DOCUMENT RESUME

ED 299 133 SE 049 657

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TITLE	Treating Mathematics as an Ill-Structured Discipline.
INSTITUTION	Pittsburgh Univ., Pa. Learning Research and Development Center.
SPONS AGENCY	Office of Educational Research and Improvement (ED), Washington, DC.
PUB DATE	29 Feb 88
NOTE	55p.
PUB TYPE	Reports - Research/Technical (143)
EDRS PRICE	MF01/PC03 Plus Postage.
DESCRIPTORS	Cognitive Structures; Educational Research; Elementary Education; *Elementary School Mathematics; *Mathematics Instruction; *Problem Solving; *Social Influences
IDENTIFIERS	Mathematics Education Research; *Meaningful Instruction

ABSTRACT

Mathematics is generally regarded as a well-structured discipline by educators and cognitive scientists. This leads to the analysis of the suggestion that practitioners begin investigating possibilities for teaching mathematics as if it were an ill-structured discipline, with more than one interpretation for mathematical statements. The paper is organized into three main headings: "The Nature of Meaning Construction for Mathematical Language"; "Socializing Mathematics Learning"; and "Collaborative Problem Solving," with research findings incorporated in the discussion of each topic. Issues for further investigation are then identified, including: (1) natural language and mathematical language; (2) social engineering; (3) integrating strategy and content in problem solving; (4) contextualizing problem solving; (5) scaffolding supports for problem solving; and (6) socializing problem solving. (MNS)

Treating Mathematics as an Ill-Structured Discipline

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To appear in R. Charles and E. A. Silver (Eds.), *Research agenda for mathematics education: Teaching and assessment of mathematical problem solving*. Hillsdale, NJ: Erlbaum.

Educators and cognitive scientists commonly think of mathematics as *the* paradigmatic "well-structured discipline." Mathematics is regarded as a field in which statements have unambiguous meanings, there is a clear hierarchy of knowledge, and the range of possible actions in response to any problem is both restricted and well defined in advance. Cognitive scientists frequently contrast mathematics and mathematical logic problems--which must and can be solved within the narrow constraints of accepted postulates and transformations--with problems in such domains as the social sciences, where large amounts of external knowledge must be brought to bear, texts draw alternative interpretations, and conclusions can be defended rationally but not always strictly proven. Educators typically treat mathematics as a field with no open questions and no arguments, at least none that young students or those not particularly talented in mathematics can appreciate. Consider in evidence how little *discussion* occurs in typical mathematics classrooms compared with English, social studies, and some science classrooms. Even when we teach problem solving, we often present stereotyped problems and look for rules that students can use to decide what the *right* interpretation of the problem is--so that they can find the single appropriate answer.

One result of this common way of teaching mathematics is that many children come to think of mathematics as a collection of symbol manipulation rules, plus some tricks for solving rather stereotyped story problems. They do not adequately link symbolic rules to mathematical concepts--often informally acquired--that give symbols meaning, constrain permissible manipulations, and link mathematical formalisms to real-world situations (Resnick, 1987,a). Widespread indications of this problem include children's use of buggy arithmetic algorithms and algebra malrules and their general inability to use mathematical knowledge for problem solving. There is some evidence, however, that strong mathematics students are less likely than other students to detach mathematical symbols from their referents. These students seem to use implicit mathematical principles and knowledge of situations involving quantities to construct explanations and justifications for mathematical rules, even when such explanations and justifications are not required by teachers.

Research in other fields of learning supports this conjecture. Studies show, for example, that good readers are more aware of their own level of comprehension than poor ones; good readers also do more elaboration and questioning to arrive at sensible interpretations of what they read (e.g., Brown, Bransford, Ferrara, & Campione, 1983). Good writers (e.g, Flower & Hayes, 1980), good reasoners in political science and economics (e.g., Voss, Greene, Post, & Penner, 1983), and good science problem solvers (e.g., Chi, Glaser, & Rees, 1982) all tend to treat learning as a process of interpretation, justification, and meaning construction. As in these other fields, students who understand mathematics as a domain that invites interpretation and meaning construction are those most likely to become flexible and inventive mathematical problem-solvers.

All of this suggests that we urgently need to begin investigating possibilities for teaching mathematics as if it were an ill-structured discipline. That is, we need to take seriously, with and for young learners, the propositions that mathematical statements can have more than one interpretation, that interpretation is the responsibility of every individual using mathematical expressions, and that argument and debate about interpretations and their implications are as natural in mathematics as they are in politics or literature. Such teaching would aim to develop both capability and disposition for finding relationships among mathematical entities and between mathematical statements and situations involving quantities, relationships, and patterns. It would aim to develop skill not only in applying mathematics but also in thinking mathematically.

To embark on such a venture requires an analysis of the various possible meanings of mathematical expressions. From one perspective the meaning of a mathematical expression is entirely contained within the formal proof system of postulates and acceptable derivations from those postulates. We establish the truth of a mathematical statement by proving it. If the initial assumptions and rules of proof are accepted, there is no ambiguity about whether a mathematical statement is true (although some statements may be "not yet proven" and, therefore, not yet of established truth). The meaning of the statement lies in its place within a system of statements--nothing less. Mathematical statements need not refer to objects or concepts as the statements of ordinary language normally do. This is the essence of the *formalist* position on the nature of mathematical knowledge (cf. von Neumann, 1985).

In its purest form, the formalist position would deny any necessary relationship between mathematics and physical reality. But viewed from another perspective, if mathematical statements really had no meaning beyond their relationship to other statements in the same formal system, mathematics could not be used to describe patterns or relations in the world or to draw inferences of new, not yet observed patterns and relations. Mathematics would be merely an intricate game, entrancing to those who loved it, but of no general value to society. Like music, it would be valued for emotional and aesthetic qualities, without reference and "utility." But mathematics *is* useful. It helps us describe and manipulate real objects and real events in the real world. Mathematical expressions therefore, must have some *reference*. Numerical expressions refer to numbers--abstract entities which, in turn, stand in some regular relationship to actual physical quantities or enumerable events. A useful essay by Carnap (1956) explores the complications of a language that refers to abstract entities. Statements in a geometry proof refer to points, lines, angles, and triangles--abstract entities which stand in some regular relationship to actual physical shapes. Algebra equations refer to functional relationships between numbers and between the quantities or events to which numbers can be reliably mapped. These *referential meanings* of mathematical statements are what allow mathematics to be used.

Accepting as the reference of mathematical statements only abstract (but still *real*) entities such as numbers, points, and lines would permit alternatives of interpretation for mathematical statements, but their number would be limited. We encounter an explosion of interpretations, however, when we include as potential referents for mathematical statements the actual things in the world to which abstract mathematical entities can be reliably mapped--what we might term "situations that we can mathematize." If we are willing to treat mathematizable situations as in some sense the potential referents of mathematical statements, we must take seriously this explosion of possible interpretations. In other words, we must recognize that there is no single meaning for a mathematical expression and no single reason the relationships it expresses are true.

