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ABSTRACT

This document is one of three related volumes. They present the rationale, background, and framework for a comprehensive monitoring system being developed for the National Science Foundation. The system is being designed to gather information about the effects of national, state, and local policy actions designed to change the teaching and learning of mathematics in the schools of America. The papers included were produced by project staff, commissioned, or reprinted from previous works. Expert reviews and critiques of sets of papers are included. Volume 1 addresses the need for a monitoring center and the new world view of what is now considered fundamental for students to know about mathematics. Specifically, part 1 outlines the issues and concerns which need to be addressed in order to develop a reasonable monitoring system. Part 2 provides support and rationale for the notion that all students should learn more and somewhat different mathematics than is in the current curriculum. (PK)

Program Report 87-1

THE MONITORING OF SCHOOL MATHEMATICS: BACKGROUND PAPERS

Volume 1: The Monitoring Project and Mathematics Curriculum

Edited by Thomas A. Romberg and Deborah M. Stewart

Report from the School Mathematics Monitoring Center

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Wisconsin Center for Education Research

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PREFACE

This set of papers, published in three volumes as a monograph of the School Mathematics Monitoring Center, presents the rationale, background, and framework for a comprehensive monitoring system being developed for the National Science Foundation. The system is being designed to gather information about the effects of national, state, and local policy actions designed to change the teaching and learning of mathematics in the schools of America.

To build the monitoring system three assumptions were made. First, as a society we are involved in a major economic revolution. This revolution, addressed in Chapter 2, directly affects mathematics, its use, and what is deemed fundamental. As a consequence we believe "that most students need to learn more, and often different, mathematics" (Romberg, 1984, p. xi). Second, in spite of the changes in school mathematics inherent in the first assumption, we believe that there is general concensus about the goals for school mathematics and about the kinds of changes needed to achieve those goals. Thus, to develop the framework for the system one must begin with an understanding of those goals and the ideas on which they are based. Only then can indicators be developed to see whether the goals are being reached. Third, the policy actions with respect to the specific goals set for school mathematics must be consistent with the more general educational goals for a free and democratic society.

The need to monitor changes in school mathematics was proposed at two conferences. The first was organized by the Conference Board of the Mathematical Sciences (the New Goals Conference, CBMS, 1984), and the second by the National Council of Teachers of Mathematics, the U.S. Department of Education, and the Wisconsin Center for Education Research (School Mathematics: Options for the 1990s, Romberg, 1984). One conclusion from both conferences was that information about the nature of proposed changes and their effects on schooling practices was needed. During the past 25 years the federal government has invested considerable funds to change the teaching and learning of mathematics in America's schools, and today it is in the process of funding several new projects. Unfortunately, evidence of the impact of past dollars on classroom instruction is lacking. The special evidence that exists was unsystematically gathered and is incomplete. As new monies are spent and programs developed, it is crucial that a systematic plan be adopted to gather information about the effects of these planned changes.

During the past year the staff of the Monitoring Center prepared a series of papers, commissioned additional papers, convinced some authors to allow us to reprint a paper they had recently prepared, and asked a few nationally recognized experts to review and critique sets of papers. In all we have collected some 30 papers that address the issues of a new world view, what is fundamental in mathematics, what implications recent research in psychology or sociology has for school mathematics, etc. The intent of gathering these papers was to assist the staff of the project in the design of a monitoring system for school mathematics. However, since they comprise a review of the current thinking about schooling by a number of noted educators, we have chosen to publish them in this three-volume monograph so that others may have access to this information.

The first volume addresses the need for a monitoring center, the new world view, and what is now considered a fundamental for students to know about mathematics. In the second volume the implications of psychology to the learning of mathematics is addressed, and the problems of assessing learning based on both the new mathematical fundamentals and our knowledge of learning is examined. The final volume is comprised of papers that are based on current sociological notions about schools and how that knowledge affects the role of teachers and instruction in classrooms.

THE SCHOOL MATHEMATICS MONITORING PROJECT

In this initial section of the monograph the staff of the Monitoring Center have outlined the issues and concerns which we believe need to be considered in order to develop a reasonable monitoring system. Chapter 1 outlines the scope of the project and the issues that need to be addressed. Chapter 2 examines in some detail the new world view that is emerging and its importance to school mathematics. Chapter 3 presents the basic causal model that we argue needs to be developed. This model includes identification of the variables for which indicators are to be developed. The final two chapters in this section were solicited as critical reviews by two noted educators. George Stanic, a historian of mathematics education, presents his comments about the first three chapters in chapter 4. Finally, Richard Shavelson and his colleagues, who are also involved in the development of indicators, give their review of the monitoring plan in chapter 5.

CHAPTER 1

THE MONITORING OF SCHOOL MATHEMATICS

Thomas A. Romberg and Marshall S. Smith

In this paper we outline the initial steps for developing a system for monitoring the health and progress of school mathematics in the United States. We have specified a set of conceptual issues related to the need for the monitoring system, the audience, and the set of strategies for gathering information. Next, we plan to prepare specifications for developing some key indicators of the health and potential change in school mathematics. Finally, we will conduct a preliminary examination of extant data sets related to those key indicators. Based on the products of this work, we will plan and propose the establishment of a monitoring center.

The conceptual issues to be addressed as we plan the monitoring system are listed below.

1--A new world view and its impact on school mathematics.
2--New fundamentals of mathematics for schools.
3--Policy information and school mathematics.
4--A causal model for school mathematics.
5--The content-conceptual network scheme for assessing mathematical performance.
6--Reasoning, intuition, and mathematical problems.
7--Attitudes toward mathematics.
8--Attainment in mathematics.
9--Analysis of curricular content in school mathematics.
10--The nature of indicators.
11--Other variables and their indicators.

Need for a Monitoring System

The need to monitor the health and progress of school mathematics is based on four beliefs. First, in order to adequately plan, the local school systems, state departments of education, and the federal government, in particular the National Science Foundation, need systematic periodic information about the health of school mathematics and the types and degree of change. In fact, the National Science Board Commission (1983) recommended that

> the federal government should finance and maintain a national mechanism to measure student achievement and participation in a manner that allows national, state and local evaluation and comparison of educational progress. (pp. 11-12)

Their more specific recommendation was that

> the National Science Foundation should lead in evaluating progress in the application of new technology, supporting prototype demonstrations, disseminating information. . . . (1983, p. xii).

The second reason for developing a monitoring system derives from the concern with the nature of the evidence currently available about mathematics education and also with the nation's ability to monitor progress. The National Academy of Sciences (1982) outlined problems and the lack of adequate information regarding teachers, enrollments, and other important issues. A later Academy report on indicators of mathematics and science education made a compelling case for more adequate information. Furthermore, Stedman and Smith (1983), in their review of recent reform proposals, implied that the "poor quality of data that are currently available" (p. 103) led to weak arguments for change and may, in fact, mislead policymakers to prepare inefficient or even counterproductive policy changes. They argued that "a longitudinal data base is needed that could be used to check assertions about the causes of achievement" (p. 103). This plan responds directly to concern about the quality of data.

The third reason for a monitoring system is based on the belief that we are in an era of radical social and economic change that must be reflected in the programs of our schools. Hence, information about how schools are responding to pressures for change is critical. For the past century, Western society has been dominated by a coherent view of how the world works and of economic activity as a result of the "industrial revolution." This view incorporated such elements as analytic thought, experimental science, the factory metaphor, the concept of hierarchical organization, and the technology of paper. It influenced schooling with such ideas as detailed curriculum segmentation, behaviorism as a model of the learning process, and scientific management. However, we are now immersed in a "second industrial revolution." Developments in electronic communications, global transportation, and increasingly complex organization of economic activity--all supported by computer technology--require that we reevaluate our conceptions of the way the world works.

Powerful new metaphors are emerging that challenge both old ways of thinking about society and traditional economic and organizational practices. For school mathematics, there are two new metaphors of importance. The first involves the mind modeled as a complex communication system (Gardner, 1985), and the second, that knowledge is seen as an economic commodity (Bates, 1978). Notions from cognitive psychology are now prevalent: learning occurs not via absorption but construction; intelligence is not an unchangeable fixed trait; activity is considered goal directed rather than being a simple matter of stimulus and response reinforcement. Similarly, ideas from critical sociology have become important: the importance of opportunity to learn and the differential distribution of knowledge. In addition, the computer provides a powerful means of organizing and analyzing information, of visualization in multiple dimensions, and of repeated revisualizations.

The combined effect of a new technology, a new vision of learning, and a complementary view of a new social order is creating a new view of the world. This view is now causing a reevaluation of the schooling process in America. Schools, as we know them, are social institutions whose primary purpose is to transmit specific knowledge and skills to our young and introduce them to our social system. If the social system is changing, then both the knowledge and skills our children need and the social institutions that deliver that knowledge will have to change. This concern about schools and whether they are consistent with social expectations is not new; it is simply becoming more urgent as the educational system becomes increasingly incompatible with the changes occurring in other areas. Stated simply, the current educational system and pedagogical ideas are based on old paradigms that are inconsistent with the emerging new world view. The first conceptual issue to be addressed is this new world view.

States and schools are now attempting to respond to the pressures for change based on problems related to the new world view. Unfortunately, most responses are patched on to the existing system and its old paradigms. Even with the best of intentions, this approach to reform is not likely to provide the kind of response to the current pressures for change that is needed. However, the new policies and their effects in certain states and communities can serve as national experiments which will be of great interest to other local and state agencies. Careful monitoring of educational progress will itself both increase the speed of change and provide a basis for coordination of effort, if needed.

The final reason for monitoring school mathematics is based on the fact that the mathematical expectations (or goals) for our students have changed in light of the current social revolution. Old procedural skills, such as computational algorithms, are no longer as important because the calculator and computer have not only freed man from the necessity of performing such tedious calculations, but have made extremely complex models and other computations possible. Thus, quantitative reasoning, mathematical modeling, statistics, and problem-solving are now more important than ever before. There seems to be general agreement that all students must have a solid basis of mathematical knowledge and that a substantial portion of the population must learn more (and somewhat different) mathematics than ever before to function in the society of the next century. The second conceptual issue to be addressed involves what is now considered fundamental in mathematics based on the new world view of society.

In summary, for each of the above reasons, information is needed that can be used to evaluate the health and progress of school mathematics. Beyond addressing the two conceptual issues noted above, to plan a monitoring system we need to consider both the audience for which the information is to be gathered and the general strategy for gathering that information.

Audience

It is one thing to design a monitoring system for educational specialists. Information from such a system could be interpretable only by such specialists. For example, it would be possible to design a system to evaluate the quality of mathematics textbooks for a committee of mathematicians. It is another thing to design a way of capturing the quality of a mathematics text in a manner that is interpretable by a state legislator or school official with little mathematics background. For the purposes of the monitoring system, we propose that the primary audience must be state and local policymakers responsible for thinking about the quality of the educational programs in their jurisdictions. For example, we want the system to be able to be used to address the following questions:

How much attention is paid to complex problem solving by the schools in our state? How has this changed over time? Is it more or less than in other states?

How well have the students in our state learned to solve complex problems? Are differences in attention to problem solving reflected in differences in performance? Has performance on problem solving improved over time?

Do some kinds of children receive more or different exposure to content than others? by race? by social class? by sex?

Are differences in exposure related to differences in level of performance? attitudes? enrollment? use of mathematics?

Do students in our schools receive the same quality of mathematics curriculum as those in other economically similar countries?

Do the state curriculum guidelines reflect current thinking about what should constitute a quality mathematics program?

How well does the actual instruction that goes on in the state relate to the state curriculum guidelines?

Have the recent state reforms in teacher education changed the content and nature of mathematics education programs in our state and other states?

This sample of questions is illustrative of both the diversity and the complexity of the questions that need to be addressed. In addition, they demonstrate the difficulty of the task of providing understandable yet valid and reliable information to this audience of educational policymakers. Thus, the third issue involves an analysis of the information needed by policymakers related to the health and change of school mathematics.

Monitoring Strategy

The papers prepared on the first three conceptual issues should make it clear that information is needed to develop an effective policy for school mathematics related to its condition. However, given that there are limitations to the resources that can be devoted to data collection, a reasonable strategy must be developed. We propose a monitoring strategy that involves four components. First, a framework for identifying an efficient set of indicators must be built; second, key indicators of health and change must be specified; third, some indicators will need to be developed; and finally, given that there is a large amount of relevant data on school mathematics regularly gathered, it should be used to fill in the framework as far as possible to provide a baseline and to help suggest what additional data need to be gathered.

From the products of our work on the first four components we will then propose the establishment of a monitoring center that would regularly gather information about current projects and relate that to the key indicators; supplement existing data bases with a representative sample of data gathered via a combined cross-sectional/longitudinal design, and conduct a series of case studies.

Developing a Framework

The base of the monitoring system must rest on notions of what components of schooling are important for examining the health and change in school mathematics, how those components are related to each other, and how reasonable indicators can be developed for those components. Such a framework should be considered as a preliminary causal model for the health of school mathematics. Thus, the fourth conceptual issue involves building a causal model.

Although it is premature to specify and to discuss all the components of such a causal model, two necessary components are briefly presented for illustrative purposes.

Outcomes. The dependent variables in any causal model of school mathematics should involve the expected outcomes of mathematics instruction. Examples of such outcomes are as follows: Students will acquire knowledge of mathematical concepts and proficiency with mathematical skills. Students will be able to use that knowledge and processes in problem situations. Students will develop favorable attitudes toward mathematics and its social utility. Students will continue to enroll in mathematically related courses or programs.

Achievement tests in the past have been used as the key indicators of mathematical instruction. However, commonly used tests rarely measure proficiency on more than a few concepts and skills. For the near future we will have to rely on existing measures such as the NAEP, SMS, standardized achievement, and SAT tests. In the long run, we believe this is an inadequate approach to assessing outcomes. Although achievement must be considered a primary indicator of outcomes of instruction in mathematics, the measures of achievement must be altered to be more useful.

One change must be in the content of the measures. New measures must tap with more precision interrelationships between the concepts and skills within a domain. The approach we will use is an extension of the notion of a "conceptual network" for specific content domains. A conceptual network is a set of situations related to a content domain, the mastering of which requires a variety of concepts, procedures, and symbolic representations tightly connected with one another (Vergnaud, 1983). This approach is being used in part because of its successful use in recent research (Romberg & Carpenter, 1985) and in part because of the inadequacy of content-by-behavior matrices that have been used in standard testing programs (such as NLSMA, NAEP, or SIMS). Those matrices have specified content topics, subtopics, and even items as if they are independent of each other. Emphasis in such matrices is not on the structure or interrelationships of concepts and procedures within the content domain. Also, the behavioral dimension of matrices are based on levels of behavior (such as those in Bloom's Taxonomy, 1956). This taxonomy simply does not reflect current knowledge from psychology about how information is processed.

To overcome limitations of standard testing we will develop the new item framework to address three problems: curricular relevance, item aggregation, and item responses. Curricular relevance is important since one aspect of the monitoring scheme is to identify changes in achievement due to changes in the curriculum. A content-conceptual network scheme will be developed for the topics that are judged to be curricularly relevant. Items from any source can then be matched to the content topics. It will serve as the basic guide for the examination of existing data and the construction of new tests. This would allow us to cross-validate time trends using different data sources.

This content-conceptual network scheme for assessing outcomes must include both topics currently taught as well as those proposed to be taught. For example, items on probability, statistics, and discrete mathematics would be categorized, and prototypic items will be constructed if necessary. The content-conceptual network scheme also can be used as a basis for item aggregation and creation of a set of indicators. Finally, frequency of correct response, errors, and strategy used are to be coded for analyses of responses. The fifth critical issue that needs to be described and illustrated is the content-conceptual network scheme we propose to use to measure achievement.

Application of mathematics is the second category of outcomes we expect to use. Students should be able to use the knowledge they acquire. Simple word problems requiring students to apply learned concepts and skills are typically used as indicators of these outcomes. Their utility has been well documented in the NAEP data where students demonstrated they knew basic arithmetic skills but had difficulty using them to solve problems (Carpenter, Corbitt, Kepner, Lindquist, & Reyes, 1981). However, the use of mathematics with more complex problems (both in applications outside mathematics and in other mathematics situations) requiring quantitative or spatial reasoning can only be assessed with any validity via interviews or complex response schemes. This is of particular concern since it is assumed that a major emphasis of new mathematics programs will be toward the use of such reasoning in problem solving. Also, one special aspect of reasoning, intuition, needs to be examined since persons with this ability have been shown to be the creators of new knowledge.

Thus, the sixth conceptual issue addresses how to assess the kind of reasoning students use when using mathematics.

Mathematical attitudes is a third outcome often stated as a goal of mathematics education. The term "attitudes" is being used here to encompass the feelings, attributions, beliefs, perceptions, etc. that students have when confronted with mathematical tasks. At a superficial level items on student attitudes toward mathematical topics, mathematics teachers, mathematics or science careers, and the usefulness of mathematics are often given. These can be used as rough indicators of attitudes. For example, a recent Canadian study on grade 7 to grade 10 students' views on mathematics, calculators, and computers indicated that most students do not particularly like mathematics, but considered it important and indicated they would take more (McLean, 1982). Similarly, many students believe mathematics is hard, or something they cannot do, or only boys are good at, etc. One would hope that contemplated program changes would change such beliefs and feelings.

The seventh conceptual issue is on attitudes and how they can be assessed.

Although favorable attitudes are considered a desired end product of schooling in and of themselves, they may also be used as a predictor for other desirable outcomes such as increased involvement with mathematics or increased achievement. However, in this latter case, answers to attitude items may not be adequate. For example, to study whether attitudes influence the growing gender gap in computer use (Miura & Hess, 1983), a more indepth examination of attitudes, perceptions, or attributions would be needed.

Attainment in mathematics is the final outcome of school mathematics we expect to consider. This includes continued enrollment in and completion of mathematically related courses, choice of college majors, choice of careers, and later career paths including life income and job satisfaction. Each is important to individual and societal goals and to the development of human resources. Each, however, is mediated by many variables other than those associated with schooling. Course completion and attendance data will be gathered. Increased enrollments in higher mathematics courses, particularly by minority students and women, is an anticipated outcome of some of the prospected changes. The eighth conceptual issue is attainment in mathematics.

In summary the key indicators of outcomes should include information about achievement on several mathematical topics, use of mathematics on applied word problems, attitudes toward mathematics, and attainment in mathematics courses. Papers on the issues related to each of these outcomes, how they can be assessed, and how indicators can be developed will be the primary product of this project in its first year.

Curriculum content. To illustrate the need to develop other indicators about both the health of school mathematics and change let us examine another component in the causal model to be developed--curricular content.

Educational practice assumes that what occurs in schools when students are taught mathematics leads to their acquisition of knowledge and skills. One thing that occurs is that teachers follow a curriculum. Note that a curriculum can be examined at four levels: an ideal level that describes the content that society would like students to learn, an intended level that specifies the content in terms of a curricular plan by a policy agency, an available level as indicated by the text and other materials used in instruction, and an actual level of teacher decisions about what to emphasize, how much time to allocate, and what problems to assign.

