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ABSTRACT

This paper gives results of a review of 41 high school, junior high school, and college algebra and mathematics textbooks. The review focuses on two facets of the curriculum: (1) the presentation of the concept of variable; and (2) word problems and the presentation of problem solving skills. With regard to the variable concept it was concluded that the review of the 41 textbooks revealed an astounding diversity in the approaches to presenting the concept of variable. Many approaches preclude other approaches, are contradictory, or are in some other way incompatible with each other. No book fully gets accross the idea of continuous variability without becoming pedantic and cumbersome. Furthermore, a majority of the texts are in some way ambiguous or confusing. Relative to the second focus of the study, problem solving, textbooks need to include more word problems and thought problems that require more than a passive, algorithmic solution. Means for developing healthy problem solving habits in students should be explored and developed. (PK)

# THE PRESENTATION OF THE CONCEPT OF VARIABLE AND THE DEVELOPMENT OF PROBLEM SOLVING SKILLS: A MULTI - TEXT REVIEW*



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This paper is a review of 41 High School, Junior High School, and college algebra and mathematics textbooks. The review focuses on two facets of the curriculum.

I. The presentation of the concept of variable.

II. Word problems and the presentation of problem solving skills.

This study was motivated by recent research done at the Cognitive Development Project at the University of Massachusetts and by Dr. James Kaput at Southeastern Massachusetts University. Their research has provided convincing evidence that many college math students have a great deal of difficulty in solving certain types of elementary algebra word problems. Underlying student demonstrated misconceptions is, in part, a very tenuous and ill-defined conception of what variables are and what part they play in word problems and equations. (For a description of some of these studies, see Clement (1980), Clement, Lochhead and Monk (1981), Rosnick and Clement (1980), Rosnick (1980), and Kaput (1980).

This textbook review was initiated to search for some of the sources and possible solutions of these difficulties. The bibliography of the publications reviewed contains recent editions from several of the large and popular publishing houses as well as some texts used in the 1960's and early 1970's. In addition, several experimental and/or innovative texts were reviewed including some from the Stanford University School Mathematics Study Group (SMSG), the University of Illinois Committee on School Mathematics, and the Madison Project. Though the bibliography is not exhaustive, it presents a wide spectrum of styles of presentation and content.

## I. THE PRESENTATION OF THE CONCEPT OF VARIABLE

Kieran (1980), Matz (1979) and Davis (1975), in addition to the above mentioned authors have all noted that many students fail to regard variables in algebraic expressions as standing for number. Just as "g" is used as a label standing for "grams", some students believe that letters are labels standing for concrete entities rather than the more abstract "number" of things.

Preliminary findings from an on-going study by this author suggests that students' misconceptions about variables go beyond the "labels" confusion. The findings suggest the hypothesis that some, and possibly, many students believe that a variable is a label for a concrete entity and simultaneously is a symbol standing for several of the attributes of that concrete entity. For example, in a problem involving both the price and quantity of books (price was given) some students said that the variable B stood simultaneously for books, the price of the books and the quantity of books. If weight and size had been relavent to the problem one wonders whether students would allow the B to take on those meanings as well. One calculus student, in the course of an interview, gave at least seven distinguishable definitions for the letter B without demonstrating any recognition that what she was doing was inconsistent and contradictory.

That students have a "low attainment" of the concept of variable has very recently been corroborated by Tonnessen (1980) at the University of Wisconsin.

One clue to this low attainment of understanding might be the fact that most textbooks (<u>Beberman's High School Mathematics</u> and SMSG's

Programmed First Course in Algebra are exceptions) spend very little space and time discussing and defining the concept of variable. The traditional texts devote at most a page to explaining the concept but usually much less. Some do not even define the concept at all. This is true despite the fact that High School Algebra is predicated on the existence of variables, and that x's predominate on virtually every page of the texts.

Another clue to why there is so much apparent confusion about variables is that the concept itself is very difficult to define. As a consequence, the 41 textbooks reviewed contain almost as many ways of presenting the concept of variable. This concept can apparently be viewed from many different perspectives.

## Five Questions for Review

The texts were analyzed with respect to the following questions and the results are summarized in Tables 1 and 2. The diversity of answers to these questions demonstrates the complexity of the concept.

Question 1. Replacement Set. What constitutes the replacement set of a variable? A replacement set is made up of all those elements that can be substituted for a variable.

In High School Algebra, the replacement set for a variable is almost always made up of numbers, usually the entire real number line. (As mentioned above, this fact is lost on many students.) However, in Statistics and elsewhere, one often talks about qualitative, as opposed to quantitative, variables, where the replacement set can be made up of words. In differential equations, the variables represent functions and in Topology, the replacement set for the variables is often sets of sets, or, sets of sets of sets.

Does one then, when introducing the concept of variable, present a general definition that encompasses all possibilities for replacement sets, or does one simply say that variables will stand for numbers, thereby making the concept less abstract? This is one way in which the textbooks differ. The following are examples of three common ways of describing replacement sets.

A. The replacement set is a set of things. One of the most generalized definitions of a variable is found in High School Mathematics by Beberman. "A variable is a pronoun" and can be replaced by "numbers, sets, points, people, teams, etc". (Beberman does make a distinction between variables that are replaced by numbers and refers to such variables as pronumerals.)

