared integers. Prime factors must come in pairs, whereas composite factors don't have to.

THE STUDENTS

Admission to Technion is competitive, based on applicants' high school graduation records, provided they took the more extensive mathematics courses offered in high school, and their achievements were above 75%.

In spring 1987, 28 students took the course, and 24 took it in spring 1988. Of the 52 students, 2/3 were females. Ninety percent were full time students, registered at Technion for an undergraduate program leading towards a B.Sc. degree in mathematics education. They had completed at least four semesters of the eight semester program. The rest were students possessing a degree in mathematics, who took the course as a part of a program leading to a high school teaching certificate. The mean age was 23 years 11 months. (In general, student population in Israel is relatively old, as they serve 2-3 years in the army prior to starting their university education).

FINDINGS - ANALYSIS and DISCUSSION

We confine ourselves here to findings obtained from the analysis of students' written responses to the task presented above (SQRT 2, SQRT 3, SQRT 4) and from the examination of the corresponding videotapes. These findings suggest a few conclusions. Some of them are task specific, others may have a more general nature. Findings based upon additional tasks and an across-task analysis will be presented elswhere (in a paper in preparation).

Analysis and Discussion of the Written Responses

Students' responses to the task, and to the challenge involved in providing an explanation to the paradoxical results about SQRT 4, were classified hierarchically. Diagram 1 presents the hierarchy. The frequencies for each category appear in parentheses. Classification criteria, and specific examples of responses, follow.

Insert Diagram 1 about here

To classify students' responses we applied the following criteria:

1. To be classified in category (1), a correct proof for the irrationality of SQRT 3 had to be provided. This was interpreted by us as 'Could follow the SQRT 2 proof'. It is necessary, albeit insufficient, to follow this proof in order to resolve the paradox.
2. As 'Could not follow the SQRT 2 proof' (category 2) we classified those who gave a wrong proof for SQRT 3, or
   were stuck in the middle of it, or
   didn't give it a try, and then didn't deal with SQRT 4 at all, or dealt with it wrongly as with SQRT 3.

Clearly, all students in this category didn't have a chance to notice the paradox resulted from the SQRT 4 'proof'. Thirteen and a half percent of the responses fell in this category. It is noteworthy that the basic assumption, that the SQRT 2 proof required no more than recall of high school knowledge, was wrong for these students. To what extent these students gained anything from the discussion that followed, this issue is addressed later.

The 86.5% of the responses, classified as Category 1, were further classified according to those who noted the paradoxical results (1.1) and those who did not (1.2).

1.1 Those who noted the paradoxical results were 75% of the total, and 87% of those who fulfilled the necessary condition to resolve it (category 1). We'll come back to them later (See 1.1.1 and 1.1.2).

1.2 Six responses (11.5% of the total, 13% of category 1) were classified as 'didn't notice the paradox'. Four of these responses consisted of a full proof that SQRT 4 is irrational, without any comment whatsoever about the fact that this contradicts common knowledge. Two others wrote a full proof for SQRT 3, and ignored the proof for SQRT 4, stating that SQRT 4 = 2. This was also interpreted by us as 'did not notice the paradoxical result'. During the discussion that followed, one of these students gave others a great opportunity to practice strategies of explanation (see details below). It should be pointed out that we saw a major difference between this response and a response which included a complete 'proof' for SQRT 4 followed by the statement "but SQRT 4 = 2". The latter were categorized as noticing the paradoxical result (1.1).

Those who noted the paradoxical result (category 1.1, 75% of the total), were furthermore classified into two subgroups: those who gave a complete resolution (1.1.1) and those who did not (1.1.2).

1.1.1 Among the 39 students who noted the paradox, only six students, that is

15% (11.5% of the total), completely resolved the paradox and were classified accordingly. They pinpointed the critical step as being the core of the trouble. Here are two quotes: "The passage from $4 \mid p^2$ to $4 \mid p$ is incorrect. For instance $4 \mid 36$ does not imply $4 \mid 6$."; "... $4 \mid p^2 \implies 2 \mid p$, therefore there exists an integer m such that $p = 2m \implies 4m^2 = 4q^2 \implies m = q$ ... and there is no contradiction".

