DOCUMENT RESUME

ED 292 682 SE 049 063

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TITLE	Learning Fractions with Understanding: Building upon Informal Knowledge.
PUB DATE	Apr 88
NOTE	41p.; Paper presented at the Annual Meeting of the American Educational Research Association (New Orleans, LA, April 5-9, 1988).
PUB TYPE	Reports - Research/Technical (143) -- Speeches/Conference Papers (150)
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	Addition; *Algorithms; Basic Skills; Computation; *Elementary School Mathematics; *Fractions; *Individualized Instruction; Intermediate Grades; Mathematics Education; *Mathematics Instruction; *Number Concepts; Subtraction
IDENTIFIERS	Mathematics Education Research

ABSTRACT

Eight sixth-grade students received individualized instruction on the addition and subtraction of fractions in a one-to-one setting for 6 weeks. Instruction was specifically designed to build upon the student's prior knowledge of fractions. It was determined that all students possessed a rich store of prior knowledge about parts of wholes in real world situations based upon whole numbers. Students related fraction symbols and procedures to prior knowledge in ways that were meaningful to them; however, there was a danger of this prior knowledge interfering when it reflected algorithmic procedures rather than fraction ideas in real world situations. (Author/PK)

LEARNING FRACTIONS WITH UNDERSTANDING: BUILDING UPON INFORMAL KNOWLEDGE

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Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA, April, 1988.

ABSTRACT

Eight sixth-grade students received individualized instruction on the addition and subtraction of fractions in a one-to-one setting for six weeks. Instruction was specifically designed to build upon the student's prior knowledge of fraction ideas. All students possessed a rich store of prior knowledge that was knowledge about parts of wholes in real world situations and was based upon knowledge of whole numbers. Students related fraction symbols and procedures to prior knowledge in ways that were meaningful to them; however, there was a danger of this prior knowledge interfering when it reflected algorithmic procedures rather than fraction ideas in real world situations.

One of the most compelling issues currently in the cognitive science of instruction is the development of students' understanding with respect to "relations between intuitive understanding and knowledge of symbolic procedures" (Greeno, 1986, p. 343). Studies concerning children's understanding of whole number arithmetic have documented that children come to instruction with a rich store of prior knowledge that may serve as a basis for instruction (Carpenter & Moser, 1983). However, based on the results of studies concerning children's understanding about fractions, it is not clear if children possess a rich store of prior knowledge for fraction ideas or if this prior knowledge may provide a basis for instruction.

Studies concerning student's understanding about fractions have primarily focused on students' misconceptions rather than on characterizing their prior knowledge of fraction ideas. Several studies have suggested that many students have little understanding of fraction ideas by documenting numerous common errors students make when performing operations with fractions, such as adding numerators together and adding denominators together when adding like and/or unlike fractions (Behr, Lesh, Post, & Silver, 1983; Behr, Wachsmuth, Post, & Lesh, 1984; Behr, Wachsmuth, & Post, 1985; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Erlwanger, 1973; Kerslake, 1986). These studies however, primarily examined students' understanding with respect to formal symbols and algorithmic procedures and have left unexamined the issue of students' intuitive understanding of fractions.

Leinhardt (in press) presented a different picture of students' understanding about fractions by documenting that students do come to instruction with some prior knowledge of fraction ideas. Her purpose however, was not to characterize the nature of this prior knowledge or to show how students may build upon this knowledge to give meaning to formal symbols and procedures, but to identify four specific classes of knowledge that evolve as a student's competence for a mathematical topic increases. Although Leinhardt's results provide a glimpse of students' knowledge of fractions ideas prior to formal instruction, the nature of this prior knowledge and whether or not it may provide a basis for instruction continue to be issues of concern.

Leinhardt (in press) and Behr et al. (1983, 1984, 1985), Carpenter et al. (1981), Erlwanger (1973), and Kerslake (1986) painted different pictures of students' understanding of fractions by examining their understanding from different perspectives; however, the results of these studies merge to raise questions about the nature of students' prior knowledge after they have received formal instruction on fractions. Questions arise such as whether or not students may possess a rich store of prior knowledge that exists in isolation of their knowledge of formal symbols and procedures and if this knowledge may provide a basis for instruction.

Examining the nature of students' prior knowledge and how they are able to build upon this knowledge to give meaning to formal symbols and procedures is relatively uncharted territory (Lampert, 1986), especially after students have received formal instruction. Hiebert and Wearne (in press) assert that in situations where students have recieved formal instruction, examinations of the development of students' understanding should not be limited to identifying ways in which students successfully build upon prior knowledge, but should also address students prior knowledge about procedures and the influence that this knowledge may have on their ability to relate pieces of knowledge. According to Greeno (1986), insights into the issues of students' prior knowledge and the ways in which they are or are not able to build upon this knowledge are necessary for gaining a better understanding of the potential of prior knowledge in the development of students' understanding of mathematical symbols and procedures.

The purpose of this study was to examine the development of students' understanding about fractions during instruction from two perspectives: (1) investigating the ways in which students built upon their prior knowledge of fractions, and (2) investigating the influence of prior knowledge about procedures on students' ability to relate new knowledge to prior knowledge. The results provide insights into the nature of students' prior knowledge of fraction ideas, the ways in which they are able to build upon this knowledge to give meaning to fraction symbols and procedures, and the influence that prior knowledge about procedures has on students' ability to relate pieces of knowledge.

METHODOLOGY

The methodology for this study emerged from three primary sources: (1) the case study (Erickson, 1986; Shulman, 1986), (2) instructional approaches that utilize instruction to influence cognitive changes (Carpenter, 1987; Hiebert & Wearne, in press), and (3) the use of students' verbal reports as data (Ginsburg, Kossan, Schwartz, & Swanson, 1983). Because of the nature of the methodology, some aspects are discussed in depth.

Sample

The sample consisted of eight sixth-grade students of average mathematical ability, who were identified as having little understanding about fractions. The students were initially identified by their classroom teachers. The teachers were asked to recommend students doing average to below average work in mathematics whom they thought would experience difficulties with fractions. Following the teacher recommendations, eight students were interviewed to screen those who demonstrated a strong understanding about fractions. All eight students demonstrated little understanding about fractions; therefore, all eight students participated in this study.

All subjects came from a middle school that predominantly draws students from middle to upper-middle income families in Madison, Wisconsin. Prior to, and during this study, none of the students received instruction on fractions in their sixth-grade mathematics class.

Initial Assessment

Instruction started with assessment. Each student's knowledge was assessed in what was referred to as the Initial Assessment. The Initial Assessment served two purposes: (1) assessing the student's prior knowledge with respect to topics related to the addition and subtraction of fractions, and (2) identifying the student's misconceptions. The student was asked questions involving characterizing fractions, estimating sums and differences for fractions, identifying equivalent fractions, partitioning a unit, and adding and subtracting like and unlike fractions.

