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ABSTRACT

This document analyzes one chapter of a textbook for college remedial mathematics. This analysis is done by one of the textbook authors. The chapter under discussion deals with fractions. The text authors, writing from a constructivist perspective, attempted to write problems which not only developed specific conceptual and heuristic objectives for students, but also problems which engaged student curiosity and challenged students to reason and communicate. The problems were intended to help students develop an awareness that mathematics is necessary for understanding and coping in the world. The document is primarily devoted to the discussion of 11 types of fraction problems. The appendices are the table of contents of the text and solutions to sample problems. (PK)

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ANALYSIS OF FRACTIONS CURRICULUM FOR CONSTRUCTIVIST COLLEGE REMEDIAL MATHEMATICS EDUCATION

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September, 1988

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Analysis of a Chapter of Math 010 Text: Fractions

The Math 010 curriculum was written jointly by myself and Dr. Deborah Schifter, currently with Mt. Holyoke College, and includes supplemental material on problem solving by Dr. Jack Lochhead, of the University of Massachusetts.

Schifter and Narode worked collaboratively in work sessions lasting two to five hours, three times a week for approximately nine months. During that time we conferred on the topics and concepts to be taught and on the methods we thought best to teach them. The separate features of the constructivist text listed above were explicitly addressed in our deliberations. Our goal was to write problems which not only developed specific conceptual and heuristic objectives for students, but also to create problems which would engage their curiosity and challenge their abilities to reason and communicate. In our deliberations, we strove to write problems which would help students to develop an awareness that, (1) mathematics is necessary for understanding and coping with their world, and (2) that they can succeed in using mathematics.

The higher-order thinking skills discussed previously were generally only implicitly treated in the text, i.e., we knew when a particular problem would present difficulties to the student and why, and we discussed the benefits and hazards of including the problem in the text. However, we did not "keep score" as to the number or name of the higher-order skills addressed by the problems. Rather, we discussed in broad terms, and based on our teaching experiences, which problems were desirable for teaching critical thinking.

Our deliberations about the conceptual and topical nature of our task was much more explicit. We often listed concepts to be taught and debated the appropriate order of their instruction. We discussed language, representations, misconceptions, structure (both mathematical structure and problem structure), and we deliberately attempted to integrate problems which could be used to promote writing assignments. The following analysis of the chapter on fractions is an attempt to recreate some of the discussion which ensued in the writing of that chapter. On reflection, it seems that this analysis would also prove useful to instructors using the text. In fact, much of the following analysis was initiated in weekly meetings, which I chaired, with instructors teaching Math 010, in preparation for their classes.

Although problem statements will be reproduced within the text of the following analysis of the chapter on fractions, the introduction to the chapter and problem solutions will be omitted for reasons of space.

Introducing the students to the idea of fractions.

Topic II, of the Workbook, entitled "Numbers And Their Uses", follows a brief chapter on "Problem Solving", and is the first treatment of arithmetic topics in the Math 010 course. The introduction to numbers is intended to afford the student an opportunity to "look ahead" at the topics of the course. Although Bruner's notion of a "spiral curriculum" is not entirelv appropriate for a one-semester remedial course, the idea of exposing students to a variety of topics which they will subsequently learn at greater levels of conceptual sophistication is considered worthwhile even in brief. Not only are the topics declared, but the short definitions allow for the introduction of specific language and terms such as: integer, fractions, decimals, percents, powers, hundredths, positive, negative, whole numbers, repeating decimal, numberline, and exponents. The introduction also serves to acquaint students with several representations: numberlines, fraction notations, negative numbers, the long-division algorithm, diagrams, and verbal descriptions of realistic situations such as dividing a cake. The brief introduction is not intended to teach either the concepts described or the meaning of the representations used to describe them. Rather, the introduction informs the student about what is going to be studied, and hopefully refreshes some terminology and representations which most, if not all, college students have at least seen before.

Problems for developing conceptual understanding of fractions

To begin, the student should be made aware of the fundamental idea of a fraction which he/she already possesses: that a fraction is a part of a whole. The first problem treats the idea by introducing a concrete representation in the context of a familiar situation. It asks the student to draw a picture of several pizzas which are supposedly eaten by different guests at a fictitious pizza party. The problem is reproduced below:

FRACTIONS I

Use pictorial representations to solve the following problems:

Tom, Sue and John went to a pizza party where pizzas were being delivered throughout the evening. (The pizzas were large, and each was sliced into twelve equal pieces.)

a. At this party, Tom ate two slices of the first pizza, one slice from the second and three slices from the third. What fraction of a whole pizza did Tom eat that evening?

Besides drawing a picture of the above problem situation, question 'a' introduces the notion of slicing a whole into equal pieces. The equal parts from different wholes are summed to a part of a whole which can then be described with different fraction names, e.g. six-twelfths or one-half. This problem also illustrates one application of the common denominator and sum of numerators idea. The number of equal parts in one whole, and representationally their size, determines the denominator, while the number of slices eaten, independent from which pizza, determines the numerator. All assumptions about equality must be articulated by the students in response to questions posed in discussion by their teacher --- particularly the assumption that the wholes are themselves of equal size, weight, flavor, or any other criteria that the students wish to consider.

b. That evening, Sue ate five pieces from the first pizza, six from the second, and two slices later on. What fraction of a whole pizza did Sue eat that evening?

