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AUTHOR	Cuneo, Diane O.
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ABSTRACT

An understanding of fraction addition can be thought to involve two quantitative ideas: (1) the understanding that adding to an original quantity increases its size, and (2) a sense of how much increase occurs. Both of these ideas should underlie or inform a child's approach to problems involving fraction addition and thereby constrain the class of possible answers to ones that "make sense." It is well known, however, that many children do not give reasonable answers when asked to compute or estimate the sum of two fractions. This problem has generated much discussion in the mathematics education community and, in general, such discussions suggest that poor understanding of fraction size is at the heart of children's difficulties. The purpose of this study was to determine what children who compute fraction sums incorrectly and, as it turns out, estimate the same sums poorly do and do not understand about fraction addition. (PK)

Understanding Fraction Addition

Diane O. Cuneo

University of Massachusetts at Amherst

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Understanding Fraction Addition

An understanding of fraction addition can be thought to involve two quantitative ideas: (1) the idea that adding to an original quantity increases its size, that is, an understanding of the addition operation, and (2) a sense of how much increase occurs that is, an understanding of fraction size. Both of these ideas should underlie or inform children's approach to problems involving fraction addition and thereby constrain the class of possible answers to ones that "make sense" (cf. Gelman & Meck, 1986). It is well known, however, that many children do not give reasonable answers when asked to compute or estimate the sum of two fractions. This problem has generated much discussion in the mathematics education community and, in general, such discussions suggest that poor understanding of fraction size is at the heart of children's difficulties (e.g., Behr, Wachsmuth, & Post, 1985; Behr, Post, & Wachsmuth, 1986; Carpenter, Coburn, Reys, & Wilson, 1976; Post, 1981).

For example, in the 1973 and 1978 mathematics assessments of the National Assessment of Education Progress (NAEP), 30 percent of the 13-year-olds in the samples found the sum of 1/2 and 1/3 by adding numerators and denominators, obtaining an answer, 2/5, that was less that one of the original addends. In their discussion of the 1973 results, Carpenter et al. (1976) recommended increased emphasis on initial conceptual work with fractions to ensure that children are "able to answer questions like 'How much is shaded?' or 'Which fraction is greater?' before there is much emphasis on formal addition with unlike fractions" (p. 140). Their view was that a firm initial understanding of fraction size would facilitate later computational learning and performance.

Similarly, Behr et al. (1985) found that many fifth grade children were unable to estimate the sum of two fractions. The task required children to make two fractions, from a given set of numerals, that when added together were as close to one as possible but not equal to one. According to Behr et al. (1986), "a large percentage of the responses (20 of 41, or 49%) suggested a processing of fraction addition that showed little understanding of fraction size" (p. 107). They recommended increased emphasis on estimation activities that focus on and develop concepts of fraction size.

Despite the logic and intuitive appeal of the above views, there is little solid evidence for thinking that children today fail to develop a general idea of fraction size by the time they leave elementary school. For example, in the 1978 assessment of NAEP, 13-year-olds did not perform all that poorly on items assessing basic ideas about fractions (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Post 1981). It is puzzling, therefore, that they performed so poorly on computational items, such as 1/2 + 1/3, and on estimation items, such as 12/13 + 7/8, in the same assessment. Clearly, fraction arithmetic remains a persistent trouble spot in the elementary school mathematics curriculum. What is not clear is why this is the case.

The purpose of this study was to determine what children who compute fraction sums incorrectly and, as it turns out, estimate the same sums poorly do and do not understand about fraction addition. Sixth grade children were asked to <u>compute sums</u> for pairs of unlike fractions. Their computational performance was classified as "high" or "low" and their performance on three estimation tasks was analyzed in terms of these groupings. The first task asked children to <u>estimate sums</u> for the same addition problems, the second presented the same problems in terms of <u>circular regions</u>, and the third asked for estimates of the size of <u>single fractions</u>. Present focus is on the estimation performance of 15 children who consistently computed fraction sums by adding across numerators and denominators.

Interpretive Framework

Figure 1 shows a tentative framework for viewing possible relationships between understanding, estimation, and computation. The arrows represent logical or previously proposed connections between these three general aspects of mathematical competence. For present purposes, connections are viewed as segments of paths to computational skill, and estimation and various types of understanding, as sites along these paths. For example, connections B, E, and G form a path that leads from Understanding Symbols (e.g., 1/8, 3/5,+) to Understanding Symbol Phrases (e.g., 1/8 + 3/5) to Estimating Results to Performing Computations. Difficulties with computation are assumed to reflect a deficiency at one or more of the sites along a path and/or weakness or nonexistence of one or more of the path's segments. The framework in Figure 1 is tentative in that some important mathematical competencies are missing, for example, informal understanding of verbal problem situations, and some of the connections may not actually exist as shown, for example, connection A between Understanding Symbols and Understanding Algorithms. It is useful, however, for understanding previous discussions of children's difficulties with fractions as well as the results and implications of the present study.

