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ABSTRACT

This document argues that qualitative graphing is an effective introduction to mathematics as a construction for communication of ideas involving quantitative relationships. It is suggested that with little or no prior knowledge of Cartesian coordinates or analytic descriptions of graphs using equations students can successfully grasp concepts of change and rates of change as well as using multiple representations to convey their understandings. Through discussion of their solutions with each other, students also observe multiple solutions which generate debate and controversy. It is argued that having students write their own thoughts on a graphing problem will encourage further development of self-monitoring and metacognition. Diagrams are included. (PK)

Qualitative Graphing: A Construction in Mathematics

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The thesis of constructivism, that knowledge is constructed rather than discovered, is extremely difficult for students to accept, especially with respect to mathematics. For experts, mathematics is a construction which describes a diverse assembly of ideas which, for the most part, are concerned with relationships among quantities. In contrast, the novice generally views mathematics as a collection of "truths" discovered by a supremely talented "elect" and broadcast by "knowledgeable" authorities. If students are to achieve conceptual understandings in mathematics and learn to reason effectively about quantities and quantitive relationships, then they must be weaned from the notion that mathematics can be learned through the memorization of facts.

The transformation of our students' mathematical epistemology can be facilitated by engaging them in activities which require them to construct solutions to problems which are open to challenging debate and heated discussion, yet result in the conceptualization of specific math content. Qualitative graphing is one such activity. Rather than graphing points designated by ordered pairs of numbers and related by the various features of formal analytic geometry (i.e., equations), students relate two quantities through graphical representations that require no numbers. Working in pairs or small groups, students can sketch and describe graphs which represent events, pictures of events, stories, geometrical properties, etc.

By exposing students to these different representations of knowledge we encourage more diverse and creative problem solving. Not only is there more than one way to solve a problem, but there are also many ways in which the solution may be discussed. Furthermore, conceptual understanding usually takes several forms in mathematics. Surely the notion of a function is only partially described with an equation. Pictures, diagrams, graphs, data tables and even computer programs are all effective representations of the idea of a function, and each representation enhances our conceptual understanding. The notion that all of these representations were invented to aid in our description of our ideas is quite beyond most of our students.

If the history of mathematics is, as Lakatos (1976) suggests, a dialogue where ideas are invented, discussed and refuted, then how can we impart to our students an appreciation for this dynamic process?

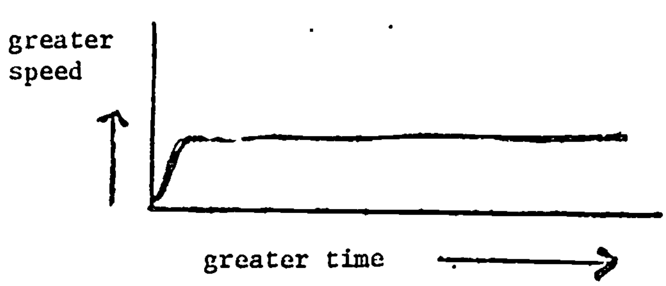
One solution is to establish dialogues in the classroom on problems that require conceptualization via multiple representations. Graphical representations are eminently suited to the task since they represent change in quantities as well as the absolute measures of quantities. Qualitative graphing does not require the use of numbers even though quantities are represented. Without numbers, students are at a loss in their search for appropriate algorithms which may be applied rotely. Instead, they must think through a solution themselves. Consider the following graphing exercise for students in an elementary college algebra course:

Graphing to Tell a Story: An Exercise for Students

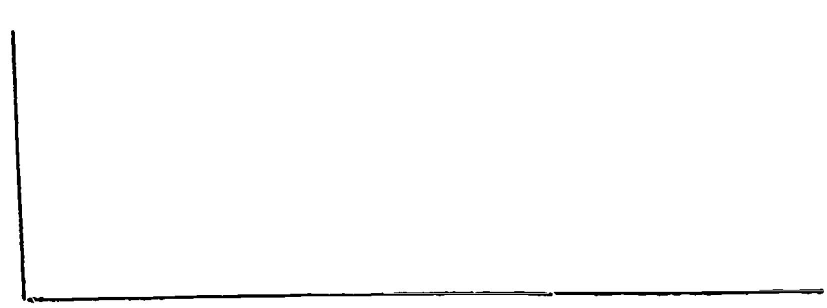
A graph is just another way in which people communicate. It can tell a story. For example, let's consider the story of the tortoise and the hare.

