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ABSTRACT

This publication contains five papers by the staff of the University of Exeter School of Education and by invited outside contributors. The focus is on the changing mathematics curriculum, instructional practices, and research findings. The papers are: (1) "Teaching and Learning Mathematics in the Primary School" (Neville Bennett); (2) "There Is Little Connection" (Kath Hart); (3) "Learning and Teaching Ratio: A Look at Some Current Textbooks" (Dietmar Kuchemann); (4) "Mathematics Teaching--Beliefs and Practice" (Barbara Jaworski); (5) "Teaching (Pupils To Make Sense) and Assessing (the Sense They Make)" (John Mason). (MNS)

TEACHING and LEARNING MATHEMATICS

Part 2

PERSPECTIVES 34

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PERSPECTIVES 34

TEACHING AND LEARNING MATHEMATICS PART 2

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TEACHING AND LEARNING MATHEMATICS PART 1

EDITOR'S INTRODUCTION

British mathematics teaching is currently in a state of ferment and change. The last five years have seen the publication of a number of major official documents recommending changes in the teaching of mathematics, most notably Cockcroft (1982) and HMI (1985); or signalling changes in the assessment of school mathematics (DES, 1985 and 1987).

At the same time as these central government initiatives, the 1980s have seen the greatest ever impact of technology upon the teaching of mathematics. Electronic calculators are universally available, microcomputers are increasingly central to the mathematics curriculum and new technologies, such as interactive video, are beginning to make their own impact.

A third strand of development has also been taking place. Research is beginning to deliver knowledge of the processes of learning and teaching mathematics. Detailed knowledge of the outcomes of the mathematics curriculum is now available, for the first time, in publications such as Hart (1981) and APU (1986).

Theories which explain the learning of mathematics, such as Constructivism, are beginning to emerge.

One outcome of these forces is a new round of curriculum developments in mathematics, including major projects at King's College, London (Nuffield Secondary Mathematics) and Exeter (Alternative Mathematics).

Perspectives 33 and 34 reflect the changes that are happening in mathematics education in Great Britain. A number of the papers focus on innovations and developments in the teaching of mathematics and give an insight into what mathematics teaching will be like in the 1990s. Some of the papers provide state-of-the-art research perspectives on the teaching and learning of mathematics, by researchers of national and international renown. Unusually, these two issues bring together expert researchers both from within the mathematics education community, and from the broader educational research community.

The range of contributions, from those of practitioners to those of pioneering researchers, shows that for all the difficulties of economic recession and cutback, there has never been a more exciting time to be involved with mathematics education than the present.

Paul Ernest

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TEACHING AND LEARNING MATHEMATICS IN THE PRIMARY SCHOOL

Introduction

In 1983 Charles Desforges and myself were asked by the Editorial Board of the 'British Journal of Educational Psychology' to edit a monograph on 'Recent Advances in Classroom Research' (Bennett and Desforges, 1985). In fulfilling this request we commissioned chapters in the major areas of classroom research from recognised leaders in the field. Reading these chapters created both optimism and concern. Optimism, because of the clear elaboration of advances and refinement of theory and method; but concern, because of what appeared to be lack of communication between researchers operating within different content areas, or from differing methodological stances. It seemed that communication tended to be within special interest groups and/or through specialist conferences, rather than across the field as a whole. Thus, for example, mainstream researchers of classroom processes, and mathematics educators, were both carrying out studies utilising similar foci and theoretical perspectives, but appeared not to know it. This issue of Perspectives, which brings together both strands, is thus a welcome addition to the literature.

The purpose of this paper is two fold. Firstly to overview, very briefly, the perspectives which have informed mainstream research on teaching-learning processes in natural classroom settings in order to identify the theoretical developments in the field over the last decade or so. Secondly, to present findings from two of our recent studies which took a constructivist perspective on mathematics teaching and learning in primary classrooms.

Changing Perspectives

Research on teaching styles dominated the 1970s (Bennett, 1976; HMI, 1978; Galton et al., 1980). This body of research indicated that more formal, didactic styles related to enhanced mathematics achievement (cf Gray and Satterley, 1981), but the perspective contained several weaknesses. Not least was its inability to identify individual teacher activities or behaviours which related to higher maths achievement. As such it was of little value in initiating improvements in teaching. In addition, the differences in scores between styles were often relatively slight, which severely limited the use of style as an explanatory variable.

Dissatisfaction with the styles approach led to the search for alternative theoretical perspectives, a search which coalesced around the concept of opportunity to learn (Carroll, 1963; Bloom, 1976; Harnischfeger and Wiley, 1976). This perspective rejected the assumption underpinning the styles approach that a direct relationship exists between teacher behaviours and pupil learning. Instead it was argued that all effects of teaching on learning are mediated by pupil activities. In particular, the amount of time the pupil spends actively engaged on a particular topic is seen as the most important determinant of achievement on that topic. The measurement of this time is generally referred to as time on task, pupil involvement or engagement. In this approach the pupil is the central focus, with the teacher seen as the manager of the attention and time of pupils in relation to the educational ends of the classroom.

Research based on this approach has spawned an extensive literature since the mid 1970s, and studies still continue.

With regard to mathematics the findings indicate that teachers spend on average some four and a half hours per week on the subject but that this varies from two to eight hours per week. In other words some pupils gain the opportunity to study maths three times more often than others. The time pupils spend actively involved in their work (i.e. on task) varies widely across classrooms and also across subjects, being lowest in maths and language activities (Bennett et al., 1980).

The most consistently replicated findings link pupil achievement to the quantity and pacing of instruction. Specifically the amount learned is related to opportunity to learn, measured, at its broadest, by the length of school day, by the time allowed for the study of different subjects in the curriculum and by the amount of time pupils spend actively engaged on their tasks (Bennett, 1976, 1982; Brophy and Good, 1986).

These findings have, in the United States, been developed into a prescriptive model of direct instruction from which teachers are urged to run structured, orderly, teacher directed classrooms with clear academic focus, frequent monitoring and supervision whilst maintaining a warm and encouraging climate. This model will not appeal to all teachers but a very similar picture is portrayed by the most recent study of junior schools in Britain (ILEA, 1986).

The major limitation of the opportunity to learn approach is that time, or involvement, is a necessary but not sufficient condition for learning. Exhortations to increase curriculum time, or to improve levels of pupil involvement, are of no avail if the quality of the curriculum tasks themselves are poor, not worthwhile, or not related to children's attainments. Consequently contemporary thinking about teaching and learning has shifted the focus from time, to the nature and quality of classroom tasks as they are worked under normal classroom conditions and constraints i.e. to the interaction of teachers, pupils and tasks in complex social settings.

This shift in focus is reflected in recent professional concern about the appropriateness of tasks to children's attainments, which has centred on the concept of matching i.e. the assignment to children of tasks which optimally sustain motivation, confidence and progress in learning. Teachers must "...avoid the twin pitfalls of demanding too much and expecting too little"! (Plowden, 1967). It is a recommendation easier to state than achieve as has been clearly demonstrated in a series of reports by Her Majesty's Inspectorate.

In a survey of over 500 primary schools in England, HMI concluded that teachers were underestimating the capabilities of their higher-attaining pupils, i.e. the top third of students in any class. In mathematics the provision of tasks that were too easy was evident in one-half of the classrooms observed. In later reports their concern has broadened to include low-attaining children. In two recent surveys of schools catering for the age range 8 to 13 years, they argued that both the more able and the less able were not given enough suitable activities in the majority of schools and concluded that "Overall, the content, level of demand and pace of work were most often directed toward children of average ability in the class. In many classes there was insufficient differentiation to cater for the full range of children's capabilities" (HMI, 1983, 1985). However, the data of which these findings are based are, from a research perspective, unsatisfactory. HMI's observations of the match of tasks and children were unstructured, unstated and undertaken without any clearly articulated view of learning or teaching.

Empirical research which has addressed this issue has been informed, at a theoretical level, by insights derived from cognitive psychology, and by theories of teaching which view classrooms as complex social settings. The adoption of cognitive psychological principles has moved the focus on learning from a behaviourist to a constructivist perspective. The assumptions underpinning this perspective are that the task on which pupils work structure what information is selected from the environment and how it is processed. Learners, then, are not seen as passive recipients of sensory experience who can learn anything if provided enough practice, rather they are seen as actively making use of cognitive strategies and previous knowledge to deal with cognitive limitations. In this conception learners are active, constructivist and interpretive, and learning is a covert, intellectual process providing the development and re-structuring of existing conceptual schemes. As such teaching affects learning through pupil thought processes i.e. teaching influences pupil thinking, pupil thinking mediates learning.

To understand learning thus requires an understanding of children's progressive performances on assigned tasks, and, to understand the impact of teaching on learning, it is necessary to ascertain the extent to which the intellectual demand in assigned work is appropriate or matched to children's attainments. Further, since classroom learning takes place within a complex soci environment, it is necessary to understand the impact of socia processes on children's task performances.

Doyle (1979, 1983) has produced the most elaborated model of classroom social processes. This views classrooms as complex social settings within which teachers and students are in a continuous process of adaptation to each other and the classroom environment. In Doyle's view the assessment system in operation in the classroom is at the heart of this process. Students must learn what the teacher will reward and the teacher must learn what the students will deliver. Mutual accommodation leads to co-operation between teacher and taught, and co-operation, in this theory, is the keystone to a classroom life acceptable to the participants. This perspective emphasises the complex social interactions involved in classroom life, assigns a crucial role to the pupils in influencing the learning processes which teachers seek to manipulate, and places in central focus the role of assessment procedures. In this perspective studying matching as it actually occurs in classrooms entails observing which tasks teachers assign, how and why they assign them, how and why pupils interpret and work on them, and how and why teachers respond to pupils work.

A Model of Task Processes

Research undertaken from this perspective in classrooms is of recent origin. Nevertheless the findings produced have been very fruitful in guiding attention to significant features of teacher and pupil behaviour, and to aspects of classroom organisation which have an impact on the quality of children's learning experiences in mathematics at primary level. In order to demonstrate this the findings of two of our recent studies are presented, one which investigated the mathematics teaching of six and seven year old children (Bennett et al., 1984), and the other which contrasted learning experiences of junior age children in mixed and single age classes (Bennett et al., 1987).

These findings can be summarised around a model of classroom task processes presented in Figure 1.

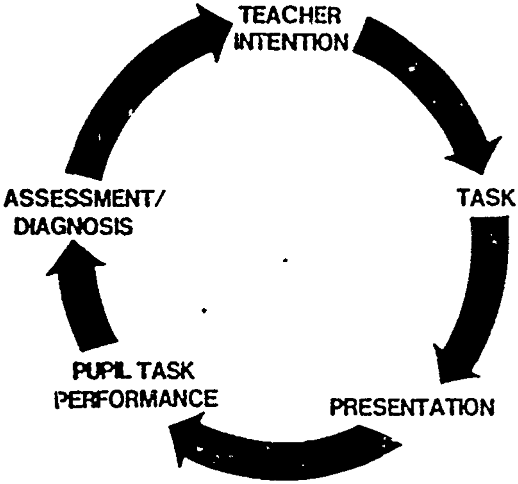

Figure 1: A model of classroom task processes

Models are, by definition, simplified versions of reality, their role is to highlight the major influential factors in an area or process. Figure 1 thus highlights the major elements in classroom task processes as delineated by our research in this area. It conceives classroom task processes as cyclic. It assumes that teachers plan the tasks they will present to pupils, or will allow their pupils to choose, on the basis of clear and specific intentions e.g. 'Angela needs work reinforcing symbol-sound relationships', or 'John is now sufficiently competent in the basic computation of Area and this should now be extended to applied or practical problems'. The tasks, once chosen, have to be presented to the child, group or class in some way. The presentation of tasks can take many forms, the major criterion being that children are clear what it is they are supposed to do. The pupils will then work on their tasks, demonstrating through their performances their conceptions and understandings of them. When the work is completed it might be expected that the teacher will assess or diagnose the work in some way, and that the knowledge of the child's understandings gained would thereby inform the teacher's next intention.

This description is deceptively simple, however, since the possibility of a mismatch or an inappropriate link is apparent between every element of the model. These links are briefly considered below, drawing on the findings of our two recent studies.

