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AUTHOR	Huynh, Cam-Loi
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ABSTRACT

Parametric measures to estimate J. Cohen's effect size (1966) from a single experiment or for a single study in meta-analysis are investigated. The main objective was to examine the principal statistical properties of this effect size--delta--under variance homogeneity, variance heterogeneity with known variance ratios, and for the Behrens-Fisher problem. Derived estimators were compared according to the criteria of their magnitudes, unbiasedness, and mean-square errors. General properties of the derived estimators were examined by means of Monte Carlo results. Results tend to confirm the recommendation of theoretical analysis that two identified estimators, "h" and "d(sub T)," should be used in conducting a meta-analytic study. These estimators better ensure unbiasedness and minimum mean-square error than do others derived. Eight tables and four graphs illustrate study data. (SLD)

A UNIFIED APPROACH TO THE ESTIMATION OF EFFECT SIZE IN META-ANALYSIS

by

Cam-Loi Huynh

Department of Psychology

University of Manitoba, Canada, R3T 2N2

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Introduction

In research synthesis, statistical methods are used to describe the findings of studies under review. Typically, the selected studies are considered as independent experiments concerning the behaviors of some common dependent variables. Each study usually consists of a control group (Y) and at least one treatment condition (X). The results of each study are summarized by an index of effect size (δ). The first estimate of effect size, developed by Cohen (1966, 1967) and modified by Glass (1976), is of the form $g = (\overline{X} - \overline{Y})/S_Y$ or the difference between means of the treatment ($\overline{X}$) and control ($\overline{Y}$) groups divided by the standard deviation of the control group (S_Y). Important contributions to the estimation theory of effect size are attributable to Rosenthal (1978), Hedges (1981), Rosenthal & Rubin (1982), Kraemer (1983) and Hedges & Olkin (1985).

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ represent some random samples of the treatment and control normal populations; μ_X and μ_Y, the population means of X and Y, respectively; and σ, the standard deviation of the response scores of all subjects in the combined treatment and control population. The present investigation addresses parameteric methods to estimate Cohen's effect size $\delta = \frac{(\mu_X - \mu_Y)}{\sigma}$ from a single experiment or for a single study in meta-analysis. The enumeration of all possible estimators of σ would result in a countless number of estimators of δ. Table 1 provides a limited list of some popular estimators of σ^2, namely S_i^2 and $\hat{S}_i^2$ for $i = 1, \ldots, 5$.

Insert Table 1 about here

Generally, an unbiased estimator of σ^2 is S^2 whereas its maximum likelihood estimator is $\hat{S}^2$. The distributions of S_i^2 and $\hat{S}_i^2$ can be summarized respectively as $fS_i^2/\sigma^2 \sim \chi^2_{(f)}$ and

$\hat{f}\hat{S}_i^2/\sigma^2 \sim \chi^2_{(\hat{f})}$, where $\chi^2_{(f)}$ and $\chi^2_{(\hat{f})}$ denote chi-squares with f and $\hat{f}$ degrees of freedom, respectively, and "$\sim$" meaning " is distributed as". The estimates of σ and their associated degrees of freedom are listed in Table 1 under various conditions of variance homogeneity and heterogeneity.

Under variance homogeneity (Case 1), as well as under variance heterogeneity with known variance ratios (Case 2), four estimates of σ are listed in Table 1. These estimates are derived after the procedures to estimate effect sizes as (i) suggested by Glass (1976) (S_1 and S_3 and their maximum likelihood estimator (MLE) versions, $\hat{S}_1$ and $\hat{S}_3$), and (ii) proposed by Hedges (1981) (S_2 and S_4 and their MLE counterparts, $\hat{S}_2$ and $\hat{S}_4$). Also listed are the estimates of σ under variance heterogeneity (Case 3) as introduced by Welch (1938) and Cohen (1966) (S_5 and its MLE counterpart, $\hat{S}_5$). Other estimates of σ^2 can be derived by modifying the values of S_i^2 and $\hat{S}_i^2$ in Table 1. For example, the values below are mathematically equivalent to $S_2^2 = [(n-1)S_X^2 + (m-1)S_Y^2]/(n+m-2)$:

$S_8^2 = [\sum (X_i - \overline{X})^2 + \sum (Y_i - \overline{Y})^2]/(n+m-2)$, and

$S_9^2 = [n\hat{S}_X^2 + m\hat{S}_Y^2]/(n+m-2)$,

where $S_X^2 = \sum(X_i - \overline{X})^2/(n-1)$, $S_Y^2 = \sum(Y_i - \overline{Y})^2/(m-1)$, $\hat{S}_X^2 = \sum(X_i - \overline{X})^2/n$, and $\hat{S}_Y^2 = \sum(Y_i - \overline{Y})^2/m$

In Table 1, the degrees of freedom for the unbiased estimates of σ (f_i, i = 1, ..., 4) are less than those for the corresponding MLE counterparts ($\hat{f}_i$). It is important to note that the estimators of effect size are non-central t statistics whose distributions are characterized by the specifiation of degrees of freedom, f and $\hat{f}$.

The main objective of this investigation is to examine the principal statistical properties of the estimators of δ under variance homogeneity (Case 1), variance heterogeneity with known variance ratios (Case 2) and for the Behrens-Fisher problem (Case 3). The derived estimators are compared according to the criteria of their magnitudes, unbiasedness and mean-square errors (MSE). The present investigation can be considered as an extension of the existing studies on the estimation of effect size in meta-analysis from three perspectives:

(i) Except for the estimators of δ listed under Case 1 in Table 1, other possibilities have not yet been examined.

(ii) A generalized approach in effect size estimation is undertaken such that common properties of estimators of δ are derived. These properties can be applied to estimates that can be considered as mathematically equivalent to those given in Table 1.

(iii) Although the estimation of treatment effects in the presence of variance heterogeneity has been discussed (Wilcox, 1985), the estimation of effect sizes for the Behrens-Fisher problem in research synthesis is addressed here for the first time. In particular, the bias of the estimators of effect size computed according to the methods of Case 1 (Variance homogeneity) in the presence of variance heterogeneity is investigated.

(iv) It will be shown that estimators of effect size are unstable, namely, variances of the estimators increase with effect size (δ). In this study, a relatively stabilized estimator with respect to δ is identified and its properties, examined.

In the following, the common statistical features of estimates of δ are treated first in general terms. Then, specific characteristics of some selected estimators, which are formed as functions of S_i and $\hat{S}_i$ listed in Table 1, are analyzed. Finally, general properties of the derived estimators are re-examined by means of Monte-Carlo results.

General Properties of Estimators of Effect Size

Model Specification and Assumptions

The properties of the estimators of effect size are studied on the basis of the following three assumptions:

(A1) The random samples X and Y are distributed normally with means μ_X, μ_Y and finite variance σ_X^2 and σ_Y^2, respectively. The population moments in the distibutions of X and Y are unknown;

(A2) $\overline{X}$, $\overline{Y}$, S_X^2, and S_Y^2 are mutually independent; and

(A3) The standardized difference between the treatment and control effects is represented by the effect size $\delta = (\mu_X - \mu_Y)/\sigma$ where the forms of σ are given in Table 1.

It will be shown that, for i = 1, ..., 5, the biased estimators of δ are of the forms $g_i = (\overline{X} - \overline{Y})/S_i$ and $\hat{g}_i = (\overline{X} - \overline{Y})/\hat{S}_i$, and the corresponding unbiased estimators are $h_i = c_f g_i$ and $\hat{h}_i = c_{\hat{f}}\hat{g}_i$, respectively; where $c_f = \frac{\Gamma[f/2]}{(\sqrt{f/2})\Gamma[(f-1)/2]}$, and $c_{\hat{f}} = [\sqrt{f/\hat{f}}]c_f$. The distributions of g, $\hat{g}$, h and $\hat{h}$ are found to be noncentral t with the noncentral parameter defined generally as $\Delta = (\sigma/\sigma_{\overline{X}-\overline{Y}})\delta$, where $\sigma^2{}_{(\overline{X}-\overline{Y})} = [\sigma^2{}_{\overline{X}} + \sigma^2{}_{\overline{Y}}]$ and $\sigma = \sigma_X = \sigma_Y$ under variance homogeneity or $\sigma^2 = (\sigma_X^2 + \sigma_Y^2)/2$ in the presence of the Behrens-Fisher problem. In addition, some linear transformations of g, $\hat{g}$, h and $\hat{h}$, such as the variance-stabilizing and shrunken estimators, are also considered.

The derivation and properties of estimators of δ are influenced by the following two properties of the functions c_f and $c_{\hat{f}}$: (i) the degrees of freedom f and $\hat{f}$ should be larger than 2, (ii) c_f and $c_{\hat{f}}$ are monotonically increasing functions of the degrees of freedom ($0.72 \leq (c_f, c_{\hat{f}}) < 1$ for $3 \leq (f, \hat{f}) < \infty$) (Hedges and Olkin, 1985, Table 2, p.80), or equivalently, the inverse of these functions are monotonically decreasing ($1.38 \geq (c_f^{-1}, c_{\hat{f}}^{-1}) > 1$ for $3 \leq (f, \hat{f}) < \infty$. For example, if $f_2 \geq f_1$ then $c_{f_2} \geq c_{f_1}$ and $c_{f_2}^{-1} \leq c_{f_1}^{-1}$.

The Distributions of the Estimators of δ

The distribution of a generalized biased estimator of δ is formalized as follows:

Proposition 1. Consider the estimators $g = (\overline{X} - \overline{Y})/S$ and $\hat{g} = (\overline{X} - \overline{Y})/\hat{S}$; where $S^2/\sigma^2 \sim X^2{}_{(f)}/f$ and $\hat{S}^2$ is a maximum likelihood estimator of σ^2 such that $\mu_{(\hat{S}^2)} = (f/\hat{f})\sigma^2$ for $f, \hat{f} \geq 3$. The distributions of g and $\hat{g}$ are $kt_{(f,k^{-1}\delta)}$ and $\hat{k}t_{(f,k^{-1}\delta)}$, respectively; where $k = \sigma_{(\overline{X}-\overline{Y})}/\sigma$, $\hat{k} = (\sqrt{\frac{\hat{f}}{f}})k$, $t_{(f,\Delta)}$ is a noncentral t-distribution with degrees of freedom f and noncentral parameter Δ.

Proof: The estimator g can be written as,

$$(\overline{X}-\overline{Y})/S = \frac{\frac{(\overline{X}-\overline{Y})-(\mu_X-\mu_Y)}{\sigma_{(\overline{X}-\overline{Y})}} + \frac{(\mu_X-\mu_Y)}{\sigma_{(\overline{X}-\overline{Y})}}}{\frac{S}{\sigma_{(\overline{X}-\overline{Y})}}} = \frac{(Z+\Delta)}{(S/\sigma)[\sigma/\sigma_{(\overline{X}-\overline{Y})}]} = \frac{k(Z+\Delta)}{\sqrt{\chi^2_{(f)}/f}}$$

where $\Delta = (\mu_X - \mu_Y)/\sigma_{(\overline{X}-\overline{Y})}$, $k = \sigma_{(\overline{X}-\overline{Y})}/\sigma$, $Z \sim N(0,1)$ and $S/\sigma \sim \sqrt{\chi^2_{(f)}/f}$.

According to Johnson and Welch's (1939), the distribution of g is of the form as specified. Moreover, the relationship $\hat{g} = [\sqrt{\hat{f}/f}]g$ is tenable since $\hat{g} = (\overline{X} - \overline{Y})/\hat{S}$, and $\hat{S} = [\sqrt{f/\hat{f}}]S$. Hence, the distribution of $\hat{g}$ is obtained a given above. ||

Corollary 1. (Biased estimators)

(i) The means and variances of g and $\hat{g}$, respectively, are:

(a) (Exact). $\mu_g = c_f^{-1}\delta$ and $\mu_{\hat{g}} = c_{\hat{f}}^{-1}\delta$, $\sigma_g^2 = (f/(f-2))(k^2 + \delta^2) - \mu_g^2$, and $\sigma_{\hat{g}}^2 = (\hat{f}/(f-2))(k^2+\delta^2) - \mu_{\hat{g}}^2$; where the expressions for c_f and $c_{\hat{f}}$ have been specified previously.