It is important to study the curriculum at each level because the choices that are made involve providing students an "opportunity to learn." That is, whether and for how long students are exposed to mathematical topics is an important variable in schooling. Opportunity to learn consists of the content of instruction (the mathematical topics included in the planned school curriculum), the time allotted to each topic in the total curriculum, and the conditions for enrollment. To a considerable extent, content is determined by the textbooks being used. Both time and enrollment are controlled by teachers, although in secondary school students themselves decide at least in part how many units of a subject to study.

The choice of what mathematical topics are included in a mathematics program and how much emphasis each receives is critical. Both Husen (1967), summarizing the first IEA mathematics assessments, and Crosswhite, Dassey, Swafford, McKnight, and Cooney (1984), in the second, found student test scores in all participating countries to be correlated with the teachers' ratings of instructional coverage of the topics on the tests. To make such comparisons the method of analysis for the content of mathematics, texts should be based on the content-conceptual network scheme developed for assessing mathematical performance.

Although the identification of a key indicator for the actual curriculum content of mathematics instruction in each classroom is not feasible at present, this does not alter the importance of this component. Surveys will be made of mathematics textbooks used at each grade level in elementary school and for all mathematics courses in secondary school. Then a content analysis of the more commonly used texts will be done. Thus, the ninth conceptual issue is analysis of curricular content in school mathematics.

Indicators

Let us consider what we mean by an indicator. The fundamental purpose of a statistical indicator is to provide information about the health of the mathematics education system. A statistic becomes an indicator when it is useful in a policy context. For example, it is not particularly useful to know that there are 2.5 million teachers in the U.S. or that there are 45 million students. These numbers describe the size of the system rather than its health. It might be more useful to put these numbers together to form a pupil/staff ratio--in this instance about 19:1. This statistic would qualify as an indicator when two conditions are met.

First, the statistic should measure something that relates to the health of the educational system. Another way of stating this is that the indicators should be related to the variables in the causal model to be developed. To make things simple we can divide indicators into two categories--actions and consequences. Thus, like an index of smoking (action) which relates to human longevity (consequence), the pupil/staff indicator (action) should be demonstrably related to an agreed upon consequence of schooling such as achievement. In general, the action variables should be considered as potential influences of consequence variables. Furthermore, each action statistic should be related to a variable in the causal model which could be changed as a result of policy. Descriptive social information about the number of students in a school who are from single-parent families, or the average income of fathers, or other conditions that vary from school to school would <u>not</u> qualify as indicators. On the other hand, pupil/staff ratio is a variable that could be changed. Obviously, the selection of consequence indicators is critical, for they are used as a test of whether an action statistic qualifies as an indicator. These are important variables which must be taken into account but cannot be changed by policy makers. In fact, the primary problematic task for this project is to reconceptualize consequence indicators for school mathematics.

Second, an indicator such as pupil/staff ratio that assesses some component of the health of a system does not have any policy meaning until it is placed into a particular context. There are four basic ways of doing this.

a. An indicator can be contrasted with a standard or criterion level. Thus, if we know that educational achievement is particularly enhanced in the type of system being monitored if the pupil/staff ratio goes below 16:1, then a ratio of 19:1 could indicate to us that the health of the system could be improved by lowering the pupil/staff ratio. "Knowledge" of a standard may be based on theory, consensus, experience from past practice, or possibly empirical findings.

b. An indicator can be contrasted with itself over time. It then takes on meaning through a combination of its relationships to the health of the system and its own direction of change: a decrease in the pupil/staff ratio from 19:1 to 18:1 may indicate an increase in the health of the system. The validity of this inference will rest in part on the degree to which other parts of the system have changed over the same time period, or more importantly, but less probable, from empirical evidence about change in outcomes.

c. An indicator assessed in two different places (systems) at the same time can be contrasted with itself. States or countries, for example, might be contrasted on the pupil/staff ratio indicator. This is a more difficult comparison for developing valid information for it requires a detailed understanding of the overall differences between the two systems. Contrasting the Japanese and U.S. systems of education on an indicator such as pupil/staff ratio makes little sense unless we know a great deal more about differences in other variables such as curricula, and teacher training.

d. A final way of obtaining meaning from an indicator is to contrast it to another indicator. This may seem odd at first, for how could we contrast pupil/staff ratio with a measure of curriculum quality? The trick is to use the relationship of each to the health of the system and then to calculate the utility of changes in each indicator in a common metric such as dollars. This particular approach is known as cost-effectiveness analysis. The comparison would then be something of the following sort: a reduction in the national pupil/staff ratio by one unit would cost $xx and would increase achievement by yy% while an increase of equal cost in amount of inservice training would raise achievement by zz%. This mode of comparison requires a strong causal model of the educational process to drive the selection of indicators.

Although we expect to construct indicators that have meaning because they are compared to a standard or to another country or are assessing cost effectiveness, our primary initial concern is to develop a system of indicators that are sensitive, reliable, and valid indicators of change in the mathematics education system of this country.

The tenth conceptual issue we face involves both the nature of key indicators and how they can be constructed. The eleventh issue is closely related, specifying and developing indicators for the other variables in the causal model. If this can be done then the indicators could become critical benchmarks for NSF and others about the effects of their reform efforts and the health of the system.

In summary, the initial tasks of this project are related to those eleven conceptual issues. Assuming they can be dealt with then a series of next steps will be followed.

Next Steps

The next steps to be taken involve an examination of existing data. It will be followed by the preparation of an outline of what new data need to be gathered.

The Use of Existing Data. There is a large amount of statistical data regularly gathered on education in general and on school mathematics in particular. We propose to incorporate some of these existing data sets into the indicators for three reasons. First, there is no need to replicate the gathering of data when satisfactory data already exist. Second, by examining existing data sets we will know what important information about health and change is missing in those collections. Third, and most important, we will be able to situate new data we propose to gather with respect to the other important data sets so that we can develop a more complete picture about the status of school mathematics.

At the national level, the National Center for Educational Statistics (NCES) publishes two major compilations annually: the Digest of Education Statistics, which provides an abstract of basic statistical information on American education, and The Condition of Education, which translates the statistical information into charts accompanied by discussion. Much of the data in these publications come from NCES-funded efforts such as the "common core" of data in elementary/secondary schools and various longitudinal studies. Although the data in these reports are not specific to school mathematics, they and other data from NCES sources provide us with excellent baseline data on the social context of schools. It should be clear that any additional data gathered from a sample of schools should be related to these data to assure that findings are generalizable to the population of schools in the U.S.

Another major body of data is the National Assessment of Educational Progress (NAEP) which has provided data on scholastic achievement and student attitudes since 1979. This is one of the few sources that involve nationally representative samples. For mathematics, data have been gathered three times during the past 15 years and are to be gathered every three years. This information provides an excellent base about achievement on current mathematical topics. For a more complete picture about change, this data set needs to be augmented with items on new topics, responses coded in different ways, and studies that examine in more depth such outcomes as the understanding of mathematical concepts and skills. Over the next few years ETS, the contractor for this project, will be working to build a better data base. We would intend to work with ETS in whatever way we could to help them toward this goal.

The NCES study "High School and Beyond" was a longitudinal study of 1980 high school graduates that has been extended to 1982 and 1984 graduates and to their experience in college and work after high school. This data set provides valuable information on student enrollment and achievement, although information specific to mathematics education is limited. This data set also is highly compatible with a Class of '72 National Longitudinal Survey. Taken together, these two major surveys offer comparisons over a decade.

Another often quoted source of information is the International Association for the Evaluation of Educational Achievement (IEA). Data on mathematics achievement in 24 countries were collected in 1981-1982 (called the SIMS battery) and an initial analysis has been reported (Crosswhite et al., 1984). This data set, like the NAEP data on mathematics, is limited in its scope (see Romberg, 1985). However, it does provide us with an excellent comparative base from which change in mathematics instruction can be more fully explored.

Three important studies were carried out in 1977-1978 with NSF support: a review of the literature on science and mathematics improvement efforts between 1955 and 1975 (Helgeson, 1977), a survey in 1977 of the current status of education in these fields (Weiss, 1978), and a series of case studies of schools (Stake & Easley, 1978). Some of the information from these NSF-supported studies and data from other sources have been compiled in a data book (also covering higher education and employment in science and engineering), which was first issued in 1980 and was revised in 1982 (NSF, 1980, 1982). Data from these studies should provide us a base on which changes in school mathematics can be portrayed.

Still other important data sets on mathematics are those collected by college entrance examinations. Although the data are not intended to be representative of any well-defined population, scores on the SAT, ACT, and other college level tests are used extensively for college admissions and placement. The degree to which they measure achievement (not aptitude) and are useful for examining change in school mathematics can be examined.

Finally, each state has its own data collection system as well, much of it devoted to fiscal, demographic, and managerial information, but generally including data on enrollments, personnel, and student assessment. There is, however, considerable variation in the types of data collected by states and in the manner of collection, matching the organizational diversity among the states (and within each state) with respect to educational systems and institutions. We propose to relate the data (particularly achievement data in mathematics) from several states to our monitoring framework. For example, the state assessment data gathered in California and Illinois provide a broad, large sample base of important information about current performance and enrollments. Again, we should be able to relate such data to the other data sets to portray the status and changes in school mathematics more clearly.

These examples of extant data sets reflect the vast amount and diversity of data that exist. Some are cross sectional (like NAEP) and others longitudinal (like "High School and Beyond"); some are drawn from well-defined populations (like state assessments) and others not (like SAT); some reflect minimal competencies (like NAEP), others aptitudes (like ACT).

In summary, this collection and analyses of extant data will have four outcomes. First, a basic picture of student performance in mathematics should be apparent; the picture will not be complete, but the gaps in available information should be identified. Second, we expect to find that extensive information is available about background variables so that key indices can be readily constructed. Third, we anticipate that very little useful data exist for schooling variables. And finally, from this analysis the needs for additional data, instrument development, and so on should be obvious.

New Data Collection. It would be naive to assume that the existing data sets examined above would contain adequate information on all of the key indicators for change that would be needed to document the health of mathematics education in America. While some data undoubtedly exist on each of the key indicators for our causal model, it can be assumed that for some indicators either the data are incomplete or the sample is not representative. It will be necessary to gather additional data to be related to the existing data on school mathematics. We will propose a longitudinal survey designed to gather specific information on the key indicators of change from a small purposive sample of schools.

Documentation of Current Reform Efforts

Another task that will be proposed is related to the overall health of school mathematics in America. This analogy to health is used to portray the assumption that school mathematics today is in poor health. However, we expect that national commissions soon will describe what it means to be healthy by stating goals, preparing curriculum guidelines, and proposing standards. Furthermore, different groups are prescribing treatments to improve the health of school mathematics.

One need is to document the characteristics of the primary efforts funded by the federal and state governments and private foundations to share information about these programs. It is clear that everyone is aware of current problems and that efforts have been mounted in every state to respond to the perceived crisis. A survey by the Education Commission of the States (1983) found that since 1982 several states had initiated task forces or commissions to study and make recommendations about school mathematics, many had changed graduation requirements, others were preparing new curriculum guides and performance standards. However, it was clear that educators in most states were unaware of what others were doing. Although some duplication of effort is expected--and even necessary if one believes in local problem solving--it would seem evident that increasing the flow of information about projects and progress would reduce duplication of effort and speed the process of change. We will propose to gather data about initiatives in school mathematics on a regular basis from these agencies and relate these activities to the model and key indicators.

A second need to document the characteristics of current reform efforts is so NSF and others can be in a position to argue that, if change does occur over time, it may be due to those efforts. Note that it is not the intent of the monitoring project to evaluate the effects of specific initiatives. However, if there are grounds to suspect that detected change is due to some project, this would call for an indepth study of that relationship.

A later addition to the proposal activities will be to conduct two types of case studies. The first type of study would investigate the concurrent validity of key indicators in order to update or to develop better indicators. The second would identify and characterize demonstrable features of successful efforts to change school mathematics.

It is premature to discuss the details of these studies. However, the studies on key indicators are important since the notion of indicators is critical to the overall monitoring design. In these studies, we intend not only to establish the concurrent validity of indices but also to examine the possibility of developing new or better indicators.

The second kind of case study would be with respect to exemplary programs. There should be significant changes in key indicators and in pupil achievement in some sites. Such exemplary sites should be examined in depth to document the activities and events that are probable causes for those significant changes. Ethnographic field studies would be carried out in such exemplary situations. Information about how change was brought about in these schools must be documented for use by other schools.

In summary, the proposed set of case studies is an important component of the overall design of a monitoring system. They should add validity and understanding to the means of data collection and to the implications of the monitoring effort for change in school mathematics.

Concluding Remarks

The problem that is addressed here is to design a system to document the health of school mathematics in America. By monitoring the type and degree of change in mathematics in light of the recent general recommendations for change, the National Science Foundation will have information that was not gathered during the last major set of reforms in mathematics education--the modern curricula era of the 1960s. The Comptroller General's Report (1976), in which the NSF-supported science education projects of the previous decade were discussed, strongly condemned the low priority and insufficient number and quality of evaluations carried out to gather information about the effectiveness of reforms. That message was echoed by the Science Board's report, the National Academy study, and numerous reports and recent articles.

A more systematic attempt must be made to gather information and to assess the impact of contemporary recommendations for change in school mathematics. Knowing the type and degree of changes should assist the Foundation in forming plans so that the intended reform of mathematics teaching and learning in American schools is realized. Thus, we believe a systematic plan for gathering, synthesizing, and reporting on the progress schools are making in implementing recommendations for change should be of high priority.

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CHAPTFR 2

A NEW WORLD VIEW AND ITS IMPACT ON SCHOOL MATHEMATICS

E. Anne Zarinnia and Thomas A. Romberg

The Information Age

Popular Perception

There is a growing consensus that Western society is moving into The Information Age (Bell, 1973; Naisbitt, 1982; Toffler, 1980). The pervasiveness of this view is reflected by the accepted use of the term in the popular media. Discussion centers on problems involved in changing from the Industrial Age to the Information Age (Marquand, 1986; American Association for the Advancement of Science, 1985) rather than on the origins of the new age or its fundamental concepts. Popular perception holds that computers are the root cause of the Information Age; many claim that everyone either is or soon will be involved with them.

At the same time, huge upheavals in industry and the economy, emphasized by industrial closure and relocation, have forced public awareness of massive change. The media focuses attention on constant developments in science and technology. Highly publicized investigations into the state of education, such as A Nation at Risk (National Commission on Excellence in Education, 1983) or Educating Americans for the 21st Century (National Science Board Commission, 1983), have ensured public consciousness of their own and their children's skills, especially in language, mathematics, and science. For many, this new social view has been accompanied by disruption, upheaval, insecurity and confusion. While chaos in science is seen as a natural part of the process of change (Prigogine & Stengers, 1984), the urge in public affairs is to restore order to chaos through immediate action. This tendency is likely to obscure clear thinking about the direction that any response should take and the means by which it is most likely to be accomplished.

Background and Attributes

Characterization of the new age as The Information Age ascribes to it a rather lofty, intellectual, cerebral image, especially when compared with the muscular, grinding, "dark, satanic mill" (Jerusalem, nd.) connotations of the Industrial Age. Early designations, such as The Post-Industrial Age (Bell, 1973) or The Super-Industrial Age (Toffler, 1980), simply recognized that the industrial economy had changed so drastically that a new description was needed. Caused by a revolution in communications started by the telegraph, it could as easily have been described as The Communications Age. However, the integration of telephone, television, and computer permits instant transfer of information between people anywhere. This, in combination with the geometric growth of knowledge, makes the Information Age a more apt label.

Information is the new capital and the new raw material. The ability to communicate is the new means of production; the communications network provides the relations of production. Industrial raw materials are valuable only if they can be combined to form a desirable product; the same is true of information. Like urbanization, which is said to have occurred when more than 50% of the population came to live in urban areas, identification of a predominantly information-based economy is usually linked to the time at which more than 50% of the population began to earn their living through the sensible linking and exchange of information. Thus, the very definition of the Information Age rests on a mathematical concept. The validity of the statistical definition is open to question; the concept is not (Naisbitt, 1982).

The works of several authors (Naisbitt, 1982; Shane and Tabler, 1981; Toffler, 1980; Yevennes, 1985) point towards some of the attributes of the new age. Naisbitt's (1982) key points characterize the shift from an industrial society to an information society:

1. It is an economic reality, not merely an intellectual abstraction.

2. The pace of change will be accelerated by continued innovation in communications and computer technology.

3. New technologies will be applied to old industrial tasks first, but will then generate new processes and products.

4. Basic communication skills are more important than ever before, necessitating a literacy-intensive society. Information has value only if it can be controlled and organized for a purpose. To tap the power of computers, it is obligatory to first be able to communicate efficiently and effectively, to be both literate and numerate. In addition, in an environment of accelerating change, the old approach of training for a lifetime occupation must be replaced by learning power, which also depends on the abilities to understand and to communicate.

5. Concurrent with the move from an industrial society to a society based on information is awareness of the change from a national economy to a global economy. This change is accompanied by the perception that the United States and other advanced societies of the West are losing their industrial supremacy. Mass production is more cheaply accomplished in the less developed parts of the world. Toffler (1980) envisioned the change as a series of waves, in much the same framework as Frederick Jackson Turner characterized the westward movement of

the frontier in North America. Thus, just as industrial society replaced agrarian society and then began to push out, so the new post-industrial age will replace industrial society in the West and then gradually expand.

Economic Change

Manuel Yevennes (1985), a labor economist, has offered a more specific interpretation of the trends in the world political economy, spelling out their significance for the economies of the advanced, industrialized core. He distinguished between the U.S. as leader of the core nations of the international capitalist economy, the newly industrialized countries, and the periphery of less-developed countries. The concept of a national economy has been dismissed as a myth, because developments in communications and transportation and the maturity of economies of scale have resulted in an interdependent world unit. The world, viewed as a single system, contains large prosperous cities, slums, industrial regions, and agricultural areas, with large areas of hunger and desolation. Yevennes envisioned continued development of industrial production in the newly industrialized and less developed countries, while the U.S. specialized in administration of the world capitalist system, communication, science and technology, leisure and education, and maintenance of peace.

The last is essential because relocation of industry to newly industrialized and less developed countries and the organization of production according to an international division of labor is contingent on controlling world conflict. Thus, the importance of maintaining a military defense force and changes in national and global economic structure are likely to dominate the U.S. labor market for decades to come (Yevennes, 1985).

This concept needs modifying in two ways. First, limiting conflict is not simply a matter of controlling war, but of controlling any violence that inhibits trade. Second, one may infer from Yevennes' (1985) statement on defense that a large, powerful, and probably conventional, rather than nuclear, military will be needed to maintain peace. An alternative viewpoint is that military spending is crippling the Western economies and that Japan's prominence as a trading nation may be directly related to the percentage of its GNP not spent on defense (Brown, 1986b). In either case, defense-related considerations will be influential in restructuring the labor market.

A clear but complex picture of economic evolution emerges. The trend started with the advanced countries building an infrastructure of transportation in the less developed countries for the export of raw materials, and subsequently building steel mills and factories near the source of raw materials. It continues with the transfer of more and more industry to the location of inexpensive labor. Mass production has been made sufficiently foolproof to be conducted anywhere. The future of the advanced industrial countries, in the lead, is to serve as financial headquarters and research and development centers for a global economy. Such international specialization requires sophisticated communications and the capability of maintaining peace, whether for the free flow of goods through the world's waterways or for the safe conduct of business. It is a potentially precarious position since it depends on scientific, technological, and fiscal supremacy.