The Initial Assessment was conducted as a clinical interview; therefore, the student's thinking with respect to each question was probed in various ways. The nature of the probing was determined by the student's response to the question and answers to previous questions (Ginsburg et al., 1983). The specific nature of the probing differed for each student, but in general, if the student was unable to answer a question presented in a "mathematics' form", the question was restated as a word problem involving the student in a real world situation. For example, if a student responded that 1/8 was larger when asked "Which of these two fractions is larger, 1/6 or 1/8?", the student was asked "If you had two pizzas of the same size and you cut one of them into six pieces and the other into eight pieces, and you got one piece from each pizza, which one would you get more from?". If the student used symbolic procedures to answer a question presented in a mathematical form, questions were asked to determine if the student was applying a rote procedure or if the student had some understanding for the procedure. For example, if a student was asked to add two unlike fractions and he or she responded that first common denominators must be found, the student was asked a question such as "Why do the denominators need to be the same?".

Not all of the students were asked the same questions during the Initial Assessment. Students were not asked questions for which their prerequisite knowledge appeared deficient.

Assessment Tasks

Each question the student was given was regarded as an assessment task. All of the tasks were based upon four central ideas that emerged from a rational task analysis for the addition and subtraction of fractions: (1) determining the relationship between the number of parts a unit is divided into and the size of the parts, (2) a fraction is a single number with a specific value, (3) different fractions represent the same amount, and (4) the addition and subtraction of fractions requires common denominators.

The specific tasks the student received were based upon the central ideas of the rational task analysis combined with the student's reponses to previous questions and his or her choice of context for the problems. The tasks were ones that encouraged the student to draw upon his or her prior knowledge and to form relationships between pieces of knowledge. For example, if a student had prior knowledge about joining and separating sets involving fractions and notions of fractions equivalent to one, the student was given a problem such as the following "Suppose you had four cookies and you ate seven-eighths of one cookie, how many cookies would you have left?".

The tasks were used to provide direction for instruction as well as to assess the student's thinking. In general, in situations where the student was unable to successfully solve a problem due to a misconception or lack of knowledge about an idea related to the problem, the student was given a simpler problem. In situations where the student successfully solved the problem by relating pieces of knowledge but the relationship appeared to be tenous, the student continued with a similar task. In situations where the student successfully solved a problem by relating pieces of knowledge and the relationship appeared to be strong, the student was given a problem that was closely related but more complex. For example, if the student successfully solved the problem above involving a real-life situation for 4 - 7/8, he or she was given a problem such as a real-life situation involving 4 1/8 - 7/8.

General Characteristics of the Instructional Sessions

Each student was regarded as an independent case study and received instruction in a one-to-one instructional session (subject and author of this paper). All instructional sessions lasted 30 minutes and occurred during regular school hours. Each student met with the author from 11-13 times over a period of six weeks, with one exception. One student, Aaron, covered the instructional content by the middle of the fifth week and his explanations reflected a strong understanding of fractions; therefore, the author decided to conclude his instructional sessions at that time.

All instructional sessions combined clinical interviews with instruction; therefore, instruction was not scripted. The majority of the problems were presented to the student verbally. The student was encouraged to think aloud as he or she solved problems. If the student failed to think aloud, the student was asked to explain what he or she had been thinking as the problems were solved.

The instructional content deviated from topics covered in chapters on fractions in traditional textbook series in two important ways: (1) the student's intuitive understanding about fractions provided the basis for instruction (Carpenter, 1987), and (2) the estimation of fractions was emphasized (Hiebert & Wearne, 1986; National Council of Teachers of Mathematics, 1980; Reys, 1984). Estimation was viewed as an intuitive skill (Hiebert & Wearne, 1986), and the specific situations in which instruction emphasized estimation consisted of three components: (1) examining individual fractions represented by concrete materials, real world situations, and symbolic representations end approximating the quantity represented, (2) estimating sums and differences involving fractions, and (3) constructing sums or differences involving fractions that are close to, but not equal to one.

Concrete materials in the form of fraction circles and fraction strips were available for the student to use, and their use was encouraged as long as the student thought they were needed. However, in situations where the student's solutions remained dependent on the concrete materials at the beginning of the fifth week, the student was gradually encouraged to make the transition to using symbolic representations for the problems. Pencil and paper were available for the student's use; however, their use was not encouraged until the student had successfully solved problems using the concrete materials in situations where misconceptions initially appeared when using pencil and paper.

After the Initial Assessment and each instructional session, a lesson was planned for the student's next session that was based upon the student's prior knowledge, misconceptions, responses to problems presented in previous sessions, and relationships between components of the instructional content. Because the purpose of the instructional sessions was to aid the student in relating pieces of knowledge, the student's misconceptions had to be dealt with; therefore, the lessons were designed to be flexible both prior to and during instruction. A rational task analysis for the addition and subtraction of fractions provided structure for the flexibility of the lessons.

The instructional flexibility also involved the specific fraction topics the student covered, the amount of time the student spent on a specific topic, whether or not the student was required to master a specific topic before moving on to another topic, and the sequence in which the student covered specific topics in the instructional content. To provide variety and motivation for the student, instruction either backed up to a simpler problem or moved to a different topic when the student began to show signs of frustration or boredom.

All instructional sessions were audio-taped. Each day I wrote out detailed notes from the student's audio-taped session and transcribed critical protocol segments. The notes and protocols were used to aid in planning instruction for the following session and in the data analysis. The student's protocols were reviewed several times during the study and after the conclusion of the study to identify relationships the student had formed between pieces of knowledge. When relationships were identified, they were compared to pieces of knowledge that instruction had attempted to relate to determine if the student had related the same pieces of knowledge, and possibly if they had related pieces of knowledge that had not yet been related through instruction. Each time I reviewed the protocols I found the same relationships that had been identified in earlier analyses, plus a few more.

Individualizing Instruction

Throughout the study, the author reacted to individual students; therefore, the specific manner in which instruction built upon intuitive understanding varied. In general, the author continually assessed the student's thinking and adjusted instruction to make the problems that drew upon prior knowledge and new knowledge more and more similar. This often involved moving back and forth between concrete materials, real world situations, and symbolic representations, as well as moving back and forth between fraction topics. Although instruction constantly moved back and forth between the student's prior knowledge and new knowledge, similar to the assessment tasks, it's movement was guided by the rational task analysis with its four central ideas and the purpose of this study.

RESULTS

Instruction was specifically designed to build upon the student's prior knowledge about fractions; therefore, instruction may have influenced the results. This section integrates individual protocols into a discussion of specific findings. Although some of the real world situations appear unrealistic, the student chose the context for his or her problems, such as cakes, pies, boards, etc. at the beginning of each session. The results are presented in two sections: (1) building upon prior knowledge, and (2) the influence of prior knowledge about procedures.

Building Upon Prior Knowledge

All eight students came to instruction with numerous misconceptions related to fraction symbols and procedures; however, they also came with a substantial store of prior knowledge about fraction ideas that enabled them to solve problems presented in the context of real world situations. Similarities existed among the students with respect to the specific ideas for which they had prior knowledge and the nature of the knowledge associated with each idea.