Just to refresh your memory it goes like this: The tortoise, who is slow and steady, is challenged to a race by the hare, a rabbit who is swift and overconfident. At the beginning of the race the hare passes the tortoise and then, feeling confident of his victory, decides to take a nap midway in the race. The plodding tortoise continues on past the hare towards the finish when all of a sudden the hare awakens and races at top speed but fails to overcome the tortoise who finishes the race moments before the hare. In order to graph this story we must first decide what are the important quantities that tell the story—basically graphs deal with relationships between quantities. Let's first see how the speed of the tortoise changed over the time of the race.

Graph of the speed of the Tortoise versus the time of the race.



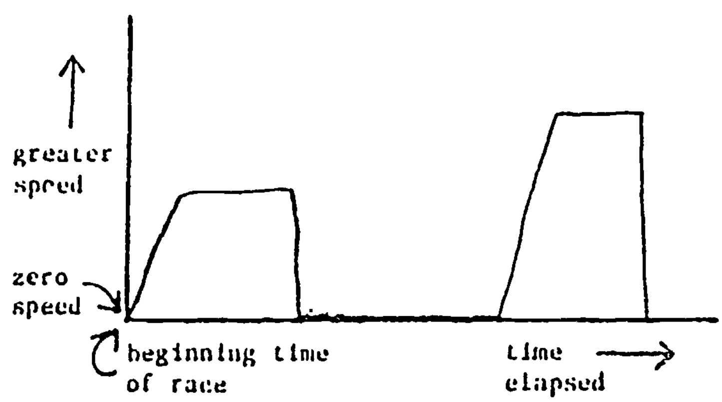
This graph shows that except for the time at the beginning of the race and at the very end of the race the tortoise's speed was unchanged with time. The straight horizontal line shows his constant speed. On the space below graph the speed of the hare for the time of the race:



Depending on how you choose to tell the story the graph of the speed of the hare versus the time of the race can vary. One possible graph is the following.

A possible graph of speed vs. time for the hare

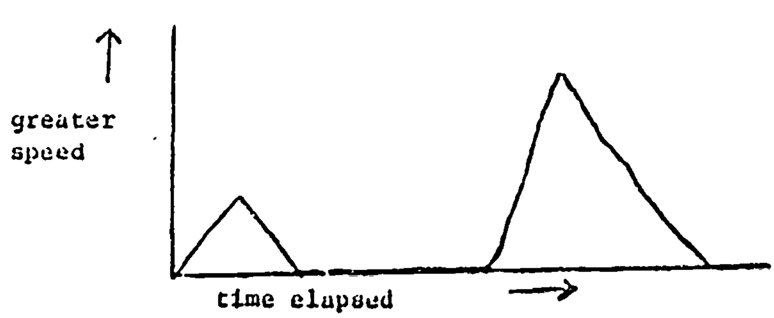
I. Hare's Speed vs. Time



This graph says that the hare jumped off to a fast start and kept a constant speed (greater than the tortoise's) until he decided to take a nap when his speed dropped to zero. When he awoke he bolted to an even higher speed--his top speed, which he kept until the end of the race.