1.1.2 The rest of the students, comprising 63.5% of the total and 73% of category 1, realised that there is a problem there, but could not figure out what was wrong. We found this number very significant as an indicator of the need for an intervention of the sort we provided.

In this category we included two types: (a) those who wrote a full proof for SQRT 4, (b) those who got stuck in it.

(a). Students who wrote the proof for SQRT 4 in full analogy to SQRT 2 and SQRT 3, expressed their confusion by either

marking the conclusion 'SQRT 4 is irrational' by 3 question marks, or

omitting this line, or

commenting that SQRT 4 = 2, or

giving a false argument to resolve the paradoxical result, or an incomplete argument, or a circular argument. E.g.: "We cannot assume that SQRT 4 is irrational to begin with, because it is rational".

(b). Students who were stuck in the proof of SQRT 4 while trying to debug it by changing the critical step. E.g.:

"... $4 \mid p^2 \implies 4 \mid p$ ... $4m^2 = q^2 \implies q = 2m$, therefore

$p/q = 4m/2m = 2$ and hence p and q are relatively prime and SQRT 4 is rational".

Another student wrote a complete proof for SQRT 4, crossed off the ctritical steps and wrote above them:

"$p^2 = 4q^2 \implies p^2/2 = 2q^2 \implies 2 \mid p^2/2$...

$2m^2 = q^2 \implies 2 \mid q^2$". At the end, this student put a question mark next to the statement: "The assumption that SQRT 4 is rational is nevertheless true".

The findings described above are indicative of five major issues: 1.Task validity. 2.Motivation. 3.Lesson format usefulness. 4.Knowledge fragility. 5.Appication of cognitive and metacognitive skills.

Task validity: These findings provide a validation of the particular task as a generator of a challenge to our preservice students. It exposed a vast majority to a discrepancy in a part of their knowledge, which has for long been considered by them as elementary and well known. We conclude that our externally imposed mathematical paradox produced an internal conflict, a particular type of a conflict between two competing ideas, as suggested by Sigel (1979).

The question marks put on many worksheets, usually next to the words: " hence SQRT 4 is irrational", we interpreted as expressions of hesitation. "Hesitation is one of the hallmarks (if not the hallmark) of psychic conflict" (Cantor 1983 p. 48). Thus we get a validation of the task as a conflict generator, through this aspect too.

Response latency was suggested by Zimmerman and Blom (1983) as another measure of internal conflict. Cantor (ibid.) found it invulnerable to criticism as an index of psychic conflict. In the present study the relative order of turning in the worksheets was used as a qualitative measure of response latency. In both years of the study, it was found that students, whose answers were later classified in category 1.1.1, were the first ones to return their handout back. They were the ones who were confident in their resolution, which indeed was found correct. We interpreted it as saying that they were in a state of conflict, if at all, for the shortest time. Others may have kept their sheets longer, for other reasons and not only because they faced an unresolved conflict, of course. Our measure of latency is therefore no more than suggestive as a measure of relieved subjective uncertainty.

Motivation: For 75% of our students (category 1.1) the 'proof' that SQRT 4 is irrational was a disturbing force that impinged upon their equilibrium state of knowledge that 2 is rational. Being confident that the latter must remain unaltered, they searched for an explanation to adjust the disturbance. The high percentage of those who tried to work out a resolution, no matter how successfully, indicates that the discrepancy, as theoretically predictable, indeed created an inner need for change, namely a great motivation for knowledge modification.

Lesson format usefulness: For at least 86.5% of the students (those in category 1) the task was valuable, as they understood its premises. For at least 63.5%, those in category (1.1.2), the discussion which followed was necessary, as they were unable to provide a resolution on their own. Only 11.5% successfully resolved the paradox. None of them said during the discussion that followed, that they found the task absolutely trivial. However, in similar populations, this may be the case for a few. Therefore, a measure of caution should be taken by the instructor. For those who may find it trivial, very little change is to be anticipated in their mathematical knowledge. This does not exclude, of course, other benefits these students may get out of this activity, as the findings from the video-taped discussions revealed. (See below).