The students' prior knowledge about fractions was knowledge about parts of things in real world situations familiar to the student. This prior knowledge focused on the size of specific parts of a whole rather than on a whole partitioned into equal-sized parts with a specific number of parts designated, such as for one-fourth, cutting a whole into four equal-sized parts and designating one of them. Teresa illustrated the general nature of this knowledge during her Initial Assessment as she drew pictures to represent various fractions. The following protocol illustrates how Teresa partitioned a whole and designated one part when the problem was presented in a mathematical form and how she focused on the size of the part when the problem was presented in a context familiar to her.

I: Draw me a picture of 1/2.
Teresa: (drew

Insert Figure 1 here

(points to one side of circle) This is 1/2.
I: Draw me a picture of 3/4 of a pizza.
Teresa: I know that's close to one. (drew

Insert Figure 2 here

Other students responded in a manner similar to Teresa's when asked to represent fractions pictorially. When presented with a problem in a mathematical form, the other students also partitioned a whole into equal-sized parts and designated the appropriate number of parts; however, when presented with a problem in the context of a real world situation, they too, focused on the size of the parts.

The specific kinds of prior knowledge the students possessed included knowledge about: (1) identifying various parts of a whole, (2) estimating quantities and estimating quantities when joining and separating sets involving parts of wholes, (3) the relationship between the number of parts a whole is divided into and the size of the parts, (4) the number of parts needed to make a whole, (5) the number of parts needed to make half of a whole, and (6) joining and separating sets involving parts of wholes. The students most frequently drew upon these latter four kinds of knowledge in their attempts to give meaning to fraction symbols and procedures. Therefore, these kinds of knowledge, although not independent of one another or of students' other prior knowledge about fractions, are discussed separately in this section to provide examples of the kinds of prior knowledge students possessed about fraction ideas, and to illustrate how students built upon this knowledge to give meaning to fraction symbols and procedures.

Knowledge About Relationships Between Parts of Wholes

Students' prior knowledge about the relationship between the number of parts a whole is divided into and the size of the parts involved wholes of the same size that are divided into an unequal number of parts. This knowledge involved recognizing that the more parts a whole is divided into the smaller the parts become. This knowledge however, was constrained to situations in which an equal number of parts were considered for each whole.

One situation that illustrates students' knowledge about the relationship between the number of parts a whole is divided into and the size of the parts of a whole involves comparing unit fractions, fractions with numerators of one. Julie presents one example of a student who utilized her prior knowledge to successfully compare unit fractions. The following protocol was taken from Julie's Initial Assessment and her third and ninth instructional sessions. Prior to Julie's third session, the only experiences she had had with comparing fractions in this study involved comparing unit fractions in the Initial Assessment. Prior to her ninth session, Julie's experiences comparing fractions included those in her Initial Assessment and third session as well as determining if a fraction is greater than or less than 1/2 or one. The following protocol illustrates how Julie utilized her prior knowledge to compare unit fractions, as well as how she built upon this knowledge to compare more complex fractions. The protocol also illustrates the constraints upon her prior knowledge.

Initial Assessment

I: (asked Julie to write 1/3 and 1/4 on her paper) Compare 1/3 and 1/4, tell me which is bigger.

Julie: One-fourth.

I: Why is 1/4 bigger?

Julie: It's a bigger number I think. Four is bigger than three.

I: Suppose you had two pizzas of the same size and you cut one of them into three pieces of the same size, and you cut the other one into four pieces of the same size. If you get one piece from each, which one do you get more of? . . .

Julie: . . . You get more from the one with three pieces, so 1/3 is bigger.

Third Instructional Session

I: Which fraction is the smallest, 4/9 or 4/12?

Julie: Four-twelfths . . . because you need more pieces.

Ninth Instructional Session

I: Tell me which of these two fractions is the smallest, 4/5 or 5/6?

Julie: Four-fifths.
I: Why four-fifths?
Julie: 'Cause the denominator, wait, would you say them again?
I: Four-fifths and 5/6.
Julie: They're the same.
I: Why are they the same?
Julie: Because there's one piece missing from each, one sixth missing from 5/6 and there's one fifth missing from 4/5.
I: (wrote 4/5 and 5/6 on Julie's paper) Okay, so you've got one fifth missing from here (pointing to 4/5) and one sixth missing from here (pointing to 5/6), are they still the same?
Julie: Yea.
I: Think about your pizza, if you have 1/5 of a pizza or if you had 1/6 of a pizza, which one would you have more of?
Julie: Oh, the 1/5... 5/6 is smaller 'cause sixths are smaller than fifths.

Julie's response that 1/4 is larger than 1/3 revealed one of her misconceptions related to fraction symbols, the larger the number in the denominator, the larger the fraction. Her response to the question about the pizzas revealed that she had prior knowledge about the relationship between the number of parts a whole is divided into and the size of the parts that was unrelated to her knowledge of fraction symbols. She suggested the she had related the symbolic representations for fractions to her prior knowledge by explaining 1/3 and 4/12 in terms of the size of pieces rather than the size of whole numbers. As Julie compared 4/5 and 5/6, she continued to respond in terms of pieces, which suggested she was attempting to build upon her prior knowledge. However, her responses also suggested that she was focusing on the number of parts missing from a whole rather than the size of the parts and had reached the limits of building upon this prior knowledge on her own.

All of the other students responded in a manner similar to Julie's when comparing unit fractions. The students' common reponse when initially compering fractions represented symbolically was that the larger fraction was the one that had the larger number in the denominator. The other students also had prior knowledge about the relationship between the number of parts a whole is divided into and the size of the parts that they related to symbolic representations when comparing unit fractions and other unlike fractions with like denominators. Like Julie, they also focused on the number of parts missing from a whole when asked to compare fractions such as 4/5 and 5/6 and responded "they're equal". With some assistance from instruction that questioned students about the size of the pieces that were missing, four of the students overcame the misconception that fractions are equal if they are an equal number of pieces less than one whole. As the students overcame this misconception, they invented alternative algorithms for comparing fractions that were based upon prior knowledge about the relationship between the number of parts a whole is divided into and the size of the parts.

Teresa and Bob were two of the students who invented alternative algorithms for comparing fractions that were based upon prior knowledge. The following protocols were taken from Teresa's tenth instructional session and Bob's eleventh instructional session. Both students responded in a manner similar to Julie's prior to being questioned about the size of the piece missing when comparing fraction such as 4/5 and 5/6. The protocols illustrate how both Teresa and Bob were able to build upon prior knowledge to invent alternative algorithms for comparing complex fractions.

Teresa - Tenth Instructional Session

I: Tell me which of these fractions is the largest, 2 2/3 or 2 5/6?

Teresa: Two and five-sixths, well because umm 2 2/3, 2/3 is close to one, and 5/6 is close to one, but sixths are smaller than thirds, and so you have littler, less way, or less umm (pause), littler pieces to get to one.

Bob - Eleventh Instructional Session

I: Tell me which fraction is the biggest, 5/6 or 6/8?

Bob: Biggest, that'd be 5/6, well six pieces, sixths are bigger than eighths, and if you have 5/6 that's almost a whole and if you have six, wait, 5/6 is 1/6 away from a whole and 6/8 is 2/8 away from a whole, so I just thought that would be bigger.