B. The replacement set is a set of numbers. An example is given by Traves et al in Using Algebra who defines a variable as representing "any number in a replacement set" of numbers. These two definitions are further contrasted by:

C. Variables are replaced by one number. Nichols et al in Holt Algebra I simply says that a variable 'n' "takes the place of a number." There is no reference to sets and there is no sense of the multi-valued nature of a variable. Tables 1-a and 2-a catalogue the texts in terms of how, in the definition of variable, replacement sets are described. Included in the tables are categories D and E.

Category D includes those texts with unorthodox approaches that cannot be categorized into A, B, or C above. For example Davis, in work done at the Madison Project, uses symbols like □ which he calls "a placeholder for a thing". Krause in Mathematics I: Concepts, Skills and Applications defines a variable only in the context of probability. Haber-Schaim et al in Prentice-Hall Mathematics (books 1 & 2) uses a "window" approach similar to Davis' above while also introducing variables via arguments involving combinatorics.

Category E is made up of those texts which have no explicit definition of variables.

Question 2. Variability. What impression of the manner in which variables vary does the book convey? There are at least three different possibilities.

F. Nonvarying variables. This is what is often referred to as an unknown. It is a way of conceptualizing a variable that is similar to when one solves a word problem for which there is a unique answer or at most, finitely many answers. Consider the following problem.

"Find the length of the side of a square whose perimeter is 20 in." One might solve this problem as follows: "Let x be the length of one side. Then 4x = 20. So x = 5." At the stage where on thinks "x is the length of one side" and despite the fact that the variable x has an infinite replacement set (all of the positive real numbers) one conceptualizes a unique square and the side of that square does not change.

Many textbooks only present word problems that have a single numerical answer. In these the student may infer that the only appropriate replacement for a variable is a single number. Either variables are not replaced when found in expressions like "x + 5" (where there is no solution) or the x is only replaced once. Rucker, et al in Heath Mathematics provides an example. The expression (A + a) - n is given along with the following table.

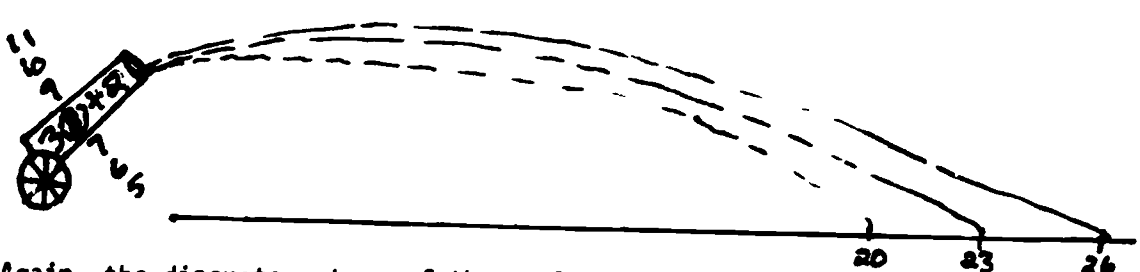
| A | a | n |
|---|---|---|
| 5 | 2 | 4 |

The replacements are then shown; vis. (5 + 2) - 4.

Again, the implication is that each variable has exactly one appropriate replacement.

G. Discretely varying variables. Considering variables in functional relationships differs from the "static" situation described in F above. The following equation relates the number of eggs one has to the number of egg cartons one has ( a dozen eggs per carton). $12x = y$ where x is the number of cartons and y is the number of eggs. The expression calls for one to conceptualize infinite (or at least a large number) of possible extensions of x.

One way of demonstrating the discrete nature of the variability of a variable is with tables, a technique used by several texts. Another example is the following: (Note that it is not necessary to have an expression in two variables to demonstrate discrete variation.) Pearson et al in Modern Algebra — A Logical Approach used a "cannon" to solve the expression $3x + 2 = 26$. The idea is that as different values for x are fed into the cannon, the cannon shoots out a value for the expression, aiming for 26.




Again, the discrete nature of the replacements is emphasized.

H. Continuously varying variables. This is similar to the discrete situation except that one must conceptualize the variable changing continuously. Examples of continuously varying variables are those in the equation $D = 40 \cdot t$ where D is distance and t is time. To understand that a variable can change continuously is an essential prerequisite to understanding the concept of the limit in calculus; and thereby calculus itself.

It can be confidently stated that no text reviewed fully explained the idea of a continuously varying variable. A few that came close did so by repeatedly using graphic representations for the functions of continuous quantities; or by emphasizing the use of variables in tautologies, stressing the fact that the letters could be replaced by any number.

The difficulty in getting the concept of continuous variation across is compounded by the fact that the very act of replacement must be done discretely. That is, one would be hard pressed to demonstrate the continuous nature of the variable D in D = 40t by plugging in values for t. This dilemma is further exasperated by the fact that though distance and time are conceptually continuous, any measurement of them must be discrete. What is continuous is the image of a moving point, and this is hard to symbolize on paper. Beberman comes close to dealing with this issue when he defines a "variable quantity". A variable quantity is a function whose range is quantitative. An example of a variable quantity is the function A. A is the set of all ordered pairs (x,y) where x is an element of the set of all squares and y is an element of the positive real numbers such that y is the area of x. (Beberman calls both x and y variables, y being the type of variable called a pronumeral.) Because A is viewed as a function that changes as x changes, the continuous nature of A can be inferred. However, the continuous nature of y may still be lost.

It is an open question as to whether dealing with continuous entities discretely simplifies the problem or whether it is an oversimplification that loses the essence of continuity. In fact, it is possible that both approaches are too abstract for many High School students, thereby implying the need for an approach similar to F above. Tables 1b and 2b catalogue the texts in terms of how variability is presented.