Knowledge fragility: This notion has recently been introduced by Steiner (1989) to describe the intermediate state of knowledge during the process of its construction, before it becomes fully crystallized and stabile. Some of our students' knowledge about rational and irrational numbers was found highly fragile, as it was susceptible of disturbance triggered by the paradox. Those in category 1.2, who did not even see anthing wrong with arriving at the conclusion that SQRT 4 is irrational, suffered from a more severe knowledge fragility than those who acknowledged the existence of a paradox there, but were unable to resolve it (category 1.1.2). However, both needed to undergo an appropriate treatment before they enter a classroom as teachers. Needless to say, this is true of those in category 2 as well, and altogether of 88.5% of the students. Clearly, compared with the others, category 2 students were less likely to benefit from a group discussion such as the one that followed the individual work stage in our class teaching design.

The video-tape of the individual work parts of the session, shows signs of tension --- students looking left and right, peeking at neighbors' work, holding their pen in their mouth, etc. Many looked agitated, moved nervously on their chairs. These signs of uneasiness were probably a result of getting aware of one's knowledge fragility.

Cognitive and metacognitive skills application: In the written responses there was evidence of cognitive as well as metacognitive strategies. Among the metacognitive ones, particularly managerial strategies (monitoring own thinking) were noticed easily. In several worksheets, traces were found of erased critical steps in the 'proof' of SQRT 4. Students re-wrote them to avoid the fallacious arguments. They must have decided at a certain point that their path had not lead them where they wanted to. They then decided to abandon it, erased whatever they had written, and started all over again.

Among the cognitive skills, analyzing ones were the most appropriate to employ (Marzano et al., 1988, p. 91). Evidently, students check marked each step in the given proof for SQRT 2, identified the critical step by underlining it, thus showing traces of a thinking process typical of analyzing arguments, of identifying errors (ibid p.97) and of verifying (ibid p. 111).

Analysis and Discussion of the Video-taped Group Discussions

As mentioned earlier, the discussion part of each class session focused on two central issues: mathematical resolution of the paradox, and reflections on the processes involved. The goal of this study was to examine the potential of dealing with paradoxes as a vehicle for preparation of student-teachers for their future professional life as mathematics educators. Consequently, evidence from the videotaped discussions was collected to support five dimensions of this potential: 1.Review and refinement of mathematical concepts. 2.Cognitive skills application (thinking skills). 3.Metacognitive skills application (thinking about thinking). 4.Awareness of the role of paradoxes in the history of mathematics and of their potential in mathematics education. 5.Reflections on the psychological state of a cognitive conflict. Under each of these, we bring several quotes and a few inferences we feel at liberty to draw.

Review and refinement of mathematical concepts. The mathematical discussion went back and forth between those who resolved the paradox (category 1.1.1, above) and the others. At this stage, the conflict caused by the paradox could be regarded as an external-internal one (Sigel 1979) for those who wittnessed an unresolved conflict (category 1.1.2 above), and an external-external one by the others (categories 1.2 and 2 above). Students of the latter categories were in fact observers of the discussion and rarely participated in it. This does not mean, of course, that they did not gain anything by doing just that.

The videotaped discussion in this part revealed very little resistance to change views. Many non-resolvers accepted resolvers' logic very quickly, and joined them to convince those who still had difficulties. One clash was observed between two students whose views were in disagreement as for the very nature of the paradox. One of them said:

"...You assume that SQRT 4 is rational, which is true, as well known. Then you reach a contradiction and you conclude that it is irrational, which is impossible because we know that SQRT 4 is 2. This is all nonsense".

This student did not look content till the end of the session.

Those students who made a quick and easy change, during the discussion, from being unable to resolve the paradox to being able to explain it to others, made a clear transition from what Skemp (1976) termed instrumental to relational understanding of the divisibility facts associated with this paradox.