The students all possessed prior knowledge about the relationship between the number of parts a whole is divided into and the size of the parts that was initially unrelated to knowledge of fraction symbols and procedures. This knowledge allowed students to focus on the size of specific fractions. The students demonstrated that they could relate fraction symbols to this prior knowledge within limits, and that occasionally these constraints could be overcome with a minimum of assistance from instruction.

Knowledge About the Number of Parts Needed for a Whole

Students' knowledge about the number of parts needed for a whole was knowledge about distinct parts of the same size that could form one whole. Teresa explained this knowledge as

> "Well it kind of gets bigger in number and smaller in size... Like 1/2 is half of a circle or half of one, but then it gets smaller when the numbers get higher... It take two halves to make a whole, but then with four, so it takes four fourths, and then for three, 1/3 it takes three thirds and things like that.".

All of the students frequently utilized this knowledge to alleviate misconceptions they had when working with fraction symbols and to invent alternative algorithms. The specific situations in which students utilized this knowledge included: (1) selecting and partitioning a whole, whether the whole was composed of a single or multiple units, (2) estimating sums and differences, (3) subtracting a fraction from a whole number or like mixed numeral, such as 4 - 7/8 and 4 1/8 - 7/8, and (4) converting mixed numerals and improper fractions.

Teresa presents one example of how she was able to draw upon her knowledge about the number of parts needed to make a whole to alleviate a misconception when working with symbolic representations for mixed numerals and improper fractions and to invent an alternative algorithm for converting these fractions. The following protocol was taken from Teresa's fifth instructional session. Prior to this session Teresa had solved subtraction problems such as 4 - 7/8 by drawing upon her prior knowledge to rename 4 as 3 8/8, but this was her first experience with mixed numerals and improper fractions.

I: Tell me some other names for "3".
Teresa: Umm (pause).
I: Can you use a whole number and a fraction to tell me?
Teresa: Oh yea! . . .'Cause you have 2 4/4, or 2 3/3, and ummm, 2 5/5.
I: Suppose I told you you have 3 1/8 cookies, how many eighths is that? (pause) If you have 3 1/8 cookies?
Teresa: Twenty five, wait! (pause) Twenty five-eighths, because 8/8 go into three, I mean 8/8 go into one, and eight times three equals, wait eight times, wait eight times . . .
I: (interupting Teresa) You have 3 1/8 cookies.
Teresa: (wrote on her paper as she talked) And then you have, you have to do it, 8, 16, 24, that's three, plus another eighth, that's 25. . .
I: . . . Suppose you have 11/8 cookies, how many cookies is that?
Teresa: Umm.
I: Do you have more than one whole cookie?
Teresa: Yea, . . . 1/3 more, . . . I mean 3/8 more . . . (wrote on her paper as she talked) I meant to say 1 3/8.
I: (pointing to 25/8) We call that an improper fraction, that's where the numerator is larger than the denominator, and we call that (pointing to 1 3/8) a mixed numeral, that's where there is a whole number and a fraction. Now I want you to write 2 4/5 as an improper fraction.
Teresa: (wrote 8/5) Eight-fifths, well four plus four is eight, and you need two sets of four to equal two.
I: Now if I told you you had 2 4/5 cookies, how many fifths did you have, what would you tell me?
Teresa: Ummm (pause) I don't know.
I: You have two, two whole cookies and 4/5 more of a cookie, and I want to know how many fifths there are.
Teresa: . . . (got out fraction circles on her own to represent 2 4/5) . . . I have 14/5.
I: Why did you say you had eight?
Teresa: I don't know . . .
I: How come you could do the problems just a few minutes ago when were were using cookies, but now you aren't so sure? Which answer do you think is right?
Teresa: Fourteen-fifths. I can see it now. . .
I: . . . I want you to write 3 5/8 as an improper fraction.
Teresa: Twenty nine-eighths, eight go into three, so it's 8, then 16, then another 24 plus five is 29. . .
I: . . . Now write 14/3 as a mixed numeral.

Teresa: (wrote 3/3 3/3 3/3 3/3 2/3) Four and two-thirds, I had to write it down or else I'd get it mixed up in my head.

Teresa's response for expressing 3 1/8 cokies in terms of eighths suggested that she utilized her knowledge about the number of parts needed for a whole to solve the problem. Her response to writing 2 4/5 as an improper fraction suggested that she had not related the symbolic representation to her prior knowledge but was applying an incorrectly learned procedure. After Teresa used the fraction circles in conjunction with her prior knowledge to overcome her misconception, her responses suggested that she had invented an alternative algorithm for converting mixed numerals and improper fractions that was based upon prior knowledge about the number of parts needed to make a whole.

Teresa's initial experience with mixed numerals and improper fractions was characteristic of the other seven students' experiences. They also invented alternative algorithms based upon prior knowledge, and eventually all of the students discovered on their own that they could use multiplication and division to convert mixed numerals and improper fractions.

Mixed numerals and improper fractions was only one of the many situations in which students utilized prior knowledge about the number of parts needed to make a whole to give meaning to fraction symbols and procedures. Examples involving estimating sums and differences and solving subtraction problems such as 4 - 7/8 appear in the following sections. In all of these situations, students' prior knowledge was only constrained temporarily when misconceptions existed with respect to fraction symbols and procedures. The students' flexibility in utilizing this prior knowledge suggested that this knolwedge played a critical role in the ability to relate formal symbols and procedures to prior knowledge.

Knowledge About the Number of Parts Needed to Make Half of a Whole

Students' knowledge about the number of parts needed to make half of a whole was based upon prior knowledge about the number of parts needed to make a whole. Similar to the knowledge discussed in the preceding section, this knowledge was knowledge about individual parts comprising half of a whole. The students frequently utilized this knowledge in a variety of situations that included: (1) identifying and finding fractions equivalent to 1/2, (2) estimating sums and differences, and (3) adding and subtracting unlike fractions.

Ned and Bob present examples of how they utilized their knowledge about the number of parts needed to make half of a whole to estimate sums and differences involving fractions and to add and subtract unlike fractions. Ned's protocol was taken from his ninth instructional session. Prior to this, Ned was asked to estimate 9/10 + 1/25 during his third instructional session. When examining 1/25, he asked, "How do you say it's small?"; therefore, instruction suggested that he initially use 0, 1/2, and 1 as references for estimating individual fractions when working with symbolic representations. Ned had also done some work with adding and subtracting like and unlike fractions. Bob's protocols were taken from his sixth and tenth instructional sessions. Prior to these sessions Bob received instruction similar to Ned's for estimating sums and differences and adding and subtracting fractions. During Bob's fifth session, he was asked to estimate 8/9 + 3/7 in the context of a cake problem. Bob rewrote the problem as 4/5 + 1/2, and then said, "I don't know how to do that."; therefore, instruction suggested that he initially use 0, 1/2, and 1 as references for estimating individual fractions.

Ned - Ninth Instructional Session

I: I want you to solve this problem, 5/8 plus 1/2.