Extending the concept of a fraction being a part of a whole, the idea of a fraction being a part of a whole which is greater than the whole is introduced in 'b'. It is appropriate here for the instructor to ask students if they know the name of fractions whose numerators are larger than their denominators and if they know another name for such fractions. The terms "improper fraction" and "mixed number" should be discussed in brief.

c. At one point in the evening, John shared a pizza equally with two other people. He then shared a second pizza equally with three other people. Later, he ate another slice. What fraction of a whole pizza did he eat that evening?

By dividing whole pizzas among different people, problem 'c' requires students to discover equivalent fractions, i.e. that 1/3 of a pizza is the same size as 4/12 of a pizza and that 1/4 = 3/12. The further realization, that in order to add the various portions some common unit must be identified, motivates the need for developing the concept of common denominator. Both the term and connecting the concept to the diagram should accompany a brief discussion as to how one could determine a common denominator in this instance.

d. Five pizzas were totally eaten that evening, and each person ate at least one slice. What is the maximum number of people that could have attended the party, including all of the people in the three previous parts of this problem?

Because some students will finish the first three parts of this problem much more quickly and easily than others, 'd' is intended both to challenge the quick student and to force the student to reflect on previous work. This is an ill-structured problem in that it can generate more than one answer. Depending on how people were assigned to eating pizza from the previous parts of this problem, several answers are possible. A comparison of answers among students should generate considerable controversy. Moreover, students will begin to understand that they will be expected to answer ill-structured problems, and that such problems do not always lead to one "right" answer, even from their teacher.

FRACTIONS II

Bob took a trip in which he traveled by car, foot and bicycle. He rode in a car for two hours at an average speed of 48 miles per hour. He then walked 12 miles in three hours and bicycled the remainder of the 144 mile trip at an average speed of 12 miles per hour.

Use line diagrams to answer the following questions.

1. What fraction of the time of the entire trip did he spend:

a. in the car?

b. on the bicycle?

c. walking?

2. What fraction of the distance of the entire trip did he travel:

a. by car?

b. by bicycle?

c. by foot?

Although Fractions II is placed in a familiar context and with a familiar conceptual construct (average speed), the juxtaposition of questions about time and distance forces students to build not just one picture of the trip but two related pictures.

The use of the line diagram is yet another representation of fractions. A fraction can be viewed both as a line segment and as a point on a line. Furthermore, the same line segment can be used to represent two different fractions. The fraction of the time spent as illustrated with a particular line segment is different from the fraction of total distance traveled though they both may be represented with the same segment. The critical awareness is the need for identifying the whole when referring to any fraction. In this manner, the convention of the number line may be introduced in a context different than that first presented in the introduction to the chapter. The novelty of the different uses of a new representation and the solution of a multistep problem incorporates many problem solving skills and fractions concepts which make this problem more difficult than the first.

The following problem uses fractions for the practicable conversion of liquid and volume measurements between U.S. and metric units. It also introduces students to the concept that liquids occupy volume, and that volume is a three-dimensional construct. The problem introduces new terminology, such as "milli" and "centi", as well as linking this terminology to the decimal system and to two-dimensional representations of three-dimensional geometric ideas. Again, concepts of common denominators and equivalent fractions are applied and reinforced in a new context --- the context of units conversion.

It should be noted that the conversions were written without use of letter symbols and equal signs to avoid the common misconception that letters may be used interchangeably as labels and as variables. In this instance, the use of language is as important for what it does not state as for what it does state.

FRACTIONS III

U.S. liquid-volume measurements:

1 gallon contains 4 quarts
1 quart contains 2 pints
1 pint contains 2 cups
1 cup contains 8 fluid ounces

Metric liquid-volume measurements:

1 liter contains 1000 cubic centimeters
1 liter contains 100 centiliters
1 centiliter contains 10 milliliters

Use the above information to answer the following questions:

1. One ounce is what fraction of:
   a. a gallon?
   b. a quart?
   c. a pint?
2. One cup is what part of:
   a. a gallon?
   b. a quart?
3. One pint is what fraction of a gallon?
4. One milliliter is what part of:
   a. a centiliter?
   b. a liter?
   c. a cubic centimeter?
5. Draw a picture of a cubic centimeter.

After instruction on the various definitions of fractions which are evoked in Fractions problems I, II, and III, the text introduces operations with fractions, specifically and in the order of multiplying, adding and subtracting, and finally dividing fractions.

Fractions IV, 1, is an unusual and rather silly problem situation whose comic character generally lends levity to the classroom. It is also a problem which requires several important problem solving and critical thinking skills. Parts 'a' through 'd' are most easily solved by the students with the use of a box diagram. In this way multiplication of fractions is literally viewed as the dividing up of fractions much as multiplying wholes by fractions operationally requires the dividing of wholes into parts. Students, by this time, have developed some facility with diagramming and with recognizing which part and which whole they are operating on. All of the fractions may be calculated without reference to any specific dollar amount with the exception of part 'e'. The problem is reproduced in its entirety:

FRACTIONS IV

1. A man left two-thirds of his total fortune to his cat and asked that one-fourth of what the cat received should be used to teach it to talk.

a. What fraction of the total fortune was used for teaching the cat to talk?

b. After two-thirds of the fortune went to the cat, what fraction remained?

c. The man left one-half of the remainder to his wife. What fraction of the total fortune did the wife receive?

d. The wife spent one-fifth of what she received on a new Italian sports car. What fraction of the total fortune was spent on the car?

e. Using the information in the problem and your own knowledge of the world, estimate the cost of the cat's speech lessons.