The task we dealt with challegned the concept of proof, and of an indirect proof, the concept of rational numbers and of irrational numbers, the concept of divisibility and of a squared integer. Students who were the most resistant to change views were the ones who had trouble with the indirect proof. This proof usually opens by the negation of the claim to be proved, and leads logically to a contradiction, which then implies that the assumption was wrong. In our case, the assumption, SQRT 4 is rational, is unfalsifiable. Knowing this, made it difficult for many students to get themselves to follow the 'proof'. This was obvious in going through the circular arguments that appeared in the written responses quoted above. It became even more obvious in observing the discussion, as the following dialogue demonstrates. This dialogue demonstrates, also, a persuasion strategy that worked after a good deal of effort:

> Students who resolved the paradox were forced to concentrate on finding a way to explain their logic. One such student approached a class mate who didn't see the point: -
>
> - "Forget for a second what you know about SQRT 4. Would you then accept this proof?"
>
> Encouraged by the instructor not to give up, students of all standpoints insisted on their "but" and "however". This particular one answered:
>
> - "What do you mean forget it?... It is what this is all about".
>
> Instructor commented to the explainer: "You'll have to try again".
>
> The camera caught the explainer and two other resolvers highly concentrated, perhaps seeking a better explanation. One of them tried a more general argument:

- "Take SQRT n".

- "It won't work. It is different for different values of n. For instance for sqaured numbers".

Smiling and leaning forward optimistically, the explainer looked straight in the other's eyes, as if transmitting a non verbalizable message, and asked:

- "In what way is it different for squared numbers?".

- "You cannot refute the assumption that it is ratioal, because it is rational for fact."

Here the resolver gave up and leaned back on her chair.

Another one carried it on, starting anew, this time from the point where the non-resolver was:

- "You are right!" he said, "This is why we are looking for the flaw in this proof".

- "But what is there to prove? SQRT 4 is rational, don't we all agree?"

Continuing the teaching strategy of 'starting from where the person is', the resolver accepted:

- "Of course. Because we all agree about that, we assume SQRT 4 = p/q. Now, a few algebraic manipulations of that, seem to lead to the contradictory end statement".

- (A slow head jesture of agreement).

- "So something must be wrong" continued the expliner "and it is <u>not</u> the assumption, right?"

- (Still, no vocal reaction, but a slow change of view is noticeable).

- "So, we are trying to see <u>what</u> is wrong,you see?" The explainer stretched up, anxiously.

- "O.K. I think I get it (talking slowly while nodding her head). I thought the flaw was at the initial assumption, but now I see it differently (her eyes focused in a point far away, concentrated)... just a minute,...then...alright, there must be a flaw somewhere along this... yes... it is, oh, that's why you were talking about divisibility before,... now I see why,... yes,(looking at her partner)... thanks, I like it (smiling)."

We infer from the social conflict experienced between those who resolved the paradox and those who did not, the existence of an internal process which resulted in learning to both parties. The negotiations that went on improved the mathematical understanding of the non-resolvers, and not less important, it enriched the pedagogical experience of the resolvers.

Judging from nonverbal facial expressions and vocal expressions like "Oh yes", or "Hmmm", we are inclined to say that the majority, if not all students but one, came out of the oral discussion feeling wiser, and wiser indeed. They accepted the views of those who resolved the paradox, not just as a matter of social conformity. For a more firm conclusion, better grounds would be required, of course.

Cognitive skills application (thinking skills). In discussing the means they employed in order to find the roots of the paradox, students exhibitted the application of the following skills:

Verification by deduction. (E.g. "By the unique factorization theorem, each prime factor of n is a double-prime factor of n^2, but this is not the case for composite factors... of n, I mean".)

Verification by instanciation (E.g., "2 | 36 implies 2 | 6; 3 | 36 implies 3 | 6; but 4 | 36 doesn't imply 4 | 6").

Step by step examination. (E.g., "I checked step by step and realised that the only place where there can be a difference is here" (pointed at the critical argument)).

Analogical thinking. (E.g., "While I was writing down the proof for SQRT 3, I already thought about SQRT 4...")

These heuristics are quite different from the ones Schoenfeld (1980) described, which he borrowed from Polya (1954). It seems that resolving a mathematical paradox is not the kind of problem usually considered with reference to problem solving. A more comprehensive list is required in order to account for this type of problems. Anyhow, it seems clear that the acquisition of these skills is crucial for the development of critical thinking.

