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ABSTRACT

Higher order thinking skills are inevitably developed or exercised relative to some discipline. The discipline may be formal or informal, may or may not be represented in a school curriculum, or relate to a wide variety of domains. Moreover, the development or exercise of thinking skills may take place at differing levels of generality. This paper is concerned with how new uses of information technology can profoundly influence the acquisition and application of higher order thinking skills in or near the domain of mathematics. It concentrates on aspects of mathematics that relate to its representational function based on the beliefs that: (1) mathematics itself, as a tool of thought and communication, is essentially representational in nature, and (2) information technology will have its greatest impact in transforming the meaning of what it means to learn and use mathematics by providing access to new forms of representation as well as providing simultaneous access to multiple, linked representations. This report describes a few examples of novel software environments from the representation perspective, points to more novel approaches to curriculum reform in mathematics that will encourage the cultivation of higher order thinking skills and relates these to unresolved research questions and educational policy issues. (CW)

INFORMATION TECHNOLOGY AND MATHEMATICS: OPENING NEW REPRESENTATIONAL WINDOWS

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Introduction.

"Higher order thinking skills" are inevitably developed or exercised relative to some discipline. Put simply, one cannot learn thinking without thinking about something. The discipline may be formal or informal, be represented or not represented within a school curriculum, or relate to a wide variety of domains: mathematics, electronics, Logo programming, summer camp recreation programming, Volkswagen engine repair, meteorology, cognitive psychology, and so on. Moreover, the development or exercise of the thinking skills may take place at differing levels of generality - with differing levels of involvement with the knowledge base comprising the discipline itself. Given the breadth of the issue, it would be wise quickly to select a sector for attention.

As the title indicates, we have chosen mathematics as the discipline of concern for this paper. But mathematics is not just any discipline. It is a discipline which has often been regarded as embodying the essence of thought, of logic, of abstract structure. It is also a *language* in which the

1. We wish to acknowledge helpful conversations with Judah Schwartz regarding the ideas of this paper - although he should not be held accountable for all the views herein offered.
2. An earlier draft of this paper was presented at the Conference on Computers and Complex Thinking, jointly sponsored by the Office of Educational Research and Improvement and the Wisconsin Center for Education Research, Washington, D. C., November, 1985.

major features of other domains can be represented. It is not enough, however, to acknowledge the dual nature of mathematics, as a body of knowledge and as a language, because, as a body of knowledge, mathematics contains both procedural as well as conceptual knowledge, and these at various levels of abstractness (Hiebert, 1986).

And it is not merely a single language, but rather a network of representational systems, which interlock not only with each other, but interact differently with different kinds of mathematical knowledges as well as with nonmathematical representation systems, such as natural language and pictures, (Kaput, 1986b). As a network of languages, it has two interlocking functions, one as a tool to think with, and another as a tool to communicate with. Failure to acknowledge the richness and multi-faceted nature of mathematics has been a source of much misstatement and overstatement regarding the relation among mathematics, cognitive skills development, and information technology.

Having acknowledged the issue's real complexities, we will now go on to use a series of concrete examples to illustrate how new uses of information technology can profoundly influence the acquisition and application of higher order thinking skills in or near the domain of mathematics. Our concentration on aspects of mathematics that relate to its representational function is based on the following twin beliefs.

- Mathematics itself, as a tool of thought and communication, is essentially representational in nature (Kaput, 1986a).

- Information technology will have its greatest impact in transforming the meaning of what it means to learn and use mathematics by providing access to new forms of representation as well as providing simultaneous access to multiple, linked representations.

Our approach will be
(1) to describe a few examples of relatively novel software environments from the representation perspective,
(2) to point to even more novel approaches to curriculum reform in mathematics that will support the cultivation of newly appropriate higher order thinking skills, and
(3) to relate these to research questions and educational policy issues as yet unresosved.

New Software, New Representations.

We will examine two existing pieces of software, one in geometry and the other in algebra, and one software environment under development intended to support learning of ratio and proportion in grades 4-9:

(A) The Geometric Supposers, by Judah Schwartz and Michal Yerushalmy (newly commercially available for current school microcomputers from Sunburst Communications) is a collection of four programs that radically change the relationship among the teacher, student, and plane geometry by subtly changing the representational character of geometric diagrams/constructions.

(B) An algebra/graphing software environment has been developed in prototype form by the same authors. Aspects of this software are being developed independently in slightly different forms by several others, including Richard Lesh's group at WICAT (Lesh, 1985, 1986) and Ronald Wenger's group at the University of Delaware (Wenger, 1984). This software makes accessible an entirely new geometric meaning for operations on algebraic equations which has the potential for changing how students think about solving algebraic equations and the relation (often confusing) with operations on expressions.

(C) The ratio-proportion software is intended (1) to "ramp" students upward from their concrete, situation-bound representations of intensive quantities (generalized rates) and operations on them, to more abstract, flexible representations, and (2) to render explicit, hence more learnable, the connections among different representations by making more than one of several linked representations visible and useable simultaneously.

We shall now examine each of these learning environments in more detail.

A. Geometry

The Supposers have three features, two central features and a third support feature that utilize the technology. The first central feature is the electronic straight-edge, compass. Anything Euclid allows can be done quickly, cleanly. This has the facilitative effect analogous to that provided by word processors. One can create constructions much more rapidly and easily than with the crude instruments of paper and pencil, straight-edge and compass. Just as the real power of word processors, especially those tailored for use by students, has not yet been adequately plumbed, we are not yet certain what the ultimate meaning of this kind of feature actually can be. It is likely to appear all over the software landscape with a variety of improvements and elaborations.