For example, the above recommendation by Carpenter et al. (1976) can be viewed as a suggestion for instructional focus on the path leading from Understanding Symbols to Understanding Algorithms to Performing Computations and, within that path, on the initial site of Understanding Symbols. Similarly, the above recommendation by Behr et al. (1986) can be viewed as a suggestion for instructional focus on the connection between Understanding Symbols and Estimating Results, with particular emphasis on the site of Understanding Symbols. Both of these recommendations locate the deficiency in children's competence with fractions at the site of Understanding Symbols,

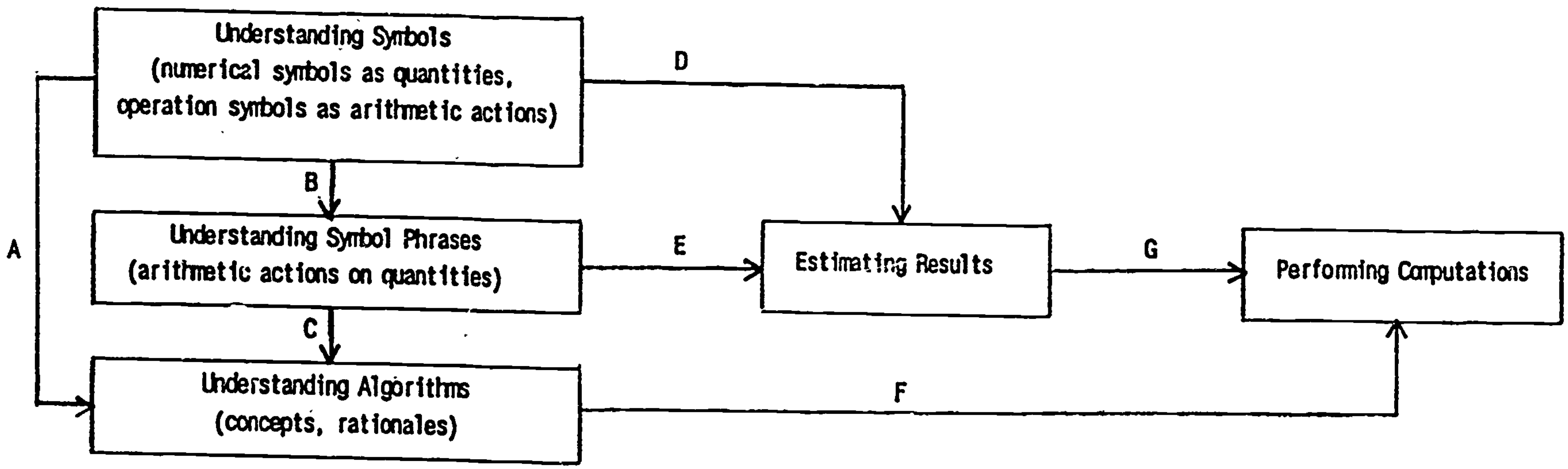


Figure 1. Tentative Interpretive Framework.

specifically, numerical symbols.

In terms of the framework in Figure 1, the present study can be viewed as an attempt to verify a particular path between understanding fraction symbols and performing fraction computations and to locate children's difficulties along that path. We were interested in the path involving Understanding Symbols, Understanding Symbol Phrases, Estimating Results, and Performing Computations for three main reasons. First, this path includes estimation as an intermediary site between understanding and computation, and thus exemplifies recent thinking about estimation and its role in children's mathematics learning (e.g., National Council of Teachers of Mathematics, 1980; Reys, 1985; Shoen, 1986). Second, this path bypasses the site of Understanding Algorithms, an important but well-known source of difficulty for students (e.g., Payne 1976). Indeed, even students who do well in fraction computation generally do not understand the algorithms they apply (e.g., Carpenter et al., 1981; Post 1981). Finally, this path was of interest because it includes Understanding Symbol Phrases as an intermediary between two sites, Understanding Symbols and Estimating Results. Previous work by the Rational Number Project showed little relationship between competence at the latter two sites (Wachsmuth, Behr, & Post, 1983). Since the ability to estimate results logically requires quantitative understanding of symbols, it must be the case that some additional competence is required. The importance of distinguishing between individual symbols and symbol phrases becomes clear in the present study.

Method

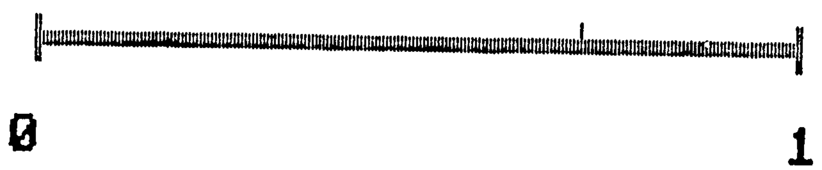
Computation task. Twelve addition problems were generated from a 2 x 6, $fraction_1$ x $fraction_2$ factorial design. $Fraction_1$ values were 1/8 and 1/4. $Fraction_2$ values were 2/9, 2/7, 3/9, 2/5, 3/7, and 3/5. Problems were presented one at a time on a computer screen. Subjects were asked to write out their solutions on paper. Order of presentation of the 12 problems was randomized separately for each subject for each of three replications.

Computational estimation task (numerical symbols). Subjects were asked to make graphic estimates of the results of the same problems used in the computation task. A 16 cm horizontal line served as the response scale (see Figure 2). Left and right ends were labelled "0" and "1", respectively, so that the response scale was essentially an unpartitioned number line from zero to one. Subjects responded to a given problem by moving a short vertical line, positioned at the zero end, along the response scale until they thought its position corresponded to the sum of the displayed addition problem. There were 99 possible positions on the response scale, corresponding to numerical responses of 0.01 to 0.99. Order of presentation of the 12 problems was randomized separately for each subject for each of three replications.

