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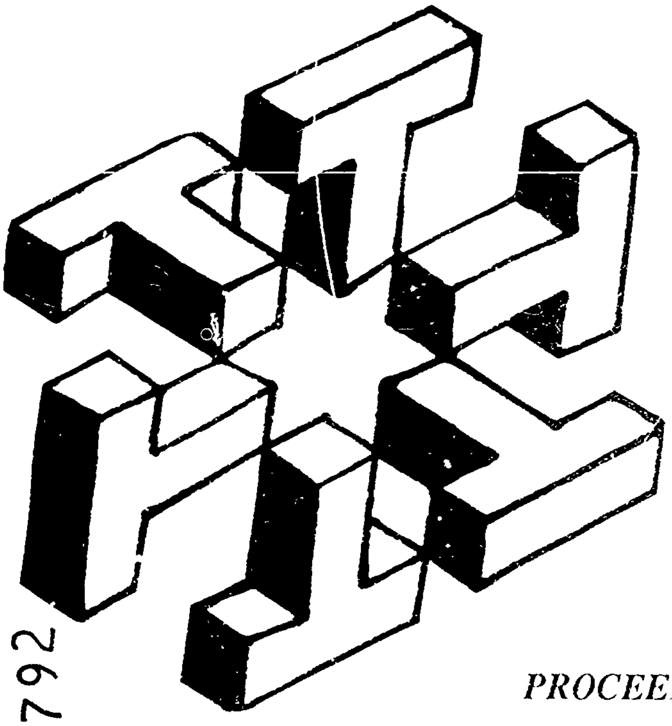
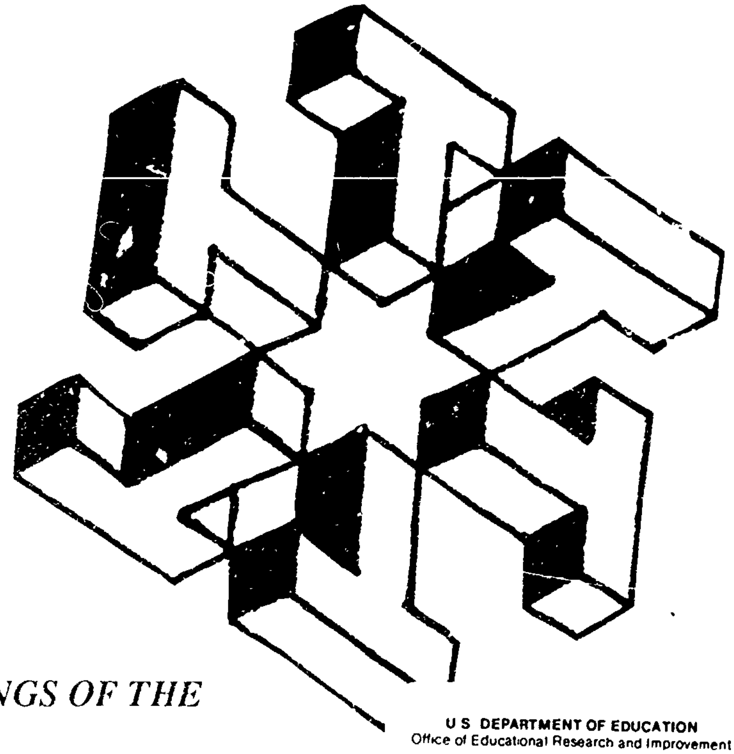
ABSTRACT

Twenty-five outstanding middle grade mathematics teachers were selected by the staff of the Middle Grades Mathematics Project (MGMP) at Michigan State University to participate in an Honors Teacher Workshop during the summer of 1987. During the final week of the encounter, 50 colleagues joined the workshop. The participants came from 20 different states and represented a cross-section of the best mathematics teachers throughout the nation. A major focus of the workshop was the development of instructional leadership. The potential for leadership weighed heavily in the selection process. The effects of the workshop will be assessed in a follow-up poll in September, 1988. This document contains general information relative to the workshop, as well as 14 presentations: "Critical Issues in Middle School Mathematics"; "Destroying Boxes"; "Some Issues in Teaching Secondary School Mathematics"; "Problem with a Purpose"; "Teaching Math with Manipulatives"; "Probability"; "Changing the Negatives to Positives: Transforming General Mathematics Classes"; "Gender Bias and Negative Numbers"; "Proportional Reasoning"; "Math Anxiety"; "Teaching Gifted and Talented"; "Algebra"; "Mathematical Paradoxes in School"; and "Equations." (PK)



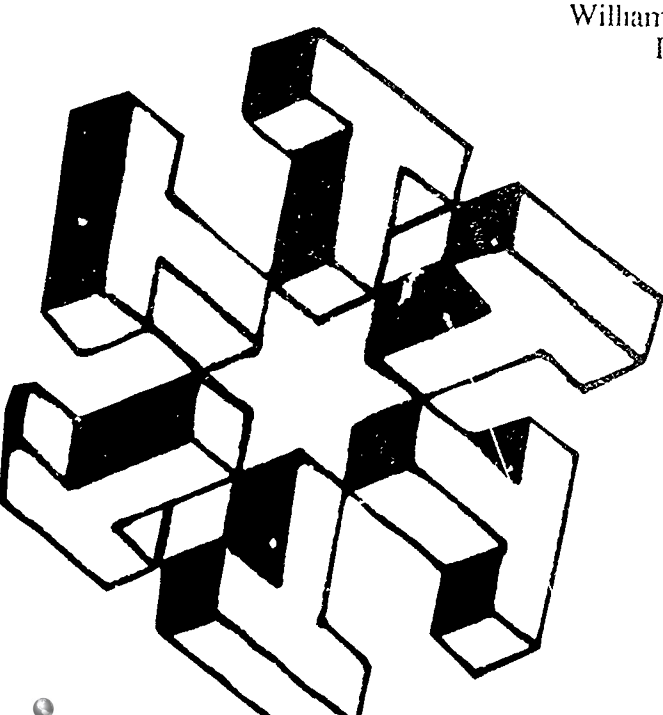


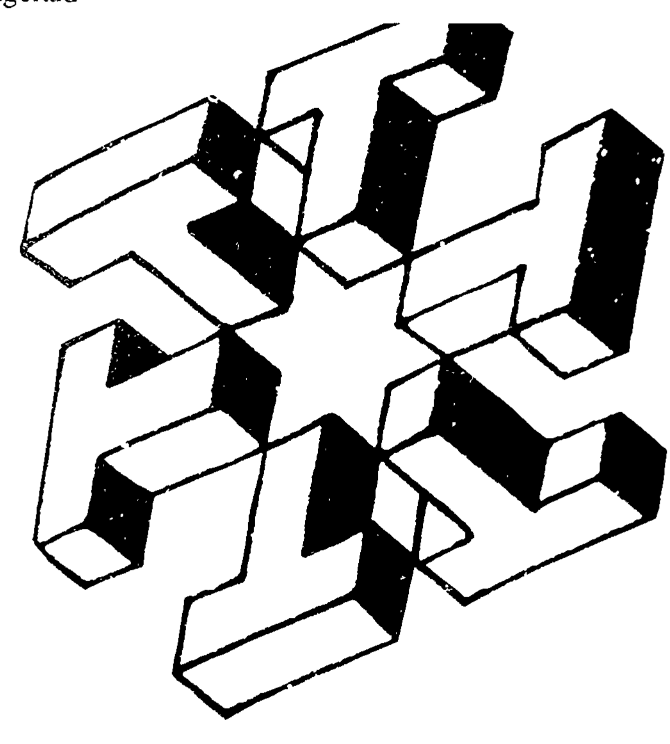
PROCEEDINGS OF THE HONORS TEACHER WORKSHOP OF MIDDLE GRADE MATHEMATICS

JUNE 21 - JULY 17, 1987

Department of Mathematics
Michigan State University
East Lansing, MI 48824

William M. Fitzgerald
Director





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Edited by Vivian Pedwaydon

Document prepared by Mary Reynolds

I'd like to sing a song for you,
to tell you that I care.

To tell you that I'm grateful for the
times that we can share.

I've been told that in this life
the friends we have are few

And I'd just like to thank you
that the friend I have is you.

A Participant

Mathematics

is a

Social

Experience

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APPLICATION

HONORS WORKSHOP FOR MATHEMATICS TEACHERS IN GRADES SIX, SEVEN, AND EIGHT

Michigan State University
June 22 - July 17, 1987

1. Name ______________________ Social Security No. ______________
2. Home Address ______________________ Home Phone ______________
   ____
3. School Address ______________________ School Phone ______________
   ____
4.

Educational History	Institution	Degree	Year	Major	Minor
Secondary School					
Undergraduate					
Graduate					

5. Describe your present assignment in your school.

6. Describe other relevant past or present work experience.

7. List the mathematics content and pedagogy courses by name that you have completed and the grades you earned to prepare you to be a middle grades mathematics teacher.

Content	Pedagogical

8. In what ways have you been involved in in-service activities?

9. In what ways have you influenced the mathematics program in your school?

10. What is the name and address of your local newspaper?

11. If you are accepted as a participant in the workshop, you will need to select two peers to join you in the workshop during the last week, July 13-17. They will receive the same stipend and expenses and two hours of graduate credit. You will be released from your teaching duties during the 1987-88 school year for sufficient time to coach your colleagues as they each teach two of the MGMP curriculum units (e.g., one hour each day for twelve weeks). Describe the preparations and agreements which have been made to accomplish this.

My two colleagues are: (Tentative)

Name (1) ____________________ Name (2) ____________________

Address ____________________ Address ____________________

____________________ ____________________

Phones (Home) ____________________ Home ____________________

(School) ____________________ School ____________________

Soc. Security No. ____________________ Soc. Security No. ____________________

12. You and your colleagues will be expected to conduct a workshop during the summer of 1988 for at least twelve teachers for at least five days in your district and possibly neighboring districts. The project will furnish the materials needed for the workshop but all other financial needs must be provided locally. We encourage you to seek some private funding for these activities. Describe steps that have been taken to provide this workshop.

We have read both sides of this document and concur with the arrangements as they have been described. They have my complete support

Principal

Additional District Officer

13. Please indicate in a statement below what is important to you in working with students in mathematics. What are the most important issues facing you in mathematics education? In what direction would you suggest mathematics education should be moving in the near future?

14. How do you feel about acting as a coach of your peers? Describe any experiences you may have had in this realm.

The commitments and conditions in this application are accurate and as complete as can reasonably be described at this time.

Applicant

Date

INTRODUCTION

Twenty-five outstanding middle grade mathematics teachers were selected by the staff of the Middle Grades Mathematics Project (MGMP) at Michigan State University to participate in an Honors Teacher Workshop from Sunday, June 21, to Saturday, July 18, 1987. During the final week of the encounter, fifty colleagues joined the workshop. The participants came from twenty different states and represented a cross-section of the best mathematics teachers throughout the nation.

Among the approximately one hundred applicants who were not selected were many very highly qualified persons. It was heartening indeed to learn of many of the commendable programs which are in practice and of the talented people who are responsible for those programs. Criteria for selection are found in Appendix D.

A major focus of our workshop was the development of instructional leadership. The potential for leadership weighed heavily in the selection process. The effects of the workshop will be assessed in a follow-up poll in September, 1988. Those results will be available from the director at that time.

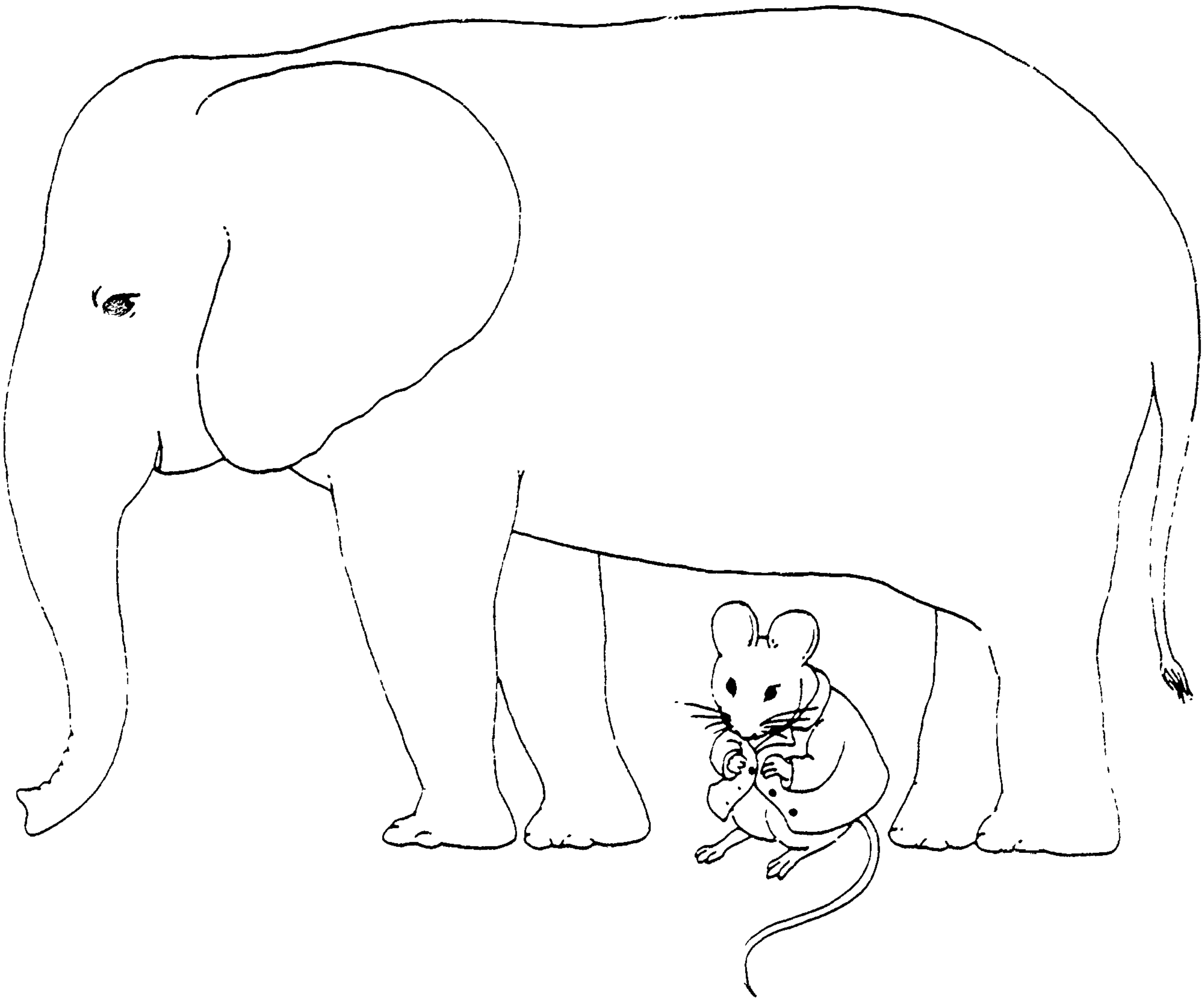
The participants came knowing very little of what to expect, except that they had been asked to work hard. They did so in twelve hour days packed to the brim with guest speakers, field trips, personal interchanges and an intensive immersion in the MGMP curriculum materials and project assignments given by workshop director.

In these proceedings, we have attempted to capture some of the excitement and enthusiasm which was ever present. This is, of course, impossible. We do hope, however, that the readings become useful to a wide audience.

William M. Fitzgerald, Director
Vivian Pedwaydon, Editor

Mouse & Elephant Unit Challenge

The mouse stands 6 cm high and the elephant stands 240 cm high.
How many mouse coats are needed to sew a coat for the elephant?
How many mice are needed on the scales to balance the elephant?



Mouse & Elephant Guess Sheet

The number of mouse coats needed to sew a coat for the elephant is ________________

The number of mice needed on the scales to balance the elephant is ________________

A HISTORICAL PERSPECTIVE
THE MIDDLE GRADES MATHEMATICS PROJECT

Department of Mathematics
Michigan State University

The Middle Grades Mathematics Project (MGMP) represents a wide-ranging effort to improve the teaching and learning of mathematics by teachers and students in grades six, seven and eight. This effort includes research on children's mathematics learning, the creation of "model" units of mathematics curriculum, the testing of the effectiveness of those units, and the inservice education of teachers including two National Honors Teacher Workshops for exemplary teachers from throughout the United States.

Prior to the 1987 workshop, the MGMP was funded by the National Science Foundation (NSF) to support two years of research on the improvement of teacher effectiveness.

The philosophy of MGMP mirrors the latest research findings from cognitive science. Resnick (1983) summarizes the fundamental view of the learner:

> First, learners construct understanding. They do not simply mirror what they are told or what they read. Learners look for meaning and will try to find regularity and order in the events of the world, even in the absence of complete information. This means that naive theories will always be constructed as part of the learning process.
>
> Second, to understand something is to know relationships. Human knowledge is stored in clusters and organized into schemata that people use both to interpret familiar situations and to reason about new ones. Bits of information isolated from these structures are forgotten or become inaccessible to memory.
>
> Third, all learning depends on prior knowledge. Learners try to link new information to what they already know in order to interpret the new material in terms of established schemata. This is why students interpret science demonstrations in terms of their naive theories and why they hold onto their naive theories for so long. The scientific theories that children are being taught in school often cannot compete as reference points for new learning because they are presented quickly and abstractly and so remain unorganized and unconnected to past experience.

History of the MGMP

The origins of MGMP are found in the dissatisfactions felt by many people in the typical mathematics education which was being provided for children in the middle grades in the mid-1970's.

With the frequent calls for "back to the basics", many programs became much more skill and drill oriented, often individualized, usually restricted to arithmetic. Students seldom had opportunities to study the interesting relationships to be found in mathematics, particularly in areas such as geometry and probability. Children in the middle grades are in a stage of adolescence where concrete experiences are essential to promote the cognitive development of abstract relationships which we wish them to learn. Although there was much written and spoken about teaching to develop problem solving skills, most teachers have not been trained to teach problem solving skills, nor to teach any mathematics in an activity oriented way.

Given these circumstances, the staff of MGMP found themselves being asked with increasing frequency to provide demonstration classes. Thus the staff began to acquire a collection of activities which worked well in the middle grades. While studying sixth grade children's ability to learn growth relationships, Fitzgerald and Shroyer (1979) developed the Mouse and Elephant unit embodying an instructional model which has been incorporated into all of the units. Concurrently, Lappan, Winter, and Phillips were developing activities in spatial visualization, probability, number theory, geometry and other activities for regular classrooms as well as extensive work with gifted student programs. These activities led to the funding of five teacher institutes conducted by the MGMP staff.

From all of these diverse activities, a plan for a proposal emerged to develop the curriculum units through a more systematic process. The funding of that proposal made possible the Middle Grades Mathematics Project.

The MGMP Units

Four new units were developed using the existing Mouse and Elephant unit as a prototype. The units are described below.

MOUSE AND ELEPHANT

If we assume that a mouse grows up to become an elephant, and we know their relative heights, how many mice are required to balance an elephant on a balance scale, and how many mouse coats can be made from an elephant coat? The unit introduces area, perimeter, surface area, and volume and their interrelations during growth.

SPATIAL VISUALIZATION

This unit involves representing 3-dimensional objects in 2-dimensional drawings and, vice-versa, constructing (with blocks) 3-dimensional objects from their 2-dimensional representations. Two different schemes for picturing buildings are developed and the relationship between the two are explored. The major objectives of the unit are for the students to build, draw and evaluate. These three themes are considered in various combinations throughout the unit.

FACTORS AND MULTIPLES

This unit focuses on the concept of primes, factors, divisors, multiples, common factor, common multiple, relatively prime, and composite. Since little in the way of computational skill is involved, students can discover these relationships by solving a variety of problems and at the same time develop their problem solving skills. They also acquire a language for reading and discussing mathematics. The calculator is used extensively. The study of fractions can follow naturally from this unit, since a sound understanding of factors and multiples is a prerequisite for understanding an operation on fractions.

PROBABILITY

This unit makes extensive use of children's interest in games to promote understanding of probability. Students continually make hypotheses and support conclusions. This unit is pertinent to the middle grades because it provides a sound understanding of the concept of fractions, equivalent fractions and comparison of fractions.

SIMILARITY

The concept of similarity is gradually developed through several concrete experiments. As a result students also gain a sound understanding of indirect measurement, scale drawing, scale models, and the nature of growth. Similarity provides an excellent opportunity to build more geometric meaning into the idea of equivalent fractions.

Development of the Units

Each unit was developed through an extensive and elaborate process which required approximately one year. Two units were developed each year. The steps for the development of a unit are as follows.

1. The staff determined an important collection of related mathematical ideas which were inadequately taught in the middle grades.

2. A set of activities were collected or developed by the staff and tried in classrooms to determine the effectiveness and desirable sequences of the activities.

3. On the basis of these experiences, a preliminary teachers guide was written.

4. A two-week Summer Teaching Institute was conducted at a local middle school consisting of forty middle school students, eight affiliated teachers and the project staff. The affiliated teachers observed the staff teach the units to the students, then joined in a critique of the materials, the pedagogy, and any other matters of concern. Items for the unit tests were also generated during these discussions.

5. After the Summer Institute, the units were rewritten by the staff and subsequently taught by the affiliated teachers and some of their colleagues.

6. On the basis of the teaching by the affiliated teachers, the units were re-written and taught by a large number of teachers with no previous exposure to the project.

The final evaluation results are found in the final report and in three doctoral dissertations listed below.

Characteristics of the Units

Each unit

* is based on a related collection of important mathematical ideas,
* provides a carefully sequenced set of activities which lead to an understanding of the mathematical challenges,
* helps the teacher foster a problem solving atmosphere in the classroom,
* uses concrete manipulatives, where appropriate to help provide the transition from concrete to abstract thinking,
* utilizes an instructional model which consists of three phases... launching, exploring and summarizing,
* provides a carefully developed instructional guide for the teacher,
* requires two to three weeks of instructional time.

The goal of the MGMP materials is to help students develop a deep, lasting understanding of the mathematical concepts and strategies studied. Rather than attempting to break the curriculum into small bits to be learned in isolation from each other, MGMP materials concentrate on a cluster of important ideas and the relationships which exist among these ideas. Where possible the ideas are embedded in concrete models to assist the students in moving from this concrete stage to more abstract reasoning.

The Instructional Model: Launch, Explore and Summarize

Many of the activities are built around a specific mathematical challenge. The instructional model used in the units focuses on helping the students solve the mathematical challenge. The instruction is divided into three phases.

During the first phase the teacher launches the challenge. The launching consists of introducing new concepts, clarifying definitions, reviewing old concepts, and issuing the challenge.

The second phase of instruction is the class exploration. During the exploration the students work individually or in small groups. The students may be gathering data, sharing ideas, looking for patterns, making conjectures, or developing other types of problem solving strategies. It is inevitable that students will exhibit variation in progress. The teacher's role during exploration is to move about the classroom observing individual performances and encouraging on-task behavior. The teacher encourages the students to persevere in seeking a solution to the challenge. The teacher does this by asking appropriate questions, encouraging and redirecting where needed. For the more able students, the teacher provides extra challenges related to the ideas being studied. The extent to which students require the teacher's attention will vary as will the nature of attention they need, but the teacher's continued presence and interest in what they are doing is critical.

When most of the children have gathered sufficient data, the class returns to a whole class mode (often beginning the next day) for the final phase of instruction, summarizing. Here the teacher has an opportunity to demonstrate ways to organize data so that patterns and related rules become more obvious. Discussing the strategies used by the students helps the teacher to guide the students in refining these strategies into efficient, effective problem solving techniques.

The teacher plays a central role in this instructional model. First the teacher provides and motivates the challenge and then joins the students in exploring the problem. The teacher asks appropriate questions, encouraging and redirecting where needed. Finally, through the summary, the teacher helps the students to deepen their understanding of both the mathematical ideas involved in the challenge and the strategies used to solve. it.

To aid the teacher in using the instructional model, a detailed instructional guide is provided for the activities. The preliminary page contains a rationale, goals for the students, an overview of the main ideas and a list of student activity sheets, materials and transparencies. Then a script is provided to help the teacher teach each phase of the instructional model. Each page of the script is divided into three columns:

Teacher Action	Teacher Talk	Expected Responses
Includes Material needed... Display on overhead... Explain Ask...	Includes Important questions which are needed to develop understandings and problem solving skills.	Includes Correct responses responses with suggestions for handling these.

At the end of each activity are black and white copies from which transparencies and student activity sheets can be made. Answer sheets are also provided.

This instructional model has been used successfully to structure the Elementary Science Study, Peas and Particles unit in a manner helpful to elementary teachers. (Beaver, 1982).

The Honors Teacher Workshop - 1984

Twenty-five middle grade mathematics teachers from sixteen states were selected by the MGMP staff to participate in an intensive nineteen-day workshop on the campus at Michigan State University during November and December, 1984. Funding for the workshop came from NSF.

To be considered, a teacher had to be nominated by their principal and two colleagues. Their district had to agree to provide a paid leave of absence for the teacher and also to develop specific prior plans to use that teacher as an instructional leader upon return. Approximately 130 applications were received from 31 states and provinces.

While in the workshop, the teachers heard a variety of guest lecturers, studied the history of important mathematical problems, took field trips, examined contemporary curriculum materials, and exchanged ideas among themselves. They received six quarter hours of graduate credit in mathematics education.

Much workshop time was used to familiarize the teachers with the MGMP units. When they left the campus they took with them a complete set of the materials needed to begin teaching the units in their classrooms immediately.

A published copy of the Proceedings of the MGMP-HTW-1984 workshop is available from MGMP.

MGMP - Phase II

In the evaluation of the MGMP materials the staff focused on student content gains as a measure of the success of the units. Affiliated teachers were used in the development stage of the materials to help improve and refine the units both in terms of content and in terms of the teaching model. These teachers also helped generate questions for the tests designed to evaluate each unit. The major vehicle for this interaction between the affiliated teachers, the staff and the materials was an intensive two week half-day session where the staff taught the units to children, with the teachers observing and assisting, followed each day by a reaction session. The staff's observations were that these affiliated teachers were much more successful in implementing the units into their classrooms in the true spirit of the instructional model than were the teachers who only received two to four hours of workshop on the unit before teaching it. In addition, many of the affiliated teachers seemed to transfer the teaching model in some ways into their teaching after the experience with MGMP. Many of them have talked to the staff about ways in which the summer experience and the follow-up of teaching the units changed the way in which they thought about teaching and about students. Some of these teachers mentioned the colleagial relations with other teachers and the staff as being an essential part of what they considered as the best in-service experience they have had.

In all there were approximately 50 teachers involved in the evaluation of the units. Informal comparison between the 16 teachers involved in the intensive two-week sessions and those who only had 2 to 4 hours of workshop prior to implementing the units has convinced the staff the full elaborated training system - - theory, demonstration, practice and feedback -- advocated by Showers (1983), holds the most promise for implementing the units. The additional component, coaching, seemed necessary to achieve a transfer of the teaching model and its relevant skills into the teachers' repertoires.

In the two-year study we designed instruments needed to collect data and evaluate the effects of three models of implementation: a fully elaborated training system (9 teachers), a fully elaborated training system plus a coaching component (8 teachers), and a fully elaborated training system with a coaching component conducted by peer teachers (3 teachers).

Levels of Impact

Within the research design, the staff has attended to the potential levels of impact of training as described by Joyce and Showers (1980).

> * Awareness - At the awareness level we realize the importance of an area and begin to focus on it. With inductive teaching, for example, the road to competence begins with awareness of the nature of inductive teaching, its probable uses, and how it fits into the curriculum.
>
> * Concepts and Organized Knowledge - Concepts provide intellectual control over relevant content. Essential to inductive teaching are knowledge of inductive processes, how learners at various levels of cognitive development respond to inductive teaching, and knowledge about concept formation.
>
> * Principles and Skills - Principles and skills are tools for action. At this level we learn the skills of inductive teaching: how to help students collect data, organize it, and build concepts and test them. We also acquire the skills for adapting to students who display varying levels of ability to think inductively and for teaching them the skills they lack. At this level, there is potential for action - we are aware of the area, can think effectively about it, and possess the skills to act.
>
> * Application and Problem Solving - Finally, we transfer the concepts, principles, and skills to the classroom. We begin to use the teaching strategy we have learned, integrate it into our style, and combine the strategy with the others in our repertoire.

Only after this fourth level has been reached can we expect impact on the education of children.

The minicourse, demonstration sessions, and detailed teachers' guides to the MGMP units address very specifically the first three levels of impact. The provision for practice with and without coaching should help answer the question of how much and what kind of help is needed by teachers to integrate a new instructional model into their repertoire, that is to reach the stage of application and problem solving.

MGMP - Honors Teacher Workshop - 1987

Twenty-five middle grade mathematics teachers were selected from twenty states by the MGMP staff to participate in an intensive thirty-day workshop on the campus of Michigan State University during June and July, 1987. The twenty-five participants were joined by two of their colleagues for the final week of the workshop.

To be considered, a teacher had to be nominated by their principal and two colleagues. Their district had to agree to provide for support of a workshop to be conducted by the participant and colleagues involving a minimum of twelve other educators. Approximately 100 applications were received from thirty-six states.

While in the workshop, the teachers experienced twelve hour days packed with instruction, presentations, collegial teams, lectures, and field trips. Many hours were spent exchanging ideas, sharing enthusiasm, and experiencing exhaustion. The twenty-five participants received six quarter hours of graduate credit in mathematics and their colleagues received two.

Participants and colleagues left the workshop familiar with the MGMP units, a complete set of MGMP curriculum materials, a repertoire of new ideas, satisfaction, excitement and alive with anticipation for the changes they perceived for the coming school year.

Follow up will include visitations from the MGMP staff, surveys, and continued communication between participants, colleagues and educators throughout the nation.

References

Beaver, John B., (1982) "A Study of the Applications of the Inquiry - Script Teaching Model to the Modification of an Elementary Science Study Unit", Ph.D. Dissertation, Michigan State University.

Ben Haim, David, (1982) "Spatial Visualization: Sex Differences, Grade Level Differences and the Effect of Instruction on the Performance and Attitudes of Middle School Boys and Girls", Ph.D. Dissertation, Michigan State University.

Fitzgerald, William and Shroyer, Janet, (1979) "A Study of the Learning and Teaching of Growth Relationships in the Sixth Grade", Final Report NSF SED 77-18545.

Friedlander, Alex, (1984) "Achievement in Similarity Tasks: Effect of Instruction, and Relationship with Achievement in Spatial Visualization at the Middle Grades Level", Ph.D. Dissertation, Michigan State University.

Joyce, B. and Showers, B., (April 1981) Teacher Training Research: Working hypotheses for program design and directions for further study. Paper presented to the annual meeting of the area. Los Angeles, CA.

Lappan, Glenda, (1983) "Middle Grades Mathematics Project", Final Report NSF SED 80-18025.

Oguntebi, Zacchaeus, (1983) "Probability: Sex and Grade Level Differences and the Effect of Instruction on the Performance and Attitudes of Middle School Boys and Girls", Ph.D. Dissertation, Michigan State University.

Resnick, Lauren, (April 1983) "Mathematics and Science Learning: A New Conception", Science, pg.477-8.

Showers, B., (April 1983) Coaching: A Training component for facilitating transfer of training. Paper presented at the annual meeting of the AERA. Montreal, Canada.

CRITICAL ISSUES IN MIDDLE SCHOOL MATHEMATICS

Glenda Lappan
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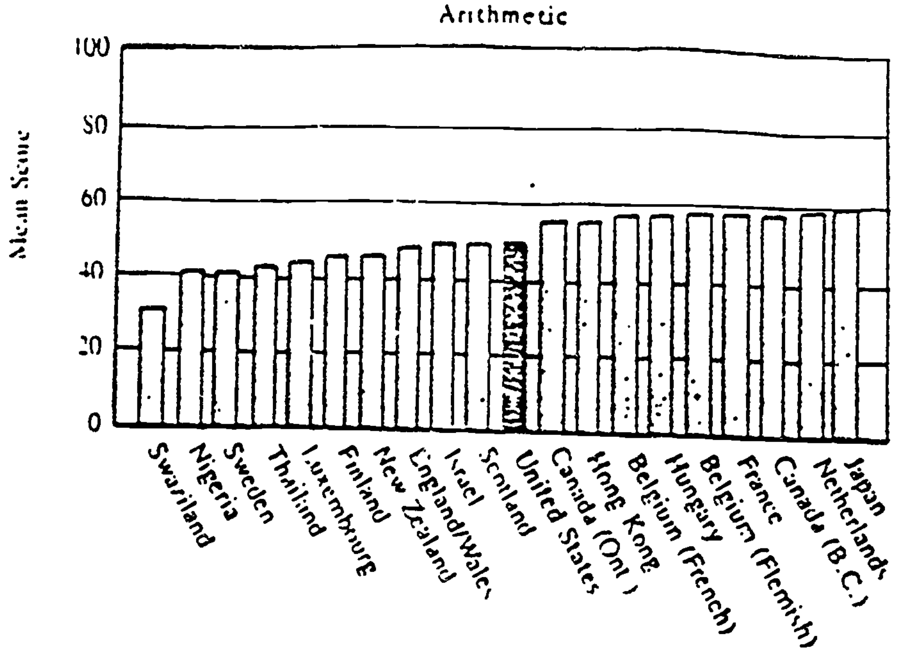
Across the nation, mathematics teaching is undergoing intensive scrutiny. A decade of declining pupil performance on standardized mathematics tests, poor comparative results on International Studies of Mathematics Education, increasing attrition from the mathematics teaching ranks, and reassignment of teachers to fill mathematics teaching positions have fueled increased criticism of the quality of mathematics instruction being given our nation's youth. At the same time an explosion of scientific and technological knowledge has increased public awareness of the importance of mathematics education in preparing young people to live and work in the society of the 1990's and beyond.

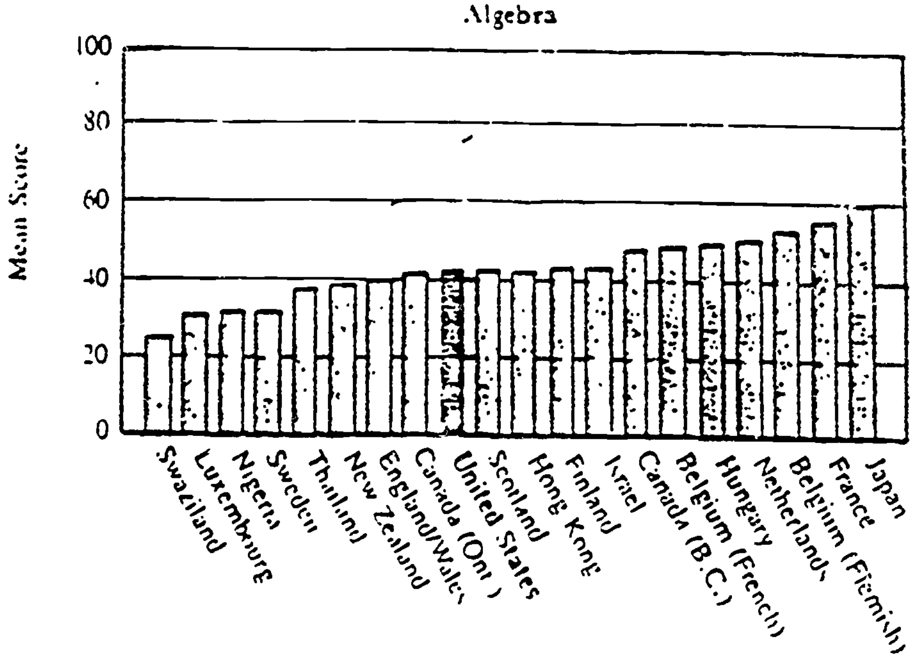
In this atmosphere of increasing public demand for accountability of schools and teachers for the effectiveness of mathematics instruction, many groups have published reports which demand that the public share in the accountability we owe to our youth. These reports call for a significant transition of teaching from an occupation with a nature, organization and status that has changed little since the middle of the nineteenth century to a genuine profession with all the emerging rights and responsibilities that that entails. The public is being asked to reward good teaching with higher pay -- sufficiently high to attract outstanding young people to teaching -- and with respect and status due those persons within our society who make a genuine professional commitment to teaching. Your presence here indicates that you are one of those committed professionals - - one that is looking for stimulating ideas - - for ways to make your teaching more effective. I am sure that you will find many.

I would like to share with you some of the emerging data from the Second International Mathematics Study (SIMS) and other studies on an international scale that can help us to stand back, look at our curriculum and identify areas of concern.

The SIMS study has been ten years in the making. Ten years from the establishment of the international study group, through the development of instruments, selection of students, gathering of data, analysis and finally the first report to the Nation released in Washington onJanuary 14, 1987. The SIMS study identified two populations, 8th grade and 12th grade, as the subjects of study. Eighth grade because it is the end in some countries of "compulsary mathematics" and twelvth grade as the end of the public school education. This allows us tolook at how well we are doing in grade 8 with all students and in grade 12 with those students who are still taking mathematics and are college bound possibly in science, mathematics, or engineering.

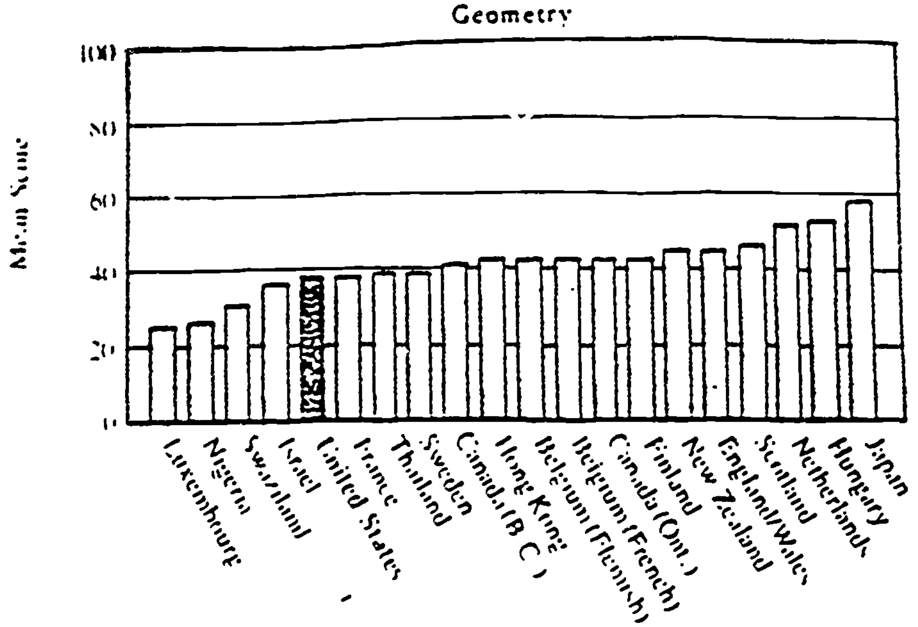
I have selected data from the study that speaks to our concerns today. The first tables show us the achievement of U.S. students in 8th grade in four of the five areas tested. As we can see the U.S. is in the middle of the pack in arithmetic and algebra performance and in the bottom quarter on geometry and measurement.

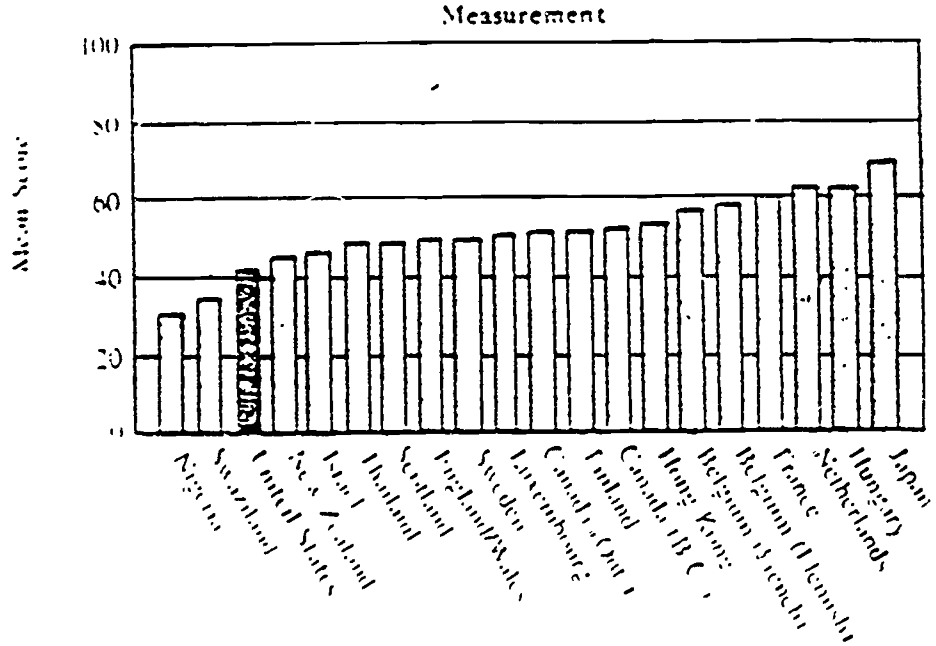




Eighth grade achievement scores in arithmetic and algebra for twenty countries show the U.S. to be in the middle group. Note, however, that countries in the middle range of achievement had very similar scores. Therefore, the ranking of those countries would be affected substantially by a change in score of only a few points.

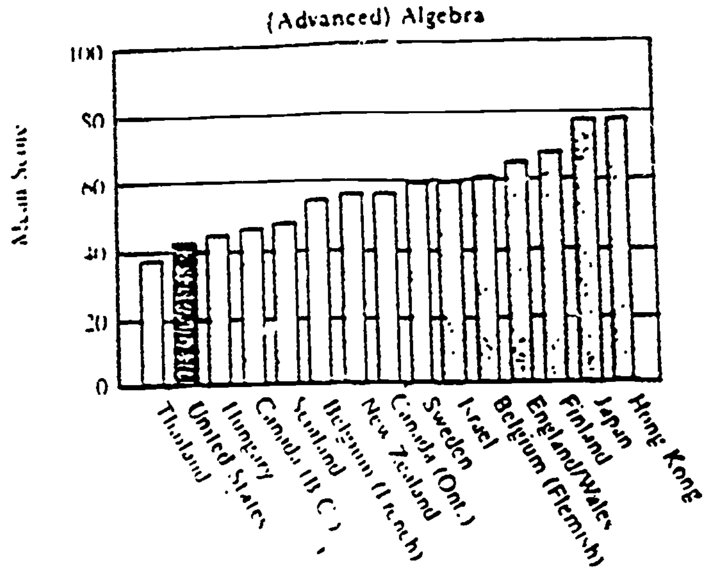
From: The Second International Study of Mathematics

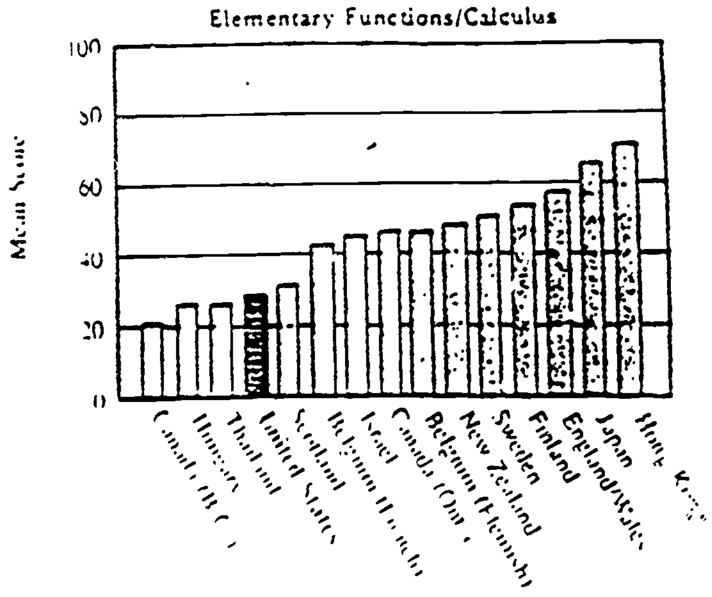




Eighth grade achievement scores in geometry and measurement for twenty countries show the U.S. to be well below the international average, among the lowest fourth of participating countries in both topics.

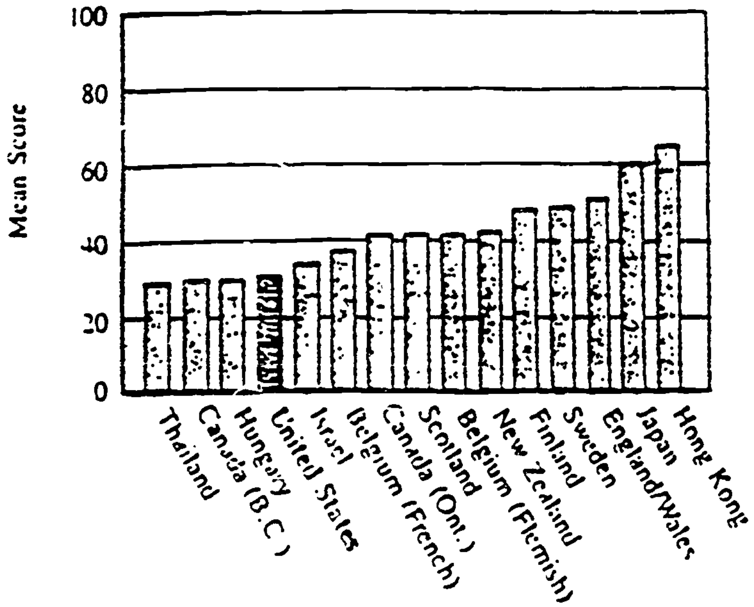
Now let's look at the twelvth grade achievement. Remember that these are the <u>advanced</u> students, the **best and brightest** mathematics students.