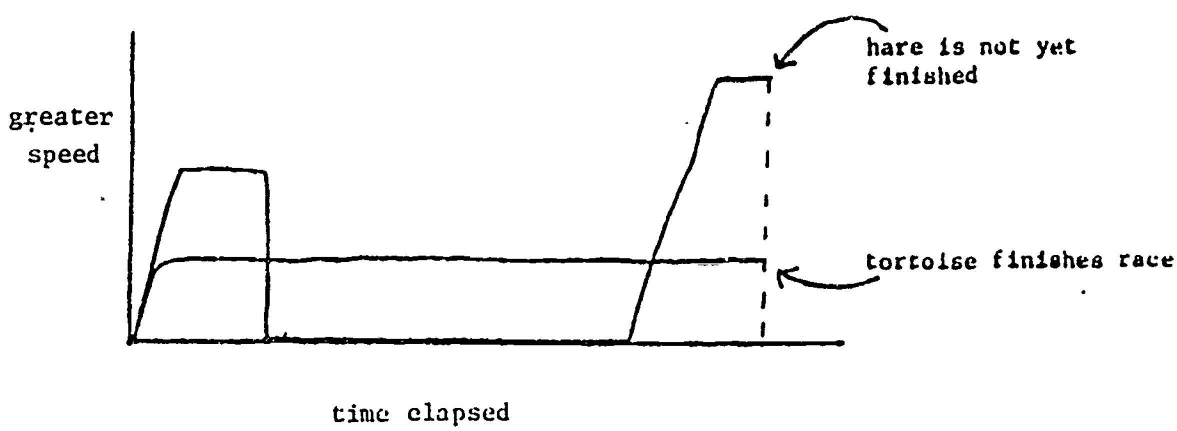
Notice that I could have drawn the graph other ways. Of course the story it would tell would be different. In your own words, describe the motion of the hare if its speed vs. time graph looked like this:

II. Hare's speed vs. Time



Let's place the first graph of the hare's speed together with the graph of the tortoise's speed.

Graph of the Speeds of the Tortoise and the Hare



Exercise I:

In the previous example we graphed the speed of the race participants as it changed over the time period of the race. Another obvious quantity that changed during the time of the race was the participants' distance from the starting line. In the space below carefully sketch the graph of both the hare's and the tortoise's distance from the starting line as it changes during the time of the race. Show your graph to one of your classmates to see if your graphs tells the story the way you intend it to be. How does my graph of their speed relate to your graph of their distance? Remember, they should tell the same story. Do they?

In the example above the students were asked first to graph a situation described in words, and then to describe in their own words a hypothetical situation given in a graph. The translation from verbal description into graphs and from graphs into words is a non-trivial task which is fraught with controversy from multiple interpretations. Students working in pairs and in small groups invariably launch into dialogues whose substance is the conceptualization of quantitative relationships and whose form is the representation of these conceptualizations. Cognitively speaking, the student dialogues exhibit most of the features described by Resnick (1986) as "Higher Order Thinking Skills", namely that their thinking is non-algorithmic, complex, uncertain, uses multiple criteria, and is self-regulated.

Writing and Qualitative Graphing

Besides having our students dialogue with each other verbally, they may also benefit from writing their solutions to graphing problems. One important benefit is that the students may examine their own thought processes while engaged in problem solving. The art of observing one's own thought processes is termed "metacognition" in the psychological literature, and has been accepted as a significant and sometimes necessary tool in successful mathematics problem solving. (Schoenfeld, 1981, 1982, 1983, 1984; Silver, 1982a, 1982b; Lesh, 1982; Garofalo and Lester, 1984).

Students in an elementary college algebra course were asked to write an in-depth description of their thoughts as-they-happen for their graphical representation of one of the following situations.

1. The distance you are from the band determines how loud it sounds to you.

2. The temperature of your cup of coffee is related to how long it has been cooling.

3. The time of sunrise depends on the day of the year.

4. The altitude of a punted football depends on the number of seconds since it was kicked.

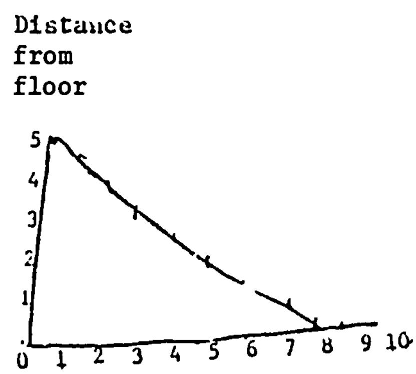
5. The distance you are from the reading lamp affects the amount of light on your book.