Intention and Task

We define purpose or intention in terms of the intellectual demand that tasks make on learners. Drawing on Norman's (1978) theory of complex learning, five types of task demand were characterised as follows:

Incremental	-	introduces new ideas, procedures and skills
Restructuring	-	demands that a child invents or discovers an idea for him/herself
Enrichment	-	demands application of familiar skills to new problems
Practice	-	demands the tuning of new skills on familiar problems
Revision	-	demands the use of skills which have not been practised for some time.

Of the 212 maths tasks observed in our first study 44% made practice, and 35% incremental demands. There were virtually no restructuring or enrichment tasks (1% and 7% respectively), a finding of some concern in the light of increasing demands for increased practical maths work. These proportions were almost identical for children irrespective of attainment level. As such low attainers received similar proportions of practice tasks as high attainers, a pattern likely to generate delays in progress for the latter and inadequate practice for the former.

Further, the teachers' intended demands were not always those which children actually experienced, and this posed a particular problem for high attainers. Overall thirty percent of all number tasks failed to make their intended demand, and high attainers suffered from this three times more often than low attainers.

Task and Pupil Performances

This is the link which HMI refer to as matching. The findings from both our studies support HMI judgements in this area. Table 1 shows the findings derived from the study of six and seven year olds, but the pattern was identical in our study of single and mixed age classes in junior schools.

Table 1: Matching of number tasks to children of differing attainment levels (% rounded) (Bennett et al., 1984).

Attainment	Match	Too easy	Too hard
High	41	41	16
Average	43	26	26
Low	44	12	44

The table shows a clear trend for the underestimation of high attainers in the class and for the overestimation of low attainers. This trend does however hide wide differences in the level of matching in different classrooms.

In tasks judged to underestimate children, performances were quick and accurate although in many cases the teachers judged the work rushed and untidy. In most cases the production or recording routines limited the cognitive experience offered by the task. Very often therefore the children practised work with which they were already perfectly familiar, often a consequence of being directed to the next exercise in the scheme without adequate checking of their competence.

The underestimation of high attaining children is serious, but perhaps less serious than the overestimation of low attainers since this could lead to blocks in learning or understanding with the possibility of decreased motivation. As such this is dealt with in more detail in what follows.

Among six and seven year olds output was low. On average children took three minutes to complete each calculation, on tasks which tended to focus on the four rules with quantities less than 20. Most effort was spent on the production features of tasks rather than on progress through an exercise, manifested in copying, rubbing out, boxing answers, and the like. The children's errors were often due to lack of understanding, or production errors. For example one child was asked to complete simple division sums of the type $12 \div 3 =$ in order to consolidate his knowledge of the three times table. He only made one error but, when asked to recite the three times table in the post-task interview, he began '$3 \times 1 = 3$, $2 \times 5 = 10$, $4 \times 2 = 8$'. It transpired that he had completed the original task by copying all the answers from a wall chart illustrating multiplication and division by 3.

Examples of production errors occurred when children miscounted or misplaced cubes. One girl was asked to divide 48 by 3 using cubes. It took her a long time to collect sufficient cubes. She did not estimate what 48 cubes might look like but kept counting what she had and making persistent journeys for more. The cubes took up a lot of space and several were knocked on to the floor. She then allocated them one at a time to three piles. She did not know she was not starting with 48, and miscounting the quantities in the three piles she got three different answers. She was clearly puzzled and appeared to remain so.

From the tasks observed in the junior school samples four commonly occurring problems were:

i) difficulties in reading task instructions
ii) difficulties in understanding the required procedures
iii) use of inappropriate strategies
iv) insufficient knowledge to complete the task

One child's task was to complete such mathematical sentences as:

7 tens and 2 =

2 tens and 11 =

2 tens and 11 = 3 tens and

3 tens and 17 = tens and 7

A disc frame was available to help with this task.

The child was unable to make any progress because of the difficulty he had in reading the question. He built up the words phonetically but clearly had no comprehension of what was required. He attempted to use the frame for counting but this activity was irrelevant to the question being attempted. He thus resorted to copying the questions into his book. No teacher contact was recorded throughout the period of this task.

Another child had a similar problem but in this instance received help from the teacher. He thus came to a partial understanding of his task which was to count straws into bundles of ten. Nevertheless he spent much of the lesson gazing around the classroom and at the end of the period had only managed to make one bundle of ten and one bundle of twenty straws.

Lack of ability to read or comprehend the tasks set was recorded in a number of tasks. One of the outcomes of this was the often enormous amount of time that children take to complete even simple tasks. One child who found difficulty in reading the task, which required him to arrange 56 straws into bundles of ten, took 75 minutes to read and complete this relatively simple exercise. Throughout this time he received no help or comment from the teacher.

On some tasks the demands of the procedure appeared to be a great deal more difficult than the demand of the actual question set. A typical example is of the child who was asked to undertake the conversion of, for example, '13 tens' into HTU. Using Dienes apparatus he counted out with some difficulty, and many times, twenty-nine 'longs' and then laid these 'longs' on top of a 100 square ('a flat'). Having done this he wrote '20'. He was then persuaded by the teacher to select a second 'flat', and then a third, laying the 'longs' on these. Eventually he wrote HTU

30.

In addition to not understanding this procedure he encountered great difficulty in finding and retaining sufficient apparatus to carry out the procedure fully, in finding enough space on the table to lay it out, and in accurately counting the 'longs' he had in front of him.

Sometimes the general procedure is understood at a concrete level which allows for no flexibility in its use. Thus one child knew the procedure for gaining a balance on a scale but persisted slowly and conscientiously, to use 2 gram weights irresepctive of the size of the weight to be balanced.

This type of partial understanding of procedures is similar in kind to partial understanding of strategies. This was particularly noticeable in work on the four rules where children appeared to have considerable problems in deciding which rule to apply. For example one child was to undertake a task comprised of written problems which required one of the four rules to solve them. He experienced great difficulty in deciding which of the rules should apply. Thus instead of dividing he multiplied (using a number that did not appear in the question!). And when faced with the question 'a runway is 3070 metres long, how many metres is this short of 5000 metres?', he attempted subtraction with the following configuration

$$\begin{array}{r} 3070 \\ -\ \underline{5000} \\ 2070 \end{array}$$

The worst cases of overestimation were where the child appeared to have no knowledge relevant to the task. One child's lack of knowledge of multiplication tables prevented her from doing anything other than copying out the first two questions. Even this was not begun until, after twenty minutes waiting, she checked with the teacher what she had to do. The task consisted of seven sums requiring the multiplication of a three digit number by a single digit number with a 'carry' figure. She was unable to even begin the task and as such was very little involved in it.

For example on six occasions she asked another child what time it was and at the end of the lesson the work was put away uncompleted without being checked.

These case studies, presented to portray the reality of over-estimation, do not reflect a peculiarly British phenomenon. The study by Anderson et al. (1984) in the United States provides very similar illustrations.

Teacher Diagnosis

Ausubel (1968) argued that the most important single factor influencing learning is what the learner already knows - "ascertain this and teach him accordingly".

From our studies it is clear that teachers do not diagnose, i.e. attempt, by observation or interview, to obtain a clear view of pupils' understandings and misconceptions. This is a serious ommision since it simply stores up problems for later stages of childrens' learning.

The majority of the teachers observed, in both infant and junior classes, have tended to be stationary at the front of the class, marking work while children queue for attention. Because of this pressure, the time spent with any individual child was short, and interactions not extensive. It could of course be argued that this reflects a justified pragmatic response strategy to the impossible situation of one adult being expected to provide high quality instruction appropriately matched to the individual capabilities of a large group of children. In order to cope in such a situation routines or procedures are brought into play which maximise efficiency at the expense of diagnosis and pupil understanding.

Lack of diagnosis was most often accompanied by teachers limiting their attention to the products of children's work rather than focussing on the processes or strategies employed by children in arriving at their product. Thus, when faced with pupil errors teachers need to shift from a strategy which entails showing children how to do it, to one which is exemplified by the request 'show me how you did it'.

Lack of diagnosis does of course mean that teachers have insufficient knowledge of children's understandings to enable optimal decisions to be made concerning the intentions for the proceeding task. It is clear from our evidence that this, in large part, explains the provision of inappropriate tasks to children.

Conclusion

Current research on teaching-learning processes is focussing on the nature and quality of classroom tasks, the accuracy of diagnosis of children's understandings and misconceptions of concepts and content, and the quality of teacher explanations to this end. The centrality of these variables in effective teaching can be gauged from one of the conclusions of the House of Commons Select Committee Report which stated "the skills of diagnosing learning success and difficulty, and selecting and presenting new tasks are the essence of teachers' profession and vital to childrens progress" (1986).

The approach takes due account of the role of the pupil in mediating and structuring knowledge, and places great stress on teachers' knowledge of subject matter, pedagogy and curriculum. These can be exemplified by questions of the following type: How can teachers teach well knowledge that they themselves do not thoroughly understand? How can teachers make clear decisions regarding what counts as development in curriculum areas with which they are not thoroughly conversant? How can teachers accurately diagnose the nature of children's misconceptions, and provide the necessary alternative learning experiences, without an adequate foundation of knowledge in the subject matter and associated pedagogy? These are important questions in relation to primary teaching where teachers tend to be generalists and where worries are currently being expressed about the proportion of teachers who have difficulty selecting and utilising subject matter, particularly in maths and science.

Another aspect of pedagogy which current research has identified as problematic, and which impinges on the quality of matching, is the organisation and management of the classroom to provide optimal learning environments. The typical organisation in maths is to mark work in front of children at the front of the class. This has unfortunate organisational consequences however, including poor supervision of the rest of the class, queuing (sometimes at both sides of the desk), insufficient time to adequately diagnose children's learning and, often, teacher frustration. These factors, together with the role many teachers take on as the provider of instant solutions to a constant stream of problems, serve to create a learning environment which is far from optimal for teacher or taught.

Current work is addressing some of these issues. Studies are now underway on the characteristics of teacher knowledge which constitute cognitive skill in teaching in expert and novice teachers (Leinhardt and Smith, 1985) and on the manner in which teachers' knowledge of subject matter contributes to the planning and instructional activities of teaching (Shulman, 1986).

Research on classroom management appears less strong but is crucially important. Attempts to confront the issue currently include the utilisation of parental involvement in children's learning, and the use of cooperative grouping strategies. There is still much to be done however on short and long term effects and on implementation issues. They do, nevertheless, hold out much promise for improving the quality of maths teaching and learning in the primary classroom.

Neville Bennett

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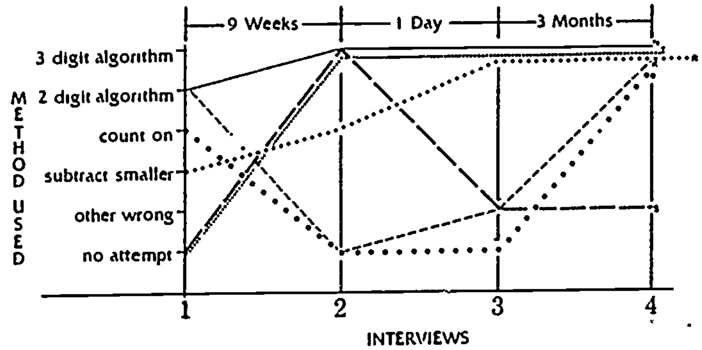
THERE IS LITTLE CONNECTION

The research project 'Children's Mathematical Frameworks' (CMF) was financed by the ESRC at Chelsea College during the years 1983-85. It followed and built upon the research projects 'Concepts in Secondary Mathematics and Science' (CSMS) and 'Strategies and Errors in Secondary Mathematics' (SESM). In the first, a description of levels of understanding in 10 topics commonly taught in the secondary school was formulated from the data obtained from both interviews and more formal testing. The test results showed that many pupils committed the same error when attempting certain questions. The reasons for some of these specific errors were investigated in the subsequent research (SESM). These misconceptions do not begin at the secondary school stage but are formed earlier. A possible learning experience during which such misconceptions can arise is when the child is required to move from a practical or material based approach to mathematics to the formal and symbolic mathematical language used in the secondary school. CMF was designed to monitor this transition to formalisation in a number of topics taught to children aged 8-13 years.