(b) (Approximate). $\tilde{\mu}_g = [(4f-1)/4(f-1)]\delta$, and $\tilde{\mu}_{\hat{g}} = [(4\hat{f}-1)/4(\hat{f}-1)]\delta$; $\tilde{\sigma}_g^2 = [f/(f-2)](k^2+\delta^2) - \tilde{\mu}_g^2$, and $\tilde{\sigma}_{\hat{g}}^2 = [\hat{f}/(f-2)](k^2+\delta^2) - \tilde{\mu}_{\hat{g}}^2$.

(c) (Asymptotic). $\mu_{g,\infty} = \delta$, and $\mu_{\hat{g},\infty} = \mu_{g,\infty}$; $\sigma^2_{g,\infty} = (k^{-2} + (\delta^2/2f))$, and $\sigma^2_{\hat{g},\infty} = \sigma^2_{g,\infty}$.

(ii) The variance-stabilizing and normalizing transformations of g and $\hat{g}$ are:

$g^* = b^{-1}sinh^{-1}(ba^{-1}g) = b^{-1}ln|bg + \sigma_g|$, and

$\hat{g}^* = \hat{b}^{-1}sinh^{-1}(\hat{b}\hat{a}^{-1}g) = \hat{b}^{-1}ln|\hat{b}\hat{g} + \sigma_{\hat{g}}|$;

where $a^2 = [f/(f-2)]k^2$, $\hat{a}^2 = [\hat{f}/(f-2)]k^2$, $b^2 = [f/(f-2)] - c_f^2$, $\hat{b}^2 = [\hat{f}/(f-2)] - \hat{c}_f^2$, $b^{-1}[sinh^{-1}(ba^{-1}g) - sinh^{-1}(ba^{-1}\delta)] \approx N(0,1)$, $\hat{b}^{-1}[sinh^{-1}(\hat{b}\hat{a}^{-1}\hat{g}) - sinh^{-1}(\hat{b}\hat{a}^{-1}\delta)] \approx N(0,1)$, and "$\approx$" denotes "is asymptotically distributed as".

(iii) The bias and mean-square errors of g and $\hat{g}$ can be derived as:

(a) $Bias(g) = [c_f^{-1} - 1]\delta$, and $Bias(\hat{g}) = [c_f^{-1} - 1]\delta$;

(b) $MSE(g) = [f/(f-2)](k^2+\delta^2) - (1-2c_f^{-1})\delta^2$, and $MSE(\hat{g}) = [\hat{f}/(f-2)](k^2+\delta^2) - (1-2c_f^{-1})\delta^2$.

Proof: The proof will be carried out for the results associated with the estimator g. Similar arguments can be derived for the results related to the maximum likelihood estimator $\hat{g}$ by means of the transformation $\hat{g} = [\sqrt{\hat{f}/f}]g$.

(i)

(a) The well-known result $Z = \frac{(\overline{X}-\overline{Y})}{S_{(\overline{X}-\overline{Y})}/\sqrt{f}} \sim N(0,1)$ implies that $\frac{S^2_{(\overline{X}-\overline{Y})}}{\sigma^2_{(\overline{X}-\overline{Y})}} \sim \chi^2_{(f)}/f$. Let $W = (\overline{X}-\overline{Y})/S_{(\overline{X}-\overline{Y})}$ be an unbised estimate of $\Delta = k^{-1}\delta$. Then W can be rewritten as:

$$W = \frac{(\overline{X}-\overline{Y})}{S_{(\overline{X}-\overline{Y})}} = \frac{\frac{(\overline{X}-\overline{Y})-(\mu_X-\mu_Y)}{\sigma_{(\overline{X}-\overline{Y})}} - \frac{(\mu_X-\mu_Y)}{\sigma_{(\overline{X}-\overline{Y})}}}{\frac{S_{(\overline{X}-\overline{Y})}}{\sigma_{(\overline{X}-\overline{Y})}}} = (Z+\Delta)/\sqrt{\chi^2_{(f)}/f}.$$

Then, from Johnson and Welch (1939), $\mu_W = c_f^{-1}\Delta$ and $\sigma^2_W = [f(1+\Delta^2)/(f-2)] - \mu^2_W$. Since $g \sim kt_{(f,\Delta)}$ implies that $\mu_g = k\mu_W$ and $\sigma^2_g = k^2\sigma^2_W$; hence, the results follow.

(b) The values of $\tilde{\mu_g}$ and $\tilde{\sigma}^2_g$ are based on the approximation $\tilde{c_f} = 1 - [3/(4f-1)]$ with an error $\leq .0003$ for $f \geq 10$ (Hedges, 1981).

(c) The limiting distribution of W is normal with mean of Δ and variance of $[1 + (\Delta^2/2f)]$ (Johnson and Welch, 1939, p. 367). Since $Z \sim N(0,1)$ and by applying the central limit theorem, it yields:

$$Z = t_{(f,\Delta)}\sqrt{1+(\Delta^2/2f)} = k(g-\delta)/\sqrt{1+(k^{-2}\delta^2/2f} = (g-\delta)/\sqrt{k^{-2}+(\delta^2/2f)}.$$

Hence, the results hold.

(ii) Since the variance and MSE of g are functions of δ, it is desirable to to transform g into variance-stabilizing estimators. The variance of g can be decomposed as $\sigma^2 = a^2 + b^2\delta^2$.

Following Laubscher (1959), the reported variance-stabilizing and normalizing transformation of g is obtained.

(iii) The results are trivial since, by definition, $Bias(g) = \mu_g + \delta$, and $MSE(g) = \sigma_g^2 - [Bias(g)]^2$. Analogous arguments apply to the derivation of $Bias(\hat{g})$ and $MSE(\hat{g})$. ||

Novick and Jackson (1974) suggested another approximation for c_f, namely, $\tilde{c}_f^N = \sqrt{1-(3/2f)}$ with an error about .01 for $f \geq 5$. Under this approach, $\tilde{\mu}_g = [\sqrt{2f/(2f-3)}]\delta$ and $\tilde{\mu}_{\hat{g}} = [\sqrt{2\hat{f}/(2\hat{f}-3)}]\delta$. The approximation given by Hedges (1981) is preferred since it is more precise and yields a smaller bias in the estimation of δ. In the limit, c_f^{-1} reduces to 1 under both approaches. When f is as small as 3, c_f^{-1} is equal to 1.38, 1.41 and 1.37 according to the exact formula and those given by Novick and Jackson (1974) and Hedges (1981), respectively.

Corollary 2 (Unbiased estimators). Consider the estimators $h = c_f g$ and $\hat{h} = c_{\hat{f}}\hat{g}$. Then $h = h$ and the following results can be applied to both h and $\hat{h}$:

(i) The distribution of h is $kc_f t_{(f,k^{-1}\delta)}$ with mean $\mu_h = \delta$ and its variance can be specified as,

(a) (Exact) $\sigma_h{}^2 = [c_f^2(\frac{f}{f-2})(k^2+\delta^2)] - \delta^2$,

(b) (Approximate) $\tilde{\sigma}_h^2 = [(\frac{4(f-1)}{4f-1})(k^2+\delta^2)] - \delta^2$,

(c) (Asymptotic) $\sigma_{h,\infty}^2 = \sigma_{g,\infty}^2$.

(ii) A variance-stabilizing and normalizing transformation of h is:

$$h^* = b_h^{-1} sinh^{-1}(b_h a_h^{-1} h) = b_h^{-1} ln|b_h h + \sigma_h|,$$

where $a_h = c_f a$, $b_h = c_f b$ and, $b_h^{-1}[sinh^{-1}(b_h a_h^{-1} h) - sinh^{-1}(b_h a_h^{-1}\delta)] \approx N(0,1)$.

(iii) The bias of h is 0 and its mean-square error is equal to σ_h^2

Proof: The unbiased estimators h and $\hat{h}$ have the same distributions because it can be written that,

$$\hat{h} = c_{\hat{f}}\hat{g} = [\sqrt{f/\hat{f}}]c_{\hat{f}}\hat{g} = (\sqrt{f/\hat{f}})c_f[\sqrt{\hat{f}/f}]g = c_f g = h.$$

The remaining results can be derived due to the proof of *Corrolary 1* as well as the fact that $\sigma_h^2 = c_f^2\sigma_g^2$ and c_f is increased to 1 as f increases to infinity. Therefore, the limiting distributions of h and g are the same by Slutsky's theorem (Serfling, 1980). ||

An unbiased estimate of σ_h^2 can be expressed in the form of $S_h{}^2 = [c_f{}^2(\frac{f}{f-2})(k^2+h^2)] - h^2$. Hedges and Olkin (1985) proposed a variance-stabilizing and normalizing transformation for the estimator $h_2 = c_{f_2}(\overline{X} - \overline{Y})/S_2$ as:

$$h_2^* = \sqrt{2N} sinh^{-1}[h_2/q] = \sqrt{2N}\sqrt{ln|(h_2/q) + (h_2^2/q^2) + 1|},$$

where $q = \sqrt{4 + 2(n/m) + 2(m/n)}$ and $N = n+m$. This equation is derived from the asymptotic distribution of h_2. Kraemer (1983) suggested another transformation method from the relationship between the distributions of h and the product- moment correlation coefficient. Both the Laubscherian transformation and that of Kraemer (1983) are based on the exact distributions of h, and thus expected to be more accurate for small sample cases than the method of Hedges and Olkin (1985). In practice, the Laubscherian equation is easier to use than Kraemer's (1983) procedure since the latter may require an additional step of converting the transformed values in terms of correlation coefficients to the original scale of measurement.

The efficiency of estimators are often evaluated according to the minimum mean-square error criterion. In the following, two additional estimators are derived from procedures that serve this purpose.

Proposition 2. (Shrunken estimators)

(i) (Thompson, 1968). Let d be an estimator of δ. The minimization of $MSE(w_T d) = \mu^2_{(w_T d - \delta)}$ results in the weight w_T of the form:

$$w_T = \mu_d^2/(\sigma_d^2 + \mu_d^2).$$

(ii) (Hedges and Olkin, 1985). Consider an estimator d of δ such that $\sigma_d^2 = a_H^2 + b_H^2 d^2$. Then the MSE of any linear transformation of d, say $d_H = w_H d + q_H$; where w_H and q_H are

some constants; is minimized by defining the weight as,

$$w_H^* = (b_H^2 + 1)^{-1}.$$

Proof :

(i) The given expression for w_T that minimizes the mean-square error (MSE) of $d_T = w_T d$,

$$MSE(w_T d) = \varepsilon(w_T d - \mu_T)^2 = w_T^2(\sigma_d^2 + \mu_d) + \mu_d^2 - 2w_T\mu_d^2,$$

can be obtained as a solution the partial derivative of

$$\partial MSE(w_T d)/\partial w_T = 2w_T(\sigma_d^2 - \mu_d^2) - 2\mu_d^2 = 0.$$

(ii) See Hedges and Olkin (1985), pp. 105-106. ||

Some applications of the results in *Proposition 2* are now explored. The distribution of d belongs to the family of non-central t distribution with degree of freedom f. When f is large, the distribution of d is approximately normal (Johnson and Welch, 1939) with mean and variance of δ and $k^{-2} + (\delta^2/2f)$, respectively (*Corollaries 1 and 2*). Since the theoretical origin of the normal distribution is zero, following Thompson (1968), a shrunken estimator of δ can be formed as:

$$d_T = (\frac{\delta^2}{k^{-2} + (\delta^2/2f) + \delta^2})d$$

or, alternatively, $d_T = [(1/k^2\delta^2) + (1/2f) + 1]^{-1}\delta$.