Question 3. Constants. Is a distinction made between constants and variables? Just as the concept of variable has a multitude of definitions, so does the term "constant". Some texts define constant as a special case of a variable. Rich defines constant in Schaum's Outline as "a variable with a replacement set of one number". Beberman takes the connection between constant and variable a step further. He sees the numeral 2, for instance, as being a variable quantity (thus a function!) whose range is the set {2} and whose domain is the real line. Thus 2 = {(x,y): y = 2}. 2 is then a "constant va iable quantity" or constant for short. In short, Beberman not only sees letters with a replacement set of one number as being a constant variable quantity, but also includes numerals in that category.

Several texts do not consider constants as variables. Constants, according to Bechenbach in Modern College Algebra and Trigonometry, are letters which have a one element replacement set, whereas variables require at least a 2 element set. Π is an example of a constant so defined.

Other texts use the term "constants" in a very different way. Banks in Algebra: Its Elements and Structure defines a literal equation (e.g., $y = ax + b$) as an equation where "constants" are represented by letters, thus distinguishing a from x. How a is different from x is not made clear.

Thus the concept of a constant in algebra, like the concept of variable, can mean different things in different texts.

Question 4. Solution sets. Is a solution set defined and is there a difference between the solution set of an expression and the replacement set of a variable? One potential difficulty with defining a variable as standing for any one of a given set of numbers becomes apparent when dealing with solvable word problems with one or two "unknowns". How, for example, can a variable stand for any one of the real numbers when there is only one or two correct solutions? At least 10 of the texts attempted to address this issue by defining a solution set that is different from the replacement set. They emphasized that distinction by noting that an open expression (e.g. $3x + 2 = 26$) is either true or false depending on which value of the replacement set one uses for x. Furthermore, any open expression is true for either some, all, or no values for x. Those values that make the expression true make up the solution set.

This approach can be contrasted with the one in Integrated Algebra and Trigonometry by Schlumpf et al. Here, unknowns are not variables. Those values for which x can be replaced are only those that make the expression true. This precludes the possibility of a false expression.

Question 5. Is the presentation of the concept of variable confusing and/or ambiguous and if so, how? Conservatively, at least 25 of the 41 books reviewed can be described as being, in some way, ambiguous or confusing concerning their presentation of the concept of variable.

I. Contradictory usage of symbols. Page in Number Lines, Functions and Fundamental Topics describes a function, b, that maps the "frame" $\square$ (a placeholder for a number) to the expression $\square + 5$. He writes $\square \xrightarrow{b} \square + 5$. (This symbolism in itself might be confusing to a beginning algebra student.) On the very next page, he introduces the symbol b as meaning "back" on a number line. That is, b2 means go back two on the line. Eventually he shows that b2 = -2. The two adjoining pages contain two completely different usages for the same symbol.

Another example of potentially confusing symbolism is the following. Throughout students' science curricula, the letter g is used as a label that stands for the word gram. Bolster et al in Mathematics Around Us: Skills and Applications, however, uses g in some problems to stand for the number of grams. This contradictory usage could conceivably result in the following scenario. A student who believes g stands for the word gram, when asked to write an equation that shows there are .0022 pounds per gram, might write .0022p=g where p means pounds. Another student, after reading Bolster's text might write .0022g=p where p is the number of pounds and g is the number of grams. Bolster should be commended on using letters to stand for numbers, but should be criticized in doing so with a letter that is most commonly used as a label.

J. Overabundant use of letters for symbols. Dolciani et al, in Modern School Mathematics: Structure and Method, use letters on one page of the text to stand for the name of a set. On the next page, letters are introduced as variables standing for elements of that set. In Dolciani's High School text, Modern Algebra: Structure and Method, letters are used in some places as names for points on a number line and as variables in other places. Beberman uses symbols copiously, including logical quantifiers like $\forall x$ (for every x) and expressions like $P\{(x,y), x \in \mathcal{S}_{1+w} : y = 2\} \cdot (1 + w)$. Though there is no formal contradiction in the multiple symbolic usage of letters in these texts, one must consider the possibility that it contributes to the confusion students have surrounding the concept of variable.

K. Lack of definition or adequate discussion. As mentioned earlier, most texts devote very little space to discussing the concept of variable. Some give no definition at all. These latter texts might be contributing to the confusion of students through an error of omission, rather than one of commission. Tables 1C and 2C indicate which texts are categorized in I, J, or K above. It should be noted that for several of the texts that are not included in any of these categories it can be argued that their approach to variables is confusing or inadequate. For example, a text that implies that a variable does not vary or can be replaced by only one number can be labelled as inadequate. Such a value judgement however, will be avoided in this paper.

## Conclusion

The review of 41 textbooks has revealed an astounding diversity in the approaches to presenting the concept of variable. Many approaches preclude other approaches, are contradictory, or are in some other way incompatible with each other. No book fully gets across the idea of continuous variability without becoming pedantic and cumbersome. Furthermore, a majority of the texts are in some way ambiguous or confusing.

## II. WORD PROBLEMS AND THE PRESENTATION OF PROBLEM SOLVING SKILLS

The second focus for this textbook review has been to analyze approaches to word problems and attempts at developing problem solving skills. As was true for the presentation of variables, there is an enourmous diversity in terms of the quantity, quality, and style of word problems and problem solving instruction.

At the top of Tables 3 and 4 are a list of eight factors that could possibly contribute to poor problem solving skills or to the development of word problem related misconceptions. This list is meant neither to be exhaustive nor conclusive. However, it does provide a means for demonstrating the diversity of approaches between the textbooks.