We usually regard mathematical problem-solving, or at least the part of it that treats real-world problems, as a process of building a mathematical interpretation of a situation and then using a formal, fully determined system to manipulate relationships that have been "mathematized" by this interpretation. Our ability to do this depends upon treating the mathematizable situation as part of the potential referential meaning of mathematical expressions and doing so opens mathematics to interpretation and meaning construction of the kind that is an inherent part of all language use. Children initially learn mathematics by interpreting mathematical symbols in terms of situations about which they already know certain defining relations. And people who become good mathematics learners continue for some time to build justifications for mathematical statements and algorithmic rules that are couched in terms other than mathematical proof (cf. Resnick, 1986). They do this even though their teachers do not demand it, in fact may even discourage it because it seems to be a form of thinking too imprecise for mathematics. Could all children be taught to regularly think about mathematics in this justificatory, "ill-structured" way? What would be the effects?

The Nature of Meaning Construction for Mathematical Language

If mathematical expressions refer to real things, both abstract and physical, not just to a system of other mathematical expressions, it makes sense to think of them as a language. In this case our knowledge of natural language understanding can guide our thinking about mathematical language understanding. People do not understand natural language statements by simply registering the words. Instead they use a combination of what is said, what they already know, and various inference processes to construct a plausible mental representation of what the statement refers to. This representation omits material that does not seem central to the message; it also *adds* information needed to make the message coherent and sensible. The process of understanding natural language is an inferential, meaning *construction* process. So is the process of understanding mathematical language. As already noted, there are two domains of reference for mathematical language: the world of abstract mathematical entities and the world of mathematizable situations. The next two sections explore ways in which children might come to treat these two domains as spaces in which to construct plausible mental representations of the meaning of mathematical statements.

Mathematical Entities as the Referents for Mathematical Language

Constructing representations that involve mathematical entities can be difficult, because the mathematical entities are themselves abstract mental constructions. Furthermore, informal discussions about mathematical entities can be difficult, because, other than the mathematics symbols themselves, we do not usually develop vocabularies for discussing mathematics. Nevertheless, children sometimes can describe the meaning of formal statements and expressions in ways that show they think of the symbols as referring to these abstractions. Here is an example of a seven-and-a-half-year-old explaining why $2 x 3$ and $3 x 2$ both equal 6:

> What's two times three? **Six.** How did you get that? **Well, two threes...one three is three; one more equals six.** Okay, what's three times two? **Six.** Anything interesting about that? **They each equal six and they're different numbers....I'll tell you why that happens.... Two has more ways; well it has more adds...like two has more twos, but it's a lower number. Three has less threes but it's a higher number....** All right, when you multiply three times two, how many adds are there? **Three...and in the other one there's two. But the two--that's two threes--but the other one is three twos, 'cause twos are littler than threes but two has more...more adds, and then the three has less adds but it's a higher number.**

In the following example, the same child explains his strategy for making the quantity, *64*, from various combinations of numbers. Asked how *23 + 41* (written vertically on paper) could be rewritten but still equal 64, he first wrote *24 + 40* and then continued:

> **I'm going one less than 40 and this one more...25 plus 39.** Tell me what you're doing now to get that. **I'm just having one go lower; take one away and put it on the other....I'm taking the 3** [from 23] **away and making that 2 and putting it on the 41 to make it 42. Like that, I was going lower, lower, higher, higher.** Okay, you gave me three examples of how you could change the numbers. Now why do all those numbers equal the same amount? **Because this is taking some away from one number and putting on the other number.** And that's okay to do? **Yes.** Why is that okay to do? **Why not?** Well, can you give me a reason? **No, anyone can do that...Because you still have the same amount. You're keeping that but putting that on something else....You're not just taking it away.**

These examples reveal an awkwardness of expression characteristic of children discussing mathematical ideas. The child struggles to find--even invent--words to express his knowledge about the mathematical objects to which the formal statements refer. In part this reflects his lack of practice in *talking about* these mathematical entities. His knowledge of them is largely implicit, expressed more in the variety of things he can *do* with numbers than in a developed ability to talk about them (cf. Gelman and Greeno, in press).

Other examples of children talking about the mathematical objects to which formal statements refer appear in Magdalene Lampert's (1986) descriptions of her fourth grade mathematics classes. In those classes, children *argue* about the meaning of mathematical expressions, attempting to convince each other (and Lampert) that various arithmetic algorithms they invent are correct. This is an important ingredient of Lampert's classes. The children are not just *doing* mathematics. They are *discussing* mathematics, arguing about it, disagreeing about it. In short they are, within some important limits, treating mathematics as an ill-structured discipline in which multiple points of view are legitimate and proposals must be justified, not so much on the basis of their being correct as of their being sensible.

Situations as Referents for Mathematical Statements

Even more room for argument and discussion exists in the relationship between mathematical language and real-world situations, because the same mathematical expressions can refer to different situations. A simple subtraction sentence ($5 - 3 = 2$, for example) illustrates the three classes of situations that have become familiar to us from research on early story problems (e.g., Riley & Greeno, in press): *Change* situations in which a starting quantity (5) is modified by removing a certain quantity (3) from it; *Combine* (and *decomposition*) situations in which a whole (5) is broken into two parts (3 and 2); *Compare* situations in which two quantities (5 and 3) are compared and their difference found. The numbers in the expression refer to different kinds of entities in each of the three situations. Decomposition situations require only cardinal numbers. That is, the 5, the 3, and the 2 in a decomposition story problem all derive from measurement or counting operations. In change situations, the 5 and 2 are both measures (cardinals), but the 3 describes an operator, i.e., a number that transforms other numbers. And in the comparison situation, the 5 and the 3 are both cardinals, but the 2 refers to a third kind of number, one that represents a difference, a relationship between measures.

We have conducted several studies examining students' abilities to interpret simple algebra and arithmetic expressions in terms of situations. In the first of these studies

(Resnick, Cauzinille-Marmeche, & Mathieu, 1987), we asked French middle school children to make up stories that could be represented by expressions such as 17-11-4 or its equivalent, 17-(11+4). Initially we were interested in whether we could use children's knowledge of the relationship between stories and expressions to help them understand the reasons for symbolic algebra rules such as the "sign-change rule." We never reached the algebra goal, however, since many of the children were not able to reliably relate arithmetic expressions to stories. The most interesting aspect of the data for the current context of discussion is that many children were reluctant to treat the numbers in the written expression as anything other than expressions of cardinals. Some of the youngest children (11-12 years old) could not construct a simple story in which a child went out to play with 17 marbles in his pocket and lost 11, then 4 of them, in two successive games. For example, they told stories in which 6 marbles were lost, so that the child could *have* 11 marbles after the first game. Older children could generally construct the two-step, marble-losing story, but many did not believe that they could combine the two operator numbers (11 and 4) to determine how many marbles had been lost in all. This difficulty, which echoes earlier findings by Vergnaud (1982, 1983) and Escarabajal, Kayser, Nguyen-xuan, Poitrenaud, & Richard (1984), suggests that by middle school the children had not reliably constructed interpretations of numbers as referring to anything other than cardinalities.

Our more recent studies (Putnam, Lesgold, Resnick, & Sterrett, 1987) attempt to confirm these surprising findings and extend the work on relationships between expressions and story situations to multiplicative as well as additive numbers. We interviewed 28 students from each of grades 5, 7, and 9. Both studies included three interview phases. In Phase 1 the child read several sets of three story situations. In each set, two storie ould be represented by equivalent formal expressions, but the expressions were not presented. The child was asked to decide which stories were equivalent and why, without actually solving the problems. Phase 2 assessed the child's knowledge of the equivalence of the formal expressions. The child was asked to judge as equal or not equal, pairs of expressions representing correct and incorrect transformations and then to justify each judgment. Phase 3 examined the child's ability to link the story situations with the formal expressions. Given a story situation and a set of three expressions, the child was asked to choose the expression best describing the situation and justify that choice. For the other expressions, the child was asked to modify the situation to fit the expression.