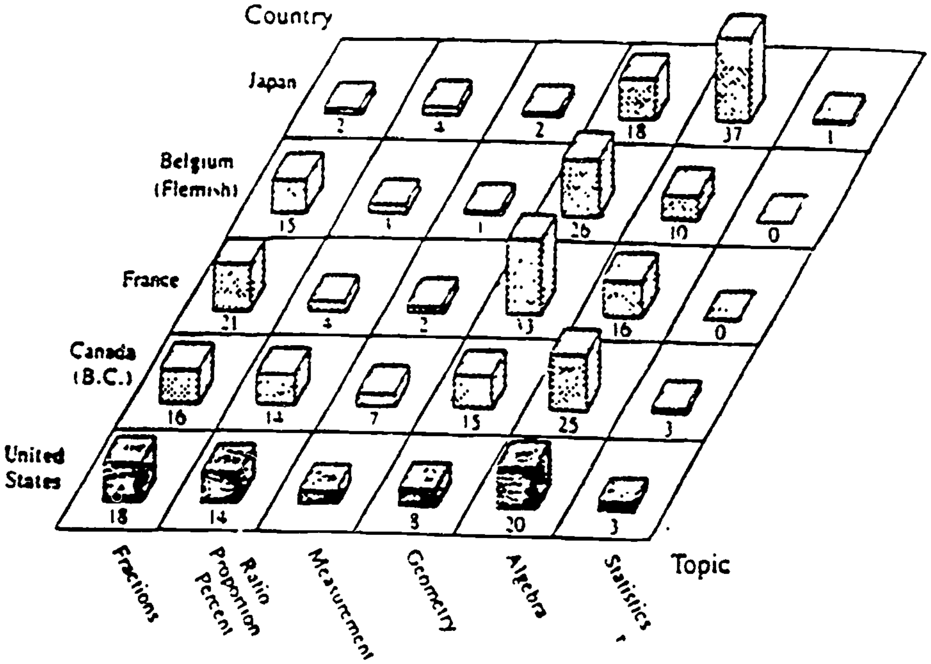
Population B (Twelvth Grade) achievement in algebra and elementary functions/calculus for fifteen countries shows the U.S. to score among the lowest fourth of participating countries. Much scientific and technical work, in college and elsewhere requires competence in these basic areas of mathematics.

GEOMETRY

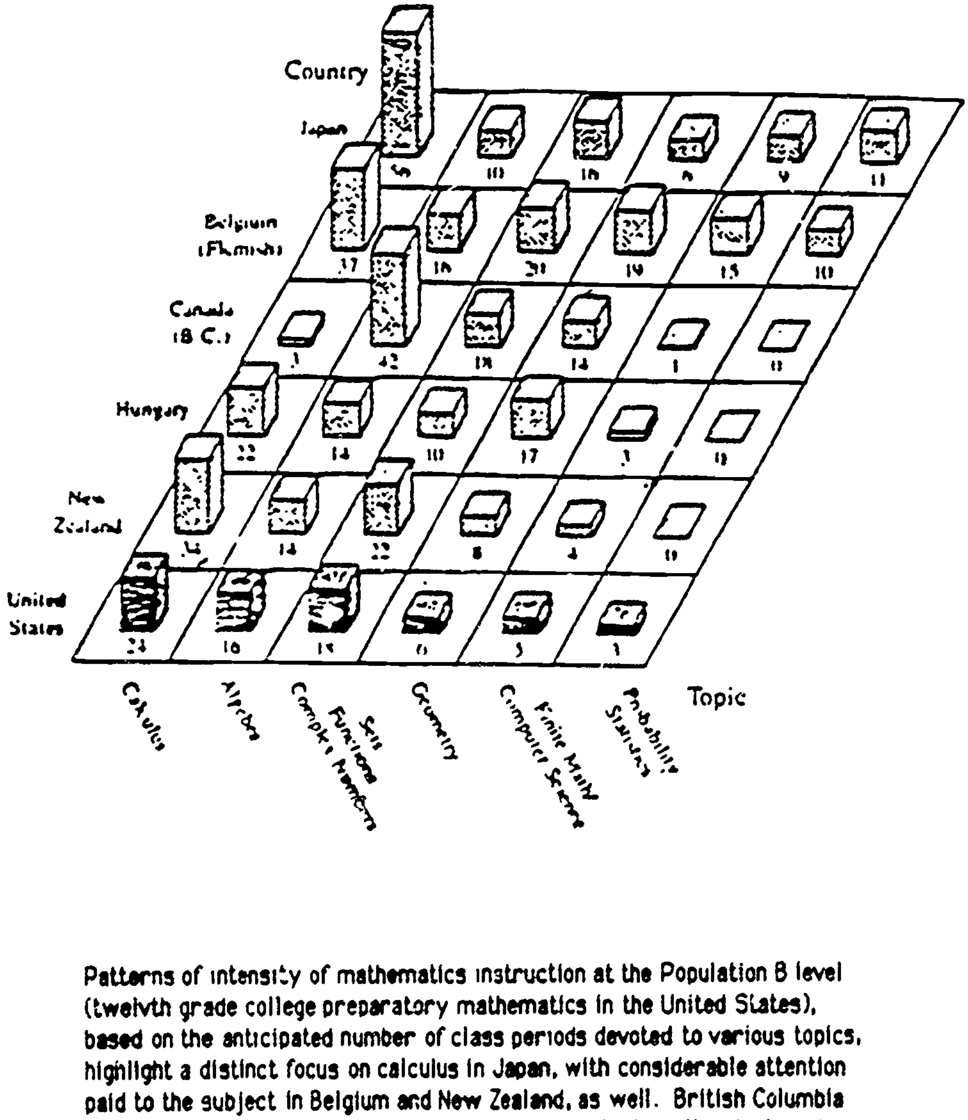


Because the population at grade 12 varies in selectivity across the countries in the study a comparison was made of the top 1% and top 5% of the age group in each country. The results showed that the U.S. came out as the lowest of any country for which data was available. The algebra achievement of our most able students was lower than any other country. In function and calculus the top 1% of U.S. students exceeded only British Columbia (and by a very few points). But calculus is not even included in the curriculum in British Columbia.

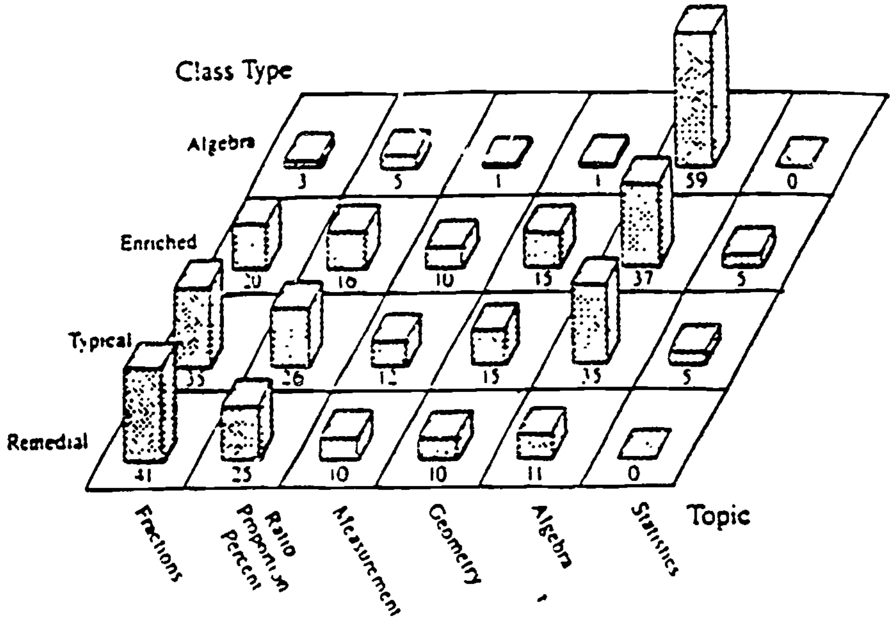
In order to get an idea of fit of the tests used to the curriculum taught "opportunity to learn" data was collected. Let's look at algebra and geometry in grade 8 and advanced algebra grade 12.



Intensity of mathematics instruction at the Population A level (eighth grade in the U.S.), based on teacher reports of anticipated percentage of class periods devoted to various topics, shows distinct patterns across countries. The Japanese curriculum provides their seventh graders with an intensive introduction to algebra. The Belgian and French programs focus on geometry and extensive work on common fractions (decimal fractions were dealt with in the elementary grades.) Correspondingly dramatic growth in achievement during the school year takes place for these topics in Japan, France and Belgium. By contrast, the U.S. curriculum has little intensity, with little sustained attention paid to any aspect of mathematics.



Patterns of intensity of mathematics instruction at the Population B level (twelvth grade college preparatory mathematics in the United States), based on the anticipated number of class periods devoted to various topics, highlight a distinct focus on calculus in Japan, with considerable attention paid to the subject in Belgium and New Zealand, as well. British Columbia provides an intense approach to algebra, but does little with calculus. In the United States, coverage appears to be spread over a variety of topics, with little concentrated attention devoted to any (with the exception, of course, of theAdvanced Placement Calculus Program, not reflected here).



Four dramatically different eighth grade mathematics programs are provided students in the United States, as this figure suggests. Students in remedial classes are offered content that is predominantly arithmetic. The typical eighth grade program has a slightly greater exposure to geometry and considerably more coverage of algebra than offered to remedial classes. Enriched classes receive still more algebra and less exposure to arithmetic. The Algebra classes are provided a great deal of instruction in algebra - usually a standard high school freshman algebra course. Statistics receives scant attention in any of the four programs.

One of the explanations for these achievement results that one often hears is "yes, but the Japanese go to school 5 1/2 days a week for 240 days compared to our 5 day week 180 day school year". This is true, but it does not reflect the correct picture. The Japanese students only attend mathematics classes 3 days a week! The comparative times are 144 hours per year in the U.S. allocated to mathematics versus 102 hours per year in Japan. However, there is another joker. Of that class time in Japan over 90% is spent on mathematics compared to just over 60% in this country.

Two other studies focusing on a comparison of Japan, China and the U.S. give us additional cause for reflection. These studies (based at the University of Chicago and at the University of Michigan) show us some real differences in instruction in the three countries. One thing stands out for me as being very relevant to our concerns today. It is not unusual in Japan or China for a mathematics teacher to spend the entire class period discussing one (yes I said one!) problem with his/her students. The problem will be presented first in a word setting and then analyzed from many different points of view. Different strategies for solution will be discussed. Different problem settings for the same mathematical model will be looked at. The over all picture that emerged is one of in-depth studying of a much more focused set of related topics.

Looking at the SIMS Data reminds us that whether directly or by default school mathematics programs make decisions about the following issues:

What mathematics will be taught
To what level of understanding
To which students
In what way
To enable them to do what
To be evaluated how

We are presently facing critical decisions relative to each of these issues. Mathematics at the middle grades is not serving many of our students well. We can look at the Second International Study Data, or our own National Assessment of Educational Progress data. We can listen to the business community. We can observe the users (or non users!) of mathematics around us. We all have our horror stories about store clerks doing arithmetic.

One of my favorite is the clerk who sold me 4 pairs of hose each costing $2.98. She wrote down

$2.98
$2.98
$2.98
$2.98
———

and added the column very laboriously from top to bottom. Then to my amazement she wrote $2.98 down four times again - but this time added from bottom to top. Producing the same number both times she turned to me and said, "I hope I got this right." I said, "yes, that's correct. It is $12.00 minus 8 cents - - $11.92." Her response -- said with a fair amount of indignation - - was, "how did you know that?" That is a good question. I certainly was never taught or encouraged to think about numbers in that way in school.

There are two other ways to know that many students are failing to learn mathematics. Look at the entering freshman at many universities and most importantly of all, spend time in schools with students in mathematics classes.

We do seem to be standing at a time when we have public attention for school mathematics. We have to collectively look to the future and do our very best to chart a course (complete with processes for course corrections) for school mathematics that has real promise to make mathematics accessible, usable and empowering to kids. The time for action is now. The class of the year 2000 enters school this fall.

I would like to look at each of the kinds of decisions we face and talk about the critical issues surrounding these decisions.

WHAT MATHEMATICS WILL BE TAUGHT

NUMBER

- Number systems/properties/structure
- Operations
- Ratio and Proportion
- Number Theory

MEASUREMENT

- Segments, plane regions, space regions
- Angles, liquid measure, mass (weight)
- Scales (time. temperature) systems of measurement

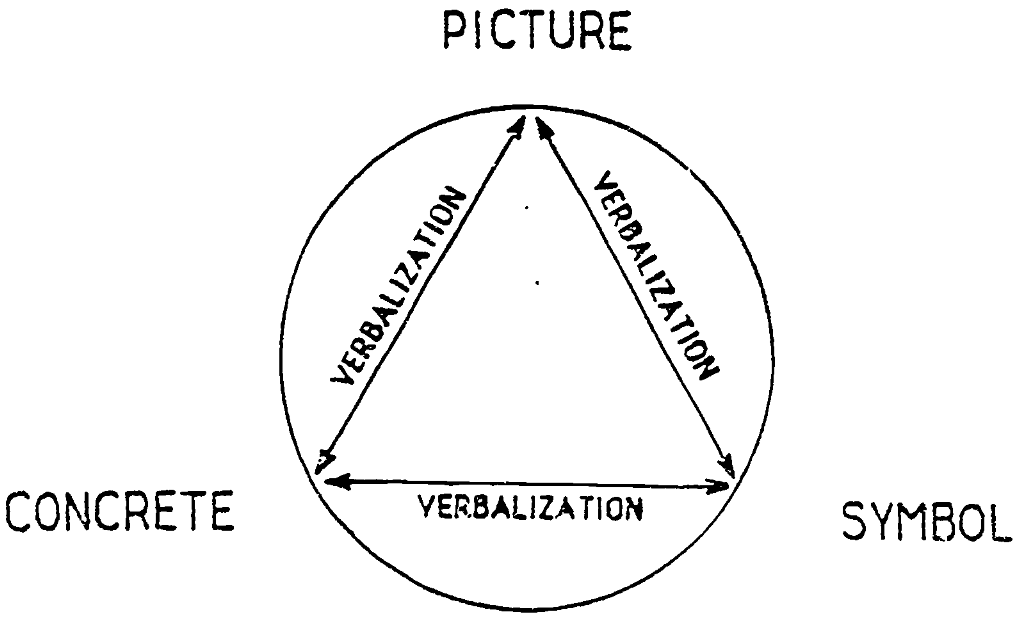
PROBABILITY/STATISTICS

- Planning, gather, organizing and using data
- Decision making (description-inference)

You have integrated the new concept, idea, etc. into your knowledge base so that you can use the new understanding in a creative way - - you can talk about it. You can see its relation to other things you know about. You can represent it in different ways. You can "think" with it. You can visualize it.

The other thing we must keep in mind is that we never reach a stage when we cannot add new shading or dimensions to our understanding of a concept or idea in mathematics. We should expect a middle school student's "understanding" of function to be quite different from a college math student or a mathematicians. But we must strive for creating a classroom situation in which "understanding" is the goal.

The following diagram illustrates how my colleague Elizabeth Phillips and I think of the process leading to understanding. The concrete-picture - symbol interaction in all directions is at the heart of teaching for understanding. Verbalization provides the specific means and evidence for moving from one to another.



TEACHING FOR UNDERSTANDING

TO WHAT LEVEL OF UNDERSTANDING

Skills

Concepts

Applications

Problem Solving

Creation of new knowledge

Another aspect of this level of understanding is that we need to stretch students in mathematics to see themselves as creators or constructors of knowledge. Marilyn Burns recently sent me a copy of an article written by Eleanor Duckworth entitled "Having a Wonderful Idea" - My goal is to help middle school students have wonderful mathematical ideas.

TO WHICH STUDENTS

All the students deserve to see the richness, beauty, and utility of mathematics.

We have many myths to be overcome...

* You can't do that! These students don't know their basics yet!
* They aren't old enough.
* They can't think abstractly.
* They don't need manipulatives or concrete experiences.
* There is not enough time.
* It is not in the textbook/It is not important.

We must see that while it is not spoken out loud, mathematics is a filter just the same. These

students are poor or minorities or women.

Our answer must be that all students deserve an opportunity to learn, not just computation, but all spects of mthematcs described earlier. One of the things that emerged from the SIMS data was a picture of the differentiated curriculum in eighth grade in this country. We <u>must</u> provide opportunities to learn to all students - recognizing that we have a responsibility to provide for students that are interested, talented or truly gifted in mathematics and for students that have learning problems.

Now let's consider -

IN WHAT WAY

Technology of the Classroom

Textbooks; other printed material
Calculators
Computers
Video Discs
Concrete Manipulatives

Organization

Group work
Individualization

Experiences

Oral discussion
Writing, listening, observing
Projects, investigations
Problem solving, applications

The present technology of the classroom is limited almost exclusively to textbooks. We must explore other technologies such as calculators, computers, manipulatives, video discs and others, yet to be invented ways to help students learn mathematics.

Understanding is enhanced by opportunities to discuss, describe, evaluate, conjecture, and validate. Group work, projects and written reports should be a part of mathematics classrooms. We must provide opportunities.

TO ENABLE STUDENTS TO DO WHAT...

Count, compute

- Mental computation
- Estimation
- Counting principles
- Calculators, computers

Measure

- Comparison
- Cover and Count
- Fill and Count
- Successive approximations
- Scales and scaling
- Formal systems

Compare

- Equal
- Less than, greater than

Mode' or represent

- Algebraic, arithmetic (symbolic)
- Geometric (graphical)
- Probabilistic, statistical
- Physical (concrete)
- Algorithms

Reason

- Inductively
- Deductively
- Generalizing
- Extending

These descriptions of student actions are the heart of a changing view. Of what students need from mathematics to the productive citizens in the 21st century. These processes and products of mathematics are all the tools of problem solving -- the major goal of mathematics.

AND TO BE EVALUATED HOW

Standardized tests

Teacher tests

Questions asked and student responses.

My main message here is that much evaluation should take place through teachers questions and student responses in classrooms! We must get smarter about evaluating our goals. The responsibility is ours as professionals to help our students learn mathematics. Evaluation is a tool to help us accomplish this goal.

Let me summarize my remarks today. We have ample evidence that our present mathematics curriculum is not serving our students well. In order to build a better mathematics curriculum we must consider critical issues that we deliberately or by default make decisions about relative to our mathematics programs:

What mathematics will be taught

To what level of understanding

To which students

In what way

To enable them to do what

To be evaluated how.

References

McKnight, Curtis, C., Crosswhite, Joe F., Dossey, J., Kifer, E., Swafford, J., Travers, K., Cooney, T., Second International Mathematics Study, The Underachieving Curriculum Assessing U.S. School Mathematics From an International Perspective, Stripes Publishing Co., (1987), Champaign, IL 61820.

DESTROYING BOXES

John Masterson
Michigan State University

At lunch, sitting with teachers in the early part of a workshop in Flint, there were two different remarks that I find very difficult to deal with. One concerned the amount of mathematics a teacher "needs to know" or ought to know or would be helpful to know. The other was about a teacher listening to a professor that was so smart that the teacher didn't understand a word the professor was talking about. Comments continued. One question was, "why do I want to study all these ridiculous things like group and ring theories?" It's a good question. It's the whole question of what should teachers know before they begin teaching.

One reason that this is a difficult question is because we moralize over things.

I'd like to talk about this somewhat uncomfortable topic without moralizing. As you know, we have different skills and have been educated in different ways. If you put me in front of a middle school class for two or three weeks in a row, dealing with the same kids, and dealing with teaching on a day-to-day basis, I'd probably be bananas at the end of the first week. The parents would be calling up and the kids would be crazy. Of course if you were asked to lead discussions on group and ring theories or other mathematics that I know, you would go through the same uneasiness.

I'd like you to consider the concept that a lot of education in a lot of areas does good things for kids.

Two years ago, I gave a talk to middle school teachers. It was somewhat interesting but I also found it disappointing. I talked about children and mathematics and relevance of history of mathematics in teaching. I talked about the following boxes

BASIC COMPONENTS

	Children	Math
Psychological		
Historical		
Theoretical		

P
E
R
C
E
P
T
I
O
N
S

I brought this here so I could rip it up and throw it away because it doesn't do anything that I want it to do. This is what I mean about destroying boxes. The teaching I do at the university fits the box very well. I mean I stuff boxes as well as anybody. When I teach calculus, I have to. We all use books that are big learning destroyers. We destroy learning with uniform examinations and routine ideas and talk, talk, talk.

I found that stuffing a box does not work with middle school kids. It's easy to keep college kids interested because they know they're going to get a grade. It may be unclear what you're teaching them, but they do shut up and listen. When teaching middle school kids on Saturday morning, I found I could drop an idea in front of these ten to twelve kids and let them go in many directions, instead ot trying to keep them in the box of learning I had prepared. I decided that I could be a guide. Of course, being a guide really does make sense, but it takes a lot of work. First, we have to know a lot about the mathematics; second, we must be willing to learn a lot about kids; and third, we must be flexible enough to go in directions that we may not have thought of.

Before I move into the area where the kids start experiencing abstract thinking, say from arithmetic to algebra, I would like to read a quote from the marvelous mathematician philosopher, David Hawkins. He's talking about his ideal teacher in a paper called "Nature, Man, and Mathematics". He says,

> "for such a teacher a limiting condition in mapping a child's thought into his own is of course the amplitude of his own grasp of those relationships in which the child is involved. His mathematical domain (meaning the teachers) must be ample enough or amplifiable enough to match the range of the child's wonder and curiosity, his

operational skills, his unexpected ways of gaining insight. But a teacher of children of the kind I postulate must be a mathematician - what I would call an elementary mathematician, one who can at least sometime sense when a child's interests and proposals, what I have called his trajectory, are taking him near to mathematically sacred ground."

I like most of that. Perhaps I'm not really big on the word "saved" but I do think mathematics includes areas where something profound is going on and places where you can see something profound going on.

Now let's consider where our middle school kids start experiencing abstract thinking. I would like to share with you some ideas on sequences, factoring algebra symbolism and some particular interests in the history of mathematics.

Before we talk about content, we should remember that kids invent fantasy worlds quicker than we do and they are willing to invent fantasy numbers quicker than we do. We also need to keep the atmosphere as non threatening as possible and try to realize what mode of learning they are bringing to the study of mathematics, which of course is very concrete.

Let's look at some interesting problems.

First, find all primes in a sequence of numbers one less than a square. The sequence would be:

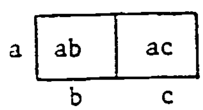
3, 8, 15, 24, 35, 48,...

Take a look at it. See what interesting thing you see. What you'll find is that many students will give you interesting relationships, often not about primes. You are looking for the fact that you will not generate another prime after 3. Kids will say they all factor. As a matter of fact the kids came up with the idea that each number factors with one above each number and one below each number.

Another idea would be to work with a whole unit on factoring. We could get into topics like least common multiple, greatest common factor, fractions as factors, many games, complex numbers, square roots, the distributive law and a whole fantasy world of symbols. There are some wonderful problems to give kids on square roots and imaginary things.

We could also look at geometric representation for algebraic formulas. We can make pictures of ideas of the distributive property.

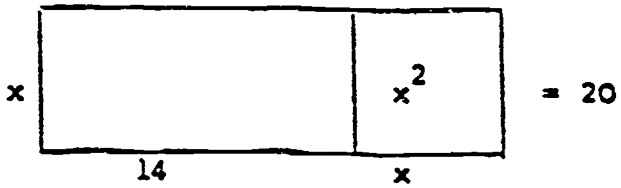
Example: (1) $a(b + c) = ab + ac$



Example: (2) $(a + b)(c + d) = ac + ad + bc + bd$.

	a	b
c	ac	bc
d	ad	bd

The figure clarifies the algebraic formula. Then we might ask if anyone can draw a picture of $a^2 - b^2$. For sort of an extra credit problem. We could include a historical perspective about an Arabic solution of a quadratic equation. Now Arabic mathematicians did not have a process for moving the number to the right or to the left. Let's do this problem.



Arabic Solution

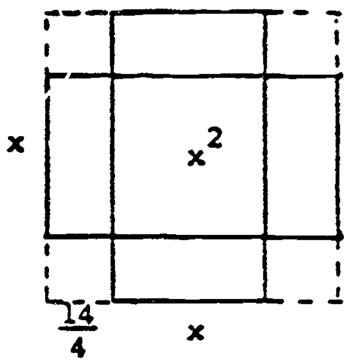
(a) Divide the rectangle into 4 equal rectangles of length 14/4 and width x and attach as in 2nd picture.

(b) If you add the corner squares, you get a large square and you have added $4(\frac{14}{4})^2$ to area.

(c) So large square has area $20 + 4(\frac{14}{4})^2$. But side length of large square is $x + 2(\frac{14}{4})$.

(d) So $x + 2(\frac{14}{4}) = \sqrt{20 + 4(14/4)^2}$;

hence $x = -2(\frac{14}{4}) + \sqrt{20 + 4(14/4)^2}$.



What I'd like to end with is the suggestion that if you get a chance to study factorization, mathematical structures, or ring theory, do so. Remember it is certainly helpful to understand more than you expect to teach to your children.

References:

Hawkins, D. (1973), Nature, Man, and Mathematics, in A.G. Howson (Ed.) Developments in Mathematics Education (pp. 115-135) Cambridge University Press.

Hawkins, LD. (1974), Finding the maximum surface area in education, New-Ways, 1.

SOME ISSUES IN TEACHING SECONDARY MATHEMATICS

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There are of course many problems in the teaching of junior secondary mathematics which could form the focus for our discussion today. For example, it is possible to consider three general areas:

the problems of instructional procedures

the problems of evaluation

the current issues.

I am personally much more interested in the last of these three categories but we may manage to focus on the other two as well.

The problems of teaching secondary mathematics are undeniably complex. This is especially so if one chooses to believe that teaching is more than ensuring that students are exposed to specified content, tested on that content and then exposed to some more specified content. Here is a caricature of teaching mathematics:

> "You put thirty kids in the same box at the same time each day with the same teacher for forty minutes at a time. They all use the same textbook, often not of the teacher's choosing, they are supposed to cover the same material at the same rate, do the same tests at the same time with the same ideal performance expected of each of them. At the end of the allotted grading period we tell some of them that they are not good enough and shrug our shoulders".

One way to begin is to decide an answer to the question: What is the main task of the teacher of secondary mathematics? Some related problems which easily come to mind are:

Problems of content:

Who should decide what mathematics should be taught? What should be the basis of the decision?

How will new mathematics find its way into the schoolroom? (e.g. discrete mathematics).

Should children use calculators and computers? If so, when? How often?

Problems of management:

How should particular students be selected for particular classrooms?

How can teachers ensure that students come to appreciate mathematics?

Responsibility for learning mathematics; teacher's or student's?

It is of course to broaden this set of issues considerably:

Is there a place for research in mathematics education? Is there only a dissemination problem or is it that the teacher can make little sense of the research process, or of the significance of findings for their practice as teachers?

Why do we have schools? To prepare children for society? How is this to be done? What are the assumptions we make about society and about children? (Perhaps we need to better prepare society for children!)

Why do we have mathematics? What should children know? What should teachers know?

Often in the research we find the comment that performance of children on a problem or part of the curriculum is unsatisfactory: what is the basis of such statements? There is the constant urge to do better: why is it presumed that this is possible? Why is it presumed that it is desirable?

What are the underlying assumptions about the nature of mathematics and teaching that drive most curricula?

If these problems or issues do not whet your appetite how about these:

How does one become a better teacher of mathematics?

Is there anything to the gender issue in mathematics education?

Where is the best place to learn to teach mathematics, in college or in a school?

Does inservice do anything for the quality of teaching? What is the best form?

Are teachers technicians administering a ready-made package? Should they be? Are there any other possibilities?

How do we know that a lesson has been successful? Is it possible to decide that a whole course has been successful? What basis should we use?

Is the problem of control in classrooms qualitatively different in mathematics classes?

What are the sources of these difficulties? Any solutions?

Is there a difference between "real" mathematics and "school" mathematics?

Will mathematics teaching ever be a profession? What would be needed?

How should we assess students in mathematics classes? Is this different from evaluati... ?

However what can be said is that most teachers share one or two particular views about the nature of mathematics; it already exists, or it has to be created; either way there are necessary skills to be practiced, the important thing is to get right answers. The process of mathematising is often seen as unimportant, teachers complain that they don't have time to allow such a process.

So I ask the consequent question:

"What does it mean to <u>know</u> mathematics?"

Do I "know" mathematics if I know the rules of the game?

Do I "know" mathematics if I understand the logical steps which take us from one line in the argument to the other?

The first of these questions is answered in the affirmative by those who think that mathematics is a formal symbol system which is internally consistent and created by men (and women, I hasten to add!) Formalists see no need to have anything to do with the "real world" whatever that is!

The logicist answers yes to the second question.

Another fundamental question is:

"Where does mathematics come from?"

Essentially there are three answers,

i it is invented in the minds of people,

ii it is discovered because it is already there,

iii it is a social product which results from what might be described as mathematical experience.

The first two of these seem to be unable to be reconciled; however the third is compatible with both! We first invent and then the objects invented take on an independent existence for those who have already constructed the idea of them. For example it is not common to meet the cube root of two walking down the street; but it is possible to talk about the properties of such a number with another person who has formed the same concept. That is, we construct elements of a mathematical universe and can then agree about the meanings and relevance of the elements in that universe. The objects we envisage may well have properties that we do not know but which can be determined by another person, and then we can agree about the veracity of such claims. So there is invention and then other properties are discovered after a kind of admission into the universe!

My reason for dwelling on the nature of mathematics is a conviction that the teacher of mathematics should have some idea about the nature of the mathematics, because it is possible to teach very differently if one chooses to believe differently!

I have become a firm believer in the importance of mathematical experience as the source of intuitions and inventions. Thus I believe that all children are initially "mathematicians" and that the processes of schooling work against the necessary provision of experience so that many children do not continue to develop and think about their own intuitions. We mistake the capacity of some children to return rote responses to routine questions for "ability" in mathematics. Expereince is gained only through children learning to model situations, first with objects, perhaps later by drawing pictures, certainly at all levels through language describing their views of situations.

It is for this reason that I insist that our job as teachers of mathematics is to work much harder at providing suitable contexts or environments, and then to develop questioning skills. These are very difficult roles for teachers who already know the mathematics they wish children to learn! There is an almost irresistible urge to tell, rather than to allow time for the ideas to form. (There is a substantial literature on the dominant use, especially in secondary schools, of the "recitation", where the principal method of instruction is for the teacher to lecture to the students. If questions are asked by the teacher they are usually used as a control device; if children ask questions they are usually directed at getting direct information about administrative or routine procedures; "Do we have to do all these?" "Can we finish these up for homework?", "Are we allowed to use calculators for these problems?")

Another avenue which might be fruitful for the group to consider is the general area of questioning in mathematics classes. How should questioning proceed?

In mathematics we can identify different kinds of knowledge; should all these be treated in the same way? What different kinds of knoweldge am I talking about?

Logicomathematical knowledge: e.g. "If $a > b$, and $b > c$, then $a > c$." This is not something one can be convinced of by telling, it is a mathematical idea constructed by each of us, that is, it is in the area of conceptual knowledge.

Arbitrary knowledge: The mark we make for the word "three" is "3", the word we say when we see "Σ" is "sum".

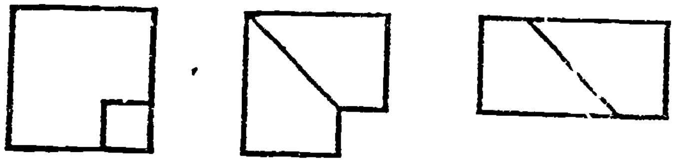
Should these types of mathematical knowledge be taught in the same way? Can you think of other examples in each category? Where would you place the operations of addition, subtraction, multiplication, and division? The ideas of fractions and decimal fractions? The angle sum of all triangles?

Now a consequence of the idea that mathematics is learned through experience is that mathematics must be a social construction; we learn it as we do other human knowledge; that is, in the company of other humans and through the interactions we have with those others. It is possible to conjecture that other societies will have developed different mathematics, in fact there is plenty of evidence that this is the case. Many language groups of Australian Aborigines have no expressible concept of relation in the quantitative sense; it is not possible to think about "longer than" in the way that Europeans do! Imagine trying to come to grips with ideas of relation and function if one were to be constrained by such a language! (I hasten to add that these languages have extraordinary complexities and can be used to describe other non-quantitative relationships with remarkable degrees of sophistication; it is an error to regard them as "primitive".) They have mathematics, but it is different, it was developed as ours was, in response to social pressures and needs.

In the workshop which follows I intend to illustrate the major idea of this presentation that we create mathematics through the social interactions taking place between people. I hope that you will recognize this as a potentially useful way to teach the children in your classes.

Workshop Activities

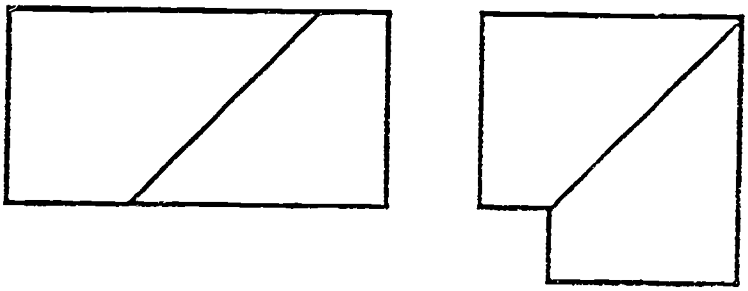
1. The first activity involved people in folding paper into a square and removing a smaller square from that square; folding along the part-diagonal, cutting and rearranging produces the following. What observation can be made?



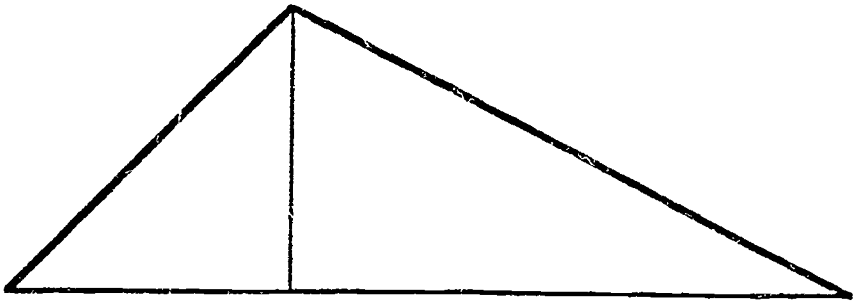
After discussion among themselves, group members decided that we had an example of the factorization of difference between two squares:

$$a^2 - b^2 = (a - b)(a + b).$$

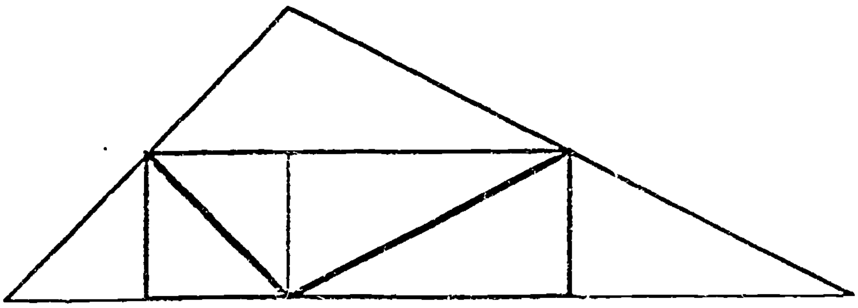
2. The second activity asked people to think about how many different ways there are to fold a paper rectangle into half. After some discussion about what "half" meant, it was determined that there were infinitely many ways, and that the resultant folds were not all axes of symmetry. However every fold through the "center" of the rectangle produced "halves" in that the two shapes so produced were congruent. The congruence could always be established by turning one piece over and matching, which led to a further discussion about symmetry.



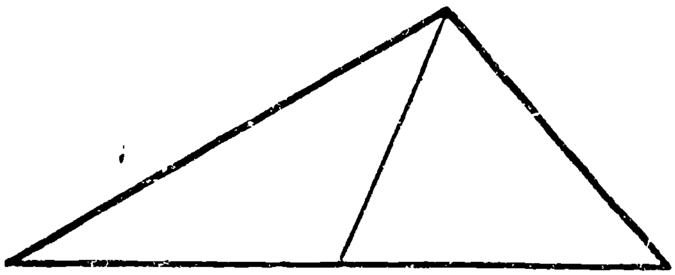
3. Each member cut out a non-special triangle. They were asked to fold to make a right angle at the base; then to fold the top vertex down onto that point. They were then invited to fold in the other two vertices.



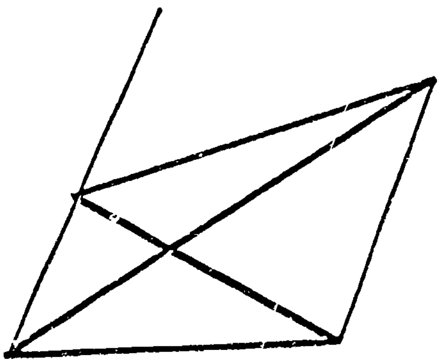
The result is shown below. The questions for dicsussion centered around the nature of the shape, and any generalizations the members of the group found. They decided that they had a very neat "proof" for the angle sum of the triangle, and that there would always be a rectangle formed in this way. The area of the rectangle is half that of the triangle (congruent triangles), and so knowing the formula for the area of a triangle it is possible to determine the area of a triangle.



4. An extension to the second problem, from the square to the triangle was discussed, which led to the notion of the median as that line which produces two triangles of equal area from the original triangle. Members of the group discussed why. (The triangles both have equal bases and share a common altitude.)



Folding along the median produces another way of looking at the triangles, they now have the median as common base and so the vertices much join to produce a parallel to the median.

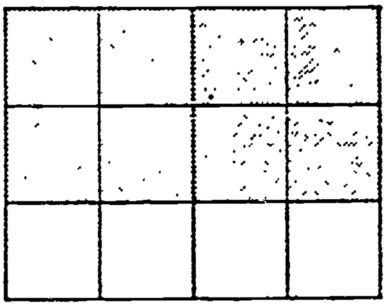


5. To make the point that it is difficult to act until there is a context, the question 1/2 - 1/3 was written on the board. When this was articulated as the situation; "I have 1/3 of a brick and it weighed 1/2 of a kilogram. How heavy is the whole brick?"Members of the group agreed that they solved the problem by thinking about the context and tripled the 1/2 without thinking about "turning upside and multiplying."

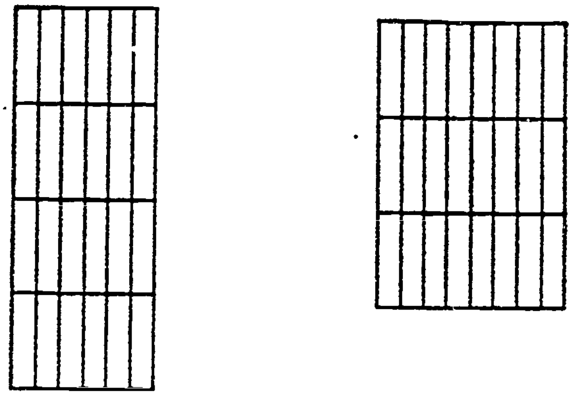
6. The members of the group were then invited to determine the decimal equivalents to 1/7, 1/17, 1/27, 1/37, and 1/47 by using a calculator. After determining that there were 6 places in the repetend for 1/7 and 16 places in the repetend for 1/17, the group was invited to predict what would happen for 1/27, and 1/37. It was expected that these would be longer, "more complicated" was the phrase used. When they were calculated and found to be relatively simple and even more surprisingly related, the group was challenged to determine why.

7. Members were invited to enter three digits into a calculator, and then to repeat the three, so that the calculator displayed a number of the form "abcabc". They were then asked to divide this number by 7, the quotient by 11, and the new quotient by 13. They were asked to explain the result.

8. The next challenge was to show "3/4 x a = 2/3" with a diagram. One group conceptualized the problem as "How many 3/4 will fit into 2/3 of a cake?"



The argument went that since only 8 of the small squares were equivalent to 2/3, and 9 squares were needed to make 3/4, the 2/3 contained only 8/9 of a three-quarters. A second attempt took a strip and showed both 2/3 and 3/4 on it and also measured the 2/3 with the 3/4, although the members of this group subdivided the strip into 36 pieces, when they might have made do with 12. Both attempts were interpreting the context as a measurement division. Some groups worked at an area interpretation.



Here we see that an area of 2/3 x 1 can be re-arranged into an area of 3/4 x 8/9.

The workshop concluded with emphasis on the notion of mathematics as a social construction, resulting from action on the part of those constructing the mathematics when engaged in making sense of contexts, and dependent upon language as a way of making meaning. Symbolization is the process whereby the mathematics is summarized.

PROBLEM WITH A PURPOSE

Thomas Butts

University of Texas at Dallas

Wall of Fame

1) billfold
2) Thoreau's home
3) Alabama governor
4) Washington city
5) famed highwire walker
6) Austrian president
7) seal relative
8) rural target

Have you answered the "Wall of Fame" yet?

What's a billbold?	wallet
Thoreau's home?	Walden Pond
How about Alabama's governor	George Wallace
Number four?	Walia Walla
Number five?	Wallynda's
six?	Waldheim
seven?	walrus
eight?	Wal Mart

The title of this talk is Problems with a Purpose. I traditionally give this type of "problem" at the beginning of class. They are often not about mathematics at all. Students who are sometimes reluctant to guess at mathematical problems, are often eager to solve such problems that do not necessarily have a right or wrong answer. Today I'd like to talk about six types of problems.

The following are somewhat arbitrary classifications:

Opening/closing problems
Process problems (demonstrate heuristics)
Skill reinforcement problems
Discussion problems
Surprise problems
Competition problems

Let's first take a look at a process problem.

> In how many ways can Alice, Marvin, and Colin share 25 pieces of candy so each person gets at least one piece of candy?

Think about this problem in small groups, for about five minutes. Try to decide how many different ways are there to divide up the candy.

Since my purpose is to expose you to a number of different things, in a short amount of time, I must break one of the cardinal rules of teaching problem solving - ALLOW SUFFICIENT WAIT TIME. In this talk, there will not be time to wait long enough to discuss and analyze everyone's responses.

Most of you made a table to record your counting of the number of ways. For example, the first three entries in several tables were

23	1	1
1	23	1
1	1	23
22	2	1
22	1	2
1	22	2
2	1	22
.		
.		
.		

Make a Table

A	M	C	
23	1	1	(1)
22	2	1	
	1	2	(2)
21	3	1	
	2	2	(3)
	1	3	
	etc.		
1	23	1	
1	22	2	
	.		
	.		(23)
	.		
1	1	23	
			276

Your point of view might be to say, "I'll take 23 and then distribute the other two. Next I'll take 22 and distribute the other three." Continuing down the table, how many entries are there in the last block? There are 23, so the total number of ways is

$$1 + 2 + 3 + 4 + 5 + \ldots + 21 + 22 + 23 = 276.$$

An important part of problem solving is looking back to see if we can arrive at the solution another way. Stop and think about how you would usually divide up candy. You probably wouldn't make up an organized table. You would divde it into piles. How would we analyze the problem doing it this way? To make it a little easier to see, we'll use a row of twenty-five dots.

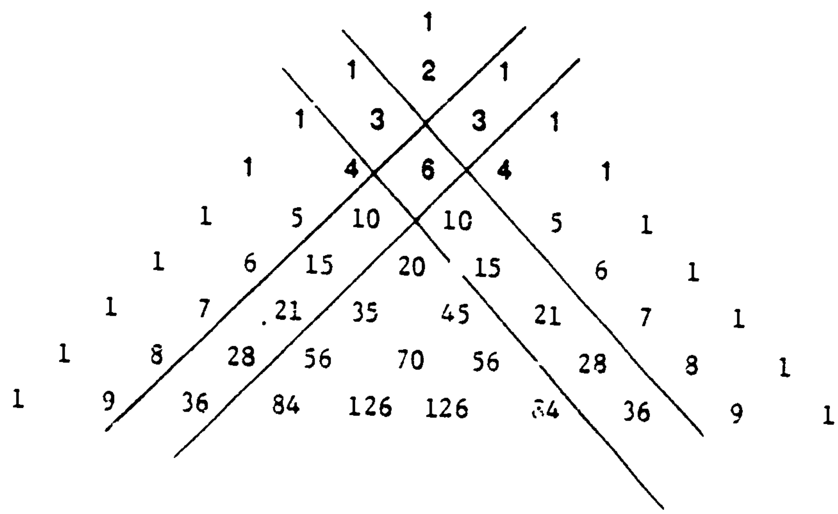
. . . .

What would you do to divide up the candy? We would try to consider how many different ways we would have of dividing the candy up among three groups? We might use marks to indicate where we would divide the candy into groups. In how many different ways can I put the two marks? Remember to consider the marks that are "mirrors" of one another. We would then see that we have 24 ways for the first one and 23 ways for the second one. Then we must divide by two for duplicates.

Summarizing, we can either add up the numbers from 1 to 23 or we can think of the solution as 24 times 23 divided by 2.