As another application of Thompson's result (1968), let d = h then a minimum MSE estimator ($d_T^{\sim}$) can be expressed as:

$$d_T = w_T h = (\mu_h^2/\sigma_h + \mu_h^2)h = (\frac{c_f^{-2}\delta^2}{[f/(f-2)][k^2 + \delta^2]})h.$$

An unbiased estimate of w_T, denoted as $w_T^{\sim}$, is obtained upon replacing δ by h. The corresponding shrunken estimator in terms of the moments of h under Hedges and Olkin's (1985) procedure is of the form:

$$d_H = w_H h = (\frac{c_f^{-2}}{f/(f-2)})h.$$

So far, five generalized estimators of effect size, namely $g, \hat{g}, h, d_T$ and d_H have been discussed. Note that these estimators have the same signs. They are now compared according to the following three criteria: the size of their absolute values, biasedness and MSE magnitudes. The comparisons are performed on the basis that all relevant estimators are computed with the same sample sizes.

Proposition 3 (Comparing the estimators). Given that $f > 2$ and $\hat{f} \geq f$, the relative rank-ordering magnitudes of g, $\hat{g}$, d_H, and d_T on the basis of their absolute values, degrees of bias and MSE magnitudes, respectively, are:

(i) $|\hat{g}| \geq |g| \geq |h| \geq |d_H| \geq |d_T|$,

(ii) $|Bias(d_T)| \geq |Bias(\hat{g})| \geq |Bias(g)| \geq |Bias(d_H)|$.

(iii) $MSE(\hat{g}) \geq MSE(g) \geq MSE(h) \geq MSE(d_H) \geq MSE(d_T)$.

Proof :

(i) This result is proved by means of pairwise comparisons among the estimators of δ, starting with the claimed smallest estimators, namely d_T and d_H. Since $d_T = [h^2/(k^2+h^2)]d_H$ and $[h^2/(k^2+h^2)] \leq 1$, we have $|d_T| \leq |d_H|$. Now, $d_H = w_H h$ implies that $|h| \geq |d_H|$ if $w_H \leq 1$ or $(f-2)/f \leq c_f^{-2}$. But this condition is tenable since $c_f^{-2} \geq 1$ and $(f-2)/f \leq 1$. Next, $h = c_f g$ indicates that $|h| \leq |g|$ since $c_f \leq 1$. Finally, $|\hat{g}| = |(\sqrt{\hat{f}/f})g| \geq |g|$ when $\hat{f} \geq f$.

(ii) With $Bias(h) = 0$ already obtained, the pairwise comparisons is conducted first with $Bias(d_H)$ and Bias(g) where $Bias(d_H) = (\frac{c_f^{-2}}{[f/(f-2)]} - 1)\delta$ and $Bias(g) = [c_f^{-1} - 1]\delta$. Since $(\frac{c_f^{-2}}{[f/(f-2)]}$ is much smaller than 1 than c_f^{-1} is larger than 1, $|Bias(d_H)| \leq |Bias(g)|$. Next, $|Bias(\hat{g})| \geq |Bias(g)|$ because $\mu_{\hat{g}} \geq \mu_g$ when $\hat{f} \geq f$. Finally, $|Bias(d_T)| \geq |Bias(\hat{g})|$; where $Bias(d_T) = [\frac{c_f^{-2}}{[f/(f-2)]}(v_T) - 1]\delta, v_T = (\delta^2/(k^2+\delta^2)$ and $Bias(\cdot) = [c_f^{-1} - 1]\delta$; since $|c_f^{-1} - 1|$ is smaller than $|(\frac{c_f^{-2}}{[f/(f-2)]}v_T) - 1|$ due to the fact that both $f/(f-2)$ and v_T are less than 1.

(iii) Consider $MSE(d_T) = \varepsilon(d_T - \mu_{d_T})^2 = \varepsilon(v_T d_H - v_T \mu_{d_H})^2 = v_T{}^2 MSE(d_H)$. Then, $MSE(d_H) \geq MSE(d_T)$ because $v_T \leq 1$. Analogously, $MSE(d_H) = w_H^2 \varepsilon(h-\delta)^2 = w_H^2 \sigma_h^2$ is less than $MSE(h) = \sigma_h^2$ because $w_H \leq 1$. Next, $MSE(g) = \sigma_g^2 + (Bias(g))^2$ is larger than $MSE(h) = c_f^2 \sigma_g^2$ since $c_f^2 \leq 1$ and $(Bias(g))^2 \geq 0$. Finally, $MSE(\hat{g}) = \varepsilon(\hat{g} - \mu_{\hat{g}})^2 = (\hat{f}/f)\varepsilon(g - \mu_g{}^2) = (\hat{f}/f)MSE(g)$ is larger than $MSE(g)$ when $\hat{f} \geq f$. ||

For a given value of f and $\hat{f}$, the biased values for each of the five estimators under consideration are smaller than its relevant MSE values. As f increases, the biased values of these estimators would rapidly converge to zero when their expected values approach δ. In the limit, the corresponding mean-square errors (MSE) would not always converge to zero and when they do, not as rapidly. In particular, the mean square errors of g, $\hat{g}$, h and d_H would be equal to $\sigma_{g,\infty}^2$ which are expected to be larger than $\sigma_{d_T}^2$ since the latter is of the form $v_T^2 w_h^2 \sigma_{g,\infty}^2$ where v_T and w_H are less than one. Note that the results in _Proposition 3_ can applied to all cases listed in Table 1 since $\hat{f}_i \geq f_i$ (i = 1, ..., 5), except that $\hat{f}_5 \leq f_5$ when $n > m$. The exception will be illustrated in a Monte-Carlo study.

The findings above can be generalized to other estimators of δ. In general, bias and MSE values grow with the increase of δ and reduce as the associated degrees of freedom increase. Values of Bias(g) and Bias($\hat{g}$) share the same signs with δ whereas those of Bias(d_H) and $Bias(d_T)$ have the oppisite signs to δ. From the results in _Proposition 3_, it is clear that the popular estimator g, proposed by Glass (1976), tends to overestimate the effect size (δ). According to the criteria of minimum bias and mean square error, the estimators h, d_H and d_T should be preferred to g and $\hat{g}$. Moreover, d_T is more favorable to d_H on the basis of the variance stabilization criterion. In terms of their absolute values, h is unbiased whereas d_T would tend to underestimate δ . Therefore, both h and d_T should be computed in a meta-analysis study. The former is used in the analysis of effect size of the experiment under consideration and the latter, as an indication of its lower bound.

Properties of Some Estimators of Effect Size

The distributions of some specific estimators of effect size (δ) are now studied with respect to the general properties presented above. In particular, some specific forms of the five estimators $g_i, \hat{g}_i, h_i, d_{H_i}$ and d_{T_i} are examined (The subscript i represents the case in which S_i, listed in Table 1, is used in the expression of the estimator under consideration). According to *Proposition 3*, the biased estimators (g_i and $\hat{g}_i$,i = 1, ..., 4) are relatively less effective than the unbiased and shrunken estimators (h_i, d_{H_i} and d_{T_i}, i = 1, ..., 4) in the estimation of effect sizes.

Estimators under variance homogeneity

The main properties of estimators listed in Table 1 for the case of $\sigma_X^2 = \sigma_Y^2$ can be summarized as follows:

Corollary 3 (Variance homogeneity)

(i) (Biased estimators)

(a) $|\hat{g}_1| \geq |g_1| \geq |\hat{g}_2| \geq |g_2|$ if $S_X^2 \geq S_Y^2$ and $|\hat{g}_2| \geq |g_2| \geq |\hat{g}_1| \geq |g_1|$ if $S_Y^2 > S_X^2$ when n = m.

(b) $|Bias(\hat{g}_1)| \geq |Bias(g_1)| \geq |Bias(\hat{g}_2)| \geq |Bias(g_2)|$.

(c) $MSE(\hat{g}_1) \geq MSE(g_1) \geq MSE(\hat{g}_2) \geq MSE(g_2)$.

(ii) (Unbiased estimators)

(a) $|h_2| \geq |h_1|$ if $S_Y^2 \geq S_X^2$ when n = m whereas $|h_1| \geq |h_2|$ if $S_X^2 \geq S_Y^2$.

(b) $MSE(h_1) \geq MSE(h_2)$

(iii) (Shrunken estimators)

(a) $|d_{T1}| \geq |d_{T2}|$ if $|h_1| \geq |h_2|$ and vice versa if $|h_1| \leq |h_2|$.

(b) $|Bias(d_{T2})| \geq |Bias(d_{T1})$

(c) $MSE(d_{T_1}) \geq MSE(d_{T_2})$

Proof:

(i) (Biased estimators)

(a) The results in *Proposition 3* implies that $|\hat{g}_i| \geq |g_i|$ (for i = 1, 2). From the definition of S_i^2 and $\hat{S}_i^2$ in Tables 1, it only requires to find the condition for $\hat{S}_2^2 \geq S_1^2$ so that $|g_1| \geq |\hat{g}_2|$. Since $\hat{S}_2^2 - S_1^2 = \frac{(n-1)S_X^2+(m-1)S_Y^2}{(n+m-2)} - S_Y^2 = \frac{S_X^2-S_Y^2}{n+m-2}$, the resulting condition is as stated in this Corollary. Analogously, it is necessary to show that $\hat{S}_1^2 \geq S_2^2$ to obtain $|g_2| \geq |\hat{g}_1|$. Since the condition $\hat{S}_1^2 \geq S_2^2$ implies that $(n+m-2)(m-1)S_Y^2 \geq m[(n-1)S_X^2 + (m-1)S_Y^2]$ or, upon simplification, $S_Y^2 \geq [m(n-1)/n(m-1)]S_X^2$, the result is obtained.

(b) and (c) From *Proposition 3*, it can be seen that $Bias(\hat{g}_i) \geq Bias(g_i)$ and $MSE(\hat{g}_i) \geq MSE(g_i)$, for all i. Moreover, since biased and MSE values are monotonic decreasing functions of f_i and from the fact that $\hat{f}_2 \geq f_1$ (Table 1), we have $Bias(g_1) \geq Bias(\hat{g}_2)$ and $MSE(g_1) \geq MSE(\hat{g}_2)$. Therefore, the results hold.

(ii) (Unbiased estimators)

(a) The condition for $h_2 \geq h_1$ can be readily specified from the facts that $h_2 - h_1 = c_{f_2}g_2 - c_{f_1}g_1$, $c_{f_2} \geq c_{f_1}$ (because c_{f_i} is a monotonic increasing function of f_i, $f_2 \geq f_1$, and $|g_2| \geq |g_1|$ for the case in which $S_Y^2 \geq S_X^2$ when n = m). On the other hand, to find the condition for $h_1 \geq h_2$, a simple proof can be carried out by using the approximation $\tilde{c}_f^N$ (Novick and Jackson's,1974). The resulting forms of approximate unbiased estimators are expressed as $\tilde{h}_1 = [\sqrt{(2m-5)/2\sum_X}](\overline{X}-\overline{Y})$ and $\tilde{h}_2 = [\sqrt{(2n-7)/2(\sum_X+\sum_Y)}](\overline{X}-\overline{Y})$, where $\sum_X$ and $\sum_Y$ are defined as $\sum_X = \sum(X-\overline{X})^2$ and $\sum_Y = \sum(Y-\overline{Y})^2$, respectively. Then, from the difference

$$h_2^2 - h_1^2 = \frac{(\overline{X}-\overline{Y})^2}{2(\sum_X+\sum_Y)\sum_Y}\{(2m-5)(\sum_X -2(m-1)\sum_Y)\},$$

one can conclude that $\tilde{h}_2^2 \geq \tilde{h}_1^2$ if $S_X^2 \geq [2(m-1)/(2m+5)]S_Y^2$. It is clear that $h_2^2 - h_1^2 \geq 0$ if $(2m-5)(\sum_X -2(m-1)\sum_Y) \geq 0$. Since 2(m-1)/(2m+5) is less than one, the condition $S_X^2 \geq S_Y^2$ would yield the result $h_1 \geq h_2$.

(b) For the unbiased estimators h_i and h_j, we have

$$MSE(h_j) - MSE(h_i) = \sigma_{h_i}^2 - \sigma_{h_j}^2.$$

Therefore, the result holds since σ_h^2 is a monotonic decreasing function of f and the fact that $f_1 \geq f_2$.