Ned: (wrote 5/8 + 1/2). This (5/8) is about close to a half isn't it?

I: Yes, it's real close to a half, so what do you think the answer's going to be?
Ned: About one, . . . well 5/8 is close to 4/8 and 4/8 is, four plus four is eight so if that (indicating 5/8) was a half [plus] a half, one whole.
I: That's a good estimate and now what is the exact answer? . . .
Ned: . . . (wrote 5/8 + 4/8 = 9/8) Oh, well you could make one whole and have two pieces left over.
I: Two pieces?
Ned: One piece. . . well 'cause 8/8 make a whole pie and you have 9/8.

Bob - Sixth Instructional Session
I: I want you to estimate the answer to this problem, 8/9 plus 3/7.
Bob: Well, 8/9 is about two whole.
I: Are you sure?
Bob: No, it's about one whole because 9/9 is one whole, and 3/7 is about a half . . . 3 1/2 is half of seven and three is only half away from a half. . . so it's about 1 1/2. . .
I: . . . Suppose you have about 2 2/3 cakes and you and your brother eat about 1 3/4 cakes, about how much cake do you have left?
Bob: Two-thirds is about a whole.
I: Is it closer to a whole than a half?
Bob: No, it's closer to a half, because 1 1/2 is half of three, and 3/3 is a whole, and two away from 3/3 is one whole number, and these two away from 1/2 is only a half, so that'd be 2 1/2 (rewrote 2 2/3 as 2 1/2). Subtract one, and 3/4 is like 2/3, wait, no it's not, wait 2/3 . . . 3/4 is closer to, it's on the line between two and four . . . [it's] almost 2/4 or almost 4/4 . . . so it's equal . . . I'll go down (rewrote 1 3/4 as 1 2/4) ... 2/4 is the same as a half, so I can write it 1 1/2, okay, so that makes it easier, then you get one whole.

Bob - Tenth Instructional Session
I: Suppose you have 2 5/6 Hershey Bars and I give you 1 1/2 more Hershey Bars, how many Hershey Bars do you have?
Bob: I'll estimate it first because then I can tell if my answer is right. . . Three, no four (pause) and 2/6 exactly.
I: What's another name for 2/6?
Bob: One-third.
I: Now how do you know that's exactly 4 1/3? I can't see that.
Bob: Yea, you go three plus one is, two plus one is three, and you need 3/6 for a half, and you already have a half here so 5/6. . .
I: (interupting Bob) Wait, wait! Why do you need 3/6 for a half?
Bob: Well, because 3/6 is half (referring to 5/6) and there's a half over here (referring to 1 1/2), so that'd make a whole (wrote 2 1/2 2/6 1 1/2 4 2/6).
I: Well can't I just add the six plus the two?
Bob: No because then that'd be changing the number of (pause) pieces you have . . . or what they're divided into . . . But 5/6 is more than 3/6, it's 2/6 more, so what we have here is 1/2 and 2/6 (referring to what he'd

> written) . . . so then two plus one is three and a half and a half is four and 2/6.

Both Ned's response concerning 4/8 and Bob's responses concerning 3/7 and 2/3 suggested that the students had related fraction symbols to their knowledge about the number of parts needed to make half of a whole to invent an alternative algorithm for finding fractions equivalent to 1/2, addding the numerator twice to obtain the denominator. Bob's response that 5/6 is composed of 1/2 and 2/6 suggested that he applied his prior knowledge to the problem to invent an alternative algorithm for adding unlike fractions.

All of the other students utilized their knowledge about the number of parts needed to make half of a whole to invent algorithms similar to those of Ned and Bob for finding fractions equivalent to 1/2. The other students also utilized this prior knowledge in a manner similar to Bob's when estimating sums and differences involving fractions such as 2/5. The students experienced no difficulty in this situation, but simply stated that "2 1/2 is half of five and two is 1/2 away from 2 1/2"

Bob's example concerning 2 5/6 + 1 1/2 was only one of three isolated examples where students applied thisprior knowledge to problems involving adding and subtracting unlike fractions. Students' prior knowledge about the number of parts needed to make half of a whole was largely constrained when students encountered problems of this type.

Teresa presents one example of the constraints placed upon this knowledge when adding and subtracting unlike fractions. Teresa's protocol is taken from her seven instructional session. Prior to this session, Teresa received instruction similar to that of Ned and Bob, and she responded in a similar manner when estimating sums and differences.

She was also able to quickly answer any question she had been given involving equivalent fractions.

I: I want you to work this problem, 1/4 plus 3/4.
Teresa: (wrote 1/4 + 3/4 4/4 1, left out =).
I: Now try this one, 3/4 minus 1/2.
Teresa: That's what I don't know. I'm just trying to guess. This is just a guess. Don't you like two can go into four two times, then you make it two or something?
I: Remember our equivalent fractions?
Teresa: Uh huh (pause).
I: We can use them to help us work the problem.
Teresa: (pause).
I: What's another name for 1/2?
Teresa: (immediately) 2/4.
I: So we can rewrite the problem as 3/4 minus 2/4. (Teresa wrote 3/4 - 2/4 1/4, left out =) . . . We're renaming 1/2 as 2/4 and when we do that we're finding equivalent fractions so we can have the same denominators . . . Try this one, 1/4 + 1/2.
Teresa: (wrote 1/4 + 1/2 =) (pause). I think the denominator is going to be four, but I don't know how to change it.
I: What did we rename 1/2 as in that problem (indicated 3/4 - 1/2)?
Teresa: Oh (wrote 1/4 + 2/4 = 3/4).

Teresa's response of "that's what I don't know" suggested that she was not drawing on her prior knowledge about the number of parts needed to make half of a whole. Her response also suggested that she was focusing on a procedure for adding unlike fractions rather than her prior knowledge.

Other students responded in a manner similar to Teresa's when they encountered addition and subtraction problems involving unlike fractions. Even when the problems were presented in the context of real world situations, the students did not focus on their prior knowledge, but focused on procedures for adding and subtracting fractions. Because of the influence of this prior knowledge about procedures, which will be discussed more fully in a following section, it is not clear whether or not constraints would have been placed upon students' knowledge about the number of parts needed to make half of a whole or not if they did not have prior knowledge about procedures. Nevertheless, the students' knowledge in this study was constrained by prior knowledge of procedures.

The students demonstrated that they came with prior knowledge about the number of parts needed to make half of a whole. They further demonstrated that they were able to utilize this knowledge to invent alternative algorithms for finding fractions equivalent to 1/2 and for estimating sums and differences.

Knowledge About Joining and Separating Sets

Students' knowledge about joining and separating sets was knowledge about joining and separating parts of wholes. Six of the students utilized this knowledge to remind them of the correct procedure for adding and subtracting like fractions, thus, alleviating the common misconception that numerators are added together and denominators are added together when fractions are added. All of the students utilized this knowledge in conjunction with prior knowledge about the number of parts needed to make a whole to alleviate misconceptions related to renaming whole numbers as fractions and to correctly subtract a fraction from a whole number or like mixed numeral when the fractions were represented symbolically.