The 2 x 6 factorial design is shown in Figure 3A ($fraction_2$ values are equally spaced along the x-axis for convenience). Each point represents the value of one of the resulting $fraction_1$ + $fraction_2$ sums. Note that the points form two parallel curves. By definition, estimates of these sums should show a similar graphical parallelism even if they are not precisely accurate. Using the logic of functional measurement theory (Anderson, 1981), this parallelism serves as an index of correct understanding of the operation of addition.

The six $fraction_2$ values were generated from the 2 x 3, numerator x denominator factorial design shown in Figure 3B. Numerators were 2 and 3. Denominators were 5, 7, and 9. Each point in Figure 2B represents the value of one of the resulting fractions. Note that the curves form a slight linear fan. By definition, implicit estimates of the size of single fractions, that is, estimates not actually made but implied by subjects' estimates of sums, should show a similar linear fan pattern. Such an obtained pattern would

$$\frac{1}{8} + \frac{3}{5} = ?$$



Adjust the marker's location,
then press the <RETURN> key.

Figure 2. A computational estimation trial.

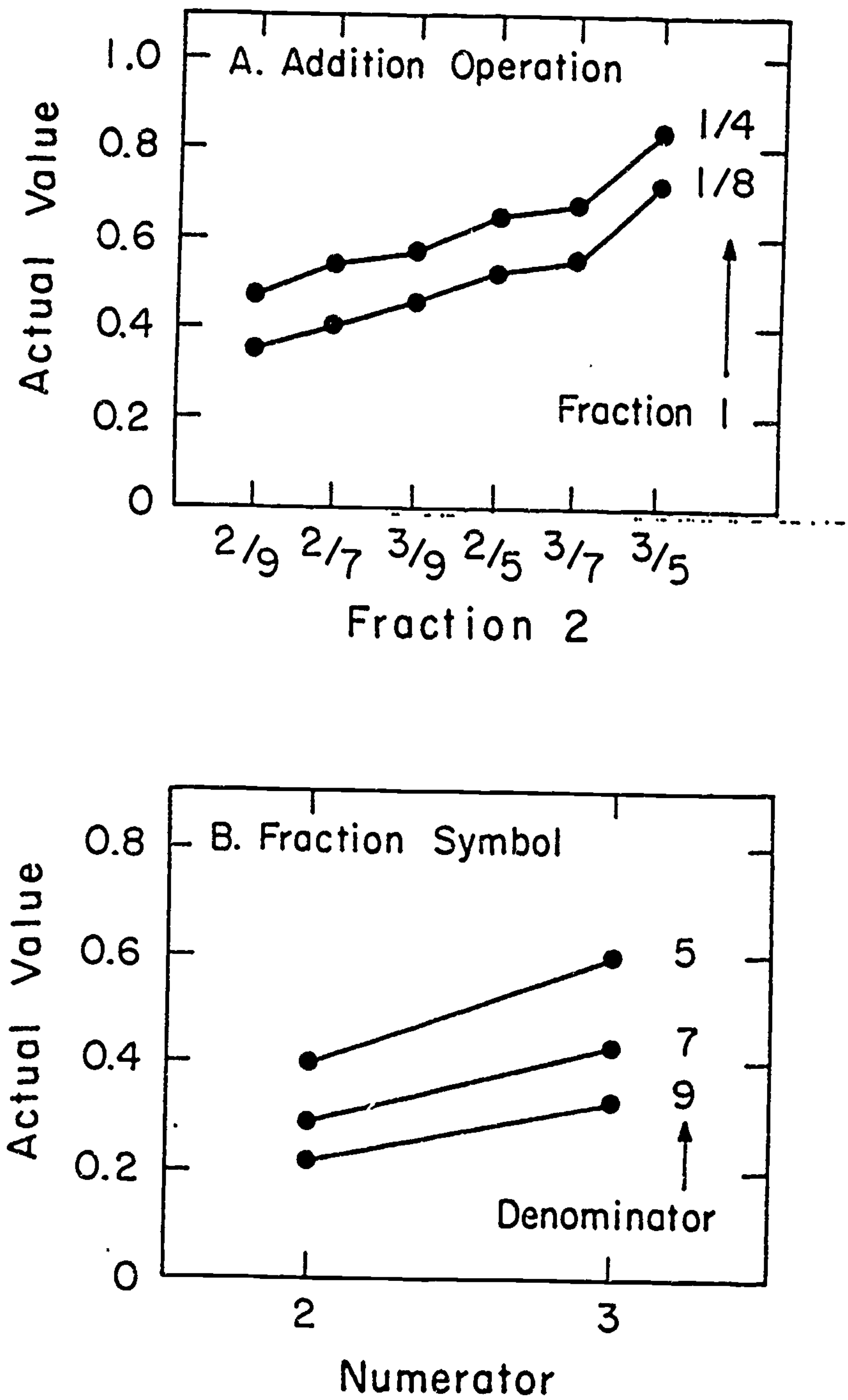


Figure 3. Theoretical curves for the computational estimation tasks.

indicate correct understanding of fraction size (Anderson, 1981; Cuneo, 1987).