Metacognitive knowledge and experience (Thinking about thinking). Metacognitive skills involve the managing of one's own cognitive resources and the monitoring of one's own cognitive performance. (Nickerson et al. 1985 p. 142). According to Flavell (1987), "Metacognitive knowledge can be subdivided into three categories: knowledge of person variables; task variables; and strategy variables" (ibid. p.22). As an example of metacognitive experiences he says: "...if one suddenly has the anxious feeling that one is not understanding something and wants and needs to understand it, that feeling would be a metacognitive experience"(ibid. p. 24). Many quotes from the videotaped discussions adhere to this last example of a metacognitive experience. E.g.:

"I had a moment of despair, then I decided I couldn't afford it, I had to 'collect' my thoughts".

"I said to myself: How come? How come? I must find the answer".

We have already mentioned decision making as to strategic change, evidenced by crossed off and changed parts in the written responses. The oral reflections included statements like:

"The indirect method of proof fooled me completely. I decided to correct the error in the SQRT 4 proof, but I couldn't get out of it".

"I knew this was wrong, but I could not tell why... I went over and over again and again".

"I knew that I must discriminate between SQRT 2, SQRT 3 on one hand and SQRT 4 on the other... Beware of overgeneralization, I told myself, that's the key, but how?".

"I went through the proof in haste, evaluating arguments by... sort of, hand waving... It didn't work, so I held back and examined it more closely, more slowly, more carefully."

(Awareness of) the role of paradoxes in the history of mathematics and of their potential in mathematics education. In summing up the educational value of this paradox, students said:

"This activity gave me a great pleasure. In particular, I believe that many young students follow the Pythagoreans' line of thought, and believe that any length can be measured by a natural number or a quotient of two."

"If I introduce them (the children) to a line segment of the measure SQRT 2, they'll deduce that SQRT 2 = a/b for some irreducible fraction and they'll reach a contradiction.. a paradox.. much like the Pythagoreans.".

"This paradox sets a very good ground for me to introduce the notion of irrational numbers... instead of defining them in a rather arbitrary fashion as commonly done".

Other students' reflections testify that they were surprised to realise how insightful the acitivity was for them:

"If I was told this could happen to me, I wouldn't believe it".

"I arrived now at a totaly different understanding of the whole matter. I am truely amazed at what I went through.".

Reflections on the state of a cognitive conflict. Students' self reflections on their being in a state of a cognitive conflict, included expressions of curiosity arousal, expressions of an inner drive to resolve, expressions of frustration, expressions of satisfaction in coping with inability to proceed, expressions of content from feeling self confident about a shaky state. Here are a few examples:

"You think you understand something, and it turns out to be wrong. It is kind of a shock... It's fun... No... It's ... mindstretching.."

"I felt so stupid I could not bear it. That's impossible, I thought, absolutely incredible".

"I felt ridiculous. There must be a flaw here, but where is it?".

"Everything seemed so reasonable, yet wrong...".

"... it was a form of revision...it made me think hard".

"I was threatened in the beginning and controlled it, then I was able to start thinking and worked it out".

"It was as if my mathematics betrayed me. I wish we were allowed to talk and discuss it instead of working separately... in solitude... alone with our worksheets."

"I was helpless. I could not wait to hear the solution".

"I felt cheated. It realy upset me, irritated me that I didn't find it. Then I went over it again, and suddenly it became so obvious, I did not know if I ought to laugh or to cry."

One should keep in mind that the task we focused on in this paper, was the very first one in the course. As the semester progressed, in the course of being exposed to more paradoxes, the number of negative expressions reduced. Our students gradually gained confidence in dealing with their subjective uncertainties. We consider such experiences as having an utmost importance to future teachers. As Mason expressed it: "To persist with thinking to the point that you can learn from it requires considerable perseverance, encouragement, and a positive attitude to getting stuck" (Mason et al. 1985 p.150). The videotaped discussions allow us to believe that during the course our students went through such processes.