(iii) (Shrunken estimators)

(a) Consider $d_{T_i} = w_{H_i}[h_i^2/(k^2 + h_i^2)]h_i$ for i = 1, 2; where $w_{H_i} = c_{f_i}^{-2}[(f_i - 2)/f_i]$. By applying the approximation $c_{f_i}^N$ (Novick and Jackson, 1974), it can be written that $w_{H_i} = [(f_i/(f_i - 1.5)][(f_i - 2)/f_i] \approx 1$ for $f_i \geq 3$. Hence, the magnitude and direction of d_{T_i} are essentially determined by h_i.

(b) Without loss of generality, assume $\delta \geq 0$. The bias of d_{T_i} can be expressed as $Bias(d_{T_i}) = w_{H_i}\{[\delta^2/(k^2 + \delta^2)] - 1\}\delta = -w_{H_i}k^2\delta/(k^2 + \delta^2)$. Now, $Bias(d_{T2}) - Bias(d_{T1}) = (w_{H1} - w_{H2})k^2\delta(k^2 + \delta^2) \geq 0$ since $f_2 > f_1$ and w_{H_i} is a monotonic decreasing function of f_i.

(c) Since $MSE(d_{T_i})$ is a monotonic decreasing function of f_i, the result follows for $f_2 \leq f_1$. ||

The results in _Corollary 3_ imply that, with the exception of the shrunken estimators, the pooled variances (used in the computation of $g_2, \hat{g}_2, h_2$ and d_{T_2}) are relatively more efficient than the unpooled variances [illegible]elding estimators with smaller biased and MSE values. The same observation applies to the following case of heterogeneous variances with known variance ratios.

Estimators under variance heterogeneity with known ratios

Corollary 4 ($\sigma_X^2 \neq \sigma_Y^2$, assuming $r = \sigma_X^2/\sigma_Y^2$)

(i) (Biased estimators)

(a) $|\hat{g}_3| \geq |g_3| \geq |\hat{g}_4| \geq |g_4|$ if $S_X^2 > [r(n + rN + 1)/(n - 1)]S_Y^2$ and $|\hat{g}_4| \geq |g_4| \geq |\hat{g}_3| \geq |g_3|$ if $S_Y^2 \geq S_X^2$

(b) $|Bias(\hat{g}_3)| \geq |Bias(g_3)| \geq |Bias(\hat{g}_4)| \geq |Bias(g_4)|$.

(c) $MSE(\hat{g}_3) \geq MSE(g_3) \geq MSE(\hat{g}_4) \geq MSE(g_4)$.

(ii) (Unbiased estimators)

(a) $|h_4| \geq |h_3|$ if $r \geq 1$ or $S_Y^2 \geq S_X^2$ and $|h_3| \geq |h_4|$ if $r \leq 1$.

(b) $MSE(h_3) \geq MSE(h_4)$.

(iii) (Shrunken estimators)

(a) $|d_{T3}| \geq |d_{T4}|$ if $|h_3| \geq |h_4|$ and vice versa if $|h_3| \leq |h_4|$.

(b) $|Bias(d_{T4})| \geq |Bias(d_{T3})$

(c) $MSE(d_{T3}) \geq MSE(d_{T4})$

Proof:

(i) (Biased estimators)

(a)Since $\hat{S}_3^2 - S_4^2 = \{[(N-2)(r+1) - 2r]/2(N-2)\}\sum_Y - [(N-2)^{-1}]\sum_X$ the condition for $\hat{S}_3^2 \geq S_4^2$ so that $|g_4| \geq |\hat{g}_3|$ is $S_Y^2 \geq \{2(n-1)/(m-1)[(N-2)(r+1) - 2r]\}S_X^2$ or $S_Y^2 \geq S_X^2$ as stated above. Similarly, since $\hat{S}_4^2 - S_3^2 = [\sum_X/nr] + [N^{-1} - (r+1)/(m-1)]\sum_Y$, the condition for $\hat{S}_4^2 \geq S_3^2$ so that $|g_3| \geq |\hat{g}_4|$ is $S_X^2 \geq [r(Nr+n+1)/(n-1)]S_Y^2$.

(b) and (c) The proofs for the order-ranking bias and mean square error values require the same arguments as presented for the biased estimators in *Corollary 4*.

(ii) (Unbiased estimators)

(a) Without loss of generality, the proof can be simplified by means of the approximation $\tilde{c}_f^N$ (Novick and Jackson, 1974). The corresponding approximate forms for the unbiased estimators h_3 and h_4 are $\tilde{h}_3 = \sqrt{(2m-5)/2(m-1)}(\overline{X} - \overline{Y})/S_3$, and $\tilde{h}_4 = \sqrt{(2N-7)/2(N-2)}(\overline{X} - \overline{Y})/S_4$, respectively. Consider the difference $\tilde{h}_3^2 - \tilde{h}_4^2 = \{[\frac{(2m-5)}{(r+1)(m-1)}] - \frac{[2(n+m)-7]}{2(n+m-2)}\}(\overline{X} - \overline{Y}) = C_{34}/D_{34}$; where $C_{34} = [(1-r)(2nm + 2m + 9m)] + 2n(r-4) + (13 - 7r)$ and $D_{34} = 2(n+m-2)(r+1)(m-1)$. Since $D_{34} \geq 0$, the sign of $\tilde{h}_3^2 - \tilde{h}_4^2$ is the same as that of C_{34}. The results hold since $C_{34} < 0$ if $r \geq 1$ and $C_{34} \geq 0$ if $r < 1$.

(b) and (c) The proofs are trivial, following the same arguments used in *Corrolary 3* with respect to the unbiased estimators.

(iii) (Shrunken estimators)

The proof follows the same arguments as given in *Corollary 4* upon replacing d_{T1}, and d_{T2} by d_{T3} and d_{T4}, respectively. ||

The results so far imply that estimators h_2 and d_{T_2} must be preferred to h_1 and d_{T_1}. The choice betwen h_3 and h_4 should depend on whether r is larger than one or not.

Estimators under the Behrens-Fisher condition

Corollary 5 (Biased estimators): The distributions of $g_5 = (\overline{X} - \overline{Y})/S_5$, $\hat{g}_5 = (\overline{X} - \overline{Y})/\hat{S}_5$, and $h_5 = c_{f_5} g_5$, respectively, are $kt_{(f_5, k^{-1}\delta)}$, $\hat{k}t_{(\hat{f}_5, \hat{k}^{-1}\delta)}$, and $c_{f_5} kt_{(f_5, k^{-1}\delta)}$ where:

(a) $k = \frac{\sigma_X^2/n + \sigma_{Y}^2/m}{\sqrt{(\sigma_X^2 + \sigma_{Y}^2)/2}}$,

(b) $\hat{k} = (\hat{f}_5/f_5)$, and

(c) f_5 and $\hat{f}_5$ are specified in Table 1.

Proof: The distributions of g_5, $\hat{g}_5$ and h_5 can be derived from the results of *Proposition 1*. The expressions for k, f_5 and $\hat{f}_5$ are obtained as follows:

(a) The effect size suggested by Cohen (1977) when $\sigma_X^2 \neq \sigma_Y^2$ is of the form $\delta = (\mu_X - \mu_Y)/\sigma$ where $\sigma = \sqrt{(\sigma_X^2 + \sigma_Y^2)/2}$, a root-mean-square value (Cohen, 1977, p.44). Hence, from *Proposition 1*, the expression for k is obtained.

(b) An unbiased estimate of σ^2 is $S_5^2 = (S_X^2 + S_Y^2)/2 = \frac{\sigma_X^2 X_{(n-1)}^2}{2(n-1)} + \frac{\sigma_Y^2 Y_{(m-1)}^2}{2(m-1)}$. Then, $S_5^2 \sim (\sigma^2/f_5)X_{(f_5)}^2$ where

$$f_5 = \frac{\{\sigma_X^2/2 + \sigma_Y^2/2\}^2}{\{\sigma_X^4/4(n-1)\} + \{\sigma_Y^4/4(m-1)\}} = \frac{(\sigma_X^2 + \sigma_X^2)^2}{\{\sigma_X^4/(n-1)\} + \{\sigma_Y^4/(m-1)\}},$$

according to Welch's (1938) procedure. Upon simplification, the expression of f_5 is given as

$$f_5 = \frac{(n-1)(m-1)(\sigma_X^2 + \sigma_Y^2)^2}{(m-1)\sigma_X^4 + (n-1)\sigma_Y^4}.$$

In Table 1, an unbiased estimates of f_5 are obtained upon replacing σ_X^2 and σ_Y^2 by S_X^2 and S_Y^2, respectively.

(c) The MLE of σ^2 under the Behrens-Fisher condition can be expressed as $\hat{S}_5^2 = \{\hat{S}_X^2 + \hat{S}_Y^2\}/2 = \frac{\{[(n-1)/n]S_X^2 + [(m-1)/m]S_Y^2\}}{2}$.

Then, $\hat{S}_5^2 \sim \frac{(\sigma_X^2/n)X_{(n-1)}^2 + (\sigma_Y^2/m)X_{(m-1)}^2}{2}$. Hence, from Welch's (1938) procedure, one gets:

$$\hat{f}_5 = \frac{[(n-1)\sigma_X^2/n + (m-1)\sigma_Y^2/m]^2}{\{(n-1)\sigma_X^4/n^2\} + \{(m-1)\sigma_Y^4/m^2\}}.$$

The above expression can be simplified as

$$\hat{f}_5 = \frac{[m(n-1)\sigma_X^2 + n(m-1)\sigma_Y^2]^2}{m^2(n-1)\sigma_X^4 + n^2(m-1)\sigma_Y^4}.$$

An unbiased estimate of $\hat{f}_5$ are given in Table 1. ||

By applying the results in *Corollary 1*, moments of the exact, approximate and limiting distributions of g_5 and $\hat{g}_5$ can be derived. The expected values and variances of g_5 and $\hat{g}_5$ are equal to those computed for g and $\hat{g}$ (*Corollary 1*) when f is defined as f_5 or $\hat{f}_5$, respectively. Similarly, it is possible to obtain moments for the unbiased estimators (h_5) and the corresponding shrunken estimators (d_{T5}) by means of *Corollary 2* and *Proposition 2*, respectively. As will be seen in the following simulation study, $\hat{f}_5 \geq f_5$ when $m \geq n$. Given this condition, the relative magnitudes, biased and MSE values of $g_5, \hat{g}_5, h_5, d_{H_5}$. and d_{T_5} follow the same patterns presented in *Proposition 3*. For the case in which $\hat{f}_5 \leq f_5$ (when $m \leq n$), the relationships expressed in *Proposition 3* still hold upon letting g and $\hat{g}$ switch places. In other words, the rankings among the unbiased and shrunken estimators are not affected by the relative sizes of n and m since they are computed on the basis of h_i (which is equal to $\hat{h}_i$ for all f_i and $\hat{f}_i$).

Monte-Carlo Results

Comparisons of Estimators of Effect Size Under Balanced Designs

Properties of the five estimators $g, \hat{g}, h, d_H$ and d_T were studied by means of a Monte-Carlo method under the balanced designs ($n = m$) as well as under several configurations of sample sizes (n and m) and variance ratios (r) for the three cases presented in *Table 1*. Under the balanced designs, the same computational expressions for the above five estimators are found in both Case 1 (Variance Homogeneity) and Case 3 (The Behrens-Fisher problem) as implied by the following results:

Corollary 6. (Balanced designs). Let $\hat{r} = S_X^2/S_Y^2$ represent an unbiased estimator of σ_X^2/σ_Y^2. When $n = m$, the following properties are observed:

(a) $k = \sqrt{2/n}$ in all three cases presented in *Table 1*.

(b) $S_2^2 = S_5^2$, $\hat{S}_2^2 = \hat{S}_5^2$, $S_3^2 = S_1^2\sqrt{(\hat{r}+1)/2}$ and $S_4^2 = \sqrt{(S_X^2/2\hat{r}) + (S_Y^2/2)}$.