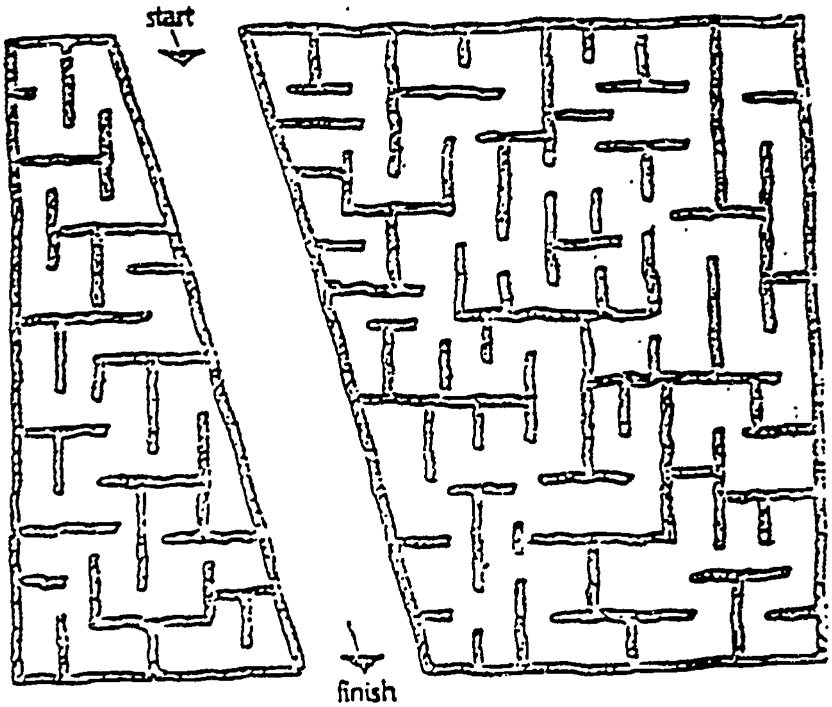
Another interesting solution can be found using two diagonals of Pascal's Triangle.

Pascal's triangle



Here's another problem to work on.

get through the MAZE!

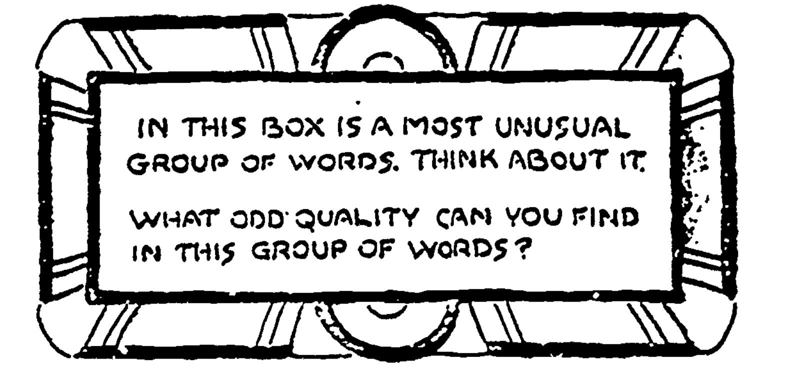


see if you can find your way through this perplexing maze.

Another example of an opening or closing problem might be:

Name ______________________________

ALPHABET GRAPHING



Answer: No e's.

You could use such examples as springboards to do problems about frequency of letter distributions, or probability of just quickies at the beginning or end of class.

A second classic process problem is the Locker Problem. It is an excellent problem to use to introduce a unit like Factors and Multiples that you'll be doing toward the end of the workshop.

One Thousand Lockers
Locker Problem #1

> In Long Middle School there are 1000 students and 1000 lockers numbered from 1 to 1000 in a long hall. All the lockers were closed. The students line up in alphabetical order. The first student (S_1) opened every locker. The second student (S_2) went to every other locker, beginning with the second locker (L_2) and closed it. The third student (S_3) went to every third locker, beginning with L_3, and changed it (if it was open, he closed it, if it was closed, he opened it). In a similar manner the fourth (S_4), the fifth (S_5), the sixth (S_6)..., student changed every fourth, fifth, sixth, ... locker. This process continued until all 1000 students had passed by all 1000 lockers.

We will simulate the problem of "Which lockers are open?", in two ways. The problem is simulated and the numbers of lockers that are open are: 1,4,9,16,25. Why are the open lockers the ones whose numbers are squares? Because only squares have an odd number of factors. Some other problems you can consider are:

1. List the lockers touched by S_{180}.
2. List the students who touched L_{36}.

3-6. Each of the following questions concerns one of the concepts in this chapter. Answer the question and give the concept.

3. Which lockers were touched by exactly two students?

In particular, what kind of question can you ask to "get at" the concept of greatest common factor? (Answer, question #4).

4. Who was the last student to touch both L_{60} and L_{84}?

What kind of question can you ask to "get at" the concept of least common multiple? (Answer, question #5).

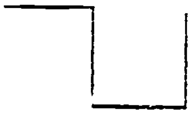
5. What was the first locker touched by both S_{12} and S_{20}?
6. Which lockers were touched only by students with even numbers?

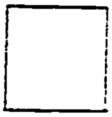
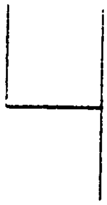
The next four questions concern whether certain lockers were open or closed after all 1000 students had passed by all 1000 lockers.

7. Was L_{12} open or closed?
8. Was L_{1000} open or closed?
9. Which of the first 25 lockers were open?
10. Which of the 1000 lockers were open? Give reasons for your answer.
11. Which lockers were touched by exactly a) 3 b) 4 students?
12. Which locker was touched by the greatest number of students?

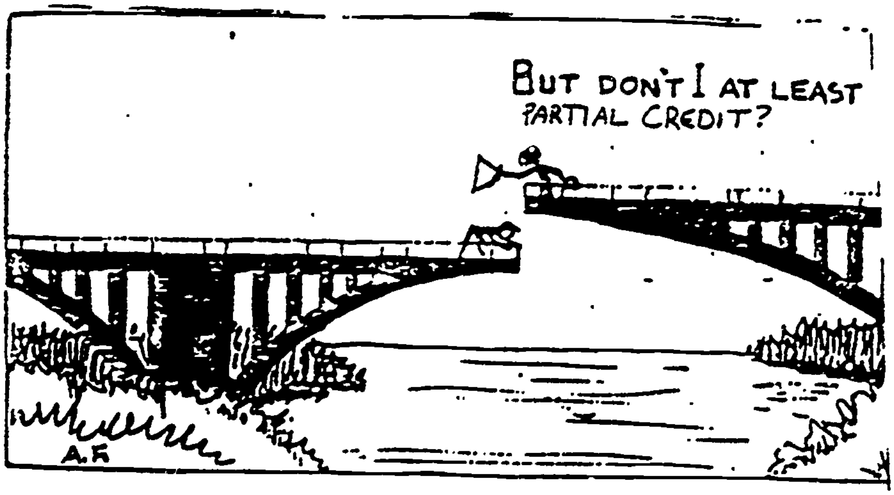
Here's another five minute question.

Move one toothpick to form a square



answer:  or  (four is a perfect square)

I guess those of you that got the first answer ought to get partial credit.



The next category of problems is reinforcement problems. Here is an example.

Choose 5 different non-zero digits and form a 3-digit and a 2-digit whole number.

1. How many such pairs are possible?
2. a. What is the maximum sum of a pair?
   b. What is the probability of randomly choosing a pair with this maximum sum?
3. What is maximum difference of a pair?
   a. positive difference
   b. negative difference

4. What is the maximum product of such a pair?
   Now choose 4 of the 5 digits.
5. How many ways?
6. Insert one of the digits in each square so the

$$\frac{\square}{\square} \circ \frac{\square}{\square} = \frac{\square}{\square}$$

a. maximize the sum
b. minimize the sum
c. maximize the product
d. minimize the product

You could spin off all sorts of reinforcement activities on fractions, decimals and whole numbers from this model. Here's another opener.

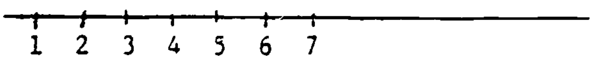
<u>Question of the Day</u>

Have in common:

1) drum, ache, ring
2) sand, news, wall
3) grease, room, tennis
4) handle, room, none
5) fly, truck, spit
6) fair, ground, double
7) power, sea, sense
8) tone, magic, down
9) easy, person, high

Answers: 1) ear 2) paper 3) elbow 4) bar 5) fire 6) play 7) horse 8) touch 9) chair.

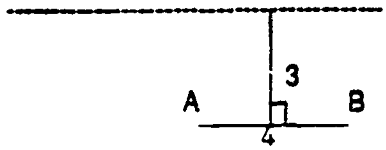
Another little category that I have is called discussion questions. These are questions whose purpose is to raise issues that merit discussion. Number 3 illustrates the use of a good generic question. Do you expect the answer to be bigger or smaller than the number?

1. 

   You are designing a form in which someone needs to write his name. Space 1 is for the first initial. Space 3 is for the middle initial. If you want to allow 16 spaces for the last name beginning with space 5, what is the number of the last space for the last name?
2. If the sum of three whole numbers is greater than 6000, then at least one of the numbers must be greater than _______. How about 4 whole numbers, 5 whole number,... Generalize and explain your reasoning.
3. In each case, write L if you expect the solution to be larger than 72, S if you expect the solution to be smaller than 72: a) $6y = 72$, b) $y/6 = 72$, c) $y/3 = 72$, d) $4y = 72$, e) $2y/3 = 72$, f) $9y/5 = 72$.
4. Is the product of any two fractions less than 1?

5. What number has the a) smallest b) largest reciprocal?
6. Which is larger: 23% of 78 or 78% of 23? Generalize and explain.

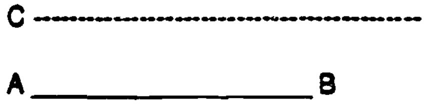
7.



On the dotted line, locate all points C for which triangle CAB is a) right b) acute c) obtuse d) isosceles e) scalene.

8. Can you draw quadrilateral with exactly a) 1 b) 2 c) 3 d) 4 right angles? Compare your drawings with those of another student. If you cannot draw a figure, explain why you think it cannot be.

9.



As point C moves along the dotted line, does the perimeter of triangle CAB increase, decrease, or stay the same?
As point C moves along the dotted line, does the area of triangle CAB increase, decrease, or stay the same?

10. Give an example of a set of 5 whole numbers for which the a) mean > median b) median > mean c) mode > mean.
11. T F If false, give an example where the statement is false.
    1. The sum of three positive numbers is positive.
    2. The sum of two positive numbers and one negative number is positive.
    3. The sum of two negative numbers and one positive number is negative.
    4. The sum of three negative numbers is negative.
12. Is there a least positive integer? Is there a greatest positive integer? Is there a least negative integer? Is there a greatest negative integer?

<u>Here's another five minute one</u>:

"Make" 20 using four nines and
any mathematical symbol(s) known
to an 8th grader.

One answer: 99/9 + 9.

Now make 20 using exactly three 9's. Answer (9 + 9)/9.

I'd like to do this next problem in groups of two. <u>It's a good probability problem.</u>

Take N people (7 < N < 13 is a good size) and give each one a standard deck of cards. At a given signal, each person randomly chooses a card. What is the probability that at least two people will choose the same card?

Solve the problem: a. empirically
b. theoretically

This problem was an example of a **Surprise Problem** - one whose solution is surprising to most people. In this case the theoretical probability is approximately .75 for N-12 people. Most of you guessed between .1 and .3.

If you've heard of the birthday problem, this is the birthday problem except with decks of cards. The advantage with the cards is that you can do more than one trial.

Here's another five minute problem.

Question

1. Mississippi Taleteller
2. Cinema's Enforcer
3. Bee Gee
4. Athos' Delineator
5. Author Cum Doctor
6. Humble Circumlocutionist
7. Energy Finder
8. Panther Star

Answers: 1) Mark Twain 2) Clint Eastwood 3) Barry Gibb 4) Alexander Dumas (author of the Three Muskateers) 5) Arthur ConanDoyle 6) Howard Cosell 7) Enrico Fermi 8) Peter Sellers.

The last category is constructing problems for contests. Below is a list of standard techniques for constructing contest problems. These techniques lend themselves to multiple choice format and timed tests.

Constructing problems is like painting pictures or composing symphonies - it's very personal and difficult to "teach". Herewith are a few suggestions arbitrarily arranged into two groups called "Types" and "Techniques". Ways to repose existing problems are left for another time.

"Techinques"

C1. Short way, long way
C2. Optimize
C3. Estimate
C4. Make a mistake
C5. All but one (nonexample)
C6. Give an example
C7. Parameterize
C8. Reverse the "usual"
C9. Do "one of three"
C10. Read Carefully
C11. Define a new operation
C12. Find the error

I'll leave you an example or two from each of the categories listed above:

C1 Compute: $\frac{597 + 598 + 599 + 600 + 601 + 602 + 603 + 604 + 605 + 606}{5}$

A) 1201 B) 1203 C) 1205 D) 1206 E) 1208

"Everyone" can solve this problem, but the insightful problem solver will solve it quickly and have more time for the other problems. There is a reward for insight, and some success for everyone.

C1 2301 + 4708 + 8086 =

A) 15095 B) 14985 C) 14885 D) 14875 E) 14095

C2 Find the smallest rational number $\frac{a}{b}$ so that the products $\frac{24}{35} \cdot \frac{a}{b}$ and $\frac{36}{55} \cdot \frac{a}{b}$ are both whole numbers.

C3 The product 737 x 767 is closest to which of the following:

a) 50,000 B) 55,000 C) 60,000 D) 500,000 E) 550,000

C3 A crowd watching a parade fills the sidewalks on both sides of Fifth Avenue for a distance of 2 miles. The sidewalks are 10 feet deep and an average person needs 4 square feet to stand on. A good estimate of the size of the crowd is

A) 25,000 people B) 50,000 people C) 100,000 people D) 250,000 people E) 500,000 people

C4 Using a calculator, a student mistakenly multiplied by 10 when he should have divided by 10. The incorrect answer displayed was 600. The correct answer is

A) .6 B) 6 C) 60 D) 6000 E) 60,000

C5 Decide which one of the following is not true. Three straight lines can be drawn in a plane so that there

(1) is no intersection point
(2) is one intersection point
(3) are two intersection points
(4) are thre intersection points
(5) are four intersection points

C6 Give an example:

100 consecutive integers of which exactly two are squares.
Four different non-zero integers whose sum exceeds their product.
Four different non-zero integers whose sum is a factor of their product.

C7 Find all integers k for which the equation $kx - 12 = 3k$ has an integer solution.

C7 In the product shown, B is a digit. The value of B is

$$\begin{array}{r} B2 \\ \underline{x\ \ 7B} \\ 6396 \end{array}$$

A) 3 B) 5 C) 6 D) 7 E) 8

C8 Write 45 as the sum of consecutive integers in as many ways as you can.

C8 Give an example of two rational numbers whose product is between 1/2 and 2/3.

C10 At a school, all students are given a questionaire asking if they approve of the President. Suppose 60% of the students answer the questionaire and 55% of these say they approve of the President. What is the maximum percent of the entire student body that approves of the President?

A) 33% B) 55% C) 73% D) 78% E) none of these

C10 Which of the following numbers has the largest reciprocal?

A) $\frac{1}{3}$ B) $\frac{2}{5}$ C) 1 D) 5 E) 1986

C11 If A*B means $\frac{A + B}{2}$, then (3*5)*8 is

A) 6 B) 8 C) 12 D) 16 E) 30

C12 Find the error and correct:

$$\frac{1}{x} - x^2 = \frac{1}{x} - \frac{x^2}{1} = \frac{1 - x^2}{x - 1}$$

$$= \frac{(1 - x)(1 + x)}{(x - 1)} = -(1 + x)$$

Problems C1 (both), C3 (both), C4, C7 (second), C10 (both), and C11 have appeared in the American Junior High School Mathematics Examination.

I'll close with this one final problem for you to work out. Some of you, I trust, will be acting this out later on today - at least one part of it.

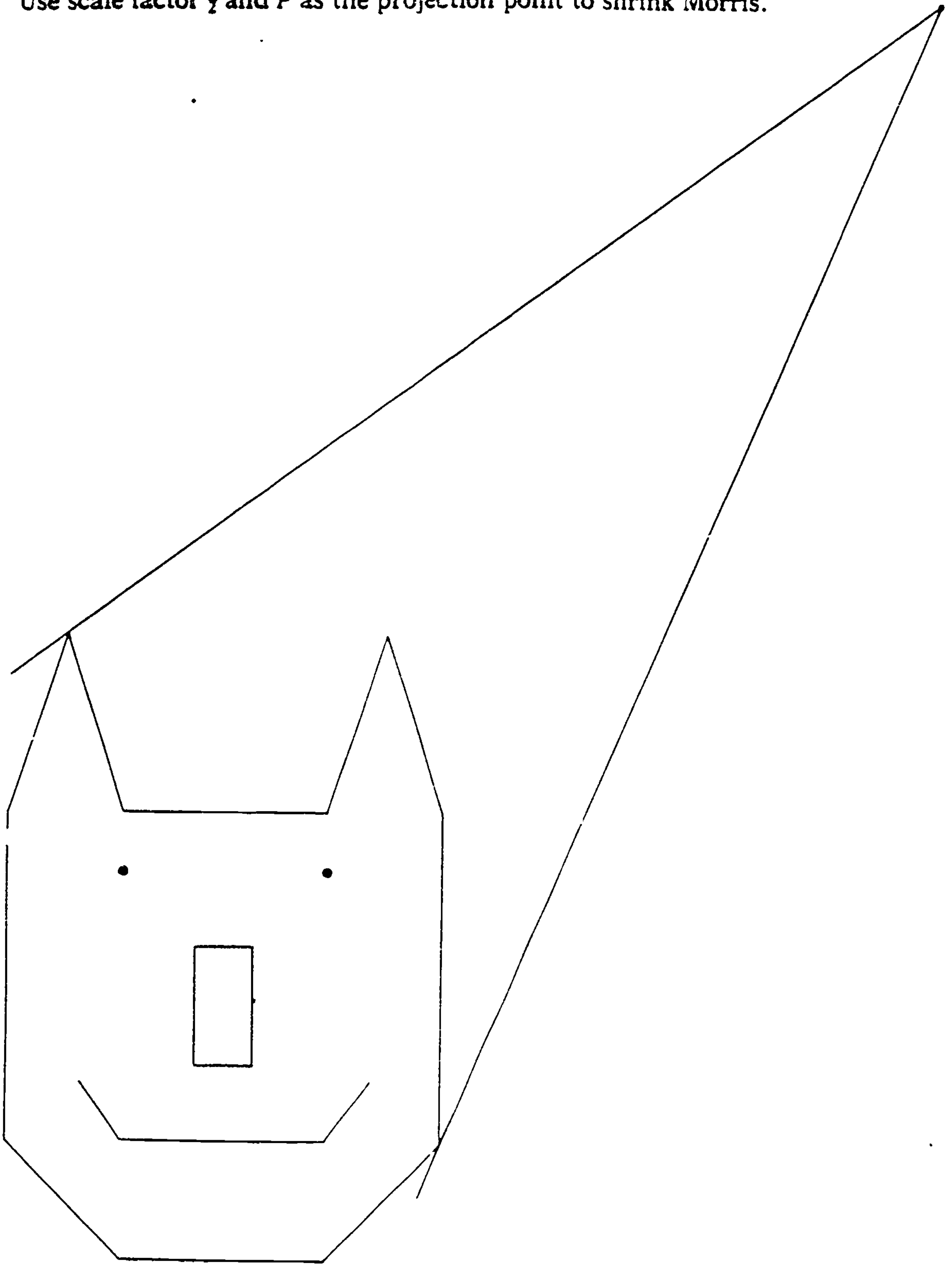
AN OLD CHESTNUT

Speedy Gonzales has a large circular track in his backyard of radius t kilometers. Each day Speedy runs n laps and for each kilometer he runs, he drinks s liters of beer. What is the probability that Speedy drinks at least one liter of beer per day?

Incredibly the probability is 0!
He drinks $2\pi tns$(km/lap*lap*lts/km)
$= 2\pi nts = .95$ liters.

Shrink Morris

Use scale factor ½ and *P* as the projection point to shrink Morris.



TEACHING MATH WITH MANIPULATIVES

Ed Antosz
Mercy College, Detroit

Preamble

The notion of working with manipulatives in mathematics is neither a new nor a novel idea. Tac-Tiles which is the name I've given to the material that I've been developing over the last ten years is but one manifestation of the idea of using concrete material to teach mathematical concepts. Some of my ideas have grown out of notions presented by Zolton Dienes, who has been concerned with the use of manipulatives for over thirty years.

When I began my teaching career I taught mathematics exactly the way I had learned it. My students had the usual difficulties with algebra and I couldn't understand why. They could do the algebra in that they manipulated symbols correctly but there was no deep structure to what they were doing. I was very perplexed. At that time I began looking at how people learn mathematics. Zolton Dienes says you have to play with mathematics. Dewey and Montessori say you have to touch it. These people say that as a result of touching the material you learn images. These images are built up, one upon the next. From these images, the student can translate concrete facts into symbolic representations of these will happen to the concrete material.

Tac-Tiles grew out of teaching adults who were coming back to the university. These students were working either on a Bachelor of Science Degree or a Bachelor of Education. Not one of these adults had the equivalent of high school mathematics. Since every university requires its students to have at least basic mathematics, my job was to teach these people mathematics. And so Tac-Tiles was born. As the material developed and modes of presentation were refined I found that I could teach any adult to solve any linear equation in less than three hours.

Today Tac-Tiles consists of squares and rectangles which come in different sizes and shapes. Each piece is coloured blue on one side and yellow on the other. The various pieces are used to represent number, positive and negative as well as fractions, variables in algebra and polynomial expressions.

The most natural progression when using Tac-Tiles is to let children play with them. Allow the student an opportunity to become familiar with the material at hand. The next step is to develop the rules for arithmetic and finally the rules for algebra and polynomial arithmetic.

Arithmetic Operations (addition & subtraction)

There are two rules for arithmetic. The first rule is that a number can be represented by only one colour. The second rule is that at any time you can add or remove a blue and yellow pair from the representation of the number without changing the value of the number. This second rule allows you to add one blue and one yellow but not two blues nor two yellows. When working with children more time might be spent developing the rules.

To develop arithmetic skills using Tac-Tiles we begin by playing a game called Blue and Yellow Numbers. Using the two rules from above we try the following examples.

2B + 3B
4Y + 1Y
3B + 2Y
4Y + 6B

Taking the first one, we place 2 blue tiles on the table, then place 3 blue tiles on the table and count them up. Of course, we get 5 blue tiles. The process looks something like this.

2B + 3B = 5B

In the second example we would place 4 yellow tiles on the table and then place 1 yellow tile and count them up.

4Y + 1Y = 5Y

The third and fourth examples appear to be different but they require the use of the second rule. We place 3 blue tiles on the table, then place 2 yellow tiles on the table. This time before we can count them up we must use the second rule and remove one blue and one yellow tile, and then remove another blue and yellow. (Remember the first rule that requires the result to be one colour only.)

3B + 2Y = (3B+2Y) = 1B

And our final example of addition looks like this.

4Y + 6B = (4Y+6B) = 2B

The next step is to ask the student for an alternative notation. The usual procedure is to assign one of the colours the value of 1. Perforce the other colour represents the opposite of 1 or -1. For our purposes let blue be 1. The examples just used can now be rewritten as shown below. (In a classroom situation you might prefer to allow the students to decide which colour represents 1. The only confusion is when one student uses blue as 1 and another students chooses yellow to be 1. They will arrive at the same arithmetic results however, the representation of the numbers will differ.)

2B + 3B	2 + 3 = 5
4Y + 1Y	-4 + -1 = -5
3B + 2Y	3 + -2 = 1
4Y + 6B	-4 + 6 = 2

Subtraction and addition can be introduced at the same time using Tac-Tiles. Playing the same game using the two rules above and performing a number of examples where we "take away" gives us an introduction to subtraction. Consider the following examples:

6B - 4B
5Y - 3Y
4Y - 2B
2B - 3Y

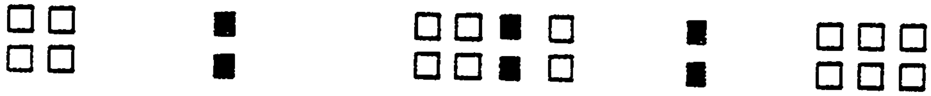
Looking at the first example, we place 6 blue tiles on the table, then remove 4 blue tiles on the table and count them up. Of course we get 2 blue tiles. The process looks something like this.

6B - 4B = 2B

The next example looks like this.

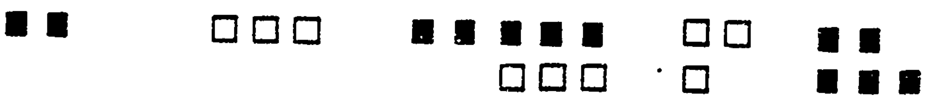
5Y - 3Y = 2Y

The next two examples again require use of the second rule. The third sample requires that you place 4 yellow tiles on the table and remove 2 blue tiles. Since there are no blue tiles we must add a pair and then add a second blue and yellow pair. Now we can remove the 2 blue tiles and count up the remainder.



4Y - 2B = 4Y + (2B +2Y) - 2B = 6Y

And in the final example we place 2 blue tiles on the table. In order to remove 3 yellow tiles we must add three pairs of blue and yellow tiles. We then remove the 3 yellows and count up the remainder.



2B - 3Y = 2B + (3B + 3Y) - 3Y = 5B

And should we follow the same procedure that was introduced earlier where the blue tile represents a 1 and the yellow tile represents -1, the examples above can be rewriten as shown below.

6B - 4B	6 - 4 = 2
5Y - 3Y	-5 - -3 = -2
4Y - 2B	-4 - 2 = -6
2B - 3Y	2 - -3 = 5

A complete development of arithmetic skills requires that we attend to the operations of multiplication, division, powers and roots, however, for this presentation I will skip over them in order to deal with the ideas of polynomial arithmetic and the solution of linear equations.

Variables

To represent a variable we use a tile of length different than that of a 1. The tile looks like  . Its width is the same as the one tile but its length has been increased. The tile is used to represent a quantity which at present is unknown. To solve an equation we must first divide our work space into two parts. The rules for addition and subtraction as outlined earlier still hold. We now require an additional rule. That rule states that in solving the equation we perform the same operation in each of our workspaces. (Obviously, we're setting up the theory of an equation here.) For example, if you choose to add one tile to one side you must also add one of the same tiles to the other side. (In a sense we are working with two rules: 1) you can add or subtract any number of these tiles from one side, and the other rule is 2(whatever we do to this side we do exactly the same to the other side.) Our goal is to isolate the variable tile by itself in one of the work spaces. There may not be variable tiles on both sides when our work is done.

As examples let's look at the following equations:

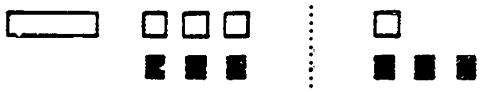
$$x + 3 = 1$$

$$2(x + 4) = 3x - 1$$

A representation of the first equation would be:

The division between the two work spaces is noted by the dotted line.

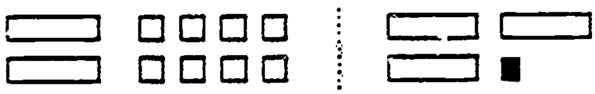
To solve we can add three blue tiles to the left side or we can remove three yellow tiles from each side. Let's add three blues. (Remember that we must then add three blues to the right side as well. The operation looks like this.



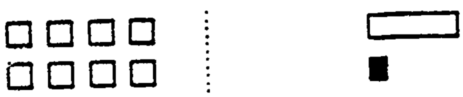
When we use the earlier rule of permitting only one colour in the solution we get the following.

The solution to the first equation is $x = -2$.

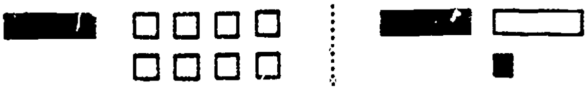
To represent the next equation requires that we develop the notion of representing a polynomial. (It is important to note that when working with children the pace of activities will differ from that of this presentation.) The equation is $2(x + 4) = 3x - 1$. The representation of a $x + 4$ is  and systematically $2(x + 4)$ would be represented by two such configurations and the equation looks like this.



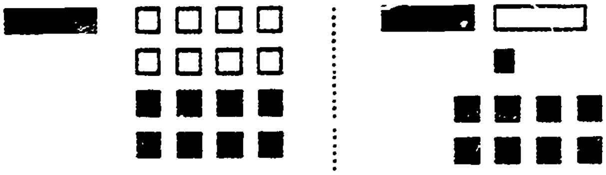
There is more than one way to solve this equation by the solution I'll use will demonstrate several features of Tac-Tiles. First let's take two x's away from each side. We get



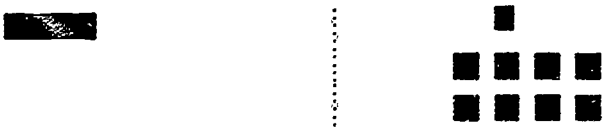
Now let's add a blue variable to each side.



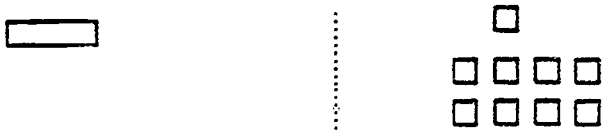
And let's add eight blue tiles to each side.



Upon collecting and removing all the tiles that annihilate each other we get the following.



We're close to the goal of isolating the yellow variable tile by itself. All we need is to flip it over. We must flip the other tiles as well. This gives us the solution $x = 9$.



After solving equations using Tac-Tiles the task becomes one of asking the students how to write the procedure used in manipulating the tiles. Children have no difficulty suggesting the following sequence for solving the equation above.

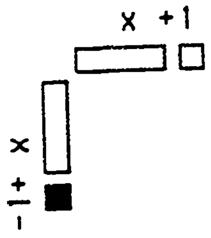
$$
\begin{aligned}
2(x+4) &= 3x - 1 \\
2x + 8 &= 3x - 1 \\
2x - 2x + 8 &= 3x - 2x - 1 \\
8 &= x - 1 \\
8 + -x &= x + -x - 1 \\
8 + -8 + -x &= x + -x - 1 + -8 \\
-x &= -9 \\
x &= 9
\end{aligned}
$$

This sequence clearly describes the steps that we took in solving the equation. After a little practice using Tac-Tiles the child can move on to solving equtions with paper and pencil referring to Tac-Tiles as need be. Experience has shown the children will refer back when working with a new situation and will soon grow to the point where they need only refer to mental images of the tiles. Polynomial arithmetic can be performed by adding tiles of varying sizes and shapes.

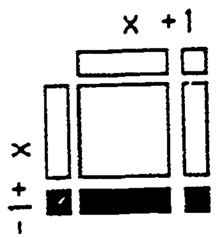
Polynomial Multiplication

The final concept that will be demonstrated here is that of multiplying polynomials. I realize that I've skipped over polynomial addition and subtraction but that can be developed naturally from what we know of addition and subtraction of integers. Polynomial multiplication is an ideal example of a situation where it is easier to show the student an idea than it is to explain what to do.

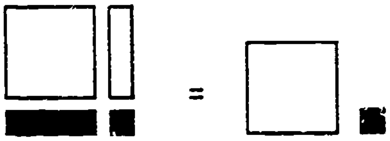
To find the product of two binomials using Tac-Tiles we will construct a rectangle with length equal to one binomial and width equal to the second binomial. This is easier done than said. Let's take the product of (x + 1) and (x - 1). The dimensions of the rectangle will be similar to the diagram below. (The diagram has been exploded so to make it easier to see the process.) Note that I've shown the dimension x - 1 as -1 + x.



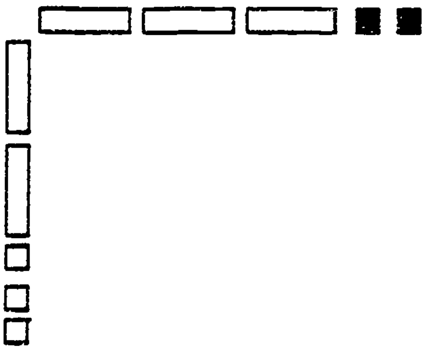
Our task now becomes to fill in the rectangle with the correct shapes and to simplify the result using the rules as developed above.



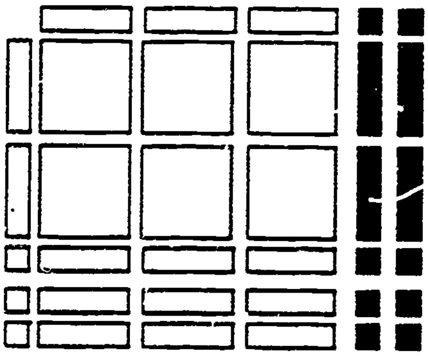
Notice the result. We get an $x^2 + x - x - 1$. When we simplify we are left with $x^2 - 1$ which looks like this.



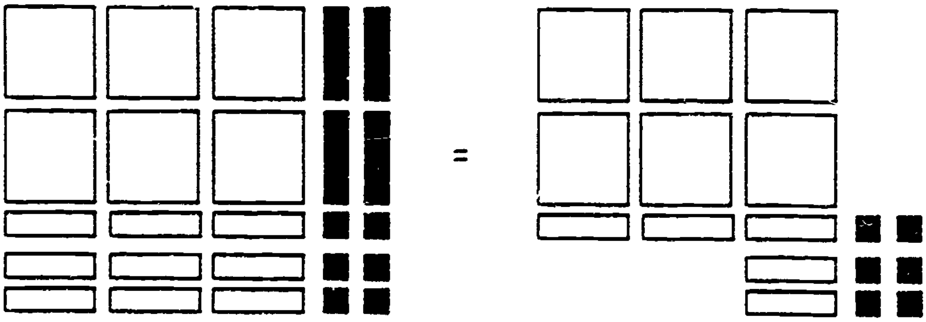
Let's look at one last example. This time we'll multiply (3x - 2) by (2x + 3). The boundary of the rectangle looks like this



When we fill in the area we get



Now when we remove the boundaries and simplify we are left with the product



When we ask the children to write the process that we've just demonstrated the result is often similar to the usual notation used for the multiplication of two binomials. That is we get:

$$(3x - 2)(2x + 3) = 6x^2 - 4x + 9x - 6$$
$$= 6x^2 + 5x - 6$$

And this is the very idea that we are trying to convey.

The symbols we use to write polynomials and show polynomial arithmetic are not always clear. The success of symbolic notation is a function of the understanding the teacher and student have of the symbol just as in spoken language where the use of words is functions to determine how the speaker communicates the message to the listener.

In mathematics we often require the student to keep a series of instructions in mind. If we provide the list correctly and the student can remember the instructions and how they inter-relate when the teacher has completed the instruction the student should be able to "see the picture". The difficulty with a process of this nature is that the student can lose instructions or interpret them in a fashion that was not originally intended. The strength of manipulative material is that we can demonstrate the instructions as well as the inter-relationships simultaneously by showing the student. The best analogy for this is a photograph. Try to describe a picture. The relationships of line, colour, form, shading, texture all contribute to the final product. It is impossible to explain the picture without listing the many relationships. Only upon completing the list can your listener reassemble the picture. More likely than not the picture they hear is not the picture you describe. One role of manipulative material is to provide the picture without words.

As mentioned at the opening this notion is neither new nor novel. However, the most appreciative audience seems not to be the mathematics teacher. Math teachers seem to be very unreceptive to the notion of using manipulatives. They're approach is to just <u>tell</u> students how to perform mathematical operations rather than to devleop the concepts behind the operation.

Montessori, Piaget, Bruner to name but a few state emphatically that children must first manipulate material. Then they formulate images with which they work. Finally, children can use symbols to manipulate the images they have. The transfer does not seem to be there when the teacher begins with the symbols. Although some children can succeed when they begin with symbols the suggestion is that their success is due to their good memories or to the fact th t they have already develo-ec vstem of images for themselves. Often mathematics teachers do not realize how dif c it is to start at an abstract level since they can operate with facility at that level. The transition is the one that draws the line in separating our population and identifying who has a love for mathematics.

PROBABILITY

Michael Shaughnessy
Oregon State University

I want to talk a little bit with you about probability and statistics. What I'd like to do is have people gather their own statistics by running some kind of a probability experiment. We'll look at probability from an experimental point of view and use the data that you have gathered.

Just to start off I want to share with you some of the most important work in statistics that's ever been done. It's summarized by a few of these statements here.

> ***If you torture data long enough, it will confess.***
>
> ~ *R. Coase*

> ***You can prove anything with statistics except the truth.***
>
> ~ *G. Canning*

> ***Don't trust any statistic you have not falsified yourself.***

> ***If there is a 50-50 chance that something can go wrong, then 9 times out ten it will.***
>
> ~ *Murphy's law of large numbers*

If you confront a satistician with a man with one foot in a bucket of boiling water and the ou r foot in a bucket of ice cold water, he will say that, on average, the subject is comfortable.

~ H. Paulsen

If your experiment needs statistics, you ought to have done a better experiment.

~ Lord Rutherford

Lies, damned lies, and statistics.

~ Disreali

I want to talk about why I think it's important to do probability. You've gotten some phi ophical overview of the MGMP units; but even beyond just the MGMP units, I think it's important to expose kids to probability experiments. They need to gather their own data, analyze their own data, and organize their own data. I'm going to share with you some of the materials to do that. Some are my own and some are commercial ones.

Why Probability and Statistics?

1) Masses of statistical information.
2) Misconception of probability and misuse of statistics.
3) Involves problem solving and decision making.
4) Experimental not theoretical treatment of probability and statistics in schools. (from a consumer's point of view)

I feel very strongly about probability and statistics. The question is, when and where do we do it? I think that we should do it in seventh and eighth grade for a half of year. Pick one quarter of the seventh and eighth grade package, do probability and statistics. If you can't do that, weave it in, and spiral it throughout the rest of the material you do. Probability and statistics involves many things. For example, computations are used in gathering data and computing averages. Ratios and proportions are used in probability. I would like to see kids get exposure. Here is a list of potential places.

Probability and Statistics

-- When and Where??

A) In 7th and 8th grade for at least 1/2 year, or spiraled throughout

B) Throughout a general math course (GEMS)

C) As the required second year mathematics course after Algebra I

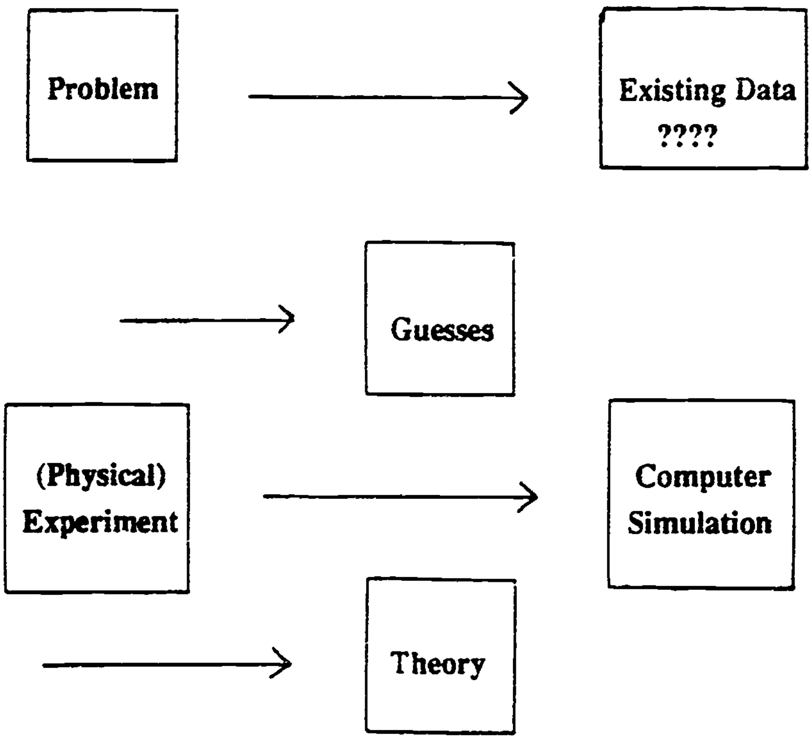
(D) As 1/2 of all of a second math course after Algebra II. (Better for most kids than Calc-Anal)

(E) Interspersed throughout a usual Algebra I or Algebra II course.

These are places where teachers that I've worked with in the Northwest have found that they can make it work. Remember, this is not just me saying, "I think I should do this". There are teachers who have been involved in an experience similar (Honors Teacher Workshop) to yours who have gone back and done this for the last two years.

I would like to share my philosophy of "How to get your hands dirty with this stuff". We talked about why probability, when and where probability. Now, let's talk about how. I'd like to do it through simulation. You've done a lot of simulations already. You've used the fair and unfair games. When I say simulation, I mean carry out an experiment that represents a problem situation.

PROBABILITY & STATISTICS
Via
SIMULATION



First you're going to look at some data. After you have looked at the data, you'll want to look behind the data. That is when you're asking, "What is really going on here?" You may then wonder why things are happening the way they're happening. If you don't have data, you have to get your own data.

I happen to have some data on what I call the 1970 Draft Lottery, even though it really occurred in December of 1969. Here are the mean numbers that were chosen.

For each month:

1970 Draft Lottery
Monthly Means

January	201.2	July	181.5
February	203	August	173.5
March	225.8	September	157.3
April	203.7	October	182.5
May	208	November	147.7
June	196.7	December	121.5

Now look behind the data. Is there anything you notice about the data? November, December and September don't look good. Why would anybody have a better chance of getting a higher number every year. The question could be, "Where these well mixed?" It comes out they didn't have good mixing.

A newspaper article came out explaining how the mix occurred. First they took December and put it in a box, then November was put in same box and mixed, then October and mixed it and so forth through all the months. Then the box was dumped upside down. December was now on top. It had not been mixed very well after all. So you can see why it's important to look behind the data.

Interesting data from "Exploring Data" can be used in middle school grades. An example is exploring amount of soft drinks consumed one year by an individual.

We are now going to do some experiments.

I want you to break into small groups. I'll give you each a card. Wait until I give you your card, then you'll find four people that match your denomination and pick a table to sit at.

I'm going to put a problem on the overhead and let you take a look at it for a minute. You're going to make a bet. Here is the situation:

Birthmonths

You and four new friends are talking at a party. One of your new friends starts to try and guess everyone's birthmonths. Then, another friend says she is willing to bet anyone a dollar that at least two of you have the same birthmonth.

Should you take the bet?

A) Make a guess (gut level).

B) Do you think that there is a 50% chance, or better that there is a match of birthmonths?

C) How could we simulate? Gather groups of 5 people to get some data, and check it out?

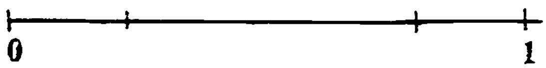
(Problem is actually done with experiment in small groups. Data is compared during groups).

Here are some other situations you could present:

Forming Triangles

Two points (numbers) between 0 and 1 are chosen at random. The points divide the interval from 0 to 1 into three line segments.

What is the probability that the three segments will form a triangle?

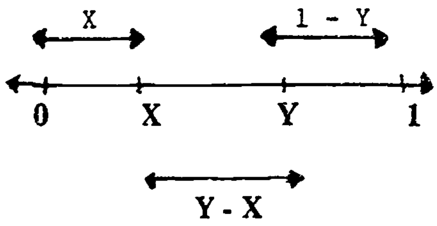


A) Make a guess.

B) Let's let a computer simulate this one.

C) (Hint!!) How could we represent two points in the interval [0,1], say by using algebra?

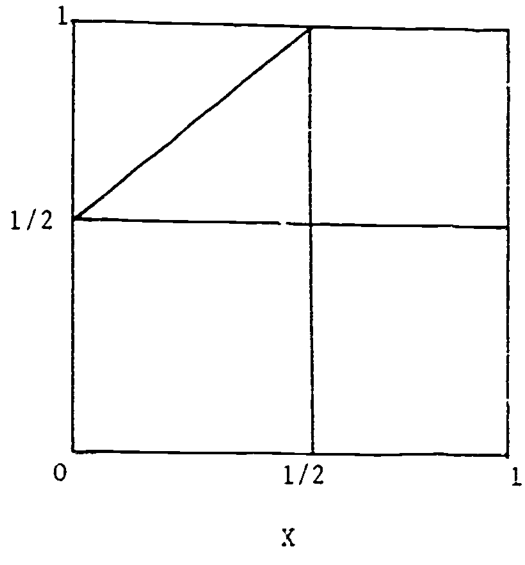
Probability of Forming A Triangle



In this case, x y, add the three pieces are of length
x, (y - x), and (1 - y)

These three equations must hold:

1) x + (y- x) (1 - y) ⟶ y 1/2
2) x + (1 - y) (y - x) ⟶ x 1/2
3) (y - x) + (1 - y) x ⟶ y x + 1/2



Cereal Boxes

Recently General Mills company included bike racing stickers in their boxes of Cheerios. There were 5 different sets of stickers.

How many boxes of Cheerios would you expext to have to buy in order to collect all 5 sets of stickers?

A) Make a guess.

B) We can't eat that much cereal! (Rather not) So...

C) How can we simulate buying the boxes and checking the stickers?

THE HARE AND THE TORTOISE

THE TORTOISE FLIPS A COIN AND MOVES EITHER 1(H) OR 2(T) STEPS.

THE HARE ROLLS A DIE AND MOVES FROM 1 TO 6 STEPS.

WHAT'S A "FAIR" LENGTH (# OF STEPS) FOR THE RACE IF THEY HAVE TO WIN BY EXACT COUNT?

(NOTE: THEY "REST" IF THEY ROLL TOO LARGE A NUMBER.)

BREAKING STICKS

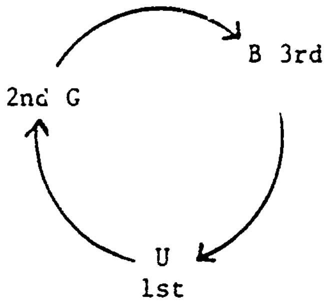
A (unit) wooden stick is randomly broken into three pieces. What is the average length of the small, medium, and large pieces?

A final example will be a simulation with the use of a computer. It's called "The Good, Bad & Ugly".

THE GOOD, THE BAD, AND THE UGLY

Three cowboys are locked in mortal combat in a 3-cornered dual. Ugly hits his target 30% of the time, Bad 50%, and Good never misses.