CONCLUDING DISCUSSION

We wish to close this paper in a discussion of three questions, of a more general nature:

To what extent the problem raised is valid?

The section devoted to 'Preparation in mathematics of prospective secondary teachers of mathematics', at the Internatioal Conference of Mathematics Education, held in Budapest in summer 1988, was summarized as follows: (Dorfler 1988): "There is a tension between the well defined aim of many mathematics departments (to get students to the research frontier as fast as possible) and the less well defined aim of teacher trainers (to develop learning and process skills together with content and to stress understanding rather than rote learning). The tension can be resolved only by joint efforts of mathematicians and mathematics educators to construct adequate programs. These programs must ensure that students experience mathematics and not merely reproduce it. (Ibid. p. 182). There seemed to be an international consensus about the need for more integration between the pedagogical and the mathematical preparation of future teachers. (Ibid. p. 181)

According to the Notices of the American Mathematical Society (August 88, p.790), research mathematicians in the U.S. are getting more involved in mathematics education, recently. College and univerity mathematics departments are starting to strengthen ties to education departments.

The problem of bridging the gap between the two disciplines, mathematics and education, seems to be widespread. Solutions are sought in many universities over the world.

To what extent the solution under study (paradoxes) seems appropriate?

Alvine White (1987) summarized six humanistic dimensions of mathematics, discussed during a 3-day conference devoted to the examination of mathematics as a humanistic discipline. One of the six dimensions was:" ....The opportunity for students to think like mathematicians, including a chance to work on... and to participate in controversy over mathematical issues". (p. 1). Many will raise their eyebrows at that. Mathematics is not perceived in general as controversial. General public belief, and the view of many teachers too, is that mathematics is a well sorted out topic, at least at school level (Pimm 1987 Ch. 2), and therefore there is no room for a discussion in mathematics. Our course materials demonstrate, as a matter of fact, that many discussion provoking activities, like the ones suggested at the Humanistic Mathematics conference, do exist and their implementation is worthwhile.

Romberg (1988) answers the question: "Can teachers be professionals?" by saying: "Teaching for long-term learning and the development of knowledge structures requires ... teachers who can diagnose difficulties and devise questions to promote progress through cognitive conflicts;..." (Ibid. p.240). Self-confrontation with a cognitive-conflict through dealing with mathematical paradoxes seems to be a way of educating teachers to this end. Such an experience seems to bring about change in student's existing conceptual frameworks, mathematical ones as well as educational ones. Activities fostering thinking about mathematical issues and about didactics are likely to make future teachers more professionals.

In the section on 'Preparation in mathematical education and pedagogy of prospective secondary mathematics teachers', at ICME 6, it was noted that a process of unpacking one's own ideas of goals, methods, and the nature of mathematics is required for many prospective teachers. (Dorfler p. 183).

In light of the above, and based upon the results presented earlier, it seems reasonable to conclude that the solution proposed in this paper, has many facets: its mathematical component is rich, both culturally and conceptually; its psychological component is enlightening and empowering. In other words, it has a potential to solve, at least in part, the problem of bridging the gap between mathematical preparation and educational preparation for future teachers.

Finally, there is one more question, that has to do with the professional attitudes of those who educate future teachers:

Do we perceive our role as teacher educators as including the responsibility to challenge students' elementary math knowledge? Is testing knowledge fragility/stability, possibly through paradoxes, moral?

We believe, as many Piagetians do, that experiencing of conflict is essential to the occurance of what Piaget termed 'true learning', that is the acquisition and modification of cognitive structures. In trying to resolve, at least partially, the so-called "Learning Paradox", Bereiter (1985) explains that the paradox applies where " - as in being introduced to rational numbers, for instance, - learners must grasp concepts or procedures more complex than those they have available for application". (p.202). It seems as if the "constructivists" view of learning, that people construct knowledge for themselves, runs into a circularity here. This view implicitely presupposes that people possess a cognitive structure, which is responsible for generating new structures, more complex than the generating structure itself. It puts into question the supposed role of teaching and of education. Without going deeper into this paradox, it is clear that mathematical concepts are complex constructs, which are not developed overnight. Even people who have a strong mathematical background, may be subject to deficiencies in understanding of concepts of a more elementary level than theirs. One can proceed in the pursue of mathematical studies, while prior knowledge still suffers from gaps in its understanding. Facing a paradox brings such unconscious gaps to the surface, and might aid in making them more accessible to rational consideration, and in turn narrow them down. Because the learner's own efforts are so crucial in constructing one's own knowledge, there is an obvious need for teaching efforts that promote self-confidence and security, and that are conducive to concentration and experimentation.