6. You pour some popcorn into a popper and turn it on. The number of pops per second depends on how long the popper has been turned on.

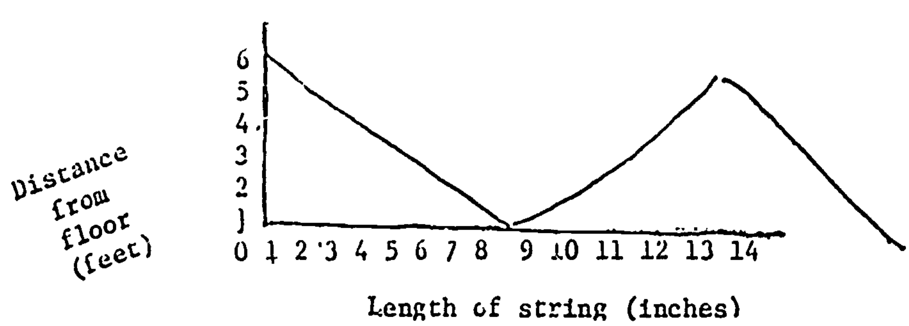
7. As you play with a yo-yo, the yo-yo's distance from the floor changes with time.

The following solution to problem number seven above was written by a student who demonstrates the critical thinking skills evident in the metacognitive activity of writing. The reader may observe how the student asks questions of herself and pursues a solution that both makes sense and instructs her in the correct conceptualization of rate-of-change and the graphical representation of that concept.

> I am going to write a thought protocol and graph number seven. "As you play with a yo-yo, the yo-yo's distance from the floor changes with time." I can label the graph with the distance from the yo-yo to the floor, and for time...When you let the yo-yo go down more string is being let out - it could be the length of string, or it could be counted for each time you let it go. It's in your hand at zero - it's on the ground at 10 - numbers 0 through 10 are the time in between your hand and dropping to the ground. Ok, assuming each time you let it go, you are doing it at the same speed you could use the seconds that are elapsing. For example, at 1 second it has just left your hand, at 5 seconds it's all the way down, at 8 it's half way up, etc. I am going to sketch a few graphs to see which works better.



> Ok. When it's in your hand lets say at shoulder level (we'll say 5")and <u>no</u> (feet) string has been let out yet, the coordinates are (5, 0) and as you let it out more string appears and it goes close tothe ground. Now the yo-yo starts back up.
>
> As in this graph - the yo-yo goes down, up and down again.



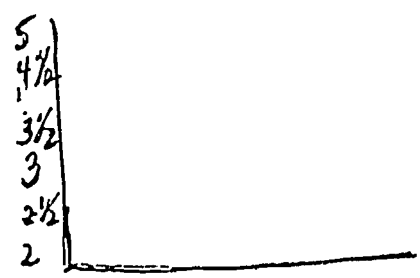
Now let's assume that the yo-yo takes 5 seconds to fully extend itself, and 5 seconds to recoil so, at 1 second it has just begun, at 3 seconds it's almost down, at 5 seconds it's down, at 8 seconds it's almost back up, etc.

Distance in feet from floor

seconds

I guess the number I have put on the left -- for distances - are inaccurate if it starts at 5 feet above the ground - the most a yo-yo goes is probably 2 feet. So, in fact I should put different numbers. Can I start the numbers at 2 feet, or do they always have to begin at 0? If they can start at 2 feet then perhaps I can put 2 feet, 2-1/2 feet, 3 feet, 3-1/2 feet and so on. 2 feet would indicate that the yo-yo is 2 feet from the ground, when fully extended. The graph would look more like this.