In both studies, the students successfully judged the equivalence of the story situations and justified the equivalences (Phase 1). Students were very poor, however, at judging the equivalence of the formal expression pairs (Phase 2). When explicitly asked (in Phase 3) to link stories with expressions, students could often do so, even when they had not spontaneously used stories to help them reason in Phase 2. This pattern of results suggests that many students have relevant informal knowledge that they do not normally draw upon in thinking about formal expressions, although difficulties like those in the earlier Fr nch study persisted. For these students, instruction focused on the task of interpreting mathematical expressic as mathematizations of possible real-world situations seems essential to their development as mathematical problem-solvers. In such instruction, as in practice on more typical problem solving in which students are given situations to interpret mathematically, the key is learning to identify the mathematical entities that map to elements in the situation. In all such instruction, both the processes of meaning construction and the relevant situational and mathematical knowledge should be the focus of attention.

Socializing Mathematics Learning

Several lines of cognitive theory and research point toward the hypothesis that we develop habits and skills of interpretation and meaning construction through a process more usefully conceived of as *socialization* than *instruction*. Psychologists use the term *socialization* to refer to the long-term process by which personal habits and traits are shaped through participation in social interactions with particular demand and reward characteristics. Theorists such as George Herbert Mead (1934) and Lev Vygotsky (1978) have proposed that thought is an internalization of initially social processes. Mead refers to thinking as *conversation with the generalized other*. Vygotksy describes learning as a process in which the child gradually takes on characteristics of adult thought as a result of carrying out activities in many situations in which an adult constrains meaning and action possibilities.

A small but growing number of psychologists, anthropologists, linguists, and sociologists have begun to study the nature of cognition as a social phenomenon. (See Resnick, 1987,c, for a review and interpretation of some of this research.) In education, the best developed line of work on socialized learning is in the field of reading. Palincsar and Brown (1984), broadly following a Vygotskian analysis of the development of thinking, propose that extended practice in *communally* constructing meanings for texts will eventually produce an internalization of the meaning construction processes in each individual. They used a highly organized small-group teaching situation in which children took turns playing the teacher's role by posing questions about texts, summarizing them, offering clarifications, and making predictions. These four activities are thought to induce the kinds of self-monitoring of comprehension characteristic of good readers. The adult's role in these reciprocal teaching sessions, although informal in style, is highly structured. In addition to facilitating the general process, the adult is expected to model problem-solving processes and provide careful reinforcement for successively better approximations of good self-monitoring behaviors on the part of the children.

Using a social setting to practice problem solving is a method shared by other investigators, at least some in the field of mathematics learning. (See Resnick, 1987,b, and Collins, Brown, and Newman, in press, for a more general review.) I have already mentioned the work of Magdalene Lampert, who conducts full-class discussions in which children invent and justify solutions to mathematical problems. Lampert's discussions, like those in reciprocal teaching, are carefully orchestrated by the teacher and include considerable modeling of interpretive problem solving by the teacher. Schoenfeld's (1985) work with college students shares many features of the Lampert class lessons. In Schoenfeld's problem-solving sessions, groups of students work together to solve mathematics problems. The instructor works with them, often stepping in when students reach an impasse to restart the problem-solving process. The instructor's special role is to "think aloud" while solving problems, thereby modeling for students heuristic processes usually carried out privately, hidden from view. To facilitate this modeling, students sometimes generate problems for the instructor, and the instructor occasionally pretends more puzzlement than actually experienced in order to show how several candidate solutions may be developed and evaluated. In contrast, Lesh's (1982, 1985) problem-solving sessions share reciprocal teaching's small-group format for collaborative problem solving but have no teacher present. This means that Lesh's problem-solving groups benefit from children's debate and mutual critiquing, but children do not have the opportunity to observe expert models engaging in the process and are not taught any specific techniques for problem analysis or solution.

Another line of work, this one rooted in a convergence of Piagetian and European social psychology theory, offers further support for the idea that collaborative problem-solving experience ought to promote general cognitive development. The research of Genevan social psychologists (Mugney, Perret-Clermont, & Doise, 1981) has shown that peer discussion of certain classical Piagetian problems (e.g., conservation) can improve performance *even when both discussants begin at the same low level.* The importance of this finding is that it eliminates the possibility that a more advanced child simply taught a new response to a more backward child. Instead something in the conflict of opinions apparently sets constructive learning processes in motion (cf. Murray, 1983).

Socially shared problem solving, then, apparently sets up several conditions that may be important in developing problem-solving skill. One function of the social setting is that it provides occasions for *modeling* effective thinking strategies. Thinkers with more skill--often the instructor, but sometimes more advanced fellow students--can demonstrate desirable ways of attacking problems, analyzing texts, constructing arguments. This process opens to inspection mental activities that are normally hidden. Observing others, the student can become aware of mental processes that might otherwise remain entirely implicit. When Palincsar and Brown compared modeling alone with modeling embedded in the full reciprocal teaching situation, however, modeling alone did not produce very powerful results. Thus there is more to the group process than just the opportunity to watch others perform.

Something about *performing* in social settings seems to be crucial to acquiring problem-solving habits and skills. "Thinking aloud" in a social setting makes it possible for others--peers or an instructor--to *critique and shape* a person's performance, something that cannot be done effectively when only the results, but not the process, of thought are visible. It also seems likely that the social setting provides a kind of *scaffolding* (Wood, Bruner, & Ross, 1976) for an individual learner's initially limited performance. Instead of practicing bits of thinking in isolation so that the significance of each bit is not visible, a group solves a problem, writes a composition, or analyzes an argument together. In this process, extreme novices can participate in solving a problem that would be beyond their individual capacities. If the process goes well, the novices can eventually take over all or most of the work themselves, with a developed appreciation of how individual elements in the process contribute to the whole.

Yet another function of the social setting for practicing thinking skills may be what many would call *motivational*. Encouraged to try new, more active approaches, and given social support even for partially successful efforts, students come to think of themselves as capable of engaging in interpretation. The public setting also lends social status and validation to what may best be called the *disposition* to meaning construction activities. Here the term *disposition* does not denote a biological or inherited trait, but rather a *habit* of thought, one that can be learned and, therefore, taught. Thus, it seems possible that engaging in problem solving with others may teach students that they have the ability, the permission, and even the obligation to engage in a kind of independent interpretation that does not automatically accept problem formulations as presented (cf. Resnick, 1987,b).

There is good reason to believe that a central aspect of developing problem-solving abilities in students is a matter of shaping this disposition to meaning construction. There is surprisingly little research linking cognitive skills and disposition to use them. On the whole, research on cognitive ability has proceeded separately from research on social and personality development, and only the latter has attended to questions of how dispositions--often labeled *traits* in the social and personality research literature--develop or can be modified. Some recent work takes important steps toward creating links between the quality of thinking and dispositions. For example, Dweck (in press; Dweck & Elliot, 1983) proposes that individuals differ fundamentally in their conceptions of intelligence and that these conceptions mediate very different ways of attacking problems. She distinguishes between two competing conceptions of ability or "theories of intelligence" that children may hold. One, called the *entity* conception, treats ability as a global, stable quality. The second, called the *incremental* conception, treats ability as a repertoire of skills that can be expanded through efforts to learn. Entity conceptions orient children toward performing well so that they can display their intelligence and toward not revealing lack of ability by giving "wrong" responses. Incremental conceptions orient children toward learning goals, seeking to acquire new knowledge or skill, mastering and understanding something new. Most relevant to the present argument, incremental conceptions of ability and associated learning goals lead children to analyze tasks and formulate strategies for overcoming difficulties. We can easily recognize these as close cousins of the interpretive, meaning construction activities discussed here. Such analyses suggest that participation in socially shared problem solving should, under certain circumstances, produce dispositional as well as cognitive ability changes.