In the time frame of economic competition, millennia, centuries, even years are inadequate measures; minutes may be more appropriate. Any society wishing to stay minutes ahead of its competitors needs to be constantly at the leading edge of scientific and technological developments and to be innovative. The British have shown that it is possible to be at the leading edge technologically while failing to reap that position's full benefits (e.g., rotary engines, VTOL, the Rolls-Royce Nene engine used in MIG 15's); the Japanese have demonstrated that a country may reap the benefits without being at the forefront of knowledge (e.g., shipbuilding, cameras, television, automobiles, Mitsubishi business jets). Thus, both the pursuit of knowledge <u>and</u> innovation, "the generation, acceptance, and <u>implementation</u> [emphasis added] of new ideas, processes, products, or services," (Kanter, 1983, p.20) are crucial to leadership of the world economy.

Constant creativity and innovation is costly. The social benefits of spin-off research are the basis of an argument used repeatedly to justify the high costs of defense-related research. However, the cost effectiveness of military research for economic advantage is doubtful, and those countries allocating a large percentage of their GNP to military spending are at a crippling disadvantage. Consequently, China has almost halved the percentage of its GNP spent on the military since 1967; it is now around 7 %, comparable to the U.S. By comparison, Japan spends only 1 % of its GNP on defense (Brown, 1986b). Thus, even the maintenance of peace requires innovative thinking, if the effect is not to generate a self-defeating burden.

Finally, beyond the pursuit of competitive advantage is the most compelling reason of all for creativity and innovation, survival. To illustrate with but a few of the myriad of interrelated problems to be resolved in order to attain a sustainable society:

1. Groundwater in the United States is being steadily depleted; one-fifth of the amount pumped each year is nonrenewable (Postel, 1986).

2. There is a growing consensus regarding an irreversible warming trend brought about by pollution, which will have a visible impact within the next 50 years (Osterlund, 1985; Cowen, 1985).

3. Soil, water, vegetation, and fisheries, the bases for life, are

being mined globally, although the economic and human consequences are most visible in Africa and India (Brown, 1986a).

4. Nuclear reactors are aging, yet there is no viable decommissioning policy (Pollock, 1986).

It is already apparent that independent subject disciplines working in isolation are incapable of providing solutions. In the case of global warming, for example, meteorologists are joining forces with biologists, geologists, and other scientists because the problems are too broad to fit within the auspices of atmospheric science or even ecology (Cowen, 1986). A sense of linkage is needed that ties, for example, preventive health measures to population control and water supply, international debt to almost everything (Brown, 1986a).

The view of the future outlined here points to some immediate necessities for the schools (Shane & Tabler, 1981):

1. Students must be educated for survival in an atmosphere of change and encouraged to contemplate alternative views of the future.

2. In a global system with glaring inequities, the quest for economic advantage may trigger turbulence. Students need to face the complex relationships that exist in a closed system between the environment, social justice, and survival.

3. The content and structure of the curriculum should not indoctrinate students with past values and rigidity but should be derived from images of the future.

4. Most of all, students need a sense of consequence.

Innovation and Integrative Thinking

The most important single attribute of the Information Age economy is that it represents a profound switch from physical energy to brain power as its driving force, and from concrete products to abstractions as its primary products. Instead of training all but a few citizens so that they will be able to function smoothly in the mechanical systems of factories, adults must be able to think. While creative intelligence is the driving force, innovation depends on communal intellectual effort rather than on the activity of a small cadre of elite thinkers.

> Whereas short-term productivity can be affected by purely mechanical systems, innovation requires intellectual effort. And that in turn means people. All people. On all fronts. (Kanter, 1983, p. 23)

This is significantly different from the concept of an intellectual elite having responsibility for innovation while workers take care of production.

Kanter (1983) identified, among other important conditions for innovation:

1. Fluid communication and network-forming devices. This refers to the development of lateral communication across groups, the development of relations with people in other geographic areas, the formation of project teams.

2. A culture in which the individual is not told what to do but given the authority to do it. Respect for the individual is not a simply a matter of human dignity but essential to the leap of faith required for innovation.

3. Complexity as an essential ingredient of an innovative environment.

Kanter's keys to innovation which also stress the intellectual approach most likely to encourage innovation include:

1. An "integrative" approach to problems, including the willingness to move beyond received wisdom, to combine ideas from unconnected sources, to embrace change as an opportunity to test limits. To view problems integratively is to see them as wholes related to larger wholes, rather than as isolated experiences; this challenges established practice.

2. The habit of operating at the edge of competence, focusing more resources and attention on what is not yet known than on controlling what is already known.

3. The measurement of accomplishment not by the standards of the past but by visions of the future.

"Segmentalism," the contrasting approach from the past industrial age, is oriented against change and prevents innovation. It is concerned with compartmentalizing actions, events, and problems and keeping each piece isolated from the others. Segmentalist approaches define problems as narrowly as possible, independent of their context and isolated from connections to any other problems. Segmentalism applied to problem solving holds that any problem is best factored into subproblems. In the integrative mode, people do just the opposite; they aggregate problems into larger problems to recreate a unity that provides more insight into required action. This helps make possible the creative leap of insight that redefines a problem so that new solutions can emerge (Kanter, 1983).

The holistic, or integrative, approach to innovation and problem solving reflects general-systems theory. It is also consistent with the growing body of knowledge about the differences between expert and novice thinkers (Chi, Glaser, & Rees, 1982; Resnick & Gelman, 1985). While the novice plunges in, the expert dances around the problem, considering it from different viewpoints. This difference in approach suggests that experts not only have more complete mental models based on greater knowledge, but are also more apt to construct alternative models from different perspectives. Identification of the common attributes of several different models permits the development of a new concept. Cooperative sharing and testing of those new concepts and the building of shared schemas result in the formation of new knowledge (Skemp, 1979).

Fischbein's argument (1975) that human cognition is fundamentally unitary is also consistent with the holistic approach. He argues that intuition and intelligence address the same reality, and that intuition is analogous to a cognitive map, either accompanied by spatial representation or consisting of a global synthesis in which visualization is secondary. Its essential quality is direct articulation, which serves action better than explicit reasoning; it gears knowledge into action. Integrative, intuitive, expert thinking is a vital ingredient of the required pace of innovation.

In summary, innovation depends on the cooperative application of creative intelligence. It is stimulated by dense communication, complexity, and integrative rather than segmentalist thinking. This suggests some priorities for schooling:

1. Environments that promote cooperation and respect for the individual and that support risk taking are also likely to promote and support innovation. The implication is that cognitive and affective aspects of schooling are inextricably interrelated.

2. Interdisciplinary approaches that encourage children to anticipate, comprehend, and cope with complex relationships are also likely to promote fluid, cross-group communication.

3. Learning situations that require the creative application of intelligence through both critical and creative, constructive, and generative thinking (de Bono, 1986; Sternberg, 1985); these are also more likely to promote the spontaneous transfer of relevant information to new problems (Bransford, Franks, Vye & Sherwood, 1986).

4. Devices that break down the artificial barriers between school learning and learning in life should be stressed, because true cross-group endeavour would not preclude the real world. Furthermore, to anybody seriously involved in the pursuit of anything, formal hours can be an unreasonable constraint.

5. Resolution of "the Japanese dilemma" is essential. Through intense discipline, Japanese students have outperformed Western students in some key academic areas of schooling. However, the

Japanese are equally conscious of the need for innovation and creativity. The Japanese dilemma is to change from a uniformist to an individualistic education in order to promote creativity, but to do it without losing the lead attained through intense discipline (Sneider, 1986). A possible answer may lie in the relationship between self-discipline and creativity. The problem is one that all the advanced countries must address.

Unfortunately, existing architectures--present habits of schooling, traditional divisions of knowledge, established approaches to curricula and assessment--are all designed to eliminate as much risk as possible. They echo the Scientific Management approach of industrialism, breaking age groups, knowledge, instruction, and learning into tiny controllable parts. They reflect conformity to meticulous specification rather than innovation, and they place more emphasis on the cognitive than on the affective. Such practices are deeply entrenched and represent a profound barrier to change.

Learning: The Key to Adaptation

The analogy of biological evolution and the selection of the fittest has pervaded many perceptions of economics and sociology. However, learning rather than evolution is the fastest way of adapting to change (Skemp, 1979). Some hard, sociological facts underscore the need for rapid adaptation through proficiency in learning:

1. With rapid economic change, people must anticipate multicareer lives in which they may experience structural unemployment and require reeducation (Virgo, 1984). This highlights the education/training interface (Smith & Sage, 1983), education for adaptability and continued learning, on-the-job training for the specific task.

2. The application of robotics suggests a steady decline in the number of jobs available. People may be expected to create their own jobs through entrepreneurial activity, whether inside or outside a corporate framework. This kind of activity requires, in addition to innovation, a level of self-assessment and social confidence that encourages productive risk taking.

3. Over 20% of the nation's children live in poverty, a condition that seriously stunts their education. This childhood poverty is in large part attributable to the changed structure of the family and the significantly lower income of women (Sitomer, 1985).

Constant reeducation for all raises the question of cost, especially for those who are not in the work force. Efficient, effective learning is most inexpensively accomplished by those who enjoy learning and who see learning as a recreation (Yevennes, 1985). These people are confident in their ability to learn and have had pleasurable and productive learning experiences in which they directed, tested, and evaluated their own learning (Skemp, 1979) in a congenial and supportive environment.

The implications for schooling of constant reeducation are profound:

1. Reeducation further emphasizes the affective aspects of learning. Such aspects may be not only an indicator of the probability of taking further coursework, but crucial to the whole process of intrinsically motivated, self-directed, lifelong learning.

2. Reeducation suggests student involvement in directing the curriculum and assessing their own progress, since this is precisely what they must do for themselves throughout life.

3. It suggests the need for learning systems and technology to support the process of personalized, cost-effective, lifelong learning (Shane and Tabler, 1981).

4. Reeducation suggests that students should experience the pleasure and satisfaction of completing an extended, major learning project of their own, and a similar endeavour in which they have been part of a team.

5. It suggests that equity of outcomes is crucial, not as a charitable matter but for the well-being of society (Reich, 1983).

The notion that the ability to learn is central to adaptation and survival increases the urgency to more fully understand the processes of learning and the nature of intelligence. It is no accident that there is a growing sense that intelligence is complex and multifaceted, goal-directed, and susceptible to improvement.

Complex and multifaceted. Walters and Gardner (1985) proposed seven basic intelligences, each with a biological foundation, an identifiable core of operations, and a symbol system for encoding. Intelligences possessing these characteristics are musical, bodily-kinesthetic, logical-mathematical, linguistic, spatial, interpersonal, and intrapersonal. Each is to a significant extent independent of the others. Each begins as a raw patterning ability and is glimpsed through different lenses at different points in development. Symbol systems, such as tonal or verbal expression, are in later phases enhanced by the acquisition of notational systems and exhibited in vocational or avocational pursuits in adulthood. Traditional views of intelligence have largely ignored all but the logical-mathematical and linguistic intelligences.

In both cognition (Sternberg, 1982) and artificial intelligence (Schank, 1980), the ability to cope with novelty and to extract from new situations generalizations that are likely to be useful for future needs has been perceived as intrinsic to the development of intelligence. In any new task, it is necessary to recognize the problem, decide on cognitive strategies, and monitor progress. Thus, Sternberg's (1982) triarchic view of the mechanisms for intellectual development drew attention to the crucial role of metacognitive processes.

Goal-directed. In addition to the multifaceted aspect of intelligence, there is a growing sense of the importance of purpose, both as establishing an end and as a force in the attainment of that end. Contemplating the intuitive process, Skemp (1979) speculated that strong desire and a continued preoccupation with a problem gradually, or suddenly, forges a pathway between related concepts within the reach of reflective thought for the development of mental models. In this sense, intelligence is creative and self-creative, and goal-driven. Creativity results from pursuit of a goal, and the resultant increase in complexity of conceptual connections provides a broader foundation for further development. The role of emotions is to provide information about progress toward or away from the goal state.

In a more global approach to intelligence, Jaques' (1985) quintave theory on the maturation of cognitive power viewed the development of intellectual capability as reflected by the time frame of an individual's goals. Essentially, the goal's time frame is regarded as indicative of cognitive power; the greater the cognitive power, the more extended the planning time frame. This approach drastically revised the higher end of the Piagetian framework, already revised upward by Perry's (1970) "period of responsibility" and by the SOLO taxonomy (Biggs & Collis, 1982). Thus, the concept of purpose is integral to the way intelligence is acquired, exercised, and exhibited.

Susceptible to improvement. Thinking and the development of the intellect increasingly are viewed as integral to the concept of intelligence and, moreover, teachable (de Bono, 1986; Sternberg, 1985). Concern over measurement has given way to contemplation of the characteristics of intelligent behavior and experimentation with curricula intended to improve the intellect (Sternberg, 1985).

A slightly different approach has resulted from investigation of the differences between expert and novice problem solvers. This, spurred by efforts in artificial intelligence, has shed some light on the nature of intelligent behavior. Experts are not necessarily intrinsically better, faster thinkers than novices. However, they do have a superior, better-structured knowledge base. This permits more successful pattern recognition and a more sophisticated approach to the problem (Chi, Glaser, & Rees, 1982; Larkin, 1985).

In Larkin's (1985) experience with physics problems, a superior understanding enabled by the expert's data base was central to the difference in performance between novices and experts. Novices worked backward from the desired answer; they proceeded from a naive understanding derived from parsing the problem statement directly to a means/end selection of seemingly appropriate mathematical equations. Their understanding was based on an analysis of form. Experts, by contrast, used their knowledge of major principles from the discipline and were, therefore, more likely to address function and to generate a scientific understanding, after which the mathematical calculations were relatively straightforward. In other words, experts worked inductively with the benefit of well-founded mental modeling; novices, deductively from naive perceptions.

Thus, the recent work on intelligent behavior and cognition underscores the need for children:

1. To have a firmly founded understanding of content as a foundation on which to build.

2. To be literate in the appropriate notational systems.

3. To be involved in the planning, direction, monitoring, and evaluation of their own activities.

4. To be conscious of their own goals and involved in setting them for projects with increasingly longer time frames.

5. To improve their intellect.

In addition, recent work draws attention to the importance for understanding of the interplay between the internalized structure of knowledge and the generation of mental models:

> Good teaching requires that the goal of learning is a schema. . . . Excellent teaching requires . . . an ability to compare the merits of alternative schemas, both for present use as a source of plans . . . and also as a tool for future learning by the assimilation of new concepts. (Skemp, 1979, p. 253)

Computers: Amplifiers, Catalysts--and Mirrors

Popular perception of the importance of computers is appropriate, for computers and their applications are inextricably intertwined in the development of the Information Age. They are part of both the solution and the problem, handling large amounts of data but generating drastic change through that very capability. Computers were created in response to the deluge of data in a society which became vastly more complex with World War II (Toffler, 1980). They have made it possible to perform tasks faster and often more reliably; and to do things that could not be done before (Joseph, 1984). Their use in factories disposes of the problem of "Monday" or "Friday" cars, while the implantation of identification devices in the necks of cows makes possible individually tailored diets.

Through mathematical modeling and simulation, the computer has made it possible to replace scientific experimentation with numerical analysis and simulation, a cheaper, safer and more easily modifiable method of investigation. Consequently, it is possible to do much more in the way exploration and testing (Cross & Moscardini, 1985). While the most obvious use of the computer has been to test ideas and develop conjectures based on numerical evidence, Jaffe (1984) suggests that the computer may do more than simply serve as an experimental laboratory or modeling tool. As in the proof of the four-color theorem, a computer can check that a finite number of cases holds for a combinatoric statement. In addition, he suggests that, although computers may not outline the architecture of a proof, they will assist by establishing a large number of identities or inequalities.

However, changes precipitated by computers stem from several different facets of their capabilities. First, programs for the early generations of computers were linear and algorithmic, causing users to think about or structure their field in ways that made problems amenable to computer solution. This forced development of models and algorithms, assignment of numerical values, and the use of mathematics in areas to which it had not previously been applied. This process was responsible not only for the development of expert systems, which threaten to replace professionals, but for speculations about the improvement of knowledge through "knowledge refineries" (Michie, 1984). Hofstadter (1982), for example, considers the question of whether inspiration can be mechanized. Such questions prompt consideration of the ways in which mathematics may be applied to the process of inspiration and also about the process of mathematical conjecture and concept formation.

The graphic display of information is ancient. However, the use of abstract, rather than representational, images is recent, requiring a diversity of artistic and mathematical skills. It conveys powerful advantages (Tufte, 1983):

> Of all methods for analyzing and communicating statistical information, well-designed data graphics are usually the simplest and at the same time the most powerful . . . . Modern data graphics can do much more than simply substitute for small statistical tables. At their best, graphics are instruments for reasoning about quantitative information. Often the most effective way to describe, explore, and summarize a set of numbers--even a very large set--is to look at pictures of those numbers. (p. 9)

Computer graphics offer obvious benefits in the conveyance of complex information; that much is evident to all who watch television weather reports. Perhaps this very benefit allows another possibility, that of promoting more rapid concept formation. Three-dimensional graphics have obvious applications in computer-aided design. However, the end product of science is communication, and there is a growing sense that interactive, three-dimensional graphics significantly enhance the communication process.

Davis and Hersh (1981) argue that kinesthetic activities and hands-on experience encourage development of the kind of preverbal conceptualization that is fundamental to mathematical intuition:

> We have intuition because we have mental representations of mathematical objects. We acquire these representations, not by memorizing verbal formulas, but by repeated experiences (on the elementary level, experience of manipulating physical objects; on the advanced level, experience of doing problems and discovering things for ourselves). (Davis & Hersh, 1981, p. 398)

Description of their experience with software designed to generate a hypercube suggests that experience with three-dimensional graphics software could help develop mathematical intuition.

> What is to be sought in designs for the display of information is the clear portrayal of complexity. Not the complication of the simple; rather the task of the designer is to give visual access to the subtle and the difficult--that is, the revelation of the complex. (Tufte, 1983, p. 191)

Intuition is fundamental to the creation of mathematical concepts (Wilder, 1967). Computers make it more feasible to test intuitive concepts through visualization empowered by numerical analysis. Digital image-processing illustrates the possibilities: Quantification in geography, for example, moved away from the kind of intuitively perceived coherence exemplified by Vidal de la Blache (1979) as the basis for identifying regions; and yet even a first grader looking at atlases of satellite imagery (Images of the world, 1983) intuitively recognizes regional boundaries that may be less obvious on the ground or on the map.

Although initially stand-alone tools, computers have progressively been connected with each other and with a wide range of other communications devices. This networking of computers into integrated communications systems increases contact between people and provides another powerful model for thinking. ARPANET, one of the earliest resource-sharing networks, originally connected only four institutions. The concept has since expanded so that worldwide networks linking millions are underway. Hiltz and Turoff (1978) observed of the Electronic Information Exchange System (EIES) that the network increased communication between people at least ten-fold. Considering means of engineering serendipity, Charles Hitt (personal communications, October 1985) speculated that it is most likely to occur when supposedly incompatible people are connected on a compatible system. Such conditions are created when there is multidisciplinary participation in computer conferences.