Aaron presents one example of a student who utilized prior knowledge about joining and separating sets and the number of parts needed to make a whole to solve subtraction problems represented symbolically. The following protocol was taken from Aaron's Initial Assessment and his second instructional session.

Initial Assessment

I: Now I want you to solve this problem (showed Aaron piece of paper with 4 - 7/8 printed on it).

Aaron: (wrote 4 - 7/8 on his paper) Well, you change this (4) . . . to 4/4.

I: Why 4/4?

Aaron: 'Cause you need a whole, so you have to have a fraction and that's that fraction, and then you have to reduce, or whatever that's called, that (4) times two, so you'll have 8/8. Eight-eighths minus seven, so it's 1/8.

I: Now suppose I told you you have four cookies and you eat 7/8 of one cookie, how many cookies do you have left?

Aaron: You don't have any cookies left. You have an eighth of a cookie left.

I: If you have four cookies...

Aaron: (interupting I.) Oh! Four cookies!

I: And you eat 7/8 of one cookie, how many cookies do you have left?

Aaron: Seven-eighths of one cookie? Three and one-eighth.

I: Now how come you got 3 1/8 here (referring to what Aaron had just said) and you got 1/8 there (referring to paper)?

Aaron: (pause) (looked over problem) I don't know. (comtemplated problem, repeated problem). Well because on this you're talking about four cookies, and on this you're talking about one.

Second Instructional Session

I: Last time we were working on the problem 4 - 7/8.

Aaron: (immediately wrote 4/4 - 7/8) That's impossible! ... This (4/4) is smaller than that (7/8), in fraction form it is. This (4/4) actually equals one.

I: Suppose you have a board four feet long and you cut off a piece 7/8 of a foot long to make a shelf. How much of the board do you have left?

Aaron: (looked at the problem he had written earlier, 4/4 - 7/8).

I: Don't look at your problem [on paper]. (repeated board problem).

Aaron: (drew line for board, first thought 7/8 of the whole board, I. repeated the problem, Aaron marked off the board to show four feet). Oh, I know now, 3 1/8 feet...

I: Very good. Now you said the problem couldn't be worked.

Aaron: You have to multiply to find the same denominator which is eight, so four times two is eight and this four times two is eight so it's 8/8. (wrote 3 8/8 - 7/8 = 3 1/8 on his paper).

I: Now where'd you get this 3 8/8?

Aaron: This used to be 3 4/4, and 4/4 is one, and I need that so I can take a piece away...

I: You couldn't figure that problem out last time.

Aaron: I thought four was the same as 4/4, but it's really the same as 3 4/4, 3 8/8, 3 2/2, 3 1/1...

I: ... I want you to solve this problem, 4 1/8 minus 5/8.

Aaron: (immediately wrote 3 8/8 - 5/8 = 3 3/8).

I: Let's use the pieces to see if that's right ...

Aaron: ... (put out three and one-eighth circles) ... Wait, 3 1/8, (looked at his paper), I can't change it to 8/8 for some reason.

I: Why not?

Aaron: It's gotta be changed to 9/8. (changed 3 8/8 to 3 9/8 on his paper).

I: Why 9/8?

Aaron: Because there's one piece over there and you have to add it.

I: How'd you figure that out?

Aaron: It just seems like that's the right thing, 'cause if there's 8/8 and there's one over here, you have to add it. You can't just forget about it, and that'd make 4/8, 3 4/8, well 3 1/2. (wrote answer on his paper).

Aaron's response during the Initial Assessment that 4 was the same as 4/4 illustrated a misconception related to symbolically representing fractions as whole numbers. Aaron's response of 3 1/8 when working with cookies and 1/8 when working with symbolic representations suggested that he had prior knowledge that was unrelated to his knowledge of fraction symbols. During his second instructional session, Aaron's responses suggested that he had related the fraction symbols to his prior knowledge both when he decided that 4 should be renamed as 3 8/8 and that 4 1/8 should be renamed as 3 9/8 to correctly solve the problems with pencil and paper.

All of the other students utilized prior knowledge in a manner similar to Aaron's when they encountered problems such as 4 - 7/8. When working with symbolic representations, the students initially renamed 4 as 4/4, but they correctly solved the problem when it was presented in the context of a real world situation. The students then utilized this knowledge to solve problems such as 4 1/8 - 7/8 and 4 1/8 - 1 7/8. All of the students successfully solved these problems prior to his or her fourth instructional session. Thus, all of the students were able to solve difficult subtraction problems early in the study by drawing upon prior knowledge about joining and separating parts of wholes.

Although the students were able to use prior knowledge about joining and separating sets to solve some difficult problems, this knowledge was constrained to situations involving like fractions, or for some students, where various combinations of 1/2 and 1/4 were involved. This knowledge did not extend to situations involving problems related to 1/2 + 1/3. Teresa's example involving 1/4 + 1/2 in the preceding section illustrates the constraints upon this knowledge. Again, the constraints may have been imposed by students' prior knowledge about procedures, but whether they were or not, these constraints did exist. Nevertheless, the students demonstrated on several occasions that they were able to relate fraction symbols and procedures to prior knowledge about joining and separating parts of wholes to solve some difficult problems.

Summary to Building Upon Informal Knowledge

All of the students came to instruction with misconceptions related to fraction symbols and procedures. The students also came with prior knowledge of specific fraction ideas that was knowledge about parts of wholes in real world situations. The students built upon this prior knowledge to give meaning to fraction symbols and procedures, thus, alleviating many of their misconceptions related to formal symbols and procedures and frequently inventing alternative algorithms.

Influence of Prior Knowledge About Procedures

All eight student came to instruction with prior knowledge of fraction ideas that was unrelated to knowledge of fraction symbols and procedures. The students frequently built upon this knowledge to give meaning to fraction symbols and procedures. All eight students also came with prior knowledge about procedures for performing operations with fractions that were represented symbolically. Students' prior knowledge of procedures included procedures for finding equivalent fractions and adding and subtracting like and unlike fractions. This prior knowledge frequently reflected common misconceptions when working with symbolic representations, such as adding numerators together and adding denominators together when adding and subtracting like and/or unlike fractions.

In situations where students possessed prior knowledge of procedures, they tended to focus on the procedures, whether correct or incorrect, rather than prior knowledge of real world situations. Jason presents one example of a student who focused on prior knowledge of procedures when addding like fractions. The following protocol was taken from Jason's Initial Assessment.