Computational estimation task (regions). This task was the same as the previous estimation task except that the 12 addition problems were presented in terms of the region model for fractions (see Figure 4). Subjects were asked to estimate the sum of two shaded parts of a region. Since the actual values of the sums correspond to the points in Figure 3A, estimates of these sums should exhibit a similar graphical parallelism.

Estimation task (single numerical symbols). This task employed the same response scale as the previous estimation tasks but asked subjects to estimate the size of single fraction symbols. Fifteen fractions were generated from the 3 x 5, numerator x denominator factorial design shown in Figure 5. Numerators were 1, 2, and 3. Denominators were 5, 7, 9, 11, and 13. Note that this design includes the one used to generate $fraction_2$ values in the computational estimation tasks. Each point in Figure 5 represents the value of one of the resulting fractions. By definition, estimates of the size of these fractions should show a linear fan pattern similar to the one shown in Figure 5.

Order of tasks. Subjects completed the computation task, computational estimation task involving numerical symbols, computational estimation task involving regions, and estimation task involving single numerical symbols, in that order, in a single session. A final task involving estimates of the size of single regions is not reported here. Session length varied from 30 to 75 minutes.

Subjects. Twenty-nine sixth graders and 27 undergraduates participated in the study. Children were recruited through newspaper ads and flyers. Adults were recruited from an introductory psychology course.

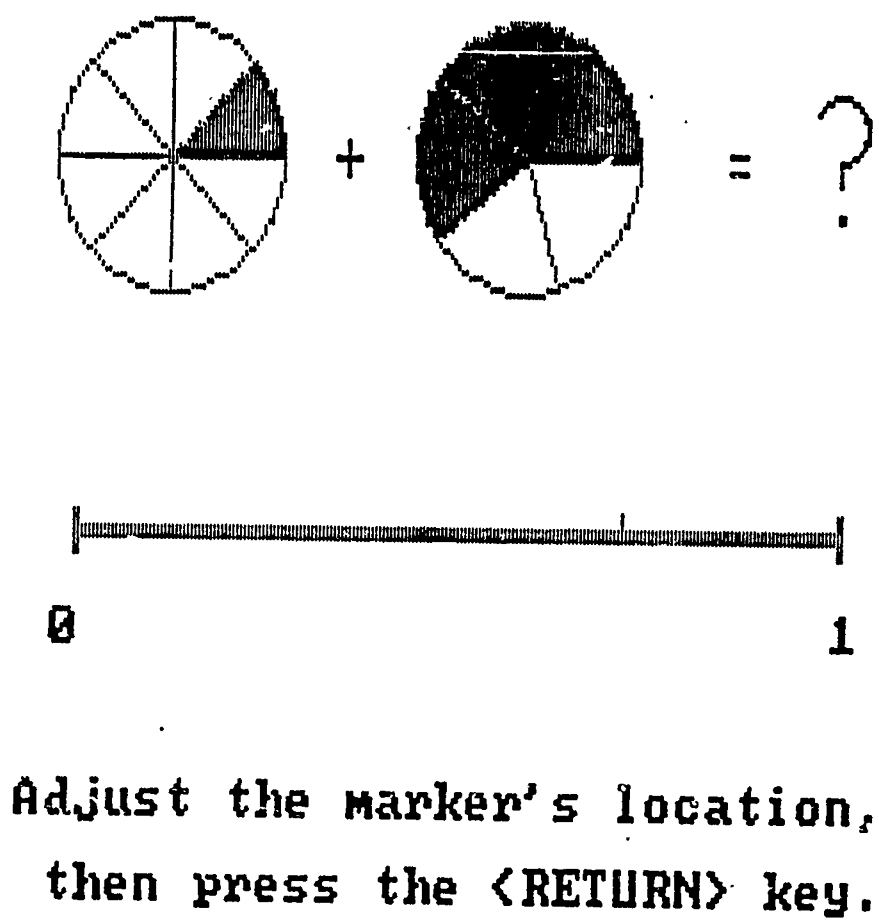


Figure 4. A computational estimation trial.