If they shoot in the rotation Ugly - Good - Bad, who should Ugly shoot at?



This is the shooting order

THE UGLY-BAD MATCH

Ugly lives if one of these outcomes occurs:

1) B_mU_h $P(B_mU_h) = (.5) \times (.3) - .15$

or

2) $B_mU_mB_mU_h$ $P(B_mU_mB_mU_h) = (.5)(.7)(.5)(.3) = .05$

or

3) $B_mU_mB_mU_mB_mU_h$ $P(B_mU_mB_mU_mB_mU_h) = (.5)(.7)(.5)(.7)(.5)(.3)$

$= .02$

or

4) $B_mU_mB_mU_mB_mU_mB_mU_h$ $P(B_mU_mB_mU_mB_mU_mB_mU_h) =$

$(.5)(.7)(.5)(.7)(.5)(.7)(.5)(.3)$

$= .006$

or

5) $[(.5)(.7)]^4(.5)(.3) = .0002$

Or

6) $[(.35)]^5(.15) = .00067$

.15 + .05
+.02 + .006
+.0002
+ 00067
= .22687

and so forth. In general, $(.15) \times \quad (.35)^{n-1} = .23$ as k .

We've had a fun afternoon. Thank you for sharing it with me. I have two more sayings to leave with you.

"This branch of mathematics (probability) is one,
I believe, in which good writers frequently get
results entirely erroneous."

Charles Sanders Pierce

Dizzy Dean
"I hate statics."

(Old Dizzy wasn't too smart.)

REFERENCES

How To Lie With Statistics. Darrell Hoff, W.W. Norton.

Statistical Excorist. Hollander (Daker) publishing Co.

Exploring Data Analysis. Turkey, J. (Addison Wesley) Publishing Co.

CHANGING THE NEGATIVES TO POSITIVES: TRANSFORMING GENERAL MATHEMATICS CLASSES

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If they had their druthers they wouldn't be there.
It has nothing to do with me, it's just that it's
general mathematics class. (A project teacher)

Researchers at MSU's Institute for Research on Teaching, in collaboration with mathematics teachers, studied the problems of teaching and learning in general mathematics. They targeted three strategic areas in critical need of improvement. These strategic areas included communication, social organization, and the mathematical content. Once identified, the staff began its review, study, and synthesis of the literature related to improving learning and instruction within the strategic areas.[1] It is the purpose of this paper to present the problems of learning and instruction in general mathematics, describe some of the instructional interventions that were implemented by the project teachers, report the outcomes for teachers and students related to the instructional improvement areas, and to discuss the implications of the findings for others interested in improving general mathematics classes. Although each strategic area will be presented separately, it was the simultaneous implementation of interventions in all three strategic areas that created a force powerful enough to overcome the massive and chronic problems found in general mathematics classes.

[1] The General Mathematics Project's reading list is included in Appendices B, and C.

Improving the Patterns of Mathematics Communication

They (students) weren't interested in understanding the math, they wanted to know what to do so they could get on with it.
(A project teacher)

As researchers studied general mathematics classes, they found the mathematics communication between teachers and students was sparse and minimal. Students simply wanted to know what the assignment was going to be and how to work the problems. The teachers responded by giving their students a set of directions or procedures for working the daily problems then followed this with the assignment. Typically, teachers spent less than 10 minutes per class period in mathematical communication with their general mathematics students (in contrast to over 25 minutes spent in their respective algebra classes). In addition, the teachers used less precise mathematical language in general mathematics classes than in their algebra classes. For example, the decimal .79 was read as "seventy-nine hundredths" in one of our teacher's algebra classes and as "point seven nine" in her general mathematics class.

Researchers were aware that teachers and students needed to engage in more mathematical communication. There needed to be more talking about math and more importantly, the talk needed to be much richer. Researchers and teachers developed methods from the readings that focused on improving patterns of communication in general mathematics - to increase the quality and quantity of communication. Some of the methods that the project teachers implemented included the following:

1. Questioning students' responses, both right and wrong answers. (E.g., "Tell me how you got that answer.")

2. Eliciting student explanations of their answers or solutions to problems. (E.g., "What were you thinking about as you worked this problem.")

3. Increasing the amount of time for students to respond to a question or to make a statement.

4. Encouraging student participation in discussions or analyses of commonly made errors on assignments.

5. Requiring that students always talk and think about several different ways to solve problems. (E.g., "Can you also show the sum of two fractions using pictures and your fraction bars?")

The following classroom example illustrates how one of our general mathematics teachers (Ms. Kaye) encouraged students to think and talk about multiple ways of representing the addition of unlike fractions.

Ms. Kaye: Suppose I wanted you to combine one-half, plus one-third, plus one-sixth. (On the chalkboard she has written)

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

Kenneth: Oh, I know, I know! It is three-twelfths...no, no, that's not right...wait... I can redeem myself!

Ms. Kaye: All right, I'll give you a chance to redeem yourself.

Kenneth: It's three-elevenths...or...one whole.

Ms. Kaye: Kenneth! We just said over here that you can't add numerators and denominators!

Kenneth: Well then how can you add that?

Melanie: I did it differently. I put one-yalf, one-third, and one-sixth in a line and I got a common denominator of six and then I added them.

(Ms. Kaye writes on the chalkboard what Melanie has told her)

$$\frac{1}{2} = \frac{3}{6}$$
$$\frac{1}{3} = \frac{2}{6}$$
$$\frac{1}{6} = \frac{1}{6} \qquad \frac{6}{6} = 1$$

Ms. Kaye: All right, so you did it using arithmetic. You are correct, but can anyone do it using a picture?

Randy: I can, I can!

(Randy goes to the chalkboard and draws the following)

Randy: Here is the one-half (pointing to it), here's the one-third (pointing to it), and here is the one-sixth (pointing to it). You CAN SEE they are equal to one whole!

Ms. Kaye: Yes, you could do it that way also. I don't care if you do it this way or use arithmetic -- I do care that you use a mental picture. Because if you draw a picture you may avoid adding across and getting three-elevenths.

(Ms. Kaye draws the following on the board)

Ms. Kaye: You see, if we draw a hexagon like Randy's and divide it into twelfths...we can say that elevenths is really close to twelfths, can't we?... (the students agree). If we took this hexagon and shaded three of those twelfths that is about as much as three-elevenths. Does that even look close to the answer Randy gave us, one whole?

Students: No.

Ms. Kaye let Kenneth "live with" his mistake until Randy and Melanie had finished their explanations for how they arrived at the answer. When they were finished, Ms. Kaye returned to Kenneth's answer and compared it to Randy's. This showed the students the importance of representing mathematical ideas or operations with pictures and the importance of communicating their thinking about these problems.

The method of asking the students to think and talk about the multiple ways to represent mathematical concepts (concretely, pictorially, and symbolically) was implemented throughout the year and across all the units of mathematics that were covered. One teacher noted significant improvement in her students' ability to think about mathematics, in their willingness to try new mathematical problems, and in their ability to verbalize their thoughts about mathematical ideas and content. Students in one project class mentioned the value of talking more mathematics in their comments about the class at the end of the year. A few of these student comments are included as follows (typed as written):

> ...you explain things, other teachers give you a book and page number and tell you to read the directions.

I new most all of the things we did not never understood it real well. You made things clear and helped me all the time...

...our teacher would give us a page number and say good luck. It is more if someone explains it to you. The only suggestions I have are to make more teachers do it this way.

Both the students and teachers noted that the quality and quantity of communication about mathematics had changed across the year.

The amount of time spent in mathematical communication increased from the average of 10 minutes at the start of the project to an average of 27 minutes per period at the end of the project. The nature of this time was also changed -- it was now enriched with dialogues, discussions, thought-provoking questions, and error analyses. One project teacher, reflecting on the changes that had been made in her general mathematics class with respect to communication, commented:

You keep asking the question, "Why?" over and over again and it finally gets to the point where the students know what it is you are looking for. For example, when you say to them, "You have four-sixths and that equals two-thirds -- why does it equal two-thirds?" eventually the students will begin saying, "Because you can divide the numerator and denominator by two-halves."

They (students) start answering in complete sentences. They are not giving you yes/no type answers, because if they do they know I will ask them, "How did you get that?"

I noticed that particularly when the semester changed. I had 10 new students. I was reviewing fractions with the class and I noticed the students I had first semester answered my questions differently than did the new students. The new students just gave an answer. My old students gave an answer and an explanation.

I think I am asking them to do a lot more of that all the time and I think that is really important. Questioning helps me find out where students' misconceptions are. It helps the students realize that they aren't alone with their misconceptions and wrong answers.

Facilitating Mathematics Learning Through the Social Organization of the Class

Drill and practice exercises are needed to cement the mathematics into their little minds. (A project teacher)

Project researchers found students spent an average of 30 to 35 minutes per class period on routine seatwork assignments. This seatwork consumed the major portion of time in the general mathematics classes. The teachers thought lengthy drill and practice exercises served two purposes: first, to let them know if the students understood how to work the problems, and second, to "cement it into their little minds." They believed that without sufficient drill and practice the students would forget how to work the problems. The students were frequently given textbook or worksheet assignments containing numerous and repetitive basic computational drills. These assignments were so routine and simple that students used this time to socialize with one another as they worked on them. One student was observed chatting with a classmate about amovie she had just seen when she turned around to another classmate and said, "I am working on my math, too. It just helps when I talk."

The amount of time general mathematics students spent in off-task socializing (particularly during seatwork and at the start and end of the class period) was a common problem in the general matheamtics classes studied at the beginning of the project. This was one characteristic of the social organization of the general mathematics classes that contributed to the problems of teaching and learning mathematics in these classes.

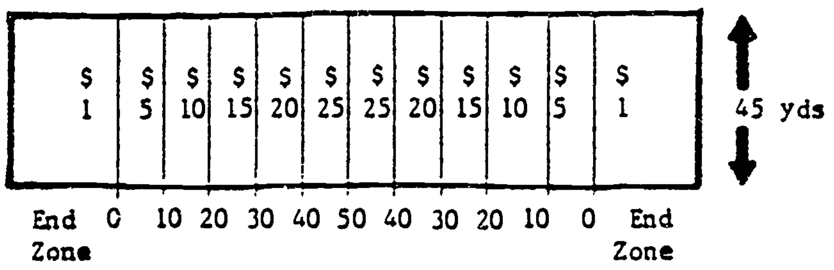
The social organization of the general mathematics classes was an important strategic instructional area which researchers believed needed to be improved. The social organization consists of the organization of students for instruction and tasks, the class routines and procedures, and the instructional methods which support and maintain the goals and objectives of the class. The social organization in general mathematics classes was characterized by a notable lack of whole group instruction, instructional planning, and interest in the mathematics content by the students and the teachers. Substantial amounts of time, however, were given to mundane seatwork assignments, endless amounts of student papers, and off-task socializing. Improving the social organization of these classes required that greater amounts of time be spent in whole class direct instruction and in the time given to planning for instruction. It also required a decrease in mundane seatwork assignments, the amount of student work, and off-task socializing. Social organization techniques would have to encourage student involvement and interest in mathematical content and tasks that fostered the development of mathematical ideas and concepts. In addition, social organization methods needed to focus classroom procedures and the learning of math. The social organizational methods that were developed by the project staff were intended to facilitate the learning and instruction of mathematics in these classes, are listed below:

1. Using student groups to work on mathematical activities.
2. Using student volunteers at the chalkboard or overhead projector to solve problems or to talk about common errors that had been made on the daily work.
3. Providing a short paper-and-pencil review activity when the students enter the room prior to the start of the daily lesson.
4. Placing an agenda on the chalkboard to focus student attention on the content just covered and that content which will be covered.
5. Providing students with more feedback on their mathematical progress through units of content.
6. Writing more long-range instructional plans that focus on the development of mathematical concepts across and within mathematical units.
7. Using controlled practice activities during direct instruction to encourage student participation.

The use of student groups for various units and activities encouraged the students to interact and communicate with one another as they worked together to solve the mathematical problems. The following is a classroom example of such a group activity:

Ms. Kaye: You have a group project to do for the rest of the hour. The Athletic Boosters are trying to raise extra money and are selling square yards of the football field. I want you to figure out how much the Boosters are going to make.

(Ms. Kaye draws the following diagram on the board)



Ms. Kaye: The Boosters are going to put the names of the buyers in a large drawing in the cafeteria. I want us to find out how many yards we can get out of the football field and how much money could be made. I want you to figure out how many square yards and the total amount of money this will generate.

(Ms. Kaye writes the directions on the board)

1. Draw the figure.
2. Figure total yards.
3. Figure total amount of money.

Ms. Kaye: I want you to put a group total of money on your paper. I want each of the groups to figure this out. Make sure you write your answer in a group.

After Ms. Kaye presented the problem and outlined the group task, the students were given calculators and the rest of the period to solve the problem. At the end of the period, each group's answer was recorded on the chalkboard with one member from each group selected to explain to the rest of the class how the group arrived at the solution.

One project teacher was convinced that some students would refuse to work in groups. She commented, "It was hard for me to let go of my control of the class." After trying student groups in her general math class she was convinced it could work. She described a review day on which she had her students work on review worksheets in their assigned groups:

> They were drilling each other and they were working. I mean, I went around the room and they almost said, "Leave me alone. We're gonna get him to learn this..." It was so much fun, I loved it. And I had one student phone another student at home asking if he had his ditto finished.

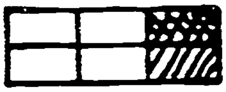
The teacher had the groups competing for the honor of leaving the room without having to put up their chairs for the custodian. She was amazed at their zeal. She now believes that the most significant benefit of group work is the increased amount of mathematics commuication that goes on when the students work together. She notes that this not only helps them develop their mathematics language but it also loosens them up for class participation during whole class direct instruction.

Using controlled practice exercises during whole-group direct instruction was a second social organizational method used to encourage students to participate actively in the mathematical content of the lessons. The following is an example of controlled practice that was undertaken during a lesson on multiplication of fractions in one of our project teacher's classes.

Ms. Kaye: Divide a rectangle into thirds like this.

(Ms. Kaye shows the students how to draw and divide their rectangles by her example on the board.)

Ms. Kaye: I want you to look at the third -- shade it in. (She waits for the students to shade in their thirds). Now, cut the third you shaded into half. (She waits a moment then shades her half of the third.)

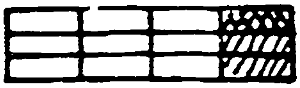


Ms. Kaye: Multiplication means you have groups of one-half of a third. Now, I want you to tell me what you have.

The students: One-sixth.

Ms. Kaye: Write this one, please. One-third times one-fourth. I want you to start with one-fourth of a rectangle and shade it in. Then I want you to take one-third of that fourth and shade that in.

The students start working and Ms. Kaye waits for a moment then draws the following on the chalkboard:



Ms. Kaye: That's good. One-fourth cut into thirds tells yu to take one-third of a fourth and you now need to find out how many equal pieces you have.

Sally: So you have twelve equal pieces and one-twelfth of that is shaded.

Ms. Kaye: Now I want you to do two-thirds of three fourths. I want you to start out with the three-fourths and then shade it in like this.

(Ms. Kaye draws the following on the chalkboard.)

Ms. Kaye: So, now we are talking about twelfths. We have six-twelfths and can that be reduced?

The students: Yes, one-half.

(Ms. Kaye has the following on the chalkboard.)

Controlled practice exercises, starting-of-class reviews, and the use of student group work encouraged active student involvement and interest in mathematics. It also communicated to the students that there indeed were mathematical goals, objectives, and purposes to general mathematics. In responding to a question about the rewards of teaching general mathematics using these methods one project teacher commented:

> When I think about how hard my job is now, I sometimes think it would be a whole lot easier to say, "Here is how you add folks." Then give them the worksheet with fifty problems on it. Then go around and work one-on-one with each student.
>
> It is a lot harder to be up front going at it for the entire class period. That's a lot more work for me. But it is a lot more enjoyable. It takes a lot more time and work to teach the class the way I'm teaching it now, but it is definitely more rewarding.
>
> It is much more rewarding and enjoyable this way.

Modifying the Content and Tasks of General Mathematics

> There's not the excitement. I'm not sure they're really getting all that much out of it [the math] because they're not that interested. (A project teacher)

Researchers found that students and teachers alike thought the content of general mathematics was neither interesting nor challenging. Teachers realized that general mathematics would be the last chance for many students to become computationally competent and master the basic arithmetic skills needed for life. Students expected the content of the class to be more of the same reviews they had throughout their years in mathematics. Therefore the content of general mathematics consisted of algorithmic reviews of the arithmetic operations of whole numbers, fractions, and decimals, with an occasional attempt at teaching percent problems.

The project staff developed instructional improvement strategies from their readings of the literature for this third strategic area that focused on improving the teaching and learning of mathematical content. These methods included the following:

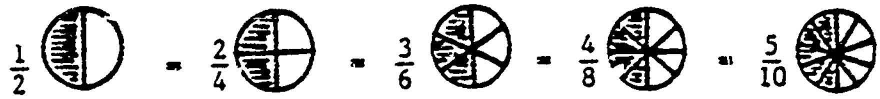
1. Focusing on the development of the students' mathematical understanding through conceptually oriented instruction.

2. Introducing new units of content that were interesting and challenging and enhanced the development of mathematical concepts.

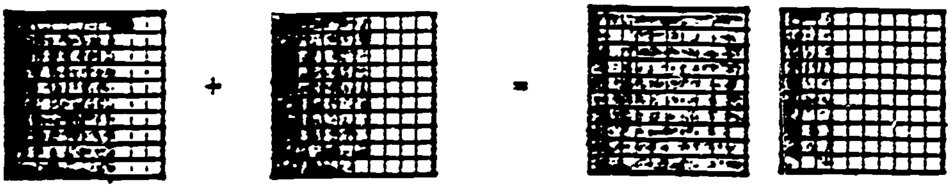
3. Modifying tasks in previously taught content to foster concept development and promote the problem solving-goals of school mathematics.

4. Emphasizing the linkages across mathematics topics and between manipulable materials, pictorial representations, and symbols.

Improving the content and tasks of general mathematics required our project teachers to move from their computational mode of instruction to one which was conceptually oriented. Units that previously consisted of arithmetic reviews of whole numbers, fractions, decimals, and percents were replaced by units of instruction designed to develop understanding of mathematical concepts. Activities were concrete manipulative materials and pictorial representations of the mathematical concepts replaced drill and practice worksheets. For example, when the teachers began a unit on fractions, their students made fraction kits consisting of fraction pieces cut from colored circles in the following divisions: halves, thirds, fourths, fifths, sixths, eighths, ninths, tenths, and twelfths. After her students had used the fraction pieces for a while one teacher said, "I don't think they realize at this point that three-thirds is one whole without using the manipultives." The students used the fraction kits throughout the fraction unit to help them concretely understand the mathematical concepts they were learning (e.g., fraction inequalities, equivalent fractions, fractional operations). In addition to using the manipulable materials to show math concepts, the students were also required to draw a picture of that concept. For example, when asked to name equivalent fractions for one-half they would be required to use their fraction pieces and then to draw the pictures to illustrate that the fractions were indeed equal:



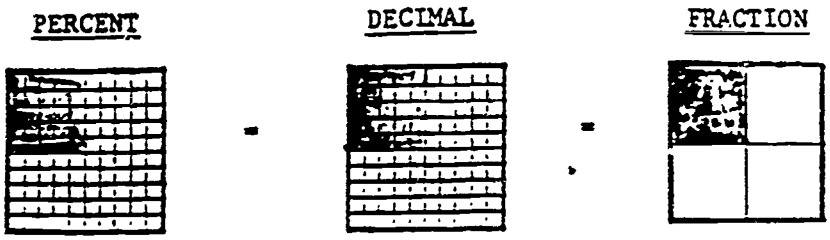
This strategy wa' continued in the decimal unit when the teachers had students pictorially represent decimal concepts on a 100-square grid. The students were expected to show an addition of decimals problem in the folowing ways using a picture, the equivalent fraction, and the decimal value:



$$.71 + .59 = 1.30$$

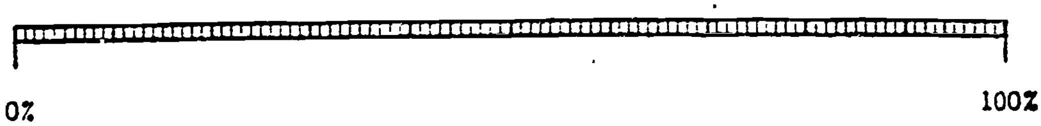
$$\frac{71}{100} + \frac{59}{100} = \frac{130}{100} = 1\frac{30}{100}$$

When the class studied percents, the 100-square grid used in the decimal unit became a 100-percent grid. Instruction during the percent unit was linked to the decimal and fraction units through the continued use of pictorial representations and activities in which the students integrated the fraction, decimal, and percent concepts together. As an example, the students in one lesson were asked to draw a picture of given percents, and then to show their decimal and fraction equivalents:



25% = .25 = $\frac{25}{100}$ or $\frac{1}{4}$

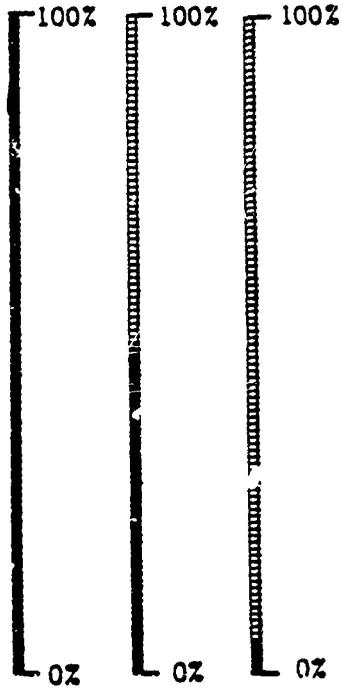
One project teacher showed her students how to make a 100-percent stick from their 100-percent grids by cutting a 100-percent grid into rows of 10 squares each and laying them end-to-end. The following is an example of the 100-percent stick:



The 100-percent stick was used to enhance the development of the conceptual understanding of percents. In addition, the 100-percent stick linked the part/whole relationships established in the fraction and decimal units to those in the percent unit. The following observation shows the interaction between the teacher and her students during a lesson in which they were locating percent values on their 100-percent sticks.

Ms. Kaye: All right ladies and gentlemen, I wanted you to color in a 100-percent stick, a 50-percent stick, and a 5-percent stick.

(The students did the following on their papers.)



Randy: How would you do a five-tenths-percent stick?

Ms. Kaye: How would you do a five-tenths-percent stick?

Holly: That would be just one-half of a square.

Ms. Kaye: Right. All right, do me a 25-percent stick and then follow it with a 33 1/3-percent stick.

Christine: (Looking at the 33 1/3-percent stick). Well, that's just one-third of a stick!

Ms. Kaye: Oh, my gosh! That's a third of it? So, what you are telling me is that 33 1/3 plus 33 1/3 plus 33 1/3 equals 99 3/3? Well, that can't be right. Your answer would have to be one hundred...you have only 99 3/3.

Christine: Yeah, but 99 3/3 is the same as 100!

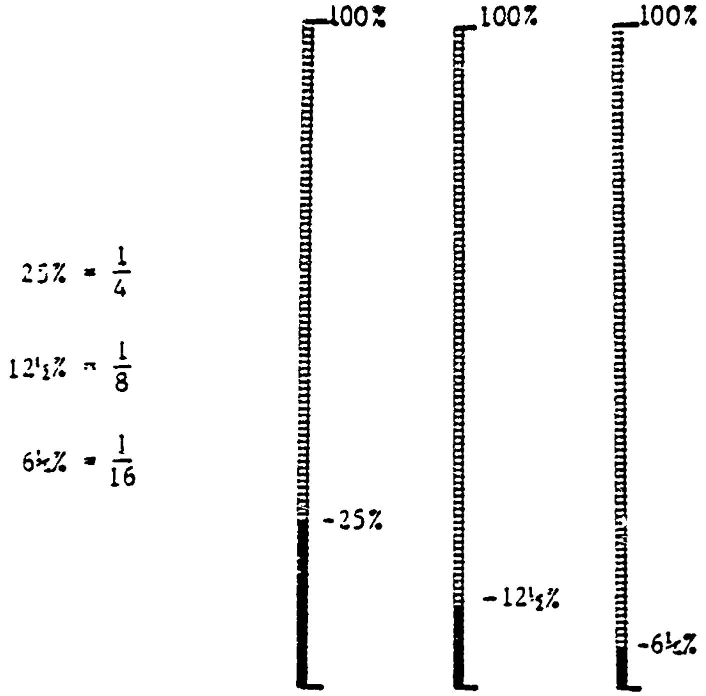
Ms. Kaye: Right. How many percents are there in 2/3?

Kevin and Randy: 66 2/3.

Ms. Kaye: Does that mean that 12 1/2 percent is one-half of a 25 percent stick?

Randy: Yeah, and it is one-eighth because it is half of a fourth. Does that mean that 6 1/4 is one-sixteenth?

(As Randy speaks, Ms. Kaye writes a picture of his description on the chalkboard)



The interaction between Ms. Kaye and the students in the observation indicated that the students had made significant linkages between percents, decimals, and fractions. The interaction also indicated that the students were developing a conceptual understanding of these interrelationships. Ms. Kaye described her percent unit in an interview:

> The percent unit was a real total amazement to me this time. Part of the reason is because I did stick with common percents. We did very few things with percents like 17% and 4%, but most of it was conceptual.
>
> I spent two and a half weeks on percents, and two weeks of that was really in terms of the concept of percent, not dealing with moving the decimal back and forth. I did do that one day to show them that there was another way to approach it.
>
> The biggest effect of teaching for the concepts this way was on my role as a teacher. It made it easier to teach percents and decimals once the students had gone all the way through the fractions. It gave them a much better understanding of percents because they had the fraction base.
>
> It all seems so simple now!

New units of content were added to the curriculum of general mathematics. Activity-based units on probability, similarity, and factors and multiples provided the students with the opportunity to interact in groups as they worked on problem using a variety of manipulable materials.[1] Other units on graphing, statistics, problem solving, and estimation provided the students with a reprieve from the "same old stuff" they had expected to get; and yet, these units still gave them as much computational practice as they would have had on their old drill and practice worksheets. At the end of the year one teacher was asked about the improvements she would think of making in the content of the curriculum for the coming year. She replied,

> I will probably approach it in much the same manner as this year. I feel pretty good about what went on this year.
>
> I definitely like the Probability Unit, although I would try to shorten it up. It may mean removing some material, or it may mean just moving faster through it.
>
> I think that I would probably not alter the percent unit. I would like to make it more workable.
>
> I would definitely do the fractions unit again, working with the hands-on types of things. I will probably take a long look at tying fractions, decimals, and percents together closer than I have.
>
> I will do estimation again. I think that's important. It goes along with problem solving and needs to be done intermittently throughout the year.
>
> Problem solving is the final goal we're trying to reach. If I can get through the other things then that's what I'm trying to get at in the end.

The content and tasks of general mathematics had been modified to focus on the learning of mathematical concepts through activities which involved student participation in hands-on activity-based math experiences. In addition, teachers and researchers selected content and tasks that encouraged more student interaction and communication about mathematical ideas, problems, and concepts. Finally, these modifications of content and tasks which were implemented changed the general mathematics classes from ones that were unrewarding and unchallenging to those where there was a great deal of interest in the learning and teaching of mthematics. Some student comments made at the end of the year in one of the project's classes reflected this view (typed as written):

[1]The activity based units are references in the appendix.

This year I learned how to add, subtract, multiply and divide fraction. I never really understood them before.

Decimals were my down fall until you explained them in simple turms.

Last year I didn't understand fractions but now I understand them perfictly.

I learned how to do guess and check. How to read decimals more easily. How you make a percent into a fraction or decimal.

I now understand fractions. I learned an easier way to do percents.

I learned more about decimals than I did before. And about percents, I learned alot more than I use to about degrees and whey they are called.

I didn't understand how to do changing %'s to decimal and dractions but now I think I do. I knew how to do fraction but I didn't know how to reduce very good but I think I do now.

I was always lost on decimals in this year they all been expalined to me in a language I can understand.

Discussion and Conclusion

Overall on on a scale of 1 to 10 this class
gets a 9 becous no class is perfact. [sic]

(typed as written by a general math student)

The General Mathematics Project finished its three years of field work in 1984 and completed the write-up of its preliminary findings this fall. The purpose of the project was to improve the conditions that make general math classes unchallenging and disliked by teachers and students alike.

Researchers and teachers used literature, exemplary materials, and classroom consultation to modify three strategic instructional areas which were conjectured to be necessary and sufficient to improve the learning and instruction in general mathe classes. These strategic areas included the following:

- Selection of mathematical content/tasks
- Communication of the content/tasks with students
- Organization of students to facilitate instruction.

Project results tend to bear out early conjectures that modifying the set of three strategic instructional areas would improve general mathematics classes; however, teachers had to modify all three--as changing only one or two was insufficient. Toward the end of the project the teachers became aware of the interdependence of the strategic areas and were able to develop instructional improvements that were powerful enough to overcome the many problems of the general mathematics class. Even when the teachers' instruction did not integrate the three tasks, their talk about the successes or problems in a particular lesson indicated they were aware of the interdependence of organization, communication, and task selection; "I should have had the students work in pairs so they would talk more about the proportional relationship of the sides of similar triangles."

Over the two-year intervention period, researchers found the teachers were implementing instructional modifications in the three strategic areas in ways that fostered their students' thinking about and understanding of mathematical concepts, principles, and generalizations. Descriptions of general mathematics classes prior to and after instructional modifications illustrates how the general malthematics classes had been reformed from computation-oriented to concept-oriented instruction and learning:

> Direct instruction of mathematics was changed from simple procedural demonstrations by the teacher into interactions between class members that included discussions of multiple ways of representing math concepts, error analyses, and student-generated explanations and examples of ideas and concepts.
>
> Seatwork, once consisting of numerous reviews of basic computational skills, changed into group activities that integrated manipulable materials, pictorial representations, and symbolic abstractions and linked mathematical concepts across and within units of instruction.
>
> Class time, once given to the students to socialize with one another, now was spent in various mathematical activities such as summarizing the mathematical concepts from the daily activity, or reviewing the mathematical concepts of the unit.

Based on student participation, teacher assessment, and our observations, the researchers judged that in each target class worked in, the students and teacher experienced educational success--a clear improvement. If general math is to be improved a form of instructional leadership rarely available today is needed. Researchers did far more than hold workshops and seminars for the teachers. It took collaborative planning with them and going into their classrooms, sometimes teaching with them, observing their math lessons and giving them feedback (sometimes not so subtly) about the effects of their instruction. One teacher said,

> To have someone else, another adult, in the room who can say, "Yes, I saw the same things happening", or "No, I didn't see those things happening; here's what I saw happening", I think that's very valuable. For teachers, that kind of thing rarely happens. We rarely have anyone in our room but us and the kids.

The teachers wanted and needed extensive discussion and consultation. They were, after all, being asked to teach in a completely new way. Continual input is important: You can't just work with a teacher for one year and quit.

Who should provide the continual input? Principals simply don't have the time. We believe consultants should come from the ranks of teachers. Perhaps the chair of the mathematics department who would spend 50% of his/her time in classroom consultation with teachers. It seems to be an effective way to improve the quality of instruction in general mathematics and probably all of school mathematics.

APPENDIX A: Improving the Quality and Quantity of Mathematical Communication

There were three readings which were of particular importance for the project's teachers as they thought about ways to improve the patterns of communication in their general mathematics classes. These readings are briefly described below. The other readings that were read and discussed by the staff which focus on improving communication are also referenced in this appendix.

(1) Rudnitsky's "Talking Mathematics with Children", discussed the value of dialogues with children in helping the teacher understand what the child knew. It emphasized the diagnostic value in talking mathematics with children.

(2) Driscoll's "Communicating Mathematics", considered the significance of the language of mathematics and effective communication. He concluded that there were teacher behaviors (i.e. monitoring and listening) that would promote such effective mathematical communication.

(3) Jencks', "Why Blame the Kids? We Teach Mistakes!", discussed the misconceptions of children's thoughts about fundamental arithmetic operations. He emphasized teachers should focus on teaching for conceptual understandings of the arithmetic operations in order to help children guide their thinking.

Readings on Improving the Quality and Quantity of Content Communication

Anderson, L. (1981). Short-term student responses to classroom instruction. The Elementary School Journal, 82 (2), 100-103.

Bishop, A.J. (1975). Opportunities for attitude development within lessons. A paper presented at the International Conference on Mathematics Education, Nyiregjaza, Hungary.

Copeland, W.D. (1980). Teaching-learning behaviors and the demands of the classroom environment. The Elementary School Journal, 80 (4), 163-170.

Dillon, J.T. (1983). Teaching and the Art of Questioning, Fastback 194. Bloomington, IN: Phi Delta Kappa Educational Foundation.

Driscoll, M. (1983). Communicating Mathematics. Research Within Reach: Secondary School Mathematics. National Institute of Education, Washington, D.C., 31-39.

Evertson, C. (1982). Differences in instructional activties in higher - and lower-achieving junior high English and Math classes. The Elementary School Journal, 82 (4), 329-350.

Hart, K.M. (1982). I know what I believe: Do I believe what I know? A paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mahtematics Education, Athens, Georgia.

Jencks, S.M. (1980). Why blame the kids? We teach mistakes? The Arithmetic Teacher, 28 (2), 38-42.

Manning, B. (1984). A self-communication structure for learning mathematics, School Science and Mathematics, 84 (1), 43-51.

McCaleb, J.L., & White, J.A. (1980). Critical dimensions in evaluating teacher clarity. Journal of Classroom Interaction, 15 (2), 27-30.

McLaughlin, M.W., & Marsh, D.D. (1978). Staff Development and School Change, Teachers College Record, 80 (1), 1-14.

Miltz, R.J. (1982). Development and evaluation of a manual for improving teachers' expectations (Tech. Rep. No. 26). Palo Alto: Stanford University, Stanford Center for Research Development in Teaching.

Olson, J. (1981). Teacher influence in the classroom: A context for understanding curriculum translation. Instructional Science, 10, 259-275. Printed in Amsterdam, Netherlands.

Rowe, M.B. (1978). Wait, Wait, Wait. School Science and Mathematics. 78 (3), 207-216.

Rudnitsky, A. (1981). Talking mathematics with children. The Arithmetic Teacher, 28 (8), 14-17.

Wearne-Hiebert, D., & Hiebert, J. (1983). Junior high school students' understanding of fractions. School Science and Mahtematics, 83 (2), 96-106.

APPENDIX B: Using the Social Organization of the Class to Facilitate the Learning of Mathematical Concepts

There were several readings which the teachers found particularly useful in helping them think about ways to improve the social organization of the classroom in order to facilitate student learning. Brief summaries of these articles are included below, other readings are referenced also.

(1) Fisher and Berliner's "Teaching Behavior, Academic Learning Time and Student Achievement", suggested that small group work provided a useful compromise for individualizing content, maintaining efficiency and task engagement, and providing social experiences.

(2) Slavin's Using Student Team Learning noted that heterogeneous student groups promoted greater on-task behavior, higher academic achievement and cooperation than did situations where these groups were not used.

(3) Good and Grouws' "Missouri Mathematics Effectiveness Project: An Experimental Study in Fourth-Grade Classrooms", reported it was possible to improve student performance in mathematics through an organized system of instruction. A summary of "Instructional Behaviors" used by teachers in their study included: Daily Review, Lesson Development, Seatwork, and Homework.

(4) Emmer and Evertson's "Effective Classroom Management at the Beginning of the School Year in Junior High Classes", noted that more effective managers had a more workable system of rules, monitored student behavior more closely, were more task-oriented, gave clearer directions, and actively instructed the whole class more often than having students do seatwork.

Readings on Improving the Social Organization to Facilitate Learning and Instruction

Artzt, A. (1979). Student teams in mathematics class. Mathematics Teacher, 72 (7), 505-508.

Blanchard, K., & Zigarmi, P. (1982). Models for change in shcools. In J. Price, & J.D. Gawronski (eds.), Changing school mathematics. Reston, VA: National Council of Teachers of Mathematics, 36-41.

Brophy, J. (1983). Successful teaching strategies for the inner-city child. Phi Delta Kappan, 63 (8), 527-529.

Brophy, J.E. (1982). Classroom organization and management. A paper presented at the National Institute of Education conference of "Implications of Research on Teaching for Practice", Airline House, Warrenton, VA.

Brophy, J.E., & Evertson, C.M. Chapter 2: Teacher Expectations, Student Characteristics and Teaching, 8-24. New York: Longman

Brophy, J.E., & Good, T.L. Chapter 2: Teacher Expectations, Teacher-Student Relationships: Causes and Consequences, 30-41. New York: Holt, Rinehart and Winston, Inc.

Cusick, P. (1972). Inside high school. New York, NY: Holt, Rinehart, and Winston.

Davis, J. (1972). Teachers, kids and conflict: Ethonography of a junior high school. In J.P. Spradley, & D.W. McCurdy (eds.), The cultural experience: Ethnography in a complex society. Chicago, IL: Science Research Associates, 103-119.

Doyle, W. (1979). Making managerial decisions in classrooms. In D. Duke (ed.), Classroom management. 78th yearbook of the National Scoeity for the study of Education. Chicago: University of Chicago Press, 42-75.

Emmer, E., & Evertson, C. (1982). Effective management at the beginning of the school year in junior high classes. Journal of Educational Psychology, 74 (4), 485-498.

Emmer, E.T., & Evertson, C.M. (1981). Adjusting instruction for special groups. Organizing and Managing the Junior High Classrooms. The University of Texas at Austin: The research and Development Center for Teacher Education, 151-173.

Emmer, E.T., & Evertson, C. (1980). Effective classroom management at the beginning of the school year in junior high classes (R & D Report No. 6107). Austin: University of Texas at Austin, The Research and Development Center for Teacher Education.

Emmer, E.T. & Evertson, C.M. (1981). Maintaining your management system. Organizing and Managing the Junior High Classroom. The University of Texas at Austin: The Research and Development Center for Teacher Education, 101-121.

Everhart, R. (1979). The fabric of meaning in a junior high school. Theory Into Practice, 18 (3), 152-157.

Finkel. D.L., & Mong, G.S. (1983). Teachers and Learning Groups: Dissolution of the Atlas Complex. In C. Bouton & R.Y. Garth (eds.), Learning in Groups: New Directions for Teaching and Learning, 14, San Francisco: Josey-Bass, 83-97.

Fisher, C.W., & Berliner, D.C. (1981). Teaching behaviors, academic learning time and student achievement. Journal of Classroom Interaction, 71 (1), 2-15.

Gibson, M.A. (1982). Reputation and respectability: How competing cultural systems affect students' performance in school. Anthropology and Education Quarterly, 13 (1), 3-26.

Good, T., & Grouws, D. (1979). The Missouri Mathematics Effectiveness Project: An experimental study in fourth-grade classrooms. Journal of Educational Psychology, 71 (3), 355-362.

Kounin, J. (1977). Discipline and group management in classrooms. New York: Holt, Rinehart and Winston.

Levin, T., & Long, R. (1981). Feedback and corrective procedures. In R. Brandt, & N. Modrak (eds.), Effective Instruction. Alexandria, VA: Association for Supervision and Curriculum Development.

Moos, R.H. (1979). Chapter I: Framework for Evaluating Environments, Evaluating Educational Environments, San Francisco, CA: Josey-Bass, 1-21.

Moos, R.H. (1979). Chapter 7: Social Environments of Secondary School Classes, Evaluating Educational Environments, San Francisco, CA: Josey-Bass, 136-158.

Moos, R.H. (1979). Appendix B: Classroom Environment Scale Scoring Key, Evaluating Educational Environments, San Francisco, CA: Josey-Bass, 287-296.

Pereir-Mendoza, L., & May, S. (1983). The environment: A teaching aid. School Science and Mathematics, 83 (1), 54-60.

Slavin, R. (1978). Using student team learning. Baltimore, MD: Center for Social Organization of Schools, The Johns Hopkins University.

Thelen, H. (1982). Authenticity, legitimacy and productivity: A study of the tensions underlying emotional activity. Journal of Curriculum Studies, 14 (1), 29-41.

Turner, L.M. (1979). GMC + TC + PR = A formula for survival in a nonacademic mathematics class. Mathematics Teacher, 72 (8), 580-583.

APPENDIX C: Modifying the Content/Tasks of General Mathematics to Enhance the Learning of Mathematical Concepts

Several readings from this selection were useful to the teachers as they set out to select the mathematical tasks for their classes. These are briefly described below. Other readings studied by the teachers regarding the mathematical content are referenced also.

(1) Driscoll's "Understanding Fractions: A Prerequisite for Success in Secondary School Mathematics", noted that students did not see the flexible nature of fractions, expressed as measures, quotients, ratios, or operators. Teachers must encourage students to verbalize and engage in classroom dialogues to develop a full understanding of fractions.

(2) Berman and Friederwitzer's "Teaching Fractions Without Numbers", emphasized the importance of using concrete materials during the development of the concept of fractions. The use of fractional circles was suggested as a way to broaden the concept development of fractions.

(3) Carpenter;s "N.A.E.P. Note: Problem Solving", recommended that specific attention should be given to the teaching of problem solving strategies. In addition, problem-solving should be an integral part of all instruction, new mathematical topics should be cast in a problem solving framework, and students should be guided into problem solving by the teacher asking a number of unobtrusive questions.

(4) Driscoll's "Estimation: A Prerequisite for Success in Secondary School Mathematics", suggested teachers teach students to value estimates in their own right as distinct from exact answers. Estimation skills should be taught on a regular basis.

(7) The Middle Grades Mathematics Project's units on Similarity, Probability, and Factors and Multiples provided Pamela with materials and strategies which were modified and implemented as new content units for her students.

Readings on Modifying the Mathematical Content

Anderson, R.D. Arithmetic in the Computer/Calculator Age, National Academy of Science, Washington, D.C.

Berman, B., & Friederwitzer, F.J. (1983). Teaching fractions without numbers. School Science and Mathematics, 83 (1), 77-82.

Bruner, J. (1971). The relevance of education. New York: Norton, 108-117.

Butts, T. (1980). Posing problems properly. In S. Krulick, & R. Reys (eds.), Problem solving in school mathematics. Reston, VA: Association for Supervision and Curriculum Development, 23-33.

Carpenter, T. (1980). N.A.E.P. Note: Problem Solving. Mathematics Teacher, 73 (6), 427-432.

Driscoll, M. (1983). Estimation: A prerequisite for success in secondary school mathematics. Research Within Reach: Secondary School Mathematics, Washington, D.C.: National Institute of Education, Department of Education, 58-90.

Driscoll, M. (1983). Understanding fractions: A prerequisite for success in secondary school mathematics. Research Within Reach: Secondary School Mahtematics, Washington, D.C.: National Institute of Education, Department of Education, 107-115.

Juraschek, W. (1983). Piaget and middle school mathematics. School Science and Mathematics, 83 (1), 4-13.

Resnick, L.B. (1975). Task analysis in instructional design: Some cases from mathematics. In David Klahr (ed.), Cognition and Instruction. Hillsdale, NJ: Lawrence Erlbaum Associates, 149-198.

Mathematical Materials Reviewed

Dolan, Daniel & Williamson, James. Teaching Problem-Solving Strategies. 1983. Addison-Wesley Publishing Co.: CA.

Lappan, G. (1986). Probability. Similarity. Factors and Multiples. Middle Grades Mathematics Project units. Available from Menlo Park, CA: Addison-Wesley Publications.

GENDER BIAS AND NEGATIVE NUMBERS

William J. Frascella
Indiana University at South Bend

Anyone who has tried to help students understand the concepts of absolute value and square root in an algebra class knows how difficult it is to get them to see that '$|x| = -x$' and '$\sqrt{x^2} \neq x$' are not always true. The resistance of the mind to see that the symbol 'x' can denote a negative number is so strong that algebraic slips of this kind are still being made by students in college mathematics and science courses.

This belief--that 'x' must represent a positive number--is often a deeply entrenched one and the factors which are responsible for its appeal do not easily surface for critical examination. I think the time to deal with the problem is early in the students' acquaintance with the language of algebra when they begin to develop a belief about what the simple letters 'x', 'y', etc. can stand for. To begin a discussion of this topic I would like you to consider the following paragraph about a certain brain surgeon:

> Dr. Hardcutt, a brain surgeon for some twenty-three years, had performed this operation over fifteen times; yet, the five hour procedure never really became routine. It was always gruelling, and almost always involved several on-the-spot decisions upon which the successful outcome of the patient depended. Today's operation was an especially tough one: much that was unexpected turned up. Everyone in the OR knew Hardcutt was under extreme pressure almost every minute of those five hours. The fact that the patient was the doctor's best friend loomed large in everyone's mind--not the least in Hardcutt's. Nonetheless, the operation was a success and Hardcutt, breathing a profound sigh of relief, went directly to the doctors' lounge and poured herself a richly deserved gin and tonic.

I suggest that for many readers of the above paragraph the last sentence offers at least a mild surprise as it reveals Hardcutt to be a woman. That Hardcutt is often assumed to be male is connected with a large set of beliefs about gender and social roles. What is going on here is not that different from the expectations of many students that '$|x|$' can never equal '$-x$' and '$\sqrt{x^2}$' can always be identified with 'x'. In these cases students assume that the symbol 'x' must represent a positive number. From this assumption all manner of mischief follows.

I should mention that alongside the assumption that 'x' must represent a positive number is the just as fallacious premise that '-x' must represent a negative number. To enlighten students on these matters the algebra teacher must get them to see that these assumptions are active and alive in their thinking. This task would be much like getting readers, who couldn't envision Dr. Hardcutt as a woman, to see that they are laboring under unwarranted assumptions. For example: All brain surgeons are male.