(c) $f_1 = f_3 = (n-1)$, $\hat{f}_1 = \hat{f}_3 = n$, $f_2 = f_4 = 2(n-1)$, $\hat{f}_2 = \hat{f}_4 = 2n$, $f_5 = \hat{f}_5 = (n-1)(1+\hat{r})^2/(1+\hat{r}^2)$.

Proof:

(a) In Case 1 (Variance Homogeneity) $k = \sqrt{(n+m)/nm} = \sqrt{2/n}$ for $n = m$. In Case 2 ($r = \sigma_X^2/\sigma_Y^2$), $k = \sqrt{2(mr+n)/nm(r+1)} = \sqrt{2n(r+1)/n^2(r+1)} = \sqrt{2/n}$. Similarly, in Case 3 ($\sigma_X^2 \neq \sigma_Y^2$), $k = \sqrt{2n(\sigma_X^2+\sigma_Y^2)/n^2(\sigma_X^2+\sigma_Y^2)} = \sqrt{2/n}$.

(b) and (c) By setting $n = m$ in the expressions for S_i^2, $\hat{S}_i^2$, f_i and $\hat{f}_i$ (i= 1, ...,5), the results are obtained upon simplifying similar terms. ||

Values of the five estimators of effect size mentioned above were generated with increasing sample sizes (5(1)105) and increasing effect sizes (0(.2)2). In Case 1 ($\sigma_X^2 = \sigma_Y^2$) and Case 2 ($r = \sigma_X^2/\sigma_Y^2$), the degrees of freedom f_1 and $\hat{f}_1$ were used in the estimation of the moments for g and $\hat{g}$, respectively; whereas the degree of freedom f_2 was used in association with the remaining three estimators (h, d_H and d_T). In Case 3 ($\sigma_X^2 \neq \sigma_Y^2$), since $f_5 = \hat{f}_5$, the degree of freedom f_5 were used in computing the moments of all five estimators. Under Cases 2 and 3, the condition of variance heterogeneity was specified by setting the values of $\hat{r}$ as 2(1)10.

For a given value of effect size, the distributions of the means and variances of the five estimators ($g, \hat{g}, h, d_H$, and d_T) in any configurations of sample sizes under consideration can be typically represented by the plots in *Figure 1*. The corresponding biased and MSE values are generally performed as the distributions plotted in *Figure 2*. For $\delta \geq 0$, the following relationships are observed across all simulated configurations of sample sizes (n = m), effect sizes (δ) and variance conditions (r):

Insert Figures 1 and 2 about here

(a) $\mu(\hat{g}) \geq \mu(g) \geq \mu(h) \geq \mu(d_H) \geq \mu(d_T)$, and

(b) $\sigma^2_{\hat{g}} \geq \sigma^2_g \geq \sigma^2_h \geq \sigma^2_{d_H} \geq \sigma^2_{d_T}$.

(c) The relative magnitudes of biased and MSE values for the five estimators followed the patterns described in *Proposition 3*.

The above relationships can also be applied to the case of negative effect size provided that values of the means in (a) are expressed in absolute terms. As sample sizes increase, the main properties of d_T are: (i) minimum variance, (ii) stability (values of $\sigma^2_{d_T}$ are quite small even for small degrees of freedom), and (iii) minimum mean square error, despite the large absolute values of $Bias(d_T)$.

The typical behaviors of the five estimators of effect size under consideration with respect to the constraint of fixed degrees of freedom and varied effect sizes are depicted in *Figure 3* and *Figure 4*.

Insert Figures 3 and 4 about here

In general, the moments of the estimators are monotonic increasing functions of δ. The notable exceptions are $Bias(h)$, which is always equal to zero, $Bias(d_T)$ for δ around zero, and $Bias(d_H)$. As δ increases, the means of the estimators grow much faster than other moments. In this case, the main properties of d_T can be summarized as: (i) minimum variance, (ii) minimum mean square error and (iii) largest bias. For $\delta \geq 0$, as effect size becomes larger, the absolute value of $Bias(d_T)$ is reduced toward zero. On the contrary, the same absolute values of the other estimators depart further from zero. In short, the shrunken estimator d_T is most efficient, as compared to g, $\hat{g}$, h and d_H, in the estimation of effect size when the sample sizes are small and δ is large.

General Comparisons of Estimators of Effect Size

The specifications of the simulation parameters (n, m, r, f and $\hat{f}$) used to generate values of the five estimators in several combinations of sample sizes and variance conditions are reported in Table 2. In each combination of $\hat{r}$, f (or $\hat{f}$) and variance conditions (denoted

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Insert Table 2 about here

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as Case 1, Case 2 and Case 3), six data sets containing values of the five estimators were generated, corresponding to the six configurations of sample sizes (n and m) (denoted as ID in Table 2). Under Case 1 ($\sigma_X^2 = \sigma_Y^2$) and Case 2 ($\sigma_X^2/\sigma_Y^2 = $ r), the degrees of freedom f_1 and $\hat{f}_1$ were used to derive the moments of g and $\hat{g}$, respectively; whereas moments of the estimators h, d_H and d_T were computed on the basis of f_2. Similarly, under Case 3 ($\sigma_X^2 \neq \sigma_Y^2$), properties of $\hat{g}$ were derived basing on $\hat{f}_5$ and moments of the other four estimators were computed by means of f_5. To generate data under Case 3 ($\sigma_X^2 \neq \sigma_Y^2$), it is assumed that $\hat{r}$ is an unbiased estimate of r where $\hat{r} = S_X^2/S_Y^2$. Therefore, the estimates of f_5 and $\hat{f}_5$ in Table 1 are revised respectively as:

$$f_5 = \frac{(n-1)(m-1)(\hat{r}^2+1)^2}{(m-1)\hat{r}^2+(n-1)}$$

$$\hat{f}_5 = \frac{[m(n-1)\hat{r}+n(m-1)]^2}{m^2(n-1)\hat{r}^2+n^2(m-1)}$$

As shown in Table 2, $\hat{f}_5 \geq f_5$ when $m \geq n$ and the reverse is true when $m \leq n$. The resulting expected values of the means, variances, bias and mean-square errors of the five estimators over all configurations under study are reported in Tables 3 to 8.

Insert Tables 3 to 8 about here

On the basis of simulated data, principal characteristics of the distributions of the five estimators under consideration can be summarized as follows:

(a) When $\delta = 0$, all estimates of effect size are unbiased. Similarly, when $\delta = 1$, the biased values are very small.

(b) Expected values of the means, variances, bias and mean square errors of estimators of effect size are monotonic increasing functions of δ. Except $\mu(d_T)$, the above moments are monotonic decreasing functions of sample sizes.

(c) As δ increases, expected values of the means grow faster than expected values of other parameters, with $\mu(g)$ and $\mu(\hat{g})$ increase fastest. The values of MSE(g) and $MSE(\hat{g})$ increase more rapidly than mean square errors of the other estimators. While the biased values of other estimators increase in absolute terms, $Bias(d_H)$ and $Bias(d_T)$ reduce to zero.

(d) As sample size increases, the biased values reduce faster than other parameters. The magnitude of reduction (R) can be arranged in descending order as R(Bias), R(MSE), $R(\sigma^2)$ and $R(\mu)$.

(e) Within a given configuration of δ, sample sizes (n and m), and variance condition (r), values of MSE are substantially larger than those of Bias. Moreover, the relative magnitudes of Bias and MSE values for the five estimators in each simulated configuration conform to the relationships described in *Proposition 3*.

(f) In Case 2 ($\sigma_X^2/\sigma_Y^2 = r$), as r increases moments of the estimators of effect size change very little in most configurations (ID). In Case 3, ($\sigma_X^2 \neq \sigma_Y^2$), with the exception of $\sigma_{d_T}^2$ which tends to stabilize, variances of the other parameters generally increase (reduce) for small (large) degrees of freedom.

In the presence of the Behrens-Fisher condition, variances of the estimators of effect size were inflated if the degrees of freedom under the assumption of variance homogeneity were used. However, this adverse effect is minimal with respect to the shrunken estimator d_T. For example, for the configuration of $\delta = 1$ and ID = 1, the values of $\sigma_{\hat{g}}^2$ are .97 in Case 1, .44 (when r = 2) and .65 (when r = 10) in Case 3. The corresponding values of $\sigma_{d_T}^2$ are .21, .20 and .22. In general, as r increases, moments of the estimators increase for small degrees of freedom (ID from 1 to 3) and reduce when the degrees of freedom are large (ID from 4 to 6). In all cases, the values of $\sigma_{d_T}^2$ are small, ranging from zero to .22, and quite stabilized as r increases.

Conclusions

The Monte-Carlo results tend to support the previous recommendation, based on theoretical analysis, that the estimators h and d_T should be used in conducting a meta-analytic study. The use of both estimators will ensure the properties of unbiasedness and minimum MSE in the estimation of effect size (δ). Moreover, on the basis of simulated data, values of both h and d_T were not affected as severely as g and $\hat{g}$ in the presence of violations to the assumption of variance homogeneity. The moments of d_T were quite stable across the different variance conditions under consideration. Clearly, due to its properties of minimum MSE and variance stability, d_T perform better than Glass-type estimators (g, $\hat{g}$) and Hedges-type estimators (h and d_H) when sample sizes are small and effect size is large In general, balanced designs should be attempted whenever possible, to minimize the effects of variance inflation caused by violations to the variance homogeneity assumption. Further research is needed to assess the relative effectiveness of estimators of δ in the context of statistical inferences and data analysis.

Table 1

Some population variances (σ^2), noncentrality parameters (Δ) and estimators of σ.

Case 1: Variance homogeneity

(a) Population variances: $\sigma = \sigma_X = \sigma_Y$.

(b) Noncentral parameter: $\Delta = k^{-1}\delta = sqrtnm/(n+m)\delta = \tilde{n}^{-1/2}\delta$.

(c) Estimators of σ:

$$S_1 = S_Y,$$
$$S_2 = \sqrt{\frac{(n-1)S_X^2 + (m-1)S_Y^2}{(n+m-2)}},$$
$$\hat{S}_1 = \hat{S}_Y = \sqrt{(m-1)S_Y^2/m},$$
$$\hat{S}_2 = \sqrt{[n\hat{S}_X^2 + m\hat{S}_Y^2]/(n+m)} \text{ where } \hat{S}_X^2 = [(n-1)/n]S_X^2 \text{ and } \hat{S}_Y^2 = [(m-1)/m]S_Y^2.$$

(d) Degrees of freedom of the chi-square distributions for S_i^2/σ^2 and $\hat{S}_i^2\sigma^2$, i = 1, 2; respectively:

$$f_1 = (m-1),$$
$$f_2 = (n+m-2),$$
$$\hat{f}_1 = m,$$
$$\hat{f}_2 = (n+m).$$

Case 2: Variance heterogeneity

(with $\sigma_X^2/\sigma_Y^2 = r$, where r is a known constant)

(a) Population variances: $\sigma^2 = \sigma_Y^2(1+r)/2$

(b) Noncentral parameter: $\Delta = k^{-1}\delta = \{\sqrt{nm(r+1)/2(mr+n)}\}\delta$

(c) Estimators of σ:

$$S_3 = S_Y\sqrt{(r+1)/2}$$
$$S_4 = \sqrt{\frac{[(n-1)S_X^2/r] + (m-1)S_Y^2}{(n+m-2)}},$$
$$\hat{S}_3 = S_3\sqrt{(m-1)/m},$$
$$\hat{S}_4 = \sqrt{[(n/r)\hat{S}_X^2 + m\hat{S}_Y^2]/(n+m)}.$$

(d) Degrees of freedom in the chi-square distributions for $f_i S_i^2/\sigma^2$ and $\hat{f}_i\hat{S}_i^2\sigma^2$, i = 3, 4; respectively: $f_3 = f_1$, $f_4 = f_2$, $\hat{f}_3 = \hat{f}_1$ and $\hat{f}_4 = \hat{f}_2$.