Collaborative Problem Solving

We have begun a line of research that attempts to adapt the principles of reciprocal teaching, as developed by Palincsar and Brown (1984) to teaching mathematics problem solving. Using the reciprocal teaching procedure for mathematics problem solving is not the straightforward process it might initially seem, primarily because mathematics problem solving is more strictly knowledge-dependent than reading. Part of what makes reciprocal teaching work smoothly in reading is that the same limited set of activities (summarizing, questioning, predicting, clarifying) is carried out over and over again. Finding repeatable activities of this kind in mathematics is not easy. Polya-like heuristics (Polya, 1973), as Schoenfeld (19xx) points out, are so general they provide little guidance for people who are not already good at solving mathematics problems. In reading activities, furthermore, children are rarely totally *wrong* but are more likely to be just weak--that is, while responses may not enhance comprehension very much, they do not turn it off course, either. In mathematics problem solving, however, children frequently come up with incorrect formulations that do actively interfere with problem solving. Another difficulty lies in our inability to calibrate mathematics problems to increase the value of practice and assessment, because nothing equivalent to readability or grade-level difficulty allows us to group problems according to difficulty, as we do texts for reading.

We have, then, three main problems to solve in adapting the reciprocal teaching strategy to mathematics problem solving. First, we need to find a set of repeatable (thus general) yet adequately constraining (thus specific) activities that children can use and develop over many practice problems. Second, we need to find an appropriate balance between attention to general problem-solving strategies and processes and attention to the specific mathematical knowledge required for problem solving. Third, we need to find ways of grouping and calibrating problems for instructional and assessment purposes. My colleagues and students and I have been working on these problems in the context of a series of exploratory studies discussed briefly below. Although none of the work is definitive at this time, it is helping us to refine questions and develop research methods that will allow us to move into the relatively uncharted waters of research lying at the intersection of social and cognitive processes.

Knowledge-dependence of Mathematical Problem Solving

In a series of four sessions, we asked a group of five fifth-grade children to solve collaboratively word problems involving some aspect of rational numbers, with children alternating as discussion leader. Sessions were tape recorded, with full transcriptions prepared. Study of the protocols revealed that two fundamental problems must be resolved if we are to adapt the principles of reciprocal teaching to mathematics. Both problems are rooted in the fact that mathematics problem solving is more strictly knowledge-dependent than reading.

First, in our problem-solving sessions, children frequently foundered because they lacked knowledge of relevant mathematical content, despite our efforts to match session content to what children were studying in their regular mathematics classes. Insecure basic mathematical knowledge at times dramatically blocked successful problem solving in our group. In one such instance the children drew a "pizza" and divided it into six parts, each called "a sixth"; they shaded three of those parts and then asserted that each shaded part was "a third." In such situations, the adult must either interrupt problem-solving processes to teach basic mathematics concepts or allow the children to continue with fundamental errors of interpretation. Neither choice seems likely to foster the proper development of appropriate meaning construction abilities.

Second, since part of what makes reciprocal teaching work smoothly in reading is the repetition of the same limited set of activities (summarizing, questioning, predicting, clarifying) and since it is not as easy to find such activities in mathematics, we had the adult leader introduce and repeat some very general questions, e.g., "What is the question we are working on?" "Would a diagram help?" "Does that [answer] make sense?" and 'What other problem is like this one?" As is also often the case for more mathematically sophisticated Polya-like heuristics, however, these questions appeared to be too general to adequately constrain the children's efforts. They did not know *what* diagram to draw, for example, or they drew it incorrectly, or could not decide if an answer was sensible because they had misunderstood basic concepts.

Using Strategies Versus Talking About Them

In a second effort, we attempted to respond to each of these problems in a systematic way. The children involved were fourth graders who were asked to work in a group of 5 for 13 sessions, each led by the same adult. To control for children's lack of specific relevant mathematical knowledge, we used problems that invoked concepts from the previous year of mathematics instruction rather than the current year. With this control for unmastered mathematical content, we encountered very few occasions in which fundamental mathematical errors or lack of knowledge impeded problem solving.

Based on cognitive theories of problem solving, we identified four key processes that should be repeated in each new problem-solving attempt: (a) planning--i.e., analyzing the problem to determine appropriate procedures; (b) organizing the steps for a chosen procedure; (c) carrying out those steps; and (d) monitoring each of the above processes to detect errors of sense and procedure. For each problem to be solved, these functions were assigned to four different children. The Planner was charged with leading a discussion of the problem in order to set forth applicable strategies and procedures. Once the group chose a procedure, the Director's task was to state explicitly the steps in the procedure. These steps were then to be carried out by the Doer at a chalkboard visible to all. The Critic's role was to intervene if any unreasonable plan or error in procedure was detected.

The tactic of dividing mental problem-solving processes into overt social roles was not initially successful. Although our research community has specific meaning for such terms as *planning, directing,* and *critiquing/monitoring,* with the exception of the Doer role, these meanings were not conveyed to children by the labels, and we were not successful in verbally explaining them to the children. As a result, the roles became instruments for controlling turn-taking and certain other social aspects of the sessions, but did not give substantive direction to problem solving. And while children discussed the roles a great deal, they did not become adept at performing them. This points to a fundamental problem with certain metacognitive training efforts that focus attention on knowledge *about* problem solving rather than on guided and constrained practice in *doing* problem solving. Such efforts may be more likely to produce ability to talk about processes and functions than to perform them.

In session 6, we attempted a modification of the Critic role in order to deal with this problem. The Critic's function was now shared by two children, who received cue cards to use in communicating their criticisms. The cue cards read:

1. Why should we do that? [request for justification of a procedure]

2. Are you sure we should be adding (subtracting, multiplying, dividing)? [request for justification of a particular calculation]

3. What are we trying to do right now? [request for clarification of a goal]

4. What do the numbers mean? [insistence that attention focus on meanings rather than calculation and symbol manipulation]

The cue cards served to scaffold the Critic function by limiting the possible critiques and providing language for them. At first the children used the cue cards almost randomly and rather intrusively. During the seven succeeding sessions, however, children's use of the cue cards became increasingly refined, i.e., used on appropriate occasions and in ways that enhanced rather than disrupted the group's work. Nevertheless, at the end of 13 sessions there was no strong evidence that the overall level of problem-solving activity had improved substantially. It seemed appropriate, therefore, to turn away from this global approach to collaborative problem solving and try to develop more targeted forms of scaffolded problem-solving experience.