The Pedagogic Problem

The problem we are facing is to get our students to see that the simple leters of algebra, 'x', 'y', 'a', 'b', etc., do not have to represent just positive numbers and, likewise, these same letters prefixed by a minus sign, i.e., '-x', '-y', '-a', '-b', do not have to represent negative numbers. The key to understanding this problem is to observe that these latter symbols are seen as being constructed from a simple letter by the linguistic operation of prefixing a minus sign. We could say, just in terms of notation, that the symbol '-x' is constructed from the symbol 'x'. In other words, the existence in the language of algebra of the symbol '-x' depends upon the prior existence of the symbol 'x'.

Positive and Negative Numbers in the History of Mathematics

With this in mind it is easy to see why 'x' gets associated exclusively with positive numbers while '-x' with only negative numbers. The history of the concept of negative numbers in mathematics is an account of an idea that usually makes its appearance only within frameworks where positive numbers are taken for granted. And even in these circumstances negative numbers are still denied the more important status that their positive counterparts enjoy. Negative numbers make one of their earliest appearances in mathematics as unexpected and unwanted intruders into the process of solving algebraic equations. Nonetheless, they keep turning up as roots of these equations intimately connected with the sought-for positive solutions. They could not be ignored. In keeping with the male/female analogy introduced earlier, one might say that the persistent presence of negative numbers were forcing mathematicians to create a kind of 'women's auxiliary' number system for negatives that could be attached to the standard system for positives. It is not until the 19th century, with the advent of the new field of modern abstract algebra, that number systems are developed and accepted that treat both positive and negative numbers as logically commensurate.

But the key word here is "logically". Though regarded logically equal for the past century negative numbers are still taught and thought about as second class citizens whose existence depend upon the positive numbers. This view is reinforced by the fact most definitions of the negative numbers rely on a prior definition of the positives. The positive numbers are seen as intuitively simple, having a more immediate connection to reality, while the negative numbers are viewed as more complex and abstract whose connections to the world must be mediated and justified by the positives. In general, one constructs the negatives from the positives.

I also contend that our students internalize this relationship of dependency between positive and negative numbers. When you consider the fact that our students see both the x/-x and positive number/negative number relationships of dependency as profoundly similar it is little wonder that the symbol 'x' comes to be associated exclusively with a positive number while '-x' with a negative one. I think this kind of identification is largely an unconscious process that is responsible for many student missteps in the language of algebra. The fact that our students are often taught to read '-x' as "negative x" doesn't help the situation.

The problem we are facing in the classroom can now be looked at as one of breaking a misleading association. A way of doing that would be to go to the other extreme and assign certain exercises that require that the simple variable letters of algebra refer only to negative numbers. This would be like making an author use 'she' as the generic third person singular pronoun throughout all of her writing. I think these kinds of exercises could be periodically assigned over an entire semester with the expectation that the symbol 'x' will eventually come to be thought of as something which refers, without bias, to both positive and negative numbers.

Prime Puzzle

There is a message hidden below. Cross out the letters in the boxes containing numbers that are *not* prime numbers to discover the message in the remaining boxes.

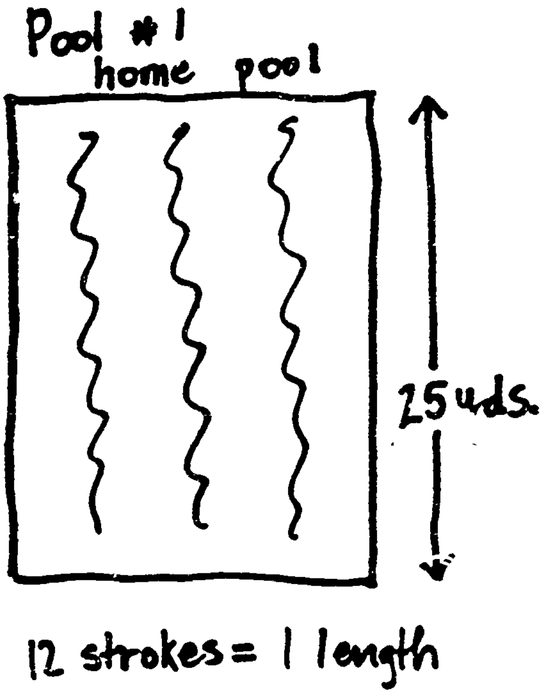
D 7	P 6	I 2	R 8	V 19	I 11	M 12	P 60	S 3	K 9	S 14	O 59	Z 35	R 11	S 37
Q 4	A 3	R 31	M 25	E 23	S 10	D 29	M 12	I 41	V 97	H 100	I 23	N 83	E 13	A 12
B 71	U 2	R 35	T 3	T 27	F 43	O 42	A 37	I 64	C 7	T 5	R 45	O 13	R 11	S 71
N 9	E 14	U 69	M 32	A 17	S 87	F 48	G 75	O 20	R 19	K 9	E 97	Q 8	T 27	D 57
F 67	R 2	C 16	I 89	M 18	E 7	T 12	K 9	N 17	D 73	L 67	N 49	I 59	E 29	R 83

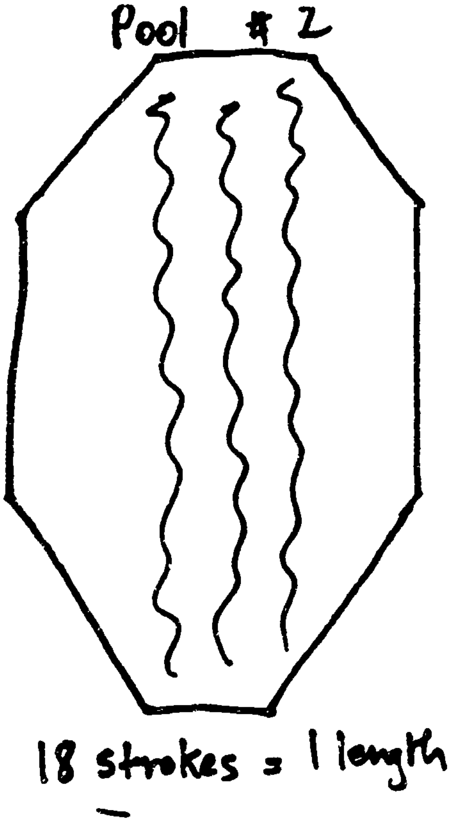
PROPORTIONAL REASONING

Magdalene Lampert
Michigan State University

I want to tell you a little bit about what I do so that you know where what I have to say is coming from. I've been a math classroom teacher for the last 5 years. I also taught several years, at a point earlier in my life. Before I came to Michigan State I was the curriculum coordinator in a small school outside of Boston. This meant I taught three groups of children and I oversaw all the teachers who taught the same grade. I also, on the side, taught some college courses to pre-service math students, who wanted to be teachers at the middle school level. When I came to Michigan State, I realized that although there were certain things that appealed to me about being a university professor, I did not want to give up either teaching kids or working directly with teachers, preferably in a school setting over time. Since that is a popular thing to do these days, I was able to negotiate with my dean to consider teaching fifth grade an hour a day as part of my university load. A lot of what I have to say about teaching and learning is quite grounded in the fact that I teach kids everyday. I also run something called the teacher study group on mathematics in East Lansing. It consists of elementary and middle school teachers who get together once a week and do math . My assumption in running that group is that one of the reasons why math is taught poorly is the people who teach it never have the opportunity to learn very much about it. Rather than starting with how to teach math, I'd like to give teachers the experience of what math is. I'm sure as a lot of you know, some beginning teachers have a rather limited notion of what math is.

Some of the work on proportion and proportional reasoning that I am going to talk about today comes out of my work with that teacher group. I decided to focus on that partly because you are people who will leave here to work with other teachers. We're going to start by doing math problems. This problem is particularly appropriate for today's weather. I'd like you to put your pencils down. Take problem number 1 and the diagrams that are on the pages, and try only to do problem number one, without using a paper or pencil. Try to <u>think</u> it through. See where you go with it.





Drawings are not accurate to scale.

Problem #1: In the "home pool" a regular swim session consists of swimming 60 lengths. What would be the equivalent distance in lengths (approximately) in Pool #2?

Once you've thought it through, and you feel quite content with what you've done, I'd like you decide if there is another strategy for figuring out the problem. Or, after you finish figuring it out, you can just sit and daydream about being in a swimming pool.

We'll take time to discuss this, but for now take time to work on it on your own. This problem comes from the experience I have. I swim everyday, and ordinarily I swim in Pool number 1. I swim back and forth, back and forth, and I consider a reasonably good swim session, 60 laps. I travel a lot and I find myself in a lot of different swimming pools, wanting to have an equivalent swim. Many swimming pools don't have a clock. It's not possible to just swim for half an hour or 45 minutes. It's surprising how little the people who run swimming pools know about the dimensions of the swimming pool they run. You might ask them, how many yards in this pool? They'll say I don't know. I couldn't depend on swimming the same length all the time. I confronted this problem in Pool number 2. I tried to figure out how far I should swim in order to have the same amount as I did in Pool number 1. Obviously, I was swimming along trying to figure this out and didn't have paper and pencil. I found there are lots of different ways of figuring this out. I'd like you to discuss with the person next to you, what you thought about when you looked at this problem. [Several minutes of discussion in groups of four].

I hate to cut off a discussion like this because I hear a lot of good math t... going on. People are looking at things from a perspective other than the one they started with. Although that is not the focus of my talk right now, I want you to realize that this is a very useful thing to do if you're a math teacher. Remember that part of what you're having to do all the time is to try to figure out what a child might be thinking when he gives you a certain answer that isn't what you expected, but might in fact make sense. As I was walking around, I heard a variety of things. I would guess, if I wrote some of these things down on a piece of paper and showed them to a stranger, he might think that all of you were working on a different problem. That's kind of interesting, because often times when math problems are presented in school, the teacher will say, this chapter is about "x" and the problems at the end of the chapter are about how to apply whatever "x" is to word problems. All the problems are about the same thing.

Let's look at our problem. Here are some of the things I heard people say:

"The whole swim takes 720 strokes in this pool, so 120 divided by 3 is 40... well, I got 37.5 but that doesn't make sense..."

"This pool is smaller, so it has to be less than the other pool...so..."

"If you divide this, you go 20 and 20,..."

"This one is half bigger than this one..."

A lot of later discussion that I heard as I walked around the room was people trying to explain to other people why they said this one's half bigger when other people said, this one is 2/3 of this one. It seems like it should be either halves we're talking about here or thirds. I wondered where you got in your discussion about justifying how you could be talking about halves and thirds in the same problem. Here's an issue: Is it legitimate to divide one of the pools into thirds and the other pool into halves? What's that about? Why can you do that or not do that?

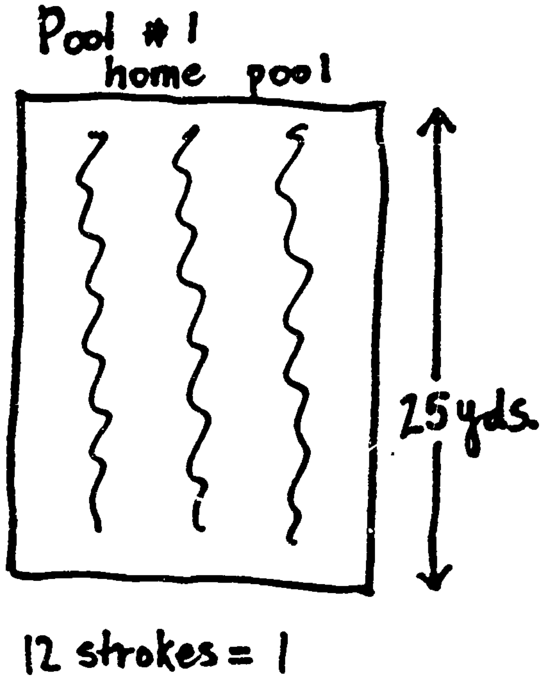
It seems 20 was an important number in many of your solutions. Twenty is half of pool #1 and 20 is a third of pool #2. We have 20 and 20 and 20 in sombody's head and 6 and 6 and 6 in sombody elses head. How can they both be right? Now you said in one of these pools you do one and a half times as much, as much what, exercise? So does that mean you do more exercise or less exercise? The unit itself keeps changing. That's what we're talking about. What about this person who said, "I got 37.5 but that doesn't make sense"? What do you think the person was thinking? John just explained that it seems that the second pool is 37 and a half or 37.5 yards and you can figure that out because, half again as many strokes. Did anybody actually set up a proportion at any point in this problem? One of the reasons I said don't use paper and pencil is because I wanted you to approach the problem without even thinking of this as a proportion problem. John explained that once you get 37 1/2, you're not finished. What did he mean?

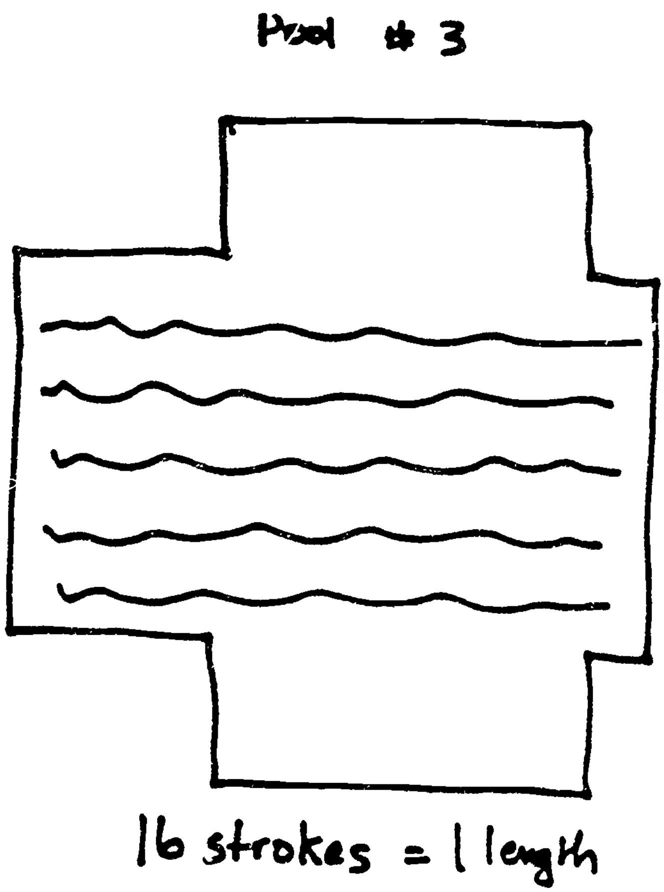
One of the problems with the way problem solving is taught in school is that kids will get 37 1/2 and stop. That is more likely to occur when there is no context around the problem., if there are no feet, inches, laps, yards, swimming pools, anything to think about. It is the <u>context</u> of the swimming pool that helps you think about the question: "Why doesn't 37 1/2 make sense, as the answer"? For one thing, you'd be in the middle of the swimming pool, since the answer is supposed to be in laps. You might think, does that mean I'm wrong if I come out in the middle of the swimming pool? Not necessarily. Is there anything else there that might make sense to you? If we have a pool and you think it's 37 1/2 and you say, that's the length of the pool. That makes sense. The pool could be 37 1/2 yards long.

Another bit of important piece besides context is the value of having discussions in math class. There are some studies that have found one reason why kids drop out of math classes in junior high and high school is because there is never any discussions and they never get to hear what other people think. There is an assumption that everybody should be thinking the same way and that way ought to be the way the teacher thinks. We know this room is full of very competent and successful math teachers. Yet different thinking did occur. How would you feel, if I came in here and said there is one way to do this problem, and explained to you how to do it and it wasn't the way you would have approached the problem. You think: "I don't know what this lady is talking about, I had it all figured out doing it this way." For this reason, discussions in math class can be productive, interesting and learning experiences. There is also a lot of research on classroom management and organization, studies about what students remember from discussions, and how the style of the discussion and the status of the students who speak contributes to others learning from the discussion. It has been found to be the case in classrooms that students target who they're going to listen to and who they're not going to listen to based on their sense of who the teacher thinks is smart. One of the things that a teacher can do about that is to be less public about who the smart kids are and who the not so smart kids are. The research also shows that in classrooms where any answer was acceptable, students didn't learn to listen to anybody. They figured there's no point in listening to the discussion because there's nothing I'm supposed to learn from it, since every answer is acceptable. On the other hand, if the discussion occurred in the classroom where the teacher asked a question and there's obviously only one right answer, then the student only listened to the kids they knew always gave the right answers. If the discussions occurred in classes where the discussion was both focused in the sense that the teacher didn't accept everything, then there was a broader range of listening and learning in the classroom. I think that's a really important distinction to think about when you try to have a discussion of a math problem.

Now back to the proportion problem. Somebody said when they got 37 1/2, "That's not a nice number". The assumption is that problems are always supposed to turn out with nice numbers. As you can see, the next problem 15 strokes equals 1 length. Since you've worked through one problem, you can also figure it out that this is going to come out to a nice number. Anybody want to talk that through. Have you thought about the next problem? How would you bring what you discussed on problem number 1 onto problem number 2? Why don't you do problem number 2 at your table? [Several minutes of discussion in groups of four].

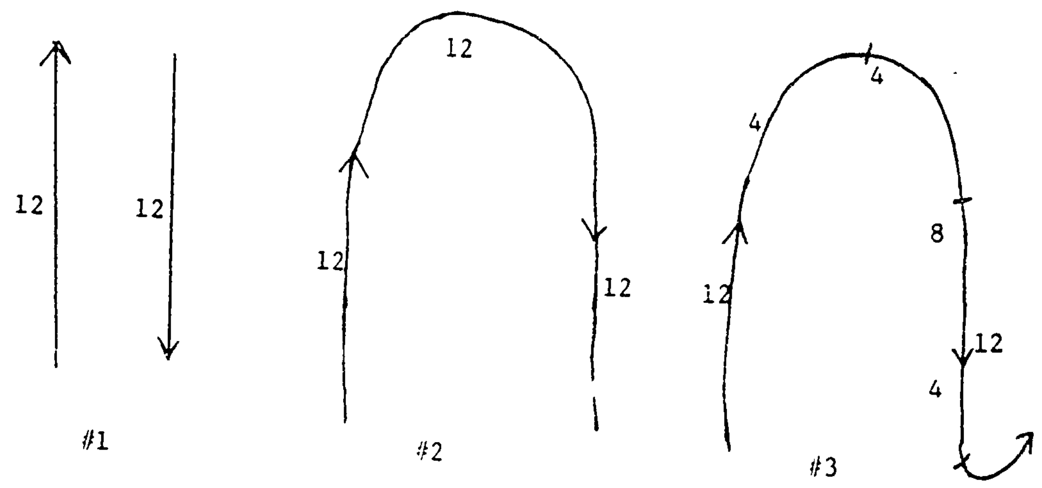
Drawings are not accurate





Problem #2:
What would be the equivalent in Pool #3?

Let's do an overall summary of some of the things that all of you were talking about. In the first problem, you can see that we swam one 12-stroke length (see diagram) and then half a length again to get to 18.



Where in the third pool you swam one 12-stroke length and a third of a length again. When you came back, in the next lap, to get to the other end, like you did in pool 2, you have to fill a little part of a lap, which nicely works out to be 4. It's a third again as much. Now you have another whole lap that takes you 8 up and then you have to do 8 again and 4 and so on, until you get a third. So in pool #3. the proportions again lead to a solution that is a "nice number".

I would like to challenge you to take these problems home, and suppose there was another pool, pool 4, that took 17 strokes. I would hunch that listening to what you did, that those same steps will not work for 17. You would not be able to use the same strategies as you did for the "nice number" proportions.

I now want to talk to you more about <u>kids thinking about properties</u>. I want to tell you a little bit about some work that has been done about kids doing proportional reasoning and where their reasoning breaks down. I think you can see that moving to 17 raises a lot of difficult issues here. I think we should at least put the question on the table, whether we teach kids in a way that enables them to solve this kind of problem or not, to do proportions, whether the numbers are nice or not. A large number of people could do problems with pools 1,2 and 3, where the numbers are "nice", but when you throw 17 in there, they would say: "I don't know how to do that." You can't reason it through in the same easy way as you reasoned through the first 2 problems. When there are nice numbers you can say, "It's just 1 1/2 bigger, and the answer is just 40". There's something really basic and reasonable about that. But when you throw 17 in there, it can't be reasoned that way; you have to do some pretty complicated arithmetic in order to solve it.

In England, researchers have done a study of several thousand 11 to 15 year old kids, testing them on a lot of different topics including what they could do with graphing, decimals, operations on whole numbers, fractions, proportions, algebraic expressions. I'm going to talk to you on the proportion part. They gave written tests to 2277 students, all over that country and the world, suburban and urban schools, who had been taught in everything from the most traditional to the activity centered, open classroom type of situations. They tested them year after year, to see how they improved or not and how they did on these proportion problems. The purpose of this was essentially to learn more about what do kids know when they get through middle school in England. The researchers were interested not only in how many questions did they get right or wrong, which seems to be the only kind of test that we give in this country, but what did they actually do, why did they do it, where did their thinking break down and where did they start making mistakes.

The following problem is a type included in the test on proportional reasoning. Given this recipe for serving 8 people, how would you adjust the ingredients to make the

6 cups flour
2 oz. butter
7 tablsp. sugar
1/2 can milk

same recipe for 4 people? They found students did pretty well on that kind of question. They went through all the ingredients and they were able to know proportionally that you'd use half. Because you half the number of people, the amount of each ingredient should be half. The next question on the test was, what if you were doing the same recipe for 6 people. It's still a pretty nice number. They found the kids did not use the proportion $\frac{a}{b} = \frac{c}{d}$. Instead what they did was similar to what you did on the swimming pool problems. They said well, 4 is half, so what I have to do is take half, and then take half of that again and that will give me how much you need for 2 people and then you add what you need for 4 people to what you need for 2 people and you'll get the answer. They did reasonably well. Eighty per cent of the kids got it right for the whole number ingredients. But 20% of the students were able to do those operations with fractions. So the reasoning is there. The mathematical thinking is there, but computational skills were not there. Many students simply did not know how to take half of a half of a half and even if they got that far, they then had two fractions that had different denominators and they didn't know how to add 1/4 and 1/8. Although they had all the thinking, they couldn't do it. They couldn't figure out what to do with the 1/8 and 1/4.

There are other examples in this research that show that once the numbers are not so nice, once the fractions look more like 5/3 and 1/2 or 2/3, then these really nice reasoning skills don't work. A lot of this work is done by a British woman named Kathleen Hart. A French researcher, Gerard Vergnaud, is interested in the same sorts of issues. He has also done an enormous amount of work watching how students do proportion problems and studying how you can teach them to do proportion problems successfully. What he finds is a lot of good reasoning, especially if kids get to the point where they can set up a chart. With the help of a chart, they can think the problems through. But when they get to big numbers and their computation skills break down, the chart doesn't help much. If they can't multiply 682 by 530 and then divide by 72, they can't figure it out, no matter how good their reasoning is. Even with the use of calcuators they found they were not able to use big numbers. Not because they didn't know how to use calculators, but once you throw out a big number, they couldn't figure out what to multiply by and what to divide by. It was no longer a situation where reasoning capabilities served well enough to work through the numbers. Now, that raises a million interesting questions about what you want kids to know and what we want to teach them in order to be successful in proportional problems. I certainly am an advocate of having to think through problems rather than simply picking numbers and performing algorithms on them. I think we all should be aware, however, that thinking serves a certain set of problems but not all. As teachers have to be able to make decisions as to how far you want to go with letting students think it through, versus using a formula and teaching them computation.

I have summarized this research on how students think about proportion and what strategies are successful in teaching them how to do proportion problems for you on the following page. I hope that your own experience with the swimming pool problem here today helps you to take sense of these findings:

FACTORS THAT HAVE BEEN FOUND TO CONTRIBUTE TO STUDENT SUCCESS IN SOLVING PROPORTIONAL REASONING PROBLEMS

Students are successful when they have:

NUMBER SENSE -- i.e., they are able to figure out and talk about relationships among numbers, able to approximate and have good judgement about the appropriateness of answers to computations.

COMPUTATIONAL SKILL -- i.e., they can recall number "facts" automatically and have workable algorithms at their fingertips for doing all operations with fractions, decimals, and large whole numbers.

ASSOCIATION OF NUMBERS WITH QUANTITIES -- i.e., if problems are stated in terms of a meaningful context, and students are able to explain what each of the numbers used in the process of arriving at a solution means in terms of the context.

DIAGRAMS AND CHARTS -- i.e., if students can translate the written text of a problem into a format which shows the relationships among the numbers in the problem so that known relationships can be used to find unknown quantities.

FACTORS THAT GET IN THE WAY OF STUDENT SUCCESS WITH PROPORTIONAL REASONING PROBLEMS

Students have trouble with these kinds of problems when they have:

LIMITED WAYS OF THINKING ABOUT WHAT A NUMBER WRITTEN IN FRACTIONAL FORM CAN MEAN -- students who have only the "pie model" for interpreting the relationship between the numberator and denominator of a fraction will have a difficult time using fractions to solve proportion problems; the "unit" in these problems is not a stable quantity -- it changes depending on what "unknown" is needed.

MECHANICAL APPLICATION OF A FORMULA TO TRIADS OF NUMBERS -- formulas that are not understood are not remembered and cannot be applied in novel circumstances; associating "solving a problem" with plugging a number into a memorized formula to get a numerical answer does not get students very far beyond succeeding on the test at the end of the chapter on "Ratio and Proportion". Simply learning where to plug in the numbers to $\frac{a}{b} = \frac{c}{d}$ does not get students very far in solving problems.

MATH ANXIETY

Madeline Masterson
Lansing Community College

My personal involvement with the topic of math anxiety is with the Math Lab at Lansing Community College. Our laboratory services about one thousand students per term in four developmental courses. Those courses are Arithmetic, Pre-algebra and Algebra I and II which correspond to high school algebra. We attract many adult students, the average age for students at the community college is twenty-nine. We have noticed over the years that many students display anxiety in doing mathematics. We try to address this problem. What we have found is that our adult students are afraid of math. They are afraid of modern math. They are afraid of failing. They are afraid of succeeding. They think that being good in math is a different breed of person. They idealize math. They feel that they are at fault and that they are never going to learn it. They never have and they think they never will. They feel that they are the only one with this problem. Really, it is not just their problem because obviously it started a long time before. It is a shared problem. They are the product of the public schools and their homes. We find that we have to address these attitudes they have toward mathematics as well as the deficiencies they have in their skills before they can be successful.

To understand any topic it is good to put it in a historical perspective. We will look at some research in math anxiety, some programs that have been developed and some of the preventions that can be incorporated into classrooms. Probably all the work that has been done in math anxiety has been done in the last fourteen years. The phrase was not used much before that. The term matho-phobia was used somewhat before that time by math educators, long before the feminists discovered math anxiety. It's really not a new phenomenon that people are afraid of mathematics. It's just recently become of great concern. It used to be tolerable because there were plenty of jobs available that did not require much mathematics. That is no longer the case. We find we don't have the luxury of enough people with a mathematics background. The major catalyst for this change of concern was a study published in 1973 by the sociologist Lucy Sells. It's a well known study quoted by many. She was doing her doctoral thesis in Berkeley at the time and studied entering freshmen. She found that 57% of the entering male students had four years of high school mathematics where only 8% of the female students had four years of high school mathematics. The study was done some time ago, but keep in mind that Berkeley attracts a strong academic population. What that meant was that 92% of the women had cut themselves off from three quarters of the majors at Berkeley at that time. Lucy Sells concluded that high school mathematics was "the critical job filter". The conclusions brought a lot of attention at that time. She subsequently went to the University of Maryland and five years later confirmed her thesis when she analyzed job recruiters. She found that 75% of the job recruiters would not interview students unless they had one term of college calculus. So, prior to these studies even though math avoidance was being acknowledged by math educators as characteristic of certain groups, like women and minorities, not much had been done about it. The concern for general low achievers had been producing remedial programs. There was then a more intense analysis of math anxiety and math avoidance, among women in particular. A feminist educator Shelia Tobias helped establish in 1975, the first math anxiety clinic at Wesleyan University. Later she published her well-known book Overcoming Math Anxiety. It is particularly helpful in assisting teachers that are trying to recognize their own math avoidance and the subtle ways that they communicate it to their students. Another excellent book written by Kogelman and Warren is Mind Over Math. We have frequently recommended this to our community college to read. The book is very readable and we have found that many of our students are helped by it. It is sometimes used as a textbook in math anxiety classrooms. Another book that compliments the other two is Math Anxiety: What It Is and What To Do About It written by Charles Mitchell. He emphasizes the physical and biological aspects of math anxiety. He also gives a desensitization strategy. Since then a number of other educators and researchers, those whose particular interest is in women's education, adult education, and equal access opportunities for minorities have begun to take some action to correct the difficulties in studying mathematics. The federal government has funded research on sex related differences and race-related differences in mathematics enrollment and achievement patterns. Elizabeth Fennema is a well known name in this area. She is a psychologist at the University of Wisconsin and has done numerous studies on sex related differences . I want to take a closer look at the results of these studies later.

I thought before we go any further we should examine what exactly is this phenomenon called math anxiety. It can be defined as the fear of being inadequate in mathematics, which includes avoiding doing mathematics. Matho-phobia has been defined as an "irrational and impeditive dread of mathematics". The terms mathophobia and math anxiety are used interchangeably to describe this irrational fear of mathematics which is not under a person's conscious control. Note there is no indication that the person is incompetent in mathematics.

Anxiety is a form of stress. Very little learning takes place in the face of intense anxiety. It's the anxiety that gets in the way of learning. Everyone needs or thrives on a moderate amount of stress to enhance performance. But excessive stress that does just the opposite. The stress is excessive when the person does not see any effective alternative way to react. We see some people who consider math as being distasteful and some who have a real paralyzing fear of mathematics. They will sit in misery in the classroom knowing that they are going to do badly, unable to help themselves. They develop even stronger adversions to mathematics and a vicious failure cycle takes place. The problem is not gone until some intervention takes place.

The number of people affected by some level of math anxiety is estimated to be the majority of the adult population. It includes many otherwise well-educated people, people that can handle frustrations and stress in other parts of their lives but who are unable to cope when it comes to mathematics.

There are a variety of reasons why a student will develop this emotional as well as intellectual block. Although matho-phobia or math anxiety is most apparent on college campuses, it has its beginnings in elementary school and secondary classrooms. It may be less obvious then in part due to the fact that students can get along with passing grades for a long time without ever really understanding what they do. A student who has been allowed to rely on the "memorize what to do" approach eventually runs into trouble. They reach the point where they don't have the background to understand. It becomes a problem when they have nothing to build on. I think the number one cause for math anxiety is the educational system itself, the teaching methodologies used to convey basic math skills to young people. J. Greenwood has an article in the Mathematics Teacher which he devotes to this particular aspect of the causes of math anxiety. He says, "The real source of the math anxiety syndrome is that the skills are too often taught using the 'explain, practice, memorize', teaching technique. This methodology isolates facts from reason and from the process of problem-solving itself. It concentrates on procedures of producing answers and is not particularly concerned with the developing of logical thought process nor the type of reasoning that is at the basis of computational mathematics".

For cause number two I would cite parental attitudes and expectations. They cannot be underestimated even if they are not displayed in an obvious way. If a parent does not perceive themselves as being good in math, they then do not expect their children to be good in math. After all, we expect our kids to be like us. Therefore those kids are going to fulfill those expectations. Another interesting point is that most parents have higher expectations for their sons than their daughters in mathematics. There are several studies that show parents encourage their sons more than their daughters toward the study of mathematics. In addition the consequences are even greater when the parents provide different activities outside of class for their sons such as building models, science experiments, and many others that influence what goes on in the classroom.

Somewhat surprising is that much of this same type of parental attitude occurs in the classroom teacher. In addition to an indifference toward mathematics on the part of teachers, girls are allowed to drop out of mathematics at an earlier age than the boys. Numerous studies have shown that teachers do not interact in the same way with boys as with girls. Teachers differentiate between students on the basis of sex through their behavior, their beliefs, and through their expectations. This happens in such a subtle way that many teachers are not aware of it.

Differential classroom experiences seem to influence mostly the development of the students internal motivational beliefs. The conditions of society influence the school environment and this environment in turn affects the students' belief system. It is the students belief system that is at issue here. Teachers have so much influence, putting them in a very unique position. Therefore our third cause for math anxiety is teacher influence and treatment of students.

The last major cause of math anxiety I will mention is the negative feelings about mathematics that come out of gender difference. Many studies show that in elementary schools and middle schools we have little difference in achievement in mathematics and where it does exist, the girls are better. At the secondary level there is a heavier amount of evidence that the boys out-perform the girls. There is a real turn around from middle school to high school. Even when there is no difference in terms of achievement, girls perceive themselves as not having the same ability, they underestimate their ability, they have a lower vision of their own achievement compared to boys. Girls are more prone to "learned helplessness". "Learned helplessness" is not necessarily comparable with achievement. You could have a girl that is achieving very well but has learned to be helpless and has learned to feel anxious. There are many studies to support this. Sterotyping math as a male-domain is setting the stage for adult female math anxiety. It is evidenced by the fact that most of the enrollment in math anxiety classes is female. You see very few men enrolled in math anxiety courses. An interesting study showed a relationship between general anxiety and math anxiety. It found that men's math anxiety was very much associated with general anxiety and women's math anxiety was not. In other words, if a man is math anxious it's probably because he is anxious anyway, but if a woman is math anxious it is more apt to be the math that produces the anxiety.

There are a number of programs that have resulted from these studies: programs for adults, students, teachers and integrated programs. The ones that I know best are the adult programs which I think are very interesting to examine. For adults there are literally hundreds of math anxiety clinics, courses, programs, mainly on college classes. Most have been very successful in helping adults overcome math anxiety. There are several math anxiety scales that have been used to document the success.

Most programs confront the myths about who can do mathematics and why. There are two different approaches in the adult programs. One approach is the counseling approach. In the counseling approach attempts are made to restructure the psyche of the learner. These are modeled after the "Mind over Math" workshops where one has to spend time at the start in self-analysis. You have to recognize the adverse, negative reactions you have to math and acknowledge the influences of the past in order to go on to confront the present. The negative feelings are vented and accepted. Some programs will include basic mathematical concepts or story problems. Some programs will have relaxation strategies to deal with physical symptoms of anxiety. Sometimes there are support groups that run concurrently with math classes. There is a strong effort to replace the "I can't " concept with the "I can" concept. Many of these programs are taught by two people, a learning counselor and a mathematics instructor.

Other types of programs make an effort to teach math in new ways. These programs are not trying to restructure the learner are trying to restructure the math. The idea is that if the adult never learned math the first time through, why give them a rehash of it all over again the same way. We need to find some new ways to provide positive experiences for them in mathematics. The concepts are made more interesting, spatial skills are learned, visualization ability is stressed, manipulatives are used with adults. A positive, supportive environment is provided. The psychological component or counseling is at a minimum or eliminated. This type of course is generally given through a math department. The other type is given through a counseling department. I observed the second type in California where I visited many math anxiety programs. There are evaluations showing both types of programs are very successful but I haven't seen any studies comparing the two. The enrollment in these classes are usually very low, however.

There are also a number of excellent programs directed toward teachers. A couple of programs are for preservice elementary teachers, a very crucial area to attend to. The EQUAL'S program in Berkeley, California is directed at inservice K-12 teachers and is excellent.

There are even more programs for students. To mention a few, "Project SEED" for the disadvantaged, Math for Girls, which is an after school program, Family Math, career awareness programs, and visiting women scientists.

Aside from programs, what can be done specifically in the classroom? Since math anxiety is a learned response to a negative experience then we should be able to prevent it. This is where many teachers have good ideas about providing a positive environment. Be sure to dispell the idea that only "special" people can do math. It does only seem to be an American myth that those people that can do math have some sort of innate ability. We need to eliminate the following math myths:

1. Everyone understands math - except me.
2. Men are better at math than women.
3. When it comes to math you cannot trust your intuition.
4. There is only one correct way of solving each math problem.
5. It's bad to count on your fingers.
6. To be good in math you must do problems quickly.
7. It is important to do mental arithmetic.
8. You can't learn from your mistakes.
9. Math requires a good memory.
10. Math is done by working intensely until the problem is solved.
11. Some people have the gift of a math mind.
12. There is a magic key to math.

Some things that can be done to ease math anxiety include having students work in groups, encourage all students to give answers, and emphasize the thought process. We also need to be careful about how we talk to our students. Don't say things like "that's an easy one", "you did it the wrong way", "you should know that". Students often interpret these things as a put down. When students give answers but can't explain it, sometimes it is better to ask another student how they might have gotten the answer. This would continue to encourage the intuitive mathematician.

Much discussion followed about tracking, math anxiety, improvement of skills and many other related topics.

MATH ANXIETY BIBLIOGRAPHY

Becker, Joanne Rossi, "Sex Equity Intervention Programs Which Work", School Science and Mathematics, March 1987.

Campbell, Patricia B., "What's a Nice Girl Like You Doing in a Math Class?", Phi Delta Kappan, March 1986.

Fennema, Elizabeth, Herbert Walberg and Cora Marrett (Ed.), "Explaining Sex-Related Differences in Mathematics: Theoretical Models", Educational Studies in Mathematics, August 1985.

Frankenstein, Marilyn, "Overcoming Math Anxiety by Learning About Learning", Mathematics and Computer Education, Fall 1984

Gentile, J. Ronald and Nanci M. Monaco, "Learned Helplessness in Mathematics: What Educators Should Know", Journal of Mathematical Behavior, August 1986.

Greenwood, Jay, "My Anxieties About Math Anxiety", Mathematics Teacher, December, 1984.

Hanna, Gila, "Sex Differences in the Mathematics Acievement of Eighth Graders in Ontario", Journal for Research in Mathematics Education, May 1986.

Kogelman, Stanley and Joseph Warren, Mind Over Math, McGraw-Hill Book Company, paperback edition, 1979.

Lazarus, Mitchell, "Mathophobia: Some Personal Speculations", National Elementary Principal, January-February 1974.

Mitchell, Charles, Math Anxiety: What It Is and What To Do About It, Tempe, Arizona: Action Press, 1984.

Morris, Janet, "Math Anxiety: Teaching to Avoid It", Mathematics Teacher, September 1981.

Robertson, Douglas F. and Joan Claesgens, "Math Anxiety - Causes and Solutions", a paper presented at the Minnesota Vocational Summer Conference of the Area Vocational -Technical Institutes, August 1983.

Skolnick, Joan, Carol Langbort and Lucille Day, How to Encourage Girls in Math and Science, Dale Seymour Publications, 1982.

Smolkin, Shelley, "Fear of Figuring", Working Women, January 1979.

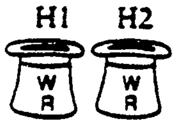
Tobias, Sheila, Overcoming Math Anxiety, New York: W.W. Norton, 1978; Houghton Mifflin, paperback, 1980.

Tobias, Sheila, "Math Anxiety: An Update", Mathematics in Michigan, Spring 1982.

Tobias, Sheila, "Math Anxiety: Who Fails Mathematics and Why", The Illinois Math Math Teacher, November 1980.

Which Is Best?

You are given two hats, two white marbles, and two red marbles. Which arrangement of the two white and two red marbles in the hats gives the best chance of drawing a white marble?



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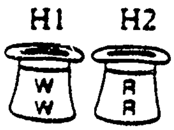

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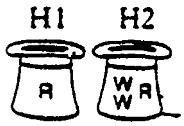

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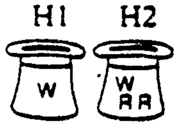

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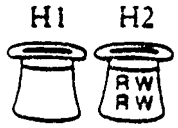

P(W) = ____

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TEACHING GIFTED AND TALENTED

Bruce Mitchell
Michigan State University

Today I would like to talk with you about teaching gifted and talented kids in middle school. I'm very much involved with honor students in mathematics. Let me start by telling you what that means. One reason that I became involved with these students is a program called Dimensions which was started last year and is in operation again this year. Criteria for involvemen in the program includes getting in the 95 percentile on the Standford Achievement Test, recommendations from their teachers, selection by their school and one other test. The students are from Ingham county and after selection is made, decide if they want to participate in math or science. Their second choice must be in writing or literature. The third choice must be taken in the arts, either art, music or drama. This is the second year that I have taught mathematics in the program. Another program that I was actively involved in consisted of two classes each morning, one at 8:00 and the other from 9:10-10:30 a.m. A third program in which I with middle school kids has been going on for four or five years with kids from East Lansing. John Masterson and I have been working together on this program with the gifted and talented on Saturday mornings in the fall, winter and spring. Kids do actually enjoy coming out, some of course get pushed out of bed by their parents because their parents think it's a good idea to be in the program. But most of them really enjoy doing mathematics. The school district pays for the program.

I plan to tell you the units that we did with the kids. That is the specific parts of mathematics that we worked on. We'll talk about the strands that run through all the units, and we'll talk about specific things that we did with the kids. John and I had some interesting days. One I remember in particular was the day John brought a peanut butter sandwich on a very hot day and we decided to use it as a two dimensional model. You can imagine what happened from there.

Let's talk about the topics. The units include number theory, counting, probability, geometry, which included basic topology and Euclidean geometry, some calculus, analysis, and problem solving. We also had strands in which we included application so the kids could see how to use mathematics, we included activities, we included recreation puzzle-type problems, and we always were working on their problem solving skills. Remember the thing we wanted was for the kids to have fun. We didn't have tests but we were able to move around to see what they were doing since John and I only had twenty to twenty-three kids. We worked on some algebra skills and the last strand was one that Bill turned me on to.

What I'd like to do now is give an example of what we did with the kids. This was a demonstration of what some of the kids can do. I'd start off with this sort of problem.

"Take 1987, keeping the numbers in the same order, using any mathematics symbols a middle school student would know, and write an expression equivalent to the numbers one through twenty".

Let's do a couple. Let's find an equivalent for eleven.

1st answer: $1 + 9 + 8 - 7 = 11$

2nd : What about the number one? Answer: $1^{987} = 1$

$-1 + \sqrt{9} - 8 + 7 = 1$

3rd : What about the number four? Answer: $19 - 8 - 7 = 4$

4th : What about ten? $1 \times 9 + 8 - 7 = 10$

Notice this is a good motivator for learning operations.

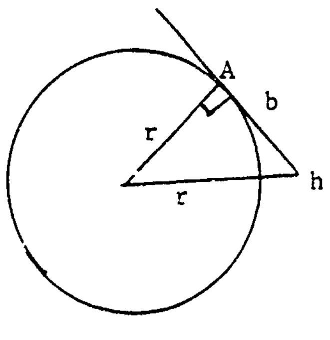
Another example that the kids liked was can you get all of the numbers out of four one's. You use the same rules. See if you can get an eight out of the four one's: 1111. The reason I have three answers is two came from the kids.

Answers:

1. x - 11 (using Roman Numerals) (I'll count it but I don't like it) (from Lorraine)
2. $1 \div .1 - 1 - 1 = 8$ (from the kids)
   or $.1^{-1} - 1 - 1 = 8$
3. $[\Sigma (1 + 1)]! + 1 + 1 = 8$ (sign is sum of divisors of a number)
   $= 2 + 1 = 3$; 3 factorial $= 3 \cdot 2 = 6$
4. I used greatest integer function
   $[\sqrt{11}]! + 1 + 1 = 8$
   (the greatest integer of the square root of eleven is three, three factorial is 6, $6 + 1 + 1 = 8$).

Let me give you other ideas of things we did with these kids. We talked about perfect numbers primes. We tried to do the proof but we found we lost them so we went back to just seeing why it works. We were also using the fundamental counting principle. We looked at composite numbers and some other proofs. A problem John thought of was "List all the primes that are one less than a perfect square and see if you can prove how many there are". This lead the kids to list them and they figured out there was the only one. We did some things with something called Armstrong Numbers, which are numbers such that the sum of the cubes of the digits equals the number itself. One number would be zero. I know there are three. You play around with that. The next topic we'll talk about is geometry. We did topology including the four color problem. We did a lot of work with networks, deciding when is a network transversable and when it isn't irraversable. We did a lot with topological equivalents. We did the Jordon Curve Theorem. Next we did some Euclidean geometry through rotations, translations and reflections. We started with motion, sliding transparencies on the board or overhead, when we wanted to turn something we turned the transparency. We would draw the triangle on the board, go back to the overhead, find the center of rotation, and turn the transparency. If we did a flip, we would just flip the transparency over. Then we used the same kind of content with dot paper; sliding, flipping, and turning, rotating. We found a lot of that material from a book, Geometry A Methodic Approach by O'Daffer & Clements.

We also investigated reflections, using mirror reflections and line reflections. We worked with magnification and similarity. We tied the Euclidean geometry together with isometries. We also included some practical problems. Here are two examples of applications of geometry that we used. The first is using the Pythogorean Theorem. The question is how far can you see? If this is the earth then there are certain things they'll need to know.



For instance, students need to know that the radius is perpendicular at the point of tangency (A above), h is how far you are standing out from the earth and b is how far you can see. Using the Pythogorean Theorem:

$$r^2 + b^2 = (r + h)^2$$
$$r^2 + b^2 = r^2 + 2rh + h^2$$
$$b^2 = 2rh + h^2$$

(We approximate the raidus of the earth as 4,000). Therefore $b^2 = 8000\ h + h^2$. (If we approximate 5000 ft/mile)

$$b^2 = \frac{8000\ h}{5000} + \left(\frac{h}{5000}\right)^2$$

If h is only our height, then $\left(\frac{h}{5000}\right)^2$ is very small and doesn't make a contribution, so our equation becomes

$$b^2 = \frac{8000\ h}{5000}$$
$$b = 1.6\ h$$

Another neat application is using the theorem in geometry that says: If parallel lines cut off equal segments of one transversal then they cut off equal segments on all transversals. If you want to divide a card into a specific number of parts you could use your lined paper and the card to divide it. Use the edge of your card as a transversal.

We also worked with students on geoboards through various activities, including slope. We also included some analytical geometry.