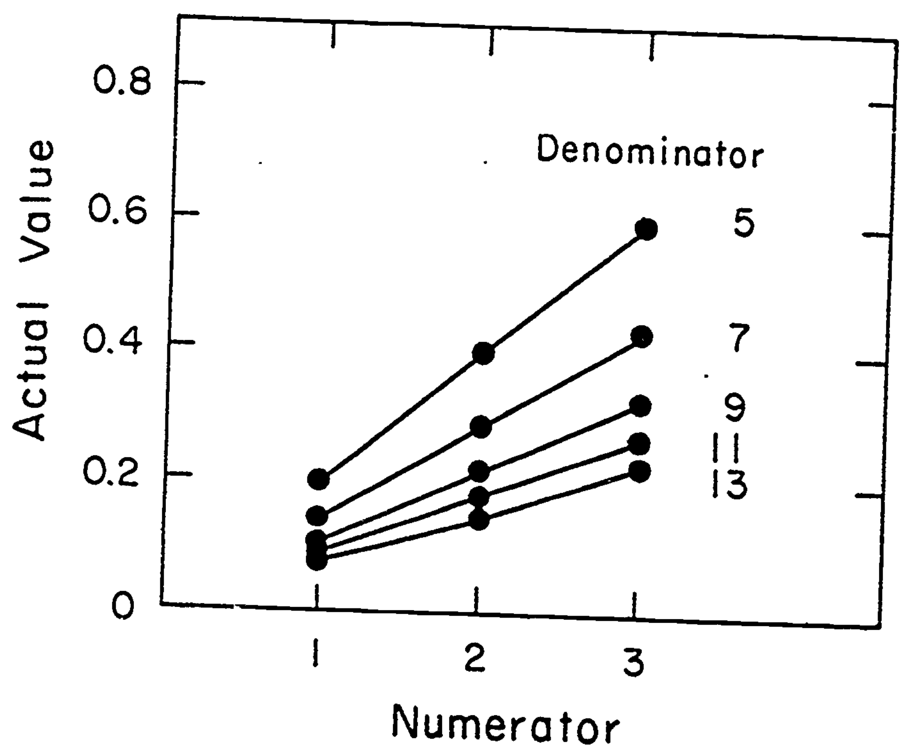


Figure 5. Theoretical curves for the estimation of fraction size task.

Results and Discussion

Computation Task

Subjects' performance on the computation task was classified as "high" if all 12 problems were scored as correct; otherwise, it was classified as "low". A problem was scored as correct if two of three solutions were correct. Due to the similarity of the problems, subjects typically were correct on all 12 problems or incorrect on all. Using this classification scheme, the computational performance of 14 children and 25 adults was classified as high, and that of 15 children and 2 adults, as low. Estimation results for the first three groups are presented below. Only overall group results are presented, so it is important to note here that results for individual subjects within a group generally mirrored the results for the group as a whole.

Computational Estimation Task (Numerical Symbols)

Understanding of addition. Figures 6A, 6C, and 6E show group results for the computational estimation task involving numerical symbols. Estimates of the children-high and adults-high groups (Figures 6C and 6E, respectively) showed the expected parallelism, indicating correct understanding of the addition operation. The graphical picture is quite different for the children-low group (Figure 6A). Deviations from parallelism occurred and suggest a problem with addition when fraction symbols are involved.

Understanding of fraction size. Figures 6B, 6D, and 6F show implicit estimates of fraction size for the three groups. Those for the adults-high group (Figure 6F) showed the expected linear fan pattern, indicating correct understanding of fraction size. Implicit estimates for the two groups of children, however, showed deviations from the expected pattern. As shown in Figure 6D, those for the children-high group exhibited graphical parallelism. This pattern suggests that these children understood the direct effect of