The rest of this section will provide the rationales for the inclusion of each of the eight factors in the set of possible causes of student misconceptions. Some examples from the texts are provided, including, for the sake of comparison, examples of texts which avoid some of . . difficulties. One highly commendable book from which some of the positive examples have been drawn is Algebra and Trigonometry: Functions and Applications by Foerster. It is the only book, of the 41 books reviewed, that successfully avoided all eight difficulties.

1. Fewer than 25% of all problems are word problems or "thought" problems. One of the clearest contributors to poor problem solving skills is the fact that a majority of the text books (over 60%) devote less than 25% of their exercises to problems requiring more than the mechanical manipulation of expressions and equations. In fact, almost

half of all of the texts reviewed devote less than 10% of their exercises to these more involved "thought" problems. Furthermore, of the eleven Algebra I and Algebra II texts put out since 1975, only one devoted more than 20% of its exercises to word or thought problems. The one exception is Foerster's book, which devoted well over 50% of its exercises to word problems and problems involving critical analysis.

Consider, as an example, the problem of finding the equation for a line connecting two points. Most traditional texts approach this problem by providing a formula (often $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$) and giving a list of exercises that consist of pairs of points. The student solves each exercise by using the formula. Foerster, on the other hand, after developing a strategy for finding linear equations from a pair of points, uses word problems as exercises. The student then not only practices the mechanical skill of finding linear equations but also continues to increase the analytical skills that go beyond the mechanical. At the same time, he is showing how linearity manifests itself in the real world. The following is an example.

Cricket Problem

Based on information in Deep River Jim's Wilderness Trail Book, the rate at which crickets chirp varies linearly with the temperature. At 59°F, they make 76 chirps per minute, and at 65°F they make 100 chirps per minute.

a. Write the particular equation expressing chirping rate in terms of temperature.

b. Predict the chirping rate for 90°F and 100°F.

c. Plot the graph of this function.

d. Calculate the temperature-intercept. What significance does this number have in the real world?

e. Based on your answer to part d, what would be a suitable domain for this linear function. Make your graph in part c agree with this domain.

f. What is the real-world significance of the chirping rate intercept.

g. Transform the equation of part a so that temperature is expressed in terms of chirping rate.

h. What would you predict the temperature to be if you counted 120 chirps per minute? 30 chirps per minute?

The constrast of this approach with that of, for example, Stockton's in Essential Precalculus is dramatic. In the algebra section of the latter book there are 1,000 problems, all of which require little more skills than memorizing formulas and mechanical techniques. There are only ten word problems, and those are only included as an appendix to the Algebra section with no instruction or introduction provided.

2. Virtually no instruction in problem solving is given. Many books that did include some word problems gave virtually no instruction as to their solution. Of those books that did include some problem solving strategies, several gave just a bare outline of steps involved in the process. Take for example Algebra: Its Elements and Structure by Banks et al. They outlined six steps.

1. Read the problem carefully.
2. Analyze the problem to determine what is to be found.
3. Introduce variable(s).
4. Write open sentences using those variables.
5. Solve the open sentences.
6. Check the solution.

However, Banks, et al, included virtually no explanation as to how to "analyze the problem", how to choose appropriate variables, or how to write appropriate open sentences.

This can be contrasted with a nine step approach advocated by Hughes-Hallett in The Math Workshop: Algebra. In addition to the steps suggested by Banks, she includes additional steps for drawing pictures and finding appropriate formulas. She also breaks the equation writing steps into three separate steps. The following is an example of how she attempts to explain each step.

> Step 4. Write down in words an equation connecting the various quantities in the problem.
>
> This equation is usually hidden in, or implied by, the wording of the problem. If you can't see it, read through again carefully, and ask yourself what the problem is telling you about one quantity being equal to another.
>
> Step 7. Write the equation in symbols, by expressing the unknown quantities in terms of the variable and substituting any numerical values given in the problem.

She then provides many examples of problems which she has solved using the nine step method. In Tables 3 and 4, an x under category 2 means that virtually no problem solving strategies were given in the text, either explicitly or implicitly.

3. Word problems are first introduced with pure number problems.

It was noted in section I of this paper that some students tend to think of variables as labels for concrete entities while others may associate variables with more than one of the attributes of that concrete entity; (B stands for "books" and/or all of the attributes of books). This excessive association of the symbol with the concrete entity may be fostered by introducing word problems with "pure number" problems.

An example of a pure number problem is the following. "Tom got an answer of twenty when he added four to two times his original number. What was the original number?" In solving this problem, one might write "let x be the original number." Here x is, appropriately, directly associated with a number. Contrast this to the following problem. "Tom bought 20 books which is four more than two times the number of books he had originally. How many did he have originally?" The concrete entity of interest in this problem is "books". However, x should not be directly associated with books but to some number (of books).

The hypothesis that an overemphasis on pure number problems fosters misconceptions concerning the role and meaning f variables in word problems is, at present, unsubstantiated. Nevertheless, it is an interesting characteristic for comparison. Texts whose first section on word problems primarily focuses on pure number problems were checked under this category.

4. <u>All word problems presented can be solved arithmetically or algorithmically.</u> That word problems can be solved arithmetically (without the use of variables) is not necessarily an indictment of a text, especially at the Junior High School level. But if all word problems can be solved algorithmically, that gives more cause for concern. (What is meant here by solving a problem algorithmically is that a set formula or pattern can be somewhat mindlessly applied). One common attribute of poor problem solvers might be that they approach word problems passively. Word problems that are not phrased in such a way as to immediately suggest the path towards a solution could prove unsolvable

for the passive student.