We had special topics like the Golden Ratio and how it relates to the Fibonacci Sequence. To introduce counting and probability problems, we asked them to figure out how many different ways they could arrange themselves? That is, suppose all eighteen of them stood in a line. How long would it take to go through the arrangements if we did one every second and didn't stop? We would first guess: a week, a year, a decade. The answer is that it takes more than 10 million years. That became a motivating problem for the kids to see how numbers grow. We worked on combinations and permutations from the vantage of using the notations and information in probability problems. We used our state lottery, picking 6 numbers out of 40 where order doesn't matter, as a problem. This turns out to be a combination problem where the odds are $\frac{40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35}{6!}$ to one which means 7,059,052 to 1 or 1 in seven million. We used probability to determine the mathematical sums and the probability of "craps". We worked on independent events and mutually exclusive events. One of the nice things about probability is that you can do experiments. The probability unit is one of the units we could find the most applications.

We did pretty standard problems for the problem-solving units. We made sure the kids realized the need to sort through information, organize the data, look for patterns, list things. We gave them problems that forced them to do those sorts of things. We perhaps would have two of those types of problems on one Saturday. We'd have them talk about how to solve the problems, discussing various techniques.

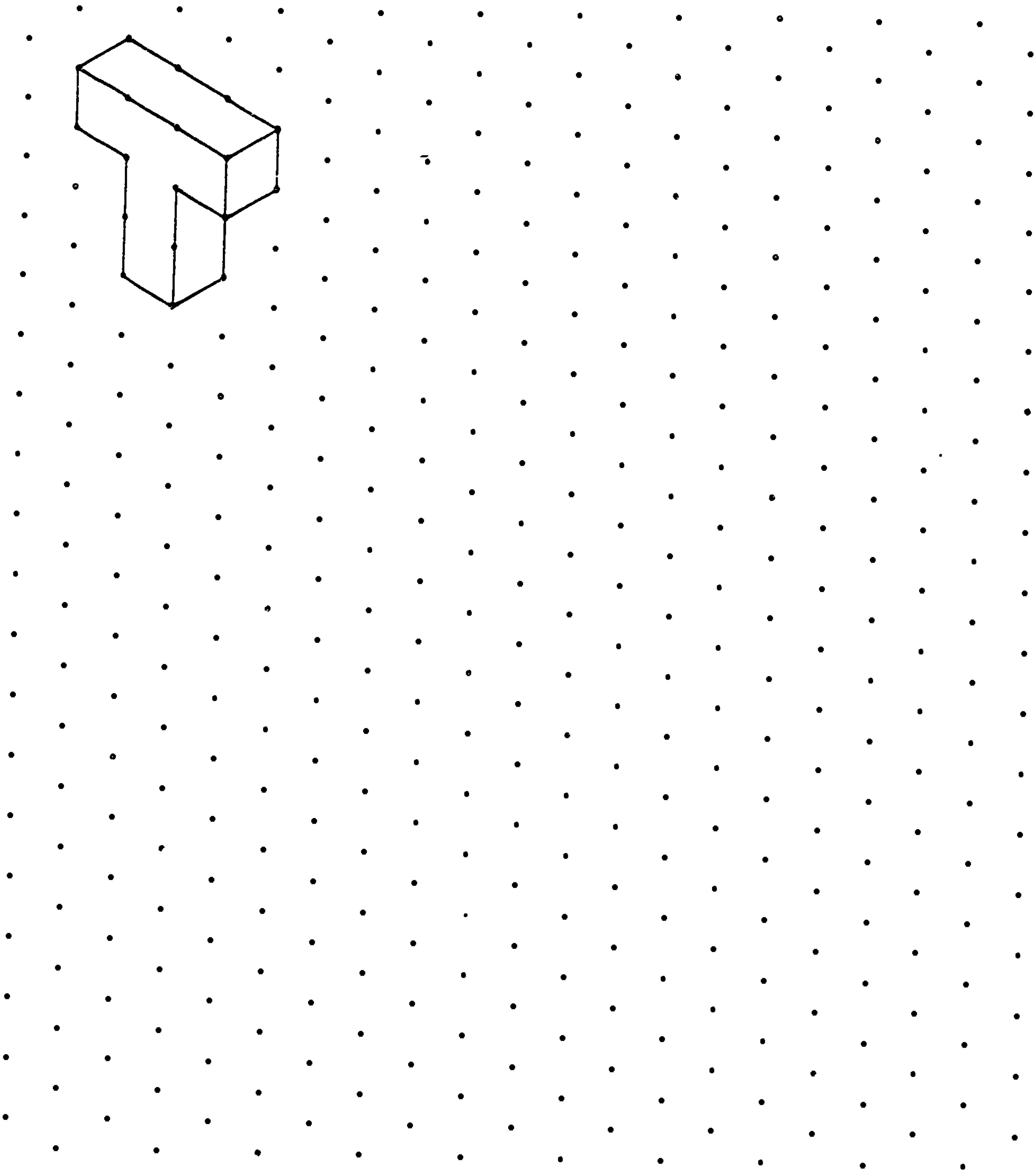
The types of examples of problems were not completed in fifteen weeks. We figured that we might have someone in there for 3 years, 6th, 7th and 8th grades, so we worked with different strands of problems throughout. This way there would not be a lot of repetition for the student that went through two or three years.

One of the important ideas that I would like to conclude with is the method of teaching which seemed to work with these kids. We would present some material, have them work on it, present a related activity or question, have them work on it and then as they were working John and I would walk around the room asking questions, finding out where the kids were at with the material or activity. We found that when the kids were working on activities and problems that were posed, when kids worked together in groups, we were able to find out more about how the kids were thinking and it felt like we were doing a better job.

Here are a few things that we found the kids needed work on: writing - we worked at trying to get the kids to write and organize information neatly; thinking on their own before they asked what should I do next; using calculators and mental arithmetic.

Drawing Ts

Make a capital T from five cubes. Pretend it is glued together so that you can turn it freely, even upside down, and look at it from every corner. How many *different* ways can you draw the T on the dot paper? An example is given.



ALGEBRA

Elizabeth Phillips
Michigan State University

I want to first share with you how I first became interested in algebra. As one of my responsibilities at Michigan State, I direct the two remedial mathematics courses; elementary and intermediate algebra courses. These are classes that students place into on the basis of a placement exam. The remedial courses are a prerequisite for college algebra, and do not count toward a degree. Therefore these classes cost students extra time, extra money, and a great deal of frustration. Many students have already had 3 and 4 years of high school mathematics, but we are not only requiring that students have had it, but also that they remember it. We prefer that they come prepared for their first college class, that is, that the students remember longer what they have learned. Therefore we have been working with high school teachers with that objective in mind. That is, if we teach for understanding, we teach for retention.

There are many topics in algebra that we could talk about today. I have decided to look at algebraic errors. I will analyze these algebraic errors and show that they are all related to one or two key ideas in teaching algebra. Before we begin I would like to share with you some of the things students say when they talk with me about their experiences with mathematics.

Have You Heard This?

- But I studied all night for the test and I still flunked.
- You didn't show us how to do that problem in class.
- Notes? What notes.
- I do all the homework but I still get it wrong on the test.
- But I understand it in class.
- No homework tonight. Just a reading assignment.
- I got the right answers. Why did you take off points?
- What do you mean I don't get any points? Look at my work.
- But I did well in my other math classes.

Let's now look at some algebraic errors:

1. $\frac{2 + x}{3 + x} = \frac{2}{3}$

2. $\frac{3 + x(x + 1)}{3x(x + 1)} = 1$

3. $-2x(x^2y) = -2x^3 - 2xy$

4. $(a+ x)^3 = a^3 + x^3$

5. $|x - 2| = |x| - |2|$

6. $\sqrt{a^2} + 4 = a + 2$

7. $-2^{-4} = +8$ or 16 or -6 or $1/16$

8. $\frac{3x^6}{x^3} = 3x^2$

9. $2 \cdot 4^x = 8^x$

10. $3 - 2(x - 3) = x - 3$ or
$3 - 2(x - 3) = 3 - 2x - 6$

Failure to recognize patterns

11. Factor:
$2x + 3x = 5x$
$ax + bx = (a + b)x$
$x(x - 1) + 3(x - 1) = ?$

12. Simplify:
$x + 2x = 3x$
$(a + b) + 2(a + b) = ?$
$/2 + 2/2 = ?$

13. Simplify: $\frac{3}{x + 1} - \frac{x + 2}{x} + 1$

$$\frac{3}{x + 1} - \frac{x + 2 + 1}{x} = \frac{3(x)}{x(x + 1)} - \frac{(x + 2)(x + 1)}{(x + 1)} + \frac{1x(x + 1)}{x(x + 1)}$$

Now what? Either students remove the common factors in each term --- or they multiply both sides by the common denominator and solve for x. In addition they may neglect to treat 1 as a term and --- forget to distribute the negative sign.

14. Solve: $\frac{3}{x + 1} + \frac{x + 2}{x} = 1$

$$\frac{3x}{x(x + 1)} + \frac{(x + 2)(x + 1)}{x(x + 1)} - \frac{x(x + 1)}{x(x + 1)} = 0$$

Now they can't get rid of the denominator.

Comments:

In error 4 compare $(a + b)^2$ with (a + b)2. What is the difference? In the students' mind this may be just a half-space! What's a half-space among friends. In error 7, students may ignore the half-space and see -2 - 4 or -2(-4). In error 11 students fail to recognize the distributive pattern in a variety of different examples. In error 12, expressions and sentences (simplifying and solving) become confused and often interchanged on tests.

When we are teaching these types of problems we need to clarify, to identify patterns, and to continue to help make connections. One idea I like to use in the case of expressions and equations is to relate these concepts to a lesson in an English class. Very often in English we are asked to work with phrases, and then put phrases together with a verb to form sentences. In math our expressions and equations are very much like phrases and sentences. We put expressions together to form sentences such as equations, inequalities, etc.

(Discussion was incorporated throughout this part of presentation about implications of each problem).

If we look back over that list of errors we find that students do not know the difference between factors and terms. Let's examine the distributive property. It is a powerful concept; it forms the backbone of algebra. Most textbooks - at most - give it one line. In my classes it might take two weeks to gain understanding of this concept. We need to emphasize that on one side it is a product of two factors. On the other side it is a sum of two terms. In one direction we call it multiplying. In the other direction we call it factoring. The dstributive property relates addition and multiplication, a very powerful affair.

<u>FACTOR AND TERM</u>
(TWO CONCEPTS ARE RELATED BY THE)
<u>DISTRIBUTIVE PROPERTY</u>

$a(b + c) = ab + ac$

product with two factors	sum with two terms

→
multiplying
←
factoring

The next source of errors stems from a lack of understanding of a concept that is the backbone of middle school mathematics -- equivalent fractions. Realize it is also very important in algebra and note the importance of factors in understanding this property.

FUNDAMENTAL PROPERTY OF FRACTIONS

$$\frac{ax}{bx} = \frac{a}{b}$$

Finally, students cannot distinguish between

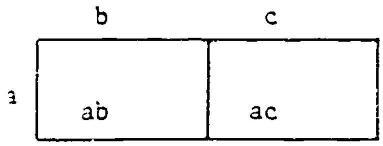
EXPRESSIONS AND SENTENCES

SIMPLIFY AND SOLVE

One thing we sometimes forget is that algebra is simply generalized arithmetic; What we did in arithmetic we do in a more generalized way in algebra using variables, expressions and sentences.

For the remainder I would like to discuss with you how we might introduce the distributive property in class and teach it for understanding.

This first thing that we want to do when teaching the distributive property is to draw a picture. We want this picture to be something on which we can hang the distributive property. A picture is worth a thousand words! For a picture of distributive property we can use area of rectangles. We can express the area of this rectangle in two ways.



Area: $a(b + c) = ab + ac$

I want to share with you a few words from Bertrand Russells' autobiography concerning his experiences with the distributive property. Bertrand Russell was a very famous mathematician. He lived in the early part of the century and was a co-author of the Principia Mathematica. He says:

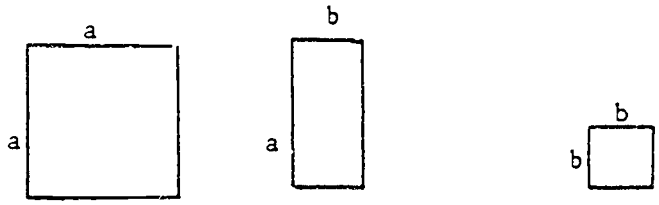
> "The beginning of algebra one I found far more difficult. Perhaps it was the result of bad teaching. I was made to learn by heart the square of the sum of two numbers is equal to the sum of the squares increased by twice the product. I had not the vaguest idea of what this meant and when I could not remember the words my tutor threw the book at my head which did not stimulate my intellect any more."

If Bertrand had been in one of our classes, this is what we would have done. He was trying to remember this rule.

$$(a + b)^2 = a^2 + 2ab + b^2$$

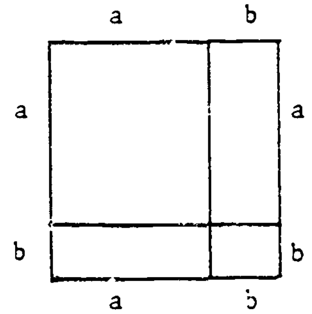
We can use these plastic tiles; a square with dimensions a x a, a square with dimensions b x b and a rectangle with dimensions a x b.

Demonstrate how to use the algebra tiles on the overhead. Several plastic tiles of the following are displayed on the overhead.



The tiles are then arranged to form a square whose dimensions are a + b. Thus the area of the larger square can be expressed in two ways: $(a + b)^2 = a^2 + 2ab + b^2$

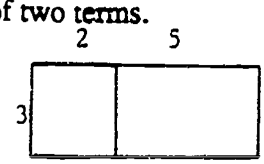
These algebraic tiles can be used to teach and practice factoring of trinomials or to multiply binomials.

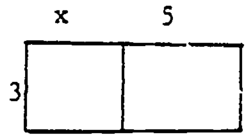


Problems:

The following set of questions are examples of questions which can be used to develop an understanding of factor, term, and expression.

1. Find the area of each rectangle in two different ways: As a product of two factors and as a sum of two terms.





2. Insert parentheses on the left side of the equality sign to make each statement true.

   a) $7 + 5p - p = 11p$ b) $7 + 5p - p = 7$
   b) $7 + 5p - p = 0$ c) $7 + 5p - p = 7 + 4p$

3. Fill in the box with a number or expression to make each statement true.

   a) $2(4 + \square) = 18$ b) $2(4 + \square) = 8 - x$

4. a) Write 12 as a product with 2 factors, 3 factors.
   b) Write 12 as a sume of 2 terms, 3 terms.
   c) Write 12 as a product of 2 factors such that one factor is the sum of 2 terms.

5. Repeat problem 4 using $12x^2$.

6. Without altering or simplifying the expressions. How many terms does each expression have? Name them.

   a) $2x$ b) $2 + x$
   c) $2x + 5$ d) $2x + 5x$
   e) $2x + 5(x + 2)$ f) $(x + 2) + 2(x + 2) + 3(x + 2)$

7. Without simplifying, in which of the following expressions is x, x + 1 a factor of the entire expression?

   a) x b) $3x$ c) $3x + 1$
   d) $3x(x + 1)$ e) $x + 3(x + 1)$ f) $(x + 3)(x + 1)$

8. Use the distributive property to write each of the expressions as a sum or difference of terms.

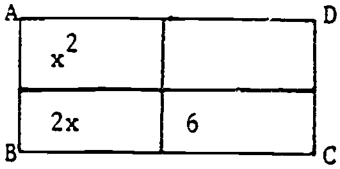
   a) $-5y(x + y)$ b) $-5y(xy)$ c) $(-5 + y)(x + y)$

9. Use the distributive property to write each expression as the product of two or more factors.

   a) $5x - 25x$ b) $5ax - 25x$ c) $3y + 5y + 7y$
   d) $(x + 1)2 + (x + 1)x$

10. At the beginning of the week Sam's checking account had a balance of \$75.50. During the week Sam wrote a check for \$15.00 on Tuesday and a check for \$20.50 on Friday. What was his balance at the end of the week? Find the answer in two different ways.

11. Rectangle ABCD has been subdivided into four rectangles. The areas of three of the smaller rectangles are given.
a) Find the area of the fourth rectangle.
b) Find the dimensions of rectangle ABCD.
c) Find the perimeter of rectangle ABCD.



12. Find the two expressions representing the amount of money collected by selling X tickets on Monday and Y tickets on Tuesday if each ticket costs \$2.50.

13. In which fractions is x a factor of both the numerator and denominator?

a) $\frac{2x^2}{3x}$ b) $\frac{2+x}{3+x}$ c) $\frac{3x(x+1)}{x+1}$ d) $\frac{3x(x+1)}{x^2+x}$

14. A student made an error in simplifying the fraction. Find the error and the correct solution.

$$\frac{x^2 + 2x}{x^2 + 3x + 2} = \frac{x(x+2)}{(x+2)(x+1)} = \frac{x}{x+1} = 1$$

15. If the variable can be any real number, which expressions represent the same value as x + 1?

a) $n + 1$ b) $1 + x$ c) $x - 1$
d) $\frac{2(p+1)}{2}$

16. Find an expression which represents the pattern in the second row.

x	1	2	3	4
y	2	5	10	17

miles	0	100	200	300
cost(\$)	20	40	60	80

17. If $x > 3$, which expression has the largest value?

$x^2 \quad x \quad 2x \quad \frac{x}{2}$

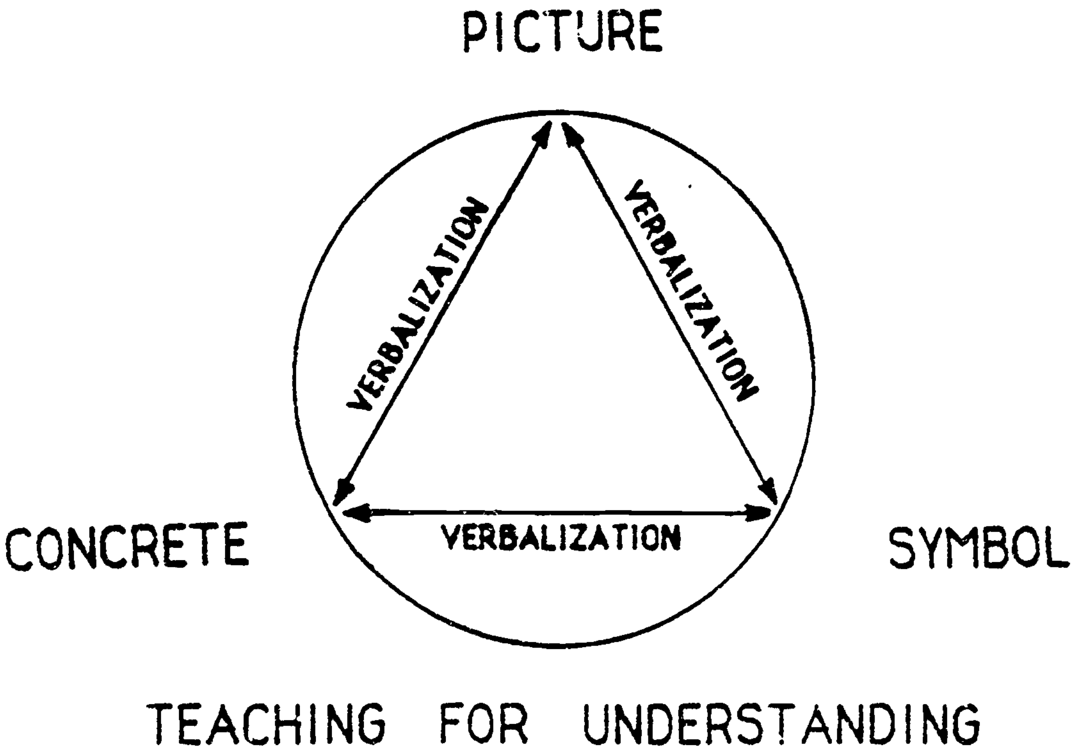
18. Write an algebraic statement describing the pattern:

$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$

In summary note the importance of the picture that we receive through concrete examples and the discussions that moved us from one stage to another. Let me end with one of my favorite overheads which illustrates the strategies we used to teach the distributive property for understanding.



References

Russell, Bertrand (1968), The Autobiography of Bertrand Russell: 1872-1914, B. Bantom, New York.

MATHEMATICAL PARADOXES IN SCHOOL

Nitsa Movshovitz-Hader
Israel Institute of Technology

OUTLINE

1. What is a paradox?
2. The role of paradoxes in the history of math.
3. The role of paradoxes in school mathematics.
4. How to collect and create paradoxes suitable to school.

1. What is a paradox?

First let's ask what are other words that are used in connection to paradoxes? A fallacy, mind benders, thought twisters, misconceptions, contradictions, tantalizing inconsistencies, loopholes, misunderstandings. Instead of defining a paradox we'll consider several classes of paradoxes.

(a) A paradox is a false statement that makes sense, a conclusion that is intuitively acceptable but that is not sound logically. The challenge is to find out the origin of the wrong expectations. An example in mathematics would be -- the sum of the exterior angles of a polygon is a function of the number of sides. (See figure 1). This statement is wrong, of course, but it makes sense as it is analogous to the correct statement about the sum of the interior angles.

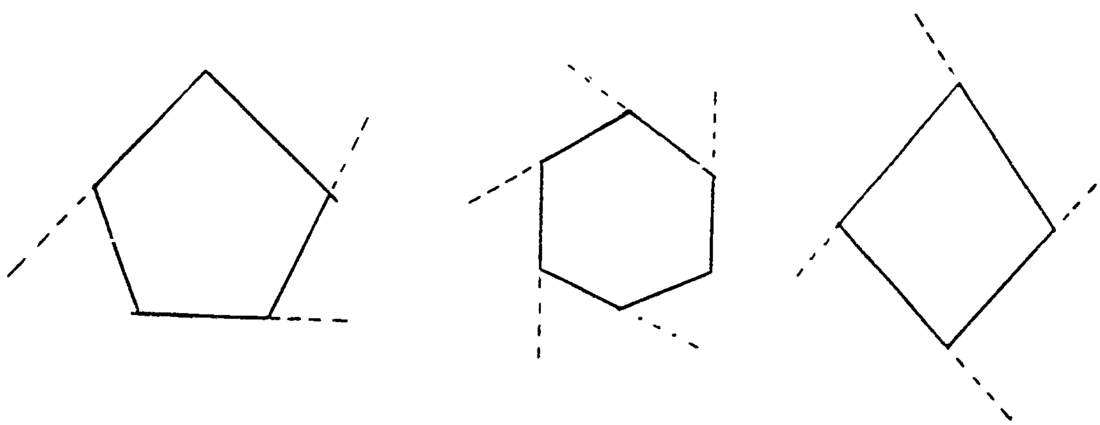


Figure 1

This class includes logically invalid conclusions which are psychologically plausible. For example: "All Cretans are liars" said Epimenides who was a Cretan himself, therefore Epimenides is a liar and hence... all Cretans are truthful. Now, Epimenides, the Cretan is truthful too and therefore all Cretans are liars as he claimed to begin with. This endless loop is based on the wrong negation of "All Cretans are liars". The correct way to negate it is "not all Cretans are liars".

(b) A paradox is also a true statement that does not make sense. Here, too, the challenge is to find out the origin of the wrong expectations. For example, consider the claim: "The sum of the interior angles in all triangles is the same". (See figure 2). For someone who does not know that this is a true statement, it may make very little sense, as triangles have many different sizes, and various type of angles.

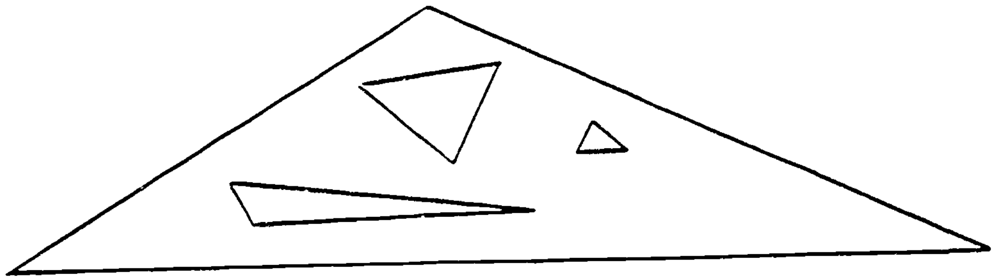


Figure 2

This class includes logically valid conclusions which are psychologically not acceptable. For example the following two sentences are mathematically equivalent yet psychologically different: "Good food is not cheap" and "Cheap food is not good".

(c) We also see a paradox as two (or more) contradicting claims that have reasonable grounds. Both or all claims make sense but at least one must be wrong. For example, let us look at the 4 cards paradox:

Introduction through a game:

To play this game you need a partner and four cards, two of each color, say red and black. If you prepare your own cards make sure that you color only one side of each card, so that all four look the same on the other side. Shuffle the four cards and let your partner choose two without looking at their color. If the two chosen cards have matching colors your partner wins a point. Change roles and repeat the game. Record your results.

Question:

What is the probability of winning a point in any round of the game?

Three different answers to this question are given below. All three seem logical yet only one is correct. Which one? (Please put x to the left f the answer you prefer).

_____ There are three equally likely results: either both cards are red, or they are both black or they don't match. In two cases the player wins a point therefore the probability is 2/3.

_____ There are two equally probable results: either the colors match (red-red or black-black) or they do not match (red-black or black-red). Therefore the probability is 1/2.

_____ Suppose the first chosen card is red. There is only one red among the remaining three cards. There is a probability of 1/3 to choose a second card with matching color.

What is wrong with the logic underlying the other two answers?

This class includes situations in which we have two claims presenting a dichotomy, yet both seem wrong, or both seem right. Look, at the following diagram (figure 3) representing areas between two equidistant lines crossing two parallel lines. The shaded areas no. 2,3 are either the same or not. This is a dichotomy. We know that 1 and 2 have the same area. Contrary to our intuition, area 2 and 3 are also the same.

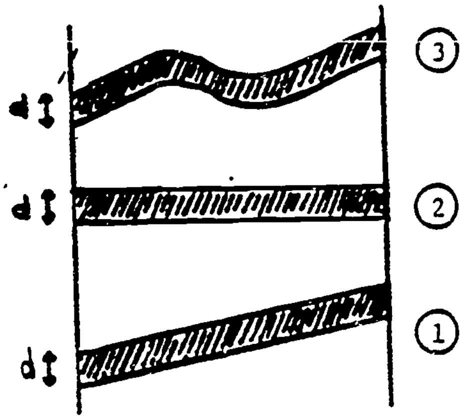


Figure 3

(d) Another class would be paradoxical questions. For instance, are there two non-congruent triangles with five congruent parts? We think, "I know that it can't be true, but since you have asked, perhaps it does afterall." (Hint: Are the following two triangles suitable: 8,12,18 and 12,18,27?)

2. The Role of Paradoxes in the Histrocial Development of Math

(a) Mathematical Defects:

These paradoxes result a contradiction unbearable in mathematics. They were solved by excluding them. They were eliminated from mathematics.

Examples:

1. $i = \sqrt{-1}$

   $-1 = \sqrt{-1^2} = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$
2. Cantor's paradox (1845-1918)

   A set with n members has 2^n subsets. Cantor's extension: The set of subsets of any set, finite or infinite, always has more members than the original set. But... what about the set of all sets?

3. **De-Morgan's Paradox:** Augustus De Morgan (1806-1871) is traditionally credited for the following fallacy:

Claim $\quad x = 1 \Rightarrow x = 0$

Proof:

$$x = 1 \Longleftrightarrow x^2 = x \Longleftrightarrow x^2 - 1 = x - 1 \Longleftrightarrow$$

$$\frac{x^2 - 1}{x - 1} = \frac{x - 1}{x - 1} \Longleftrightarrow$$

$$\frac{(x + 1)(x - 1)}{x - 1} = \frac{x - 1}{x - 1} \Longleftrightarrow x + 1 = 1 \Longleftrightarrow x = 0$$

4. The Existing Solution Paradox: An equation which should not have a solution has one. Students learned that $\frac{3}{x} = \frac{3}{2} \Longrightarrow x = 2$. In a test they were asked to solve the following equation:

$$\frac{x - 3}{x - 1} = \frac{x - 3}{x - 2}$$

Some of them wrote: "It follows that

$x - 1 = x - 2$

$-1 = -2$ contradiction. There is no solution."

Other students wrote: "Multiplying by the common denominator we get

$$\frac{(x - 1)(x - 2)(x - 3)}{x - 1} = \frac{(x - 1)(x - 2)(x - 3)}{x - 2}$$

now cancel

$(x - 2)(x - 3) = (x - 1)(x - 3)$

$x^2 - 5x + 6 = x^2 - 4x + 3$

$-5x + 6 = -4x + 3$

$x = 3$

Which answer is right ______________________________

What is wrong with the other answer? ______________________

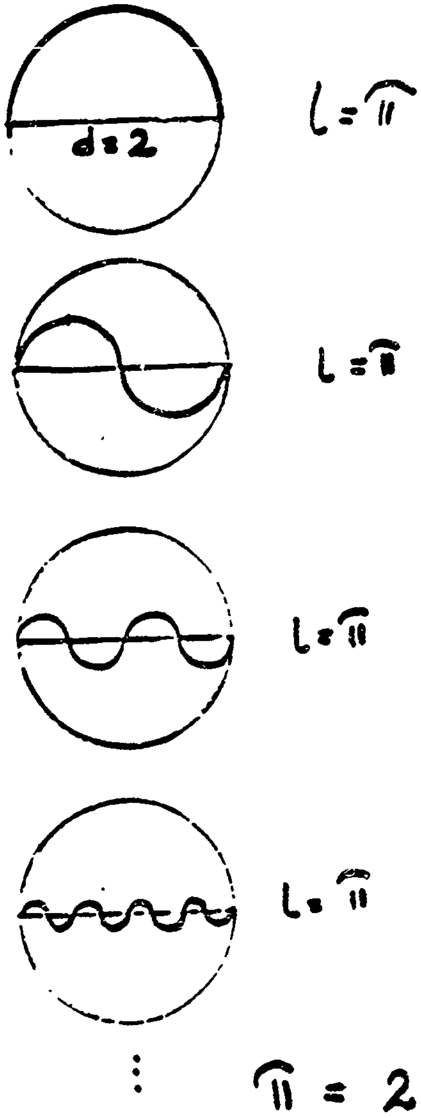
____

What about the reciprocal equation, does it have a solution or not? $\frac{x - 1}{x - 3} = \frac{x - 2}{x - 3}$

(b) Mathematician's Defects

1. Leibnitz (1646-1716) is credited for the following paradox:
   (Assuming existence of the sum of a nonconvergent infinite series)
   1 - 1 + 1 - 1 + 1... = S
   S = (1 - 1) + (1 - 1) +... = 0
   S = 1 - (1 - 1) + (1 - 1) + (1 - 1) +... = 1
   S = 1 - S ===> 2S = 1 ===> S = 1/2
2. The following set of diagrams (see figure 4) demonstrates that Pi = 2
   M. Gardner (p.58): "It is true that as the semicircles are made smaller their radii approach zero as a limit and therefore the wavy line can be made as close to the diameter of the large circle as one pleases. At no step, however, do the semicircles alter their shape. Since they always remain semicircles, no matter how small, their total length always remain pi. The fallacy is an excellent example of the fact that the elements of a converging infinite series may retain properties quite distinct from those of the limit itself.

Figure 4

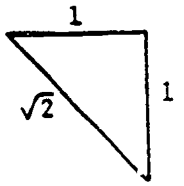


(c) Paradoxes we live with

Some paradoxes in history we learn to live with, some we build a new theory, or some we extend existing theory.

1. The Pythagoreans (500-400 B.C.) believed that any length can be measured by a natural number or a quotient of two natural numbers. They also new that there is a line segment of the measure $\sqrt{2}$ (the hyptenuse of an equilateral right angled triangle). They deduced that $\sqrt{2} = a/b$ for some irreducible fraction, and reached a contradiction thus facing a paradox. This paradox turned out later into the proof of existence of irrational numbers, more precisely that $\sqrt{2}$ is irrational.

Pythagoreans (400 B.C.)



$\sqrt{2} = \frac{a}{b} \quad (a,b) = 1$

$\sqrt{2}\, b = a$

$2b^2 = a^2 \Rightarrow 2|a^2 \Rightarrow 2|a$

Let $a = 2m \quad (m \in N)$

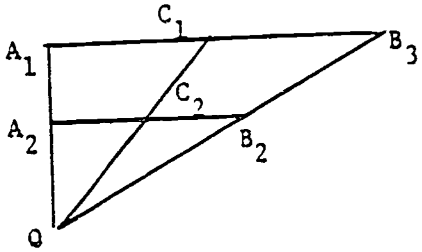
$2b^2 = 4m^2$

$b^2 = 2m^2 \Rightarrow 2|b^2 \Rightarrow 2|b$

but ... $(a,b) = 1$

2. Infinite subset of a set with 1 to 1 correspndence between them wer at the same time in conflict with the mathematical idea that the whole always is greater than its part.

   Cantor (1845-1918)



Two different line segments, same number of points.

2.1 Cantor's (1845-1918) proof that two line segments of different length contain the same number of points.

2.2 The set of even numbers and the set of integers are both enumerable, and

2.3 Finally the existence of two different infinite numbers - there are many, many more real numbers than rational numbers.

When mathematicians became convinced that these paradoxes did not result in a contradiction they were incorporated into mathematics and were no longer paradoxes.

(d) Unresolved paradoxes

We also have paradoxes upon which the mathematics community have not reached a common agreement. Please refer to the references at the end of this paper to explore these.

3. The Role of Paradoxes in School Mathematics

Paradoxes have a role in school mathematics. We can use paradoxes in the teaching-learning process in the same way that paradoxes played a role in the development of mathematics. We can use them to clarify concepts, carefully extend theories, introduce them as intellectual challenges, and to help students avoid mistakes. Paradoxes perplex us, dumbfound us, thrill us, challenge us and arouse our curiosity. When solving a paradox we better understand both possibilities and limitations of human reasoning, we better understand both possibilties and limitations of mathematics, we are invigorated, and we are delighted.

We might ask, what is the educational potential of paradoxes? What makes paradoxes educationally useful? Leone Burton in her address to the MOIFEM Conference in Quebec in July, 1986 said:

> "There is a private and a public world of mathematics. In the private world struggle, failure, incomprehension, intuition and creativity dominate. The public world is where the results of the private struggle make their appearance in a formal, conventional, abstract form from which all evidence of false trials, inadequate reasoning or misunderstandings have been eliminated. Unfortunately, textbooks give our pupils access only to the public world. As a result, they see their struggle and failure to understand as personal inadequacy rather than part of the process inherent in the development of mathematics. Our pedagogical practices in teaching mathematics deny the influence of the individual or the social context, and present young people with a pretend world of certainty, exactitude and objectivity. This pretend world is associated with power and control and given our social history, that is perceived as male by association".

Paradoxes - Educational Merits

There are many educationally useful paradoxes that can help in one or more of the following:

* Shape up mathematical ideas
* Refinement of basic concepts
* Clarification of nuances
* Replace endless preaching
* Provoke a desirable imbalance
* Surprise and motivatio
* Problem solving
* Challenge
* Provide an indication of intellectual function

* Learning to (read a) prove
* Debugging

4. How To Collect or Create Paradoxes

Consider the following five paradoxes. (They are prepared as worksheets for school use).

(1). $\sqrt{4}$ Is ... Irrational

By definition we call a number r irrational if for any two integers a,b $r \neq a/b$. Assume that $\sqrt{4}$ is rational. We'll show that this assumption leads logically to a contradiction. According to our assumption there exist two integers p,q relatively prime such that

$p/q = \sqrt{4}$

$\Longrightarrow$

$p^2/q^2 = 4$

$4q^2 = p^2$

$4 \mid p^2 \Longrightarrow 4 \mid p \Longrightarrow$ there exists an integer n s.t.

$p = 4n$

hence

$4q^2 = 16n^2 \Longrightarrow q^2 = 4n^2 \Longrightarrow 4 \mid q^2$

$\Longrightarrow 4 \mid q$ and therefore p and q have a common devisor greater than 1 which contradicts the initial assumption that they are relatively prime.

It follows that our assumption was false and therefore $\sqrt{4}$ is not a rational number. Q.E.D. Where is the flaw?

(2). **Triangle Similarity Paradox**

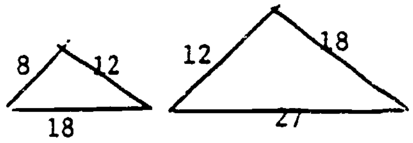
Any two triangles with five congruent elements (sides, angles) must be congruent.

<u>One one hand</u>:

This is right, of course.
Three elements are usually enough for congruency.
Here 5 elements are congruent, namely either 2 sides and 3 angles, or 2 angles and 3 sides are congruent. At any rate we have here A.S.A. or S.A.S. or S.S.S. which are sufficient conditions for congruency.

<u>On the other hand</u>:

Here are two non-congruent triangles



Each has sides of 12 and 18 and the three angles are congruent because the triangles are similar ones (with ratio 2:3).

How Come?

(3). For any natural number n

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} \le \frac{1}{\sqrt{2n}}$$

<u>One one hand</u>:

It is sufficient to prove that for all n

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} \le \frac{1}{\sqrt{2n+1}}$$

This we'll prove by induction

a. For n = 1 the left-hand side = 1/2; the right-hand side = $\frac{1}{\sqrt{3}}$, $\frac{1}{2} = \frac{1}{\sqrt{4}} < \frac{1}{\sqrt{3}}$

therefore nor n = 1 it's right.

b. Assume for any natural number k

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k} \le \frac{1}{\sqrt{2k+1}}$$

we'll prove that

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)(2k+1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k \cdot 2(k+1)} \le \frac{1}{\sqrt{2k+3}}$$

<u>On the other hand</u>:

When we try to prove the original claim straightforward by induction we get:

a. For n = 1

the left-hand side = 1/2

the right-hand side = $\frac{1}{\sqrt{2}}$

$2 > \sqrt{2} \Rightarrow 1/2 < \frac{1}{\sqrt{2}}$

therefore for n = 1 it's right.

b. Assume $\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k \cdot 2(k+1)} < \frac{1}{\sqrt{2(k+1)}}$

but

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)(2k+1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k \cdot 2 \cdot (k+1)} \le \frac{2k+1}{\sqrt{2k}}$$

$$\frac{1}{2(k+1)} = \frac{\sqrt{(2k+1)^2}}{\sqrt{2k(2k+2)} \cdot \sqrt{2(k+1)}}$$

$$\frac{\sqrt{4k^2+4k+1}}{\sqrt{4k^2+4k}\,\sqrt{2(k+1)}} > \frac{\sqrt{4k^2+4k}}{\sqrt{4k^2+4k}\,\sqrt{2(k+1}}$$

$$= \frac{1}{\sqrt{2(k+1)}}$$

proof: $\dfrac{1\cdot3\cdot5\cdot\ldots\cdot(2k-1)(2k+1)}{2\cdot4\cdot6\cdot\ldots\cdot2k\cdot2(k+1)}$

$$\le \frac{1}{\sqrt{2k+1}}\cdot\frac{2k+1}{2(k+1)} = \frac{\sqrt{2k+1}}{\sqrt{2(k+1)^2}}$$

$$= \frac{\sqrt{(2k+1)(2k+3)}}{\sqrt{4(k+1)^2(2k+3)}}$$

$$\frac{\sqrt{4k^2+8k+3}}{\sqrt{4k^2+8k+4}\cdot\sqrt{2k+3}} \le \frac{1}{\sqrt{2k+3}}$$

Therefore the claim is right.

contrary to the claim.

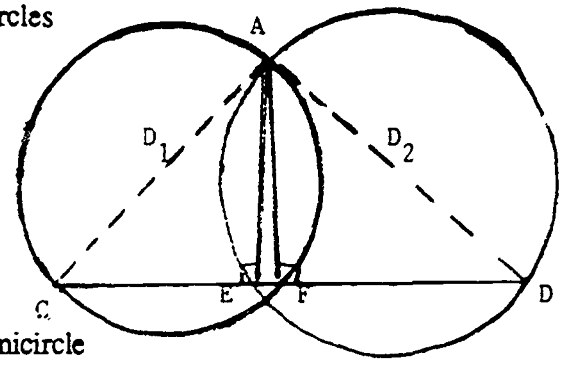
Therefore the claim is false.

How Come?

(4). The Intersecting Circles Paradox

Two distinct perpendicular lines to one line through one point.

The figure on the right presents two-circles 0_1 and 0_2 intersecting at A.



AC is the diameter of circle 0_1

AD is the diameter of circle 0_2

E,F are the intersection points of line CD with the two circles respectively

$\angle AFC$ is a circumference angle on a semicircle therefore it is right.

$\angle AED$ is a circumference angle on a semicircle therefore it is also right.

$\Longrightarrow AE \perp CD$ and $AF \perp CD$

How Come?

(5). Any Two Consecutive Integers Are Equal...

As you know, for any integer n the following identity holds:

$$(n+1)^2 = n^2 + 2n + 1$$

Therefore

$$(n+1)^2 - (2n+1) = n^2$$

Subtracting $n(2n+1)$ from both sides we get

$$(n+1)^2 - (2n+1) - n(2n+1) = n^2 - n(2n+1)$$

$$(n+1)^2 - (2n+1)(n+1) = n^2 - n(2n+1)$$

Adding to both sides $[(1/2)(2n+1)]^2$ we get

$$(n+1)^2 - 2(1/2)(2n+1)(n+1) + [(1/2)(2n+1)]^2$$
$$= n^2 - 2(1/2)n(2n+1)$$
$$+ [(1/2)(2n+1)]^2$$

Therefore

$$[(n+1) - (1/2)(2n+1)]^2 = [n - (1/2)(2n+1)]^2$$

$$(n+1) - (1/2)(2n+1) = n - (1/2)(2n+1)$$

$$n + 1 = n$$

Q.E.D. And you have always believed that $3 \neq 2$ ...

Any comments?

By the way: Are the following equations equivalent?

$x^2 = 9$

and

$x = \sqrt{9}$

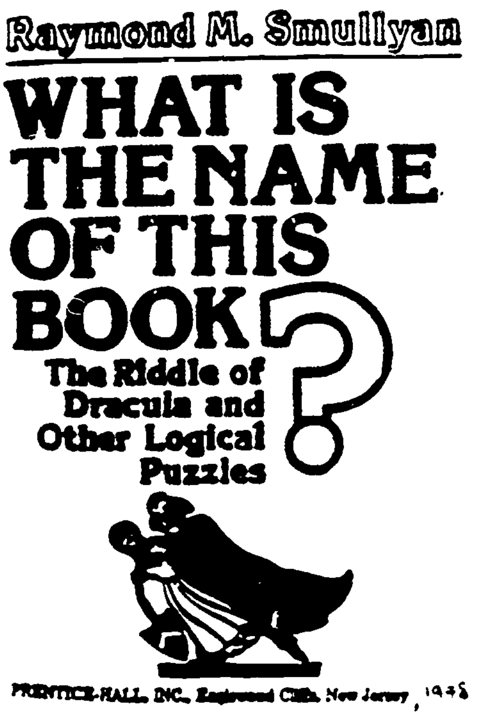
Teachers often wonder how to collect and create suitable paradoxes. I would suggest a first source as the history of mathematics. It provides many good examples. You can also create useful paradoxes from student's errors. Almost any mathematical error is a fallacy since there is an underlying logic that led to the error. You can create a paradox by bringing a mistake to an absurd, that is, make a conflict with known facts.

Examples:

1. $3 = \sqrt{9} = -3 \Longrightarrow = -3$ (the error: $\sqrt{9} = \pm 3$).
2. Error: to test divisibility by 3 it is sufficient that the unit digit is odd, (in analogy to divisibility by 2).
   Paradox: As we know divisibility by six is tested by divisibility by both 2 and by 3. In teting divisibility of 814 by 6 one child argues: "814 is divisible by 2 because it ends with an even number: 814:2 = 407. 407 is divisible by 3 because it ends with an odd number. Therefore 814 is divisible by 6." Another child calculated 814:6 = 135R4. Why?

Final comment

I would like to end with a riddle called "Fooled?" It can be found in a book by Raymond M. Smullyan titled What Is The Name Of This Book?



My introduction to logic was at the age of six. It happened this way: On April 1, 1925, I was sick in bed with grippe, or flu, or something. In the morning by broher Emile (ten years my senior) came into my bedroom and said: "Well, Raymond, today is April Fool's Day, and I will fool you as you have never been fooled before!" I waited all day long for him to fool me, but he didn't. Late that night, my mother asked me, "Why don't you go to sleep?" I replied, "I'm waiting for Emile to fool me." My mother turned to Emile and said, "Emile, will you please fool the child!" Emile then turned to me, and the following dialogue ensued:

Emile / So, you expected me to fool you, didn't you?
Raymond / Yes.
Emile / But I didn't, did I?
Raymond / No.
Emile / But you expected me to, didn't you?
Raymond / Yes.
Emile / So I fooled you, didn't I!

Well, I recall lying in bed long after the lights were turned out wondering whether or not I had really been fooled. On the one hand, if I wasn't fooled, then I did not get when I expected, hence I was fooled. (This was Emile's argument.) But with equal reason it can be said that if I was fooled, then I [illegible] get what I expected, so then, in w[illegible]e was I fooled. So, was I fooled, or wasn't I?

PARADOXES - REFERENCES

B.H. Bunch: Mathematical Fallacies and Paradoxes. Van Nostrand Reinhold Company 1982.

H. Eves: Great Moments in Mathematics (before 1650). Dolciani Mathematical Expositions (Vol. 5). The Mathematical Association of America 1983.