The investigation involved many one-to-one interviews with children as well as observation of them being taught mathematics in their normal classrooms. Volunteer teachers (often those attending Diploma or Masters degree courses in Mathematics Education) were recruited and asked to select a topic from the Mathematics curriculum scheduled for their class and involving the transition from practical work to formal mathematics. The formalisation was a rule or formula verbalised and often written symbolically as a result of the structured work which had gone before. An example of such a formalisation can be found in the teaching of area. Many primary school teachers lead up to the formula for the area of a rectangle by providing work which requires the child (i) to fill space with squares, (ii) record the number of units in the length and width of a rectangle,(iii) draw up a table which shows length, width and area and, (iv) from it deduce a relationship which can be formalised as $A = l \times w$. The formula is an example of 'formalisation' and the preceding experiences with squares and rectangles form part of the 'practical work'. The teacher, having chosen a topic, was asked to write a scheme of work which detailed the approach to the teaching, the equipment, work cards and the situations the children would meet. The interview cohort was chosen by the teacher from the class being taught. Sometimes, when the teacher was dealing with a class of 30-40 pupils, the cohort was chosen to represent different levels of attainment, on the other hand, if a group of children had been withdrawn because they were assumed to be 'ready' for the formalisation then six of these were interviewed. Each child was interviewed four times, before the teaching started, just before and immediately after the lesson in which the formalisation was verbalised, and then three months later. The questions which the children were asked were designed to test, (i) their closeness to verbalising the rule, (ii) the acceptance of the rule as the 'preferred' method and, (iii) the efficient use of the rule or formula. Very often the teacher stressed other (and more fundamental) mathematical concepts in the process of teaching the rule, so questions were asked about these.

The results will be published in book form late in 1987 and a suitable subtitle for this work might be 'Sums are Sums and Bricks are Bricks' which accurately describes the lack of connection between the two types of learning made by the children. When asked for the connection between practical work and symbolic statement of rule the best reply was to say one was a quicker route to the answer than the other. Nobody mentioned that the practical experience provided the data on which the formula was built. The teachers did not stress why this procedure was being followed, nor emphasise the generalisability the rule and thus the advantage of accepting it. Often pa. of the philosophy underlining the teaching methodology which suggests there be practical experiences leading to a formalisation, is that the children themselves will be ready to notice and verbalise the relationship apparent from their work. The teacher is the one who draws the attention of others to the phenomenon and perhaps couches the relationship in elegant terms. This presupposes some uniformity of progress amongst the pupils. One of the topics investigated was the subtraction algorithm with decomposition and two groups of eight children were withdrawn from their classes and taught by the same teacher. The teacher chose these pupils because they were considered to be at about the same level of attainment and 'ready' for the work. Figure 1 shows for these two groups, the results obtained from the four interviews.



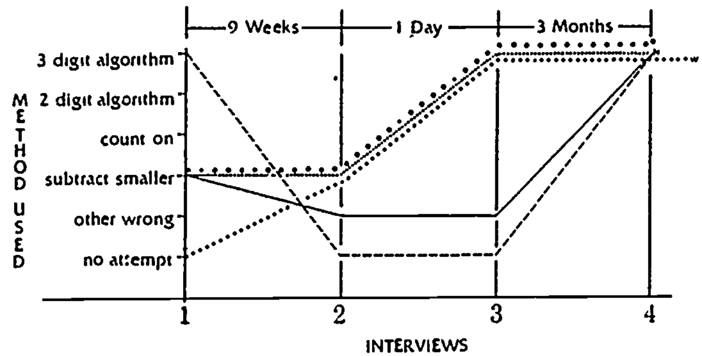


Figure 1. Children's understanding of subtraction during four interviews.

One axis shows methods used by the children when presented with a subtraction question such as

$$\begin{array}{r} 246 \; - \\ \underline{158} \\ \underline{} \end{array}$$

Note that some children were already using the common incorrect method of always subtracting the smaller digit whether it was on the bottom or top line, before the teaching started. Additionally there were children who had a workable method of 'counting on' prior to the teaching but who could not attempt the questions after the formalisation.

The diagrams show clearly that children chosen as being at the same stage of readiness, and taught by the same teacher for the same length of time, displayed very different patterns of progress. Although they nearly all appear to have achieved an ability to use the algorithm after three months, their responses to a question on decomposition display faulty reasoning. At the interview three months after the teaching the children were asked to do a subtraction question and then by looking at the 'working out' to say what number was on the top line. Further questioning probed whether they thought the top line number had changed in value; five of the six children asked this question, thought it had changed in value. During the observed lessons some teachers appealed to the evidence provided by the configuration of bricks or discs as superior or more convincing than the formula just taught. "Now, we've actually done it, in front of ourselves, with the material, we've seen it work - you've done it. There shouldn't be any mystery about it because you're the ones who've actually moved the bits and pieces around." Additionally children were encouraged to return to the materials if they were having difficulty solving problems using the taught rule. This implies that the child can provide the model to match the rule. If we consider the case of the teeching of equivalent fractions, one of the research classes used both discs and a fraction wall to build up a symbolised set of pairs of equal fractions. The teacher told the children what to use as a whole (usually a representation of 12) and then what parts to find. None of the three teachers observed teaching equivalent fractions provided a general method for setting up materials to find <u>any</u> pair of equivalent fractions. Indeed, to do so is rather a difficult task and it may be that one can only set up a convincing demonstration using bricks if one already knows the answer. This certainly seemed true of the demonstrations of equivalence using sections of a circle. Both teachers and children drew diagrams such as those shown in Figure 2 which are inadequate for demonstrating the equivalence but give a spurious veracity if one already knows the fact.

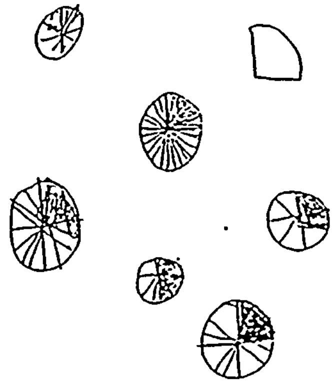


Figure 2. Mark's Cakes (to check 3/8, 10/26)

Referring children to materials because they are having trouble with a formula is of little help unless one is sure that they have a workable method of setting up objects in order to mirror symbols. The theories of Piaget have influenced the teaching of mathematics to young children and these theories have over time been translated in many cases to a belief that 'practical work' is a 'good thing'. the results of the CMF investigation show that we need to think carefully of the assumptions we make concerning the transition from practical work to formalisation and whether in fact the methods employed when using material really are translatable into the terms of the algorithm. Consider, for example, the case of a class of eight year olds who used Unifix blocks to do subtraction questions building up to the algorithm involving decomposition. A valid and much used method of solving 56 - 28 was to set out 56 as five columns of ten bricks and six single bricks and then to use 28 as a mental instruction. This was followed by the removal of three of the tens, returning two units (broken off one of the tens) to the table. Finally the collection still left on the table was counted. This is an adequate way of dealing with subtraction but it has very little connection to the algorithm which is supposed to result from all the experience with bricks.

Many of us have believed that in order to teach formal mathematics one should build up to the formalisation by using materials and that the child will then better understand the process. I now believe that the gap between the two types of experience is too large and that we should investigate ways of bridging that gap by providing a third transitional form. Nuffield Secondary Mathematics has a one year research grant to investigate possible transitional experiences, and successful outcomes will then become part of the information given to teachers in the Teachers' Resource Material.

K.M. Hart

LEARNING AND TEACHING RATIO

A LOOK AT SOME CURRENT TEXTBOOKS

Students' Ratio Strategies

School students seem to show considerable resistance to adopting formal methods such as the rule-of-three for solving ratio and proportion tasks. Hart (1981) for example, who gave a written test on ratio to over two thousand English school students, reports that an inspection of the students' scripts revealed almost no evidence of the rule-of-three being used. Carraher (1986) interviewed seventeen Brazilian 7th grade students on a set of ratio tasks; only one student used the rule-of-three even though they had all received instruction on it during the 7th grade.

Instead of formal methods, school students seem to develop a host of informal methods, of varying degrees of effectiveness, for tackling ratio tasks. Figure 1 shows three tasks from the CSMS ratio test developed by Hart (1985). On task B, for example, Hart found that students may attempt to find the missing number of sprats by adding 1 to the original number of sprats; by adding 2 (which in this case gives a better answer); by doubling; or by adding 5 (because the length of the eel has increased by 5 (cm)). This latter strategy has become known as the Addition Strategy (of which more shortly), though it is certainly not the only one that uses addition. Indeed, Booth (1981) makes the point that 'child methods' are typically additive, and in this particular case an additive approach can produce the required answer, by adopting an argument of this sort: to get from (10,12) to (15,?), add on half as much again, i.e. (10,12) + (5,6) = (15,18). Carraher (1986) calls this Rated Addition.

Task A **Onion Soup Recipe for 8 persons**

8 onions
2 pints water
4 chicken soup cubes
2 dessertspoons butter
$\frac{1}{2}$ pint cream

I am cooking onion soup for 4 people.

How many chicken soup cubes do I need? ....

Task B There are 3 eels. A, B and C in the tank at the Zoo.

A 15cm long

B 10cm long

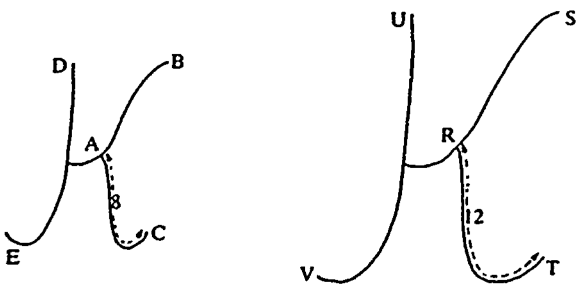
C 5cm long

The eels are fed sprats. the number depending on their length.

If B eats 12 sprats. how many sprats should A be fed to match?

A ....

Task C These 2 letters are the same shape. one is larger than the other.
The curve AC is 8 units. RT is 12 units.



The curve AB is 9 units. How long is the curve RS? ....

Figure 1. Three ratio tasks (from Hart,1985)

The Addition Strategy was identified by Piaget (see for example Inhelder and Piaget, 1958). It has since been extensively investigated by Karplus (for example, Karplus and Peterson, 1970), whilst Hart (1984) has devised teaching strategies that attempt to suppress it. Piaget regarded the strategy as an indicator of cognitive level, whilst Karplus has suggested its use is more a matter of cognitive style (Karplus et al., 1974). Whichever view one may incline to, it is clear (see for example Küchemann, 1981 and Karplus et al., 1983) that the use of the strategy also depends on various characteristics of the given ratio task, in particular the numbers involved and the context.

In Figure 2, Hart's three ratio tasks are shown plotted against facility (obtained on a representative sample of 767 students aged about 14 years). In task C, 40% of the sample gave the answer 13, commensurate with the Addition Strategy, whilst in task B only 9% gave the corresponding answer of 17. Task A was answered correctly by 95% of the sample and it is doubtful that many of the remaining students used the Addition Strategy on this task as it gives the highly dissonant answer of 0 cubes.

The numbers in task A also lend themselves to the very simple, and in this case effective, strategy of halving. No such simple strategy works for B and C, though both can be solved by Rated Addition, in both cases by taking 'half as much again': (10,12) + (5,6) = (15,18) and (8,9) + (4,4.5) = (12,13.5). Why then did C prove so much more difficult than B, and why did so many more students resort to the Addition Strategy? The reason would seem to lie in the differences in context. In task C, increasing all the lengths in the small K by 4cm can still produce a K that looks very much like the original: it could be difficult to <u>see</u> that the two Ks are not similar. On the other hand, the notion that a 5cm longer eel needs 5 extra sprats is in much more obvious conflict with the given information that a 10cm eel requires 12 sprats, not 10. This would perhaps be even clearer in the context of task A. This involves a soup recipe for 8 people, but consider the case for 9 people, say: the Addition Strategy argument, that the extra person requires an extra stock cube, strongly conflicts with the information that 8 people require only 4 cubes.