The intellectual models offered by computer networking and new computer development offer a powerful source of analogy in many areas. For example, both business and cognitive science have adopted the model and the metaphors of networking, much of the language of which is derived from mathematics and anthropology. Toffler's (1985) recommendation in 1975 to AT&T (then Bell Telephone) for organizational restructuring is similar to the notion that networks are created to serve a purpose and rapidly atrophy when that purpose no longer exists (Hiltz & Turoff, 1978). At the same time, the vocabulary of cognitive science describing networks in the brain is often identical to that used in computer networking to describe common concepts and comparable problems, such as information overload. In other words, in the practical application of computers, mathematical concepts interweave with scientific principles from academic disciplines. The result serves as a new model in a reciprocal relationship.

Computers perform work through logical analysis. That analysis is critically dependent on mathematical ideas, insights, and methods, be it binary arithmetic, boolean logic, fuzzy set theory, randomness, or network theory. Computers currently follow sequential algorithms, but future computers will emulate the more complex parallel processing of the brain (Jaffe, 1984). Thus, efforts to improve computer functioning are closely tied to efforts to understand the nature of human information processing. Intelligence (whether in man, machine, or animal) consists of the capabilities of data capture and storage, processing speed, and the flexibility, efficiency, and range of software. In the development of ultra-intelligent machines, there are only two limits: the speed of light and the internal consistency of mathematics (Evans, 1979). If Evans' argument is correct that the computer is increasingly capable of outpacing the human brain in many of the basic capabilities of intelligence, several propositions follow:

1. There is a need to understand the mathematical relationships in the structure of knowledge, the way knowledge is stored, and the way it is put together and used (Brookes, 1980), between content and process.

2. From these understandings should flow more efficient and effective computing, a greater development of machine intelligence, and perhaps more effective strategies for improving human intellect.

Computer capabilities for logical analysis may be viewed another way. Expert systems consist of a knowledge base of assertions and an inference machine that reasons purely on form, not content. Thus, the only errors are those that result from omitted knowledge or incorrect assertions (Robinson, 1984). It is possible, therefore, to leave repetitive reasoning to computers in much the same way they perform the drudgery of calculation. Telemetry suggests that even some parts of the assertion process can be handled by computer, leaving to humans those things beyond the reach of systematization. If statements in the knowledge base consist of assertions that are open to falsification, expert systems can be applied to any content area. While complete mechanization is impossible, vastly increased productivity in reasoning is attainable (Sheperdson, 1984), raising the prospect of a surge in the production of new knowledge.

Computers serve a further purpose: their use in modeling essentially caricatures our rational processes, thereby exposing the strengths and weaknesses of rationality and the limits of human knowledge.

In other words, the computer enables the development of new concepts--intuitive and verbal, global and microscopic--through the powers of calculation, comparison, visual representation and logical analysis. Unfortunately, education is not yet past the stage of applying computers to old problems; the most serious implication of this is that the computer is the instrument that enables the final domination of the educational system by scientific management (Baker, 1978). However, the implications for schools are much more profound:

1. If computers amplify, it behooves education to be deliberate.

2. If computers are capable of calculation and repetitive reasoning, children need to acquire through experience a sense of what computers can do and when it is appropriate to use them. As technology changes in size, cost, power, and ease of use, these perceptions must also change.

3. If computers can assist in the intuition of concepts, children need to make full use of them in the learning process and to use them consciously as a tool in making their own inferences.

4. If, in the process of learning to make computers behave intelligently, a new consciousness of means for improving human intelligence emerges, every effort should be made to make that available to all.

5. If computers can effectively increase the experience of children through simulation, and can broaden their contact with other people both globally and locally, advantage should be taken.

6. Finally, children deserve to experience the sense of power derived from creating their own expert systems from a combination of assertions known to be true and valid inferences from that base--and this applies both on and off the computer.

Summary

Accelerating change in the rezoning of world economic activity has drastically changed educational expectations. Instead of a standardized, punctual product that would mesh smoothly into the industrial machine, society now needs people who are creative and innovative. While creativity is inevitably individual, innovation is cooperative. Among the results of these new educational expectations are:

1. A stress on integrative, intuitive, expert thinking.

2. The notion that evaluation should be measured against a goal rather than against past performance.

3. A stress on an educated populace, rather than on selecting an elite for educating, and training the remainder.

Stress on the importance of learning, and systems for learning, stems from the prospect of multicareer lives and the demands of social welfare. Consequently, intelligence recently has been perceived as multifaceted, complex, and goal directed. The role of the emotions in goal direction suggests that the affective strongly influences the cognitive. In addition, intelligence is viewed as modifiable rather than fixed. Efforts to increase intelligence have, in part, focused on the differences between novice and expert thinking. Understanding based on a well-developed internal structure of knowledge seems central to the difference.

Computers are both a solution to old problems and a source of new ones. They are a crucial tool for testing models and intuiting concepts. In addition, they enhance communication and provide a rich source of metaphor. For mathematics, they have elucidated and expanded the discipline's role in the scientific process and, more tantalizingly, in the understanding and improvement of intelligence. Most of all, they are a catalyst, forcing reexamination of whatever area to which they are applied, and that includes the way children learn, what they learn, and why they should learn.

The Impact on Mathematical Education

Demand for Change

Popular perception holds that citizens need a better understanding of science and technology if society is to prosper. As a consequence, mathematical education has been the focus of public demand for action through changes in the curriculum. In addition, application of computers in areas ranging from linguistics to business has increased the demand for mathematics in all fields.

Awareness of the need to increase mathematical education, usually stated simply as "children need more mathematics" (Curriculum change in secondary school mathematics, 1983; National Commission on Excellence in Education, 1983; National Council of Teachers of Mathematics [NCTM], 1980), is not new. For at least the last 25 years (e.g., College Entrance Examination Board, 1959; Fehr & Bunt, 1961; Goals for School Mathematics, 1963), those responsible for mathematical education have attempted to reshape and improve the school mathematics curriculum. Activities in Western nations intended to improve mathematics education have ranged from international conferences and assessments to curricular experiments (Howson, Keitel, & Kilpatrick, 1981) and national investigations (e.g., Committee of Inquiry into the Teaching of Mathematics in the Schools [CITMS], 1982; National Advisory Committee on Mathematical Education [NACOME], 1975).

Two reports important for school mathematics are A Nation at Risk (National Commission on Excellence in Education, 1983) and Educating Americans for the 21st Century (National Science Board Commission, 1983). Reactions to these reports by mathematical sciences professionals are presented in New Goals for Mathematical Sciences Education (Conference Board of the Mathematical Sciences, 1984) and School Mathematics: Options for the 1990s (Romberg, 1984a).

Widespread public concern has resulted in responses that vary from increasing the credit hours in mathematics required for graduation to writing new curriculum guidelines (e.g., Chambers, 1986). Nevertheless, general mathematical understanding remains for many both inadequate and traumatic: a physical education teacher ordered a 6-foot medicine ball and then could not see the class because it was 6 feet in diameter rather than circumference; and a reading teacher burst into tears when she could not translate a recipe for a 9-inch pan to an 8-inch pan, (Anonymous personal communications, October 1985; CITMS, 1982). Despite enormous efforts, society's mathematical understanding does not match its needs.

The Curriculum as an Instrument of Change

A traditional assumption in education is that changing curricula is the easiest way to change school practices. However, curriculum development is more than a change in method or content; it is an effort to change the culture of schools and may involve different levels of restructuring. Ameliorative change simply substitutes one tiny part of the system, such as the current textbook, but does not challenge values or traditions. Typically, response to change which does challenge those values and traditions is nominal. If there is actual change, it is more frequently mechanical or illusory than real. Patterns of adoption range from labels and procedures to complete surface trappings, but the change can only be considered real if the values and principles also are adopted. Actual, substantive change disrupts old habits and beliefs, usually invoking resistance. Therefore, the first step toward increasing the likelihood of constructive curricular change is identification of the traditions and values being challenged (Romberg & Price, 1983).

The Power of Metaphor and Model

The most deep-seated traditions and values are personal in that they are rooted in each individual's beliefs about the world. In any effort, people are goal directed; they work best toward new goals if those goals are personal. They may appreciate the need for change but formulate inappropriate goals, unless their own model of the world matches reality. Cognitive science suggests that people constantly construct theories to make sense of things (Resnick & Gelman, 1985). Men have habitually relied on the metaphors and models offered by familiar objects and experiences to make sense of complex situations. Thus, their theories about the way the world works have been consistent with the way they work to make a living, for that is the model constantly before them. This model has, in turn, dominated the way they think about most other things. As a result, prevailing views in most fields mirror the means of production and the relations of production in the dominant economy, because that model presents a convenient source of analogy and a framework for thinking.

For the past hundred years, Western society has been dominated by the machine, the factory, rationalism, analytic thought, experimental science, and the technology of paper. These, standing as metaphor and model in their turn, prompted the development of scientific management, behaviorism, and the matrix as an organizational tool (Kilpatrick, 1979). This sequence led directly to what Howson, Keitel and Kilpatrick (1981) refer to as the "taylorization" of school mathematics, the most pressing problem of mathematical education. As long as the sequence remains unrecognized, unchallenged, and unchanged, it will continue as a source of intellectual incoherence, impeding progress towards new goals; thinking and action will be constrained by old models and metaphors. Tradition ensures that people's belief structures and work habits are not easily changed. Even when there is intent to change, if mental models remain the same, real change may not be effected, despite the illusion of change created by the trappings. Old beliefs and habits will persist and nominal, rather than real, change in the curriculum probably will continue (Romberg & Price, 1983).

Briefly, the disparity between what is desired, what is attempted, and what is accomplished rests on intellectual conflict between traditional values and practices and the need for creativity, coherence, and a sense of consequence. One alternative is to simply wait for the world of work to change and to expect the evolving environment to exert its own influence in changing current models, thus removing the intellectual conflict. However, acceleration of the pace of change makes this problematic, because early inadequate models remain entrenched (Resnick & Gelman, 1985).

Particular models are rooted in the dominant cultural metaphor in which children are raised and educated, leaving the prospect of several generations of chaos and trauma before intellectual and societal institutionalization of the old models gives way to the new.

Consequently, mathematical education for a new age requires a proactive, rather than reactive, approach (Romberg, 1984b). As long as those involved adopt a policy of adapting mathematical education to cope with current problems, time lag alone ensures perpetual inadequacy. Mathematics education today demands recognition and removal of structural and intellectual impediments through careful consideration of the possible, probable, and desirable attributes of the new age, the self-conscious formulation of new models through abstraction rather than experience. The knowledge imparted, the work of students and teachers, and the professionalism of teachers are all imbued with structures, traditions, and values that are challenged by new models (Romberg & Price, 1983). Anything less will continue an already prolonged period of painful maladjustment in which efforts to improve mathematical education meet with little real success.

Knowledge, Work, and Professionalism: Traditional Beliefs and New Models

To isolate segments of educational practice and to discuss knowledge as though it were separate from the way we work, or teachers' professionalism as though it were separate from the work expected of children, is contrary to reality. All of these things are enmeshed with one another. One can only discuss the tangle from one viewpoint and then from another, recognizing that one is looking not at separate things but at a model of a single reality in which every part and process is inextricably and often reciprocally tied to all other parts and processes. Despite that reservation, different perspectives on school practices are useful in considering the direction of desirable change in mathematics education. These varying perspectives also are helpful in examining the entrenched beliefs, values and traditions that must be addressed if innovation is to occur: knowledge, the work of students and teachers; and the professional nature of teachers (Romberg & Price, 1983).

Knowledge. One of the intentions of mathematical education is to ensure students' acquisition of mathematical knowledge, preferably as a communally accepted structure. This has typically meant a careful structuring for instruction of mathematical skills and concepts. However, a crucial distinction between knowledge and the record of knowledge, knowing and knowing about (Romberg, 1983a), is at the root of several dilemmas of mathematical education.

As a record of knowledge, mathematics has a vast content (Romberg, 1983a). Furthermore, the accepted content of mathematics changes. Davis and Hersh (1981), observing that the world is in a golden age of mathematical production, raised the possibility of internal saturation and exhaustion and suggested that there is a limit to the amount of mathematics that humanity can sustain at any one time. In other words, human beings can manage only a limited amount of information, and the field's subdivision means that some parts must inevitably be abandoned as new parts are added. For the world of mathematics as a whole, this may not be true. Experience with information overload in computer conferencing systems, for example, suggests that users should be provided with filtering devices and left to do their own filtering (Hiltz & Turoff, 1985) because overload may be crucial redundancy. What is junk to one may be information to another.

However, the content of mathematical education is necessarily restricted. This is at the root of the controversy between mathematics as a science and mathematics as a school subject, which arises when emphasis is on the record of knowledge rather than on knowing. The inevitable consequence is an attempt to identify those parts of the curriculum that are no longer appropriate, such as logarithms. Unfortunately, the obvious desirability of adding, for example, mathematical modeling, discrete mathematics, probability, and statistics needs more time than can be made available, prompting demands for an increase in the amount of time students spend studying mathematics (e.g. NCTM, 1980) to fit everything in. Reaction of the National Advisory Committee on Mathematical Education (1975) differed:

> Curriculum content, subject to the flux of accelerating change in all areas of our society, cannot be viewed as a set of fixed goals or ideas; it must be allowed to emerge, ever changing, responsive to the human and technological lessons of the past, concerns of the present, and hopes for the future. With this in mind, no definitive curriculum can ever be recommended. (p. 138)

In either approach, it becomes essential to reconsider carefully the purposes of mathematical education of children to eliminate redundancy and ensure the crucial.

Reconsideration of the intent for students to acquire a structured knowledge of mathematics is enlightening. Scientific management of the record of knowledge resulted in hierarchical classification and taxonomies of knowledge. This approach meant that mathematics to most students was, and still is, the sequential mastery of concepts and skills. For many, it is not even an aggregation. It was hoped that "modern math," with its emphasis on such organizing constructs as sets and functions, would bring coherence to the curriculum. Unfortunately, it became simply a more abstract collection of routines. The process of segmenting and sequencing mathematical ideas for instruction in models such as Gagne's (1965) learning hierarchy and in individualized programs such as IPI (Lindvall & Bolvin, 1976) and IGE (e.g., Harvey, Green & McLeod, 1972; Harvey, McLeod & Romberg, 1970) separated mathematics into thousands of independent pieces (Romberg, 1984b). While the planners and teachers had meticulously coded networks linking concepts, the resultant view for students was of isolated pieces rather than a functioning whole.

Erlwanger (1978) illustrated the problems of individualized mathematics. In a study of IPI mathematics, he interviewed several students on their notions of mathematics. One student's overall conception of mathematics was of a large collection of distinct and unrelated skills to be mastered.

Thus, the coherence of mathematical concepts and the concept of structure so essential to expert thinking remains absent. A focus on isolated parts essentially trains students in a series of routines without educating them to grasp the overall picture--a skill that would ensure their selection of appropriate tools for a given purpose. The segmenting and sequencing of mathematics has also led to the notion that there is an incontestable partial ordering of mathematics (Romberg, 1984b), usually reflected in the declaration of "prerequisites" in the curriculum, chapters in the text, and so on. Although those devising the curriculum may think of the parts as functional entities (decimals, fractions, equations), children, who view them in isolation without any real perception of the complementary relationships among them, are very likely to perceive the parts morphologically rather than functionally. This may well be a major barrier to the development of expert thinking, which usually focuses on function rather than form; naive perceptions persist (Resnick & Gelman, 1985).

Although learning inevitably occurs sequentially, all knowledge must be treated integratively. Mathematics as a discipline has not only internal structure but integral and reciprocal relationships with other disciplines, especially science, and increasingly with the social sciences and humanities. The complexities of these relationships are likely to challenge the traditional hierarchical taxonomies of content. Theories are needed to provide mental models of the relationships between concepts and topics (e.g., Jackson, 1984). Students must see and experience mathematics as a language (CITMS, 1982) and a science which orders the universe (Jaffe, 1984), as a tool for representing situations, defining relationships, solving problems, and thinking. They need to experience the powers of its language and notational system in solving problems in a wide variety of domains. The connectedness of ideas is critical, and so is the connectedness of process and concept (Vergnaud, 1982). Students must experience mathematics as part of both larger content and larger process. They need to see it as a process of abstracting quantitative relations and spatial forms from the real world of practical problems, and inventing through the process of conjecture and demonstration of logical validity. The emphasis in instruction must now be on experiences that help students to know mathematics (Romberg, 1983a).

The emphasis of the new world view stresses creativity, innovation, problem solving, and a general high level of "at-homeness" (CITMS, 1982, p.11) with mathematics. Integrative, intuitive thinking is seen as essential to the process (Kanter, 1983). In turn, creativity and innovation are integral to both the creation of new knowledge and to solving problems, which could be described as applied knowledge creation. Thus, the emphasis of the new world view suggests a kind of school mathematics that would be very close to the approaches of applied mathematics and the informal mathematics of Lakatos (1976). Whether Lakatos' or another's, all mathematical pedagogy rests on a philosophy of mathematics (Thom, 1973) that amounts to a model of mathematicians doing mathematics. Hence, explicitly (e.g., Romberg, 1983a) or implicitly as in problem solving, for example (e.g., Chambers, 1986; CITMS, 1982; NCTM, 1980), knowing and doing mathematics, as opposed to knowing about mathematics, is an important part of major current statements of purpose.

When mathematical knowledge means knowing and doing mathematics rather than knowing about mathematics, other things follow. Knowledge is personal and communal in that, while it may originate in an individual, it is validated by the community. Thus, the process of adding to mathematical knowledge through communication is an integral part of knowing mathematics. Furthermore, the criterion for knowledge is not necessarily that it be true but that it be incorporated into the general system of knowledge (Rescher, 1979). In a sense, adding to the structure of mathematical knowledge is mathematics. This view means that mathematics is, by definition, dynamic and constantly changing; it is not, as has been the case in schools, a static, bound cumulation. The implications of these views for school culture are extensive, suggesting radical change in the work of students and teachers and in the professional character of all educators.

<u>The work of students</u>. The roles, and therefore work, of teachers and students are complementary (Skemp, 1979); one teaches, the others learn. However, schools ostensibly are places where students gather to learn; thus, the role of the teacher should complement that of the student, rather than vice versa. Unfortunately, when knowledge is regarded as knowing about rather than knowing, the vocabulary reflects a reversal of emphasis: the work of the teacher is to "transmit" knowledge. Logically, this means that the student's job is to receive knowledge and to regurgitate it on demand. In fact, the real work of the student is often a matter of negative goals, meeting expectations sufficiently to pass through the system (Skemp, 1979). Welch's (1978) description of a student's work in a mathematics classroom is similar to that of Clarke (cited in Stephens & Romberg, 1985):

> She tells us what we're gonna do. And she'll probably write up a few examples and notes on the board. Then we'll either get sheets handed out or she'll write up questions on the board. Not very often. We mainly get a textbook. We'll get pages. She'll write up what work to

> do, page number and exercise. And if you finish quick you may get an activity sheet. And that's about what happens. (p. 22)

The traditional situation described is organized, routine, controlled, and predictable--an unlikely environment for the creation of knowledge.