H. Eves: Great Moments in Mathematics (after 1650). Dolciani Mathematical Expositions (Vol. 7). The MAA 1983.

M. Gardner: AHA Gotcha. Paradoxes to puzzle and delight. W.H. Freeman and Company, 1982.

S.W. Huck, H.M. Sandler: Statistical Illusions, Book 1; Problems, Boox 2; Solutions Harper & Row Publishers 1984.

Northrop Eugene P.: Riddles in Mathematics: A Book of Paradoxes. Robert E. Kreiger Publishing Company Huntington, New York, 1975.

R. Smullyan: What Is The Name Of This Book? Penguin Books 1978.

EQUATIONS

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Joan Ross has worked for several years with Layman Allen, the inventor of the Equations Game. The game is a mainstay in the Academic Olympics program which attracts students from throughout the U.S.

NOTES ON CLASSROOM TOURNAMENT STRUCTURE

Purposes of the Classroom Tournament Structure:

1) To equalize competition as much as possible between the students playing a match.

2) A "Bumping System" regularly adjusts the level of competition for each student to account for changes in performance and varying learnig rates.

3) Equalizing competition at each tournament table encourages teaching, diagnosis, and learning between students of similar abilities and understanding.

Tournament Structure - Table Seating

1) Students are arranged into playing groups or "tables" with three students per table.

2) Players at each table should be as evenly matched in ability as possible (initial assignments to tables can be done using a grade book or any other general measure of ability).

   EXAMPLE: A class of 30 total students would be arranged into 10 tables groups with 3 students per table. Table #1 would have the three most competent students, Table #2 would have the next three most competent students, and so on until Table #10 which would have the three least competent students.

3) If the number of total students in a class is not an even multiple of three, then the very bottom tables (low ability) should have two-player games instead of three player games.

EXAMPLE: If 29 total students in class: Tables #1 - #9 have three players each.
Table #10 has only two players.
If 28 total students in class: Tables #1 - #8 have three players each.
Tables #9 and #10 have only two players.

It's better to have several two-player games than a four-player game.

4) Absences: If one student is absent at a table, the remaining two players play a two player game. If two students at a table are absent, the remaining students should join the next higher table in the tournament structure and play a four player game. (It's always better to move up one than down one table). If the player who moved up because of absences wins at the new table, then that player will bump up to the next higher table for the following week's tournament. NOTE: This is the only time a player can advance upward more than one table during a single week.

Bumping Procedure

1) A "tournament round" is a specified period of time (usually a single class period) during which players at each table play as many matches as there is time for. The number of matches played each time will vary from week to week and table to table. A tournament round is usually done once each week.

At the end of each week's tournament round, the table assignments for the next week are determined as follows:

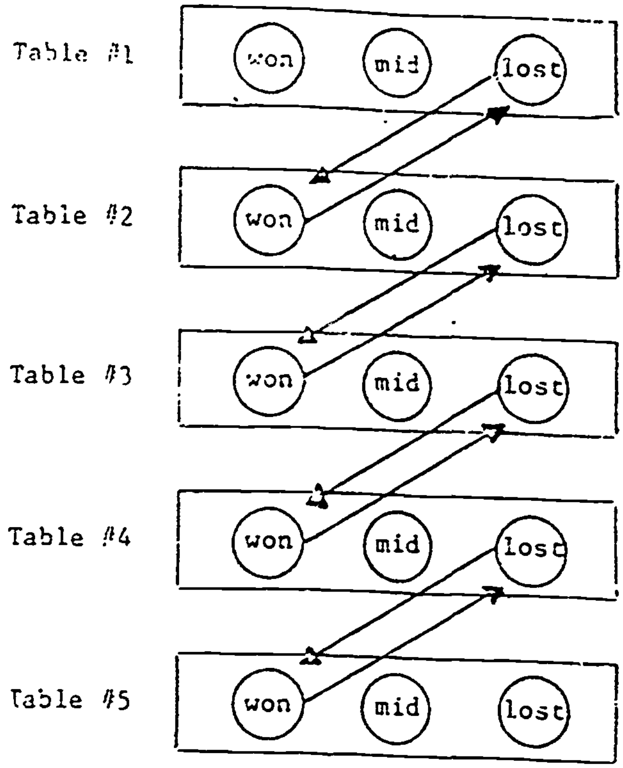
The winner at each table moves up one table.
The loser at each table moves down one table, and
The middle player stays at the same table (see EXAMPLE)

EXAMPLE: A class of 15 total students divided into 5 tables with 3 students at each table.

The winning and losing players have already been determined. The bumping for seating in next week's tournament is shown by the arrows:

won = winning player
mid = middle player
lost = losing player

The winning, losing, and middle players at each table are determined by comparing the total of the "raw" match scores earned by each player in the matches played at that table during the tournament round.



** Notice how at Table #1 (high ability), the winning and middle players both remain behind and only the losing player bumps down. At Table #5 (bottom table), the middle and losing players both remain at the same table and only the winning player moves up one table for next week's tournament.

NOTE: The "raw" match scores are only used to determine the table winners and losers. They are not used to calculate team standings or individual cumulative standing in the overall tournament.

See "Calculation of Team Points" on page describing Team Organization.

QUESTIONS

1) How do you end the tournament when the class period is over?

The teacher gives a 5 minute warning before the time the tournament round will end. Once the warning is given players should not begin any new matches. At the end of the 5 minute warning, the teacher calls a Teacher Force Out: all unfinished matches cease to make moves in the game. Each player submits a Solution that they feel is still possible. The results are scored like a normal Force-Out: those who have a correct Solution get 8 points, those who do not get 6 points.

Informal EQUATIONS Rules

You can learn to play EQUATIONS without reading the questions and answers in script. The script material concerns details of the game that can be ignored at first.

1. Q: How many people can play?
   A: Two or more. Three is probably best.

2. Q: What equipment is needed?
   A: Cubes and a playing mat. Do not use all the cubes in the box. Four of each color make a good beginners' game. The pros use six of each color.

3. Q: How does the game begin?
   A: One player rolls out the cubes. The symbols on their upward faces are called the "Resources." They are not changed during the game.
   The player who rolled the cubes builds a Goal by taking a cube (or cubes) from the Resources and putting it (or them) on the Goal line of the playing mat. The Goal is a number.

3A. Q: What are the rules about the Goal?
   A: At most five cubes may be used in building the Goal. Operation cubes may be used. If two operation cubes are used, grouping may be shown by spacing. Otherwise the Goal is interpreted according to the usual order of operations.

3B. Q: Can digits be next to each other in the Goal?
   A: Yes. 13 is an acceptable Goal. So is 4 - 726.

4. Q: How does play continue after the Goal is set?
   A: Players take turns; on his turn, a player takes one cube from Resources and puts it into the Forbidden, Permitted, or Required section of the playing mat. A cube cannot be moved after it is played.

4A. *Q: Can a player ever move more than one cube?*
   A: Yes. LWhenever he wishes, he may take a bonus move on his turn. He does so by saying "bonus", playing one cube into Forbidden and then playing one more cube wherever he likes.

4B. *Q: Can the Goal-setter bonus before he builds the Goal?*
   A: Yes.

5. Q: What are the reasons behind the moves?
   A: Each move has an effect on the solutions that are possible. Some moves work toward certain solutions, some moves wipe out certain solutions.

6. Q: What is a solution?
   A: A solution is an expression that is equal to the Goal. Resources are used to construct it. It must contain every resource that is in Required. It may contain any resource that is in Permitted. It must not contain any resource that is in Forbidden. It may also contain resources that have not yet been played.

6A. *Q: When are solutions built?*
   A: Players are secretly writing solutions all through the game. A player only shows a solution to the other players at the end of the game. He shows it in writing; he does not touch the cubes.

6B. *Q: May a player who is building a solution use grouping?*
   A: Yes. Because players write soltions rather than build them with the cubes, grouping is shown by using parentheses.

6C. *Q: Can parentheses be used to indicate multiplication with using a times sign?*
   A: No.

6D. *Q: Can a player put digits next to each other in a solution?*
   A: No. Example: When 13 is the Goal, 13 is not a solution.

6E. *Q: Can the minus sign be used to stand for negative property?*
   A: No. The minus sign can only be used to stand for the operation of subtraction.

6F. *Q: What does "*" stand for?*
   A: The asterisk or star of "" stands for "to the power of". Thus* $2*3 = 2^3 = 8$, *and* $3*2 = 3^2 = 9$.

6G. *Q: How is the root sign used?*
   A: Just like the ordinary root except that it is always preceded by a numeral that shows what root is being taken. Thus
   $2\sqrt{9} = \sqrt{9} = 3$ *and* $3\sqrt{8} = \sqrt[3]{8} = 2$.

7. Q: How does a player win?
   A: By getting into a challenge and being correct.

8. Q: Who can challenge and how?
   A: Anyone except the player who just moved. It is done by saying "challenge".

9. Q: Who is challenged?
   A: The player who just moved (the "mover").

10. Q: Why should you challenge?
   A: You should challenge for one of three reasons. Basically you challenge because
   1) you think that it will <u>never</u> be possibie to build a solution (all solutions have been wiped out); or because
   2) you know a solution that can be built right away (with just one more unplayed resource cube in addition to the appropriate cubes from the mat).
   3) You should also challenge if you think that mover could correctly have challenged the player who moved before him (which might even be you).
   When you challenge, you must tell your reason.

10A. Q: How might a player give a reason for his challenge?
   A: A player might say
   (1) "noway"
   (2) "one more cube"
   (3)"you should have challenged no way"
   or
   (4) you should have challenged one more cube".

11. Q: What happens after a challenge?
   A: The play of the game is over and the winner is determined by having the player who says a solution can be built try to write one. If he succeeds, he wins. Otherwise he loses.

12. Q: Who has to write a soution where challenger has said that it will never be possible to build one.
   A: The mover. Of course he has to use all the cubes in Required. He's allowed to use as many cubes as he likes from Permitted and from the unplayed Resources.

13. Q: Who has to write a soution when challenger has said that a solution can be built right away?
   A: Challenger. He has to use all the cubes in Required. He's allowed to use as many cubes as he likes from Permitted, but he's only allowed to use <u>one</u> of the unplayed Resource cubes.

13A. Q: Who has to write a soution if challenger says that mover should have challenged?
   A: That depends on <u>why</u> challenger says that mover should have challenged. If the challenger says that some earlier move allowed a soution to be built right away, the challenger has to show such a solution.

13B. *Q: After a challenge what should the other players do?*
A: All players other than challenger and mover have to join either challenger or mover. If you join a player who has to write a solution, you also have to write a solution in order to score.

13C *Q: What is the scoring after a challenge?*
A: A player who has to write a soution scores 2 if he writes one and 0 otherwise. A player who does not have to write a solution scores 2 if no one writes a solution and 0 otherwise.

13D. *Q: Is there any exception to the scoring in 13C?*
A: Yes. A player who joins a challenger only scores 1 for being correct (rather than 2).

14. Q: What happens if a player is in a bind where he either has to make it impossible ever to build any solution or allow a solution to be built right away.
A: A player in this position should make a move that allows a solution to be built right away. (We say he is "forced out"). However he should not be challenged for this move. Rather the next player should say "force out".

14A. *Q: When do most force-outs occur?*
A: When two unplayed cubes are left and both are necessary for all possible solutions.

14B. *Q: Suppose a player challenges rather than saying "force-out"?*
A: He will score 0. A challenger who says that a solution can be built right away has to be prepared to show that mover was not forced out. He does so by giving an OK alternative more that mover could have made. An OK alternative move is a move that does not allow a solution right away and does not wipe out all solutions.

15. Q: What happens after a player says "force-out"?
A: All players have to write a soution. They must use every cube in Required; they may use any cubes in Permitted, and they may use one unplayed cube from Resources.

16. Q: Who wins in a force-out?
A: A force-out is usually a tie game. Every player who writes a solution scores 1. All other score 0.

16A. Q: Is there any exception to 16?
A: Yes. If the player who said "force-out" doesn't write a solution, he scores -1.

Teacher's Notes or ˜ ˈorm: . EQUATIONS Rules
(The Rea: · Behina the Rules)

1. Inexperienced players may be more comfortable in a two-player game. A three-player game is usually more exciting because of the competition to be the challenger. A game with more than three players tends to be less interesting because each player has fewer opportunities to influence the game.

2. You can play a game using only the twelve red cubes -- or using eight red cubes and eight blue cubes. Such games are simpler because they do not involve the √ and * operations or the larger digits. However even young players can be introduced to the full spectrum of cubes. Symbols that are not understood can be played into Forbidden or rolled again, or even reinterpreted. For example, A√B could be taken to mean "the average of A and B".

3. After the cubes are rolled the players should be encouraged to sort them. The operation of sorting is in itself fruitful for some young players and the problem solving involved in the game is facilitated by an orderly arrangement of the resources. A reasonable way to sort is exemplified in the examples.
An unalterable Goal is one of the important psychological features of the game: After the Goal is set, the players are striving to find a variety of ways to "achieve" it.

3A. The order of operations is:
a) first, perform √ and * operations;
b) second, perform x and ÷ operations;
c) last, perform + and - operations.

When working out an expression with several operations on the same level, work from left to right. For example:

2 x 5 * 7 * 6 = 2 x [(5 * 7) * 6].

$2\sqrt{9}\sqrt{8} = (2\sqrt{9})\sqrt{8}$.

If a Goal-setter wishes to indicate 2(/9 /8) he spaces the cubes like this:

$2\ \sqrt{9}\sqrt{8}$.

4A. This rule was vitally important in the old form of the game in which a player could win by completing a solution. It is less important now but has been retained to give players an opportunity to have more effect on each turn. It also speeds up the game.

4B. This rule allows a Goal-setter to set a Goal of (for example) 4 even though there are two 4's in resources. He simply bonuses away one of the 4's before putting the other on the Goal line. If he did not bonus, he would probably be challenged for allowing a solution to be built right away.

5. Players have a variety of strategies open to them. Beginners will frequently move defensively: They try to avoid moves that can be correctly challenged; they forbid operations that they do not understand; they think of one Solution and play so that it will remain possible; they try to think of a second solution only when their original solution has been wiped out by the move of another player.

More experienced players look for many solutions. They try to figure out what solutions other players have in mind and to wipe them out (the solutions, not the players) if they can. They try to make moves that will precipitate mistaken challenges (for example, a move that makes it look as if all solutions are wiped out when a clever solution is still possible). Probably the ultimate strategy is to delibertaely make a move that can be correctly challenged in the expectation that the next player will move rather than challenge - thereby enabling you to challenge him for not having challenged you.

6A. During the play of the cubes, players are busy writing down their solutions. Of course they do not let the other players see what they are writing. An ample supply of scratch paper is a necessity for an EQUATIONS game.

6B. Of all the mathematical concepts that students gain from EQUATIONS, probably the first is the power of the parentheses. They also gain much experience in working out the combinatorial possibilities of a set of resources. For example, ask yourself how many different numbers you can make with

$$+ \div 2\ 3\ 4.$$

How about with $- \div 1\ 2\ 6$?

6C. The decision to exclude the use of place value in solutions was made in order to put more emphasis on the use of the six arithmetical operations. It also had the effect of simplifying the game: if place value were allowed the number of possible solutions would be vastly increased. A classroom teacher will of course revise these basic rules in any appropriate way to fit the curriculum. We would appreciate hearing about such revisions.

6D. This decision preserves a certain symmetry. all operations are binary and all solutions have the form: digit operation digit ... operation digit.

6E. The structure of the game necessitates an explicit symbol for exponentiation. (If no such symbol were used, the game would be much more complex because exponentiation would be available arbitrarily often). An explicit symbol for exponentiation is used in other contexts -- for example, in computer programming.

6F. See 6D. Notice also that the game encourages players to investigate roots with fractional or negative indices. Although such an investigation is not pursued in most textbooks, it can be mathematically fruitful. All the properties of the root operation follow from its definition as an inverse of star operation:

$$A\sqrt{B} = C \quad \text{if and only if} \quad C * A = B.$$

7. Inexperienced players frequently assume that winning can be achieved simply by finding a solution or simply by finding a very lengthy complicated solution. More experienced players understand that because of the challenge structure winning can be achieved only by knowing more about what solutions are possible than the other players know.

8. This rule has the effect that if there are more than two players, the attention of eve player is concentrated on the mat almost all the time: if a move can be correctly challenged, a scoring advantage goes to the challenger.

9. The mover has full responsibility for the game situation. Therefore when it is your turn, you need to be particularly careful in deciding whether to challenge.

10. Most students find it intuitively reasonable to challenge when they believe all solutions have been wiped out.

The second challenge -- that a solution can be built right away -- is less intuitively acceptable at first meeting. Its genesis lies in an earlier version of EQUATIONS. In this version a player could "go out' by completing a solution on his turn. This somewhat simpler game turned out to be unfair -- a Goal-setter could usually win -- and, worse, his winning strategy consisted of keeping the math trivial. Therefore the rules were changed so that a player could win by challenging that a solution could be completed (rather than by actually completing one).

The final challenge -- that mover should have challenged instead -- allows the game to be a very effective teaching tool. A teacher playing a student (or students) can deliberately make moves that allow an interesting solution at the appropriate level to be built with one unplayed resource. If the student challenges and wins, the teacher uses a somewhat more interesting solution and does the same thing in the next game. If the student moves instead of challenging, the teacher challenges the student for having missed the challenge, displays the solution, and in the next game again gives the student a chance to win by recognizing the existence of the same type of solution.

The official game rules describe challenges in a way that may appear very different from that used here (although logically they are equivalent). The point of view in the official rules is shifted to the mover who is understood to be making certain claims each time he moves. A move that violates a claim is called a "flub" and a challenger asserts that mover has flubbed in one of three ways.

The following page gives the usual presentation.

11, 12, 13. The formal game rules introduce the notion of "burden of proof", it being cast upon the players who assert the existence of a solution.

13A. A player who is correct and has joined a challenge is penalized in the scoring because he could correctly have challenged. See Remarks 1 and 8 above.

14. In the formal rules the word "force-out" is not used. Instead the player states "that a solution can be built with one more cube..." The "force-out" use is more common in practice.

Although the idea of the force-out complicates the game and sometimes seems difficult for students to understand, it is a necessary feature of EQUATIONS. Its purpose is to keep the game fair. It would be completely unfair to force a player to make a move that can be correctly challenged. Thus we introduce an exception to the rule that a player who allows a solution right away should be challenged. He should be challenged <u>only</u> if he had another move that did not wipe out all solutions and did not allow a solution right away.

Sample Game to go with the Informal EQUATIONS RULES

1. Dwight and John sit down to play EQUATIONS.

2. They take out four cubes of each color and the playing mat.

3. John rolls out the cubes and sorts them. The resources are:
   + - x ÷ 0 1 2 3 4 5 8 9
   \+ ÷ 1 3

   John puts the 9 cube on the Goal line.

4. Dwight plays a - to Forbidden.
   John plays a - to Required.
   The playing mat looks like this:

Resources	+ + x ÷ 0 1 1 2 3 3 4 5 8	
Forbidden ÷	Permitted	Required -
__________ Solution		= 9 Goal

5. Dwight is trying to keep the game simple. He has wiped out every Solution that contains two division signs.

   John is trying to make the game more complex. He has wiped out every solution that does not contain a minus sign. He has also begun to work toward solutions that do contain a minus sign.

6. Before Dwight's move some complicated solutions using two division signs were possible (for example, 8 + (3 ÷ 3) ÷ 1). After Dwight's move, a solution may contain at most one division sign. Some possible solutions are 8 + 1 and 8 + (3 - 3).

   John's move wipes out both 8 + 1 and 8 - (3 ÷ 3) because neither contains a minus sign. After John's move every solution has to contain a minus sign.

7,8,9,10,11,12. Dwight challenges John that his move has wiped out all solutions. Dwight cannot find any solution that contains a minus. John has to write a solution to win. John writes

3 x (4 - 1).

Of course he has to put in parentheses because 3 x 4 - 1 is not equal to 9. Dwight checks that John has used the resources properly and that his solution is equal to the Goal. John wins and Dwight loses.

Second Sample Game to go with Informal EQUATIONS Rules

Cindi, Dwight and John are playing.
The resources are: + - x ÷ 0 1 2 3 4
+ - x ÷ 0 1 2

Dwight sets a Goal of 4.
Players take turns until the mat looks like this:

Resources	x 2	
Forbidden	**Permitted**	**Required**
+ - x ÷ 0 1 + - ÷ 0 1	2 3	
______ Solution	=	4 Goal

14. It is John's turn. If he plays into Forbidden he will wipe out the one remaining solution: 2 x 2. Instead he plays the times sign into Required. Cindi says, "Force-Out". All three players write the solution 2 x 2 and the game ends in a tie.

14a. The resources are: + - x ÷ 0 1 2 3 4
+ - x ÷ 0 1 2

The goal is 4. The players have permitted - and required 0, 1, 2, + and - (just like example 2). The next player puts a + into Required. The mat looks like this:

Resources	- x 0 1 2 3 x	
Forbidden	**Permitted**	**Required**
	-	0 1 2 + - +
______ Solution	=	4 Goal

John challenges that the mover should have challenged (instead of moving) because a Solution could have been built right away. John has to write a solution that could have been built. He wins by writing 2 + 2 - 0 ÷ 1. Notice that John does not have to use the seond plus sign in Required.

Summary of Adventurous Rules

Rules that are Functions

1. Additive Inverse: AI(a) AI(0 - 1); AI(2/3) = -2/3
2. Arithmetic Mean: AM(a,b,...) AM(5,9) = 7; AM(5,5,9) = 19/3
3. Arithmetic Next: AN(a,b) AN(1,5) = 9; AN(0,0-1/8) = -1/4
4. Arithmetic Term: AT(a,b,c) AT(1,3,5) = 13; AT(0,0 - 2,6) = -10
5. Base Ten Log: LT(a) LT(2 x 5) = 1; LT(1) = 0; LT(7 + 3)*5) = 5
6. Combinations: C(a,b) C(5,3) = 10; C(5,1) = 5; C(5,0) = 1; C(5,4) = 5
7. Factorial: a! 5! = 120; 3! = 6; 1! = 1
8. General Log: aLb 2L8 = 3; 8L2 = 1/3; 2L(1/2) = -1
9. General Rounding: R(a,b) R(8 x 3,5) = 24; R(0 - 2,3) = -3
10. Geometric Mean: GM(a,b,...) GM(5,9) = 2 45; GM(2,3,1) = 3 6
11. Geometric Next: GN(a,b) GN(1,5) = 25; GN(3,0 - 1/3) = 1/27
12. Geometric Term: GT(a,b,c) GT(1,2,6) = 32; GT(0,5,9) = 0
13. Greatest Common Factor: GCF(a,b) GCF(6,8) = 2; GCF(7,1) = 1; GCF(1/2,2) is undefined
14. Greatest Integer: [a] [5 + 9/5] = 6; [0 - 1/2] = -1; [2 2] = 1
15. Least Common Multiple: LCM(a,b) LCM(6,8) = 24; LCM(7,1) = 7; LCM(1/2,2) is undefined
16. Number of Factors: FA(a) FA(8) = 4; FA(7) = 2; FA(1/2) is undefined
17. Percent: a% 5% = .05; 5%% = .0005
18. Permutations: P(a,b) P(5,3) = 60; P(5,1) = 5; P(5,0) = 1; P(5,4) = 120
19. Rational Number: RAT(a) RAT(3/7) = 1; RAT(0) = 1; RAT(2 2) = 0
20. Reciprocal: RE(a) RE(0 - 1) = -1; RE(2/3) = 3/2
21. Round to Multiple of Ten: RT(a) RT(6 x 8) = 50; RT(0 - 1/2) = 0; RT(5*4) = 630
22. Round to Multiple of Power of Ten: RPT(a,b) RT(9*2,1) = 80; RT(2*9,3) = 1000
23. Round to Whole: RW(a) RW(5 + 9/5) = 7; RW(0 - 1/2) = 0; RW(2 2) = 1
24. "Scientific" Notation: SN(a,b) SN(7,2) = 700; SN(3/2,0 - 3) = .0015
25. Smallest Prime: PR(a) PR(8) = 11; PR(7) = 7; PR(1/2) = 2
26. Solution of Linear Equation: SL(a,b,c) SL(2,2,3) = 1/2; SL(9,5,5) = 0
27. Substitution in Linear Expression: V(a,b,c) V(2,2,3) = 7; V(9,5,5) = 50

Goal Rules

28. Base n (Base six ... Base eleven, Base twelve)
    Additional Resources: Base eleven: T,T; Base twelve: T,T,E,E
29. Decimal Goal: (.) 1 + .5; 1.5; .15, but not: .50; 0.5; 1. or 01.5
30. Goal Constraint: Prime Goal; Negative Goal; O < Goal , 1; Goal > 100; Non-Integer Goal
31. Goal Pair: (Goal Pair) Additional Resources: y,y,y*2, !!, ||, =
32. Identity Rule: (Identity Rule) Additional Resources: y,y,y,y,y^2,y^2
33. Interval Goal: (,) [,] [,) [2,5]; (12,34); [1,2); (2,5]
34. Mixed Numeral Goal: (Mixed Numeral) 1 2/3; 1 2/4; but not 1 3/3 or 1 4/3
35. Percent of Goal: (n%) 5%; 100%
36. Repeating Decimal Goal: (.) .3; .30; .03; .03
37. Significant Figure: (Significant Figure)

38. Son of Interval Goal: (Son of Interval Goal)
Additional Resources: y,y,y*2, ||, ≤, ≤, <, <
39. Substitution Rule: (Substitution Rule) Additional Resources: y,y,y2
40. Unique Goal: (Unique Goal) Additional Resources: y,y,y*2 =

Other Rules

41. Extra Minus Signs: (-)
42. Extra Division Signs: (-)
43. Extra Exponentiation Signs: (*)

APPENDIX A: List of Suggested Adventurous Rules

Rules that Involve Functions
Format of Rules

Name of Rule: Symbols that indicate the function and its argument(s) and a description.
A significant attribute of the function (sometimes)
Example(s) of function
Domain of the function (types of numbers that variables range over)
Comment(s) (sometimes)

Additive Inverse: AI(a) indicates the additive inverse of a.
So, AI(a) = -a
Examples: AI(0 - 1) = 1; AI(2/3) = -2/3.
Domain: a can be any real number.

Arithmetic Mean: AM(a,b,...,z) indicates the arithmetic mean of the set of arguments.
So, AM(a,b,...,z) = (a + b +...+ z)/n where z is the nth argument.
Examples: AM(5,9) = 7 because (5 + 9)/2 = 7; AM(5,5,9) = 19/3;
AM(0,1/2,1,2/1) = 3/4.
Domain: a and b and ...z can be any real numbers.

Arithmetic Next: AN(a,b) indicates the next term in the arithmetic sequence beginning a,b.
So, AN(a,b) = b + common difference.
Examples: AN(1,5) = 9 because 9 is the next term in the arithmetic sequence that begins with 1,5; AN(0.0 - 1/8) = -1/4.
Domain: a and b can be any real numbers.

Arithmetic Term: AT(a,b,c) indicates the cth term of the arithmetic sequence with first term a and common difference b.
Examples: AT(1,3,5) = 13 because 13 is the 5th term in the arithmetic sequence 1, 4, 7, 10, 13; AT(0,0 - 2,6) = -10.
Domain: c must be a natural number. a and b can be any real numbers.

Base Ten Log: LT(a) indicates the logarithym of a to the base ten.
So, LT(a) = $Log_{10}a$, which equals c if and only if 10*c = a.
Examples: LT(2 x 5) = 1 because 10*1 = 10; LT(1) = 0; LT((7 + 3)*5) = 5.
Domain: a can be any positive real number.

Combinations: C(a,b) indicates the number of combinations of a things taken b at a time.
So, C(a,b) = a!{(a - b)!xb!}.
Examples: C(5,3) = 5!!(2! x 3!) = 10; C(5,1) = 5; C(5,0) = 1; C(5,4) = 5.
Domain: a is a natural number. b is a non-negative integer.

Factorial: a! indicates a factorial. So, a! = a x (a - 1) x (a - 2) x...x 3 x 2 x 1, except that 0! = 1.
Examples: 5! = 5 x 4 x 3 x 2 x 1 = 120; 3! = 6; 1! = 1.
Domain: a can be any non-negative integer.

Fractional Part: FP(a) indicates the non-integer part of a. So, FP(a) = a - [a] (see Greatest Integer below).
Examples: FP(5 + 9/5) = 4/5; FP(0 - 3/4) = 1/4; FP(5) = 0.
Domain: a can be any real number.

Suggested Adventurous Rules

General Log: bLa indicates the logarithym of a to the base b.
So, bLa = $Log_b a$, which equals c if and only if b*c = a.
Examples: 2L8 = 3 because 2*3 = 8. 8L2 = 1/3; 2L(1/2) = -1.
Domain: a can be any positive number except 1. b can be any positive number.

General Rounding: R(a,b) indicates a rounded to the nearest multiple of b. If two multiples of b are equally near, round up.
Examples: R(8 x 3,5) = 25 because 25 is the multiple of 5 that is nearest to 24; R(0 - 2.3) = -3.
Domain: a can be any real number.
Comment: b should be restricted to integers when this rule is first introduced. Later, the domain of b can be extended to all real numbers. (For example, the multiples of 1/2 are 0, 1/2, -1/2, 1, -1, 3/2, -3/2, 2, -2, 5/2, etc.)

Geometric Mean: GM(a,b,...) indicates the geometric mean of the set of arguments.
So, GM(a,b,...,z) = n (a x b x...x z) where z is the nth argument.
Examples: GM(5,9) = 2; (5 x 9) = 2 45; GM(2,3,1) = 3 6.
Domain: a and b and ... can be any non-negative numbers.

Geometric Next: GN(a,b) indicates the next term in the geometric sequence beginning a,b.
Examples: GN(1,5) = 25 because 25 is the next term in the geometric sequence that begins with 1, 5; GN(3.0 - 1/3) = 1/27; GN(0,b) = GN(a,0) = 0.
Domain: a and b can be any rean numbers.

Geometric Term: GT(a,b,c) indicates the cth term of the geometric sequence with first term a and common ration b.
Examples: GT(1,2,6) = 32 because 32 is the 6th term in the geometric sequence 1, 2, 4, 8, 16, 32; GT(0,5,9) = 0; GT(4,3,3) = 36.
Domain: c must be a natural number. a and b can be any real numbers.

Greatest Common Factor: GCF(a,b) indicates the greatest common factor of a and b.
Examples: GCF(36,48) = 12 because 12 is a factor of both 36 and 48, and no number greater than 12 is a factor of both; GCF(6,8) = 2; GCF(7,1) = 1; GCF(1/2,2) is undefined.
Domain: a and b must be natural numbers.

Greatest Integer: [a] indicates the greatest integer less than or equal to a.
Examples: [5 + 9/5] = 6 because 5 + 9/5 = 6.8 and 6 is the greatest integer that is less than or equal to 6.8; [0 - 1/2] = -1; [2 2] = 4.
Domain: a can be any real number.

Least Common Multiple: LCM(a,b) indicates the least common multiple of a and b.
Examples: LCM(6,8) = 24 because 24 is a multiple of both 6 and 8 and no number less than 24 is a multiple of both; LCM(7,1) = 7; LCM(1/2,2) is undefined.
Domain: a and b must be natural numbers.

Number of Factors: FA(a) indicates the number of positive factors of a.
Examples: FA(12) = 6 because the set of positive factors of 12 is {1,2,3,4,6,12}; FA(7) = 2; FA(1/2) is undefined.
Domain: a must be a natural number.

Percent: a% indicates the percentage that a is of 100.
So, a% = a/100.
Examples: 5% = 5/100 = .05; 5%% = (5/100)/100 = .0005.
Domain: a can be any real number.

Permutations: P(a,b) indicates the number of permutations of a things taken b at a time.
So, P(a,b) = a!/(a - b)!.
Examples: P(5,3) = 5!/(5 - 3)! = 60; P(5,1) = 5; P(5,0) = 1; P(5,4) = 120.
Domain: a is a natural number. b is a non-negative integer.

Rational Number: RAT(a) indicates whether or not a is rational.
RAT(a) = 1 if a is rational; 0 if a is irrational.
Examples: RAT(3/7) = 1 because 3/7 is rational; RAT(0) = 1; RAT(2 2) = 0.
Domain: a can be any real number.

Reciprocal: RE(a) indicates the reciprocal of a.
So, RE(a) = 1/a.
Examples: RE(0 - 1) = -1; Re(2/3) = 3/2.
Domain: a is a non-zero real number.

Round to Multiple of Power of 10: RPT(a,b) indicates a rounded to the nearest number that is a multiple of the bth power of 10. If two such numbers are equally near, round up.
Examples: RPT(9*2,1) = 80 because 80 is the multiple of 10 that is nearest to 9*2; RPT(2*9,3) = 1000 because 1000 is the multiple of 10*3 that is nearest to 2*9 (which is 512).
Domain: a can be any real number. b should be restricted to non-negative integers when the rule is first introduced. Later, the domain of b can be extended to all integers.

Round to Multiple of Ten: RT(a) indicates a rounded to the nearest multiple of 10. If two such multiples are equally near, round up.
Examples: RT(6 x 8) = 50 because 50 is the multiple of 10 that is nearest to 6 x 8; RT(0 - 1/2) = 0; RT(5*4) = 630; RT(1 - 4*2) = -10.
Domain: a can be any real number.
Comment: The teache. can generalize this rule. For example, RC(a) might b a rounded to the nearest multiple of 100; RTE(a) might be a rounded to the nearest tenth, etc.

Round to Whole: RW(a) indicates a rounded to the nearest whole number. If two whole numbers are equally near to a, round up.
Examples: RW(5 + 9/5) = 7 because 5 + 9/5 = 6.8 and 7 is the nearest whole number to 6.8; RW(0 - 1/2) = 0; RW(2 2) = 1.
Domain: a can be any real number.

"Scientific" Notation: SN(a,b) indicates a generalization of true scientific notation.
By definition, SN(a,b) = a x 10*b.
Examples: SN(7,2) = 7 x 10*2 = 700; SN(3/2,0 - 3) = .0015; SN(2 2,1/2) = 2 20.
Domain: In this generalization of true scientific notation, a and b can be any real numbers. True scientific notation results if the domains of a and b are restricted so that $1 \leq a < 10$ and b is an integer.

Smallest Prime: PR(a) indicates the smallest prime greater than or equal to a.
Examples: PR(8) = 11 because 11 is the smallest prime number greater than or equal to 8; PR(7) = 7; PR(1/2) = 2.
Domain: a can be any real number.

Solution of Linear Equation: SL(a,b,c) indicates the solution of the equation: ay + b = c.
Examples: SL(2,4,3) = -1/2 because 2y + 4 = 3 when y = -1/2; SL(9,5,5) = 0.
Domain: a can be any non-zero real number. b and c can be any real numbers.

Substitution in Linear Expression: V(a,b,c) indicates the value of ay + b when y = c.
Examples: V(2,3,4) = 11 because 2y + 3 = 11 when y = 4; V(9,5,5) = 50.
Domain: a,b, and c can be any real numbers.

Goal Rules

The indicators for Goal rules may be written in either the Goal Obligation(s) area of the Goal Modifier(s) area of the Augmented Playing Mat, or in both, except when specified otherwise in the particular Goal Rule.

Base n: $[\ ]_n$ indicates Base n.

$[\ ]_{six}$, $[\ ]_{seven}$,...,$[\ ]_{twelve}$ indicates Base six, Base seven,..., Base twelve.
The Goal is interpreted as a base n numeral where n can be 6 or 7 or... or 11 or 12. The value of n is specified in the Goal Obligation(s) or Goal Modifier(s) section. Two-digit numerals are permitted in the Solution. Solutions are interpreted in the same base as the Goal.

Suggested Additional Resources: If n is eleven: T, T. If n is twelve: T, T, E, E.
"T" stands for ten and "E" stands for eleven.

Examples: In Base twelve, if the Goal is $1T_{twelve}$, some possible solutions are:

$[20 - 2]_{twelve}$; $[10 + T]_{twelve}$; $[1E - 1]_{twelve}$.

If Base six is the only Goal Modifier and the Goal is 17, then the Goal must be interpreted in Base ten. A Goal of $[17]_{six}$ is not okay.

Decimal Goal (.): One or more decimal points may be used in setting or modifying the Goal when a decimal point is written in the Goal Obligation(s) area or in the Goal Modifier(s) area, respectively. Note, however, if the deletion of a decimal point or a zero does not change the value of the Goal, then the Goal is not okay. Superfluous decimal points or zeros are not alowed.

If a decimal point is written in the Goal Obligation(s) section, the Goal setter may use one or more decimal points and must use at least one decimal point in the Goal. If a decimal point is written in the Goal Modifier(s) section, a Solution writer may write one or more decimal points in the Goal expression where desired as shown in the examples below.

Examples:

If a decimal point is written in the Goal Obligation(s) area, and the numerals 1,2,4 are available in the Resources, the following would be okay Goals: .142; 1.42; or 14.2, however, 142 would not be okay because it does not contain the required decimal point. Expressions like 142, are never okay Goals because their value would be unchanged if the decimal point were deleted.

Description of a Training Workshop for
The Instructional Gaming Program

The following schedule outlines a two-day workshop that is a highly successful way to train a pilot group of teachers to use the EQUATIONS game as a regular part of the curriculum. The workshop uses 3-4 demonstration classes to give teachers the chance to see the program work with their own students. The schedule also includes extensive time for the teachers to meet together so they can practice the game themselves, discuss classroom tournament methods and learn ways to incorporate the usual curriculum objectives into the gaming sessions.

To get the benefit of working with several demonstration classes, it is important that all teachers involved in the workshop are released from teaching duties for the full two days. The demonstration classes should be selected from among a few of the teacher's regular classes. When teachers from several buildings are involved it is usually advisable to try to have at least one demonstration class in each building. It is not necessary for every teacher in the workshop to have one of her own classes selected as a demonstration class. Clearly, those who do will have a slight head start with their students, but the repetition built into the two days gives most teachers the experience and confidence needed to get the program started with their own classes. This single two-day workshop could be used to train a group of 15 or more teachers.

Schedule of Two-Day Workshop

MORNING	DAY 1	DAY 2
Period 1 (45-60 min.)*	Introduce EQUATIONS rules to demonstration class #1	Classroom tournament with demonstration class #1
Period 2 (45-6) min.)*	Introduce EQUATIONS rules to demonstration class #2	Classroom tournament with demonstration class #2
Period 3 (45-60 min.)*	Introduce EQUATIONS rules to demonstration class #3	Classroom tournament with demonstration class #3

*Classroom sessions at the Jr. High level should be a normal class period (hopefully at least 45 minutes). Sessions at the elementary level should be at least 60 minutes long, and as long as 75 minutes with very young or slow students.

LUNCH
AFTERNOON

(2-3 hours)	TEACHER'S MEETING: review game rules, practice playing, officiating and discuss classroom methods and tournament structure. Teachers with demonstration classes must organize their classes into teams and tournament tables for Day 2 sessions.	TEACHER'S MEETING: Go results of tournaments, review scoring methods and newsletters, discuss advanced EQUATIONS rules and other methods for implementing specific curriculum objectives with EQUATIONS, demonstrate the DIG Math Computer program (if included in pilot).

Selecting Personnel: Grade Level and Coordination

The most appropriate level for the training workshop is grades 6-8. If elementary teachers are to be trained, grades 4-5 would be best. The instructional Gaming Program can be successfully extended to grades four through high school.

The district should select at least one or two people in the pilot group who will be able to expand the program once they gain enough experience. An administrative coordinator or designated resource person (without extensive teaching duties) should be the primary coordinator of the program for the district. The WFF'N PROOF consultant will remain in regular contact with the coordinator to guide the expansion of the program and handle questions from the teachers as they arise.

Training teachers to use a completely new basis for a curriculum naturally involves more than a two-day workshop. The WFF'N PROOF consultant will be available through regular phone contact to provide needed support and guidance. Where feasible, it is useful to plan a second one-day session with the consultant and teachers approximately six weeks after the initial implementation. By this time, the teachers' classes should be about ready to start using advanced scoring methods and game rules that focus the gaming activity on specific curriculum objectives.

The DIG Math Computer Program

The DIG Math Computer program is a valuable tool for training teachers and accelerating the use of advanced math concepts by students in classroom tournaments. However, the computer is supplementary and certainly not necessary to have a successful EQUATIONS program. The EQUATIONS board games and classroom tournaments remain the most cost effective way to real all students.

Cost of a Training Program

The cost of the initial two-day workshop depends on the number of classrooms outfitted with games, whether the computer program is used, and the distance travelled by the WFF'N PROOF consultant.

MATERIALS COST: EQUATIONS games

To outfit a classroom for a tournament, one game is needed for every three students in the class. Assuming an average class size of 30 students, this means that each classroom set requires 10 games. The retail price of an EQUATIONS game is $12.00. School purchasing twelve or more games will receive a 10% discount on retail price.

Cost per classroom outfitted: 10 EQUATIONS games x $10.80/game = $108.00 per class.

NOTE: A complete classroom set of 10 EQUATIONS games may be shared between several teachers on different days within the same building. If teachers from different buildings are involved in the training, the district should plan to outfit each building with a minimum of one complete classroom set. It is best for each teacher to have one complete classroom set.

CONSULTING COST

The standard cost for consulting is $150.00 per day spent in the district plus reimbursement for travel and lodging expenses. All efforts will be made to select the least expensive travel methods and lodging.

EXAMPLES

#1 1 classroom outfitted
District driving distance from Detroit area

Materials: 1 x $108.00	= $108
Consulting: 2 days @ $150/day	= $300
Driving: .15/mile (600 mi rnd trip)	= $ 90
Lodging: motel, 2 nights	= $ 60
TOTAL	$558

#2 3 classrooms outfitted
District in New York Metro area

Materials: 3 x $108.00	= $324
Consulting: 2 x $150	= $300
Travel: airfare-Detroit/NY	= $160
Lodging: motel, 2 nights	= $ 80
TOTAL	$864

NOTE: For additional cost of DIG Math Computer program see DIG Math Pricing Information sheet.

EVALUATION

PURPOSE

The evaluation of the MSU Honors Teacher Workshop 1987 has several purposes. It aims:

1. to detect the participant teachers' perception of their classroom teaching before and after the workshop;

2. to detect relationship and patterns in the participant teachers' confidence in solving, confidence in teaching and level of performance in twenty tasks related to the mathematical content presented by the five MGMP instructional units, both before and after the workshop;

3. to detect the participant teachers' perception of the workshop itself.

INSTRUMENTATION

Questionnaires were given to 24 participating teachers in the first and last 45 minute sessions of the workshop. In order to assure respondent anonimity, but still allow for pre-post matching of answers, each teacher assumed a pseudo-name that was systematically used on each questionnaire.

1. Questions on teachers' perception of their classroom teaching.

This questionnaire was similar to the one that was used for the first Honors Teacher Workshop. (Friedlander, 1985). The teachers were presented with a list of mathematical topics and teaching strategies. The curriculum topics included arithmetic, number theory, geometry, spatial visualization, probability, measurement, algebra, use of calculators and computers and history of mathematics. The teaching styles and strategies included using drill and practice, short problems, and complex problems or projects; whole class instruction, group work, seat work (same assignment for all) and individualized work; posing open-ended challenges, gathering and organizing student responses and encouraging analysis and generalization; using concrete manipulatives, worksheets and games. The teachers were asked at the beginning of the workshop about their own frequency in teaching/using the above topics and strategies and at the end about their planned frequency in teaching/using them in their classes. The answer options varied on a 5-level scale from "very frequently" to "never" and were later graded accordingly from 5 to 1.

A complete version of the post workshop questions is presented in Appendix A.

Since this instrument can detect only planned change in teaching but not change in practice, we administered another questionnaire: The Middle Grades Mathematics Project Teaching Style Inventory, to the teachers at the beginning of the workshop and a similar Student Inventory to their students at the beginning of the school year 1987-1988 (see Appendix D). At the end of the school year we will administer the questionnaires again. This will help us to detect practical changes in teaching style. Reports on these findings will be prepared and published elsewhere.

2. Questions on twenty MGMP unit related tasks.

This questionnaire which was identical to the one that was used at the first Honors Teacher Workshop was adopted from an evaluation study of a summer workshop for junior high mathematics teachers in Israel (The Weizmann Institute, Rehovot) and was originally designed by Fresko and Ben Haim (1984). The questionnaire (Appendix B) presents the teachers with four mathematical problems for each one of the five MGMP instructional units: Factors and Multiples (number theory), The Mouse and The Elephant (area and volume growth), Probability, Similarity and Spatial Visualization. For each one of the 20 problems the teachers were asked (1) to grade his/her confidence in solving the item on a four-level scale varying from "I am positive that I can solve it" to "I am not familiar at all with the topic" (graded from 4 to 1); (2) to grade his/her confidence in teaching the item on a three-level scale varying from "I am certain that I could teach it" to "I am not confident that I could teach it" (graded from 3 to 1); and finally (3) to actually solve the items (graded 0/1 for wrong/right solution).

The teachers were presented with the same twenty problems before and after the workshop.

3. Questions on teachers' perceptions of the workshop.

The participant teachers' evaluation of the workshop was determined by their post workshop reactions to the question "did the workshop adequately deal with the topic for your purposes?" for each one of the curriculum topics and teaching strategies from the part on teachers' perception of their classroom teaching. The answer options, again, varied on a 5-level scale from "definitely yes" to "definitely no" and were later graded accordingly from 5 to 1.

A complete version of the questions is presented in Appendix A.

MAIN FINDINGS AND DISCUSSIONS

Graphical representations of means are used in this section to describe patterns in the participant data gathered before the workshop, and to detect changes that can be attributed to the influence of the workshop. Complete tables of means and standard deviations may be found in Appendix C. Correlation, and t-test analyses were also occasionally employed to further clarify the data.