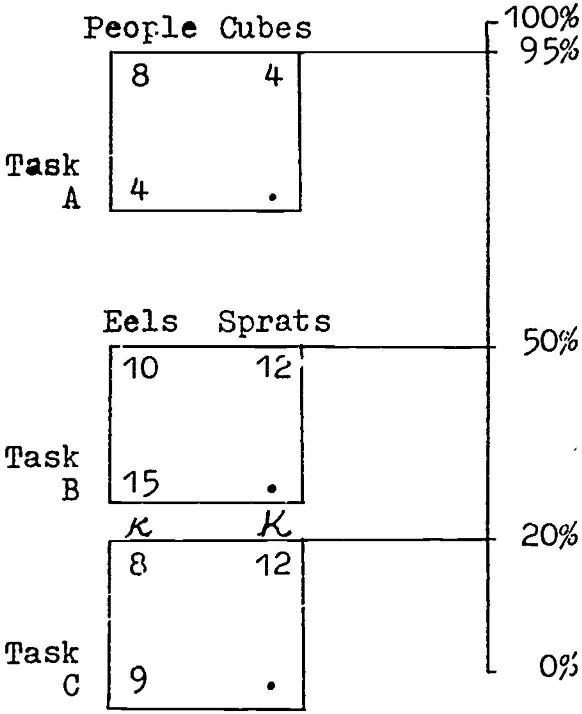


Figure 2. Ratio tasks (from Hart,1985) against facility

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Task C involves geometic enlargement. Even in the case of rectilinear figures, the Addition Strategy can seem plausible, as the standard sizes of photographic enlargments shown in Figure 3 testify.

COLOUR PRINTS (at time of processing)

Standard size prints	3½" × 5"	90 × 130mm	£0.18
*Post Card size	4" × 6"	100 × 150mm	£0.20
Enlargements	5" × 7"	130 × 180mm	£0.40
Enlargements	6" × 8"	150 × 200mm	£0.70
Enlargements	8" × 10"	200 × 255mm	£1.00

Figure 3. Standard photographic print size (Foto Inn)

Though some students may have a greater propensity towards using the Addition Strategy than others, the main point of the above has been to argue that the use of the Addition Strategy will also depend on the particular ratio task being tackled. The same argument can be made for the various other strategies in the individual student's repertoire (for example, Rated Addition is more attractive in task B than in C because it makes sense to say a 10cm eel plus a 5cm eel makes a 15cm eel, but not that two small Ks make a large one). As far as the teacher (and text-book writer) is concerned, this means students should be encouraged to engage in a wide variety of ratio tasks (in terms of number, context and overall difficulty) so that every opportunity is given for the various strategies that the student has constructed to emerge and to be made explicit - to the teacher as well as the student. In turn, this provides the opportunity for the strategies to be evaluated, be it to reconcile strategies, to (attempt to) eliminate them or to determine whether or when they are appropriate. Even if the teacher's aim is to help students adopt other, more powerful strategies (such as the rule-of-three), this is far more likely to be seen as worthwhile by the student if the limitations of his or her existing strategies are made explicit, rather than if the strategies are simply ignored (see for example Case, 1978). As von Glasersfeld puts it, "...the child is unlikely to modify a conceptual structure unless there is an experience of failure or, at least, surprise at something not working out in the expected fashion" (von Glasersfeld, 1978, p.14).

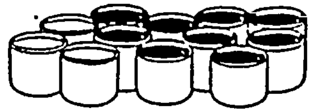
Crucially, the above approach is a constructivist one; it starts from where the students are, rather than with ideas and strategies that might hardly engage at all with the students' existing cognitive structures.

Ratio in Recent Mathematics Texts

Over the past few years I have been involved in a project that is producing mathematics text-books for 11-16 year olds. The first chapter devoted specifically to ratio (and proportion) comes in the year 2 book. In the upper track version (Harper et al., 1987b), the chapter opens with a rather simple recipe task (in retrospect, perhaps too simple). The second page involves mixtures, and part of it is shown below (Figure 4). The idea of using mixtures comes from a study by Noelting (1980), though he used mixtures of orange squash and water rather than tins of white and black paint. Mixtures do not, perhaps, embody the ratio relationship as strongly as do recipes of the kind in task A (the correspondence between people and stock cubes, say, seems very strong, whilst that between tins of white and black paint threatens to dissolve as the paints are mixed...). The attraction of mixtures, however, is the salience of the result, which can be expressed as a more primitive concept than it really is: 'Which tastes stronger?' 'Which is darker?' rather than 'Which has more orange squash compared to water?' 'Which has relatively more black paint?'

B 1 a) Glenda and Basil are painting the garden shed.
Glenda's dad wants it grey
The shop only has small tins of white paint and small tins of black paint.

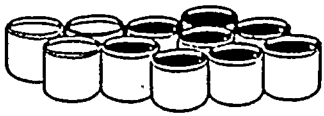
Glenda mixes 2 tins of white paint with 7 tins of black.



Basil mixes 3 tins of white paint with 9 tins of black.

Will the two sides of the shed be the same colour?
If not, whose side will be darker?

b) Glenda adds one more tin of white and one more tin of black.
Does her colour get lighter or darker?



c) Glenda keeps adding one tin of each colour.
Does she ever get to Basil's colour?
If so, how many tins of each colour has she mixed?

2 to 7,
3 to 8,
4 to 9, ..

Figure 4. Extract from Harper et al (1987b)

In the above extract, students are deliberately not offered a method of solution. Rather, they are expected to use methods of their own, which cannot be said of most of the extracts from other schemes considered later. From part b) onwards, students are given the opportunity to confront the Addition Strategy (though the strategy is not made explicit at this stage). Similar tasks, but in the context of coffee mixtures, are presented on the third page and also on the fourth, of which the lower track version (Harper et al., 1987a) is shown in Figure 5 below. Again, no specific method is put forward. Instead, by working with a friend, it is hoped that students will begin to make their own methods explicit and to look at them critically. Part c) in particular, which is reminiscent of Bishop's celebrated fractions task (Lerman, 1983), provides considerable scope for students' strategies to emerge.

With a friend

7 Here are some more recipes for coffee.

a) Decide between you which ones make stronger coffee than Basil's.

Basilian Coffee
4 cups of water
3 coffee bags

A
4 cups of water
5 coffee bags

B
3 cups of water
3 coffee bags

C
8 cups of water
7 coffee bags

D
40 cups of water
39 coffee bags

E
20 cups of water
12 coffee bags

F
2 cups of water
3 coffee bags

G
7 cups of water
6 coffee bags

b) Which recipes make weaker coffee than Basil's?

c) Write a recipe of your own.
It should make stronger coffee than Basil's,
but weaker than recipe C.
You can make it for as many cups of coffee as you like.

4 cups 3 coffee bags
8 cups. 6 coffee bags

Take note

Coffees which have the same strength use the same proportion of water and coffee

Figure 5. Extract from Harper et al (1987a)

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SMP 11-16 also uses the idea of paint mixtures. An extract from the booklet 'Ratio 1' (SMP, 1983a), which is intended for year 1 or 2 of secondary school, is shown in Figure 6. Mixtures of different strengths are presented, but students are not asked to compare them - indeed, this would be rather undemanding as the quantity of white paint is kept constant. Instead, students are to determine different quantities of the mixtures. No specific method is spelt out, but the given numbers suggest that the authors had one method very much in mind (that of scaling-up by a whole number).

F6 Here are the recipes for three different kinds of grey.

Extra light grey	Very light grey	Light grey
Mix black to white in the ratio 2 to 5.	Mix black to white in the ratio 3 to 5.	Mix black to white in the ratio 4 to 5.

(a) You are making extra light grey.
How many tins of white do you mix with 8 tins of black?

(b) You are making very light grey.
How many tins of black do you mix with 15 tins of white?

(c) You are making light grey.
How many tins of white do you mix with 16 tins of black?

(d) You mix 6 tins of black and 10 tins of white.
Which of the three kinds of grey do you get?

(e) You mix 12 tins of black and 30 tins of white.
Which of the three kinds of grey do you get?

Figure 6. Extract from SMP(1983a)

The first page of the SMP 11-16 ratio booklet is shown in Figure 7, below. Standard ratio notation is presented, which is then used to describe various (rather odd) situations. Here, from the very start, mathematical ideas are very clearly being imposed and, as with the other extract from the booklet, one feels that a number of opportunities have been lost - of gaining an insight into how students think, of connecting with their ways of thinking, and of offering a challenge. If von Glasersfeld is right, that children learn by encountering surprise or conflict, it is unlikely to happen here.

Some much more interesting tasks are to be found in the second SMP 11-16 ratio booklet (SMP, 1983b), under the heading Picture Puzzles, one of which is shown in Figure 8. It is a pity that something like this was not presented at the very beginning (though in a more appealing context and without the attempt to constrain students' thinking in part a). The tendency to <u>tell</u> students about ratio (rather than ask them), and to start with purely descriptive, and hence rather purposeless, tasks is not confined to SMP 11-16. Indeed, it seems almost universal. Thus it can be found in doughty revision series like 'CSE Mathematics' (Greer, 1978) and 'Basic Mathematics' (Elvin et al., 1979), but also in more recent series like the 'Integrated Mathematics Scheme' (IMS) (Kaner, 1982) and 'Understanding Mathematics' (UM) (Cox and Bell, 1985ab).

Figures 9 and 10 show the beginning of the first ratio chapter in IMS and UM respectively. IMS actually starts with a definition of ratio which is both abstract and vague (compare it to the 'Take note' in Figure 5). Ratio notation is then introduced, simultaneously with the idea of simplifying ratios, and this is followed by a series of descriptive tasks. The UM approach is similar, though more gradual, with simplification of ratio deferred to the third page of the chapter.

In both chapters students are presented with a method for solving ratio tasks and no thought seems to have been given to the ideas and strategies that the students may already have. In the case of IMS the method is none other than the rule-of-three (albeit made slightly simpler), which is introduced as early as the second page of the chapter (Figure 11). UM offers what is in effect the unitary method (Figure 12).

A Ratio

In this picture,
each man has 2 horses.

We say

the ratio of horses to men
is 2 to 1.

We also say

the ratio of men to horses
is 1 to 2.

(a) The ratio of girls to dogs
is 1 to . . .

(b) The ratio of dogs to girls
is. . . to 1.

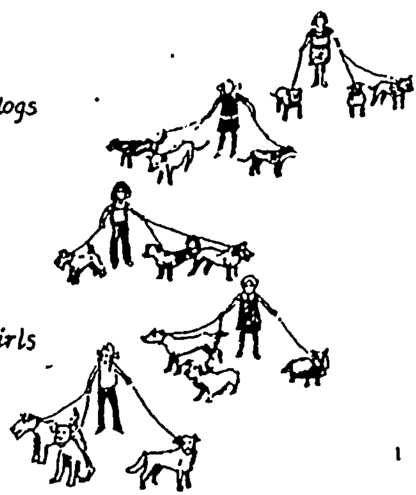


Figure 7. Extract from SMP(1983a)

C2 Measure the height and width of this doorway.

(a) Copy and complete:

Height = ? × width

(b) Which of the pictures A to G show the same doorway as this one?

A

B

C

Figure 8. Extract from SMP(1983b)

42

Unit M30 Ratio

Ratio of a pair of numbers

When two measurements in the same unit are to be compared we can use pure numbers to make the comparison. A pair of numbers used in this way is called a ratio.

Example 1:

In a club there are 20 girls and 40 boys.

The ratio of girls to boys is *20:40*.

This **simplifies** to 1:2 (dividing both parts of the ratio by 20)

Example 2:

The total surface area of the Earth is 197 million square miles.

139 4 million square miles are water and 57 5 million square miles are land.

The ratio of land to water is 57 5:139 4.

This simplifies to 1:2 4 (dividing both parts by 57 5).

Another way of saying this is, 'There is about two and a half times as much sea as land on Earth'

Exercise M97

A What are the ratios of boys to girls in these situations?

1. In our class there are 17 boys and 19 girls
2. In our club there are 25 boys and 40 girls
3. In the swimming team there are 8 girls and 6 boys

Figure 9. Extract from K:aner (1982)

43

12 Ratios; Proportional division

A Ratio, 1:*x*

These gear wheels give a gear ratio of 1:2.

The large wheel has 24 teeth and the smaller wheel has 12 teeth, so the larger wheel turning once makes the smaller wheel turn twice.

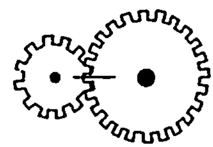


Fig. 12:1

Ratios can be written using the word 'to', using a colon (:), or as a fraction.

Example 12 to 24 = 1 to 2 = 1:2 = $\frac{1}{2}$

1 The ratio of John's money to Carol's money is 1:2. What is the ratio of Carol's money to John's?

2 Dad is 2 metres tall; Sam is 1 metre tall. What is the ratio of:
(a) Dad's height to Sam's height (b) Sam's height to Dad's height?