Case 3: The Behrens-Fisher problem ($\sigma^2_X \neq \sigma^2_Y$)

(a) Population variances: $\sigma = \sqrt{(\sigma^2_X + \sigma^2_Y)/2}$

(b) Noncentral parameter: $\Delta = k^{-1}\delta = \sqrt{\frac{nm(\sigma^2_{X\cdot} + \sigma^2_{Y\cdot})}{2(m\sigma^2_{X\cdot} + n\sigma^2_{Y\cdot})}}\delta$

(c) Estimators of σ:

$$S_5 = \sqrt{(S^2_X + S^2_Y)/2}$$

$$\hat{S}_5 = \sqrt{[\hat{S}^2_X + \hat{S}^2_Y]/2},$$

(d) Estimates of degrees of freedom of the chi-square distributions for S^2_4/σ^2 and $\hat{S}^2_5/\sigma^2$; respectively:

$$f_5 = \frac{(n-1)(m-1)(S^2_{X\cdot} + S^2_{Y\cdot})^2}{(m-1)S^4_{X\cdot} + (n-1)S^4_{Y\cdot}}$$

$$\hat{f}_5 = \frac{[m(n-1)S^2_{X} + n(m-1)S^2_{Y\cdot}]^2}{m^2(n-1)S^4_{X\cdot} + n^2(m-1)S^4_{Y\cdot}}$$

Figure 1

Typical values of means and variances for five estimators of effect size (g, $\hat{g}$, h, d_H, d_T) under balanced design with $\delta = .6$ & $n = m = 5, \ldots, 105$.

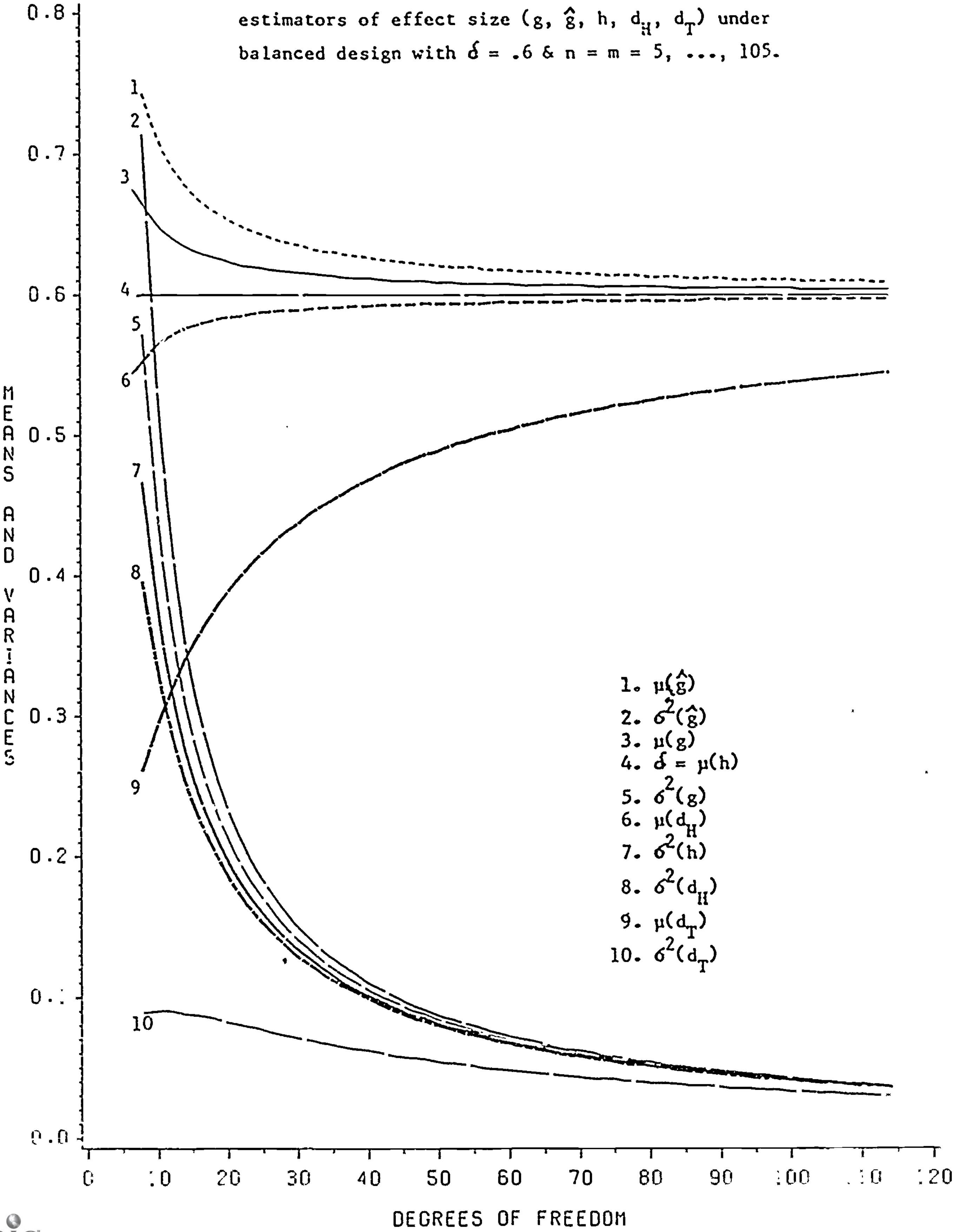

Figure 2

Typical biased and mean square error values of five estimators of effect size (g, $\hat{g}$, h, d_H, d_T) under balanced design with δ = .6 & n = m = 5,..., 105.

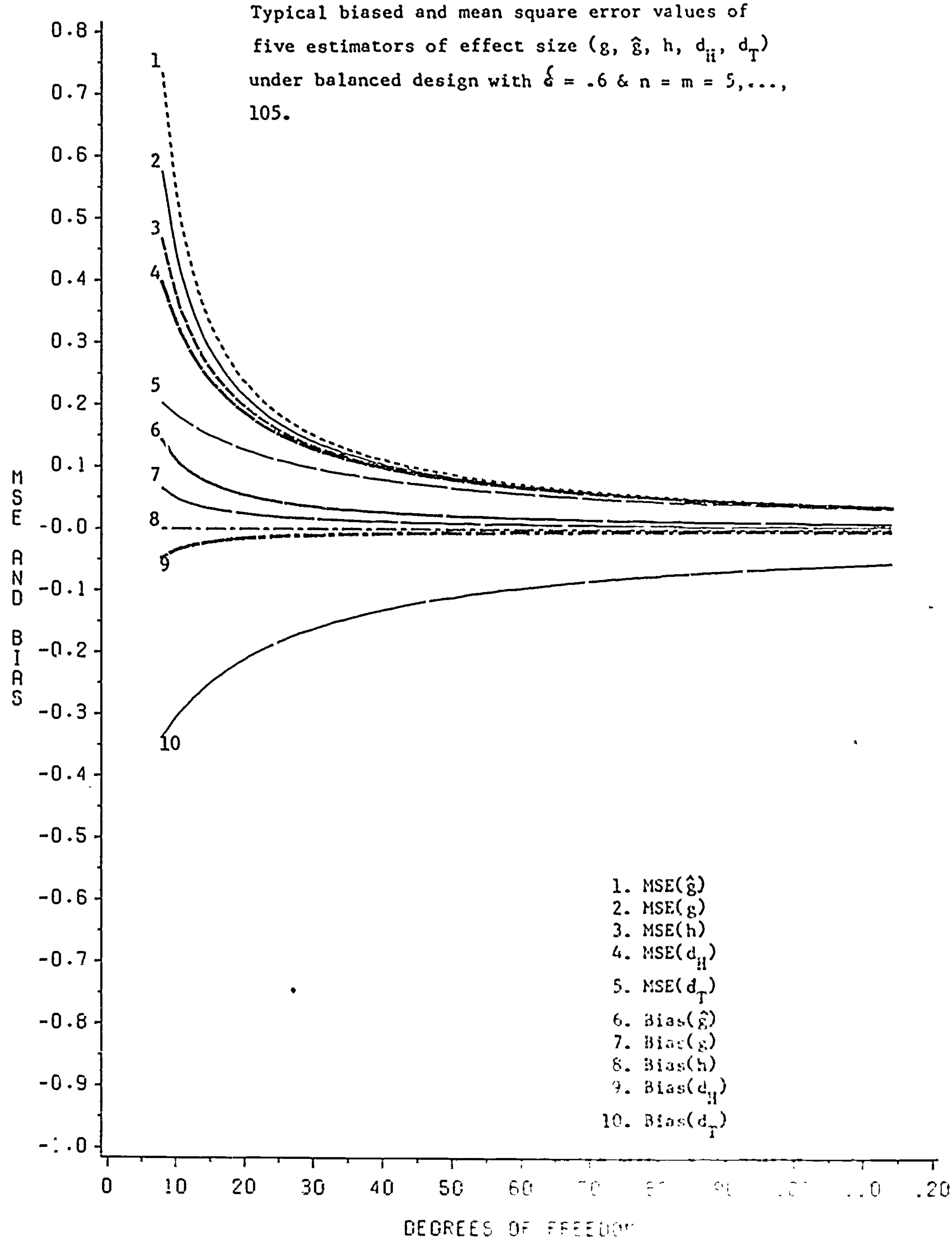

Figure 3

Typical values of means and variances for five estimators of effect size (g, $\hat{g}$, h, d_H, d_T) under balanced design with $n = m = 30$ and $\delta = 0, \ldots, 2$.

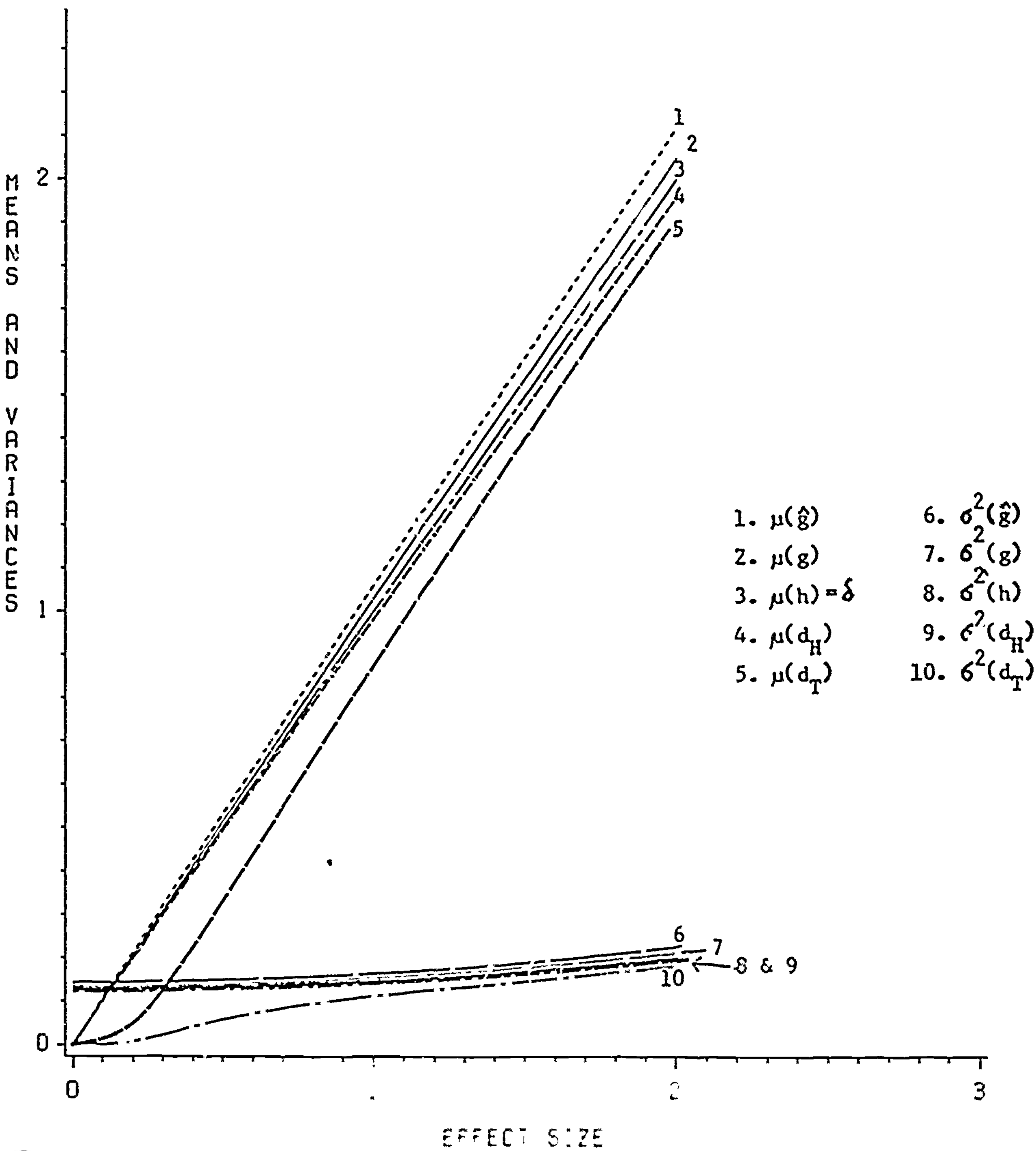

Figure 4

Typical biased and mean square error values of five estimators of effect size (g, $\hat{g}$, h, d_H, d_T) under balanced design with $n = m = 30$ and $\delta = 0, \ldots, 2$.