(Hint: First estimate the cost of an Italian sports car.)

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For those students able to work quickly through parts 'a' through 'd', part 'e' should pose a challenge. Even if the student had knowledge enough of the world to be able to estimate the cost of speech lessons for a cat, linking this to previous information in the problem is far from trivial. The intention of this problem, as indicated in the hint, is to provoke the student to estimate the cost of one familiar, or at least semi-familiar, item like the sports car, and then to use the estimate within the context of the problem statement to answer the question. To succeed within the bounds of the problem requires the student to construct the value of the whole from an estimate of one of the parts, and then to calculate the number of dollars needed for the cat's speech lessons. The final step in the problem generally involves the multiplication of fractions, but students solve the problem in a number of ways, including reference to their diagram.

A problem which requires the division of fractions is illustrated in Fractions IV, 2. Although a pictorial representation is not explicitly requested, it helps both the student and the teacher enormously with this problem, and students should be encouraged to develop one.

FRACTIONS IV (continued)

2. The Math 010 office purchases bottled drinking water. Mrs. Gazebo's Mountain Spring Water comes in containers that hold 6-3/4 liters. Two-thirds of the office container was drained last week to put out a fire in Ron's waste basket.

a. How many liters of water are left in the bottle?

b. How many 3/8 liter glasses can be filled with the water that remains in the jar?

Part 'a' of this problem requires at least two steps for its solution. First the student must find the fraction of the tank which remains, then the number of liters must be calculated. The calculation of the number of liters remaining is accomplished through the multiplication of fractions. The novelty here is that one of the fractions is presented as a mixed number. While some students in fact find a solution using the mixed number, most students convert the number into an improper fraction and then use the algorithmic rule for multiplication of fractions. Answers are presented in both mixed number and improper fraction forms.

Part 'b' is quite different from the previous questions asked in Fractions IV in that it is the first question which requires the operation of division rather than multiplication. Most students do not recognize this difference and proceed by multiplying two fractions of their choosing nevertheless. They usually need to be questioned about the reasonableness of their answer, or they need to attempt to represent the situation in a picture to discover their error. While many students will correctly calculate the answer from carefully shading 3/8 liter representations in their picture, most can connect their operations on the diagram to the operation of division, but few can connect either of these operations with the algorithm for the division of fractions.

The following problem involves a multi-step solution which requires using whole numbers and fractions with all of the arithmetic operations discussed previously. The problem is similar to a problem from the Problem Solving chapter of the Workbook [appendix a], and can be solved similarly, with the use of diagrams. The context of this problem is intended to illustrate the importance of simple arithmetic in serious, realistic situations.

FRACTIONS V

A researcher at a medical school would like to test a hypothesis that a certain drug may be administered in smaller doses than originally prescribed. He has a vial which contains 235 doses, and each dose is two and one-half milliliters. How many new-size doses will he have if he reduces the dose to three-fifths of the original dose?

Another applied fractions problem concerns domestic budgets:

FRACTIONS VI

Use pictorial representations to solve the following problems:

Typical expenses from my monthly income are:

1/3 rent
1/4 food
1/6 phone and utilities

a. How much of my monthly income does this account for?

b. Last month, I spent as much money on entertainment as I did on rent. Assuming that my other expenses were the same as usual, how much of my monthly income did I withdraw from my savings account?

c. What fraction of the savings account was withdrawn?

In some respects all of the parts of this problem are ill-structured. Part 'a' is a straight-forward addition of fractions problem so long as the

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student interprets the phrase "how much" as meaning "what fraction". The reason for leaving the language of the problem statement vague is to provoke in the students a need to interpret language within the greater context of the problem situation and information given.

Problem section 'b' uses the same device, and has the added feature of ill-structured problems in that an important assumption must be made by the student for the problem to be solved. The improper fractional solution must be interpreted to mean that more than one monthly income was spent in one month. This is possible only if one assumes that there is more money in the savings account than one monthly income.

The most ill-structured problem in this set of problems is the question in section 'c'. The answer to problem section 'c' is that the fraction of the savings account which was withdrawn to satisfy the answer in 'b' cannot be determined. Knowing the fraction of the monthly income which must be withdrawn yields no information about the fraction of the savings account withdrawn unless the dollar amounts of monthly income and of the savings account are known. The problem is intended to evoke in the students just this realization.

Fractions VII, 1, is an example of a multistep applied problem whose language is complicated, but whose solution is easily represented in a picture.

FRACTIONS VII

1. A birthday cake is to be shared. The children, who are allotted half a piece each, consume twice as

much cake as the adults, who are allotted two pieces for each man and one piece for each woman. 3/45 of the adults are male, and he is cutting the cake.

How many children are at the party?

Note: Everybody eats their share of the cake, and none remains.