Forms of Scaffolding

A continuing problem in this research has been finding adequately controlled methods of study. Just recording and transcribing conversations among five children is a daunting task. Finding systematic ways to analyze these conversations that enable us to go beyond the anecdotal without losing their essential character in the service of quantification is even more challenging. One way of solving these problems is to conduct an interrelated series of studies, each designed to answer particular questions. With such a strategy, the program of research as a whole instead of any one study should produce strong conclusions. We are now conducting several studies in this spirit, each aimed at exploring how specific aspects of scaffolded problem-solving practice, together with discussion and argument, may shape the dispositions and skills of problem solving.

An initial study examined pairs of children solving problems particularly suited to classical means-ends strategies (cf. Newell & Simon, 1972). The limitation to pairs of children allowed us to record all conversation and develop a way of analyzing the conversation that captured key aspects of both the problem-solving structure and the social interaction. The problems given to the children, highly structured arithmetic story problems, were suitable for developing our method of analysis, although they limited the range of discussion and interpretation we would eventually like to develop in problem-solving groups. For the problems used, we were able to determine the probable paths of solution and points of difficulty in advance, and this guided our data analysis.

Participants in the study were 12 pairs of children, 3 pairs each from grades 4, 5, 6, and 7. Each pair of children met 3 times for 40 minutes and solved from 2 to 6 problems. To scaffold the means-ends problem-solving strategy, children were given a Planning Board on which to work. The board provides spaces for recording what is known (either stated in the problem or generated by the children) and what knowledge is needed (goals and subgoals of the problem). Using the board, children can work both "bottom-up" (generating "what we know" entries) and "top-down" (generating "what we need to know" entries). A space at the bottom is provided for calculation. Figure 1 shows a Planning Board with some typical entries. Each child writes on the board with a different color pen to facilitate tracking the social exchange.

Insert Figure 1 about here

At each grade level, one pair of children was assigned to each of the following three conditions:

1. Planning Board with Maximum Instruction. The children solved problems using the Planning Board. The adult demonstrated use of the Planning Board during the first session and then participated in the first two sessions by providing hints and prompts to further scaffold the problem-solving process and increase use of the board.

2. Planning Board with Minimum Instruction. The children solved problems using the Planning Board. The adult demonstrated use of the board and provided hints and prompts during the first session only.

3. Control. The children solved problems without the Planning Board during all three sessions.

Since this was a pilot study, we have results on only a few problems and can draw no strong conclusions about processes of interaction under the three conditions of scaffolding. It is possible, however, to use these initial data to demonstrate some possibilities for detailed analysis of collaborative problem solving. In the following paragraphs we examine data on the final problem of the third session in the study--a problem that all dyads worked without adult intervention.

Efficiency in using the board. We rated efficiency in use of the Planning Board on a three-point scale reflecting the extent to which the children reduced the language in the problem statement to a more succinct form. Table 1 shows the results of this coding for the children who used the Planning Board with minimum and maximum instruction. As can be seen, fourth graders, regardless of amount of instruction, and fifth graders with minimum instruction mostly recopied the words of the problem onto the

Planning Board. In contrast, seventh graders with maximum training tended to reduce the information to a symbolic form that could be entered directly into calculations. Not surprisingly, older children used the planning board more efficiently. It is encouraging that even the very brief training and practice in this study seemed to produce more efficient use of the board among all but the youngest children. With more extended practice all children could conceivably become very efficient--in the limited sense thus far considered--at using the board.

Insert Table 1 about here

The board as a scaffold for goal analysis. The Planning Board was not meant simply as a recording device for students, however. It was intended to prompt and support them in identifying goals and subgoals and clarifying the relationship of the given information to these goals. We wanted to examine the extent to which children in the different groups carried out such problem analysis. We began by delineating the goal-given structure of a problem in "expert" terms. Figure 2 shows both this structure and the problem statement given to the children. Although apparently simple, this problem is deceptive. The basic structure of the problem requires *adding* the amount of money spent to the amount left at the end of the day to determine the money in hand at the beginning of the day. In basic structure the problem is a Riley/Greeno (in press) Change-3 problem, one of the most difficult additive story problems. The problem can be stated algebraically as $? - b = c$. But it is complicated by the fact that Mark also *gains* money during his outing, and this money must be *subtracted* from the amount spent before the amount spent is *added* to the amount left at the end of the day. The algebraic structure thus becomes $? - (b - d) = c$. Our research on children's ability to connect additive story problems with arithmetic statements (Putnam, Lesgold, Resnick,

& Sterrett, 1987) has shown this composition-of-transformation type problem to be among the most difficult for children to interpret. Thus this problem provides extensive complex material for fourth through seventh graders.

Insert Figure 2 about here

In the goal analysis shown in Figure 2, direct statements from the problem are shown without brackets; statements in brackets contain information the problem solver must generate (either implicitly or explicitly) to solve the problem. G_0 represents the "top goal" which, when reached, means that the whole problem has been solved. To meet this goal it is necessary to find out how much money Mark spent (G_1), how much he received in the course of the day (G_2), and how much he had left at the end of the day (G_3). G_1 has several subgoals (G_{11}, G_{13}, G_{14}), each asking for a specific portion of the day's expenditure. Information provided to help satisfy goals is coded as V (for givens); the subscripts specify the relevant goal or subgoal for each piece of information.

With this analysis in hand, it was possible to use the typed protocol of each dyad's problem-solving session to construct a record similar to a problem behavior graph (cf. Ericsson & Simon, 1984) showing the sequence in which the dyad generated and used the goals and givens of the problem. In our working records, we enter statements made by each subject in the same color as the subject's pen. This color coding enables us later to recapture information about the social aspects of the joint problem solving.

From the problem behavior graphs, different kinds of information can be extracted. For example, we examined the extent to which each of the dyads in the study noted the necessary information (goals and givens) in the problem statement and the extent to which they generated explicitly (in writing or orally) the necessary subgoals. For the problem under consideration here, all dyads specified at least 7 of the 9 pieces of given information, exhibiting no differences due to grade level or treatment. This high level of noting the given information was common to all problems studied. The children were not so good, however, at generating goals. As shown in Figure 2, six subgoals could be specified for this problem. Table 2 shows how many of these each dyad specified. As can be seen, no dyad was complete in its goal specification. It must be noted, however, that it is possible to properly use the information in V_{111}, V_{112}, V_{131}, V_{132}, V_{141}, and V_{142} without specifying the intermediate subgoals, i.e., without explicitly asking how much was spent on the subcategories of bus rides, puzzle books, and lunch. Thus it is only absolutely necessary to specify three subgoals (G_1, G_2, and G_3) in order to solve the problem. Both the Planning Board and the level of training apparently affected the number of goals specified. Groups with the board and with maximum training on it generated a much higher proportion of the goals.

Insert Table 2 about here

Solution structures. Despite the greater proportion of goal generation, no dyad gave exactly correct responses to the Saturday Shopping problem. Our goal analysis and algebraic analysis of the problem allow us to examine the points of difficulty and nature of errors for each group. Table 3 shows a structural analysis of the responses to this problem and characterizes each dyad's responses.

Insert Table 3 about here

We have described the problem as being of the form $? - (b - d) = c$, where $b =$ the amount spent, $d =$ the amount received along the way, and $c =$ the amount left at the end of the day. Since the amount left at the end of the day is directly stated in the story, the problem can be rerepresented as $? - (b - d) = \$4.43$. Correct solution requires adding the quantity $(b - d)$ to $4.43. We look first at which dyads correctly *added* the result of their calculations on spending and receiving to the $4.43, despite the minus sign in front of this quantity. All responses above the midline in Table 3 were of this kind; those below the midline subtracted rather than added or carried out no arithmetic linking b and d to 4.43. Table 4 shows the data organized by grade and treatment. While we cannot determine statistical significance, it appears that the Planning Board with maximum instruction may have had some effect in supporting students' analyses of the problem at this level.