However, by far the most important and novel feature of the Supposers is its second major feature – the ability to remember a construction *as a procedure that can then be executed on other objects chosen or constructed by the student.* Thus an observed regularity in a given diagram can be investigated regarding its generality – does it hold more generally, and if so, *how* generally? Having constructed the three medians of your initial triangle and noticed that they all intersect at a point, you are strongly pulled to see if this holds for other triangles, including some that perhaps lack the regularity of the first one. (See Figure 1.)

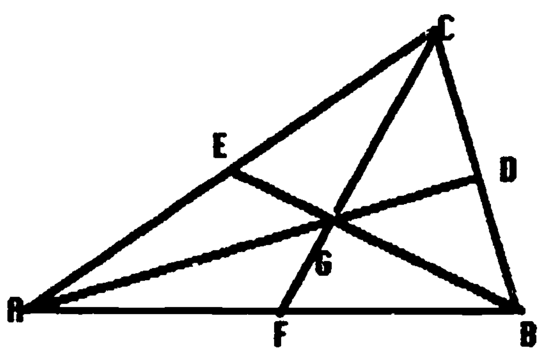


Figure 1

We have already seen students produce new theorems, theorems that have not appeared in the literature dating back over two thousand years (Kidder, 1985). One is not limited to the simple constructions that one's unsteady hand and limited patience allow – or that the clay, sticks and feathers of earlier times allowed.

There is an important representational breakthrough here, embodied in the repeat feature: the type/token relationship between a particular diagram or picture and the class of geometric objects that it may be presumed to represent has been fundamentally altered. Until now, a particular construction could only represent itself – it could not act as the token of a general type in a principled way. Any generality of the resulting figure was strictly a matter of assertion on the part of the creator or observer of the item. This is in strong contrast to the way algebraic entities represent numbers or quantitative relationships: $Y = 2X$ represents, with no ambiguity and in a principled way (via specification of the domain of X), a potentially infinite collection of number pairs.

A third feature supporting the other two is a measuring and calculating utility that allows students to measure lengths, areas, and angles as well as to compare and calculate with the resulting measures. However, accuracy is limited to two decimal places (necessity turns out to be a true virtue in this case) so that measurement can play only a suggestive function, not a conclusive one.

The combination of the three briefly described features changes - or can change - the experience of doing plane geometry. No longer need it be the tightly controlled museum trip, where the student is asked to "prove" the carefully displayed artifacts of history, and where the style and substance of "proof" resembles true contemporary mathematical investigation about as much as the minuet resembles contemporary dance.

What, more specifically, is now different in practical terms from the museum trip? It is important to move beyond simple romantic cries of "power to the student" or vague claims about the learning of higher order thinking skills. We need substantive and detailed specification regarding what should be done and how it might be done. Intensive work on this type of issue is underway at ETC, the foundations for which have been provided by Michal Yerushalmy's doctoral dissertation (Yerushalmy, 1986).

One thing that can be said concerns **a radical expansion of the kinds of rational activities supported by this type of software environment** in comparison with the traditional geometry learning environment.

Consider the following scenario, closely modeled on actual in-class events. (All these results were discovered by a single class in a hour's quiz in which they were asked to tell all they believed to be true regarding the midsegments of a triangle.)

Having built the three midsegments of a triangle, (see Figure 2) and noticed by judicious repetition on other triangles that the original triangle is now subdivided into four triangles, each congruent to the original and having area one fourth of the original and perimeter one half the original, one may now wish to generalize the notion of midsegment by subdividing the sides of the triangle into three rather than two pieces, (see Figure 3).

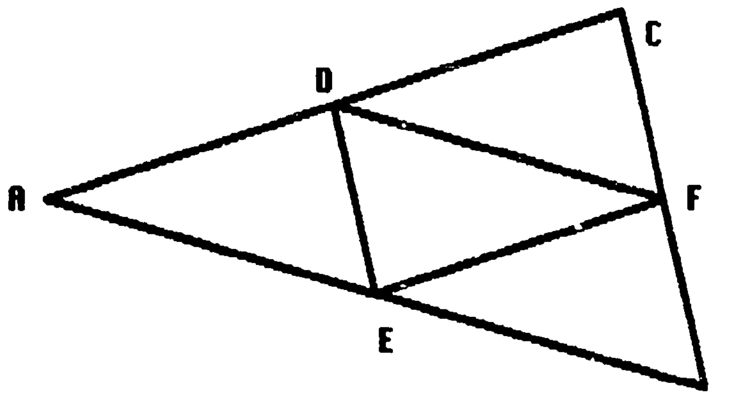


Figure 2

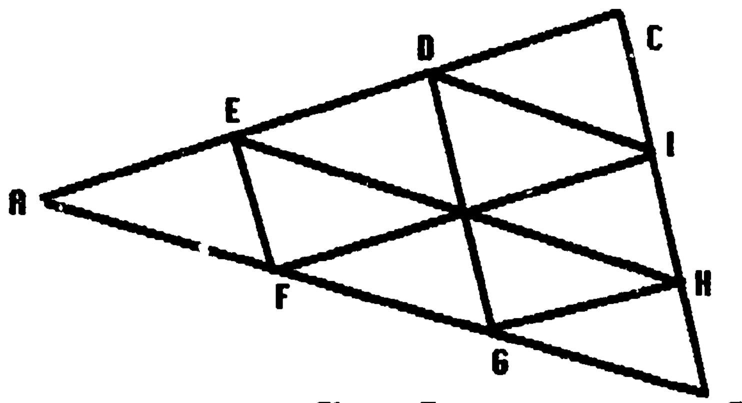


Figure 3

Or one might move to generalize the idea to squares and other quadrilaterals - see Figure 4. Do the four midsegments of a kite (perhaps an asymmetric kite) always form a parallelogram??