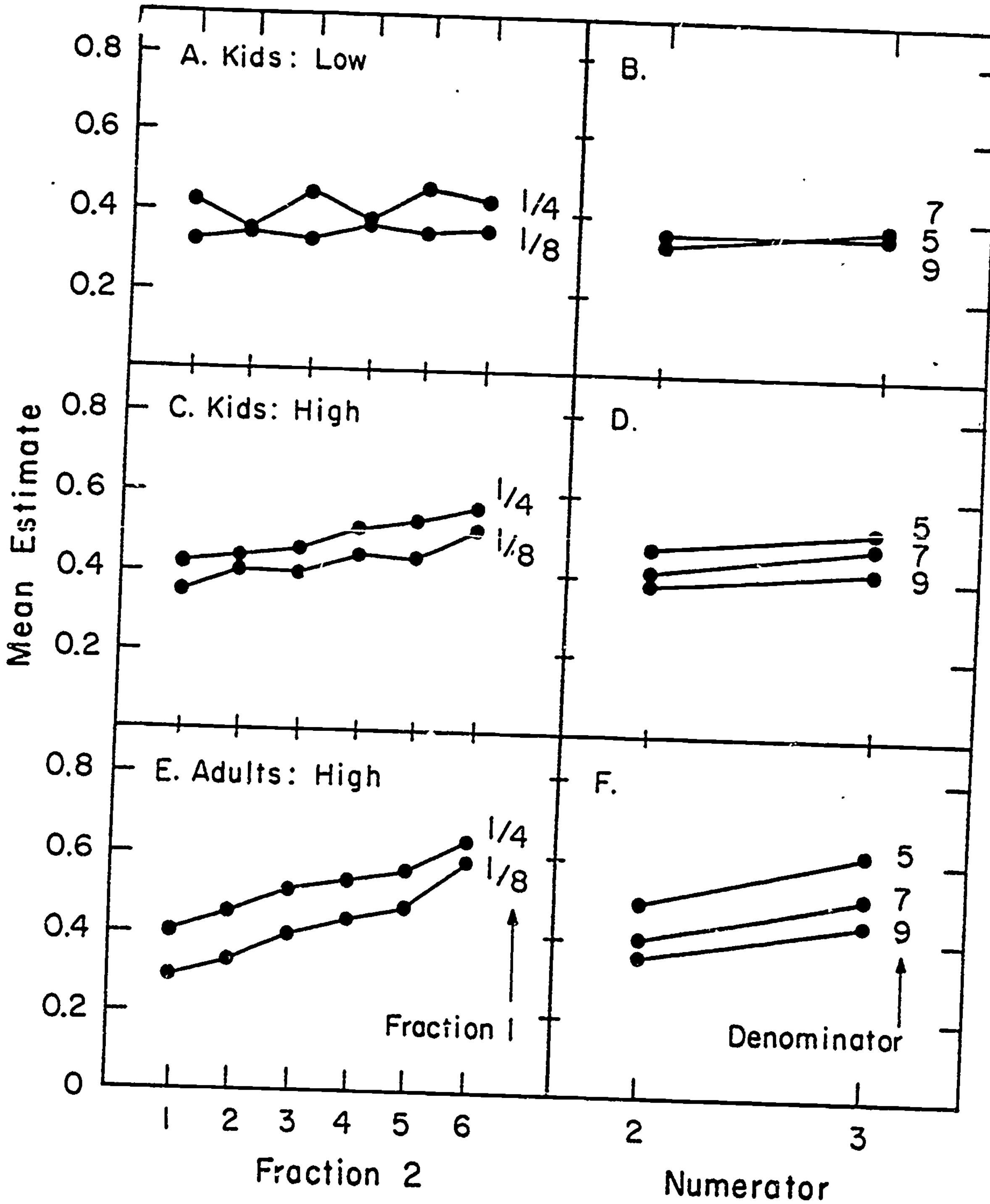


Figure 6. Estimates of the sum of two numerical symbols and implicit estimates of fraction size.

Computational Estimation Task (Regions)

Understanding of addition. Figures 7A, 7C, and 7E show group results for the estimation task involving sums of two parts of a region. In contrast to the previous task, estimates of all three groups showed the expected graphical parallelism. This indicates that all groups understood the operation of addition within the region model for fractions.

Understanding of fraction size. Figures 7B, 7D, and 7F show implicit estimates of fraction size within the region model for the three groups. Not surprisingly, implicit estimates of all groups showed the normative linear fan pattern.

Discussion. These results are important for interpreting the behavior of the children-low group on the previous estimation task. Their estimates of the sums of numerical symbols deviated from the normative pattern and it was suggested that this deviation might simply indicate that a problem in using the response scale. The appearance of the expected graphical parallelism in their estimates of region sums provides a strong case against this argument, since this pattern depends on appropriate use of the same response scale. The results also indicate that the problem was not poor overall estimation skill or weak understanding of the operation of addition per se. This group gave quite accurate estimates of region sums (mean percent error was 17, 10, and 11 for the children-low, children-high, and adult-high groups, respectively). The obtained parallelism indicates that they understood the addition operation per se, and extended this understanding to problems involving two fractional parts of a region.

This leaves the interpretation, consistent with the views of Carpenter et al. (1976) and Behr et al. (1985, 1986), that poor understanding of fraction symbols was the source of computational and estimation difficulties in the first two tasks. However, this interpretation was ruled out by the results of