Jason: . . . See if they tell me [anything] . . . I do it. I don't really care about math. If it's gonna help me I just use it in the way, I don't have to understand it, I just have to know how to do it . . . I don't really want to understand fractions. . .
I: . . . Suppose you had 2/8 of a pizza and I gave you 3/8 more of a pizza, how much of a pizza would you have?
Jason: (repeated problem) One-half probably, 1/2 or one whole.
I: There's a difference between a half and a whole. Why do you say a half, or why do you say a whole?
Jason: Because, the answer to this kind of question is usually a half or a whole.
I: . . . This type of question, what do you mean by that?
Jason: Okay, they ask you, add this and this. They're not gonna give you like half and a quarter or like one away from a whole, they usually don't do that.
I: Who's they?
Jason: The people who write the tests.
I: (repeated pizza question).
Jason: Do you have to add 'em?
I: Yes. (had Jason write problem on his paper).
Jason: I was thinking about whether adding the top part or bottom part. . . You can't add the bottom part, that's too common. . . Well see, I'm not looking at this mathematical, I'm looking about what would they think, and so I think. . . You wouldn't have me change the denominators of two and three and have me add eight and eight, no you wouldn't. . . 'Cause that'd be wierd. . . It's just, okay you
Jason: (continued) think about it. . . Now look, I kind of know (pause) you just think about what people do . . . I know what it's gonna be like. You don't add eight and eight together and change the denominators of two and three, so I guessed that eight and eight must be the bottom part and that you leave that stay and that two and three 'cause you wouldn't, I mean I know. I mean you would but that would be against some odds. . .
I: . . . Why are you always trying to out-guess people?
Jason: Because it usually works.
I: . . . But sometimes it doesn't work.
Jason: Get one wrong. . . If you don't know something it's smarter to guess than to totally flunk.

Jason's responses suggested that he clearly did not care to know any more about fractions than the procedures he needed, and that he had invented his own procedures for solving a variety of problems. His response concerning why 3/8 + 2/8 equals 5/8 illustrated that he had focused on prior knowledge of procedures rather than prior knowledge about joining and separating parts of a whole to solve the problem.

Other students focused on this prior knowledge about procedures in much the same way that Jason had; however, their explanations were not as verbose, but were stated simply as "That's what my teach taught me in fifth-grade." or "It's a rule in fractions.". Although students tended to focus on procedures in situations where they had this prior knowledge, they were able to overcome its influence and build upon prior knowledge that was unrelated to fraction symbols and procedures. For all of the students, overcoming the influence of this prior knowledge was not easy task, but one that required a great deal of time and assistance from instruction that specifically addressed this issue.

Tony presents one example of a student who had prior knowledge about joining and separating parts of wholes, but he also had incorrect prior knowledge about procedures for adding fractions, which he focused on when adding fractions represented symbolically. The following protocol was taken from Tony's Initial Assessment and his third, fifth, and tenth instructional sessions. Between each of these sessions, Tony received extensive instruction on addding and subtracting fractions using the fraction strips. He solved numerous problems involving like and unlike fractions while instruction stressed that he could easily determine the answer when the fraction strips were the same size pieces. The protocol illustrates Tony's misconceptions related to adding fractions represented symbolically, his prior knowledge about joining and separating parts of wholes, and his struggle to overcome the influence of his prior knowledge about procedures.

Initial Assessment

I: When you add fractions, how do you add them?
Tony: Across.
I: So you add the top numbers across and then the bottom numbers across?
Tony: Yea.
I: Does it make any difference if the bottom numbers are different or the same?
Tony: No. . .
I: . . . I want you to think of the answer to this problem in your head. If you had 3/8 of a pizza and I gave you 2/8 more of a pizza, how much pizza would you have?
Tony: 5/8 (went to his paper on his own initiative and wrote 3/8 + 2/8 =, gasped, stopped, then wrote 5/8). I don't think that's right. I don't know. I think this (8 in 5/8) just might be 16. I think this'd be 5/16.
I: Let's use our pieces to figure this out. (Tony got out 3/8 and then 2/8 of the fraction circles and put the pieces together.) Now how much do you have?
Tony: Five-eighths. It seems like it would be sixteenths . . . This is hard.
I: (wrote 23 + 42 and led Tony to see adding like place values) It's the same thing with fractions, we add things that are the same, or we have to have the same size pieces.

Third Instructional Session

I: Suppose you have 3/4 of a pancake and I give you 1/2 more of a pancake, how many pancakes do you have?
Tony: (got out fraction circles, three 1/4's and one 1/2, put two 1/4's and 1/2 together) Four and 1/4?
I: Do you see four whole pancakes?
Tony: (long pause) I do.
I: You see four whole pancakes?
Tony: No, not four, umm, four 1/3's, wait, this is not easy. . .
I: . . . (covered 1/2 with two 1/4's using fraction circles) Now how much do you have?
Tony: One whole pancake . . . Four-fourths, one, . . . 1 1/4.
I: . . . (asked Tony to write 3/4 + 2/4 =) And what does that equal?
Tony: Five-eighths. No!
I: Wait, you just told me you had?. . .
Tony: . . . One and 1/4, but how do we get the one (pointing to 3/4 + 2/4 written on paper)? There's three plus two.
I: What's that equal to?
Tony: Five.
I: Five, and what do you say the denominator is supposed to be?
Tony: The denominator's down here and that's a four, but these two are going to be five. (wrote 5/4).
I: You just told me this is going to be 5/8, why'd you write 5/4?
Tony: 'Cause you don't add the two denominators. . . 'cause it's a rule . . .
I: Let's use our pieces and see if we can figure out why it's a rule. (I. asked Tony to model several problems such as 3/5 + 1/5 using the fraction

strips or circles.) It's like with whole numbers, we have to add things that are the same, and the denominator tells us if the fractions are the same and how many parts each one is divided into. It's easy to tell what our answer is if they're divided into the same size pieces.

Fifth Instructional Session

I: Suppose I have a board 5/5 of a foot long and you give me 2/5 feet more of a board, how much board do I have?

Tony: Seven-fifths feet.

I: How come I don't have 7/10?

Tony: 'Cause you don't add the two denominators . . .'Cause it's a rule . . .

I: Let's see if we can figure this out. (repeated problem and had Tony model the problem with the fraction strips).

Tony: (wrote 5/5 + 2/5 , did not write = or answer the question) You can't get 7/10, (pause). Isn't it like you said, you can't add tens and ones, see they're fifths, that means umm, umm, you'd have to cut these in half each time.

Tenth Instruct: ,nal Session

I: I want you to solve this problem, 1 5/8 plus 2 7/8.

Tony: (wrote 1 5/8 + 2 7/8) . . . Thirteen-eighths. I mean (pause), ohhhh! I found out the whole problem! . . . It's 4 4/8 or 1/2. . . Well, I figured I'll add this (5/8 and 7/8) up. . . that's 12. . . and then I thought that's more than one so what if I just made that one and I took the remainder of it and made it into a fraction, so this became four and I kept the denominators the same though . . .

I: Why'd you take off some numbers to make it 8/8, to make it one?

Tony: Because it was easier for me if I did that, if I made this just a full solid number one, and then I took the rest and made it a fraction, and I kept the denominators the same because they tell me what size pieces I have . . . I think I found out how to add fractions.

I: I thought you already knew how to add fractions.

Tony: I know, but this seems to be easier.