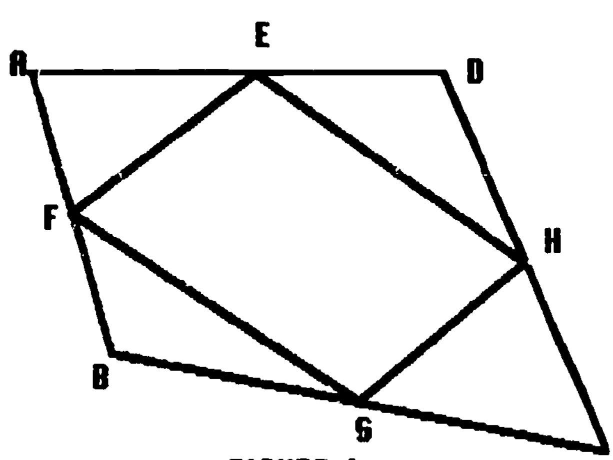


FIGURE 4

The point here is merely to illustrate the rich texture of the possibilities to set the stage for more general discussion of such an environment. The traditional geometry course was presumed to teach deductive thinking and the logic of axiomatic systems. It was also presumed to teach something about two dimensional space. Few researchers today would agree that these goals are commonly met (Wirszup, 1976; Yersushalmy, 1986). Indeed, in the typical geometry course students make few constructions on their own initiative, the deductions drawn from those constructions are almost always directed to an externally teacher or text-provided goal statement, and they almost never are allowed to explore the consequences of altering an axiom system. Contrast that situation with the one hinted at above.

(1) **The epistemological context is redefined: Epistemological authority no longer is the exclusive purview of teacher and text, but is provided by proof, convincing proof.** Given the ease with which constructions can be built, students immediately move outside the narrow range of the results normally presented in the museum's display cabinets, so that the truth of conjectures quickly becomes problematic. Another epistemological consequence of the ease with which constructions can be made and repeated is the devaluing of the currency of examples - "proof by example," after experience in this environment, loses its force, and examples come to be seen in their real logical role, as conjecture exploration devices. This increased "felt need" for truly functional proof has been consistently observed to develop during the second half of the academic year in classes where the software has played a central role.

(2) **The kinds of rational skills called into play are vastly broader in scope and much more representative of those exercised by a person actually doing mathematics.** In particular, since the making of conjectures and then the exploring of conjectures has been so greatly facilitated, a whole range of ancillary and support skills can be cultivated, especially those connected with varieties of inductive reasoning. These include:

(a) choosing confirming and disconfirming examples wisely, as well as learning the logical role of examples;

(b) systematic simplification to expose the logical core of a particular conjecture (especially important because of the richness of the constructions now possible);

(c) systematic variation in the parameters of a particular construction or entity to which the construction is to be applied, including numerical incrementing of parameters to support inductive generalization.

(3) **Following from the redistribution of epistemological authority is a redistribution of social authority and personal responsibility.** Students can now become sources of knowledge as well

as sources of important questions. The demands upon the teacher, both intellectually as well as managerially, can be substantial. A variety of support materials, written and cybernetic, will be needed. In addition to requiring much more knowledge of geometry, and confidence in the general power of that knowledge, a whole new set of intellectual and pedagogic skills is called for, skills based on an understanding and appreciation of the realities of *doing one's own mathematics, rather than reciting someone else's mathematics.* The teacher training implications of this shift are enormous.

B. Algebra

Suppose you have before you a linear algebraic equation in one variable, say

$$X + 1 = 7 - 2X$$

In solving this equation one adds or subtracts numerical constants from both sides, multiplies and divides both sides by numerical constants, and adds or subtracts linear terms from both sides. The standard argument behind this strategy is based on conditionally equivalent equations: the solution set is unaffected by such actions, provided one is careful to do the same thing to both sides. Much of this argument is lost on the majority of students because of its essential abstractness, leaving them with a deep and pervading confusion regarding what can be done with equations as opposed to what can be done with expressions, a confusion that rears its ugly head when they attempt to deal with rational algebraic expressions. And there is no easily accessible distinction between the two types of actions.

Suppose one could graph the two sides of the equation on the same coordinate axes, say

$$y_L = X + 1 \text{ and } y_R = 7 - 2X$$

representing the two sides, respectively. Then, as one subtracts 1 from both sides, the two graphs each shift vertically downward by that amount - but the X coordinate of the intersection *does not change.* (See Figures 5a,b.)

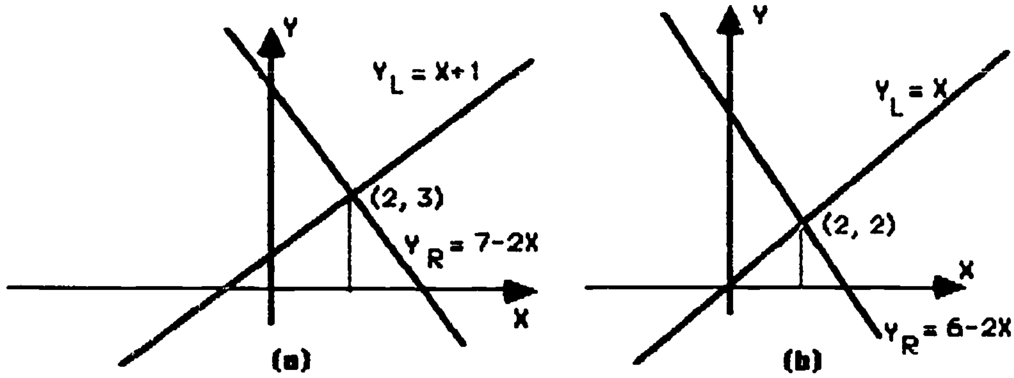


FIGURE 5

Similarly, if one adds 2X to both sides, the slopes of the respective graphs change, but the the X coordinate of the intersection does not change. (See Figure 6a.)