Insert Table 4 about here

We can also examine the extent to which students understood that the amounts received during the day (d) reduced the amount spent. They could show this understanding either by subtracting d from b before adding the result to 4.43 (following the form we have given: $? - (b - d) = 4.43$) or by converting the problem to the equivalent form $(? - b + c = 4.43)$, which requires adding b to 4.43 and subtracting d. No dyad did this correctly by either method. The arguments and elaborations children provided for each other, however, showed that for several of the dyads the question of what to do with the d quantities constituted a major point of discussion. Thus most of the dyads located the difficult aspect of the problem but could not resolve their questions successfully.

Social collaboration. We have been considering the problem-solving processes of the dyad as those of an individual. Now we turn to the examination of social interaction between the pairs of children, unpacking their performance according to which child engaged in various parts of the shared problem-solving process. This is possible because our sequential coding of each dyad's problem-solving process retained information about which child was responsible for each statement or written operation.

A question often raised in considering the effects of discussion and interaction on learning concerns whether children adopt highly specialized roles in such interactions and thus fail to practice all of the different activities that comprise successful problem solving. The result of consistent, long-term specialization in these situations might be that, while the group becomes effective at problem solving, individual members (or at least some individual members) could not function independently. According to the theory of scaffolded learning, early in their learning individuals ought to succeed in jointly solving problems that they cannot solve individually, but eventually they should take on more components of problem solution and function independently. Presumably then, in successfully scaffolded learning, specialization would be apparent during early phases but less so later.

Our present data do not follow dyads over long enough periods to track changes in specialization, but we can show how our analyses would permit us to determine such specialization. Table 5a (columns 1-3) shows overall utterances by the two children in each dyad and the number of utterances by each child (H for the child with the higher math achievement score, L for the child with the lower math achievement score). The amount of overt activity by the dyads varies considerably, with the total number of utterances ranging from 5 (dyad 7 Con) to 106 (dyad 6 Min). We examined the percentage of talk by the dominant member of each dyad, but for this problem no clear pattern seemed attributable to either grade level or treatment condition. Table 5b shows that there was somewhat more total talk by the two Planning Board groups, especially by the more intensively trained group.

Insert Tables 5a and b about here

The remaining columns of Table 5a break down the interaction more specifically according to who states the givens in the problems (columns 4-5), who makes the goal statements (columns 6-7), and who does calculations (columns 8-9). There were wide variations among the dyads, but again no clear pattern seemed attributable to grade level or treatment condition. Given more problems, more stable assessments for each dyad might be made, and some pattern might emerge. In addition, studies following subjects over many sessions of shared problem solving could track changes in degree of specialization.

We are interested not only in role specialization, but also in the quality of the shared interaction over problem solving. One way to examine this is to identify statements by one member of the dyad showing some level of direct response to the other's problem-solving effort. These direct responses can then be contrasted with simple division of labor (such as we examined in the last paragraph) and purely parallel work, where each student solves the problem separately, despite the shared work space. In one analysis, we identified three kinds of direct response--repetition, argument, and elaboration. Of the three, arguments and elaborations correspond roughly to the two main ways in which socially shared problem solving is thought to facilitate learning--peer conflict and peer scaffolding.

Only a small minority of all utterances made while working on the problem were coded as arguments or elaborations. This is characteristic of other data we have examined and represents one of the difficulties of both studying shared intellectual activity and using it as a pedagogical method. There is a very low density of the kind of activity we believe is most instrumental in producing learning. Shared problem solving looks inefficient by usual pedagogical standards. These children, for example, could very probably work on more problems in a similar amount of time if they worked alone. Shared problem solving also requires tremendous patience from the researcher who would study it. Not only are the data harder to collect and transcribe and the sessions longer, but also the density of "interesting" events--events probably worth detailed scrutiny and qualitative analysis--is very low. Nevertheless much can be learned from such scrutiny. Important to our research agenda will be establishing what patterns of challenge and elaboration exist and how givers and receivers of challenges and elaborations benefit from the exchange.

Issues for Further Investigation

In this chapter I have suggested a broad point of view on the nature of mathematical problem-solving and illustrated some of the research questions and strategies such a point of view generates. To explore more fully the possibilities for and the difficulties of teaching mathematics as an ill-structured discipline, we need continuing research of several kinds. My concluding statements identify some of the questions requiring additional research.

Natural Language and Mathematical Language

At the heart of the suggestion that we teach mathematics as an ill-structured discipline lies the proposal that *talk* about mathematical ideas should become a much more central part of students' mathematics experience than it now is. This will inevitably entail greater use of ordinary language, rather than the specialized language and notation of mathematics, in mathematics classrooms. The ways in which ordinary language expresses mathematical ideas have been little studied. We know that under current teaching conditions students have little opportunity to develop a vocabulary that expresses their implicit knowledge of mathematical concepts. In what ordinary language terms can mathematical ideas be discussed? What complications can we expect as we begin to talk more with students about mathematics?

Recent work by Kintsch and Greeno (1985) uses the rich body of cognitive theory about how people interpret and make sense of texts to explain how children understand arithmetic story problems. Kintsch's work shows how children at different levels of linguistic and mathematical competence use story problem texts to construct representations of quantities and their relationships. This work represents a valuable first joining of cognitive research on mathematics problems and on language understanding. It focuses on a narrow band of problems, however, and on textual forms that are so stereotyped as to function almost as quasi-formalisms. To solve these problems, students must learn to interpret the special linguistic code in which story problems are expressed. But this kind of special code is unlikely to be the vehicle for active discussions of mathematical relationships or concepts.

When students themselves generate linguistic expressions of mathematical arguments, we move closer to natural language discussion of mathematics. A recent book by Eleanor Wilson Orr (1987), a teacher who has for many years required students studying algebra and geometry to develop informal, natural language justifications for problems they work, describes some of the difficulties that may be encountered in such a program. Orr's book documents the ways in which some students' vernacular language may fail to encode precisely key mathematical relationships. These relationships include distinctions between distances and locations, between directions of movement, and between quantities and differences among quantities. Orr is concerned that some vernaculars--especially Black English Vernacular (BEV)--may be particularly poor at expressing these mathematical relationships. Researchers interested in mathematics problem solving should not stop with this admittedly controversial issue, however, for Orr raises a much more general and fundamental question about the relationship between natural language and mathematical thought. The successes and difficulties of a mathematics teaching program that has been grounded in natural language expression will suggest many new questions for systematic investigation.

Social Engineering

How can we profitably organize collaborative problem solving and other forms of mathematical talk and discussion, given typical teacher-student ratios of one adult to 25 or 30 students? An apparently simple answer--having groups of students work problems independently of the teacher--seems unlikely to prove successful as the sole or even the major mode in which students talk about mathematical ideas. There does not yet exist a body of research that examines patterns of students' activity over extended periods of collaborative work. It seems likely, however, that left to themselves students will often fail to generate productive ideas and may allow one or two strong students to do the group's work rather than supporting everyone's learning needs. Most cooperative learning efforts have found it necessary to carefully engineer ways of posing questions, grouping students, and providing incentives in order to develop productive patterns of cooperative intellectual behavior. Slavin (1983), for example, has developed a successful system of cooperative team learning. Children are organized into heterogeneous ability teams that study together and coach each other; individuals then compete against others of about the same ability from other teams to earn points for their study teams. The homogeneous ability competitions allow even the weaker students to earn points for their teams and thereby motivate the study teams to help each student learn. This system works well for highly structured learning tasks where students can easily tutor one another through drill and rehearsal. Might it also work for much more ill-structured problem solving, where what is to be learned is a general approach and a set of heuristics that cannot be so easily specified and rehearsed? We do not know. We must find out.