Some sense of the necessity for students to create knowledge rather than simply memorize somebody else's is reflected in current documents. The NCTM (1980), for example, recommended that problem solving be the focus of school mathematics for the 1980s, and that "mathematics curricula and teachers should set as objectives the development of logical processes, concepts, and language" (p. 8). Unfortunately, the very format of its presentation presented a checklist of discrete basic skills with no sense of coherent activity. A clearer and more focused sense was conveyed by the Department of Education and Science (1985), which stressed that

> making, testing and modifying hypotheses are parts of the thinking processes of everyone at different levels within mathematics, within the whole curriculum and in everyday life. (p. 22)

> The aim should be to show mathematics as a process, as a creative activity . . . and not as an imposed body of knowledge immune to any change or development. (p. 5)

However, if students are to create mathematical knowledge, both the kinds of new knowledge and the work involved in its creation must be more clearly defined. Knowledge may be new to the individual student, or it may be new to the global community. For mathematical education, the most exciting new knowledge is that cr ted by the student which is also new to the mathematical community, such as Stringer's Conjecture (Kidder, 1985b). Another type is the new "knowledge" generated by the successful application of mathematics to problems, each instance of which either revalidates existing knowledge or prompts efforts to expand the domain (Jaffe, 1984). This suggests for students a process of continually expanding and applying the system of personal knowledge and validating it against the domain as a whole.

The process of knowing is complex, involving both conscious and unconscious thought. Russell (1978) posits a cognitive unconscious that mirrors the processes of conscious thought but, without the constraint of reflexivity, works much faster: "All knowledge begins with 'intuition'" (p. 30). While this is not knowledge, it is the starting point for the conscious, logical processes of creating mathematical knowledge. The conscious work of mathematics is that of logically reconstructing, representing, and validating intuited abstraction, while recurrently drawing on the unconscious.

Briefly then, students' work consists of extending the structure of the mathematics that they know by making, testing, and validating conjectures which may originate as postulates of conscious thought or be derived intuitively. As long as students are making the conjectures, their mathematical knowledge will be structured, consciously or unconsciously, because conjectures cannot be created from nothing. Those conjectures may be abstract or applied; the modeling involved in the latter both tests a conjecture and develops a sense of consequence. This amounts to the process of reflective intelligence in which the structure of knowledge is constantly revised by reflecting on events, seeking ways to fit them into the existing structure, and testing the structure's predictive powers (Skemp, 1979).

Verbal (Adler, 1986) and written communication is a crucial part of the process for several reasons. First, logical argument is central to mathematical proof. Second, communication of that proof is the means by which personal knowledge is submitted for systematizing into the domain and thus accepted as new knowledge (Rescher, 1979). Third, developing competence in the categories and structures of the language system structures the child's understanding and advances it toward a public mode of consciousness (Russell, 1978). Fourth, there are indications that the process of repeating something aloud (Berk, 1986) or verbally explaining (Hart, 1981) assists children's understanding.

For the child, this conception of work suggests making and testing conjectures, building supporting arguments which are tested on peers. Whatever is new knowledge for the group is checked against the record of mathematical knowledge, interweaving the structure of personal knowledge with the structure of the domain. This personally built structure and the conscious linking of it to the domain is intrinsic to the vital notion of "at-homeness" with mathematics (CITMS, 1982). This view of work also implies that practical applications, which inevitably require cross-disciplinary activity, are essential. Obviously, it is radically different from absorption of the record of knowledge.

The work of children is not a matter of memorization, nor of following algorithms, even though these play a part. The creation of knowledge, whether at the personal or global level, involves a constant process of deliberately moving beyond what is known into the realm of disorganization, repeatedly guessing at connections and mental models until new and definable structures, objects, relationships, and processes emerge (cf. Skemp, 1979). It is important to note that this is not the same as the kind of discovery learning that plans for the acquisition of particular knowledge through discovery rather than exposition. It requires creativity, fluent verbal and written communication, and constructive, critical thinking (in the epistemological sense).

One consequence, for example, is the imperative that students be fluent correspondents in their native tongue and in a level of mathematical notation commensurate with the ideas they wish to express. Another is the integral nature of evaluation and reevaluation by students of their own efforts and, as the creation of knowledge is eventually a communal matter, cooperative assessment of the arguments of peers. Clearly, the work of students is no longer a matter of acting within somebody else's structures, answering somebody else's questions, and waiting for the teacher to check the response. Nor is it a matter of evaluating knowledge according to right or wrong answers. In the creation of knowledge, there is only that which fits the structure of mathematical knowledge already created by the student and that which does not, and therefore should prompt conjecture.

The work of teachers.

> The "median" classroom is self-contained. The mathematics period is about 44 minutes long, and about half this time is written work. A single text is used in whole class instruction. The text is followed fairly closely, but students are likely to read at most one or two pages out of five pages of textual materials other than problems. It seems likely that the text, at least as far as the students are concerned[,] is primarily a source of problem lists. Teachers are essentially teaching the same way they were taught in school. Almost none of the concepts, methods, or big ideas of modern mathematics programs have appeared in this median classroom. (NACOME, 1975, p. 77)

The NACOME description of work in a composite classroom is supported by other researchers (Welch, 1978; Stephens & Romberg, 1985) and is echoed by the observations of Romberg (1983b):

> Teachers modified the program, selecting parts to be taught and how much time was to be spent on each part. Furthermore the dominant pattern was not to select activities that encourage discovery and exploration, but to emphasize skill development and practice via worksheets. (pp. 21, 28)

Attempts to prevent emasculation of curricula include a focus on textbook selection and conscious efforts to bypass teachers by relegating them to the role of manager (Berliner, 1982). The single greatest consequence of this was to invalidate the teacher's skills (Apple, 1979). Another is that the textbook rather than the teacher is the classroom authority on knowledge (Stake & Easley, 1978).

However, the persistent neutering of curricula arises because the primary work of teachers is to maintain order and control (Romberg & Carpenter, 1985; Wisconsin Education Association Council, Curriculum and Instruction Committee, 1986). An inexorably logical sequence occurs when the acknowledged work of teachers is to transmit the record of knowledge. The most cost-effective way to transmit the record of knowledge is through exposition to a captive audience. Theoretically, the child could read and cover the same material, but that would require a voluntary act which is unlikely as long as children are not setting their own goals. Consequently, that exposition cannot occur unless there is control, which is easier if children talk as little as possible and stay in one place. It is essentially a system for "delivering" knowledge to the group by controlling the individual. This simple sequence has dictated work, furniture arrangement, architecture, etc., for the last hundred years, and it is the tradition most challenged by any attempt at change. The result is that

> the traditional classroom focuses on competition, management, and group aptitudes, the mathematics taught is assumed to be a fixed body of knowledge, and it is taught under the assumption that learners absorb what has been covered. (Romberg & Carpenter, 1985, p. 868)

If one regards the roles and work of students and teacher as complementary, and if the emphasis is on creating knowledge rather than absorbing the history of other people's knowledge, then the teacher's work is to support, promote, encourage, and facilitate the creation of knowledge by students. In order to know, students must educate themselves. In flight training, for example, where knowing can make the difference between living and dying, the intrinsic motivation that ensures that the student takes responsibility for his own learning is crucial (unless one regards death as an extrinsic motivation). Thus, according to Dawson (1983), "The teacher and the content are not paramount: THE LEARNER IS" (p. 592).

One way in which knowledge is acquired is through imitation and play (Russell, 1978). Imitation and invention are parts of the same process. That which children imitate, they invariably adapt, perhaps creating new combinations. An implication of imitation as a source of knowledge is that teachers display for imitation intuitive and logical ways of extending their own knowledge and encourage children to imitate others as a means of acquiring knowledge. Unfortunately, many patterns of children learning from peers through imitation are regarded as cheating or as failure to learn for themselves and are actively discouraged.

The importance of "the higher-order mental processes of logical reasoning, information processing, and decision making" (NCTM, 1980, p. 8) is considered basic to mathematics and is thus accepted as part of the work of teachers. However, "human cognition is fundamentally unitary" (Fischbein, 1975, p. 6); these higher order cognitive processes are but the second, conscious part of creating knowledge (Russell, 1978) and are, moreover, typically taught as a series of skills. If teachers are to support the total, integrated process of knowledge creation, they must provide an environment in which those skills are part of a continuous and coherent cycle of intuition, logical argument, and evaluation. In addition, because the network of knowledge built by the child is more important than those portions of the formal domain acquired by the child, the organizing principle is the cohesion of the child's knowledge, and not that of the domain.

Because the work of the student is to create knowledge, which begins with intuition (Russell, 1978), and the role of the teacher is to complement the child, teachers must support the unconscious, cognitive processes of intuition in children. Long considered important to mathematics (e.g., Fischbein, 1975; Freudenthal, 1983; Hadamard, 1945), intuition is poorly understood. Although automatic, ballistic, and autonomous, it is not random (Russell, 1978). The groundwork is intense effort over a long period, although flashes of intuition usually occur in a period of alert nonarousal after relaxation (Goldberg, 1983). Such insight is promoted by diversity of experience, which encourages analogies and the productive pairing of polarized concepts. It is inhibited by reliance on acquired routines (Freudenthal, 1983) and by anxiety (Goldberg, 1983). Tactile and kinesthetic experience, abstraction, and communication all encourage its development (Wittman, 1983; Henkin, 1983; Davis & Hersh, 1981).

The second part of the cycle is consciously cognitive and involves conjecture, modeling, logical reasoning, and communication. In this respect, Adler (1986) argued:

1. That the work of students is to read and discuss, integrating the private experience with the public;

2. That this should be so from kindergarten to college;

3. That the worst thing you can do is "cover the ground."

Although inherently pragmatic and practical in its approach to curricular change, the Department of Education and Science (1985) drew similar conclusions regarding the importance of discussion: "The quality of pupils' mathematical thinking as well as their ability to express themselves are considerably enhanced by discussion" (p. 39). If this is so, Adler (1986) reasoned, the teacher's role is to provide a seminar framework, backed by tutorials and an occasional lecture.

The third part of the essentially iterative process is evaluative. The role of the teacher is two-fold: to ensure that assessment, both individual and cooperative, is a part of the process of knowing; and to diagnose children's difficulties.

Briefly, this suggests that the essential work of teachers includes:

1. Ensuring successful experience for children;

2. Providing for extended and cooperative project work, whose final product is a report;

3. Bringing an informal and interdisciplinary approach to

mathematics;

4. Encouraging verbal and written eloquence in arguing intuitions;

5. Encouraging self-evaluation and providing for group evaluation of new knowledge and reference to the formal domain;

6. Demonstrably exercising intuition and adding to their own personal knowledge;

7. Providing an emotional and physical environment supportive of student work. This includes, for example, recognition of the need for cessation of conscious effort or a change of activity, or of an urgency to immediately capture a thought on paper. It also includes providing for student experience with both physical and intellectual modeling;

8. Changing from structural authority based on negative or extrinsic goals of students to sapiental authority (Skemp, 1979) founded on intrinsic goals. This may be an answer to the problem, characterized earlier as the Japanese dilemma, of moving from regimented uniformity to individual and collaborative creativity;

9. Monitoring the structure of knowledge being created by the child;

10. Using technology appropriately in the processes of intuition, play, acquisition and manipulation of information, logical argument and communication, evaluating new knowledge against the domain, and tracing the development of the student's network of knowledge.

In short, it is essential that the teacher provide the environment, act as a mentor, and get out of the way.

The professionalism of teachers. The legitimacy of schooling is derived from the professional status of teachers (Popkewitz, Tabachnick, & Wehlage, 1982), which vests them with the authority to mold children and bestow a "social identity" which frequently channels their entire adult life (Romberg & Price, 1983). A profession is recognized because it has specialized knowledge, a corporate bond which supports the development of collective wisdom, and sovereignty in its field (Otte, 1979). Competency testing of teachers by school districts suggests that, for at least the first and last categories, teaching is not respected as a profession. This is a significant impediment to the changes in knowledge and work argued above, for they rely heavily on the professional ability of teachers.

Hall (1968) identified five attributes necessary to professionalism:

1. Collegiality: Belief in the formal organization and informal exchange with colleagues as the major source of ideas, judgment, and identity;

2. Public service: A belief that the profession performs an indispensible public service;

3. Self-regulation: A belief that only colleagues are qualified to judge performance;

4. Vocation: An inner compulsion to the profession;

5. Autonomy: The freedom to make professional decisions without pressure from other professions, nonprofessionals, or employing institutions.

On every count, teachers are under pressure--sometimes fairly, sometimes not. Their professional organizations have diluted their own professional power by acting as collective bargaining units. They bargained not only for teachers but for other professions, such as librarians, who had a professional association but no bargaining organization. Furthermore, whereas teachers may believe that only other teachers are competent to judge their performance, competency-based testing of teachers makes it patently obvious that the community at large, having made its own assessment of student performance, is judging and finding teachers lacking.

The lack of control over the judging of teachers arises largely because the profession is not self-regulating. Teachers participate in establishing state certification requirements, for example, but those standards are administered by state bureaucrats and not by the profession. Although teachers say it is impossible to conceive of teaching unless one has a sense of vocation, the aphorism that "those who can, do; those who can't, teach" evinces a high level of general scepticism about the avocation of many teachers. Teachers have, in one sense, a high degree of autonomy inside their own classrooms, but as a group, they are in danger of losing whatever professional autonomy they once enjoyed.

In all fairness, the problems of the profession are external as well as internal. Teachers are part of society. They were educated by the system, and they function within the system. The perceived inadequacies of students are relative to the expectations of society, which have changed. Teachers' levels of expertise and subject knowledge are directly attributable to the system that educated them. If they retain order, control, scientific management, and student ranking as their conception of work (Romberg, 1985), the entire system is at fault, including their preservice experience.

The working environment must allow teachers to implement change. In this regard, superintendents are pivotal in that they serve as intermediaries among board, community, parent, child, and teacher. It is, for example, absurd for a district administrator to espouse educational excellence, curriculum development by teachers, and the concept of the Information Age and then to ask teachers presenting ideas if they have gone through channels (cf. Kanter, 1983). However, superintendents, in their turn, usually are products of the system that produces administrators. Hence, some of the blame for the working environment of teachers is directly attributable to the graduate schools of education.

The community at large must comprehend that the methods for judging children, and therefore teachers, need revision (Adler, 1986). If society expects to mandate standards for teachers and children through objective testing and vetted texts, it is unrealistic to simultaneously expect creativity and innovation from children and a high level of professionalism from teachers.

On the other hand, if the role of teachers is to support the creation of mathematical knowledge by students, the professionalism of teachers should support and enhance their role. In fact, if belonging to a profession fosters specialized knowledge, collegiality, collective wisdom, and sovereignty in the field, membership endows teachers with sapiental authority. This suggests that if teachers do not have the attributes of professionals, they either are not members or the profession itself is lacking.

Approached from a different view, the professional backing needed by teachers is that which would:

1. Ensure excellent preservice and inservice education, congruent in style with the quality of teaching expected;

2. Provide for teachers to constantly expand their own knowledge of the domain through such things as sabbaticals, summer scholarships for foreign study, inservice, and computer conferencing with experts, placing no restrictions on the directions of investigation;

3. Provide for constant electronic collegiality;

4. Educate the entire system of education to support the efforts of the teacher, including superintendents, boards, professional bodies, and parents;

5. Provide a framework for rigorous self-regulation;

6. Abolish the intolerable constraints under which they now operate; namely, standardized testing, standardized texts, and the cover-the-ground philosophy.

Implications for Monitoring

Information on the System, Not Just Parts

It is patently obvious that society bears on education and all parts of the educational subsystem bear on one another. One part cannot be considered and monitored in isolation. The system's values affect architecture, curriculum, materials, etc.; materials and architecture affect curriculum; and so on. Consequently, the whole is not simply the sum of the parts. This means, for example, that placing computers in the classroom, or increasing time spent learning, cannot be averaged into the system as a proxy for improved understanding. Much depends on such aspects of the system as philosophical context, the materials used, the diagnostic approach of the teacher. Some parts of the system may have a stronger impact than others. Values, for example, are very powerful and could act as catalysts throughout the system. Thus to monitor changes in values without considering other system factors may provide an erroneous view of the system as a whole.

What is needed is a sense of holistic functioning, a sense that:

1. The performance of a whole is affected by every one of its parts;

2. No part has an effect on the whole that is independent of the other parts;

3. Any subgroups that are formed are also subject to the first two constraints.

Thus, for example, it is not possible to make recommendations about the way teachers teach without considering how they were taught, the materials to be used, the educational philosophy of the school district, etc. . . . Therefore, to monitor curricular change meaningfully, the educational system must be monitored as an entity. For example, it is relatively meaningless to assess teachers for minimum competency, thus conveying a value, if the purpose of mathematical education is more than minimum competency.

Causal Model

One problem with monitoring curricular change is that to monitor parts of the educational system in isolation is inadequate. The purpose for monitoring frames the questions asked nd the means of assessment. Consequently, it is important to develop a causal model of mathematical education (and education in general) which in essence says: The challenge of mathematical education is for children to create mathematical knowledge and to use mathematics in the process of creativity and innovation.

Information is already gathered on some of the variables of mathematical education. However, some elements not regarded by policymakers as essential to the purpose for monitoring, such as attitudes, may share relationships with others, such as knowledge. Consequently, any effort to monitor elements independently will provide an incomplete and misleading picture. Thus, the first task of monitoring is to create a purposeful and causal model of the educational system which identifies the contributing elements and specifies their relationships. Such a model would provide a number of immediate benefits:

1. It would focus attention on the purpose for change;

2. It would force the clarification of contributary elements and their relationships;

3. Comparison of simulation with the results of monitoring would further clarify conceptual relationships;

4. It would be congruent with other facets of the Information Age and therefore help lessen intellectual conflict;

5. It would help clarify the extent and validity of information about the system;

6. It would expose through its own inadequacies those parts of the system about which more information is needed.

New Vision of Outcomes

One major benefit of the causal modeling approach is that the monitoring process would inevitably focus attention of all parts of the system on the new vision of outcomes, communicating and clarifying the purpose for demanding change in mathematical education. Clarity of vision will help states and local boards decide what it is they want of teachers, and teachers what they want to inculcate in children. A new vision of outcomes, when combined with a modeling approach, is likely to clarify the relationships, or lack thereof, between, for example, a stress on drilling minimum competency in skills and the need for creativity. When mathematical modeling was applied to economics, it led to the wisecrack that economics is a little bit better than astrology, but not quite as good as meteorology. The point is not that economists are incompetent, but that their conceptual models are inadequate. This led to a greater appreciation of the importance of some relatively neglected human factors. Thus, the mathematical modeling and monitoring of outcomes, if treated as a means and not an end, will assist progress toward those outcomes. Its *sine qua non* is clarity of purpose, and that derives from clarity of vision.

Conclusion

This paper is essentially an argument for lucid thought and coherence of purpose, tools, action and knowledge. It outlines recent views on the world and their significance for the mathematical education of children. It argues for a service concept of education, with children as the recipients. Time is important as one measure of effort. It is, more fundamentally, a measure of a person's lifespan. If, through failure to take necessary steps to improve the mathematical education of children, we waste their time, we also waste a part of their lives.