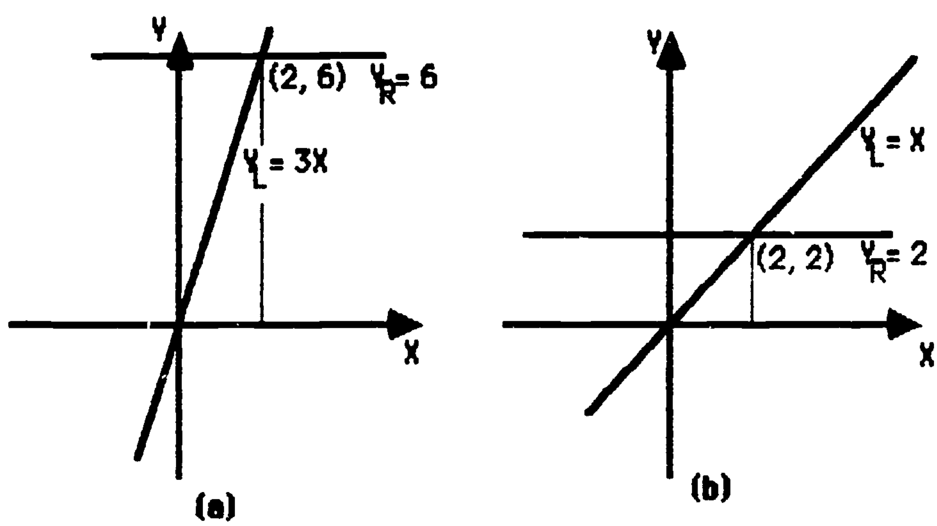


Figure 6

And finally, if we multiply both sides by a constant, 1/3, we reach the standard form,

$$X = 2$$

which, graphically, is the intersection of a diagonal and a horizontal line (Figure 6b).

We are here doing more with Descartes' remarkable invention than was previously feasible simply because of the quickness and ease by which the graphical representations are generated. The essence of this software is to provide a new representation, a visual representation, of actions on equations. With this new representation come new meanings for such events as the introduction of extraneous roots. When one multiplies both sides of the linear equation

$$X = 6 - 2X$$

by X, the lines are transformed into parabolas, the graphs of

$$Y_L = X^2 \text{ and } Y_R = 6X - 2X^2$$

which no longer need intersect at a single point! (See Figures 7a,b.) One can literally *see* the extraneous root come into existence.

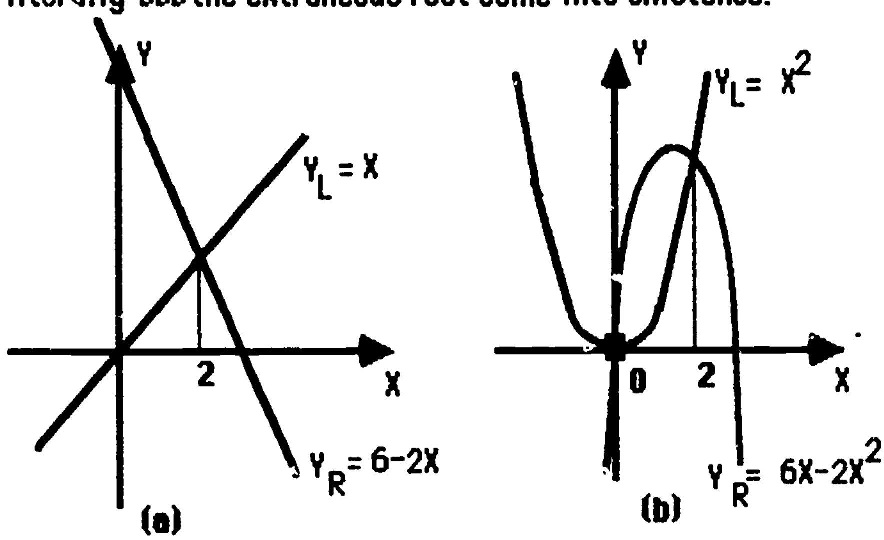


Figure 7

On the other hand, if one combines like terms on one side of an equation or does arithmetic, the graphs of the equations do not change at all. *One is performing actions on expressions, not equations.* Actions on expressions do not change their graphs.

And on the other hand, one can initiate changes in the opposite direction: from the graphical to the algebraic. Hence one can modify equations by acting on their geometric representations - translating, "unbending" (dividing an equation through by x), and so on, with the goal of tranforming the graph of the original equation into its "canonical" diagonal-crossing-horizontal form. Naturally, a readout is available that one can observe the algebraic consequences of one's geometric actions. A fuller environment will contain means by which one can support predictions regarding the effects of actions across representations by appropriately hiding and uncovering such.

We are a long way from understanding how to exploit this new representational tool pedagogically, but, given an appropriate curriculum, we have every reason to expect it to have significant impact on the way students think about equations and expressions. Indeed, we have given visual meaning to conditions on expressions more generally: equality is represented by the X-coordinate of the intersections of graphs (which can be further highlighted via a projection to the X-axis), but inequality conditions likewise have visual meaning, as intervals on the X axis corresponding to intervals where one side's graph is above or below the other's graphically. We have also given visual meaning to *actions* on conditions, which should further alter the way students think about equations and inequalities. These are likely to assume a new concreteness and reality in the same way and for the same reasons that Gauss' invention of the visual representation of the complex numbers gave the latter an acceptability among mathematicians that they had previously lacked.

They could now see them and also see their actions on them.

C. Multiple, Dynamically Linked Representations for Ratio Reasoning.

Current mathematics education literature reveals a strong consensus on the sources of student inability to solve multiplication and division word problems, especially those involving intensive quantities (Schwartz, 1984; Kaput, 1985). (For our purposes here, one can think of an intensive quantity as a ratio or "per quantity," which also includes such quantities as rates, e.g., miles/hr.)