Tony's response of 5/8 to the pizza problem during the Initial Assessment suggested that he had prior knowledge about joining and separating parts of wholes. His response that 5/8 should be 5/16 when working with symbolic representations illustrated the influence of his prior knowledge about incorrect procedures. With some assistance from instruction to help Tony see why he needed like denominators when adding and subtraction fractions, Tony alleviated his misconception for adding fractions; however, he continued to focus on his knowledge of procedures rather than his other prior knowledge when adding and subtracting fractions, which was illustrated by his frequent response of "it's a rule". Tony's responses in his fifth session suggested that he was beginning to shift his focus from his knowledge about procedures when he explained why the answer to 5/5 + 2/5 could not be 7/10. During his tenth session, Tony's explanation for why he kept the denominators the same and his comment that he had discovered how to add fractions suggested that he had overcome the influence of his prior knowledge about procedures.

Other students responded in a manner similar to Tony's when they encountered situations in which they possessed prior knowledge about procedures. Many times their knowledge was characterized by misconceptions. Even after instruction specifically addressed these misconceptions, students continued to focus on this knowledge rather than on prior knowledge of fractions in real world situations. With a great deal of time and care from instruction, all but one of the students overcame the influence of this prior knowledge about procedures and built prior knowledge of other fraction ideas to give meaning to formal symbols and procedures.

DISCUSSION

This study presents a different picture of students' understanding about fractions than has been presented by previous studies. Whereas previous studies suggested that many students have little understanding of fraction symbols and procedures, this study shows students coming to instruction with a substantial store of prior knowledge about fraction ideas that they are able to build upon to give meaning to formal symbols and procedures.

Although the students demonstrated that they were able to relate fraction symbols and procedures to prior knowledge to give them meaning, the results suggest that there is a danger of this knowledge interfering when it reflect algorithmic procedures rather than fraction ideas in real world situations. Students' focus on symbolic manipulations, whether correct or incorrect, in situations where they had prior knowledge about procedures suggested a dominating influence of this knowledge. Although the results show that this prior knowledge often interfered with students' attempts to give meaning to fraction symbols and procedures, they do not suggest that its influence cannot be overcome. However, they do suggest that overcoming the influence of this knowledge requires a great deal of time and sensitivity on the part of the teacher. Therefore, the results add more evidence to the argument in favor of teaching concepts prior to procedures, and suggest that in doing so, students can build upon prior knowledge in ways that are meaningful to them.

As the students related fraction symbols and procedures to prior knowledge of fractions ideas, their responses suggested that they think about fractions in a way that differs from what is traditionally taught. Their responses suggested that they focused on the size of specific fractions, and that they had a clear understanding of fractions such as one-fourth of a pizza. These responses furthered suggested that students viewed fractions as distinct parts of a whole rather than as a whole partitioned into equal-sized parts with a specific number of parts designated. The students' frequent references to numbers of pieces, such as "they both have one piece missing from a whole" or "four one-fourth pieces make a whole cake", suggested that their prior knowledge was quasi-whole number in nature. Their misconceptions however, suggested that students knowledge of fraction symbols and procedures was initially unrelated to their rich store of prior knowledge.

Therefore, the results suggest that students think of fractions as distinct parts of wholes in real world situations rather than as a whole divided into equal-sized parts.

The students' explanations also suggested that the "natural development" of their fraction ideas differs from the traditional sequencing of fraction topics. Their explanations for symbolic manipulations in terms of cakes, cookies, boards, etc., and their inventions of alternative algorithms suggested that they were building upon prior knowledge in a way that was meaningful to them. Thus, they were able to solve problems early in the study that have traditionally been considered very difficult, such as 4 1/8 - 7/8 or converting mixed numerals and improper fractions. Therefore, the results suggest that by building upon prior knowledge of fraction ideas, the development of students' understanding about fraction symbols and procedures may proceed in a very non-traditional manner.

This study is a beginning in examining the role that students' prior knowledge may play in the development of their understanding about fraction symbols and procedures. The results of this study suggest that its role is critical. The results further suggest that students' prior knowledge of fractions can provide a basis for instruction; however, they caution that if instruction is based upon this prior knowledge, it must consider that the nature of this knowledge and its natural development may be very different from what is traditionally taught about fraction ideas. More insights are needed to determine ways in which students, who have and have not received formal instruction on fractions, in regular classroom settings can be encouraged to relate fraction symbols and procedures to prior knowledge to give them meaning. Nevertheless, this study demonstrated that in an individualized setting students were able to relate fraction symbols and procedures to prior knowledge in ways that were meaningful to them. Thus, the picture of students' understanding of fractions has been enlarged to show that students who have received formal instruction on fractions possess a rich store of prior knowledge that they are able to build upon to give meaning to fraction symbols and procedures.

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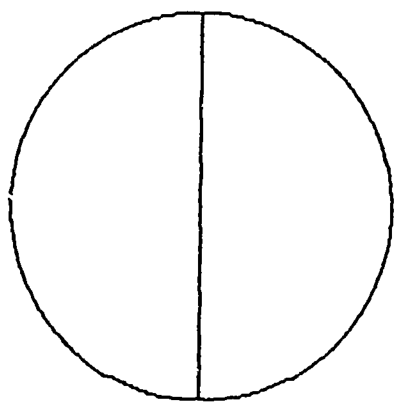

Figure 1: Teresa's Drawing of 1/2

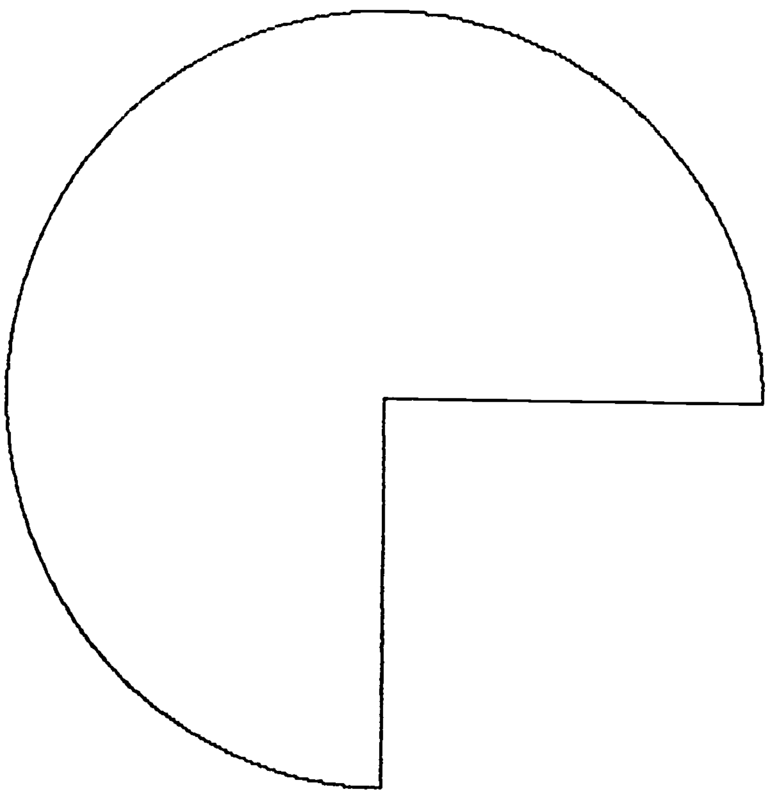

Figure 2: Teresa's Drawing of 3/4