Many books provide what amounts to algorithms for solving particular kinds of word problems. Holt Algebra I by Nichols et al is an example. Problems are separated as to their "type" (i.e., coin, perimeter, or distance, rate and time problems). A method of solution for each type is provided and only exercises for which that method can be directly applied are given. No attempt is made to compare types of problems. No transformations of any kind are required of the student. For example, there are many "rate" problems that are analogous to distance, rate and time problems but these are not included in the text. A student merely has to plug in to the model chart and/or model formulas provided. It is highly questionable that this method of presenting word problems would help to develop generalizable problem solving skills.

On the other hand, in Foerster's "Cricket Problem", the student is forced to actively analyze the situation. Each problem is significantly different from the next. The questions asked of the students require a non-algorithmic approach to at least part of each problem. Furthermore, Foerster encourages the use of qualitative analytical reasoning as seen by the following problem.

> Draw a graph that shows how "as you play with a yo-yo, the number of seconds that have passed and the yo-yo's distance from the floor are related."

This was one in a set of forty problems. The graph of each problem was different in a significant way. Each problem required active thought and qualitative processing on the part of the student. A student could not remain passive and still be successful on this type of problem.

A text was so categorized in #4 if a large majority of the word problems (over 75%) could be solved algorithmically.

5. The wording of problems is contrived, apparently to avoid mental effort. Many textbooks seem to take the "path of least resistance" in terms of the wording and structure of word problems. This was seen in #4 above where presenting problems which fit into an algorithmic mode allowed the student to succeed passively.

It is also apparent in the wording of the word problems. Consider the following two sentences. "The first number is twelve less than the second number." "The first number is equal to the second number decreased by twelve." A correct equation for both sentences is $x = y - 12$ where x represents the first number. Notice that the latter sentence more easily maps into the equation. However, it is more awkward and contrived than the former. Many texts opted for the latter type of sentence over the former. They protect the student from a confrontation with the meaning of words, thereby reducing the mental effort required of the student.

Clement (1980) and Clement, Lochhead and Monk (1981) first documented what is referred to as the "reversal error". This reversal error manifests itself with problems like the following (called the Students and Professors Problem).

> Write an equation which represents the following statement. "At a certain university, there are six times as many students as there are professors." Let S be the number of students and P be the number of professors.

The correct answer is $6P = S$. The reversed answer is $6S = P$ (over 50%

of all of the business and social science calculus students who tried this problem answered it incorrectly). However, when the wording of the original problem is changed to "the number of students is six times the number of professors" the error rate drops dramatically.

Almost all of the texts avoided wording of problems in a way that would induce reversal errors, opting for the latter wording. This is true even though language like "there are six times as many students as professors" is common and comfortably colloquial. When it is found in the text, it is usually found in its additive version (there are six more of this than of that). Otherwise, it is found in the context of one variable problems, which circumvents the reversal difficulty (see #6 below).

Most of the text books reveiwed (Foerster being a notable exception) focused on word problems with short quantitative answers. Problems whose final answers are equations or verbal analysis were very rare.

By taking these paths of least resistance, the student is probably more successful (in terms of the number of correct answers) and therefore the teacher can (naively) feel more successful. But one must question whether, by reducing the need for mental effort, a text book is avoiding those situations where conceptual learning and development really do take place.

An x was placed under category 5 if a large majority of the word problems (over 75%) were worded so as to fit into a pattern or to, in some other way, reduce the need for mental effort on the part of the student.

6. <u>One variable problems are emphasised over two.</u> Many of the word problems that are presented in algebra texts can be done both with one or two variables. Rich, in <u>Modern Elementary Algebra</u> (a Schaum's Outline) demonstrates both methods side by side. An abridged example is as follows.

> The larger of two numbers is three times the smaller. Their sum is eight more than twice the smaller. Find the numbers.

Let s= smaller number
3s = larger
Then 3s + s= 2s + 8
So s= 4
3s= 12

Let s= smaller number
l= larger number

Then l=3s
l + s= 2s + 8
So s= 4
l= 12

Most texts opt strongly for the one variable solution. In these texts, all problems that can be, are done with one variable, even when a two variable approach would be less awkward. In so doing, the explicit functional relationship between the two variables is circumvented. It is this functional relationship that brings out the dynamic quality of variables. That is, in the language introduced in section I, to understand the functional relationship between two variables, one must be able to perceive the discrete variability (if not the continuous variability) of the respective variables.

To further explain how a one variable solution to a problem is significantly different from the two variable solution, consider the following version of the Students and Professors problem. "There are six times as many students as there are professors. If there were 10 professors, how many students are there?" (The answer is 6x10 or 60 students.) The solution to this problem is analagous to that of a one-variable solution of the original problem, which might look as follows.

Let P= number of professors
If there are P professors, then there would be 6P students.
So 6P= number of students.

Note that, though "6P= number of students" is equivalent to 6P= S (since S is the number of students), the former representation avoids the explicit comparison of two discretely varying variables.

Among the same people who did poorly on the original Students and Professors problem, over 95% were able to solve the arithmetic version. It appears that the "path of least resistance" would be to opt for the one-variable approach to word problems because by doing so, one may be cognitively "closer" to the more simple arithmetic approach. However, by avoiding functional relationships, one avoids one of the fundamental concepts of mathematics.