The conclusion of much research is that students lack suitably rich and flexible models of these operations and intensive quantities, and are thus unable to recognize situations, except in the most well rehearsed contexts, in which these mathematical concepts apply (Fischbein, 1985; Greer & Mangan, 1984; A. Bell, Fischbein & Greer, 1984; Usiskin & M. Bell, 1983; Greer, in press). Other research in both the problem solving and rational number literature (e.g., Lesh, 1985; Lesh, et al, in press; Behr, et al, 1983), as well as in the algebra learning literature (Clement, 1982;

Clement, et al, 1981; Kaput & Sims-Knight, 1983) has shown the depth of student difficulty in translating concepts between different representations within mathematics, as well as in translating quantitative relationships into mathematics.

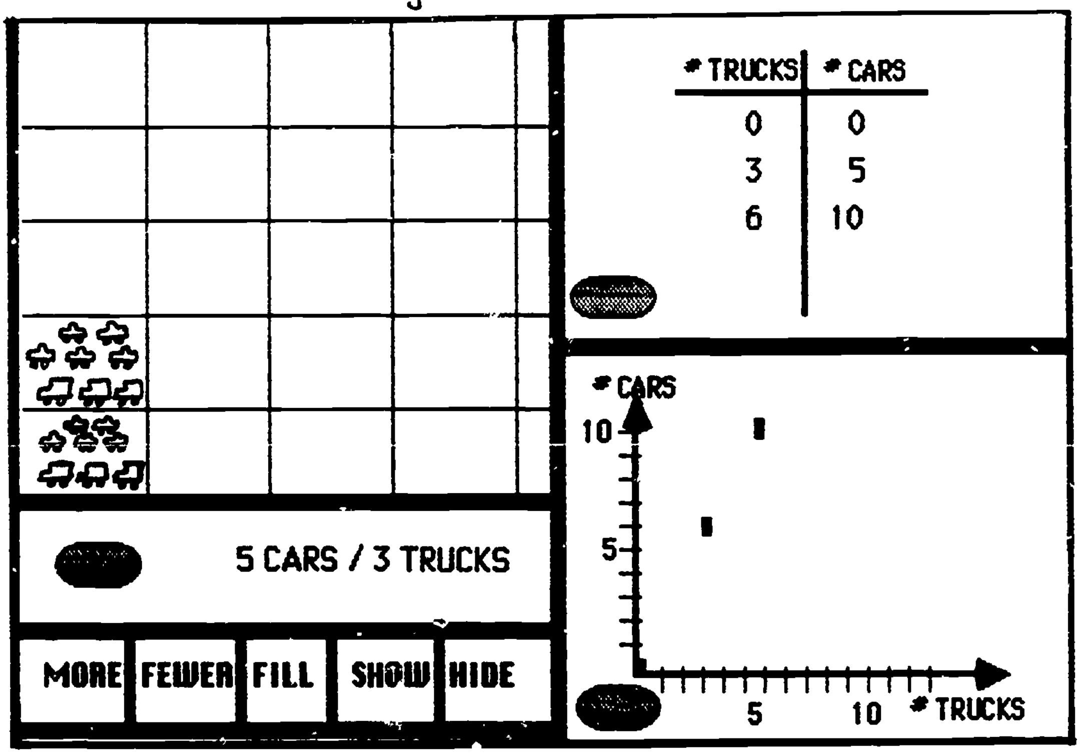
One research team at the Educational Technology Center is in the process of developing a series of dynamically linked representations for intensive quantities that would support a variety of actions in such a way that a student could see the results of manipulations in one representation ramify across other simultaneously visible representations in a multiple window environment. This environment is being implemented on a Macintosh, which provides the necessary graphics and computing power as well as a user interface that supports, via the mouse-based manipulation of screen objects, the necessary kinesthetic reality to the manipulations of the screen objects.

The full environment consists of four representations of intensive quantities, which, for purposes of illustration, we will assume to be the ratio between the number of cars and trucks produced by a factory, say a ratio of 5 to 3, respectively. These are:

(1) iconic representations, the simplest of which consists of a window tesselated with rectangular boxes, each of which contains 5 of one kind of icon and 3 of another;
(2) a numerical, tabular representation, with one column labeling the number of cars and the other the number of trucks;
(3) a coordinate graphical representation with one axis representing the number of cars and the other the number of trucks; and
(4) a "quantity calculator" that allows one to do arithmetic on quantities (numbers with referents) with the referents of the quantities appropriately tracked during the calculations.

Three of these coordinated representations are visible simultaneously when desired, so that the consequences of actions in one can be viewed in the others. Given, for example, the cars-trucks situation, the student begins by constructing an iconic representation of the intensive quantity by picking an appropriate number of car and truck icons from a menu, putting these into a box, and then systematically filling the icon window with such boxes, each of which contains 5 cars and 3 trucks. As the student adds boxes in the iconic representation (by clicking on MORE), the numerical data table likewise fills out in coordinated fashion, and corresponding points are added automatically on the coordinate graph. (See Figure 8.) This initial default condition can be altered, however, by choosing to hide, in either or both of the other representations, the consequences of adding icons in the icon window.

Figure 8



Similarly, the student can initiate actions in either of the other two representations and inspect (perhaps after first hiding) the consequences of these actions in any of the other representations. The "virtual infinitude" of available representational items also helps attack a most pervasive difficulty students have with such ratio situations: a failure to appreciate the typicality, or representativeness, of the 5 to 3 ratio. We believe that this may also be tied to a lack of cognitive models for the critically important idea of variable, so the software attacks this serious and well documented curricular and pedagogical problem as well.