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CHAPTER 3

A CAUSAL MODEL TO MONITOR CHANGES IN SCHOOL MATHEMATICS

Thomas A. Romberg

This paper describes the causal model proposed to document changes in school mathematics. The model has been designed to be a basic framework in a system developed to monitor the changes in school mathematics in the U.S. and the impact of those changes on the health of the system (see chapter 1). We have assumed that the development of a valid monitoring system must rest on notions of which schooling components are important for examining the health and change in school mathematics and how those components are related to one another. In particular, the model and the monitoring system have been designed to reflect the proposed (or anticipated) changes in the mathematical content, sequencing and segmenting of lessons, the job of teaching, and methods of instruction. Furthermore, it has been assumed that, if key components of school mathematics can be identified, it may be possible to construct reasonable measures for those components. If measures can be constructed, perhaps indicators of system health can be developed. However, we believe that constructing measures and indicators without such a framework would be futile.

Expected Changes in School Mathematics

This paper is based on

> a belief that today, in most classrooms at all school levels, mathematics instruction is neither suitable nor sufficient to adequately equip our children with the mathematical concepts and skills needed for the 21st century. Furthermore, unless something is done to alter current schooling trends, conditions are likely to get worse in the coming decade. (Romberg, 1984a, p. 1)

Assuming that the schools of America will respond to the current perceived crisis in school mathematics, the nature of those anticipated changes needs to be identified.

Changes in school mathematics during the next decade should reflect the new and different goals currently being proposed for our students. At present, school mathematics has been geared to preparing a minority of students to take calculus. Topics were included (or excluded) based on assumptions about their relevance to that goal. For college-bound students who were disinclined toward calculus, or deemed incapable of achieving competence, some basic knowledge of algebra and geometry has been considered sufficient. For noncollege-bound students, only arithmetic competence has been deemed essential.

Today, these goals are being challenged. Calculus, while still of major importance in most fields, no longer holds its preeminent position in mathematics (Ralston & Young, 1983). The calculator and, in particular, the computer have expanded the utility of other mathematical ideas, such as mathematical modeling, algorithmic analysis, discrete mathematics, matrix algebra, coordinate geometry, statistics, and applications in various fields. In particular, the new technology has freed us from the cumbersome calculation routines of arithmetic, algebra, statistics, and calculus and, in so doing, has expanded our ability to carry out even more complex computations.

A variety of recommendations have recently been made (Conference Board of the Mathematical Sciences, 1982; Conference Board of the Mathematical Sciences, 1984; Romberg, 1984a). Anticipated changes include:

1) changes in course content and structure,
2) changes in course requirements,
3) changes in the sequencing and segmenting of mathematical topics,
4) changes in the use of technology,
5) changes in methods of assessment,
6) changes in the knowledge and professional responsibility of teachers,
7) changes in the way mathematics is taught, and
8) changes in the policy environment within communities.

The assumption underlying these prospective changes is that their implementation will cause students to know more mathematics, to be able to use mathematics more effectively, and to be productive citizens of tomorrow's world.

This is not the place for a detailed discussion of the basis and the nature of each category of possible change (see Romberg, 1984a). The papers comprising this three-volume monograph will document and summarize the rationale behind each of these anticipated changes.

For monitoring purposes, three questions must be raised: Have these anticipated changes actually occurred, and, if so, to what degree? What is the effect of these changes on students? Do these changes improve the health of school mathematics? To approach the design of a monitoring system that would gather appropriate information to answer these questions, a causal model has been developed.

Causal Modeling

To bring conceptual order into the chaotic world in which we live, man has invented myths, developed cultural traditions, and proposed rational frameworks, models and theories. During the past decade, an important development in the social sciences has been the development of causal models. A model is not a completely accurate representation of a real situation; it is only an attempt to capture some key components and their interrelationships with respect to some phenomena. The utility and status of a model depends on social agreement and the empirical evidence that validates the model.

Figure 1 illustrates the basic elements in model building. It implies that model building involves several stages. The development of a model starts in some empirical situatio that presents a "problem" for which an "answer" can be very misleading.

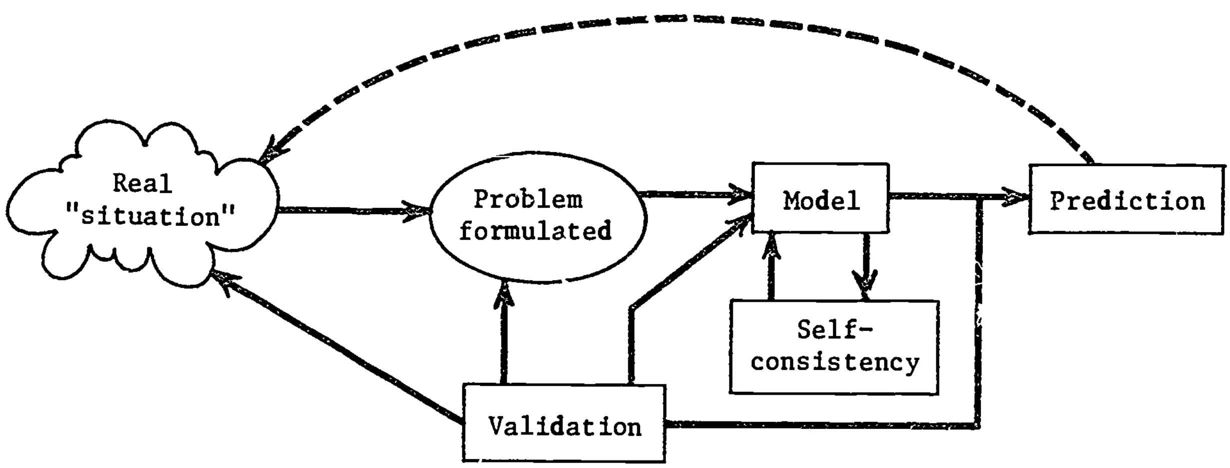

Figure 1. The basic elements in model building.

Real situations are rarely well defined and are often embedded in an environment that makes a clear statement of the situation hard to obtain. Formulating the problem involves specifying the assumptions, concepts, and principles one believes are operating in the real situation. Such specification must, of course, be selective in its bias. Simplification or idealization is a crucial stage, since the general problem is usually exceedingly complex and involves many processes. Some features will appear significant, many irrelevant. In fact, identification of situational features should be done by a group knowledgeable about the real situation. Once the significant features have been identified, they are translated into a causal model. The model itself contains a list of variables, and a list of relationships (or equations) specifying the links of any type that are hypothesized to exist between the variables. Obviously, a model is mathematical if the relationships are expressed using the language of mathematics. However, the real power of causal modeling lies not in the representation process, but in what one can do with those representations. The statements can be viewed as a set of premises from which other sets of consequences can be deduced (predictions can be made). When a model is constructed, it needs to be validated. Indeed, some form of validation is usually carried out throughout the formulation. In particular, a model's validity rests on its ability to represent the situation initially described. Although a model may have to represent reality, it is not itself reality. Situations are modeled for various purposes, foremost among which is the need to predict new results or new features. Since the conditions to be predicted are likely to exist at some future date, the predictions themselves can be empirically validated.

A model is called <u>causal</u> if there are reasons to believe there is a causal order which relates the variables in the model. In particular, if for two variables X and Y, one can logically argue that X might influence Y but Y does not influence X (conventionally represented as $X \rightarrow Y$), then one can assert that "X is a possible cause of Y," but not that "X is <u>the</u> cause of Y." A suggested relationship alone does not prove causation. Causal claims must rest on other, persuasive evidence about X and Y or on appropriately controlled experiments. However, when experiments are impossible, such as in a study of schools' response to pressures for change, causal modeling allows researchers to develop causal propositions supported by data and logic.

Finally, the variables in a causal model are typically referred to in blocks as <u>prior</u>, <u>independent</u>, <u>intervening</u>, <u>dependent</u>, and <u>consequent</u>. Those variables within a block are called <u>parallel</u>. Thus for $X \rightarrow Y$, X is the independent variable and Y is the dependent variable. Usually, if X is an independent variable, it can be changed or manipulated. For example, in mathematics classes, textbook content (X) can be changed to include statistics. Y then would refer to the outcome one would expect from such a change, such as student performance on a statistics test. Prior variables occur before X; intervening between X and Y and consequent, after Y. For example, the teacher's knowledge of statistics would be a prior variable, the time allocated to teaching statistics an intervening variable, and enrollment in a later statistics course a consequent variable.

In the following sections of this paper, a causal model has been developed for the "real situation"--the anticipated changes in the teaching and learning of mathematics in schools.

Variables for a Model

Given the anticipated changes described earlier in this paper, key variables which reflect those changes and their effects need to be specified.

Dependent Variables

The typical expected outcomes of mathematics instruction are that students will acquire some knowledge of mathematical concepts and some proficiency with mathematical skills, will be able to use that knowledge in problem situations, will develop favorable attitudes toward mathematics and its social utility, and will continue to enroll in mathematically related courses (or programs) if appropriate and if options are available. These should be considered as three dependent variables (knowledge, attitude and application) and one consequent variable (further enrollment). (See Figure 2.) However, simply specifying these four variables for the causal model does not reflect the changed meaning of those variables.

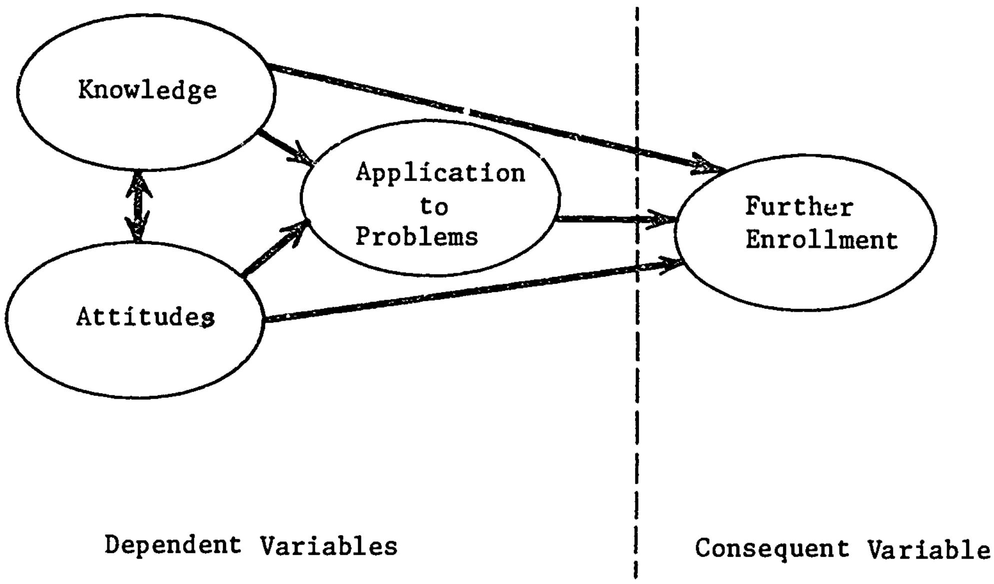

Figure 2. Outcome variables for school mathematics.

In the past, paper-and-pencil tests (usually multiple choice), where frequency of correct responses was calculated, have been used to judge acquisition of knowledge. Note also that neither attitudes, applications nor further enrollment have usually been assessed and used as dependent variables.

Knowledge. Different goals obviously imply different specific outcomes; hence, there are different expectations of knowledge in a causal model. Increasingly important are quantitative and spatial reasoning; strategies used on problems; knowledge of the relationships between concepts, or between concepts and procedures; means of representing situations; and means of transforming expressions. Unfortunately, many of these outcomes cannot be assessed simply by counting the number of correct answers on a multiple-choice test.

As a start to overcoming limitations of standard testing procedures, an item framework is being developed to address three problems: curricular relevance, item aggregation, and item responses. Curricular relevance is important since the monitoring scheme is to identify changes in achievement due to changes in the curriculum. A content conceptual network scheme will be developed where the content is judged to be curricularly relevant. Items from any source can then be matched to the scheme. This would allow us to cross-validate time trends using different data sources. It will also serve as a useful guide for incorporating new items and assessment techniques (e.g., from other NSF projects, NAEP, and OERI research centers) into the monitoring scheme.

For analyses of items, not only correct response but also use of strategies and errors must be coded. Item-level data are very useful for dissemination purposes, so the public can have a concrete sense of students' level of knowledge. The content network scheme also can be used as a basis for item aggregation and creation of scores for comparison purposes. Item responses can be aggregated by content category, strategies, errors and other dimensions.

Attitudes Toward Mathematics. A second dependent variable often stated as a goal of mathematics education is the development of favorable student attitudes toward the subject and its utility. At a superficial level, items on student attitudes toward mathematical topics, mathematics teachers, mathematics or science careers, and the usefulness of mathematics are often given. These can be used as rough indicators of attitudes. For example, a recent Canadian study on views of students in grade 7-10 on mathematics, calculators, and computers indicated that most students do not particularly like mathematics but consider it important, and indicate they would take more mathematics courses (McLean, 1982). One would hope that contemplated program changes would influence responses to such questions.

Application to Problems. The third category of outcome variables relates to the expectation that students will be able to use the knowledge they acquire. It is assumed that the greater the familiarity students have with the concepts and procedures of mathematics, and the richer the relevant relationships between and among those concepts and procedures they have experienced, the more readily they will be able to solve problems. However, knowledge alone is not sufficient. One succeeds in solving problems to the extent that one can construct mental models that represent the relevant information in an appropriate fashion and use these mental models with flexibility.

Simple word problems requiring students to use learned concepts and skills are typically used to assess these outcomes. Their utility has been well documented in the NAEP data, where students demonstrated that they knew basic arithmetic skills but had difficulty using them to solve problems (Carpenter, Corbitt, Kepner, Lindquist, & Reyes, 1981). However, the use of mathematics with more complex problems (both in applications outside mathematics and to other mathematics situations) requiring higher order reasoning can only be assessed with any validity via interviews or complex-response schemes. This is of particular concern, since it is assumed that a major emphasis of new mathematics programs will be toward problem solving.

Specifying these dependent variables for the causal model actually pose two problems for monitoring. The first is the need to develop procedures to validly assess these outcomes, and the second is to document the degree to which other commonly used assessment procedures (standardized tests, NAEP, state assessments, classroom tests, etc.) reflect the changes in goals. Content of other assessments becomes one of the independent variables in the causal model.

Consequent Variables

The final outcomes of school mathematics generally considered include continued enrollment and completion of mathematically related courses: choice of college majors; choice of careers; and later career paths, including life income and job satisfaction. Each is important to individual and societal goals and to the development of human resources. Each, however, is mediated by many variables other than those associated with schooling.

For purposes of the causal model, only course completion and continued enrollment data are included. Increased enrollment in higher mathematics courses, particularly by minority students and women, is an anticipated outcome of the proposed changes.

Independent Variables

Educational practice assumes that what occurs in schools when students are taught mathematics leads to their acquisition of knowledge and the development of favorable attitudes. Some schooling activities can be deliberately varied (manipulated) by actions of educational policymakers. These must be the independent variables in the causal model.

Content. The changes in goals for school mathematics as described earlier in this paper primarily involve changes in the mathematical content of school programs. These changes should be reflected in changes in the curriculum. The curriculum can be viewed at four levels: the ideal curriculum (what mathematics educators would like to see taught); the intended curriculum (what is recommended in state and local guidelines); the available curriculum (what exists in texts); and the actual curriculum (what is covered and emphasized in classrooms). In particular, new programs (the available curriculum) should be more congruent with the new goals (the ideal curriculum). For this causal model, five variables will be used to reflect the anticipated changes in content goals described earlier. These are guidelines, technology, course requirements, texts, and tests (see Figure 3). Because of the interdependence of these five variables, it is not clear whether they should be considered separate variables or as a composite variable.

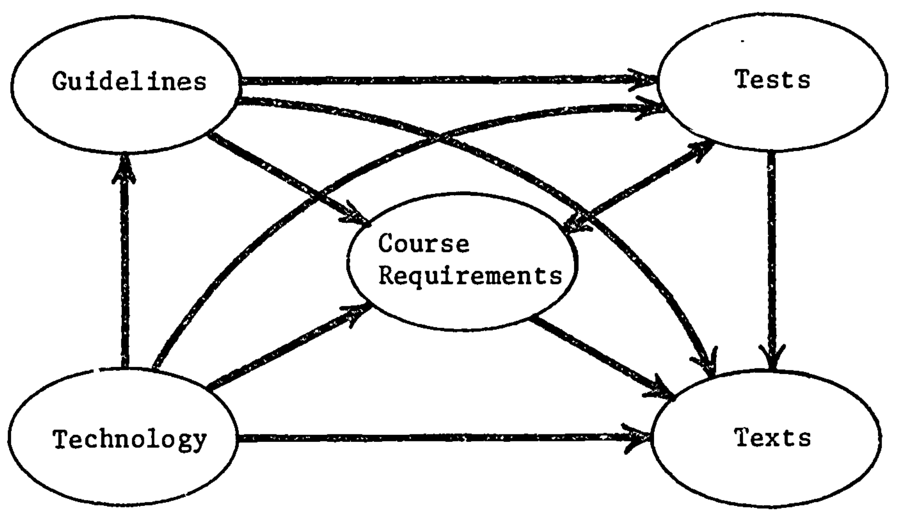

Figure 3. Independent variables for changes in content goals in school mathematics.

These five related variables should reflect the changes in content goals in the following manner. First, state and local guidelines (frameworks) which outline the general mathematical content to be included in the total school curriculum (the intended curriculum) should reflect the anticipated changes in the content of school mathematics and the time allocated to each topic in the curriculum (the ideal curriculum). For example, in such guidelines, one would expect less emphasis on procedural skills and more on conceptual development; less on algorithmic drill and practice and more on application and use of mathematics in open situations; less arithmetic and more statistics and probability, mathematical modeling, estimations and approximations.

Second, changes in the use of technology in instruction will involve a decrease in paper-and-pencil routines and an increase in the use of calculators and computers as tools for simulation and problem solving.

Third, both the new guidelines and technology should influence the requirements and structure of courses for all students. While one or two years of mathematics have been required of all students for high school graduation, three years soon will be common. Furthermore, the course structure will no longer involve eight years of arithmetic followed by an algebra-geometry-advanced algebra sequence. Instead, it is expected that most courses from at least grade 4 will integrate several mathematical strands.

In turn, the changes in goals should be reflected in new tests or assessment procedures, as discussed above. Finally, to a considerable extent, many of the proposed changes in goals, technology and courses will be reflected in the textbooks and other instructional materials (e.g. manipulatives, software, etc.) being used in the available curriculum. These materials should reflect changes in mathematical strands and the segmenting and sequencing of topics within strands. The strands will focus on conceptual domains such as assigning numbers to objects and sets, additive structures, multiplicative structures, representing problems algebraically, exploring data sets, geometric transformations, etc. These conceptual domains probably will be segmented into instructional units to be taught in two or three weeks, and the units will be spirally sequenced. Furthermore, the dependence on a printed book with each child having an identical copy for instruction undoubtedly will change.