Very few texts made a connection, as Rich did, between one and two variable solutions to the same problem. If that connection is not made, it is possible that those word problems that come later in the curriculum that do require the use of functional relationships may appear foreign and unreasonably difficult to the students.

Tables 3 and 4 were filled out conservatively for this category (#6) in that only those texts that virtually excluded any two-variable representations in their solutions to word problems were included.

7. <u>Translations are demonstrated via a key word match</u>. In #5 above, an example was given of a translation that can be called a "key word match". In a key word match, key words in the English sentence are directly mapped onto symbols in the algebraic expression. The order of the words is preserved in the symbols, as seen with the previously cited example.

| The first number | is equal to | the second number | decreased by | 12 |
|---|---|---|---|---|
| $x$ | $=$ | $y$ | $-$ | $12$ |

At least 12 of the texts used this method to demonstrate translations.

The inadequacy of this technique is made obvious when one tries to apply it to the Students and Professors problem. In the sentence, "there are six times as many students as professors, the verb "are" does not come between the "students" and the "professors". As a result, subjects solving this problem often erroneously associate the equal sign with "as". They also use S and P to be the labels for students and professor, respectively. This results in the reversed equation, 6S= P, as shown below.

| There are | six | times | as many | students | as | professors. |
|---|---|---|---|---|---|---|
| | $6$ | $\times$ | | $S$ | $=$ | $P$ |

The teaching of a direct mapping from words to symbols is an attempt at an algorithm for translations. When it works, (and those texts who use this method usually only present problems for which this method would work) the student can perform the problem passively.

Furthermore, in the physical and social sciences, the results of experiments are often displayed in tabular or pictoral forms. It is often necessary to translate those representations into algebraic expressions. The only text that gave problems requiring the student to move from a tabular or pictoral representation to an algebraic one was Foerster's.

By focusing on only simplistic methods of English to algebra translations, textbooks again may be taking the "path of least resistance", thereby failing to foster general problem solving abilities.

8. The symbolism is confusing or ambiguous. This issue was addressed in part I. It is included here simply for the reason that if the symbolism used in an attempted solution to a word problem is not understood, the problem becomes difficult, if not impossible, to solve.

As an example of a positive approach to the meaning of symbolism in word problems, consider the following exercise found in the Stanford University School Mathematics Study Group's (SMSG) Programmed First Course in Algebra.

> Let us consider the open phrase 7w. Which of the following meanings might w and 7w have in a problem?
>
> A. w is the water in the well and 7w is seven times the water in the well.
>
> B. w is the number of wolves in a zoo, and 7w is the number of wolves in seven zoos.
>
> C. w is the number of dollars I paid for one bushel of wheat and 7w is the number of dollars I paid for seven bushels of wheat.
>
> Answer
>
> A. Don't you mean the number of gallons of water or the number of pounds of water? Remember that the variable must always represent a number...

B. This is a possible choice, but would be a correct one only if you knew that each zoo had the same number of wolves...

C. You are correct... It is more reasonable to expect each of several bushels of wheat to cost the same amount than to expect each of several zoos to contain the same number of wolves.

Throughout its instruction on word problems, this text continually reminds the student that the letters stand for numbers, and forces the student to actively assign a meaning to the letters. The woman mentioned in section I who accepted seven different meanings for one letter in a word problem, could well benefit from this approach.

Those texts with marks in any of columns I,J, or K of tables 1 and 2 were categorized in #8 in tables 3 and 4.

## Conclusion

Tables 3 and 4 indicate the diversity among the texts in terms of what were their possible drawbacks and positive points. Each text scored negatively on the average in four of the eight categories. As mentioned earlier, only Foerster's text scored negatively in no categories. Considering only noninnovative Algebra I and Algebra II texts published since 1975, each text scored negatively on an average in close to five of the eight categories.

An important outcome to note is that in categories 4, 5, and 6, a majority of texts scored negatively. These three categories all described "paths of least resistance" which enabled the problem solver to be more passive. It is possible that the passivity of the problem solver, coupled with a lack of experience (related to category 1) and a confused conception of variable (related to category 8) can combine to significantly influence a student's ability to solve word problems.

In addition to factors encouraging passivity in problem solving, other factors (categories 3, 5, and 7) may have direct bearing on some of the problem solving misconceptions cited by Clement, Rosnick, et al.

## Recommendations.

The research cited in this paper demonstrates that there seems to be a significant problem with regard to students' problem solving skills and their conception of the concept of variable. That these issues have become a pertinent and crucial concern of math educators is demonstrated by the fact that the National Council of Teachers of Mathematics (NCTM) has devoted its 1980 Yearbook to Problem Solving in School Mathematics.

This textbook review provides some insight into the complexity of the problem. There is an enormous diversity in ways of conceptualizing variables and word problems and there is an enormous diversity in teaching approaches to those issues.

As is always true in education, there is much room for improvement. A first step might be to understand the role that variables play in the curriculum, to better understand students' present conceptions, and to develop more clearly defined goals as to what we, as math educators, would like those conceptions to be. If a substantial gap exists between what students' conceptions are and what we would like them to be (which, based on current research, appears to be true), we m develop teaching units whose purpose would be to more fully develop students' understanding of variable.

Secondly, more emphasis snould be placed on problem solving skills (as implied by the NCTM). Word problems and thought problems should be presented that require more than a passive, algorithmic, solution. Means for developing healthy problem solving habits in students should be explored and developed.