We can also provide for traditional "missing value" tasks across representations: For example, the question "How many trucks are produced when 25 cars are produced?" can be approached in each representation. In the iconic representation the student can be asked to decide how many truck icons are needed to fill the appropriate number of boxes. In the data table, the question amounts to filling in a missing entry, and in the graph it amounts to naming a point and its coordinates – or naming a place on the truck axis that corresponds (via a function-like mapping) to 25 on the car axis.

One can similarly cross representations in other traditional proportional reasoning activities, e.g., ratio comparisons. If experience is any guide, however, the environment will suggest a whole new generation of ratio and proportional reasoning activities which are not feasible in a static paper and pencil-like environment.

Fr, our purposes here, this software environment renders three very important goals of mathematics education far more attainable than previously:

1. By presenting a family of iconic representations varying in concreteness (only one was illustrated above) one can accomplish three important subgoals:
(a) provide a variety of beginning-level representations that increase the chance of linking with a student's existing, but primitive cognitive representations;
(b) provide schematic representations that allow for a closer fit, hence more cognitively accessible representations, for differing problem situations, including the important differences between discrete and continuous quantities; and, most importantly,
(c) expose, through multiple instantiation and by tying with more general and abstract representations (i.e., the coordinate graph), the deeper mathematical commonalities that underly superficially varying situations.
2. By making visually explicit the relationships between different representations and the ways that actions in one have consequences in the others, the most difficult pedagogical and curricular problem of building cognitive links between them becomes much more tractable than when representations could be tied together only by clumsy, serial illustration in static media.
3. By providing a series of carefully chosen representations that begin with the students' primitive and inflexible ones and ramp upward to ever more powerful and abstract mathematical ones, a new level of longitudinal coherence in the mathematics curriculum is possible, especially as those more powerful representations are introduced in earlier grades and used to represent fundamental mathematical ideas such as ratio appearing in those earlier grades. In the case at hand, we explicitly chose to represent the idea of ratio as the slope of a line in a coordinate graph, thereby providing the basis for natural extension later to nonlinear functions and the enormous corpus of mathematics beyond which uses exactly the same set of representational tools.

Building for the Future

The software environments described briefly in the previous section, together with a variety of others of similar novelty and force, will help shape the direction of mathematics teaching and learning in the near term *if* given reasonable teacher training support, curriculum material support, and hardware availability. Certainly the first item in our list, teacher support, ranks highest in priority. Several years' experience with a wide spectrum of teachers and schools with earlier versions of the Geometric Supposer has helped confirm the obvious: Limited imagination, lack of confidence, or a totalitarian or otherwise negative attitude towards student intellectual endeavor can close the shades on any representational window.

But the importance of the software environments for the purposes of this paper lies in the direction in which they point and the new issues that they raise.

Consider first geometry. The American geometry curriculum has long been regarded as in need of reform (Wirszup, 1976; Yerushalmy, 1986). And the new graphics capabilities of microcomputers make them an ideal medium in which to carry out that reform at all levels (Fey, 1984). Logo-based environments, constructed to include wisely chosen primitives, can help students learn many geometric ideas now neglected and tie those ideas to other domains, including, but not certainly not exclusively, mathematics. The "PreSupposer," one of the Geometric Supposer series, is aimed at middle school students with a view to bringing active discovery to the basic concepts of plane geometry. Similarly, highly flexible electronic Geoboards will soon become available.

But beyond this we can list other areas in which these modest eight bit beginnings might be extended without straining the likely capabilities of the next generation of school microcomputers. While many of these extensions are already being considered, the fact that they may be attainable in software is not, of course, sufficient reason for creating them. That is an educational decision.

1. Consider, for example, a non-Euclidean Supposer, where certain axioms are selectively altered and one inspects the consequences on one's constructions. The only way to learn about axiomatics is to investigate a range of axiom systems. One cannot learn from a single instantiation. And certainly not by laboriously, with crude tools, attempting to prove results declared by authorities to be true, but requiring "proof" only *because the same authorities request it.*

2. Or consider a locally Euclidean Supposer which supports constructions on a manifold of constant curvature - where one can vary not only the objects on which the constructions are performed, but the curvature as well. What happens as one decreases positive curvature and then passes from positive (through zero curvature - the flat plane) to negative curvature?
3. Or a Euclidean Supposer in three or even more dimensions?
4. Another way to extend the Supposer would be to allow the naming, concatenating, and embedding of constructions, treating them as building-block procedures, procedures which can take arguments as well. In this case, one might name the midsegment construction seen earlier (Figure 2), and then, upon seeing its generalizability to "trisegments" (Figure 3), define a new "SEG (n)" construction that for n = 2, 3 yields the given constructions for triangles. The largest leaps in mathematical power have historically and developmentally accompanied the systematic naming and resultant reification of procedures and processes (Kaput, 1986b), and it would appear that entirely new understandings and operations with geometric constructions could follow in this case as well.
5. Given sufficient memory (surely less than two megabytes), one could support simultaneous accessibility of a range of primitive figure types so that, for example, a result determined for triangles, could be examined for validity among quadrilaterals - where some surprising generalizations occur. (See Figure 4 and Schwartz & Yerulshamy, in press, for a whole host of such conjectures.) Very deep insights into the nature of planar space are thus revealed.
6. A straightforward elaboration of the original environment would record a series of constructions in a visible sequence, where either the construction, or the objects on which they operate, are systematically varied. Perhaps the next step would be to allow continuous variation of the objects, yielding dynamic representations and new sources of insight - Klein's Erlanger Program concretized.
7. The whole area of coordinate geometry begs exploration. The only real question is the choice of primitives. But this is **the** central design question in mathematics software design.
8. Even more fascinating geometric issues arise if one allows the **coordinate system** to vary while holding either the visual object or its locus definition constant - the best known such comparison involves rectangular and polar coordinates, but others are possible (Kavanaugh, 1983).