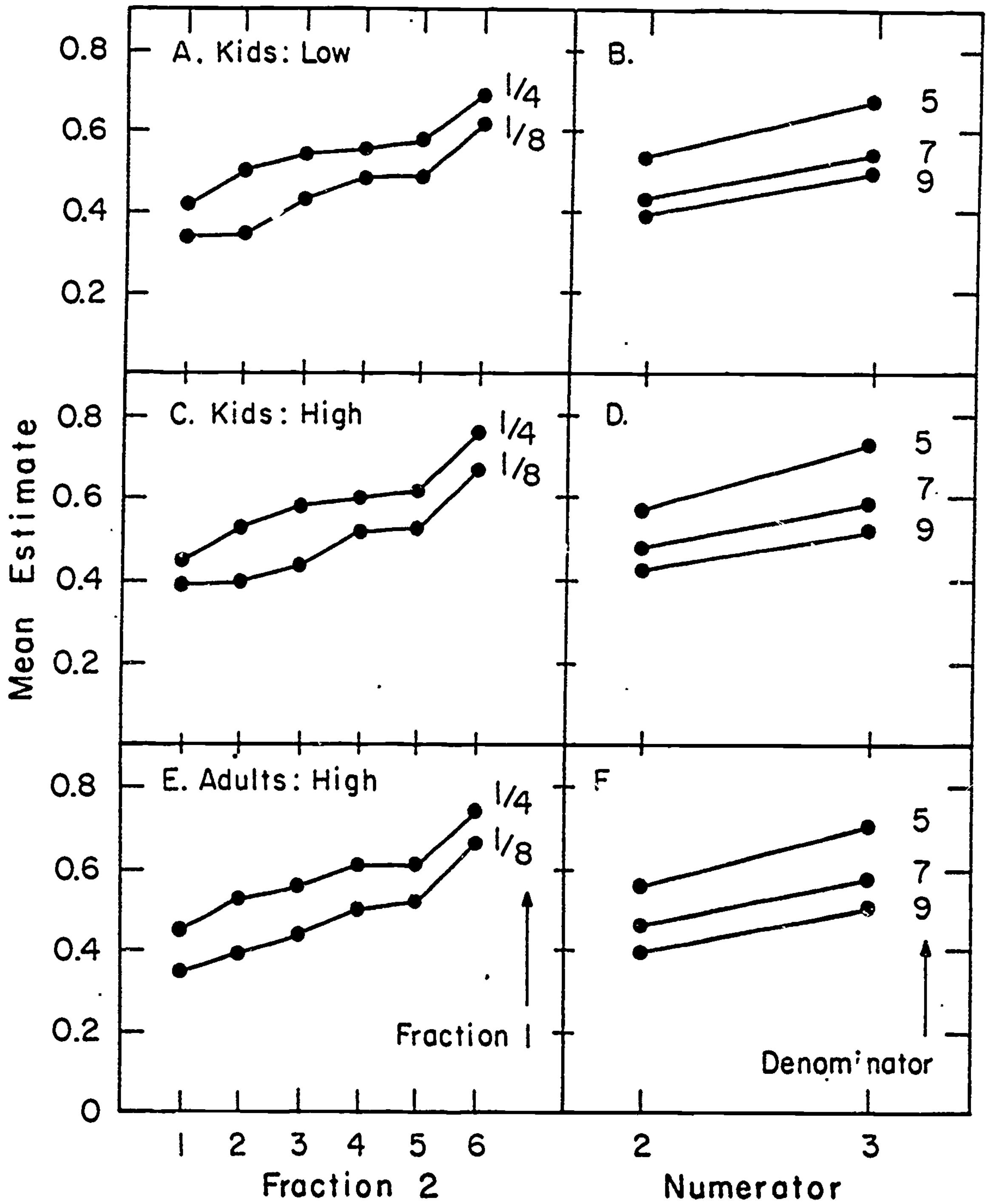


Figure 7. Estimates of the sum of two parts of a region and implicit estimates of fraction size.

the following task.

Estimation Task (Single Numerical Symbols)

Figures 8A-C show mean estimates of fraction size for the three groups. Estimates of all groups showed the normative linear fan pattern. This suggests that all groups had reasonably good understanding of fraction size, at least for the numerical symbols used in this study.

The above diagnosis is puzzling in the case of the children-low group. Recall that the estimates of fraction size implicit in their estimates of fraction sums in the computational estimation task suggested poor, if any, understanding of fraction size. This kind of between-task discrepancy in diagnosis is fairly common in the literature on mathematics education, and has led to the general notion that much of children's difficulties with school mathematics derives from a failure to access and apply relevant intuitive or conceptual knowledge (e.g., Ginsburg & Yamamoto, 1986; Hiebert & LeFevre, 1986; Lampert, 1986; Resnick, 1986; Resnick & Omanson, 1987). In the present case, children in the low-computation group had knowledge of fraction size but failed to apply this knowledge when estimating fraction sums. The discrepant diagnoses obtained in this study also serve as a reminder for us to exercise caution in inferring conceptual knowledge states from performance in computational tasks, even those that ask for estimates rather than computed answers.

Conclusions and Implications

Main interest revolves around the pattern of results for children who computed fraction sums that did not "make sense." These children also did not give reasonable estimates of fraction sums. However, they gave good estimates of single fractions, indicating that they understood that fractions represent quantities and have magnitudes. They also understood the meaning of "+" in addition problems involving fractional parts of a region. This pattern of