Edward A. Silver (personal communication, June 1987) has suggested another form of classroom social engineering that seems promising and particularly well suited to open-ended problem solving. In Silver's plan, a problem is posed to an entire class. Students initially try to work it alone. Then they work in pairs; then pairs are joined to make quartets of students who compare, share, and rework ideas. A whole class discussion of the problem can next be used to merge and rework the quartets' ideas, but the final solutions are left to quartets, pairs, and, eventually, individual students. This organization forces individual students to formulate initial ideas but uses successively larger groups as vehicles for confrontation and enlargement of ideas. The whole class discussion allows the teacher to help students organize ideas, suggest new approaches, raise questions, and otherwise orchestrate and nurture the problem solving of groups and individuals. Effective use of this and similar schemes will require detailed investigation of the nature of student interactions in various work phases and of the outcomes of extended participation in such activities for children of different characteristics.

Integrating Strategy and Content in Problem Solving

Future research must also concentrate on how to integrate teaching fundamental mathematical concepts with teaching problem solving. As indicated earlier, if students do not know key concepts on which a solution might be based, their problem-solving efforts can go badly awry. In our research, we found that one solution was to base problem-solving sessions on concepts that had been well learned a year or two earlier. This allowed students to focus on strategies of problem analysis and interpretation, key components of problem solving. Yet long-term or exclusive reliance on such a pedagogical technique might lead students to view problem solving as a game or application that is optional for those who have learned the "real" content and not as an integral part of mathematics.

One way to avoid this difficulty and to integrate problem-solving and sense-making activities fully with the main body of students' mathematics learning is to use discussion and sense-making activities to introduce and develop basic mathematical content. Lampert has taken this approach in her classroom work. It is also a regular feature of Japanese elementary school mathematics teaching. It is typical there to base an entire class lesson on working out only one to three problems. In the course of a class period, children propose and evaluate multiple solutions under the teacher's careful guidance. In classes I have observed, the discussion was punctuated with periods in which children worked out solutions individually and recorded some of the solutions agreed upon by the group. We need research on lessons such as Lampert's and those of the Japanese, research that tells us how expert teachers direct and manage discussions, how children of differing abilities and learning characteristics participate in the discussions, and what children of various types learn from their participation. We also need experimentation and analysis that will provide sharper pedagogical guidelines for conducting lessons in this discussion/problem-solving mode, including a careful working out of the kinds of problems and concepts that can profitably be taught in this way. We need, in other words, a much more extensive base of pedagogical lore for discussion-based conceptual teaching than now exists, supported by the kind of theoretical analyses that researchers can provide.

Contextualizing Problem Solving

Another approach to integrating conceptual and problem-solving activities is to try to develop more contextualized problems for classroom use. Some of what appear to be conceptual deficiencies may derive more from children's difficulty in working in a decontextualized classroom situation than from complete lack of mathematical understanding. As Lave (this volume) has shown, people often engage in successful mathematical reasoning when working within a specific context of action and decision. In such cases, they often use mathematical knowledge that they do not--perhaps cannot--bring to bear on the kind of decontextualized problems presented in the classroom. Can ways be found to make classroom mathematics approximate the contextualized mathematics in which people engage outside school?

Traditionally story problems have been used with this intent. In such use, the stories are intended to evoke familiar situations in which mathematics might be applied.

Students are, in effect, invited to imagine themselves elsewhere and consider how the mathematics they know might be used in that situation. Considerable evidence now exists, however, that story problems do not effectively simulate the out-of-school contexts in which mathematics is used. As already noted, the language of story problems is highly specialized and functions as almost a quasi-formalism, requiring special linguistic knowledge and distinct effort on the part of the student to build a representation of the situation described. Furthermore, this representation, once built, is a stripped down and highly schematic one that does not share the material and contextual cues of the real situation.

If we are to engage students in contextualized mathematics problem solving, we must find ways to create in the classroom situations of sufficient complexity and engagement that they become mathematically engaging contexts in their own right. Several of the approaches to mathematical problem solving described in this chapter and elsewhere in this volume represent efforts in this direction. Some of Lesh's extended and not fully defined problems, for example, can be thought of not as stories containing mathematics problems, but as settings in which planning a project (e.g., wallpapering a room) engages a substantial amount of mathematical knowledge and strategy. Similarly Bransford's (this volume) proposal for using videodisc presentations can be viewed as an effort to bring complex situations into the classroom. The realism and engaging character of the filmed sequences should more fully contextualize mathematical activity than verbal presentations of the same story line would. They should also permit students to develop questions, not only solve problems posed by others.

Computerized simulation environments can also provide settings for highly contextualized mathematical activity. College students who work in simulated microeconomic environments, for example, do mathematical work in the context of the simulation world. To the extent that they engage with this world, accepting its rules and constraints, they are doing contextualized mathematics in much the same way that Lave's supermarket shoppers and weightwatcher cooks are. Computer and board games in which calculation, estimation, or other mathematical processes are required also can be thought of as contextualizing devices. Such games do not so much simulate external environments as provide fully engaging environments in their own right. Children who play computer games such as "How the West Was Won" or board games such as "Pachisi"--both games that require strategic use of number combinations--are engaging in highly contextualized arithmetic problem solving. In much the same way, students captivated by the equation/graph problems of Dugdale's (1982) "Green Globs" game engage in highly contextualized algebra problem solving. These various forms of contextualized mathematical problem solving in the classroom need further development and study.

Scaffolding Supports for Problem Solving

As discussed earlier in this chapter and elsewhere in this volume, students can often engage successfully in thinking and problem solving that is "beyond their capacities," if their activity receives adequate support either from the social context in which it is carried out or from special tools and displays that scaffold their early efforts. In current cognitive theory, scaffolding is a provocative but not fully developed idea. As introduced by Wood and Middleton (1975), the term *scaffolding* referred to the support for a child's cognitive activity provided by an adult when child and adult performed a task together. In that original use, scaffolding described a natural way that adults interact with children; little could be said prescriptively. Brown and others have expanded the meaning of the term to include the support provided by other children in joint problem-solving activity. This is a significant extension, for it implies that several individuals, none of whom are expert at a task, can nevertheless scaffold each other's inexpert performance, eventually resulting in independent performance by all individuals. This idea has not yet been rigorously tested. In the Palincsar and Brown (1984) interventions, an adult worked with each group of children. Research on groups of children working without an adult has not usually included pretests that establish the entering competence of individual members of the group. As a result, the group members may not be strictly peers; instead, some children may serve as expert scaffolders for the less expert in the group. These various forms of socially mediated scaffolding need thorough investigation if the notion of scaffolding is to move from a description of a natural phenomenon to a prescription for teaching in which details of scaffolding strategies and their conditions of application can be specified.