3 Write the ratio 1 to 3 as a fraction.

4

Fig. 12:2

CD is twice as long as AB, so CD:AB = 2:1 and AB:CD = 1:2.

EF is three times as long as AB, so EF:AB = 3:1 and AB:EF = 1:3.

Copy Figure 12:2, then write a sentence like the ones above about GH and AB.

Figure 10. Extract from Cox and Bell(1985a)

Using a known ratio

If we know the ratio of two measurements and also one of the measurements we can find the other very easily.

Example 3:

The ratio of women to men in the army is 1:8 If there are 360 000 men. how many women are there?

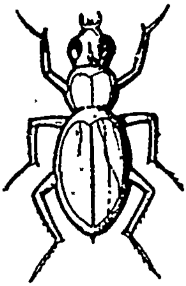
1:8 must be the same as n:360 000 so $n = 360\,000 \div 8$
$= 45\,000$.

There are 45 000 women in the army.

Example 4:

In this picture the insect is enlarged in the ratio 3:10

Measure the length of the insect in the picture and find the actual length of the insect.



Length of picture 42 mm
Ratio 3:10 = 1:3·333
so 3:10 = length:42
1:3 333 = length:42
⇒ length = 42 ÷ 3 333
= 12·6 mm
Length of insect is 12·6 mm

Figure 11. Extract from Kaner(1982)

Using ratios

For Discussion

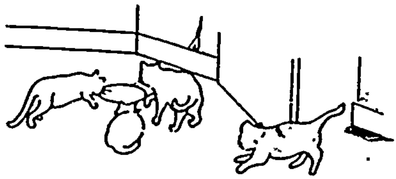
Example Ann's pay to Tom's pay is in the ratio 4:5. Ann earns £100. What does Tom earn?

Divide Ann's pay into the 4 parts of her ratio, giving £25 a part. Tom receives 5 of these parts, giving: *Answer* 5 × £25 = £125.

Fig. 12:11

Figure 12. Extract from Cox and Bell(1985a)

26 Ratios: proportion



Example 10 tins of food a week are required to feed 3 cats. How many tins would be required for 4 cats?

Ratio Method
The number of cats has increased in the ratio 4:3.
The number of tins must increase in the same ratio.
To increase in the ratio 4:3, multiply by $\frac{4}{3}$.
Answer: $10 \times \frac{4}{3} = 13\frac{1}{3}$ tins.

Unitary ('one') Method
3 cats require 10 tins
1 cat requires $\frac{10}{3}$ tins
4 cats will require $4 \times \frac{10}{3} = 13\frac{1}{3}$ tins.

Questions for Discussion

(a) Three eggs cost 15p. What do eight eggs cost?

(b) Three packets weigh 5 kg. What will seven of the same packets weigh?

(c) A spring increases by 36 mm with a 7.2 kg load. What is the increase with a 9.2 kg load?

(d) Four men on a raft have enough food for 21 days. If three more survivors are picked up, how long will the food last at the same rate?

(e) £14 will feed two cats for 12 weeks. For how long will £21 feed them? If a stray joins the two cats, for how long will the £14 feed them?

1 A bus travels 100 km in 4 hours. How far would it travel in 3 hours?

2 A man pays the same tax each week. He pays £500 in 8 weeks. How much will he pay in 10 weeks?

3 Six exercise books cost £1. What will fifteen exercise books cost?

Figure 13. Extract from Cox and Bell(1985b)

Which ratio is the larger, 6 : 5 or 7 : 6?

(We need to compare the sizes of $\frac{6}{5}$ and $\frac{7}{6}$ so we express both with the same denominator.)

$$\frac{6}{5} = \frac{36}{30} \text{ and } \frac{7}{6} = \frac{35}{30}$$

so 6 : 5 is larger than 7 : 6

Figure 14. Extract from Bostock et al (19c4)

Find the missing numbers in the following ratios:

1. 2 : 5 = 4 :	6. : 15 = 8 : 10
2. : 6 = 12 : 18	7. 9 : 6 = : 4
3. 24 : 14 = 12 .	8. $\frac{}{4} = \frac{15}{10}$
4. $\frac{6}{} = \frac{9}{3}$	9. $\frac{6}{8} = \frac{}{12}$
5. 3 : = 12 : 32	10. 6 : 9 = 8 :

Figure 15. Extract from Bostock et al (1984)

A book of 250 pages is 1.5 cm thick (not counting the covers).

a) How thick is a book of 400 pages?

Method 1 (using algebra):

a) If the second book is x cm thick, then $\frac{x}{1.5} = \frac{400}{250}$

$$\cancel{1.5} \times \frac{x}{\cancel{1.5}} = \overset{0.3}{\cancel{1.5}} \times \frac{\overset{8}{\cancel{400}}}{\underset{5}{\cancel{250}}}$$

$$x = 2.4$$

The second book is 2.4 cm thick.

Figure 16. Extract from Bostock et al (1984)

Method 2 (unitary method):

a) 250 pages are 15 mm thick

1 page is $\frac{15}{250}$ mm thick

so 400 pages are $\frac{\overset{3}{\cancel{15}}}{\underset{5}{\cancel{250}}} \times \overset{8}{\cancel{400}}$ mm thick

that is, 24 mm or 2.4 cm thick

Figure 17. Extract from Bostock et al (1984)

The unitary method is presented again at the start of the second year UM ratio chapter, together with a 'ratio method' (Figure 13). The latter is highly formal and sits oddly with the claim in the book's preface that "The development of each topic was planned with reference to the findings of CSMS". The chapter continues with an interesting discussion section and it is a pity that this does not come first. In the exercise that follows, many of the questions can be solved by Rated Addition and it seems likely that many students will use this rather than the less accessible methods that they have been given.

As with IMS, the first ratio chapter of the 'ST(P) Mathematics' series (Bostock et al., 1984) starts with notation and with the idea of simplifying ratios. However, before embarking on the usual descriptive ratio tasks, two exercises on simplifying ratios and one on comparing ratios have to be gone through. This part of the chapter eschews virtually all context, and it is interesting to compare the worked example above (Figure 14) with the coffee recipe tasks in Figure 5.

Interestingly, the chapter includes an exercise (Figure 15) for which no specific method is advocated (the exercise is introduced by the statement "Some missing numbers are fairly obvious"). Clearly this allows students some scope for airing their own ideas, but it is not long before the rule-of-three and the unitary method are presented, both in a flurry of symbols (Figures 16 and 17).

In this brief look at published ratio materials, I have focused on some of the more recent, more widely used, secondary school mathematics schemes, and have looked only at the introductory work on ratio. It would be of interest to see how the ratio work is developed in the later books, though it seems unlikely that there would be any marked shift to a less didactic approach. I have argued that the materials need to engage with students' existing ideas and strategies, even if the aim is to move students to other ways of thinking. The materials that I have been involved in make a start in this direction, but the approach still seems to be rare, and in the schemes considered above, almost nonexistent.

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MATHEMATICS TEACHING - BELIEF AND PRACTICE

I have been a mathematics teacher for many years, and have recently spent much time observing lessons and talking with other teachers about their teaching. This article is about some of my observations and related thinking.

A Teaching Reality

I had made plans to observe a lesson of a particular teacher one Friday morning, and then to talk with him afterwards during his 'free' period. The lesson was immediately after break so I arrived before break and walked over to the staffroom with him for a cup of coffee. On the way over, a number of pupils stopped to ask him about the skiing trip that they were all taking over half-term, and he told them to meet him during the lunch hour to talk about it.

As we dodged through the crowded corridors, he raked his hair and said that he felt terrible. It had been a parents evening the night before and he had not gone home until 10pm. He said that he had talked to so many parents that it was hard to remember what he had been saying by the end of the evening. One of his comments was:

> "You know, if only we could come in (to school) an hour later, or something, the next morning - so that you could just get an extra bit of sleep - it wouldn't be too bad. As it is you just feel under constant pressure."

As we drank coffee, he suddenly swore as he found on the noticeboard that he had been asked to cover a class for the lesson in which we were supposed to talk. He said:

> "I have three free periods a week, and this is the third time this week that I've had to cover."

The deputy head walked past at that point, and he challenged her. Her reply was that she sympathised, but they happened to have 12 staff away that day, and so what could she do?

As we walked back to the Maths area at the end of break the conversation was about how with the best will in the world you could not find all the time that was needed for preparation of lessons, assessment of work, and exciting teaching!

The Effectiveness of Teaching

When going into schools to observe their lessons, a number of teachers have apologised to me with words such as

"I'm sorry. I'm afraid that its only KMP today."

The implication has been that I should be looking out for something more exciting than KMP, or SMP or whatever scheme or text was in use; that resorting to a text or scheme was something to be regretted. In discussion with these teachers they have admitted that they value more the lessons into which they are able to put the most energy in terms of planning and originality, but that it is impossible to be exciting and original for 36 lessons a week, and a buffer in the form of a text or scheme is essential to them. Moreover, the pupils very often like the schemes. They provide a comfortable environment for pupils in many respects.

All of this raises questions about what it means to teach, and, in particular, what is **effective** teaching?

I can easily get a sense of the effectiveness of a can of defrosting foam by noticing how well it disperses the ice on my windscreen. As a teacher, to get a sense of the effectiveness of my teaching I have somehow to be able to judge how well it affects my pupils. To put a measure on this I have first to define the affecting that I consider valuable.

I believe that in every lesson a teacher asks, either implicitly or explicitly,

'What sense are the pupils making of the mathematics that they encounter in this lesson?'

This question is often very difficult to answer and it is possibly in finding ways of answering it that a measure of effectivity might be found for any teacher.

I observed a class of 11/12-year olds, working on area and perimeter of rectangles. The exercise in their text book gave them the values of two of the following four variables - length, width, area, perimeter - and pupils had to work out the other two. Two girls asked me a question about the resulting area of 0.9, when they multiplied a length of 1.8 by a width of 0.5. As one girl put it,

> "We can't see why it's 0.9. 0.9 is about the same as 0.5, not much bigger. We expected the answer to be bigger."

In our subsequent conversation, they drew a rectangle and talked about the sizes of the sides, what area they would expect, and the result of multiplying the two numbers. I had the impression that this was the first time that they had really thought of the question in terms of area, rather than just in terms of multiplying numbers.

I knew that a previous lesson had been spent on an activity called 'Half-and-Half'. I could see the results of it up on the wall; some very impressive two-colour squares in which pupils had imaginatively split up the area into two halves and shaded half the area in each colour. The teacher had told me excitedly about how pupils had constructed and justified the construction of their squares. I asked the two girls which of the squares were theirs and how they had constructed them, and they showed me what they had done. I was impressed both with their creativity and with their explanation of why the result was half and half. They went on to tell me about a homework problem, which the teacher had set, in which someone started with 100m of fencing and wanted to enclose the maximum possible area within it. They had some ideas as to how to tackle the problem, but had not got very far with it yet.

As I thought of the three separate activities, I wondered what links the pupils were making between them, if any. I asked the two girls what they thought today's lesson had to do with the work on half-and-half, and with the homework problem. They told me that "they're all to do with area, aren't they", and I wondered what else I might have expected!

I mentioned it to the teacher afterwards. I wondered, even if the pupils were articulate enough to express what they felt about it, whether the sense they were making, the links that they saw, would have coincided with the expectations of the teacher, or if they would have surprised or disappointed him. We wondered together, whether asking the pupils frequently to try to explain any links that they saw might eventually lead to a more explicit making of such links. He decided to ask the class to write down a few sentences to say what they thought the topics had been about, and how they were connected, and then said wryly,

> "That's hard isn't it? I'm not even sure that I could do it myself!"

This experience led me to think about what, as a teacher, I actually want to know when I ask, 'What sense are pupils making of this?' and how I can find out. I recalled a lesson which I taught myself some time ago at the end of which I asked pupils to write a sentence or two on what they thought the lesson had been about. The replies included:

(i) "We drew a shape and described it to our partner who drew it any way he thought right."

(ii) "This lesson was about discussing and drawing your thoughts and trying to use other peoples descriptions."