The second sentence of this problem is the longest sentence to this point in the Workbook. It must be analyzed in pieces, and then reorganized in such a fashion as to discern the relationship between the pieces as well as the differences between various clauses. For instance, many students confuse the part eaten by all the children and the part eaten by each child. There is another difficulty with language inherent in the sentence "3/45 of the adults are male, and he is cutting the cake." Students assume that the fraction '3/45' implies that there are 45 adults, three of whom are male. The phrase "and he is cutting the cake", serves to disequilibrate the students, and usually succeeds in provoking them to reduce the fraction to 1/15. It is a "trick" which most students moan about, much as they would to a pun --- the vocal response may be interpreted as a sign that the pun or "trick" is understood. Most often, students see the value of interpreting fractions within the greater context of the problem. The use of the ill-structured problem is deliberately intended to cause the student to become more reflective of their problem solving generally.

A well-structured problem can also expose misconceptions and confusions, which while appearing to be related to the language used in the problems, may be traced to deeper misconceptions about the topic of instruction. The translation difficulties encountered with the Student and Professors Problem is an example of such a well-structured problem with language related misconceptions. The following problem illustrates a similarly well-structured problem which exposes deep-seated misconceptions about fractions through the use of a problematic phrase:

FRACTIONS VII

2. Four people share a pizza in the following way: Tom got a third and Mary got a third of the remainder while Dick and Harry shared equally what Tom and Mary did not get.

What fraction of the whole pizza did Harry receive?

The above problem presents a difficulty to students in their interpretation of the phrase "a third of the remainder". Many students interpret the problem to state that both Tom and Mary each received 1/3 of the whole. That Mary received only 1/3 of 2/3 (the part remaining), often goes unnoticed. Some students, when reading the problem aloud, often fail to read the words "of the remainder", literally reading past them to the next word in the sentence. However, even when asked to reread with special attention on this particular phrase, many students declare that the phrase makes no difference to the problem's solution. To these students a third of the whole is not distinguished from a third of the remainder. Furthermore, their pictorial representations of the pizza often reflect this misconception. Many students draw a circle or a rectangle, divided first into three equal pieces and labeled with Tom and Mary each in one of the sections. This indicates that pictures do not in themselves obviate all conceptual difficulties with regard to fractions. Some guided discussion is also required.

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The collection of problems in Fractions VIII are designed to encourage students to draw pictures to aid them in the solution of the problems and also to help them visualize the various arithmetic operations performed with fractions. Fractions VIII, 1, combines ratio and fractions ideas without explicitly mentioning any fractions.

FRACTIONS VIII

1. A company of 266 persons consists of men, women and children. There are four times as many men as children, and twice as many women as children. How many of each were there?

The above problem is similar to typical algebra word problems. Students in Math 010 do not yet know algebra well enough to be able to use it to answer this problem. However, a multi-step solution which uses a diagram helps students to realize that the most convenient unit for dividing the whole is the group of children. Once realized, the operations which are carried out in diagram form are an excellent predecessor to understanding the algebraic solution. While not yet instructing the students in algebra, the experience of symbolizing the problem with a diagram, representing the relative sizes of groups based on a unit group (generally though not exclusively the group of children), and incorporating the ratio information into the solution, are all useful tools for solving problems with or without algebra.

The problem below, Fractions VIII, 2, is similar to the preceding problem. Instead of stating the total number in the whole and the ratio of the parts, Fractions VIII, 2, states the fraction of the whole constituted by each part and a number of individual units in the remaining fraction of the whole. The problem appears below:

FRACTIONS VIII

2. In Sarah's flower garden, one-third of the plants are marigolds, one-fourth are petunias, one eighth are zinnias, one twelfth are mums, and there are ten other plants each of a different variety. How many plants are there of each variety?

A detailed solution to this problem is included in the Workbook (appendix b) in the form of a thought-process protocol. The protocol is a seven-page reconstruction of a student's solution to the problem. Included are diagrams, self-posed questions, metacognitive musings and many qualifying statements, dead-ends, and self-corrected errors. The solution is an excellent model for effectively using diagrams, and for metacognition through writing.

Fractions VIII, problems 3, 4, and 5, are intended to aid the students in their understanding of algorithmic operations with fractions by having students perform the operation on a diagram and then relate the diagram to the algorithm.

FRACTIONS VIII

3. To convert 3-1/4 to an improper fraction, you multiply the 3 times the 4 and add 1 to get 1-3/4. Using pictures to illustrate, explain in words what you are doing when you "multiply 3 times 4 and add 1", and why this gives the correct result.

4. Use pictures to show that 2/3 of 3/4 equals 1/2.

5. Is it also true that 3/4 of 2/3 equals 1/2?
Again, use pictures.

The above problems are intended to demystify algorithms and to make concrete their operations. Students are expected to associate steps in an algorithm (changing a mixed number to an improper fraction in problem 3) with drawing lines in box diagrams. Problems 4 and 5 reinforce picture drawing with yet another algorithm, multiplication of fractions, and also exhibits the commutative property of multiplication as it applies to fractions.

FRACTIONS VIII 1/2, as may be inferred from the numeration, was inserted into the text in an attempt to persuade students to read the thought-process protocol. It is almost identical in form to Fractions VIII, 2, with the added feature of presenting the unit quantities in pairs rather than single units. Asking students to work Fractions VIII 1/2, and informing them of the similarity of this problem to the problem solved in the thought-process protocol, should provide incentive for reading the protocol solution and using the solution as a model for thinking and for diagramming. The problem appears below:

FRACTIONS VIII 1/2

A limited number of tickets were available for a jazz concert. Ruth bought one-fourth of the tickets, David bought on-sixth, George bought two-ninths, and Linda bought one-twelfth of the tickets. The remaining 50 pairs of tickets were bought by Nina.

a. How many tickets did George buy?

b. How many tickets were there altogether?