The geometry list could be extended even farther, but instead, we shall move on to extending our algebra ideas. A basic algebra question raised by Pollack (1983), and echoed by Fey (1984) is What algebra is important to know in a world where symbol manipulators are inexpensive and easy to use? Thinking in terms of the MuMath model, education researchers have begun to analyze the content of the existing algebra curriculum to determine the root knowledge that is needed by a user first to decide what to "punch in" and then to interpret what the machine outputs in return. New curricular options based on symbol manipulators specifically designed by educators for educational use, rather than by computer scientists for use by algebra experts trained in the 50's, 60's, and 70's, suggest that this question needs to be updated.

Consider, for example, a symbol manipulator with certain parsing and grouping primitives built in, e.g., "combine like terms," "multiply," "factor out ...," and so on. Suppose further that these primitives can be treated as procedures that can be assembled into more complex procedures such as "build a common denominator," and so on. Then "to do a set of exercises" can be reinterpreted to mean build a procedure, or set of procedures, that will solve the given set of problems.

There appear to be two fundamental virtues of this tools-based approach: First, it respects the fact that the student will be living in a world where such tools will be a way of life; and second, it will build exactly the kinds of algebraic insight that a tool user will need. After all, imagine the kinds of thinking needed to put together a procedure that will add pairs of algebraic fractions, and especially imagine the reasoning and understanding that goes into the testing and revision of the procedure to handle troublesome cases - where the procedure crashes on, say, a complex fraction. Again, it appears that the central question is the **choice of primitives**, including the primitives that control assembly of procedures.

Another example of this manipulable multiple representation approach in currently available software is provided by "Math Path" (Kleiman, 1985). Here the student or teacher can build, modify, and otherwise pedagogically engage a machine-based representation of arithmetic and algebraic expressions: an expression is represented as an appropriate combination of the four operations, each of which is an input-output device, joined together by wires. When one inputs numerical values, they travel along the wires as a mouse in a snake, passing through the various components and eventually landing in an output bin. Depending on the activity engaged in, the computer will display or hide a given machine's corresponding formal expression. Although relatively modest in scope, this software uses the computer's ability to provide a dynamic and manipulable set of procedural representations that provide a new perspective on the structure of formal expressions. One can easily envision workbench extensions to allow more complicated machine primitives (e.g., whole new classes of functions beyond the strictly rational) and new means for joining and manipulating entire machines so that the act of combining (including composition) and simplifying expressions can be represented. Indeed, such environments are under construction as of this writing, and certain professional tools of this genre are already available, e.g., STELLA (1986). This last example listed is an entirely new type of modeling tool that uses any combination of numeric, iconic, algebraic, and graphical representations, e.g., one can enter data graphically via "freehand" drawing if one chooses.

The linked representation software environment for learning ratio and proportion can likewise be extended to connect with algebra software. The study of ratio and proportion can be regarded as the study of linear patterns, and the representational tools used in that study, as already indicated, can be applied far more broadly. Without going into detail, we simply note that algebraic and graphical representations of ratio and proportion can be introduced relatively early, can be tied to the concept of variable, now very poorly handled in the school curriculum (Kaput, Clement, & Sims-Knight, 1985), and be naturally extended to more general functions. Similarly, looking to the earlier grades, students' impoverished cognitive models of multiplication (limited mainly to repeated addition) can be enriched by providing a much expanded set of visually accessible meanings for multiplication that include combinatoric models, as well as a variety of "acting across" models (Usiskin & Bell, 1983).

Another extension of tools-based software can be envisioned for arithmetic algorithms. Through the use of elementary grouping and decomposition primitives applied to Dienes-like screen objects linked to formal representations, students are able to build their own algorithms for the four basic operations on integers in a base ten placeholder system. Extension to other bases is another tempting possibility - an idea which, when first suggested and tried in the 1960's, may have failed because of the limited medium in which it was attempted. Think of it as an arithmetic algorithm toolkit, or workbench for use in the elementary grades. Here the student would have a horizontally split screen with a lower screen consisting of formal representations of arithmetic statements (in an adjustable base) and an upper screen consisting of a Dienes Blocks style of representation paralleling and coordinated with that appearing below. The student would be able to manipulate either side of the screen and the computer would track the corresponding changes in the opposite representation. In such an environment a student could construct "homemade" algorithms, eventually comparing them to the standard ones. In fact, steps towards this type of environment have been taken by several groups, but (to my knowledge) without the open character implied by the workbench metaphor. Again, the point is that learning environments need to be drastically modified to prepare students for the tools-rich environment that they will most certainly inhabit. The kinds of insights into mathematical procedures appropriate for such a world need to be reanalyzed, as do the means for learning such insights.

Reflections

Two fundamental characteristics pervade the current and projected software ideas described in the previous sections. The first is a **radical enrichment in the kinds of rational activities associated with learning and doing mathematics.**

The kinds and levels of cognitive skills cultivated and exercised stretch far beyond the norm of today's school mathematics classrooms. This may be an especially important new development because of the parallel expansion of intellectual activities that information technology tools seem to be demanding in a wide range of professions and domains, e.g., spreadsheets call for new forms of experimentation and generalization, and the example-based data base world opened by new PROLOG systems likewise seem to call for a new kind of cognitive skill in dealing with samples and examples. Incidently, the kinds of example manipulating skills fostered in the Supposer environment seem to parallel the example-based thinking needed in such new data-base systems.