Teaching. Two variables have been chosen to reflect changes in teaching in this causal model: increased teacher knowledge and teacher professionalism (see Figure 4). Many teachers at both the elementary and secondary levels undoubtedly have an inadequate or somewhat outdated knowledge of mathematics. It is unreasonable to expect that teachers who have little knowledge or preparation in mathematics, or in the teaching and learning of mathematics will perform adequately in classrooms. Yet we have little direct information on the actual mathematics knowledge base of most teachers. Nevertheless, new pre-service training programs and particularly inservice courses are being funded and developed and soon will be implemented.

Figure 4. Independent variables for changes in teachers.

One problem addressed in most commission reports is the professional status of teachers. At present, teachers tend to be isolated in their own classrooms. They have little opportunity to share information with other staff members and little access to new knowledge (Tye & Tye, 1984). Thus, the role of teachers in the traditional classroom often is managerial or procedural in that "their job is to assign lessons to their classes of students, start and stop the lessons according to some schedule, explain the rules and procedures of each lesson, judge the actions of the students during the lesson, and maintain order and control throughout" (Romberg, 1984b, p. 13).

In such situations, mathematics is too often taught without care or reflection. The job of teaching is perceived to be procedural or managerial, and not adaptive. Too many teachers feel obligated to cover the book. They may adapt instruction so they can better manage the diverse group of students in their class, or so their students will earn higher test scores. Too few teachers see student learning of mathematical methods and their use in solving problems as the primary goal of instruction.

To meet these and related problems of providing adequate instruction in schools where the teaching force is inadequately prepared, a variety of proposals have been developed. One recommendation is that elementary schools adopt a differential staffing pattern so that mathematics instruction is carried out or directed by adequately prepared, capable teachers. This involves creating new specialist teachers of mathematics, who would teach or supervise all mathematics instruction (see Romberg, 1984a). Another proposal is to organize support networks (as suggested by Conference Board of the Mathematical Sciences, 1984) which would link teachers with their colleagues at every level and provide ready access to information about all aspects of school mathematics.

Policy environment. To complete our examination of independent variables, it is necessary to add another. From the information gathered about initiatives, one should be able to keep track of changes in the policy environment. For example, changes in state requirements for licensing of teachers, level of proficiency in mathematics required to teach in schools, standards for graduation, and so on should be tracked to provide insight about this impact. In fact, even the notions about what is fundamental for all (or some) students will vary with communities.

Finally, one category of recommended change, methods of instruction, will not be treated as an independent variable in this model. This separates those actions that actually occur in classrooms from those that persons other than teachers can initiate. Variables associated with these changes will be included as intervening variables.

Intervening Variables

Conceptually, this category of variables is very important. Intervening variables occur between the independent variables and the dependent variables. They are not directly manipulable by policymakers, but they significantly affect outcomes. For purposes of this causal model three variables (pedagogical decisions, classroom events and pupil pursuits) have been identified (see Figure 5).

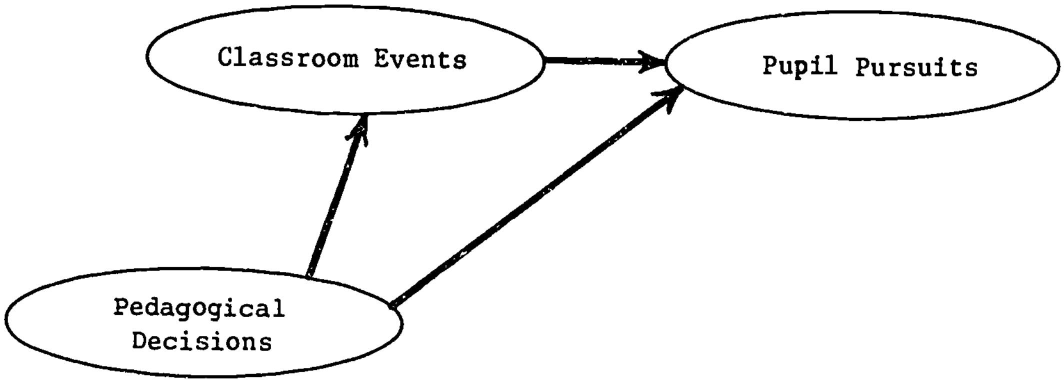

Figure 5. Intervening variables for school mathematics.

Pedagogical Decisions. This variable refers to the decisions teachers make in order to carry out instruction. Such desicions include: time allocated to mathematical activities, the adaptation of intended activities, and the emphasis given to any lesson. This variable should be considered either as the last step in defining this intended curriculum for it reflects the teacher's instructional intent, or the first step in describing the actual curriculum. For example, the average amount of time spent at the elementary level on mathematical topics each week can be reliably estimated from time logs filled out by teachers. This measure could serve as an indicator of length of exposure to pertinent content. Values can be compared for different years. Hence, if proposed curricular changes actually occur, we would expect a steady decrease in time allocated to computational skills and an increase in time allocated to statistics. However, the variable of allocated time by itself is not sufficient. It has been shown that teachers frequently adapt or change lessons. Sometimes such changes are made to increase learning or to provide for individual differences. However, too frequently adaptations are made for managerial reasons, or because teachers fail to see the significance of an activity (Stephens, 1983). Such adaptations reflect a teacher's intent. On the other hand, several researchers (Bishop, 1985; Brousseau, 1984; Donovan, 1983) have found that teachers vary in the ways they demonstrate the importance of various lessons through explicit or implicit actions. Such actions reflect the actual curriculum as experienced by students.

Classroom events. Teacher behaviors during instruction and other aspects of the classroom environment undoubtedly influence student achievement. Such discrete variables as the structuring of lessons, asking questions, and grouping are important (Waxman & Walberg, 1982). Unfortunately, they can only be documented with certainty in small studies through extensive observations. It may be possible to document more general instructional strategies such as "active or direct instruction." Finally, teachers may take a number of actions, such as structuring lessons, which have been shown to substantially improve achievement (Good, Grouws, & Ebmeier, 1983).

Pupil pursuits. As with classroom events, there are several things students do in classrooms that have been shown to be related to achievement. For example, time students are engaged in learning, time spent on homework, strategies used in working assignments, perception of the importance of an assignment or lesson, and degree of peer interaction on assignments have all been shown to be important.

In summary, these classroom variables--pedagogical decisions, classroom events, and pupil pursuits--seem reasonable for this model.

Prior Variables

The teaching and learning of mathematics does not occur in a vacuum. The background and prior knowledge of students and teachers influence instruction. The social context or culture in that the school operates and which supports or hinders change is critical. For this model, we have identified three prior variables that influence the independent variables in the causal model: pupil background, teacher background, and social context.

Pupil background. Survey techniques can be used to gather data on fixed characteristics to classify students in terms of gender, age, socioeconomic status, ethnic heritage, etc. Such classification is important to determine whether, for example, more girls are enrolling in mathematics classes or Hispanic children's performance is improving. For pupils' prior knowledge, previous outcome data can be used as a base for the study of growth. In particular, given the detailed information about outcomes in this monitoring scheme, we can examine the actual effects of prior knowledge on new learning in mathematics.

Teacher background. As for pupils, data on several fixed characteristics can be readily gathered from surveys. In particular, such additional data as mathematics preparation, years of teaching experience, or familiarity with computers can be gathered.

Social context. Schools operate within a cultural or social context. Features such as the demographic location of the school, average socioeconomic status of the students, and the percentage of limited-English-speaking students can be easily determined. The average wealth of districts, and the tradition and contribution of the states to schooling can also be assessed.

In summary, the prior background variables of pupils, teachers, and community can be used to classify the population in order to contrast degree and types of change in mathematics instruction.

A Model for Monitoring

The complete causal model proposed for this School Mathematics Monitoring Project is shown in Figure 6. It contains the variables identified in the past sections and the hypothesized causal relationships between variables.

This model provides the basis for gathering and interpreting information about mathematics instruction and anticipated changes in school mathematics during the next decade. It demonstrates the belief that mathematics teaching is complex and that changes cannot be simplistically studied. If appropriate indices can be developed to capture the variability associated with each variable, then useful information should be available for policymakers, school personnel, and researchers.

Indices, Indicators, and Implications

The causal model illustrated in Figure 6 is proposed as the basic framework for the monitoring system being developed. However, it should be made clear that, as pictured, it is a static model. It can only be used to investigate the relationships between variables at a particular point in time, with a sample from particular populations. Furthermore, the unit of analysis must be class (or, more likely, school), and the causal links between variables refer to "averages" or "tendencies," since exceptions are to be expected. In fact, some of the exceptions (schools where relationships differ) are likely to be of interest, for it is possibly in those schools that real change is taking place.

Since the purpose of monitoring is to capture the dynamic aspect of change in school mathematics, it will be particularly important to investigate the relationships indicated in the model at several points in time. Thus, changes in independent or intervening variables over time should be related to the changes in dependent measures. From the analyses of such changes, a dynamic causal model based on the static model should be possible.

Indices. To use the causal model shown in Figure 6 for analytic purposes, one or more indices (measures) must be created

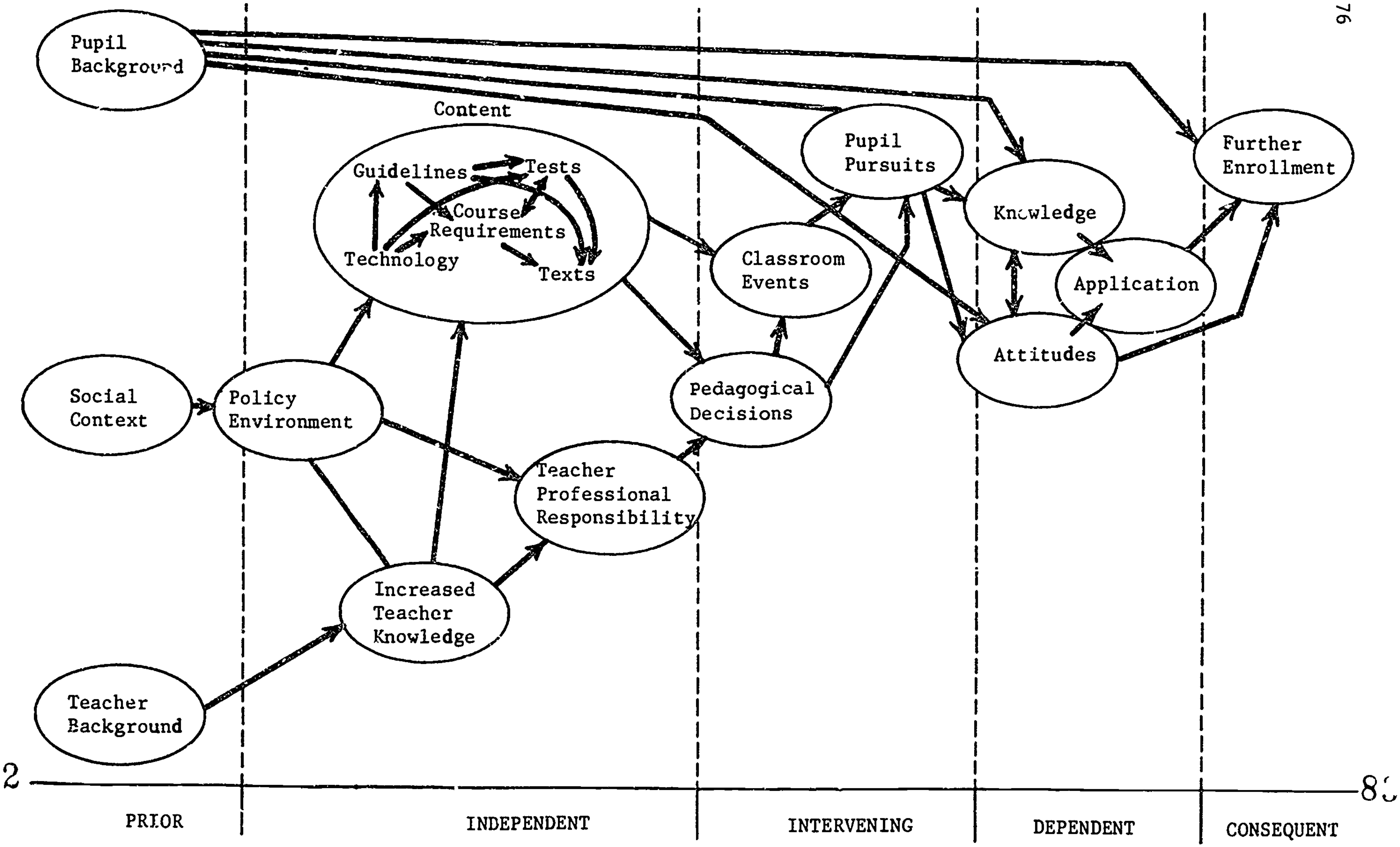

Figure 6. A causal model for the monitoring of school mathematics.

for each variable. The purpose of each index would be to capture the variability across classes (or schools) with respect to the variable. For example, to measure content, several indices (such as an index of content relevance, an index of content coverage, and an index of correspondence with assessment of knowledge) need to be developed. Content relevance includes balance between the learning of concepts, skills, and applications; emphasis given to specific topics; adherence to the logic of the discipline; incorporation of research-based knowledge about learning and teaching; and relationship to proposed content changes. Such an index would be an important development, since the current practice of accepting similar text titles as representing exposure to similar material is totally inadequate. Another example could involve using teacher logs to determine which aspects of the available curriculum are actually taught and emphasized. Another index should be created to assess the degree of correspondence between the instructional content of a text and the test items used to measure knowledge. From such indices, an overall content index should be possible.

We see as the major task of the next few years the development and validation of indices for the variables in the model.

Indicators. As described, an index is only a measure of a variable. As a thermometer only gives a measure of body temperature, a comparison of that measure with a standard (e.g., 98.6°F for normal body temperature), a prior measure, or a different measure, is needed to determine health. Indicators are new indices created from a prior index by making such ratios or comparisons.

Implications. This monitoring project is designed to evaluate the progress of school mathematics. This assumes that:

1. The schools of America will respond to the crisis in school mathematics.
2. New courses, new instructional materials (texts, tests, computer software, etc.), new instructional strategies, and new training programs for teachers are being or soon will be developed and implemented.
3. New course requirements; new entrance and exit requirements for special courses, high school graduation, and college; and new requirements for teachers of mathematics are being initiated by local and state agencies throughout the nation.

If these assumptions are correct, this causal model can be used as a basic framework to gather information about national, state, and local efforts to change school mathematics. Furthermore, it is assumed that data either are available or can be collected over time. Having such data would be useful to the Foundation, other federal government agencies, state education agencies, local school districts, and researchers to evaluate efforts to improve practice, to formulate plans, and to identify effective school programs.

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CHAPTER 4

COMMENTS ON A PLAN TO MONITOR SCHOOL MATHEMATICS: REACTIONS TO CHAPTERS 1-3

George M. A. Stanic

Romberg and Smith's idea of developing "a system for monitoring the health and progress of school mathematics in the United States" (p. 3) is excellent. In the first three chapters of this monograph, the authors have presented a plan for the monitoring system that matches the quality of the idea. In a time of simple solutions to complex questions, Romberg, Smith, and Zarinnia confront and deal with the complexity of trying to monitor school mathematics in the United States. This analysis of the chapters is based on the hope that the monitoring system will be fully implemented and will fulfill its great potential.

A Persuasive and Problematic View of the World

The new world view presented by Zarinnia and Romberg in chapter 2 is comprehensive and convincing. The broad conception of the social, economic, and educational contexts for change makes this work stand out from the many recent commentaries on America and its system of education. The previous reports of this decade have certainly included discussions of what the future might hold, but the future they present is in no way as clear as that described by Zarinnia and Romberg. Moreover, the authors of previous reports seem to have based many of their recommendations for school mathematics as much on a romantic view of the past as on a view of the future (Stanic, 1984a). Zarinnia and Romberg express no such romantic view of the past; indeed, they see the need to break away from past ways of looking at the world and school mathematics as the main problem to be overcome. By linking discussions on the Information Age with research on learning and intelligence, the authors make their suggestions for change in school mathematics seem not only reasonable but essential. And by recognizing that computers represent both a solution to and a source of problems, Zarinnia and Romberg highlight the need to clearly understand <u>why</u> we are doing <u>what</u> we are doing during the change process. They provide us with a helpful way to think about necessary change in mathematics programs through the distinction made between knowledge and the record of knowledge, between knowing and knowing about (Romberg, 1983). The purpose of mathematics instruction should be, according to Zarinnia and Romberg, to have children come to know mathematics by doing mathematics, by creating their own mathematical knowledge. Based on this view of the learner as the central actor in the processes of schooling and education, Zarinnia and Romberg reconstruct the roles of student and teacher and point to the causal model described more fully by Romberg in Chapter 3 of the monograph.

Although the work by Zarinnia and Romberg is clearly different in many important ways from other recent reports on American education, it is like these reports in that schooling and education are tied directly to the needs of the economy. At a general level, apart from the difficulty of predicting our economic future, this focus on the economy is a concern because we may not want to limit our justification for what is done in schools to the needs of the economy. Fortunately, Zarinnia and Romberg do not entirely limit their argument in this way, although our economic future is a significant aspect of their new world view. At a more specific level, there are elements of their analysis that should be looked at more closely. For example, referring to Naisbitt (1982), they claim that "the United States and other advanced societies of the West are losing their industrial supremacy. Mass production is more cheaply accomplished in the less developed parts of the world" (Zarinnia & Romberg, 1986, p. 22). Because industrialists in so-called advanced societies are taking advantage of cheap labor in less-developed parts of the world, we need to clarify what this shift means in terms of industrial supremacy.

The language of supremacy used to describe the economic future is itself a bit troubling. Consider the following passage from Zarinnia and Romberg:

> The future of the advanced industrial countries, in the lead, is to serve as financial headquarters and research and development centers for a global economy. Such international specialization requires sophisticated communications and the capability of maintaining peace, whether for the free flow of goods through the world's waterways or for the safe conduct of business. It is a potentially precarious position since it depends on scientific, technological, and fiscal supremacy. (pp. 23-24)

As a result, say Zarinnia and Romberg, "both the pursuit of knowledge <u>and</u> innovation . . . are crucial to leadership of the world economy" (p. 24). Although the importance of the pursuit of knowledge and innovation cannot be disputed, if we really are becoming a global community, with a global economy, we need to ask whether the concept of community is compatible with the concept of scientific, technological, and fiscal supremacy for the United States. To the extent that Zarinnia and Romberg advocate this view of the United States as supreme, they would appear to fit into the mainstream view in our country. And to the extent that they base their recommendations for schools on the need for America to maintain supremacy, they clearly fit into the mainstream of reports on education in this decade.

In part because chapter 2 is so persuasive, it is also problematic. It makes a future that human beings must struggle for and create sound like an inevitable future which must be accepted and to which we must respond. The value positions embedded in Zarinnia and Romberg's new world view are almost masked by the overpowering argument that the Information Age is here and that future trends dictate the necessary directions for school mathematics. The authors do reveal their values in the discussion; however, because the changes in our society and the concomitant changes in school mathematics are presented as though there were no reasonable alternatives, the values which underlie the changes are not really open to question.