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Fractions IX is a well-structured problem, but difficult in that it incorporates a misconception which many students seem to demonstrate. The problem asks students to consider a whole which is comprised of two distinct parts, and each part divided differently among people. Students often exhibit a specific misconception which is described after the problem statement.

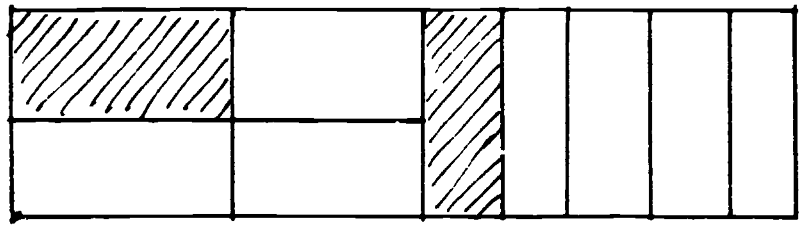
FRACTIONS IX

Use a pictorial representation to solve the following problem:

The oil bill for a group house was for the months of January and February. Four people lived in the house during January, and a fifth moved in on February 1st. Assuming that the same amount of heat was used each of the two months, what fraction of the bill should each person pay?

Note: The fifth person should not have to pay for January's heat.

By dividing a single box diagram in half, and each half into fourths and fifths respectively, students determine that each person who lived in the house for both months should pay 1/4 + 1/5 of the total bill. The diagram below illustrates their misconception:



The misconception lies in a confusion about which box constitutes the whole. Because the larger box is divided in half, each of the segments is actually one-half the fraction of the smaller box when referenced to the

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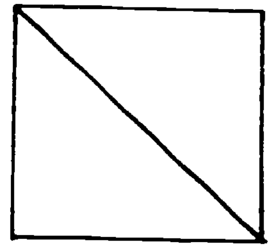
larger box. Consequently the 1/4 of January's portion of the heating bill is actually 1/8 of the total bill, and 1/5 of February's bill is 1/10 of the total heating bill. Quarters and fifths are visible in the diagram, whereas the fractions 1/8 and 1/10 are not.

Fractions X is intended to offer a brief introduction to several geometric shapes, terms, and properties. Problem 1 introduces the term and idea of perimeter. Problem 2 considers the ratio of the areas of two identical triangles. Problem 3 relates the areas of the triangles to the area of the square. Problem 4 is a simple sum of fractions. Students approximate the ratio of the diagonal to all of the sides and to one of the sides of a square in problems 5 and 6 respectively. Both of these problems are intended to illustrate that the diagonal is longer than one of the sides, and the solution to 6 is an improper fraction which teaches that a quantity can be more than 1 times larger than another quantity. The group of problems appears below:

FRACTIONS X



1. In this square, one side is what fraction of the perimeter?

2. The area of one triangle is what portion of the area of the square?

3. The area of both of the triangles is what portion of the are of the square?

4. What fraction of the perimeter of the square are three of its sides?

5. The diagonal is what fraction of the perimeter of the square? (You may approximate your answer.)

6. The diagonal is what fraction of one of the sides of the square? (You may approximate your answer.)

Fractions XI, 1, was discussed earlier in this chapter. It is an ill-defined problem in that certain assumptions are necessary for its solution even though they do not appear in the text of the problem. The problem also exposes the misconception that all problems which require the addition of fractions involve <u>only</u> finding a common denominator. The following problem also requires finding a common numerator.

FRACTIONS XI

1. In a certain population, two-thirds of all men are married, but only 3/5 of all the women are married.

a. What fraction of the population is single?

b. Are there more men or women?

The assumption needed to answer this problem is that men are married to women, and that all of the spouses are in the population. It is this one-to-one correspondence which forces the numerators in the fractions of married men and married women to be the same, i.e. common numerators. A common error, in student solutions to part 'a', involves adding the fractions 1/3, the fraction of single men, and 2/5, the fraction of single women. Finding a common denominator and adding the numerator yields the solution that 11/15 of the population is single. Students generally notice that this fraction states

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that the majority of people in the population are single which is not consistent with the situation described in the problem.

The question appearing in part 'b' is intended to provoke questions about assumptions, and to help the student initiate non-specific quantitative reasoning; that is, reasoning which gives a sense for the relative sizes of quantities rather than their specific numbers or fractional sizes. While it is a trivial question for anyone who has already solved part 'a', it can have the effect of stimulating a fruitful line of thought or a fresh attack on an otherwise confounding problem.

Fractions XI, 2, is quite complicated. It was written to introduce the Venn Diagram, and is most easily approached using this method. In addition to providing a new representation for problems involving fractions, the problem also provides an introduction to the rudiments of set theory.

FRACTIONS XI

> 2. In a certain city, four-fifths of all homeowners are men, and 2/3 of all men who own homes are married. A full 3/5 of all men are property-less, even though half of them are married. What fraction of all men are sirgle homeowners?