The replies were almost equally split between those pupils who described the activity in terms of drawing shapes (as in i), and those who talked about the communication (as in ii). I remember feeling particularly pleased that some pupils had actually perceived the communication aspect, although they may have found it difficult to put into words. However I was not surprised that many pupils had seen the activity mainly in terms of what they had done physically. In the case of the area activities, the pupils' responses to the teacher's request were mainly factual about the particular activities, and conveyed little sense of pupils' general awareness of area.

There seemed to be a parallel here between reaching generality in terms of seeing mathematical rules from a number of particular cases, as in a text-book exercise for example, and having a general sense of a mathematical topic from a number of activities in which one has engaged. Mason and Pimm (1984) discuss the meaning of generic and general in such contexts. It seems an important objective of mathematics teaching that pupils should become aware of such generality, but how the teacher might monitor this awareness is not clear.

The Teacher's Role

A teacher's judgement of the effectiveness of a lesson is bound up both in what actually occured and how this is reconciled with the teacher's own beliefs about teaching and learning. It is often true that what seems ideally desirable is difficult to achieve in practice and the teacher has to struggle with implementation. However, some of the struggles which I have perceived recently have been as much to do with belief itself as with its implementation. If the struggle is with belief, then it makes the implementation even more difficult.

The following extract is from a conversation which I had with a teacher after a particular lesson on tessellation:

> Teacher: "...my aim was to follow one of the questions that was asked at the end of last lesson, which was 'Why are some tessellating and some not?', because some people got to the stage where they saw hexagons tessellating and the quadrilaterals, but they found pentagons didn't, nor did octagons... and the other aim was to work on children explaining more fully when they were discussing things. I'm not sure whether I achieved the first one or not. ..."
>
> "...I'm not sure that it worked exactly as I'd hoped it would work or that they actually focussed on the angles meeting at a point as I'd hoped they might..."
>
> "...They kept referring to the fact that if they were able to make the shapes into quadrilaterals or rectangles that they would be able to tessellate the shapes. But yet they weren't all convinced that quadrilaterals tessellated. That was the thing I wanted them to go on to..."

This teacher was struggling with the tension between wanting the pupils to explore their own ideas and come up with their own explanations, and wanting them to perceive certain mathematical results which she thought important. One option open to her was to explain her ideas to the pupils, but for the moment she was rejecting this.

The effectiveness of teacher explanation is measured in some sense by how well the pupils understand the explanation, and for many pupils a good teacher is one whose explanations are understandable. One perspective of teaching and learning mathematics is of a transmission process where mathematical knowledge exists and may be conveyed by the teacher to the learner. The teacher's responsibility is to give a clear articulate exposition of the mathematical knowledge and the assumption is that having heard it the recipients then have the knowledge and ought to be able to say it clearly themselves, or give other evidence of understanding. Exercises are designed for pupils so that they can rehearse the ideas and provide this evidence for the teacher.

An obvious but nevertheless important consideration is that:

the teacher cannot do the learning for any pupil.

Even the transmission process depends for its success on pupils constructing their own images of what the teacher has said, and trying to make sense of them. This trying to make sense is a complex process of relating what is heard to previous experience and making links which tie in the new ideas to established understanding or belief. Von Glasersfeld (1984) describes this in terms of the 'fitting' a new experience into existing experience. We seek to explain what we encounter in terms of our existing belief, and understanding consists not of comprehending an absolute reality but of establishing such a 'fit'.

The word construal might replace the phrase trying to make sense. Construal takes place for everyone at all times in which they are awake and alert. Even though a pupil is not attending to the teacher's words she is still construing. Her construal may involve a growing awareness that she is missing whatever the teacher is saying, and some establishing of an attitude to this, maybe of panic, of unconcern or of resistance to authority. The teacher engaged in transmission hopes that pupils' construal is of mathematics, and indeed of the particular mathematics which the teacher has in mind. However, individual construal, as described above, implies that every pupil will have made a different sense of what was said according to their past experience and current thinking. So the teacher may have no assurance that all pupils believe exactly what she had in mind for them to believe.

One way for the teacher to find out what pupils are thinking is to get them to try to talk about it, to the teacher or to each other. If they talk to each other, the teacher cannot listen to everyone at once, but she can listen in to different groups and get an overall sense of what is being said.

In one classroom, after a particular activity in which pupils had described the fitting of three red and three white Cuisenaire rods to a blue rod in different arrangements, the teacher asked pupils to talk in pairs about whether they thought the different arrangements were important. As she listened to two boys talking, it appeared that they were talking about football. One said to the other,

> "If its QPR 3 and Bristol Rovers 3, it doesn't matter which order the goals were scored in, does it?"

The boy's ability to translate the situation into an example that made sense for him gave the teacher an insight into his construal.

Pupils do not always find it easy to put mathematical ideas into words, which is not surprising since adults, and even mathematics teachers often have difficulty themselves. If a teacher wants pupils to talk about mathematics then she has to provide opportunities for them to develop this skill. Activities which are designed to require negotiation between pupils can provide this opportunity in a fairly natural way; for example the activity mentioned in section 1 where one pupils had to draw a shape in secret, then describe it to a partner who had to draw the shape which he understood from the explanation. When the second drawing did not match the first the pupils were asked to discuss what would have improved the explanation. It was interesting to see how the discussions progressed from recriminations from the explainer, about the inability of the responder to understand what was said, to a realisation that the onus for clarity was on the explainer as much as on the responder. The activity was done first in pairs, but then a number of pairs volunteered to perform for the whole class and the class commented on the quality of the explanations which they heard. The teacher listening was able to get a sense of the pupils' understanding as the negotiation progressed. The role of the teacher in this activity was mostly that of facilitator and listener, but the opportunities for learning about pupils' understanding were greater than in a lesson in which the teacher had done most of the talking.

An investigative approach to teaching and learning involves providing opportunities for pupils to express and explore ideas for themselves, and encourages pupils to ask their own questions and follow their own lines of enquiry. It requires confidence and flexibility on the part of the teacher and a willingness to explore whatever comes up. It does not prevent the teacher from posing questions which alert pupils to ideas which the teacher considers to be important, but it does involve the teacher in being prepared for a wider consideration of ideas than an expository style might allow.

The daughter of one of my colleagues had talked with her father about triangles on the surface of a sphere. She approached her maths teacher the next day with a challenge which I paraphrase:

> 'You know that you said that angles of a triangle add up to 180 degrees?'

Teacher's reply: 'Yes?'.

> 'Well, what if the triangle is on a sphere? The angles don't add up to 180 degrees then, do they?'

The teacher was reported to have replied tersely that he had not been talking about triangles on spheres and that what he had told her had been quite correct for plane triangles. It is possible that the teacher felt both embarrassed and threatened by the challenge and that this conditioned his response. Being more welcoming of questions from pupils and more open to alternatives might avoid such embarrassment, but this is not always easy. One teacher, whose lessons I had enjoyed because her classroom always seemed to be a place of high energy and lots of pupil ideas and initiative, talked to me about her experience when she had to teach calculus to her fifth year O-level group.

"I've never taught calculus before, and I'm not very confident about it. I prepared myself thoroughly by reading up first in a number of text books, but I couldn't bring myself to make it open-ended. I taught it very straight from the blackboard and didn't really invite questions. When Elizabeth asked a question that I didn't know the answer to, I nearly panicked. But I was able to invite the rest of the class to comment on the question, and Janet said something that I realised was right and so I took it from there and it was OK."

How Can the Teacher Control What Pupils Learn?

It seems an inevitable step to come to the conclusion that the teacher cannot hope to prescribe what pupils learn, except in a very narrow sense; that having to work to a prescribed syllabus contradicts reason, as do the traditional forms of examination by which the syllabus is assessed.

A compromise exists in practice, and individual teachers have to establish what this compromise means for them. One teacher is currently working on a balance between encouraging pupils to generate their own ideas and ways of working and exercising his responsibility as a teacher to help them in their development of successful strategies.

"I listened to what they were saying, and it was clear to me that they were in difficulties because they couldn't organise what they had found out so far. So I said, 'well if I were doing this I would...' ".

He had set up groups of four in which the two pairs were given different problems on which they were expected to work. Although the problems were different, the pairs were asked to talk to each other occasionally about what they were doing and thinking. One group of four decided to ignore this instruction and start off all together on the same problem. When they were well and truly bogged down in masses of data with no conclusions, the teacher came across to help. They expressed their frustration as,

> "Clearly you have some result in your head which you want us to find, but we can't find it."

To ease their frustration he decided to give them some hints, but pointed out that had they followed his instructions, the other problem might have provided some of the insights which they required.

He said afterwards that there are sometimes occasions when he does want pupils to find out a result which he has in his head, and one of his responsibilities is to find ways of making this happen. Sometimes it is appropriate to tell them, at other times he needs to find other ways. He is actively working on this problem as he teaches his classes from day to day.

A group of researchers at the University of Grenoble in France are actively looking into what questions and activities will bring pupils up against particular mathematical ideas. This research could be of considerable use to teachers who are struggling with such issues.

The pressure on teachers is unlikely to change. It can only be bearable if seen in the context of pupils' success. As teachers, the way in which we use time and resources is indicative of what we most value. We need to be continually asking about the sense which pupils are making in our classrooms. No major action is called for, just an awareness that pupils are trying to make sense in their terms and teaching has to make room for this. Every teacher can be a researcher, and every lesson can provide opportunities to find out more about how we can help pupils to learn.

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TEACHING (PUPILS TO MAKE SENSE)

AND ASSESSING (THE SENSE THEY MAKE)

The purpose of this article is to make some concrete proposals about assessment in mathematics. The question is, what to assess, and how. Before answering that question, it is necessary to establish a perspective in which the suggestions are based. This leads me to begin with the question of 'making sense', and what this means when teaching. Only then is it reasonable to address the question of assessment.

My basic premise is two-fold: first, that pupils try to make sense of the world, and second, that they do so by assembling fragments of their experience into some sort of story. I arrived at this perspective as a result of marking scripts, and spending a large amount of time listening to what students had to offer at the end of a period of concerted work on a topic. I found to my surprise that even after working through the most carefully constructed exercises, students remained mostly inarticulate and highly fragmented in their attempts to account for what they were doing.

I have used two techniques in this respect. At the end of a session I invite pupils to contribute to a public list of technical terms involved in the topic, and then to try to spin some sort of story using these technical terms which describes what they have been doing in the session. I call this **Reconstruction** because pupils are explicitly invited to reconstruct their own story, rather than to 'learn' mine. The second technique involves setting up a relaxed and informal evening session in which I pose questions of a mathematical nature for discussion, and then **listen** to what the pupils bring from their recent 'learning'.

The effect of these sessions has been to bring into question what it was that I and the students think we have been about when engaged in traditional classroom activity. In particular the question comes up for me again and again - what is going on inside their heads?

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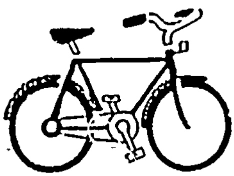
What are Pupils Attending to?

The accompanying extract came home from school in the hands of seven year old Lydia. The section shown (Fig.1) is the first of three, the others having a tricycle and a car respectively, but with the questions otherwise identical. She asked me to help her because

"I don't know what to do".

Counting in twos



There are 2 wheels on the bicycle.

1 On 4 bicycles there are 2 + 2 + 2 + 2 = 8 wheels.

Instead of ADDING equal groups you can MULTIPLY.

Write

2 4 groups of 2 = 8

3 4 × 2 = 8

4 4 multiplied by 2 = 8

5 4 times 2 = 8

6 How many wheels are there on 7 bicycles? 14

7 How many wheels are there on a 6 bicycles b 10 bicycles? 20

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Figure 1. Multiplication Tasks

I asked her to start reading to me. She got to 2 + 2 + 2 + 2 and said

"Is it eight?"

I replied with

"What do you think?"

She said in a tentative tone of voice. "eight". She skipped over the "Instead..." and went on to "4 groups of 2".

"What does that mean?" she asked.

"What does 4 groups of 2 mean?" I asked her.

"Eight?"

I pointed to the bicycle and asked her what that was doing there. She didn't know. She went on to 4 x 2, then asked

"What is 'multiplied'?"

I watched her carry on, using her fingers to do 7 bicycles, and in response to my raised eyebrows she did it again. The last bicycle question she did quickly.

"Is that all there is?"

She set to, head down, pencil tightly gripped. She worked through the bicycles, the tricycles and the cars. Each question was tackled in turn.