These recommendations are the result of the symbiosis between research into students' conceptions and the text book review. This dual approach to reviewing math education is a valuable one and implies a third recommendation. That is, that research that assesses students' mathematics conceptions must be continued in all areas, coupled with ongoing curriculum reviews. Together, the results of these studies could help to form the goals and directions of the future.

| a. REPLACEMENT SET | b. HOW VARIABLES VARY | c. AMBIGUITY |
|---|---|---|
| A. Sets of things.<br>B. Sets of numbers.<br>C. A number.<br>D. Unusual approaches.<br>E. No definition. | F. Unvaryingly.<br>G. Discretely.<br>H. Continuously. | I. Contradictory usage.<br>J. Overuse of letters.<br>K. No definition. |

| | a. | | | | | b. | | | c. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| JUNIOR HIGH TEXTS | A | B | C | D | E | F | G | H | I | J | K |
| Addison-Wesley 1975 Krause et al. | | | | X | | | X | | | X | X |
| D.C. Heath 1979 Rucker et al 7,8. | | | X | | | X | | | | | |
| Houghton Mifflin 1977 Dolciani et al 7,8. | X | | | | | X | | | | X | |
| Houghton Mifflin 1978 Duncan et al 7,8. | | | X | | | X | | | | X | |
| MacMillan 1978 Forbes et al 7,8 | | X | | | | | X | | | | |
| Prentice Hall 1980 Haber-Shaim Book 1 | | | X | | | X | | | | | X |
| Prentice Hall 1980 Haber-Shaim Book 2 | | | | X | | | X | | | | X |
| Scott Foresman 1978 Bolster et al 7,8 | | | X | | | | X | | | X | |
| TOTALS-JUNIOR HIGH | 2 | 2 | 7 | 2 | 0 | 7 | 6 | 0 | 0 | 5 | 3 |
| ALGEBRA I TEXTS | | | | | | | | | | | |
| Addison Wesley 1978 Keedy et al | | | | | X | X | | | | X | X |
| Doubleday 1977 Traves et al. | | X | | | | | X | | | | |
| Ginn and Co., 1964 Pearson et al. | | | X | | | | X | | | | |
| Harcourt Brace Jovanovich 1977 Payne et al | | | | | X | | X | | | | |
| Holt, Rinhart & Winston 1978 Nichols et al | | | X | | | | X | | | X | |
| Houghton Mifflin 1970 Dolciani et al | X | | | | | | | | | X | |
| Houghton Mifflin 1977 Denholm et al | | | X | | | X | | | | | |
| McGraw Hill 1968 Henderson et al | X | | | | | X | | | | | X |
| McGraw Hill 1973 Rich (Schaum's outline) | | X | | | | X | | | | | |
| McGraw Hill 1974 Banks et al | | X | | | | X | | | | X | |
| Oxford Books 1967 Schlumpf et al | X | | | | | X | | | | | |
| TOTALS- ALGEBRA I | 3 | 3 | 3 | 0 | 2 | 6 | 4.5 | .5 | 0 | 4 | 2 |

TABLE 1 - Variables in Junior High and Algebra I texts.

| a. REPLACEMENT SET | b. HOW VARIABLES VARY | c. AMBIGUITY |
|---|---|---|
| A. Sets of things.<br>B. Sets of numbers.<br>C. A number.<br>D. Unusual approaches.<br>E. No definition. | F. Unvaryingly.<br>G. Discretely.<br>H. Continuously. | I. Contradictory usage.<br>J. Overuse of letters.<br>K. No definition. |

| | a. | | | | | b. | | | c. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ALGEBRA II TEXTS | A | B | C | D | E | F | G | H | I | J | K |
| Addison-Wesley 1977 | | | X | | | | X | | | | X |
| Addison-Wesley 1978 Keedy et al | | | | | X | | | | | | X |
| Harcourt Brace Jovanovich 1976 Jacobs | | X | | | | | X | | X | | X |
| Harcourt Brace Jovanovich 1977 Payne et al | | | | | X | | | | | | X |
| Houghton Mifflin 1977 Denholm et al | | | | | X | | | | | | X |
| TOTALS - ALGEBRA II | 0 | 1 | 1 | 0 | 3 | 0 | 2 | 0 | 1 | 0 | 5 |
| INNOVATIVE/EXPERIMENTAL TEXTS | | | | | | | | | | | |
| Addison-Wesley 1964 Davis (Madison Proj.) | | X | | | | X | | | X | | |
| Addison-Wesley 1980 Foerster | | X | | | | | X | | | | |
| Berkeley H.S. 1970 Rasmussen | | | X | | | X | | | | | |
| MacMillan Co. 1964 Page | | | | X | | X | | X | X | | |
| SMSG 1964 | | X | | | | X | | | X | | |
| SMSG 1965 (2 books) | | X | | | | X | | | | | |
| U of Chicago Press 1976 Usiskin | X | | | | | | X | X | X | | |
| U of Illinois Press 1960 Beberman | X | | | | | | X | | X | | |
| TOTALS - INNOVATIVE/EXPERIMENTAL | 2 | 5 | 1 | 1 | 0 | 1 | 5.5 | 1.5 | 3 | 4 | 0 |
| COLLEGE - PRECALCULUS TEXTS | | | | | | | | | | | |
| Houghton Mifflin 1978 Stockton | | | | X | | | | | | | X |
| Wadsworth 1977 Bechenbach | X | | | | | | | | | | X |
| W.W. Norton 1980 Hughes-Hallet | | X | | | | | X | | | | |
| TOTALS - COLLEGE PRECALCULUS | 1 | 1 | 0 | 0 | 1 | 0 | .5 | .5 | 0 | 0 | 2 |
| GRAND TOTALS | 8 | 12 | 12 | 3 | 6 | 14 | 19.5 | 12.5 | 4 | 13 | 12 |

TABLE 2 - Variables in Algebra II, innovative, and college texts.