The above problem is ill-structured in that more information is given than is needed to solve the problem. Careful reading and monitoring is needed to filter the essential from the non-essential information.

The final problem which appears in the chapter on fractions is a writing exercise which, though appearing at the end of the chapter, can be assigned earlier during instruction. The problem incorporates a peculiar misconception which the students are asked to expose in a 400-word thought process protocol. The problem appears below:

FRACTIONS PROTOCOL

The following protocol was written by a student in response to the question below. It contains misconceptions which you should identify and clarify in your own 400-word protocol.

"Use a pictorial representation to answer the question:

3 is what fraction of 9?"

"I'll start by representing the numbers with circles.

3 =

9 =

I want a fraction so I'll put the three circles over the nine circles since I know that the three is less than the nine.

3
------- = ------------------
9

Since I can cancel equal things on top and bottom, the picture looks like this:

------------------ = -----------------

I'm left with 6 on the bottom and nothing on the top, so the answer must be 0/6 which is the same as zero.

But that can't be, since I know the answer can't be zero. Looking back at what I did, I see that I forgot that when you cancel equal things, you get ones, not zeroes, so that the answer should be 1/6."

By using misconceptions, attention may be focused on the concepts needed for reducing fractions. In particular, students should discuss the importance of finding common factors. The pictorial representation is an inappropriate over-simplification of the cancelling process. There is an additional misconception about the meaning of cancelling common factors as well as a rather meaningless treatment of the problematic zero, as discussed in this protocol. All of these errors provide a forum for discussion and concept elucidation.

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APPENDIX A

A COURSE GUIDE TO MATH 010L

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APPENDIX B

SAMPLE PROBLEM SOLUTION

PROBLEM STATEMENT

Fractions VII

2. In Sarah's flower garden, one-third of the plants are marigolds, one-fourth are petunias, one-eighth are zinnias, one-twelfths are mums, and there are ten other plants each of a different variety. How many plants are there of each variety?

PROBLEM SOLUTION:

The problem asks "How many plants are there of each variety?" Well, I wonder how many there are altogether.

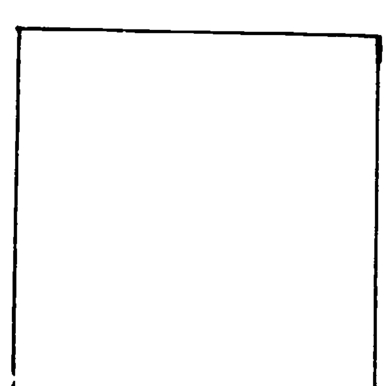
Here's the breakdown of different plants:

1/3 marigolds
1/4 petunias
1/8 zinnias
1/12 mums
10 other plants

All I know definitely is that there are 10 plants (in addition to the other flowers), each of a different variety. So, there must be more than 10 plants altogether.

OK. The fractions represent parts of the whole. For instance, 1/3 (fraction) of the plants (the whole) are marigolds. But, how many plants make up the whole?

I am going to start with an empty box to represent all the plants in the garden:



All of the plants

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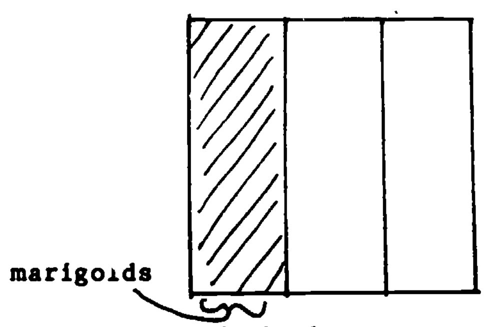
How should I divide it up? How do I know how many parts to divide it into if I don't know how many plants there are? I don't know.

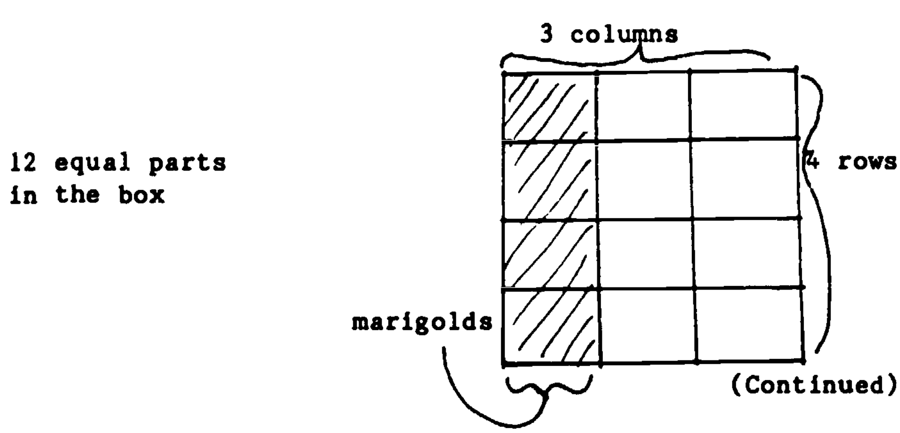
Just to have some place to star., I'll start at the top of the list, with the marigolds. Since 1/3 of the plants are marigolds, I'll divide my box into three equal parts. I'll shade in one of the three parts to show that that part contains the marigolds:



But, what does one-third of the box represent? One marigold? Obviously, there isn't just one marigold. That would mean there are only 3 plants in the garden, and I already know there are at least 10.