In Bereiter's terms (ibid. p.220), our strategy was an indirect instructional strategy. The specific paradox resolutions re not the goal of the course. Rather, it was the means for bringing about the creation of a new cognitive structure, which does not resemble any specific paradox dealt with during the course. It stems from a phylosophy of teaching mathematics through errors, conflicts, debates and discussions, that leads to gradual purification of concepts. Testing knowledge fragility is, according to this perception, not only morally alright, but immoral to ignore. It is not a luxury, but a must. Truely, there is always a danger of misusing it, thereby causing frustration and learned helplessness, instead of building up self confidence in coping with hesitation and search. This, however must not discourage us from applying knowledge fragility tests.

CONCLUSIONS

Despite the "soft" nature of the data collectd, the following conclusions and cautions can quite safely be drawn from them:

a. Mathematical paradoxes provide a convenient ground for a non-routine review and polish of high school materials, alongside an introduction to critical moments in the history of mathematics. The findings indicate that the model of dealing with paradoxes as applied in the course has relevance to such aspects of mathematics education as motivation, misconceptions and constructive learning.

b. A paradox based on high school mathematics can put an adult student, whose background includes some university level mathematics (prospective teacher), in a perplexing situation, known as cognitive conflict. Experiencing such conflicts is valuable for future teachers, in order to be able to identify with future students of theirs, when they face a parallel experience.

c. The impulse to resolve the paradox is a powerful motivator for change of knowledge frameworks. For instance, a student who possesses a procedural understanding may experience a transition to the stage of relational understanding.