What did she make of the task? What did the author intend her to make of it? I suspect that she was meant to see that the operation of multiplication is signalled and notated in a variety of ways, and that repeated addition is the same as multiplication. Did she? I doubt it. She looked at me in amazement when I asked her what it was about, as if to say

"It's just a bunch of questions, Dad (you fool!)".

I conclude that it is not easy to point people in the direction of seeing the general in the particular, the sameness in different events, but that 'seeing the general in the particular' is one of the root processes in mathematics, and probably in every discipline. Indeed, different disciplines might be characterised by the sorts of features of situations which are attended to, and the ways in which generality is perceived in particularity.

The question - what is going on inside their heads? - is endemic to teaching. At its heart lies a tension arising from what Brousseau (1984) calls the Didactic Contract. This tension arises between pupils and teachers in the following way. The pupils know that the teacher is looking to them to behave in a particular way. The teacher wishes the pupils to behave in a particular way as a result of, or even as a manifestation of, their understanding of the concepts or the topic. The more explicit the teacher is about the specific behaviour being sought, the more readily the pupils can provide that sought-after behaviour, but simply by producing the behaviour and not as a manifestation of their understanding. Tension arises because the pupils are seeking the behaviour and expect the teacher to be explicit about that behaviour, whereas the teacher is in the bind that the more explicit he is, the less effective the teaching. The question then arises as to how it might be possible to make positive use of the didactic tension rather than descending into a negative spiral in which the teacher is more and more explicit about the sought behaviour and the students more and more mechanical in their production of that behaviour. In reflecting on this question I have over the years made a number of self-evident but for me potent observations:

Observation 1 - I can't <u>do</u> the learning for my students.

Gattegno (1971) elaborates on this theme based on his memorable book title 'The Subordination of Teaching to Learning'. If I stop trying to do the learning for students, what are the implications? The ancient expression 'there is no royal road to geometry' has applied more generally to learning mathematics for 2000 years, yet for 2000 years teachers have struggled to find the educator's stone!

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Observation 2 - Students bring to class a rich experience of making sense in the past. Which of these powers do I particularly need to evoke in a given topic?

The expression 'Starting where the students are' has unfortunately become a cliché, often meaning little more than not assuming the students know very much, whereas it could focus attention on helping pupils to use their undoubted powers effectively. Gattegno offers the challenging suggestion that 'only awareness is educable', and I take this assertion to encompass my question as a special case. My question then becomes, 'How do I evoke pupils' sense-making powers, and how do I help them work on their awareness?'. Some people take Gattegno's expression to extremes, refusing to tell students anything on the grounds that it is useless unless the students discover it for themselves. Telling people facts is not in itself useless, indeed it is often essential, but the critical factor is how students go about making sense of what they are told as well as what they discover.

Observation 3 - Experience is fragmentary. We piece together bits of explanatory stories that we hear or construct, in an attempt to organise experience. We constantly probe our past experience to look for similar situations, and that similarity comprises the structure of our understanding.

As the Simon and Garfunkel song 'The Boxer' has it, "A man hears what he wants to hear and disregards the rest". "Those that have ears...". We can only notice certain aspects of events, namely those we are attuned to or which stand out for us. We are not just solipsistic robots however. We are constantly seeking resonance with ourselves and with others by expressing our stories and seeing what others make of them. The extent of that resonance provides confidence, and helps determine the company we keep.

Observation 4 - The result of making sense seems generally to consist of two elements: articulate stories which explain or account for a variety of situations; manipulative skills which are the subjects of examination questions.

The didactic tension leads to emphasis on manipulative skills, and on the conveyance-container metaphor for teaching. Students are 'given' skills, they try to 'get the point' of the lesson, they do or do not have a 'grip' on the concepts, and so on. It is interesting that in any discipline, the development of techniques or skills for answering certain kinds of questions arises as the result of people observing a similarity or commonality to a whole range of questions, thus making it worthwhile to try to find a common solution. The common solution is then taught to students as a technique. The immense amount of construal which is required in order to reach a perception of commonality is not often shared with students, who are simply 'given' the skills without any reference to, or appreciation of, the original or underlying questions.

> Observation 5 - Because as teachers we are engaged in transmitting culture to the younger generation, there is a tendency to move rapidly from a quick glimpse of an idea, to succinct, manipulable, (symbolic) recordings - definitions, technical terms, major results,...

As a culture we put a great emphasis on the written expression of what people experience or think about. Some pupils have been recorded as saying that school is a series of events to be written about by pupils - thus writing becomes an aim or purpose of school rather than merely one manifestation of making sense.

> Observation 6 - The push to written records, and to manipulating symbols and technical terms is due to a teacher's wish for pupils to automate procedures so that they can manifest the desired behaviour. These procedures are both particular, in the sense of being topic based, and general or heuristic, in the sense of being thinking skills which are used throughout a particular discipline.

Reflection on these observations suggests that between **seeing**, in the sense of a vague and fuzzy glimpse, and **saying**, in the sense of striving for succinct verbal expression, and between manipulating examples and formulating articulate stories, lies the domain of mental imagery as source for and agent of the act of verbalizing. (See Mason, 1986b, for elaboration.) Furthermore, although the act of trying to express in words on paper is helpful for clarifying one's ideas, it is often difficult to write what cannot yet be said. Thus it makes sense to spend time trying to contact any mental imagery that is involved or associated with the topic, trying to express verbally to oneself and to colleagues, and only then trying to record or express ideas in pictures and words. Written records may go through many different drafts before becoming succinct and formal. The triad of **Seeing, Saying, Recording** is a useful reminder that each contributes to the growth of understanding, and that a too rapid push to written records which omits the opportunity to modify, to try to express oneself verbally, may be so demanding as to block progress, and even to turn one against the particular topic.

The same sort of idea can be expressed from a different perspective by observing that it is generally considered good practice to invite pupils to carry out exercises, in the hope that this will literally 'exercise' their growing technical, manipulative skills, and give them access to the abstract ideas underlying the technique. There is however a good deal more hope than structure in such an approach. Whenever we get stuck on a problem or find some statement too abstract or general, we search in our experience for an example with which we are familiar. In other words we turn to entities which are for us confidently manipulable, so that we can try to interpret the unfamiliar in a more familiar context. The act of manipulating, interpreting, exploring is more than simply doing exercises, because we are trying to get a sense of what the person is talking about.

In order to 'get a sense' we summon up various forms of imagery connected with past experience of similar situations. If we are given assistance by the speaker in the form of pauses, during which we can try out our examples, or ponder examples provided by the speaker, in which our attention is drawn to the features salient for the speaker, then by this suitable stressing and ignoring we can be assisted to re-experience the generality which we first heard. If we have the opportunity to try to express this generality for ourselves, then we begin to bring to articulation the vague sense that was germinating while working on the illustrative examples. Once we become familiar with, and confident with, manipulating the succinct expressions of the generality, we have new entities which might themselves be employed in later topics as illustrative examples of further ideas. It is convenient to display this process of making sense, of construing, as a helix in which the movement from confident manipulation through imagery and the processes of specializing and generalizing in order to 'get a sense of', to expressing and articulating the general principles in an increasingly succinct form, are seen as one complete turn around the helix (See Figure 2).

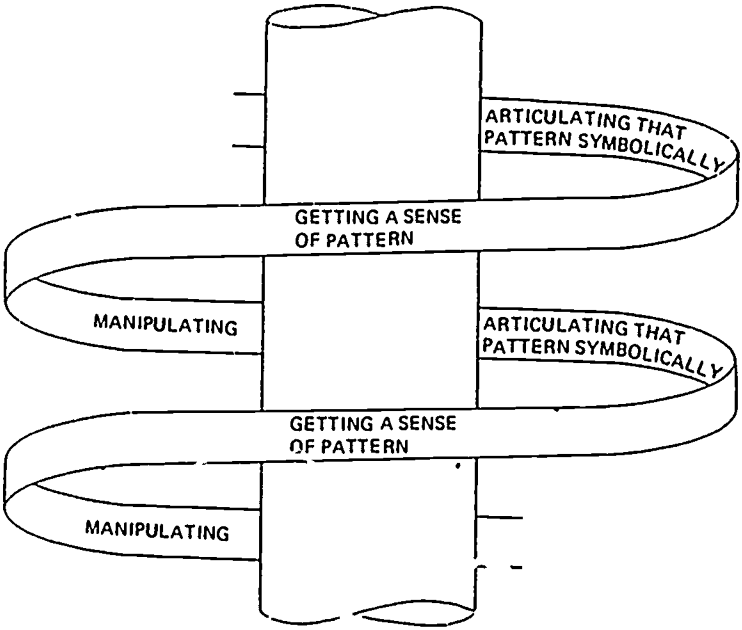


Figure 2. The Process of Making Sense

When we get into difficulty with something that someone is saying we tend to move back down the helix through our images and through sufficient turns so that we find something confident that we can use as an example. We then retrace steps up the helix to try to re-express the generality for ourselves.

Teaching and Assessing

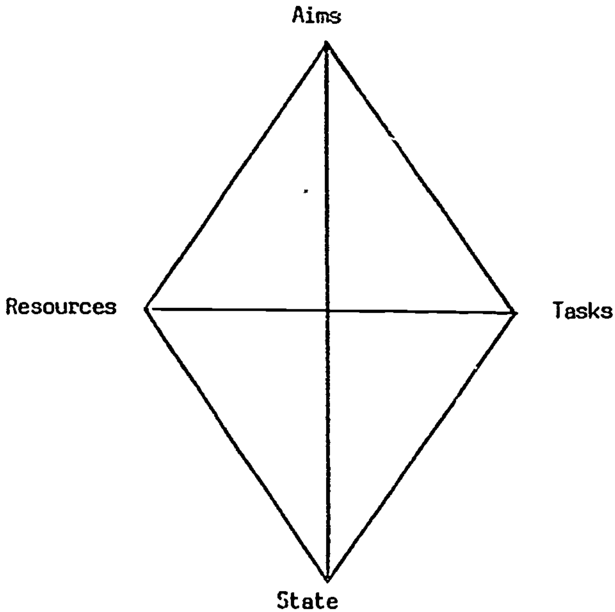
With these observations and remarks we can now address the title of this paper. What does it mean to teach by helping pupils to make sense, and how might we go about assessing the sense that they make? The word 'teaching' is a curious one, because in some sense the one thing that students do not need is to know how to make sense of the world around them. Having survived to the age they have reached, having learned to speak, to walk and to run, they have illustrated all of the powers necessary for making sense of the world. However it seems to be the case that over the centuries individual disciplines have developed specific techniques for efficiently making sense of questions which are oriented in that discipline. Therefore one thing a teacher can do is to work explicitly with pupils on the question of how you make sense in the particular discipline - in my case mathematics. Thus I need to evoke imagery, to learn to be explicit and precise about the imagery which I have inside me and from which I speak, to be explicit about the processes and methods of specializing and generalizing which pertain to my discipline, and to work explicitly with students on exposing and weaving together into stories the fragments of their experience which they recall.

Such remarks are rather general and unstructured and so the rest of this paper is concerned with offering a structure both to help students to construe in a discipline and to help teachers to assess that construal so that students at all levels of experience and ability can demonstrate what they can do rather than be penalized for what they cannot.

Before going into detail, it is important to acknowledge the fantastic tensions which are present in this approach. As a teacher there are a myriad of details and aspects to which I must attend. Is it not asking too much of pupils to be able to weave together articulate stories for complex ideas when in the past we have struggled and struggled simply to get a majority of pupils to be able to manipulate a few entities and to carry out a modicum of techniques in response to examination questions? No matter what ideals I carry as a teacher, when I find myself in a classroom with pupils in a particular state, is it not asking too much that they engage in the sorts of activities which are and will be suggested? In Mason (1986a,c) there are elaborations of these and other tensions, and of ways to work on those tensions. As a teacher I can only work with the energy which is present in the pupils. If I am required to work on topics which are not yet of direct and immediate concern to them, then I must harness their energies and evoke their interest.

Activity of Teaching

The activity of teaching can be viewed from a fourfold perspective in which the current state of teacher and pupils, the aims of both, the resources available and the tasks embarked upon, strive for balance and appropriateness.