I still don't know how many plants there are altogether. But, since 12 is the largest denominator of the fractions mentioned in the problem, I will start there. So, I will divide my box into 12 equal parts. (I have to remember that the box includes all of the plants.)

The box is already divided into 3 equal parts. What do I do to divide it into 12 equal parts? Well, 3 x 4 = 12. There are three columns; and if I make 4 rows, that will leave 12 equal parts within the box.



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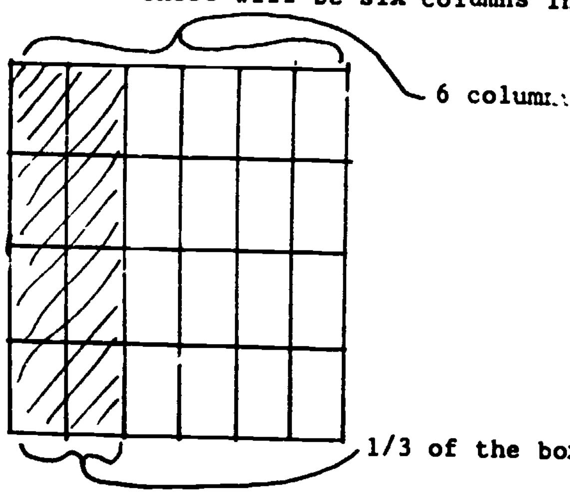
Now, one-third of the plants (or 1/3 of the box) is marigolds. What is 1/3 of the box now that it is sectioned into 12 parts? (One-third is still one part out of 3 equal parts.)

But, how many equal parts are there in the one-third of the box? 1, 2, 3, 4. There are four parts. Does that mean there are only 8 other plants? No, that can't be true. I know that there are more than 10 plants in addition to the marigolds. Each individual part must represent more than one plant, so I have to further divide the box into more equal parts.

I am going to further divide the box. Each column will be divided in half so there will be six columns in all instead of three.



I have changed the total number of equal parts in the box. Now there are 24 equal parts.

One-third of the box is still marigolds, but now there are 8 equal parts within that one-third whereas there were 4 equal parts before.

OK. I am going to continue to section off the box with the other plants. What's next? Well, the problem mentions "One-fourth are petunias." So, I need to section off one quarter of the box for the petunias. How many parts are contained within 1/4 of the box? First, I need to know how many parts there are altogether. There are 24 equal parts. Well, what is 1/4 of 24?

$$\frac{1}{4} \times 24 = \frac{24}{6} = 6$$

There are 6 parts within 1/4 of the box.

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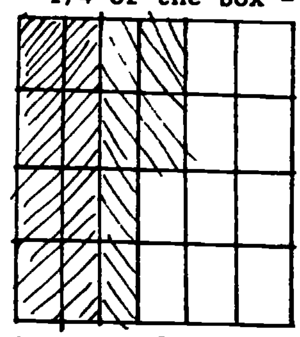
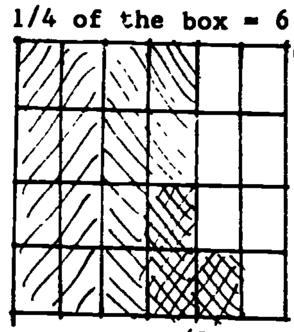
I will shade in 6 parts of my box to represent the petunias.



What's next? "One-eighth are zinnias." Now I have the same problem as before: how many parts represent 1/8 of the box? I know there are 24 parts altogether, and 1/8 of 24 is:

$$\frac{1}{8} \times 24 = \frac{24}{8} = 3$$

So, 3 parts represent 1/8 of the box or the zinnias. Now I will shade in that portion of the box:



Next? "One-twelfth are mums." OK. How many parts represent 1/12 of the box? 1/12 of 24 is:

$$\frac{1}{12} \times 24 = \frac{24}{12} = 2$$

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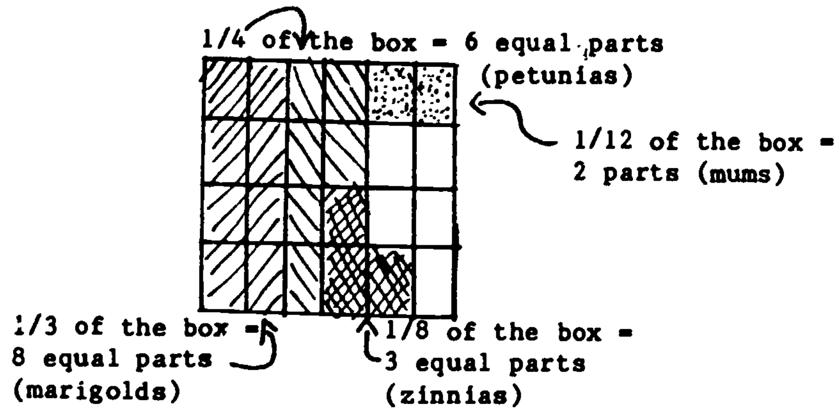
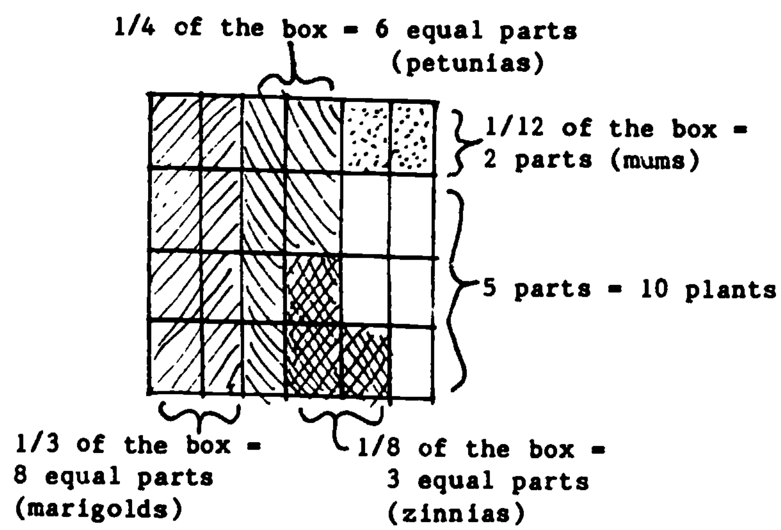
There are 2 parts out of 24 that represent 1/12 of the plants (or the mums). I'll shade in 2 parts for the mums.