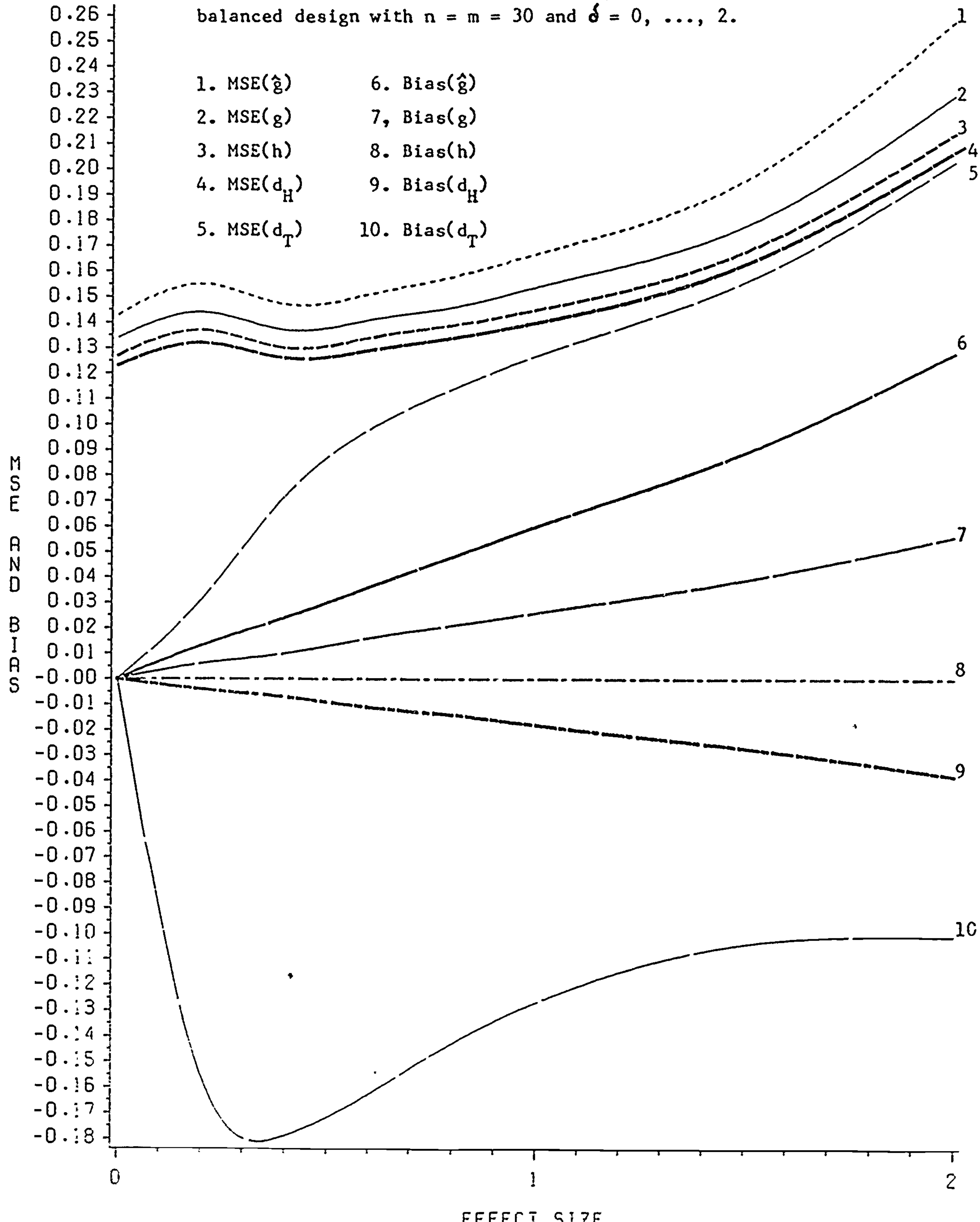

Table 2

Variance ratios ($r = \sigma_x^2 / \sigma_y^2$), sample sizes (n and m) and degrees of freedom (f) used to generate the six configurations (ID) in the Monte-Carlo study.

ID	n	m	Cases 1 and 2, r = 1, 2, 10: f_1	$\hat{f}_1$	f_2	Case 3, r=2: f_5	$\hat{f}_5$	Case 3, r = 10: f_5	$\hat{f}_5$
1	6	6	5	6	10	9.00	9.00	5.99	5.99
2	6	21	20	21	25	10.59	11.42	6.03	6.19
3	6	51	50	51	55	10.98	12.19	6.04	6.24
4	51	6	5	6	55	32.14	36.18	55.00	54.90
5	51	21	20	21	70	69.23	69.42	59.02	58.80
6	51	51	50	51	100	90.00	90.00	59.90	59.90

Notes:

(a) r = An unbiased estimate of the variance ratio ($r = \sigma_x^2 / \sigma_y^2$).

(b) In Case 1 (Variance Homogeneity, $r = 1$), the degrees of freedom f_1 and $\hat{f}_1$ were used in the computation of the parameters for g anf $\hat{g}$, respectively. The degree of freedom f_2 were ued to derive values of the parameters for h, d_H and d_T. The same procedure was applied in Cae 2 (Variance Heterogeneity with known r)) since $f_3 = f_1$, $\hat{f}_3 = \hat{f}_1$ and $f_2 = \hat{f}_2$. In Case 3 (The Behrens-Fisher condition, $\sigma_y^2 = \sigma_x^2$), the degree of freedom $\hat{f}_5$ was used in association with $\hat{g}$ whereas the parameters of the other four estimators were obtained on the basis of f_5.

Table 3

Means of some generalized estimators of effect size (δ) for six configurations of sample sizes (ID)

δ	ID	Case 1				Case 2								Case 3							
		r = 1				r = 2				r = 10				r = 2				r = 10			
		μ_g	$\mu_{\hat{g}}$	μ_{d_H}	μ_{d_T}	μ_g	$\mu_{\hat{g}}$	μ_{d_H}	μ_{d_T}	μ_g	$\mu_{\hat{g}}$	μ_{d_H}	μ_{d_T}	μ_g	$\mu_{\hat{g}}$	μ_{d_H}	μ_{d_T}	μ_g	$\mu_{\hat{g}}$	μ_{d_H}	μ_{d_T}
0	1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0	2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0	3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0	4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0	5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0	6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
.6	1	0.71	0.78	0.56	0.29	0.71	0.79	0.56	0.29	0.71	0.78	0.56	0.29	0.66	0.66	0.56	0.31	0.69	0.69	0.53	0.28
.6	2	0.62	0.64	0.59	0.37	0.62	0.64	0.59	0.34	0.62	0.64	0.59	0.31	0.65	0.67	0.57	0.34	0.69	0.70	0.53	0.29
.6	3	0.61	0.61	0.59	0.39	0.61	0.61	0.59	0.36	0.61	0.61	0.59	0.32	0.64	0.68	0.57	0.35	0.69	0.70	0.53	0.29
.6	4	0.71	0.78	0.59	0.39	0.71	0.78	0.59	0.43	0.71	0.78	0.59	0.50	0.61	0.65	0.59	0.48	0.61	0.61	0.59	0.52
.6	5	0.62	0.64	0.60	0.50	0.62	0.64	0.60	0.51	0.62	0.64	0.60	0.53	0.60	0.61	0.60	0.53	0.61	0.61	0.59	0.52
.6	6	0.61	0.61	0.60	0.54	0.61	0.61	0.60	0.54	0.61	0.61	0.60	0.54	0.60	0.60	0.60	0.55	0.61	0.61	0.60	0.54
1	1	1.19	1.30	0.94	0.70	1.19	1.30	0.94	0.70	1.19	1.30	0.94	0.70	1.09	1.09	0.93	0.73	1.15	1.15	0.88	0.67
1	2	1.04	1.06	0.98	0.81	1.04	1.06	0.98	0.78	1.04	1.06	0.98	0.75	1.08	1.12	0.94	0.76	1.15	1.16	0.88	0.68
1	3	1.01	1.02	0.99	0.83	1.01	1.02	0.99	0.80	1.01	1.02	0.99	0.76	1.08	1.13	0.95	0.77	1.15	1.17	0.88	0.68
1	4	1.19	1.30	0.99	0.83	1.19	1.30	0.99	0.87	1.19	1.30	0.99	0.93	1.02	1.09	0.98	0.91	1.01	1.01	0.99	0.94
1	5	1.04	1.06	0.99	0.93	1.05	1.06	0.99	0.94	1.04	1.06	0.99	0.95	1.01	1.01	0.99	0.95	1.01	1.01	0.99	0.95
1	6	1.01	1.02	0.99	0.96	1.01	1.02	0.99	0.96	1.01	1.02	0.99	0.96	1.01	1.01	0.99	0.96	1.01	1.01	0.99	0.96

Table 4

Variances of some generalized estimators of effect size (δ) for six configurations of sample sizes (ID)

δ	ID	Case 1, r = 1: σ^2_g	$\sigma^2_{\hat{g}}$	σ^2_h	$\sigma^2_{d_H}$	$\sigma^2_{d_T}$	Case 2, r = 2: σ^2_g	$\sigma^2_{\hat{g}}$	σ^2_h	$\sigma^2_{d_H}$	$\sigma^2_{d_T}$	Case 2, r = 10: σ^2_g	$\sigma^2_{\hat{g}}$	σ^2_h	$\sigma^2_{d_H}$	$\sigma^2_{d_T}$
0.0	1	0.56	0.67	0.35	0.31	0.00	0.56	0.67	0.35	0.31	0.00	0.56	0.67	0.35	0.31	0.00
0.0	2	0.24	0.25	0.22	0.21	0.00	0.24	0.25	0.26	0.25	0.00	0.24	0.25	0.32	0.30	0.00
0.0	3	0.19	0.20	0.19	0.18	0.00	0.19	0.20	0.24	0.23	0.00	0.19	0.20	0.31	0.30	0.00
0.0	4	0.31	0.37	0.19	0.18	0.00	0.31	0.37	0.14	0.14	0.00	0.31	0.37	0.07	0.06	0.00
0.0	5	0.07	0.08	0.07	0.07	0.00	0.07	0.08	0.06	0.06	0.00	0.07	0.08	0.04	0.04	0.00
0.0	6	0.04	0.04	0.04	0.04	0.00	0.04	0.04	0.04	0.04	0.00	0.04	0.04	0.04	0.04	0.00
0.6	1	0.65	0.78	0.38	0.33	0.09	0.65	0.78	0.38	0.33	0.09	0.65	0.78	0.38	0.33	0.09
0.6	2	0.25	0.26	0.23	0.22	0.08	0.25	0.26	0.27	0.26	0.09	0.25	0.26	0.33	0.31	0.09
0.6	3	0.20	0.20	0.19	0.19	0.08	0.20	0.20	0.24	0.24	0.09	0.20	0.20	0.31	0.31	0.09
0.6	4	0.40	0.48	0.19	0.19	0.08	0.40	0.48	0.14	0.14	0.07	0.40	0.48	0.07	0.07	0.05
0.6	5	0.09	0.09	0.07	0.07	0.05	0.09	0.09	0.06	0.06	0.04	0.09	0.09	0.05	0.05	0.04
0.6	6	0.04	0.05	0.04	0.04	0.03	0.04	0.05	0.04	0.04	0.03	0.04	0.05	0.04	0.04	0.03
1.0	1	0.81	0.97	0.42	0.37	0.21	0.81	0.97	0.42	0.37	0.21	0.86	0.97	0.42	0.37	0.21
1.0	2	0.27	0.28	0.24	0.23	0.16	0.27	0.28	0.28	0.27	0.17	0.27	0.28	0.34	0.33	0.19
1.0	3	0.20	0.21	0.20	0.19	0.14	0.20	0.21	0.25	0.24	0.16	0.20	0.21	0.32	0.31	0.18
1.0	4	0.56	0.67	0.20	0.19	0.14	0.56	0.67	0.15	0.14	0.11	0.56	0.67	0.08	0.07	0.06
1.0	5	0.10	0.11	0.07	0.07	0.06	0.10	0.11	0.07	0.06	0.06	0.10	0.11	0.05	0.05	0.05
1.0	6	0.05	0.05	0.04	0.04	0.04	0.05	0.05	0.04	0.04	0.04	0.05	0.05	0.04	0.04	0.04