## POSSIBLE CONTRIBUTING FACTORS TO POOR PROBLEM SOLVING SKILLS

1. Fewer than 25% of all problems are word or thought problems.
2. Virtually no problem solving strategies are given other than by example.
3. Word problems are introduced with pure number problems.
4. Word problems can be solved arithmetically or algorithmically.
5. Wording of problems is contrived, avoiding mental effort.
6. One variable problems are emphasized.
7. Translations are done via key word match.
8. Symbolism is confusing.

| JUNIOR HIGH TEXTS | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Addison-Wesley 1975 Krause et al. | | | | | | X | | X |
| D.C. Heath 1979 Rucker et al. 7,8. | X | | | X | X | X | X | |
| Houghton Mifflin 1977 Dolciani et al 7,8. | X | | X | X | X | X | | X |
| Houghton Mifflin 1978 Duncan et al 7,8. | X | | | X | X | X | | X |
| MacMillan 1978 Forbes et al 7,8. | | X | | X | X | X | | X |
| Prentice Hall 1980 Haber-Shaim et al — Book 1 | | | | | X | X | | X |
| Prentice Hall 1980 Haber-Shaim et al — Book 2 | | | | | | X | | X |
| Scott Foresman 1978 Bolster et al. | - | | | | | X | X | X |
| TOTALS - JUNIOR HIGH TEXTS | 6 | 2 | 2 | 8 | 9 | 13 | 4 | 11 |
| ALGEBRA I TEXTS | | | | | | | | |
| Addison-Wesley 1978 Keedy at al. | X | | X | X | | | X | X |
| Doubleday 1977 Traves et al. | X | | | X | X | | | |
| Ginn and Co. 1964 Pearson et al. | X | X | | | X | | | |
| Harcourt, Brace, Jovanovich 1977 Payne et al. | X | X | | X | | | | |
| Holt, Rinhart & Winston 1978 Nichols et al. | | | X | X | X | | X | X |
| Houghton Mifflin 1970 Dolciani et al. | | | | | X | | X | X |
| Houghton Mifflin 1977 Denholm et al. | X | X | X | | X | X | | |
| McGraw Hill 1968 Henderson et al. | X | X | | X | X | | | X |
| McGraw Hill 1973 Rich (Schaum's outline). | X | | X | X | X | | X | X |
| McGraw Hill 1974 Banks et al. | X | | X | X | X | | X | X |
| Oxford Books 1967 Schlumpf et al. | | X | | X | X | | | X |
| TOTALS - ALGEBRA I TEXTS | 8 | 5 | 5 | 8 | 9 | 1 | 5 | 7 |

TABLE 3 - Problem solving in Junior High and Algebra I texts.

## POSSIBLE CONTRIBUTING FACTORS TO POOR PROBLEM SOLVING SKILLS

1. Fewer than 25% of all problems are word or thought problems.
2. Virtually no problem solving strategies are given other than by example.
3. Word Problems are introduced with pure number problems.
4. Word problems can be solved arithmetically or algorithmically.
5. Wording of problems is contrived, avoiding mental effort.
6. One variable problems are emphasized.
7. Translations are done via key word match.
8. Symbolism is confusing.

| ALGEBRA II TEXTS | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Addison-Wesley 1977. | X | | X | X | X | | X | X |
| Addison-Wesley 1978 Keedy et al. | X | | | X | X | | X | X |
| Harcourt, Brace, Jovanovich 1976 Jacobs. | X | X | X | | | | X | X |
| Harcourt, Brace, Jovanovich 1977 Payne et al | X | X | X | X | X | | | X |
| Houghton Mifflin 1977 Denholm et al. | X | X | | | | X | | X |
| TOTALS - ALGEBRA II TEXTS | 5 | 3 | 3 | 3 | 3 | 1 | 3 | 5 |
| INNOVATIVE/EXPERIMENTAL TEXTS | | | | | | | | |
| Addison-Wesley 1964 Davis (Madison Project). | | | | | | | | X |
| Addison-Wesley 1980 Foerster. | | | | | | | | |
| Berkeley H.S. 1970 Rasmussen. | X | X | X | X | X | X | | |
| MacMillan Co. 1964 Page. | X | X | | | X | X | | X |
| SMSG 1964 | | | | | | X | | X |
| SMSG 1965 (2 books). | X | X | | | | X | | |
| U of Chicago Press 1976 Usiskin. | | X | | | | | | X |
| U of Illinois Press 1960 Beberman. | | | | | | | | X |
| TOTALS - INNOVATIVE/EXPERIMENTAL | 4 | 5 | 1 | 1 | 2 | 5 | 0 | 4 |
| COLLEGE - PRECALCULUS TEXTS | | | | | | | | |
| Houghton Mifflin 1978 Stockton | X | X | | | | X | | X |
| Wadsworth 1977 Bechenbach | X | | | X | | | | X |
| W.W. Norton 1980 Hughes-Hallet | | | X | X | | X | | |
| TOTALS - COLLEGE PRECALCULUS | 6 | 6 | 2 | 3 | 2 | 7 | 0 | 7 |
| GRAND TOTALS | 29 | 21 | 13 | 23 | 25 | 27 | 11 | 33 |

TABLE 4 - Problem solving in Algebra II, innovative, and college texts.

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