The findings are organized in three parts in respect to the three purposes of the evaluation. The first part relates to the participants' perception of their own teaching repertoire. The second relates to the twenty MGMP unit related mathematical tasks. The third part includes the particpants' perception of the workshop's contribution.

1. Teaching repertoire: Mathematical topics and teaching strategies.

a) Content. The distribution of different mathematical topics in the teachers' pre workshop class curriculum and the effects of the workshop on their planned curriculum with regard to the use of drill and practice, short problems, and complex problems or projects is presented in Figure 1. The arrow extends from the pre to the post score.

Table C.1 (in Appendix C) shows means and standard deviations of teaching distribution. The results indicate that spatial visualization, statistics and probability were taught less frequently than the other topics. Their teaching frequencies were between "seldom" and "sometimes". However, as a result of the workshop, the teachers plan a large increase in the teaching of these topics. The planned teaching frequencies of these topics, as well as the others, are more than "frequently". This planned change can be attributed to the influence of the workshop since spatial visualization, statistics and probability are among the subjects that were dealt with extensively in the workshop. Measurement, number theory and geometry were also dealt with extensively but had only moderate planned change in teaching frequency, perhaps because their pre workshop frequency was higher.

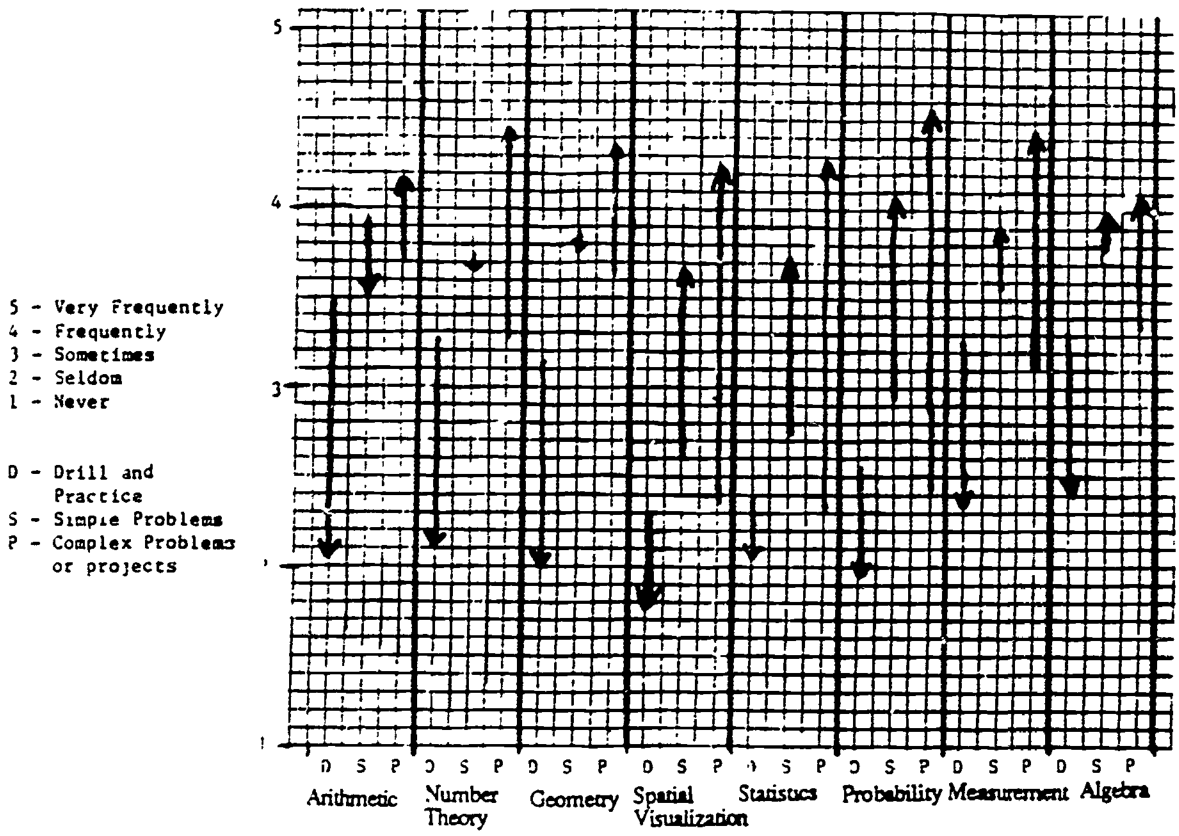


Figure 1. Teaching and planned teaching frequency

b) Strategies. A strong influence on teaching styles and strategies recommended in the workshop may be observed from Figure 1. Before the workshop, the frequencies of using drill and practice, and complex problems or projects were about the same, both a little bit less than the use of simple problems. After the workshop the participants plan to use much less drill and practice and much more complex problems or projects in each one of the mathematical topics.

The graph presented in Figure 2 indicates the frequency of use of different teaching styles and strategies before and after the workshop. Workshop's influence on planned changes in recommended teaching styles and strategies may be observed in the graph. Participant mean scores, standard deviations, and results of t-tests for pre-post workshop differences of teachers' reaction to teaching frequency with regard to teaching strategies may be found in Table C.2 (Appendix C).

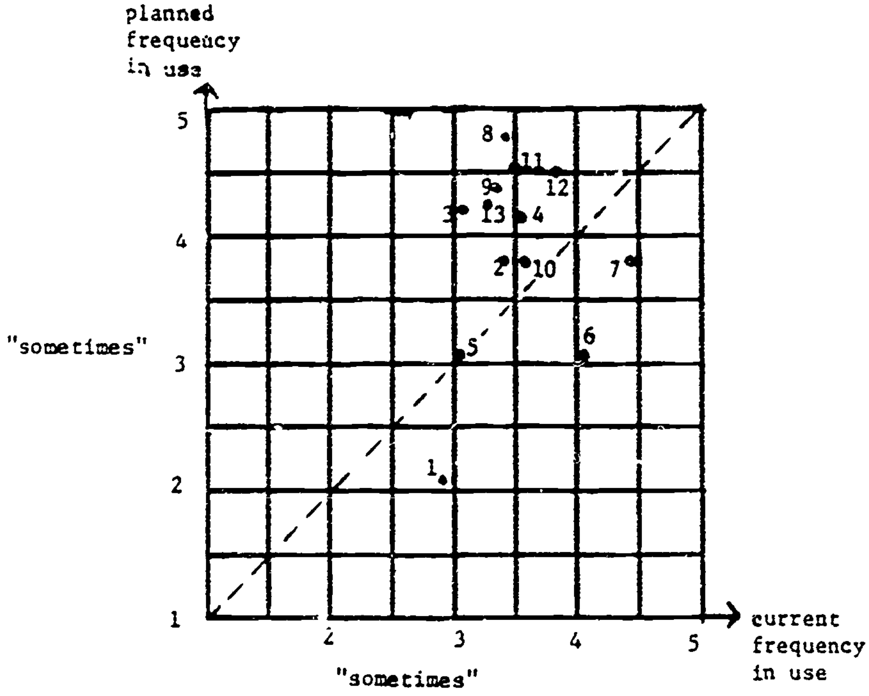


Figure 2. Frequency of using (1) drill and practice exercises, (2) short word problems, (3) projects, (4) group work, (5) individualized work, (6) seatwork, (7) whole class instruction, (8) concrete manipulatives, (9) games, (10) worksheets, (11) open challenges, (12) generalizations and (13) organizing student responses before and after the workshop.

The participants plan to increase the use of (3) complex projects and the use of (2) short problems in teaching and to decrease the use of (1) drill and practice. The most substantial planned changes are in the use of complex projects and drill and practice, but all three planned changes are statistically significant.

They also plan to increase the use of (4) group work in class and to decrease the use of (6) seatwork and (7) whole class instruction. They do not plan to make any changes in the frequency of using (5) individualized work.

The participant teachers plan to increase the use of (8) concrete manipulatives and (9) games in their classes. They do not plan to make changes in the frequency of using(10) worksheets.

They also plan to increase the use of (11) open-ended challenges, (13) organizing student responses and (12) encouraging generalizations.

c) Miscellaneous. The participants plan to increase the use of calculators as a means for solving complex problems. This fits the trend for increased use of complex projects indicated earlier. For a complete summary of teachers responses to teaching frequency with regard to different uses of calculators, computers and history of mathematics, see Table C.3 (Appendix C).

2. The twenty MGMP mathematical tasks.

About half of the workshop was dedicated to the five MGMP units: Factors and Multiples, Mouse and Elephant, Probability, Similarity, and Spatial Visualization. Therefore, the participants' level of performance on related mathematical problems, their confidence in solving, and teaching them are of particular interest. A complete list of participant scores and an item analysis may be found in Tables C.4 and C.5 (Appendix C).

All three group averages increased after the workshop. These results are presented in Table 1.

Table 1 - Participant mean scores and results of t-tests for pre-post workshop differences

	Mean Scores Pre	Mean Scores Post	t Value	D.F.	p <
Level of performance(1)	14.35 (72%)	16.83 (84%)	4.97	22	0.001
Confidence in solving(2)	3.72	3.94	4.96	22	0.001
Confidence in teaching(3)	2.66	2.84	2.79	22	0.05

(1) 20 problems on a scale of 0/1 for wrong/right
(2) on a scale of 1 to 4
(3) on a scale of 1 to 3

Relationships among level of performance, confidence in solving and confidence in teaching the twenty MGMP questions both before and after the workshop are presented in Table 2.

Table 2 - Pearson correlation coefficients among level of performance, confidence in solving and confidence in teaching on the twenty MGMP questions

	Conf. in solving pre(1)	Conf. in teaching pre(2)	Level of performance pre(3)	Conf. in solving post(4)	Conf. in teaching post(5)	Level of performance post(6)
(2)	* 0.72					
(3)	* 0.32	0.09				
(4)	* 0.21	* 0.48	0.06			
(5)	0.20	* 0.44	* 0.56	0.37		
(6)	0.23	0.36	* 0.63	0.29	* 0.64	

* Significant at = 0.05

Interesting relationships may be indicated from the correlation coefficients at 0.05 significance level. For example, before the workshop, there was a significant linear relationship between confidence in solving and confidence in teaching (i.e., teachers who were more confident that they can solve a problem were also more confident that they can teach it, and vice versa). But after the workshop, this relationship could not be indicated. This might be because the teachers realized that being able to solve a problem is not sufficient to being able to teach it to others. Another interesting finding is that no significant linear relationship between confidence in solving and actual ability to solve could be detected before or after the workshop.

The twenty MGMP mathematical tasks included four tasks for each one of the five MGMP units. Analysis of the relationships among participants' level of performance, confidence in solving, and confidence in teaching before and after the workshop, for each one of the units may clarify the results. Mean scores, standard deviations and results of t-tests for pre-post workshop differences of teachers' reaction and performance on the five MGMP unit problems are presented in Table C.6 (Appendix C).

Graphical representations of these means are presented in Figure 3.

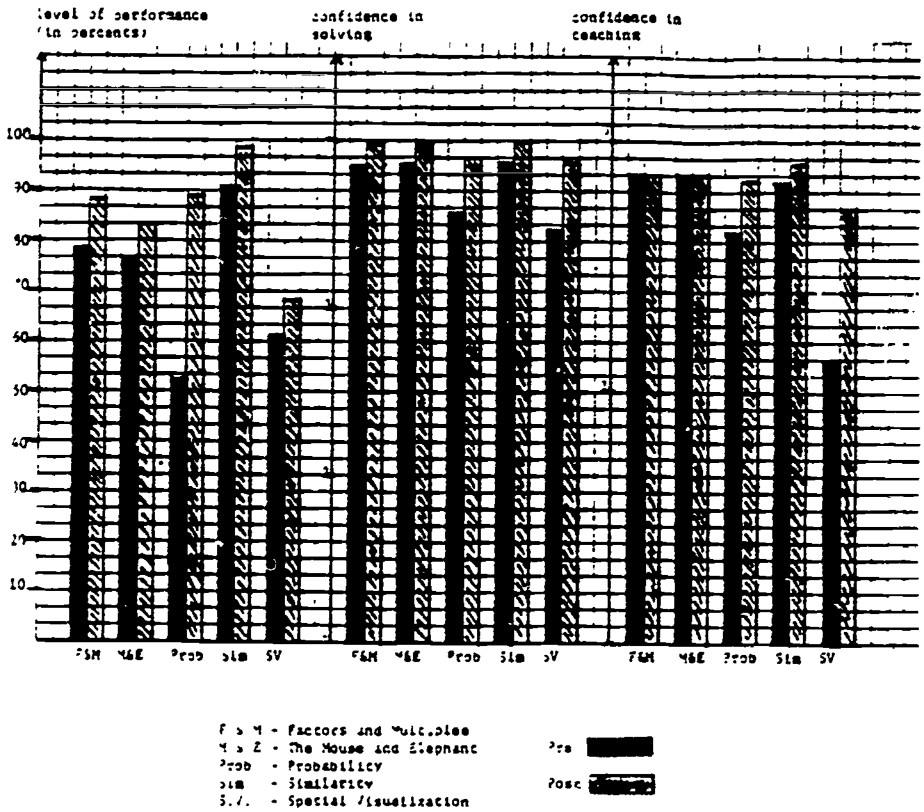


Figure 3. Pre-Post Workshop differences of teachers' level of performance, confidence in solving and confidence in teaching on the five MGMP unit problems.

The weakest topic before the workshop was probability (an average level of performance of 52%). The strongest topic was similarity (91%). The most significant change in level of solving the problems also occurred in probability (post average of 88%), while in similarity and spatial visualization there was not any significant change. (Performance on similarity was high before the workshop - 91%, and remained low on spatial visualization even after the workshop - 67%). Although the teachers were pretty confident in their ability of solving problems in each one of the topics before the workshop, there was a statistically significant improvement in their confidence in each one of the topics.

The teachers were also confident in their ability to teach the problems in each of the topics but spatial visualization, before the workshop. It is interesting to note that even though they scored very low on probability tasks they were confident in their ability to teach them. After the workshop this was corrected by their huge improvement in actual solving probability tasks. At the same time their performance on spatial visualization tasks was not improved at all during the workshop and remained low, but their confidence in teaching these tasks, which was appropriately low, became too high. Their confidence in solving these tasks was also improved. Figure 3 indicates a general tendency to overestimate one's own ability to solve the problems.

3. Teacher perception of the workshop

The teachers had high expectations (between 4 and 5 on a scale of 5) from the workshop with regard to the following teaching strategies: use of complex problems or projects, group work, use of concrete manipulatives, games, open-ended challenges, organizing student responses, and encouraging generalizations. (For a detailed list see Table C.7, Appendix C). These are exactly the teaching strategies that were considered important by the workshop organizers and were emphasized during the workshop.

The workshop evaluation, as may be found in Tables C.7 and C.8 was very positive. The teachers rated very high the workshop's treatment of the use of different teaching strategies as well of the different topics.

SUMMARY

The 1987 Honors Teacher Workshop, as the previous one in 1984 (Friedlander, 1985), was considered to be successful by its participants. The 1987 HTW was especially successful in causing a growth in the planned frequency in teaching spatial visualization, statistics and probability - topics that according to the participants own reports had been neglected in their curriculum. The workshop was also successful in causing a growth in the planned use of teaching strategies that were emphasized throughout the workshop: presenting to students complex projects, gathering and organizing student responses, posing open challenges, encouraging generalizations, group work, use of mathematical games and of manipulatives. The teachers also planned to decrease the use of strategies that were de-emphasized throughout the workshop: drill and practice, seat work and whole class instruction.

All these plans are the first stage in making a change in teacher's behavior. Their classroom implementation will be the subject of a follow-up evaluation.

The workshop dealt extensively with the five MGMP units: Factors and Multiples (number theory), Mouse and Elephant (perimeter, area and volume relationships), Probability, Smiliarity and Spatial Visualization. The teachers' performance on probability improved enormously. They have already mastered the similarity tasks before the workshop, but did not improve their low performance on spatial visualization. The teachers reported on growth in their confidence to solve and to teach probability and spatial visualization after the workshop.

In general, the teachers had high expectations from the workshop and seemed to be very satisfied at the end.

REFERENCE

Fresko, B. and Ben-Haim, D. (1984). An evaluation of two in-service courses for mathematics teachers, unpublished technical report. Department of Science Teaching, The Weizmann Institute of Science, Rehovot, Israel.

Freidlander, A. (1985). The Honors Teacher Workshop: A first evaluation. In Proceedings of the MSU Honors Teacher Workshop 1984, Department of Mathematics, Michigan State University, East Lansing, MI.

APPENDIX A

TOPIC	How frequently will you teach the topic in your classes (grades 6-8)?					Did the workshop adequately deal with topic for your purposes?				
	Very frequently	Frequently	Sometimes	Seldom	Never	Definitely yes	I would like it	I don't have an opinion	I would not like it	Definitely no
Arithmetic (whole nos., fractions, decimals										
Drill										
Short word problems										
More complex problems or projects										
Number Theory (factors, primes, odd/even, etc.)										
Drill										
Short word problems										
More complex problems or projects										
Geometry										
Drill										
Short problems										
More complex problems or projects										
Spatial Visualization										
Drill										
Short problems										
More complex problems or projects										
Statistics										
Drill										
Short problems										
More complex problems or projects										
Probability										
Drill										
Short problems										
More complex problems or projects										

TOPIC	How frequently will you teach the topic in your classes (grades 6-8)?					Did the workshop adequately deal with topic for your purposes?				
	Very frequently	Frequently	Sometimes	Seldom	Never	Definitely yes	I would like it	I don't have an opinion	I would not like it	Definitely no
<u>Measurement</u>										
Drill										
Short problems										
More complex problems or projects										
<u>Algebra</u>										
Drill										
Short word problems										
More complex problems or projects										
<u>Use of calculators</u>										
For checking answers										
As a computational aid										
As a means for solving complex problems										
<u>Use of computers</u>										
Use of software for drill										
Use of software for simulations										
Use of software for solving problems										
Learning a programming language										
Programming for problem solving										
<u>History of Mathematics</u>										

STRATEGY	How frequently will you teach the topic in your classes (grades 6-8)?					Did the workshop adequately deal with topic for your purposes?				
	Very frequently	Fre-quently	Sometimes	Seldom	Never	Definitely yes	I would like it	I don't have an opinion	I would not like it	Definitely no
Whole class instruction										
Group work										
Seat work (same assignment for all)										
Individualized work										
Posing open-ended challenges										
Gathering and organizing student responses										
Encouraging analysis and generalization										
Assigning homework										
Discussing homework										
Using concrete manipulatives										
Using worksheets										
Using games										

APPENDIX B

Honors Teacher Workshop Evaluation ______________________________
pseudoname

Indicate, by writing the appropriate number and letter in the boxes beside each question, to what extent you feel confident that you could solve the problem and teach the problem to middle school students. (You are not asked to solve the problem.)

Confidence in solving

1. I am positive that I can solve it.
2. I am somewhat certain that I can solve it.
3. I don't think I can solve it - the topic is not familiar.
4. I am not familiar at all with the topic.

Confidence in teaching

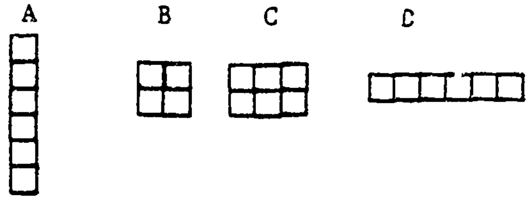
(A) I am certain that I could teach it.
(B) I am somewhat certain I could teach it.
(C) I am not confident that I could teach it.

Question	Solving	Teaching
1.	☐	☐
2.	☐	☐
3.	☐	☐
4.	☐	☐
5.	☐	☐
6.	☐	☐
7.	☐	☐
8.	☐	☐
9.	☐	☐
10.	☐	☐
11.	☐	☐
12.	☐	☐
13.	☐	☐
14.	☐	☐
15.	☐	☐
16.	☐	☐
17.	☐	☐
18.	☐	☐
19.	☐	☐
20.	☐	☐

TWENTY MATHEMATICS PROBLEMS

1. A set of blocks can be separated into 6 equal piles. It can also be separated into 15 equal piles. What is the smallest number of blocks that could be in the set?

2. Here are the 4 different rectangles that cover 6 squares on a grid.

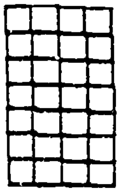


   How many different rectangles can be made which would vover exactly 30 squares on a grid?

3. 3 is the grestest common divisor of 15 and another number. What is the other number?

4. What is the smallest prime number which is larger than 200?

5. What is the perimeter of this rectangle?



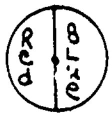
6. A rectangular field with an area of 240 square units has one edge of 20 units. What is the length of the other edge?

7. What is the surface area of a cube with an edge of 10?

8. You have a melon in your garden that is 3 inches across and weighs 3 ounces. If it grows to be 6 inches across, how much would it weigh?

9. What is the probability of getting a sum of 12 when two dice are thrown?

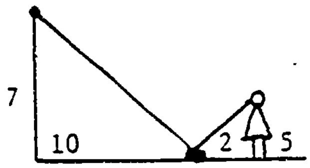
10. If the spinner shown is to be spun twice, what is the probability of getting red-red?



11. Two bills are drawn randomly from a bag containing a five dollar bill and 3 one dollar bills. If the experiment is repeated many times, what would you expect the average amount of money drawn per time to be?

12. What is the probability that a family of three children will have 2 girls and 1 boy?

13. Joan estimates the height of a flagpole by using a mirror.



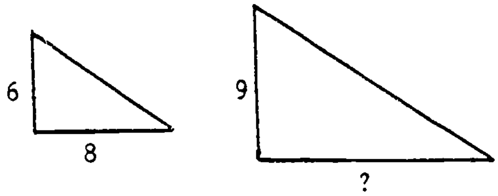
Distances

to eye level	5 ft.
Joan to mirror	2 ft.
Mirror to pole	10 ft.

How tall is the pole?

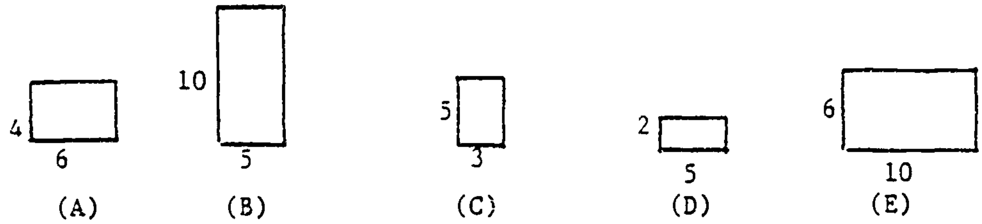
14. If the lengths of the sides of a triangle are each multiplied by 3, hw much larger is the area of the new triangle?

15. These triangles are similar:

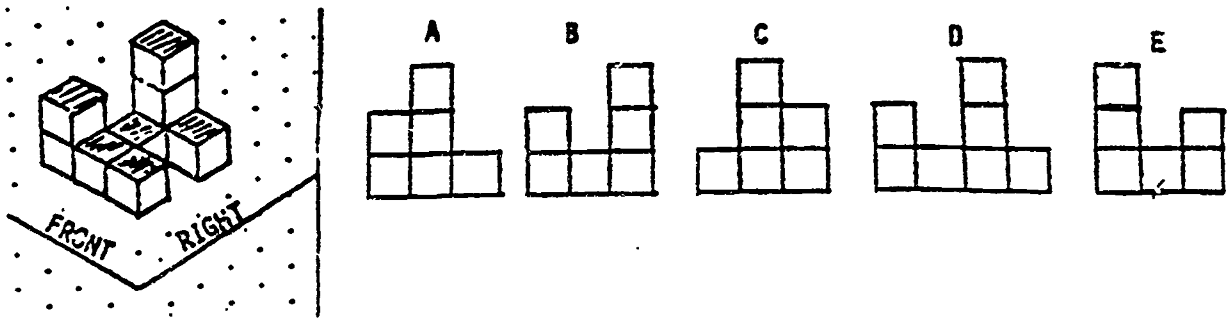


Find the missing length.

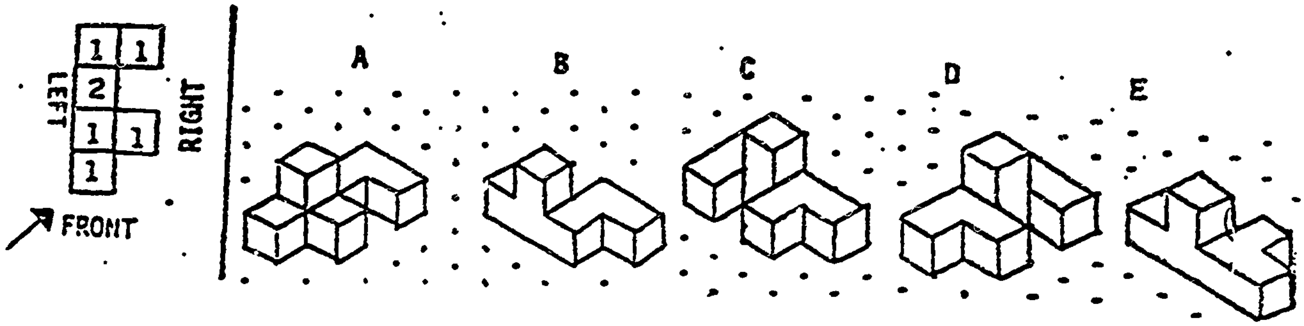
16. Mark which of the following rectangles is similar to a 10 x 15 rectangle?



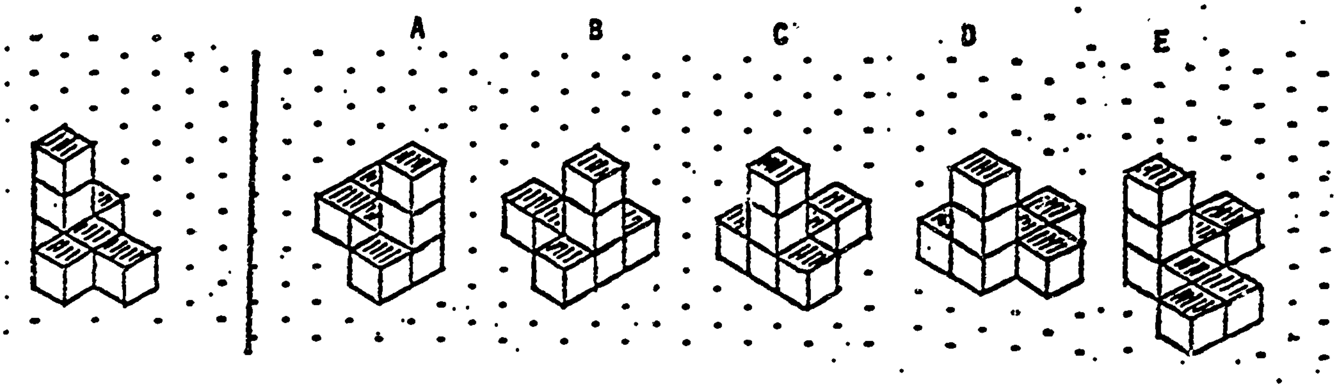
17. Mark the RIGHT VIEW. You are given a picture of a building drawn from the FRONT-RIGHT corner. Find the RIGHT VIEW.



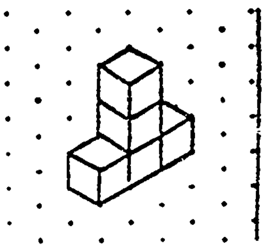
18. The number on each square indicates the number of cubes placed on that square. Mark the view from the FRONT-LEFT corner.



19. Mark another view of the first building.



20. What is the maximum number of cubes that could be used to build a building that has the given isometric drawing as a corner view?



APPENDIX C

Table C.1

Mean scores* and standard deviations of teaching and planned teaching frequency

	Arithmetic			Number Theory			Geometry			Spatial Visualization		
	D	S	P	D	S	P	D	S	P	D	S	P
Teaching Frequency	3.46 (1.10)	3.92 (0.72)	3.71 (0.75)	3.25 (1.11)	3.75 (0.85)	3.29 (0.86)	3.12 (1.15)	3.88 (0.85)	3.67 (1.01)	2.29 (0.81)	2.62 (0.92)	2.[illegible]8 (0.97)
Planned Teaching Frequency	2.08 (0.72)	3.56 (1.04)	4.19 (0.76)	2.12 (0.74)	3.69 (0.88)	4.44 (0.50)	2.00 (0.60)	3.81 (0.84)	4.35 (0.56)	1.79 (0.72)	3.65 (0.78)	4.23 (0.56)

	Statistics			Probability			Measurement			Algebra		
	D	S	P	D	S	P	D	S	P	D	S	P
Teaching Frequency	2.38 (0.92)	2.75 (0.90)	2.33 (0.96)	2.54 (1.14)	2.92 (0.97)	2.42 (0.97)	3.25 (0.90)	3.58 (0.83)	3.12 (1.08)	3.30 (1.02)	3.75 (0.94)	3.33 (1.43)
Planned Teaching Frequency	2.09 (0.61)	3.72 (0.89)	4.28 (0.75)	1.96 (0.69)	4.06 (0.76)	4.56 (0.50)	2.35 (0.93)	3.90 (0.62)	4.44 (0.58)	2.43 (0.79)	3.98 (0.76)	4.10 (0.72)

D - Drill and Practice
S - Short Problems
P - Complex Problems or Projects

*On a scale from 1 to 5

Table C.2

Mean scores, standard deviations and results of t-tests for pre-post workshop differences of teachers' reaction to teaching frequency with regard to teaching strategies

	Drill and Practice	Short Word Problems	Complex Projects	Whole Class Instruction	Group Work	Seat Work	Individualized Work
Frequency in teaching	2.95 (1.10)	3.40 (0.99)	3.03 (1.14)	4.46 (0.51)	3.58 (1.02)	4.04 (0.86)	3.12 (0.61)
Planned frequency in teaching	2.10 (0.74)	3.79 (0.84)	4.32 (0.66)	3.75 (0.74)	4.29 (0.62)	3.08 (0.88)	3.04 (0.75)
t-value	-10.29	3.75	11.08	-4.04	3.63	-5.17	-0.44
p <	0.001	0.001	0.001	0.001	0.01	0.001	N.S.

	Concrete Mani-pulatives	Work-sheets	Games	Open ended Challenges	Organizing Student Responses	Encouraging Generalizations
Frequency in teaching	3.46 (0.93)	3.58 (0.78)	3.42 (0.88)	3.50 (1.06)	3.38 (1.06)	3.79 (0.98)
Planned frequency in teaching	4.71 (0.46)	3.83 (0.76)	4.29 (0.69)	4.50 (0.59)	4.33 (0.56)	4.50 (0.51)
t-value	3.18	1.49	2.98	5.54	6.49	3.20
p <	0.01	N.S.	0.01	0.001	0.001	0.01

* on a scale from 1 to 5

Table C.3

Mean scores* and standard deviations from teaching frequency

	CALCULATORS			HISTORY OF MATHEMATICS
	Checking Answers	Computational Aid	Complex Problems	
Frequency in using	3.25 (1.19)	3.42 (1.14)	3.88 (0.95)	2.5 (1.20)
Planned frequency in using	3.88 (1.15)	4.38 (0.71)	4.71 (0.46)	3.5 (1.19)

	COMPUTERS				
	Drill	Simulations	Problem Solving	Programming	Programming for problem solving
Frequency in using	1.78 (0.95)	2.00 (1.21)	2.00 (1.24)	1.71 (1.20)	1.83 (1.34)
Planned frequency in using	2.22 (0.95)	2.83 (1.17)	2.83 (1.17)	2.29 (1.23)	2.42 (1.35)

* on a scale from 1 to 5

Table C.4

Individual scores on the Twenty Mathematical Problems

	Pre	Post
1. Bailey	19	18
2. Brenda	16	19
3. Captain	no pre-test	9
4. Deu	11	17
5. Dennis Jay	17	18
6. Flash	16	18
7. Floretta M.	17	16
8. Garrus	13	16
9. Hobbes	17	19
10. Jane Doe	14	17
11. Lady Jane	16	16
12. Mae	16	19
13. Melson David	15	17
14. Michelle	7	15
15. Michelle J.	16	15
16. Mint	10	11
17. Mohonrimoriancimur	17	18
18. Nola	11	15
19. Nolte	15	19
20. Rosalyn	15	19
21. Roy Johnson	16	18
22. Sam Jones	11	16
23. Wayne Gretzki	9	15
24. Winchester	16	15

Table C.5

Distribution of errors on Twenty Mathematical Problems

Problem	Number of Errors	
	Pre	Post
1	2	1
2	8	3
3	2	1
4	8	9
5	1	0
6	2	1
7	2	3
8	17	10
9	3	1
10	6	2
11	17	8
12	17	4
13	3	0
14	3	0
15	1	0
16	3	1
17	5	1
18	5	3
19	5	5
20	21	24

Table C.6

Mean scores and results of t-tests for pre-post workshop differences of teachers' reaction and performance on Twenty Mathematical Problems

Level of performance (in percents)

	Mean scores (s.d.) Pre	Post	t value	p <
Factors and Multiples (Problems 1-4)	78% (0.41)	87% (0.34)	2.15	0.05
Mouse and Elephant (Problems 5-8)	76% (0.43)	84% (0.37)	3.22	0.01
Probability (Problems 9-12)	52% (0.50)	88% (0.33)	5.74	0.001
Similarity (Problems 13-16)	91% (0.28)	98% (0.15)	1.66	N.S.
Spatial Visualization (Problems 17-20)	61% (0.49)	67% (0.47)	1.82	N.S.

Confidence in Solving (on a scale from 1 to 4)

	Mean Scores (s.d.) Pre	Post	t value	p <
Factors and Multiples (Problems 1-4)	3.84 (0.37)	3.99 (0.27)	3.42	0.01
Mouse and Elephant (Problems 5-8)	3.85 (0.44)	3.96 (0.20)	3.15	0.01
Probability (Problems 9-12)	3.56 (0.58)	3.90 (0.30)	3.94	0.001
Similarity (Problems 13-16)	3.85 (0.46)	4.0 (0.00)	2.26	0.005
Spatial Visualization (Problems 17-20)	3.47 (0.70)	3.87 (0.34)	2.40	0.05

Table C.6 Continued

Confidence in Teaching (on a scale from 1 to 3)

	Mean scores (s.d.) Pre	Post	t value	p <
Factors and Multiples (Problems 1-4)	2.87 (0.54)	2.86 (0.35)	1.57	N.S.
Mouse and Elephant (Problems 5-8)	2.87 (0.40)	2.86 (0.35)	1.17	N.S.
Probability (Problems 9-12)	2.63 (0.61)	2.83 (0.38)	2.18	0.05
Similarity (Problems 13-16)	2.82 (0.53)	2.91 (0.28)	1.54	N.S.
Spatial Visualization (Problems 17-20)	2.13 (0.88)	2.72 (0.54)	3.57	0.01

Table C.7

Mean scores* and standard deviations of workshop perception with regard to teaching strategies

	Drill and Practice	Short Word Problems	Complex Projects	Whole Class Instruction	Group Work	Seat Work
Expectations to deal with in workshop	3.20 (1.29)	3.92 (1.01)	4.33 (0.73)	3.25 (1.36)	4.25 (0.74)	3.25 (1.39)
Evaluation of how the workshop dealt with	4.06 (1.11)	4.62 (0.71)	4.70 (0.71)	4.67 (0.64)	4.96 (0.20)	4.46 (0.72)

	Individualized Work	Concrete Manipulatives	Work-sheets	Games
Expectations to deal with in workshop	3.92 (1.06)	4.67 (0.56)	3.29 (1.20)	4.50 (0.59)
Evaluation of how the workshop dealt with	4.46 (0.72)	4.92 (0.41)	4.79 (0.59)	4.88 (0.54)

	Open-ended Challenges	Organizing Student Responses	Encouraging Generalizations
Expectations to deal with in workshop	4.75 (0.61)	4.33 (0.76)	4.54 (0.59)
Evaluation of how the workshop dealt with	4.96 (0.20)	5.00 (0.00)	5.00 (0.00)

* on a scale from 1 to 5

Table C.8

Mean scores* and standard deviations of workshop evaluation with regard to mathematical topics

Arithmetic			Number Theory			Geometry		
D	S	C	D	S	C	D	S	C
4.00 (1.06)	4.62 (0.58)	4.79 (0.41)	4.17 (1.05)	4.71 (0.46)	4.83 (0.64)	4.22 (1.04)	4.79 (0.41)	4.83 (0.64)

Spatial Visualization			Statistics			Probability		
D	S	C	D	S	C	D	S	C
4.29 (1.04)	4.88 (0.34)	4.96 (0.20)	4.08 (1.14)	4.62 (0.77)	4.67 (0.76)	4.33 (1.05)	4.96 (0.20)	5.00 (0.00)

Measurement			Algebra		
D	S	C	D	S	C
3.92 (1.14)	4.33 (0.92)	4.50 (0.78)	3.46 (1.28)	4.00 (1.13)	4.00 (1.13)

D - Drill and practice
S - Short problems
P - Complex problems

* on a scale from 1 to 5

APPENDIX D

MIDDLE GRADES MATHEMATICS PROJECT

Teaching Style Inventory -- Summer, 1987

Name ________________________________

School ________________________________

PART I CLASSROOM PROCEDURES

Please check the point within each of the following scales which most accurately describes your math class. (If you are teaching math for the first time or your present situation is very different from previous years, please respond as you anticipate your class will be like this year.) Please respond according to what actually happens, not what you think should happen, or what you would like to have happen. There are no right or wrong answers. Please answer all the questions.

1. Almost all help is initiated by students asking for it. ____1
____2
____3
____4
Almost all help is initiated by my seeing the need for it. ____5

2. When students have trouble, I ask them leading questions. ____1
____2
____3
____4
When students have trouble, I explain how to do it. ____5

3. Almost always many different activities are going on simultaneously during math class. ____1
____2
____3
____4
Almost all the time the students are all engaged in the same activity during math class. ____5

4. In class, students frequently work together on assignments. ____1
____2
____3
____4
Students seldom work together on assignments in class. ____5

5. When studying a math unit, students spend some time working in small groups to solve a big problem.

___1
___2
___3
___4
___5

When studying a math unit, students will not be working in small groups to solve a big problem.

6. I encourage students to solve a given math problem the way I have demonstrated.

___1
___2
___3
___4
___5

I encourage students to solve math problems in a variety of ways.

7. I present a math concept first then illustrate that concept by working several problems (deductive).

___1
___2
___3
___4
___5

I present the class with a series of similar problems, then together we develop concepts and methods of solving the problems (inductive).

8. Certain topics are repeated (but in more depth) on a regular basis throughout the year.

___1
___2
___3
___4
___5

Once a topic is covered, that same topic is not covered again except during reviews.

9. When I teach a new topic, I spend a good deal of the time (1/3) trying to teach students to see similarities and differences between new and previously learned math ideas.

___1
___2
___3
___4
___5

New topics are generally taught with limited reference to previously learned math ideas.

10. The furniture arrangement is the same for every math lesson.

___1
___2
___3
___4
___5

The furniture arrangement varies according to the lesson.

11. In my math class I emphasize the basic computational skills three/fourths of the time or more.

In my math class I emphasize concept development three/fourths of the time or more.

___1
___2
___3
___4
___5

12. I seldom change my approach throughout the semester (such as lecture-discussion, discovery, etc.).

I change my approach frequently (from discovery to direct telling or from another method to something different) throughout the semester.

___1
___2
___3
___4
___5

13. Understanding why a given rule or procedure gives the correct answer is important.

Understanding the rule or procedure is not critical.

___1
___2
___3
___4
___5

14. Almost all my questions in math class can be answered with yes, no, or a number.

Almost all my questions in math class require the students to give explanations.

___1
___2
___3
___4
___5

15. In my class, I give different assignments to students with different ability levels.

In my class, I give the same assignment to all students.

___1
___2
___3
___4
___5

16. I usually use a game, story, or challenging problem to provide a context for a new math unit.

I usually do not use a game, story, or challenging problem to provide a context for a new math unit.

___1
___2
___3
___4
___5

17. I usually start a new math unit by giving examples and showing students how to work them.

I do not usually start a new math unit by giving examples and showing students how to work them.

___1
___2
___3
___4
___5

PART II STRATEGIES

How frequently do you use the strategy in your classes?

	Very Frequently	Fre-quently	Sometimes	Seldom	Never
18. Whole class instruction					
19. Whole class discussion					
20. Posing open-ended challenges					
21. Gathering and organizing student responses					
22. Encouraging analysis and generalization					
23. Assigning homework					
24. Discussing homework					
25. Using concrete manipulatives					
26. Using games					
27. Drills					
28. Story problems					
29. Non-routine problems					

PART III TEACHER OPINION

Select the appropriate choice for each statement.

A = Agree
B = Somewhat agree
C = Undecided
D = Somewhat disagree
E = Disagree

30. _____ I am an effective mathematics teacher.

31. _____ I like doing mathematics.

32. _____ My basic function as a math teacher is to convey my knowledge of math to the students in a direct manner.

PART IV COMPLETION

33. As of today, I have _____ students that are discipline problems.

34. I assign math work to be done at home about _____ times a week.

35. Think of your average student. When you make a homework assignment, approximately what percentage of the time is it:

_____ completed in class by most students.

_____ begun in class but finished at home.

_____ done entirely at home.
100%

36. When some students do poorly on tests or indicate that they have not understood a topic in math, what do you do?

37. Sometimes students have difficulty solving story problems. Briefly describe how you help your students solve story problems. (Example: I have pupils make drawings or diagrams to help clarify the problem.)

38. As of today, I have ______ students that are chronically absent.

39. When students who have been absent return to class, what do you do to catch them up?

40. List the kinds of manipulatives or educational equipment that you use.

41. How frequently and for what purposes do you use them?

42. How do your students use calculators and computers?

43. How many years (including this year) have you taught math to 6-8 grade students?

_____ years

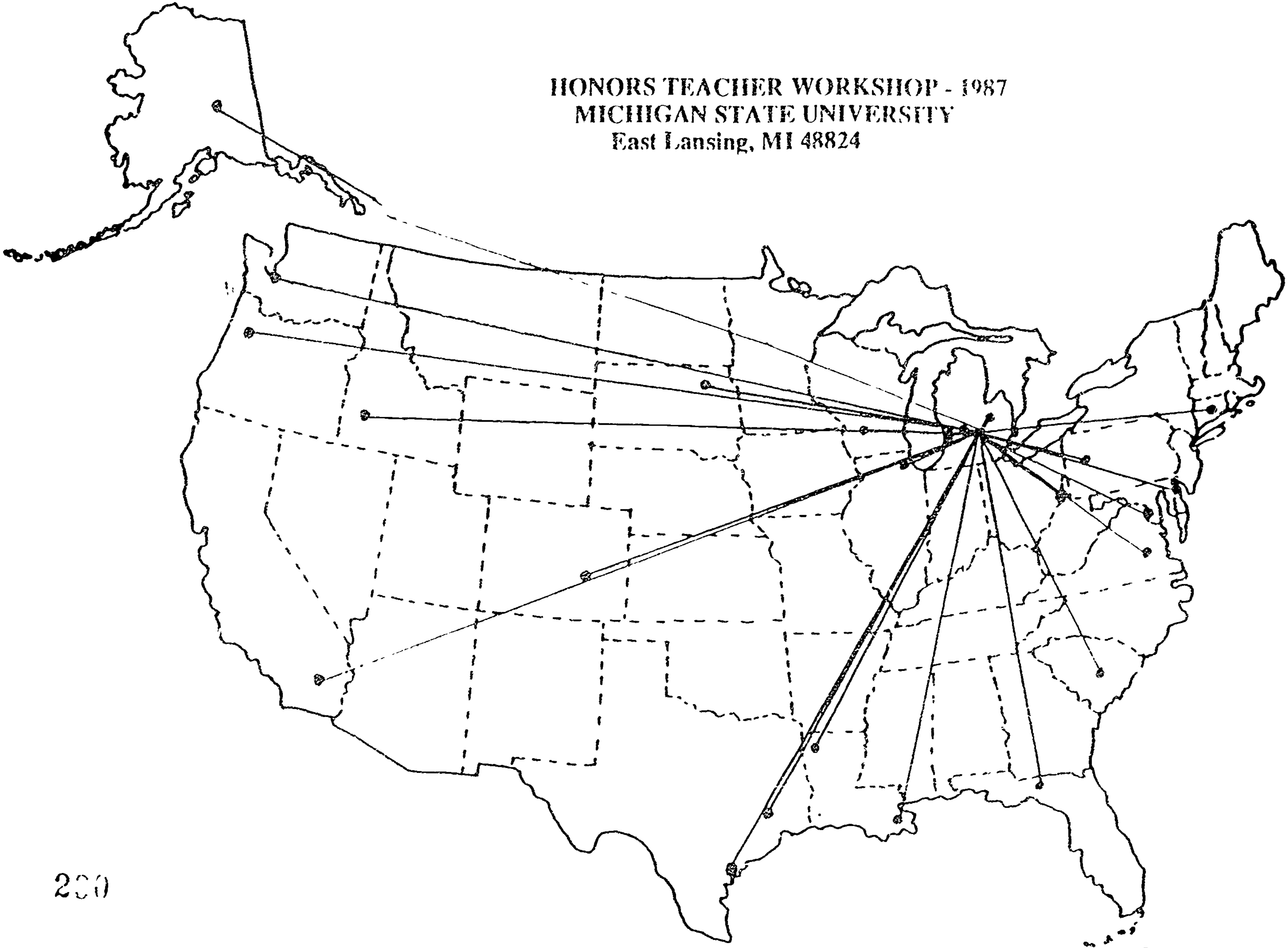
44. How many years (including this year) have you taught?

_____ years

45. How many hours of college credit in math have you completed (including math methods courses)?

_____ semester hours

_____ (quarter) term hours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