Table 5

Variances of some generalized estimators of effect size (δ) for six configurations of sample sizes (ID) under Case 3 (The Behrens-Fisher Problem)

δ	ID						r = 10				
		σ^2_g	$\sigma^2_{\hat{g}}$	σ^2_h	$\sigma^2_{d_H}$	$\sigma^2_{d_T}$	σ^2_g	$\sigma^2_{\hat{g}}$	σ^2_h	$\sigma^2_{d_H}$	$\sigma^2_{d_T}$
0.0	1	0.36	0.36	0.30	0.26	0.00	0.48	0.48	0.36	0.28	0.00
0.0	2	0.29	0.32	0.25	0.22	0.00	0.46	0.47	0.35	0.27	0.00
0.0	3	0.28	0.31	0.24	0.22	0.00	0.46	0.47	0.34	0.27	0.00
0.0	4	0.09	0.10	0.08	0.08	0.00	0.05	0.05	0.05	0.05	0.00
0.0	5	0.04	0.04	0.04	0.04	0.00	0.04	0.04	0.04	0.04	0.00
0.0	6	0.03	0.03	0.03	0.03	0.00	0.04	0.04	0.04	0.04	0.00
0.6	1	0.39	0.39	0.32	0.28	0.09	0.54	0.54	0.41	0.32	0.09
0.6	2	0.32	0.34	0.27	0.24	0.09	0.52	0.53	0.39	0.31	0.09
0.6	3	0.30	0.34	0.26	0.23	0.09	0.52	0.53	0.39	0.31	0.09
0.6	4	0.09	0.10	0.09	0.09	0.06	0.06	0.06	0.05	0.05	0.04
0.6	5	0.05	0.05	0.04	0.04	0.04	0.04	0.04	0.05	0.05	0.03
0.6	6	0.04	0.04	0.03	0.03	0.03	0.04	0.04	0.04	0.04	0.03
1.0	1	0.44	0.44	0.37	0.32	0.20	0.65	0.65	0.49	0.38	0.22
1.0	2	0.36	0.39	0.31	0.28	0.18	0.63	0.65	0.48	0.37	0.22
1.0	3	0.35	0.38	0.30	0.27	0.18	0.63	0.65	0.47	0.37	0.22
1.0	4	0.10	0.12	0.10	0.10	0.08	0.06	0.06	0.06	0.06	0.05
1.0	5	0.05	0.05	0.05	0.05	0.04	0.05	0.05	0.05	0.05	0.04
1.0	6	0.04	0.04	0.04	0.05	0.04	0.05	0.05	0.05	0.05	0.04

Table 6

Biased values of some generalized estimators of effect size (δ) for six configurations of sample sizes (ID)

δ	ID	Case 1				Case 2								Case 3							
		$r = 1$				$r = 2$				$r = 10$				$r = 2$				$r = 10$			
		BIAS g	BIAS $\hat{g}$	BIAS d_H	BIAS d_T	BIAS g	BIAS $\hat{g}$	BIAS d_H	BIAS d_T	BIAS g	BIAS $\hat{g}$	BIAS d_H	BIAS d_T	BIAS g	BIAS $\hat{g}$	BIAS d_H	BIAS d_T	BIAS g	BIAS $\hat{g}$	BIAS d_H	BIAS d_T
0.6	1	0.11	0.18	-0.04	-0.31	0.11	0.18	-0.04	-0.31	0.11	0.18	-0.04	-0.31	0.06	0.06	-0.04	-0.28	0.09	0.09	-0.07	-0.32
0.6	2	0.02	0.04	-0.01	-0.23	0.02	0.04	-0.01	-0.26	0.02	0.04	-0.01	-0.28	0.05	0.07	-0.03	-0.26	0.09	0.10	-0.07	-0.31
0.6	3	0.01	0.01	-0.01	-0.21	0.01	0.01	-0.01	-0.24	0.01	0.01	-0.01	-0.28	0.04	0.08	-0.03	-0.25	0.09	0.10	-0.07	-0.31
0.6	4	0.11	0.18	-0.01	-0.21	0.11	0.18	-0.01	-0.17	0.11	0.18	-0.01	-0.10	0.01	0.05	-0.01	-0.12	0.01	0.01	-0.00	-0.08
0.6	5	0.02	0.04	-0.00	-0.09	0.02	0.04	-0.00	-0.09	0.02	0.04	-0.00	-0.07	0.01	0.01	-0.00	-0.07	0.01	0.01	-0.00	-0.06
0.6	6	0.01	0.01	-0.00	-0.06	0.01	0.01	-0.00	-0.06	0.01	0.01	-0.00	-0.06	0.00	0.00	-0.00	-0.05	0.01	0.01	-0.00	-0.06
1.0	1	0.19	0.30	-0.06	-0.29	0.19	0.30	-0.06	-0.29	0.19	0.30	-0.06	-0.29	0.09	0.09	-0.07	-0.27	0.15	0.15	-0.12	-0.33
1.0	2	0.04	0.06	-0.02	-0.19	0.04	0.06	-0.02	-0.22	0.04	0.06	-0.02	-0.25	0.08	0.12	-0.06	-0.24	0.15	0.16	-0.11	-0.32
1.0	3	0.01	0.02	-0.01	-0.16	0.01	0.02	-0.01	-0.20	0.01	0.02	-0.01	-0.24	0.08	0.13	-0.05	-0.23	0.15	0.17	-0.11	-0.32
1.0	4	0.19	0.30	-0.01	-0.16	0.19	0.30	-0.01	-0.13	0.19	0.30	-0.01	-0.07	0.02	0.09	-0.02	-0.09	0.01	0.01	-0.01	-0.06
1.0	5	0.04	0.06	-0.01	-0.07	0.04	0.06	-0.01	-0.06	0.04	0.06	-0.01	-0.05	0.01	0.01	-0.01	-0.05	0.01	0.01	-0.01	-0.05
1.0	6	0.01	0.02	-0.00	-0.04	0.01	0.02	-0.00	-0.04	0.01	0.02	-0.00	-0.04	0.01	0.01	-0.01	-0.04	0.01	0.01	-0.01	-0.04

Table 7

Mean square errors (MSE) of some generalized estimators of effect size (δ) for six configurations of sample sizes (ID)

δ	ID	Case 1					Case 2									
		r = 1					r = 2					r = 10				
		MSE g	MSE $\hat{g}$	MSE h	MSE d_H	MSE d_T	MSE g	MSE $\hat{g}$	MSE h	MSE d_H	MSE d_T	MSE g	MSE $\hat{g}$	MSE h	MSE d_H	MSE d_T
0.6	1	0.66	0.81	0.38	0.33	0.18	0.66	0.80	0.38	0.33	0.18	0.66	0.80	0.38	0.33	0.18
0.6	2	0.25	0.26	0.23	0.22	0.14	0.25	0.26	0.27	0.26	0.15	0.25	0.26	0.33	0.31	0.17
0.6	3	0.20	0.20	0.19	0.19	0.13	0.20	0.20	0.24	0.24	0.14	0.20	0.20	0.31	0.31	0.17
0.6	4	0.41	0.51	0.19	0.19	0.13	0.41	0.51	0.14	0.14	0.10	0.41	0.51	0.07	0.07	0.06
0.6	5	0.09	0.09	0.07	0.07	0.06	0.09	0.09	0.06	0.06	0.05	0.09	0.09	0.05	0.05	0.04
0.6	6	0.04	0.05	0.04	0.04	0.04	0.04	0.05	0.04	0.04	0.04	0.04	0.05	0.04	0.04	0.04
1.0	1	0.84	1.06	0.42	0.37	0.29	0.84	1.06	0.42	0.37	0.29	0.84	1.06	0.42	0.37	0.29
1.0	2	0.27	0.29	0.24	0.23	0.19	0.27	0.29	0.28	0.27	0.22	0.27	0.29	0.34	0.33	0.25
1.0	3	0.20	0.21	0.20	0.19	0.16	0.20	0.21	0.25	0.24	0.20	0.20	0.21	0.32	0.31	0.24
1.0	4	0.60	0.77	0.20	0.19	0.16	0.60	0.77	0.15	0.14	0.13	0.60	0.77	0.08	0.07	0.07
1.0	5	0.11	0.11	0.07	0.07	0.07	0.11	0.11	0.07	0.06	0.06	0.11	0.11	0.05	0.05	0.05
1.0	6	0.05	0.05	0.04	0.04	0.04	0.05	0.05	0.04	0.04	0.04	0.05	0.05	0.04	0.04	0.04

Table 8

Mean square errors (MSE) of some generalized estimators of effect size (δ) for six configurations of sample sizes (ID) under Case 3 (The Behrens-Fisher problem)

δ	ID	r = 2					r = 10				
		MSE g	MSE $\hat{g}$	MSE h	MSE d_H	MSE d_T	MSE g	MSE $\hat{g}$	MSE h	MSE d_H	MSE d_T
0.0	1	0.36	0.36	0.30	0.26	0.00	0.48	0.48	0.36	0.28	0.00
0.0	2	0.29	0.32	0.25	0.22	0.00	0.46	0.47	0.35	0.27	0.00
0.0	3	0.28	0.31	0.24	0.22	0.00	0.46	0.47	0.34	0.27	0.00
0.0	4	0.09	0.09	0.08	0.08	0.00	0.05	0.05	0.05	0.05	0.00
0.0	5	0.04	0.04	0.04	0.05	0.00	0.04	0.04	0.04	0.04	0.00
0.0	6	0.03	0.03	0.03	0.03	0.00	0.04	0.04	0.04	0.04	0.00
0.6	1	0.39	0.39	0.32	0.28	0.17	0.55	0.55	0.41	0.32	0.19
0.6	2	0.32	0.35	0.27	0.24	0.16	0.53	0.54	0.39	0.31	0.19
0.6	3	0.31	0.34	0.26	0.24	0.15	0.53	0.54	0.39	0.31	0.19
0.6	4	0.09	0.11	0.09	0.09	0.07	0.06	0.06	0.05	0.05	0.05
0.6	5	0.05	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
0.6	6	0.04	0.04	0.03	0.03	0.03	0.04	0.04	0.04	0.04	0.04
1.0	1	0.45	0.45	0.37	0.33	0.27	0.68	0.68	0.49	0.40	0.33
1.0	2	0.37	0.41	0.31	0.28	0.24	0.65	0.68	0.48	0.39	0.32
1.0	3	0.35	0.40	0.30	0.27	0.23	0.65	0.68	0.47	0.38	0.32
1.0	4	0.10	0.12	0.10	0.10	0.09	0.06	0.06	0.06	0.06	0.06
1.0	5	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
1.0	6	0.04	0.04	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.04

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