OK. So far, so good. What's left? "...ten other plants, each of a different variety." I don't know what fraction of the plants is represented by the 10, so I can't figure this out the way I did the fractions.

How many parts are left? There are 5. And I need a certain number of equal parts to represent 10 plants. Well, if each part represented two plants, I would have the 10 plants taken care of. That means I have to divide each equal part into two parts. Or, I can just note on my diagram that each equal part represents 2 plants. Letting each part represent 2 plants sounds simpler than dividing each part in 2. I will show this on my diagram:



NOTE: Each part represents 2 plants.

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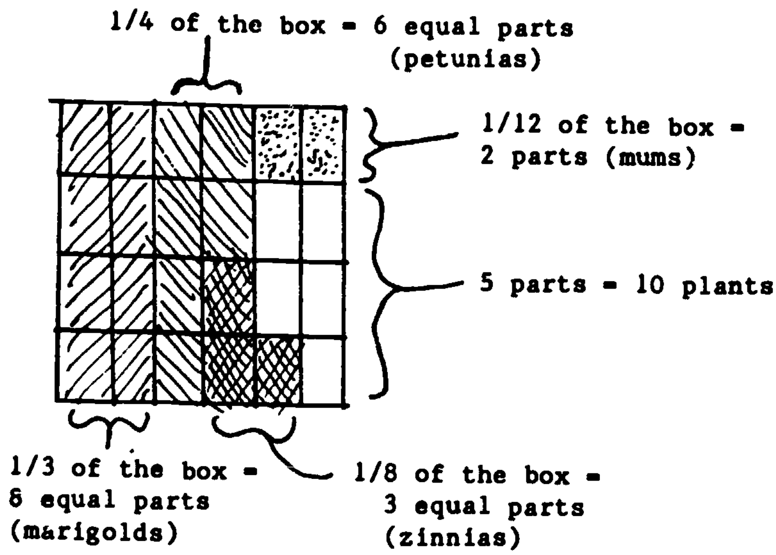
What does this mean in terms of answering the original question? The problem asks: "How many plants are there of each variety?" And, I wondered how may there were altogether. I have to know how many plants there are in total to figure out how many plants are represented by the fraction. For instance, "one-third" doesn't tell me an absolute number; it just tells me the relative portion. One-third of 12 is not the same number as 1/3 of 24. One-third of 12 is 4, and 1/3 of 24. One-third of 12 is 4 and 1/3 of 24 is 8.

And, the ten plants gave me a reference point. The 10 plants helped me determine the size of each equal part in my box. Now I know that each equal part represents two plants. With this information, I can look at my diagram and figure out how many plants there are of each variety.



NOTE: Each part represents 2 plants.

Marigolds

1/3 of the box = 8 equal parts. Each part represents 2 plants. Therefore, 8 x 2 = 16 marigolds.

Petunias

1/4 of the box = 6 equal parts. Each part represents 2 plants. Therefore, 3 x 2 = 12 petunias.

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Zinnias

1/8 of the box = 3 parts. Each part represents 2 plants. Therefore, 3 x 2 = 6 zinnias.

Mums

1/12 of the box = 2 parts. Each part represents 2 plants. Therefore, 2 x 2 = 4 mums.

Other plants, each of a different variety.

5 equal parts. Each equal part represents 2 plants. Therefore, 5 x 2 = 10 other plants, each of a different variety.

OK. That looks like the end of this problem solution. But, can I be sure? Have I accounted for each equal part within my box? There are 24 equal parts, and each part represents 2 plants so there must be 24 x 2 = 48 plants altogether. Let me add up the flowers I've already counted:

16 marigolds
12 petunias
6 zinnias
4 mums
10 others

48 TOTAL

Yes, I now feel confident that all plants have been accounted for.

Note: This solution demonstrates one way to solve this problem. There are many other ways to also arrive at the correct answer. The specific operations used in the solution may not apply directly to other problems in this section. This problem solution is intended to be a model thought process protocol. It is not intended to be memorized for future quizzes, tests, etc. Each problem is unique and must be approached as such.

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