Distance

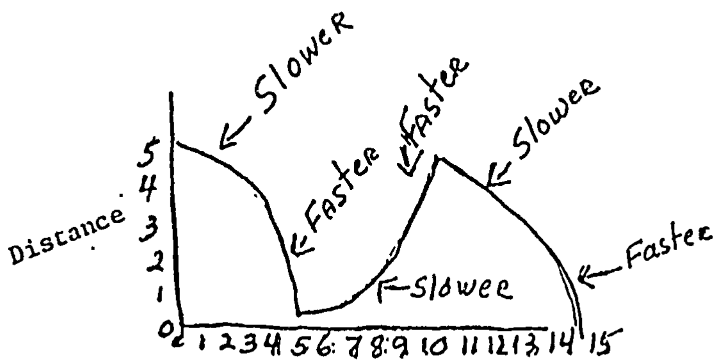


I just thought of something. I guess it doesn't matter how I number it, I just won't graph the line all of the way down to zero.

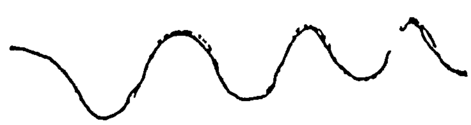
Distance from ground

Seconds Elapsing

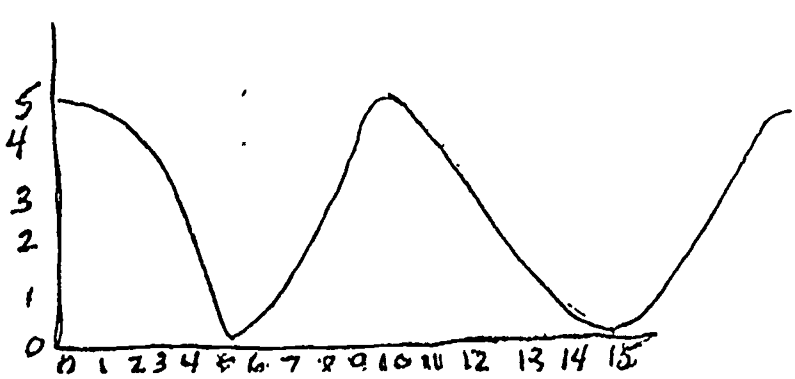
Now, I'm starting to wonder if it goes at a constant speed all the way up and down, or if it gets faster as it gets closer to the ground. It seems to me that things get faster as they fall further, but I'm wondering if this is true about a yo-yo. If that is the case though, wouldn't it mean that the speed coming back up would change too? This would put a curve in the graph if this were true. It would. Lets assume then that as it gets closer to the ground, the speed increases. Except - it seems to me that when it's re-coiling (coming back up) that it's the fastest. Let's assume that it gets faster as it gets closer to the ground, and gets faster as it gets closer to your hand. The graph would look more like this:



That looks like an awfully funny graph though. It seems like the curve should look more like this



I think my conclusion is this: That I should graph it as Distances vs. Time as opposed to Distance vs. the Length of string (could there be another way that I'm not thinking about?) and that the speed <u>does</u> change - my Final Graph:



This student demonstrates all of the higher order thinking skills mentioned earlier. After misinterpreting the problem, she caught her mistake and corrected it by identifying the correct variables of interest. After playing with several graphs (a heuristic she knowingly employed) she selected one which met the multiple criteria which she selected: Her close approximation to a sine-wave is an excellent conclusion to an admirable job of monitoring her own thoughts.

Conclusion

Qualitative graphing is an effective introduction to mathematics as a construction for communication of ideas involving quantitative relationships. With little or no prior knowledge of Cartesian coordinates or analytic descriptions of graphs using equations students can successfully grasp concepts of change and rates of change as well as using multiple representations to convey their understandings. Through discussion of their solutions with each other, students also observe multiple solutions which generate debate and controversy. Writing their thoughts on a graphing problem encourages yet further development of self-monitoring and metacognition. In summary: qualitative graphing provides a fun way for student. to positively change their perspective of mathematics.

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