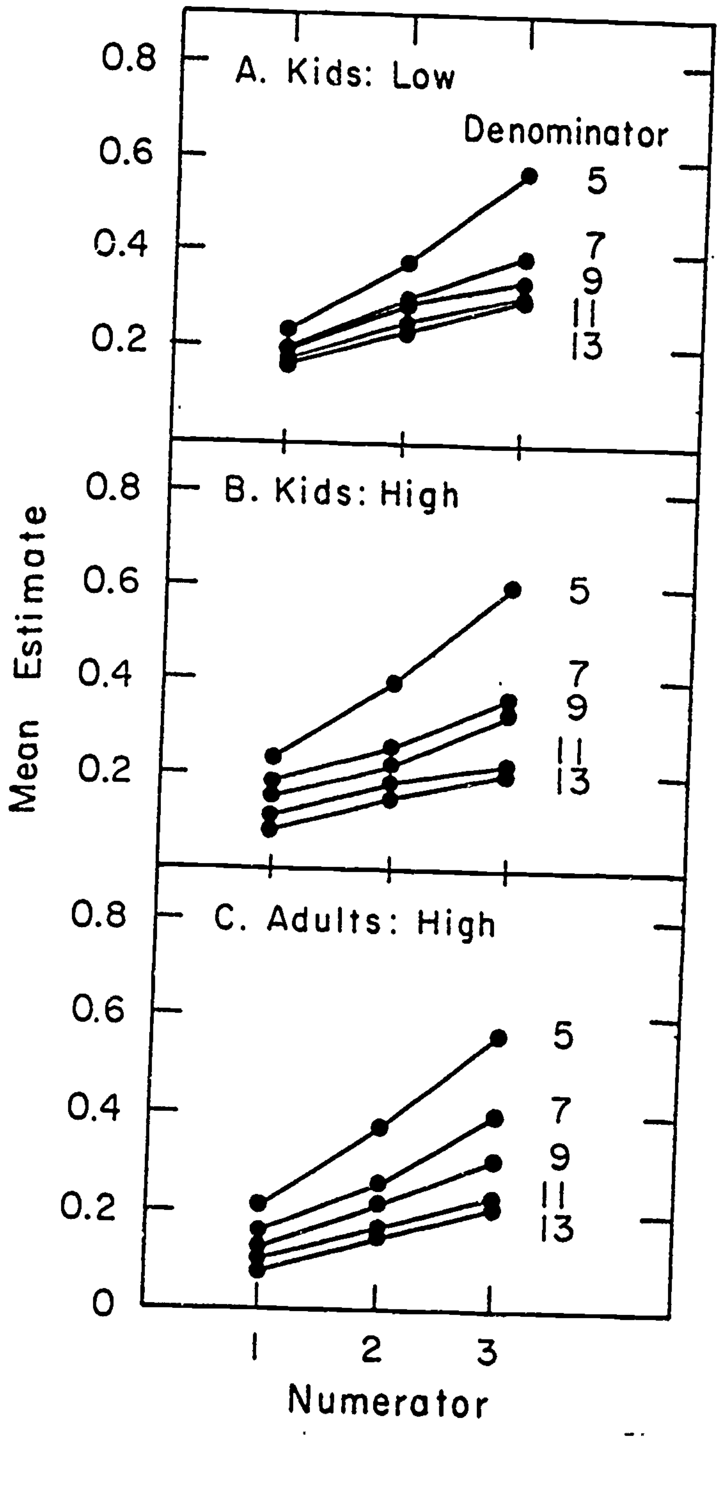


Figure 8. Estimates of the size of single fractions.

estimation performance suggests that these children had a respectable grasp of the two basic ideas involved in fraction addition. In terms of the framework in Figure 1, their difficulties at the sites of Estimating Results and Performing Computations were not due to deficiencies at the site of Understanding Symbols, at least not primarily. Instead, the problem seems to be that sites were isolated from one another. That is, the children did not draw upon their existing knowledge of addition and knowledge of fraction size when they computed and estimated fraction sums.

This "isolated knowledge" problem can be viewed and discussed in two slightly different ways, both of which make useful suggestions for instruction. One view is that the problem reflects the absence of connection between the site of Understanding Symbols, on the one hand, and the sites of Estimating Results and Performing Computations, on the other. Similar views have been offered in other mathematics domains, for example, multidigit subtraction (Resnick & Omanson, 1987), multidigit multiplication (Lampert, 1986), and decimal computation (Hiebert & Wearne, 1985). The implication is that classroom instruction should focus on building the appropriate connections. Lampert demonstrates some possible classroom strategies in the case of multidigit multiplication and, of particular relevance here, Behr et al. (1986) suggest some promising estimation activities in the case of fraction arithmetic. According to Behr et al., estimation activities focus children's attention on the quantitative meaning of a symbol or symbol phrase, and thus provides a good vehicle for developing such understanding.

A second and related way of looking at the "isolated knowledge" problem is in terms of a site of Understanding Symbol Phrases. Note that the framework in Figure 1 distinguishes Understanding Symbols from Understanding Symbol Phrases. This allows us to view estimation and computational difficulties as arising when children do not understand or view the computational problem format, that is, symbol phrases, in terms of an arithmetic action involving two quantities. However, this need not and, according to present results, does not mean that they do not understand the individual symbol components. The suggestion is that children may fail to integrate their separate knowledge of numerical symbols and knowledge of arithmetic operational symbols. It seems likely that this would result in their not applying either one to situations that require such integrated knowledge, that is, situations involving symbol phrases. Asking them to estimate rather than compute does not seem to facilitate this application.

According to this second view of the "isolated knowledge" problem, instruction should focus on building both conceptual and quantitative understanding of symbol phrases. One way to accomplish the former for fraction symbol phrases would be to use concrete or pictorial models to extend the basic ideas or meanings underlying arithmetic operations to the domain of fractions (see Post (1981) for a useful demonstration and discussion). For example, in the present study, children understood the computational problem format when regions rather than numerical fraction symbols were used. The idea would be to help them use and extend this understanding to the corresponding numerical problem. Another way to build conceptual understanding of fraction symbol phrases would be to use verbal or story problems, as suggested by Hiebert (1984). Previous work on addition and subtraction with whole numbers (e.g., Carpenter, Hiebert, & Moser, 1983; Carpenter & Moser, 1984) and addition with fractions (VanLeuvan, 1988) shows that many children adopt a "sense-making" approach to story problems but resort to "symbol-manipulation" in dealing with the corresponding number sentences, that is, symbol phrases. The idea would be to use children's informal knowledge of verbal problem situations to help them build meaning for the formal symbolism of fraction arithmetic.

The above mentioned strategies for building conceptual understanding of fraction symbol phrases focus more on the arithmetic operation than on the fraction symbols. It might be expected that they would have limited success unless combined with a strategy for focusing more directly on the quantitative meaning of fraction symbols. One such combination would result from embedding conceptual activities involving regions or verbal problems within an estimation response context. For example, one of the tasks in this study asked children to estimate the sums of fractional parts of a region. The suggestion here is that this kind of task may be an important tool for building understanding of fraction symbol phrases because it taps into children's intuitive knowledge of arithmetic operations as well as their understanding of fraction size.

The two different ways of discussing the "isolated knowledge" problem in children's mathematics learning are perhaps more similar than they are different. Both suggest instructional efforts aimed at building children's intuitive understanding of mathematics, and both would attach importance to the use of estimation activities. In the case of fraction arithmetic, both views would endorse classroom activities that involve estimating fraction sums or other fraction computational results. The difference seems to lie in the particular abilities and activities that would be expected to facilitate such computational estimation. According to the first view, classroom activities should also involve estimating the size of single fractions in order to build upon and develop children's understanding of fraction size. In contrast or perhaps in addition, the second view would recommend the joint use of activities that involve estimating, say, region sums in order to tap into, develop, and coordinate children's understanding of addition and their understanding of fraction size. In either case, the estimation tasks used in this study suggest themselves as possible classroom activities.

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