Our own recent work has introduced yet another extension of the scaffolding metaphor. We have proposed that record-keeping and other tools can also be viewed as scaffolds for learning. This conception, thus far tried with only the simple planning board tool, suggests that many pedagogical devices can be considered and treated as learning scaffolds. Representational devices that display an underlying theoretical structure, for example, can be treated as tools for supporting problem solving in early phases of learning. A number of *microworlds*--graphic displays that "objectify" mathematical entities such as numbers and operators (e.g., Ohlsson, 1987; Peled and Resnick, 1987)--can function as scaffolds. These displays can also support conversation between two or more individuals about mathematical ideas, thus allowing two forms of scaffolding--social and tool--to function together. In an example drawn from physics rather than mathematics, Behrend, Singer, and Roschelle (1988) have described in some detail the growth of concepts about projectile motion in two 9-year olds as the result of their joint investigation of a graphic system that represents trajectories as a function of force vectors. Scaffolding tools of these types need to be developed and explored more fully as part of a full agenda of research on mathematical problem solving.

Socializing Problem Solving: A Long-Term Agenda

As suggested earlier in this chapter, the reconceptualization of thinking and learning that is emerging from the body of recent work on the nature of cognition suggests that becoming a good mathematical problem solver--becoming a good thinker in any domain--may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies, or knowledge. If this is so, we may do well to conceive of mathematics education less as an instructional process (in the traditional sense of teaching specific, well-defined skills or items of knowledge), than as a socialization process. In this conception, people develop points of view and abilities of mathematical thought in much the same way that they develop points of view and behavior patterns associated with gender roles, ethnic and familial cultures, and other socially defined traits. When we describe the processes by which children are socialized into these cultural patterns of thought, affect, and action, we describe long-term patterns of interaction and engagement in a social environment, not a series of lessons in how to behave or what to say on particular occasions. If we want students to treat mathematics as an ill-structured discipline--making sense of it, arguing about it, and creating it, rather than doing it according to prescribed rules--we will have to socialize them as much as instruct them. This means that we cannot expect any brief or encapsulated program on problem solving to do the job. Instead we must seek the kind of long-term engagement in mathematical thinking that the concept of socialization implies. This challenge is larger than those normally confronted by any single discipline of the school curriculum. It is, however, a challenge whose time has come. The theoretical bases for a conceptualization of mathematics as a way of thinking rather than a set of skills now exist, along with an emerging body of research and theory on the links between social and cognitive aspects of thought and learning. Building on these foundations, research and development on teaching mathematics problem solving may well lead the way in stimulating important changes in educational practice.

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PLANNING BOARD WITH TYPICAL ENTRIES

FIGURE 1

What We Know	What we want to Know
He rode the bus for 35¢.	How much money did he start with? $8.37
Mack bought a race car for $1.12.	How much money did he have after he ate lunch $7.52
He found a question outside the store	How much did the pen cost?
He ate lunch he bought a sandwich for $1.25 and milk for .40¢	~~How much did his friend pay for ...~~
Mark's friend bought a pen from mark for 1.50.	How much did the 3 puzzle books cost for one?
We know that he had 4.43 left back.	How much did the books cost all together? 235
He bought 3 puzzle books for 75¢ each.	

Figure 2. SATURDAY SHOPPING ANALYSIS

Problem Statement:

One Saturday, Mark went shopping with some friends. He rode the bus to the shopping center for $.75. Mark bought a miniature race car for $1.12 and 3 puzzle books for $.75 each. He found a quarter outside one of the stores. The boys ate lunch while they were out. Mark bought a sandwich for $1.25 and milk for $.40. Mark's friend, Denny, talked him into selling his pen. Denny paid Mark $1.50 for it. After lunch, Mark rode back home on the bus. The ride cost $.75. After Mark got home, he counted his money. He had $4.43 left. How much money did he start out with?

Goal Analysis:

G_0: How much money did he start out with?

[G_1: How much did he spend?]

[G_{11}: How much did he spend on bus rides?]

V_{111}: He rode the bus to the shopping center for $.75

V_{112}: [He] rode back home on the bus [for] $.75

V_{12}: [He] bought a miniature race car for $1.12

[G_{13}: How much did he spend on puzzle books?]

V_{131}: [He bought] 3 puzzle books

V_{132}: Puzzle books [cost] $.75 each

[G_{14}: How much did he spend on lunch?]

V_{141}: [He] bought a sandwich for $1.25

V_{142}: [He bought] milk for $.40

[G_2: How much did he receive along the way?]

V_{21}: He found a quarter outside one of the stores [$.25]

V_{22}: Denny paid Mark [him] $1.50 for [his pen]

[G_3: How much did he have left?]

V_{31}: He had $4.43 left

G = Goals
V = Givens
[ ] = implicit

Table 1

Mean Planning Board Efficiency by Level of Training and Grade

	Grade			
	4	5	6	7
Level of Training				
Minimum	0	0	1	1.5
Maximum	0	1	1.5	2

Scale

0 - children expressed information on planning board in full sentences, or, as worded in the problem;
1 - children paraphrased, or expressed phrases with quantities and referents;
2 - children expressed information symbolically

Table 2

Number of Goals Generated by Dyads on Saturday Shopping

	Grade				
Condition	4	5	6	7	Mean
PB-Maximum	2	1	2	3	2.67
PB-Minimum	0	0	0	3	.75
Control	x	0	1	0	.33
Mean	1	.3	1	2	

Note: x = data not available

Table 3

Structural Analysis of Responses to "SATURDAY SHOPPING"

Problem Structure: ? - [b-d] = 4.43

Solution Structure	Dyad 4M	4I	5B	5M	5I	6B	6M	6I	7B	7M	7I
[b+d] + 4.43								X			X
[b+d_1] + 4.43	X[3]		X[2]								
[b+d_2] + 4.43					X						
[b] + 4.43							X			X	
[b-d] + 4.43											
[b+d] 4.43									X		
[b+d] (-4.43)				X[1]							
[b+d] - 4.43		X									
[b] 4.43						X					

Table 4

Dyads That Added Their [bc] Result to 4.43 in Saturday Shopping

	Grade				
Condition	4	5	6	7	Mean
PB-Maximum	0	1	1	1	.73
PB-Minimum	1	0	1	1	.67
Control	x	1	0	0	.33
Mean	.5	.67	.67	.67	

Note: x = data not available

Table 5

Analysis of Verbal Instructions

a. By Dyads

Dyads	Number of Utterances H	L	Total	Givens H	L	Goals H	L	Calculations H	L
4 Con	x	x	x	x	x	x	x	x	x
4 Min	7	4	11	0	1	0	2	2	3
4 Max	43	31	74	8	3	7	5	6	0
5 Con	8	2	10	0	2	0	1	1	1
5 Min	21	13	34	8	10	1	2	0	2
5 Max	20	31	51	4	0	1	1	2	12
6 Con	6	22	28	4	3	1	1	0	7
6 Min	49	57	106	11	18	1	0	14	4
6 Max	9	15	24	4	5	1	1	6	8
7 Con	5	0	5	0	0	1	0	2	2
7 Min	20	11	31	3	8	0	3	0	8
7 Max	7	4	11	7	3	4	1	3	0

b. Total Utterances by Condition by Grade

	Grade 4	5	6	7	Mean
Control	x	10	28	5	14.33
Minimum	11	34	106	31	45.50
Maximum	74	51	24	11	40.00
	42.5	31.67	52.67	15.67	