What would I like students to achieve at the end of a topic or course? I would like students to have 'seen' connections, to have experienced some sort of integration or crystallization of disparate experiences which are subsumed under some general concept. I would like them to have 'gained a sense of' some coherence of a topic and how the techniques, technical terms, and

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'facts' fit together. I would like them to 'be articulate' about the meaning of various technical terms, of how standard and novel examples illustrate the ideas of the topic (see Michener, 1978, for elaboration); I would like them to be articulate about their own story of what the topic is about and what sort of questions it answers or deals with. I would like them to get to the point of being able to employ succinct articulations confidently in the future as components or examples or tools in new topics.

These aims are teacher aims. Does 'subordination of teaching to learning' and 'starting where the pupils are' mean that my aims should be the same as the students' aims? I suggest not. Students quite naturally often wish to minimize their effort, and many are reluctant to stand out from their peer group. The didactic tension comes into play. But the aims I have outlined here are aims connected with making sense, not with showing off or with making extra effort Consequently, bearing in mind my initial assumption about students wanting to make sense of the world, my aims and students' aims are at least confluent, if not identical.

The wishes outlined contain automated skills, general impressions and articulate stories, experience and familiarity with examples, and more generally with the effects of specializing and generalizing in this particular topic area. The extent to which pupils succeed in all of these aims will depend not only on the teaching style adopted, but on a host of factors including

predisposition/interest/involvement in problematic questions at the heart of the topic;

peer group attitude to learning;

teacher attitude to learning and teaching, interest, commitment, ...;

facility with assumed automated skills;

the extent to which pupils' <u>own</u> powers are evoked and employed in the teaching and learning;

the extent to which pupils share the teacher's goals.

These are some of the factors which make up the Didactic Situation (Brousseau, 1984). With so many influences, it is clear that there is no royal road to learning or to teaching. Suggestions, such as

reconstruction and listening,
pausing,
attending to the spiral through manipulating, getting a sense, articulating,
attending to the back and forth flow between particular and general,

are merely fragments of an ethos or Weltanschauung. They are devices intended to promote a perspective, and do not comprise a 'method'. The words are the results of attempts to draw distinctions which teachers have found useful, because in the midst of an event, suddenly becoming aware of the distinction reminds them that there may be some aspects they may have neglected or some alternative ways of engaging pupils and evoking their powers. In other words, Gattegno's assertion holds equally for students and teachers: 'Only awareness is educable'.

Since assessment is currently such a major concern for pupils, teachers and educators, it seems sensible to face the tide, and deal directly with how one might assess the sense that pupils make of lessons. This will in turn indicate a perspective which informs classroom practice so as to better prepare pupils for assessment.

Assessment

Taking the view expressed in the National Criteria (SEC, 1986) and the Cockcroft Reports (1982) that assessment should provide pupils with an opportunity to show what they can do, and not to hide what they cannot, I suggest that assessment might sensibly look for:

evidence of what a pupil can do, at a functional, behavioural, technique level;

evidence of facility at various levels of explanatory coherence, showing a sense-of, and where appropriate an articulateness about particular concepts;

evidence of carrying out various mathematical thinking processes such as resorting to particular cases, seeking and expressing general patterns, convincing oneself and others;

evidence of participating constructively in group work and discussions, both as participant and as leader.

In the language of Skemp (1971), these aspects might be described succinctly as:

instrumental understanding of content;

relational understanding of content;

participation in process;

participation in social roles.

Consider a topic, such as Density, the Norman Invasion, Solving Triangles, Macbeth, or any other that comes to mind. I submit that the nature and purpose of formal education is to facilitate movement of attention to and fro between particularities and generalities. In other words, it is to become aware of, and articulate about, patterns or generalities which encompass a variety of contexts and situations (Mason, 1984b). (Note this same movement in the use of particular examples like Density and Macbeth in order to indicate the general). Successful pupils can move from the particular to the general, and from the general to the particular. Pupils who are process-aware (but not necessarily articulate about it) quite naturally evoke both movements **automatically** as appropriate. In some situations pupils can operate only **reflexively** in the sense that they can employ both movements appropriately when reminded, but not always without being reminded. In some situations pupils can only **react** to explicit suggestions. They need specific help in invoking fundamental thinking processes which they have already used in order to learn to walk and talk, but which for some reasons are not employed in the particular lesson. The triad of **Reactive, Reflexive** and **Automatic** helps some teachers to recognise differences in pupil responses that might otherwise have been overlooked, and thus enables them to extend pupils appropriately.

Movement from explicit and detailed work employing thinking processes (and thus developing and refining them), to reflexive triggering off processes by a word or gesture, to automatic invocation, is subtle. It can be assisted by the use of explicit vocabulary, but words can also become superficial jargon. Mason (1984a, 1986c) elaborates the tension between words as superficial jargon and as precise technical terms, and suggests techniques for developing and maintaining richness and meaning for didactic frameworks.

The following six levels of performance are based on an analysis of the particular-general movement, together with a distinction between being able to give an account of what a topic is about, and being able to account for various features or anomalies that appear in the topic. The six levels provide both a basis for designing assessment and a technique for helping pupils make sense of a topic for themselves - in short, to construe and verify their own meanings. An overall picture of the six levels is given in Figure 3, which can usefully be read from right to left, as a flow from the functional to the perceptive, from left to right as an unfolding of the essence into the functional, or as levels developing clockwise from bottom right round to top right. Elaboration with examples follows the diagram.

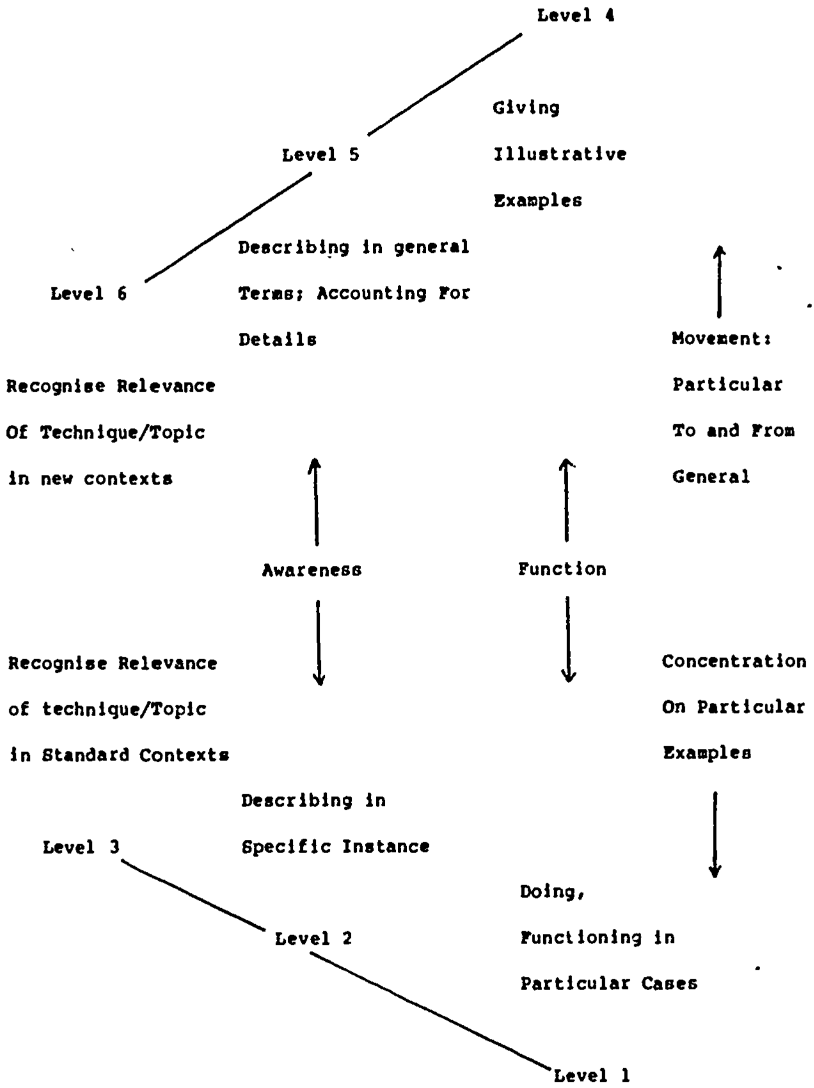

Figure 3. Six Levels of Mathematical Process

Most examinations test facility at level 1 and level 2 with respect to techniques like solving equations. Rarely are pupils called upon to show what they can do at higher levels. Similarly, pupils are frequently put to work on sets of exercises which are intended to develop facility at level 1 or 2. Rarely are they encouraged to give their own account <u>of</u> a topic, or to account <u>for</u> how topics fit together. Yet it is in the act of explaining to others ('you only really learn when you have to teach it') that most people really make a topic their own by constructing their <u>own</u> story of how all the details fit together.

<u>Levels 1 to 3</u> - Giving an Account of (Describing)

Level 1 Doing specific calculations, functioning with certain practical apparatus (e.g. add fractions of a particular type; make measurements; read tabular data; find a solution to a given pair of equations; find the reflection of a point in a line...).

Recalling specific aspects of a topic, and specific technical terms (e.g. fractions can be added, multiplied, compared; equations sometimes have solutions and sometimes not...).

Level 2 Giving an account of how a technique is carried out on an example in own words; describing several contexts in which it is relevant (e.g. you multiply these together and add those...; you measure perpendicularly here...; fractions arise as parts or shares of a whole...).

Giving a coherent account of the main points of a topic in relation to a specific example (e.g. fractions can be compared by subtracting or by dividing...).

Giving a coherent account of what a group did, in specific terms (e.g. we tried this and this, we noticed that...).

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Level 3 Recognising relevance of technique or topic/idea in standard contexts (e.g. if two-thirds of a team have flu... - recognise fractions; two kilos of coffee and one kilo of tea cost... - recognise simultaneous equations).

Levels 4 to 6 - Accounting For (Explaining)

Level 4 Giving illustrative examples (standard and own) of generalizations drawn from topic, or of relationships between relevant ideas (e.g. the simplest denominator is not always the product - give an example; sometimes simultaneous equations have no solutions - give an example).

Identifying what particular examples have in common and how they illustrate aspects of the technique or topic (e.g. what does 5/6 + 3/8 = 23/24 illustrate about adding fractions?; when this point is reflected in this line, and its image reflected in the same line, you get back to the starting point - what general rule is being illustrated?).

Level 5 Describing in general terms how a technique is carried out; to account for anomalies, special cases, particular aspects of the technique (e.g. to add two fractior you...; simultaneous equations with no solution arise because...; a triangle reflected in a line cannot be translated and rotated back to its original position - why?).

Level 6 Recognising relevance of technique or topic in new contexts.

Connecting topic coherently with other mathematical topics (e.g. fractions are one way to get hold of certain kinds of numbers...; reflections are examples of transformations...).

The six levels are intended to suggest ways of structuring tasks for pupils which will provide impetus and opportunity for them to make ideas their own, as well as providing a format for assessment of content. The proposed format includes assessment of the major mathematical processes - specializing, generalizing and convincing. Other processes such as the use of, and switching between, representations will arise in the context of particular mathematical topics. Problem solving and investigational work could be assessed using the same format, in which pupils are called upon to report (orally or in written form) on their own work and the work of their group, at the various levels.

It is important to bear in mind constantly what the final assessment record might look like. It is intended to be positive, to state what the pupil has achieved, to give pupils something short-term to aim for, to provide reward and thus a boost to confidence. Any recorded evidence of completion of some task will have to be relevant and clearly understood by agencies who might later see the certificate - employers, colleges, etc. Assessment records will have to be simple enough for such agencies to comprehend, yet complex enough not to reduce 12 years of schooling to a few numbers. Performance on a single day under unusual circumstances is a peculiar way to sum up someone's potential. Giving pupils many opportunities to demonstrate competence is fairer to all concerned.

In Mason (1984a), it is suggested that the brick-by-brick construction metaphor for scientific epistemology is wildly inappropriate in (mathematics) education, that a more appropriate metaphor might be a forest, which is constantly changing, yet also unchanging; that Plato, and all teachers before and after, have had to face the same basic didactic tensions; that each generation struggles to express their questions and solutions in idioms appropriate to the times. This paper illustrates the point, for its roots can be found in Sankya philosophy, pythagorean qualities of numbers, Plato and Dewey - to name only a few. Pedigree is, I believe, much less important than vividness and resonance in the current didactic context.

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