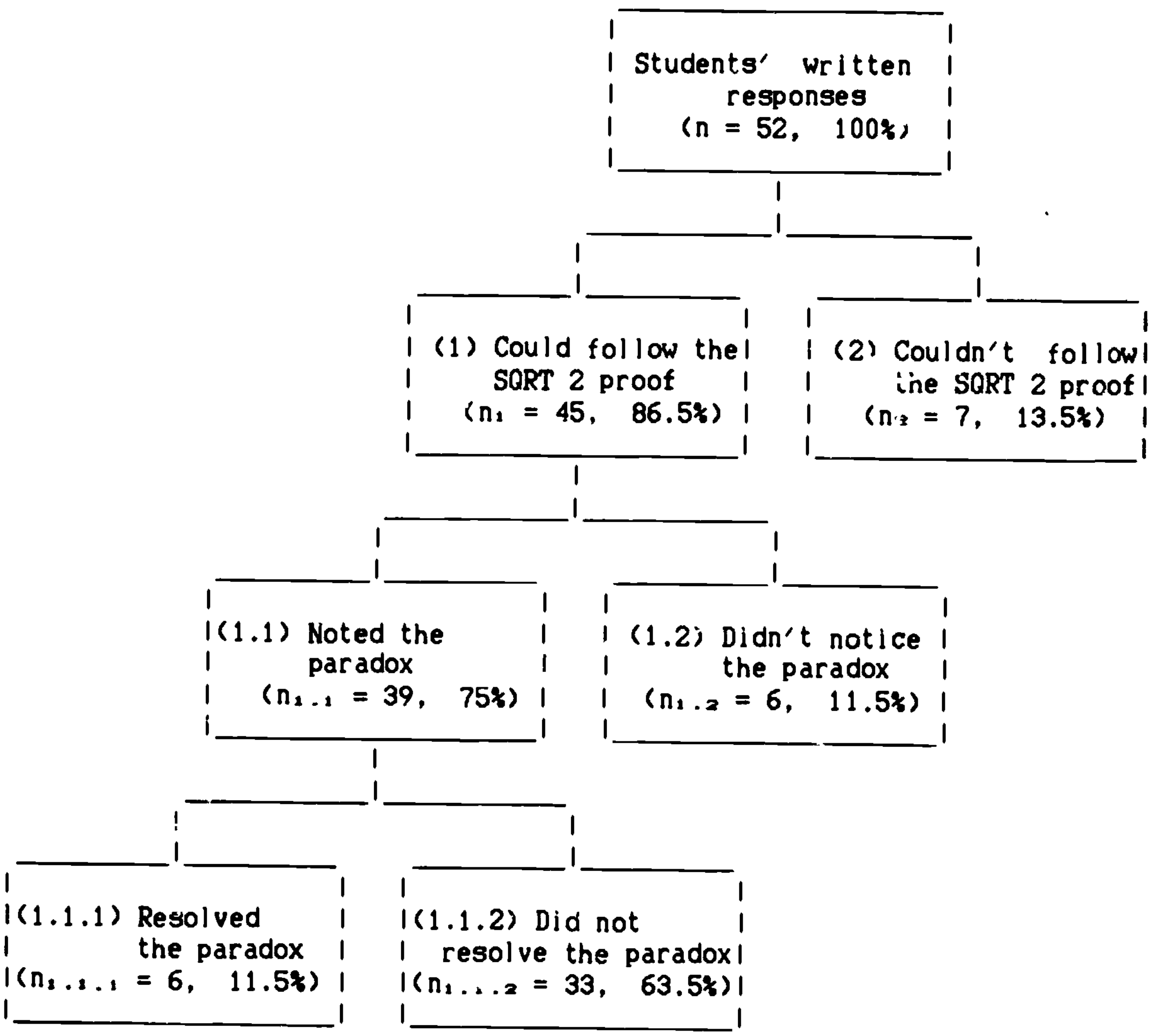
d. Working on paradox-resolutions can sharpen student-teachers' sensitivity to mathematical loopholes, mistakes, inaccuracies etc., and to the crucial role of error detection as a learning opportunity. Paradox clarification activities provide ample scope for preservice teachers to study critical issues in mathematics, and its history, along with critical issues of math education, particularly concept formation. This is a step forward in the search for cultural enrichment combined with beneficial pedagogical tools.

e. Dealing with the challenge embedded in a paradox can improve students' awareness of problem-solving heuristics and metacognitive strategies.

f. Such training probably works only for individuals who are ready for it, that is to say, who have the necessary cognitive foundations upon which to build.

g. The teaching method adopted in this course is not necessarily a good practice to be immitated blindly in school. Incorporating paradoxes in high school mathematics deserve a serious and carefully planned study.

Diagram 1: Classification and Frequency of Responses



Appendix

ANOTHER SAMPLE HANDOUT

The development of this handout was based upon Gardner M. (1983, p.42)

THE 2x2 CARDS PARADOX

Introduction through a game: To play this game you need a partner and four cards, two of each color, say red and black. (If you prepare your own cards, make sure that you color only one side of each card, so that all four look the same on the other side). Shuffle the four cards and let your partner choose two without looking at their color. If the two chosen cards have matching colors, your partner wins a point. Change roles and repeat the game. Record your results for at least 10 rounds.

Problem: What is the probability of winning a point in any round of the game?

Three different answers to this question are given below. All three seem logical, yet only one is correct. Which one? (Please put x to the left of the answer you prefer. Notice: As long as you cannot make up your mind, there is a paradox).

____ There are three equally likely results: either both cards are red, or they are both black or they don't match. In two cases the player wins a point, therefore the probability is 2/3.

____ There are two equally probable results: either the colors match (red-red or black-black) or they do not match (red-black or black-red). Therefore the probability is 1/2.

____ Suppose the first chosen card is red. There is only one red among the remaining three cards. There is a probablity of 1/3 to chose a second card with a matching color.

What is wrong with the logic underlying the other two answers? (Express your thoughts in writing on the other side, please).

Have you heard about the "Principle of Indifference"? Yes/No (Please circle one).

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