The second fundamental characteristic of the new learning environments that we have been discussing is the way that **they are organized around the student as an active agent using a potent tool for the investigation of important mathematical ideas.**

Indeed, it is not a long step from what has been briefly described to an entire mathematics curriculum based on Tools, Workbenches, and Challenges. The Challenges are environments where the student uses the tools, and the Workbenches are environments where the student uses carefully designed primitive tools and assembly procedures to build specific cybernetic devices to solve problems in the Challenge Environments. In fact the author and John Richards have outlined a system of this type in some detail (Kaput & Richards, 1985).

Challenge environments are not hard to come by. The practical mathematics education literature of the past thirty or more years is a virtually unlimited repository of Challenge ideas, many of which are more congenial in a microcomputer setting than in their original formats. One such example already mentioned is the Geoboard and the many activities introducing not only mathematical ideas, but problem solving strategies and systematic investigation techniques, e.g., control of variables in the development of the Geoboard classic, Pick's Theorem.

But, of course, the cybernetic utopia is not yet at hand. Nor will anything approaching imaginative, empowering use of information technology occur without hard thinking and careful analysis on a broad set of research agendas. These include rethinking teacher training and credentialing processes. Consider, for example:
How much geometry does a teacher need to know to make confident, intelligent use of a Supposer type of geometry learning environment, especially given the newer options now opening up via the emergence of new representational tools? What attitudes and understandings must such a teacher have about the nature of mathematics, and about the nature of learning? (See Yerushalmy, 1986, for an in-depth discussion of these issues.

Another critical agenda item is the more complete parsing of the cognitions associated with using microcomputers as tools in different domains. And there is the recurring underlying question of how much sense does it make to speak of "higher order thinking skills" as if they were domain-transcendent? Are these theoretical artifacts or are they psychological realities? And, given a new access to multiple representations, might not new cognitive skills be called for?

These are not new questions. And many of them have been the subject of extended study - the general ones in cognitive psychology, and the domain specific ones, for example, in fields such as mathematics education. Information technology decisions are inevitably couched within the political, economic and practical contexts that define the nature of schools, the domains deemed proper subjects of study, the distribution of the population expected to study them, and the depth to which they are to be studied. At finer levels of detail, every last decision is both a pedagogical and curricular one, and hence requires knowledge about the structure of the domain, the structure of the mind involved with that domain (including that mind's patterns of growth and development), and the practical constraints in which that decision must be implemented. No news here. These are nothing more than the traditional questions of education research.

But I have the persistent intuition that the issues are more crisply drawn when the design and implementation of information technology-based learning environments are involved. Larger issues, such as the locus of epistemological authority, can take the form, for example, of whether or not an omniscient and infallible tutor should preside over the environment. At fine levels of detail, in the building of the Ratio-Proportion learning environment for example, we are forced to push earlier mathematics education research to its limits regarding student understanding of rational numbers, proportional reasoning, multiplication, division, and graphical representations. Indeed, we have joined forces with other research teams, e.g., the Rational Number - Proportional Reasoning Project, in order to use their voluminous and high quality data.

But perhaps a more interesting development is the extent to which new general research questions in mathematics education are being raised by this kind of application of information technology:

For example, **does the simultaneous presence of manipulable linked representations support the learning of translation skills between those modes of representation?**

Given that student inability to translate concepts, procedures, or relationships across different representations is well documented and widespread, this question is of vital importance, but couldn't be raised until such learning situations became feasible and testable. And if such skills do prove to be learnable in particular domains, to what extent might they then be transferrable? Other, much more detailed mathematics education questions also arise, but there is not the space here to develop them (see Kaput, 1985).

Of Bulls, Beasts and Bugs

The previous section was intended to argue that information technology cranks up virtually every traditional education issue in intensity and immediacy, while simultaneously bringing many of those traditional issues into bolder relief. Recall the classic argument of the cognitive modelers that the process of instantiating one's theory in a computer program forces a clarity and precision about one's hypotheses and their consequences that informal modeling does not. In some sense, information technology seems to have a similar kind of forcing effect more broadly across the educational scene. And we saw how it raises new issues as well. Moreover, it challenges the hypotheses of carefully crafted research positions. For example, what happens to the relevance of spatial ability research in the presence of graphics capabilities that supplant the previously required cognitive actions? The phenomenon goes well beyond education. In philosophy, for example, the traditional questions have either come to new life, e.g., in epistemology, or have taken new forms, as with the mind-body problem. Indeed, information technology is a bull in everybody's china closet. Hence we should not shy away from its destabilizing effects in education and instead treat them as the windows of opportunity that they are.

Certain prophets of microcomputer doom in education are declaring the curricular and pedagogical promise of information technology dead - this **when more than half the microcomputers in schools have been in place less than two full academic years!** A bit like declaring the fetus' life a failure at the first twinges of labor. Are the judgements being made on the basis of the environment into which such would be born? Indeed, we do not yet have any real idea of the nature of the beast, nor whether or how it may come to change the environment.

Most serious observers of this new technology regard its novelty and importance as measurable in terms of frequency of occurence as being on a scale of centuries. Indeed, detailed studies of how microcomputers have been used in education are as close to being absolutely irrelevant to the future of information technology in education as any studies could be - at least as factors that would inform policy. As marginally interesting recent history, perhaps. The hardware technology present initially in the schools is first-generation, crude, and without significant quantities of software that uses the properties and potentials of the technology. Not even a single K-12 computer based curriculum exists in the schools (although the WICAT system will be appearing commercially soon).

At least a generation of hard work and experimentation will be needed to begin to uncover ways in which cybernetically centered educational environments can be appropriately designed, built and implemented. In fact, we must become accustomed to "information technology" as representing a rapidly changing wave of potentials rather than a fixed entity. Despite the fluidity and uncertainty that such a metaphor implies, I hope that the very preliminary glimpses of near term potential exhibited earlier in this paper support the claim that a certain level of rational optimism is in order. With full apologies to Roy Blount (1984), I counter that no,

the computer is not just another bug on the windshield of education.

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