In the end, what we are presented with is much more than a monitoring system. Despite the fact that Romberg and Smith claim in chapter 1 that national commissions will describe what it means to have a healthy school mathematics program, the authors of the first three chapters have, in effect, defined what it means to be in good health, and the system they have proposed provides a mechanism to guide the progress of school mathematics rather than simply to monitor such progress.

In chapter 1, Romberg and Smith claim that "we are in an era of radical social and economic change that must be reflected in the programs of our schools" (p. 4). Romberg makes it clear in chapter 3 that he believes present school mathematics programs are not adequately preparing students for the 21st century. Now, there is nothing wrong with trying to determine what the future will look like and suggesting that changes in our society call for changes in its schools. Putting aside for a moment the rich tradition which suggests that we may not want to view schooling and education as preparation for adulthood (see, e.g., Kliebard, 1975), let us assume that it is reasonable to ask schools to prepare people for the future. The problem that remains is that we may not all agree on what the future will look like or, more importantly, what we want the future to look like. Furthermore, it is not clear that a particular future dictates the need for a particular and unique school program.

A Link with Goals of the Past

The basic goals for school mathematics espoused by the authors of the first three chapters are not new. In chapter 2, referring to the work of Shane and Tabler (1981), Zarinnia and Romberg present "immediate necessities for the schools" (p. 25). Among the necessities are that "students must be educated for survival in an atmosphere of change," that "the content and structure of the curriculum should not indoctrinate students with past values and rigidity," and that "most of all, students need a sense of consequence" (p. 25). There is a strong tradition in the history of education which is based on such goals. For example, early in the 20th century, John Dewey, among others, argued articulately for goals much like these. In addition, although we would not want to claim that what Zarinnia and Romberg are suggesting is simply discovery learning revisited--and, indeed, they tell us that "this is not the same as the kind of discovery learning that plans for the acquisition of particular knowledge through discovery rather than exposition" (p. 44)--there is clearly a link between their suggestions for how children should learn mathematics and the discovery-learning tradition.

Zarinnia and Romberg succinctly state the essence of their view when they claim that "society now needs people who are creative and innovative" (p. 36). Although they express a clear view of the future, it is not clear that their fundamental goals for education are new or that society has a greater need for creative and innovative people today than it has had in the past. We have always needed children and adults who can think creatively and solve important problems. The 21st century will present new and difficult problems, but only an ahistorical position could be used to justify the claim that our present and future problems will be so difficult and so unique that there is now and will be a greater need for creative problem solvers.

Many of the reasons given for this increasing need for creative and innovative people focus on the impact of the computer on peoples' lives and jobs. Many claim, say Zarinnia and Romberg, "that everyone either is or soon will be involved with them" (p. 21). They agree that "popular perception of the importance of computers is appropriate, for computers and their applications are inextricably intertwined in the development of the Information Age" (p. 31). Although Zarinnia and Romberg discuss much more about the computer than simply its impact on the economy, the work of Levin and Rumberger (1983) on the implications of high technology should make us think further about this issue, especially as it relates to the need for more creative and innovative people.

Levin and Rumberger (1983) discussed the assumptions that, in the future, opportunities for the unskilled will be reduced or even eliminated and that skill requirements of existing jobs will increase. Based on Department of Labor projections, Levin and Rumberger argued convincingly that "the expansion of the lowest skill jobs in the American economy will vastly outstrip the growth of high technology ones" and that "the proliferation of high technology industries and their products is more likely to reduce the skill requirements for jobs in the U.S. economy than to upgrade them" (p. 2). Unlike Zarinnia and Romberg, whose discussion takes us into the needs of the 21st century, Levin and Rumberger's discussion focuses on the relatively near future in this century. Certainly, we would expect things to change in the 21st century. But at least for the near future, it is not clear that the economy dictates a greater need for creative and innovative people than we have had in the past.

The point is not that the goals espoused in the first three chapters of this monograph are inappropriate. The point is that what the future will be is not a <u>given</u> and that a particular future does not necessarily dictate the need for a particular school program. In a sense, the authors could have argued for their goals as worthwhile in any era and for any future. That is, we may want people to be creative and innovative regardless of their jobs and their possible futures. The content of the mathematics curriculum may be relevant to a particular future, and Romberg argues articulately in chapter 3 for what that content might be. However, the overall goal of developing creative and innovative people is not dependent on a particular view of the future. The question is why, if there has been a persistent tradition based on goals much like those espoused by the authors of these chapters, there has been an equally persistent perception that the goals have not been accomplished.

The Need to Deal with the Lack of Consensus

Zarinnia and Romberg (1986) claim that "for at least the last 25 years . . . , those responsible for mathematical education have attempted to reshape and improve the school mathematics curriculum" (p. 37). In fact, the attempt to reshape and improve the school mathematics curriculum is a continuing process, going at least as far back as the turn of the 20th century, when the work of David Eugene Smith at Teachers College, Columbia University, and Jacob William Albert Young at the University of Chicago helped establish the field of mathematics education as a legitimate professional field of study at colleges and universities (Jones, 1970; Stanic, 1984b, 1986). Mathematics educators have never completely agreed on what the form and content of the mathematics curriculum should look like. Furthermore, the mathematics curriculum has never fully embodied the views of mathematics educators because their views have at times been in conflict with the views of people outside of mathematics education (Stanic, 1984b, 1986). There is no reason to assume that a consensus that we have never had before on goals for mathematics education should appear now.

This claim of lack of consensus does not depend exclusively on the evidence we have received in almost every recent issue of The Mathematics Teacher as John Saxon advertises his textbooks and his feud with the National Council of Teachers of Mathematics. The authors of the recent general reports on the state of education in our country also do not agree with each other or with mathematics educators on what needs to be done (Stanic, 1984a). Just one example of this lack of agreement in the reports can be seen in what the authors of A Nation at Risk (USDE, 1983) and Action for Excellence (ECS, 1983) have to say about minimum competency testing. In A Nation at Risk, we are told that "'minimum competency' examinations (now required in 37 states) fall short of what is needed, as the 'minimum' tends to become the 'maximum,' thus lowering educational standards for all" (USDE, 1983, p. 20). On the other hand, the authors of Action for Excellence tell us that the fact that "thirty-seven states now have some form of competency testing or assessment of student achievement to measure educational effectiveness" (ECS, 1983, p. 46) is a "hopeful sign."

Another example of the fact that even though many people believe there is something wrong with our schools, not everyone agrees about what to do, can be seen in the race for state superintendent of schools in Georgia. One of the candidates is running on the platform of getting "modern math" out of schools and making all children memorize the "multiplication tables." The candidate is not favored to win the election, but he has served as a local school system superintendent in Georgia, has collected campaign contributions, and reflects the sorts of obstacles standing in the way of implementing the goals called for in this monograph. We need to recognize that there is no consensus on goals for schooling in general or for school mathematics in particular.

This lack of consensus is not something we should be dismayed about because, unlike what the future might bring, a lack of consensus about the purposes of schooling and the form and content of the school curriculum _is_ inevitable. Even though it would be hard to find someone who does not believe that society needs people who are creative and innovative, there is not agreement on how that view should be translated into a school program. Most mathematics educators, including teachers, would probably agree that we should develop the problem solving ability of students, but not all would agree about the extent to which "technology has freed us from the cumbersome calculation routines of arithmetic, algebra, statistics, and calculus" (p. 64) or about what _problem solving_ means. For some, whatever problem solving is, it should be taught after children learn how to add, subtract, multiply, and divide whole numbers, fractions, and decimals. For others, including Romberg, Smith, and Zarinnia, finding solutions to nonroutine problems should be a central element in mathematics instruction from the beginning.

At a broader level, this inevitable lack of consensus can be seen in the existence of conflicting curriculum interest groups during the 20th century (Kliebard, 1981, 1986; Stanic, 1984b, 1986). According to Kliebard (1981, 1986), the American school curriculum represents an untidy compromise among competing interest groups with different visions of what knowledge is of most worth and of the purposes of schooling. Zarinnia and Romberg recognize the powerful influence of one of the interest groups in their discussion of the production metaphor. They convincingly argue that this metaphor has had problematic consequences for school mathematics. Although Zarinnia and Romberg present their new world view as a response to this limited way of looking at the world and mathematics education, the metaphor is so powerful that even they do not entirely escape it. Consider the following description they provide of the Information Age:

> Information is the new capital and the new raw material. The ability to communicate is the new means of production; the communications network provides the relations of production. Industrial raw materials are valuable only if they can be combined to form a desirable product; the same is true of information. (p. 22)

Even speaking, as Zarinnia and Romberg do, of what teachers and students do as work (as opposed to, say, play or artistic endeavor) is related to the production metaphor, through which schools are viewed as places where work goes on.

Despite the fact that Zarinnia and Romberg do not entirely escape the production metaphor, they do clearly point us in a different direction when they speak of schools as places where children should create knowledge. Yet even here there is some contradiction when we compare this position with what Romberg and Smith say in chapter 1 about the purpose of schools. According to them,

> schools, as we know them, are social institutions whose primary purpose is to transmit specific knowledge and skills to our young and introduce them to our social system. If the social system is changing, then both the knowledge and skills our children need and the social institutions that deliver that knowledge will have to change. (p. 5)

Apart from pointing to issues such as the extent to which schools do or should mirror society and whether the social system is ever in a period when it is not changing, this passage highlights the contradiction between viewing schools as places where knowledge is transmitted and delivered and viewing them as places where knowledge is created. In the end, however, even though there are contradictions and inconsistencies, the main message across all three chapters is that the tradition based on the production metaphor must be challenged.

As the earlier discussion on goals from the past indicates, Zarinnia and Romberg's challenge of the production metaphor does not represent a break with previous tradition as much as it represents the advocacy of a different, competing tradition which has not had as much influence on school mathematics as has the tradition based on the production metaphor. The lack of consensus about the goals for school mathematics is embodied in the competing metaphors, or competing interest groups, that exist. Zarinnia and Romberg, relying on the work of Romberg and Price (1983), recognize the difficulty of changing the views people have come to accept:

> Even when there is intent to change, if mental models remain the same, real change may not be effected, despite the illusion of change created by the trappings. Old beliefs and habits will persist and nominal, rather than real, change in the curriculum probably will continue. (p. 38)

Their answer to the problem is the "recognition and removal of structural and intellectual impediments [to change] through careful consideration of the possible, probable, and desirable attributes of the new age, the self-conscious formulation of new models through abstraction rather than experience" (p. 39). In effect, the answer lies, for Zarinnia and Romberg, in the careful presentation and application of their new world view.

Beyond the New World View

Zarinnia and Romberg have approached the problem they identify in a reasonable and well reasoned way. Their basic assumption is that the world is changing, and our schools are not adequately reflecting the change. The problem they identify is a tradition which looks at the world and schools through the eyes of the production metaphor. Their resolution of the problem is to present a new world view so compelling that people will reject the production metaphor, accept the new world view, and implement a new and very different school mathematics program.

Zarinnia and Romberg's efforts must be judged as successful in many respects. Although it is problematic to do so, they had no choice but to portray their new world view as though it represented an inevitable future. In effect, it was their purpose to present the future in this way. They reasoned that, if people are going to give up and change their beliefs, they need such a powerful new view. Yet it is exactly for this reason that problems remain. In short, a new world view is not enough. The teacher who still thinks long division with three-digit divisors is important for children to perform needs more than a description of what the 21st century might be like to change this view of mathematics education. The problem is that we still do not know enough about what the alternative program should look like. "Freedom from cumbersome calculations" is a slogan in danger of becoming a cliche.

To their credit, Zarinnia and Romberg focus half of chapter 2 on the impact of their new world view on school mathematics. To say that the new world view is not enough is not to claim that Zarinnia and Romberg should have given us more in chapter 2. It is, in effect, the ongoing task of all mathematics educators who believe in the goals for school mathematics argued for in this monograph to design school programs that reflect these goals. Furthermore, it is our crucial responsibility to involve classroom teachers in this process. Even though Zarinnia and Romberg have outlined new roles for teachers and students, classroom teachers rightfully ask for more explanation of what the goal that students be creative and innovative people says about day-to-day life in classrooms. The first three chapters of this monograph may have defined what it means for school mathematics to be in good health, but we do not yet know enough about how to get healthy.

At least part of our problem in the field of mathematics education is not thinking enough about why we should teach mathematics to anyone. The assumption in this monograph (and the assumption held by most people) is that we teach it because it is useful. People look to schools to prepare their children for the "real world," for the future, for satisfying careers. Mathematics obviously is useful for these purposes, but we need to clarify what is meant by useful. Do we mean that the <u>content</u> itself is useful? Do we mean that the creativity and problem solving abilities we hope to develop are useful apart from the content? Is mathematics useful regardless of what a person's job might be in the 21st century? How are we to decide who should learn how much mathematics?

We also need to consider the extent to which the justification for teaching mathematics is related to the reasons for including other areas in the school curriculum. What, for instance, does the new world view say about the teaching of literature, art, or history? Do the reasons for including mathematics in the school curriculum relate at all to the reasons for including these other areas? It is only as we carefully consider our justification for teaching mathematics to students that we can begin to construct an alternative program that is rich enough to lead people, especially teachers, to give up the tradition they have come to accept.

Conclusion

The first three chapters of this monograph deserve more than the limited response presented here. In particular, the causal model, which is described by Romberg in chapter 3, calls for continuing analysis. The model is excellent in terms of both what it represents and what it suggests. Although no model can capture the full complexity of the world (Lave & March, 1975), through the variables and relationships identified, Romberg has captured a great deal of the complexity that characterizes school mathematics. Not only does the model fairly represent the various factors involved in school mathematics; it extends our understanding by suggesting areas for further analysis and research. Because we can make interesting predictions based on the model, Lave and March (1975) would refer to it as being fertile. From Romberg's description of his model, we learn not only about school mathematics; we also learn about the purposes of causal modeling itself, as he describes the prior, independent, intervening, dependent, and consequent variables in his model and the relationships among them.

One example of the need for continuing analysis comes from Romberg's discussion of intervening variables in the model. He includes, as intervening variables, planning, classroom events, and pupil pursuits. Classroom events encompass "teacher behaviors during instruction and other aspects of the classroom environment [which] undoubtedly influence student achievement" (p. 74). As an example of what is being referred to here, Romberg mentions the work of Good, Grouws, and Ebmeier (1983) on active mathematics teaching and claims that this work shows how lessons may be structured to substantially improve achievement. Although it is clear that the excellent research done by Good, Grouws, and Ebmeier provides us with a way to think about structuring mathematics lessons, we need to ask whether this manner of structuring lessons will develop the creative problem solvers Romberg wants mathematics students to become. That is, the achievement that was substantially improved during the experiments described by Good, Grouws, and Ebmeier did not reflect the kinds of new outcomes Romberg seems to be calling for; instead, their research focused only on outcomes of mathematics programs that are measured by standardized achievement tests. Grouws and Good are now in the process of trying to identify teaching behavior that enhances the ability of secondary students to solve problems (NCTM, 1986) It may be that the results of this work will be more helpful in achieving the goals described in this monograph than the results of the earlier research. It may also be true that the results of the earlier research will be quite compatible with what Grouws and Good find in their research on problem solving.

Although it challenges our common sense to suggest that the way we learn certain skills may be incompatible with the goal of helping students become creative problem solvers, there is some reason for concern. Based on his evaluations of the Follow Through program, Richard Snow (1984) concluded that a limited focus on skills through direct instruction "may improve skills in reading and math at the expense of other skill developments that may be crucial for later reasoning and problem-solving ability" (p. 13). The powerful active mathematics teaching model developed by Good, Grouws, and Ebmeier is not a simple example of the direct instruction described by Snow. However, Snow's conclusion should make us ask how the means and ends of mathematics instruction are linked with each other.

The purpose of bringing up this example is twofold. First, it points to the need to consider whether the measures of health and progress included in the monitoring system are compatible with the new goals for school mathematics described in this monograph. Second, by describing how his goals for school mathematics fit with the sorts of activities that took place in the classrooms studied by Good, Grouws, and Ebmeier, Romberg has an opportunity to further clarify his goals, how we are to achieve them, and the extent to which they are compatible with the present goals of mathematics instruction.

This sort of ongoing analysis of individual variables could also be extended to the relationships posited in the causal model. Although the figure summarizing the entire model includes only one-way relationships, there are at least some points where the relationship between variables, in addition to knowledge and attitudes, would appear to be reciprocal. A reciprocal relationship makes sense especially in the connection between knowledge and attitudes (Reyes & Stanic, in press); for similar reasons, one would expect a reciprocal relationship between application and attitudes, which is posited as being one-way from attitudes to application in the model. The relationship between pupil pursuits and attitudes also would appear to be a reciprocal one. According to the model, pupil pursuits have a direct effect on student attitudes, but it is hard to imagine that the attitudes of students do not also have a direct effect on their pursuits in the classroom. Another reciprocal relationship would seem to exist between knowledge and application. In the model, the arrow goes from knowledge to application; yet in the process of using or applying the knowledge they create, students could and would create still more knowledge, implying a reciprocal relationship. The point is that, since causal modeling can take into account reciprocal relationships (Duncan, 1975), it would be beneficial to give further consideration to the relationships posited in this model. It is a clear strength of the model that it makes us think about relationships between and among important variables in school mathematics.

Beyond the need for the ongoing analysis of the causal model, it will be important, as the monitoring system is implemented, to look for unintended consequences of implementing such a system. For example, although no mention is made of the possibility in these three chapters, such a powerful system for monitoring and guiding school mathematics across the United States might have the unintended consequence of standardizing the American mathematics curriculum. Of course, for some people, this consequence may not be a concern, especially if the curriculum really can be standardized in the directions suggested by Romberg, Smith, and Zarinnia. If, however, we view the development of school mathematics as an ongoing process, if we view innovation and experimentation as good things, if we want future teachers to have more responsibility over fundamental aspects of their work such as making reasoned decisions about what and how to teach, then standardization will be a problematic consequence of implementing the monitoring system. Regardless of how one feels about the possibility of standardization, looking for unintended consequences of implementing the monitoring system is a crucial task.

It is a credit to Zarinnia and Romberg that the evidence presented in chapter 2 is so convincing that it makes the possible seem inevitable. They do not hide their value positions as much as they overwhelm us with the idea that we have little choice about the directions in which school mathematics must move if we are to survive and thrive in the future. If we agree that school mathematics programs should move in the directions so clearly outlined by the authors of the first three chapters, it is easy to be swept away by the vision for the future presented. But if, especially in terms of school mathematics, we want the possible future to become the actual future, we need to recognize that the active creation, construction, and reconstruction of knowledge we expect of our students must be matched by our own active creation, construction, and reconstruction of school mathematics programs.

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CHAPTER 5

A CONCEPTUAL INDICATOR MODEL OF CHANGES IN SCHOOL MATHEMATICS

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In chapter 3, Romberg argues that the development of a valid indicator system for monitoring changes in mathematics education rests on some notion of what the important components of schooling are and their interrelations. The paper goes on to assert that such a model should be a causal model, one that demonstrates "a causal order that relates the variables in the model."

We agree that an indicator system for monitoring mathematics education should be firmly grounded in a model of the education system (Hall, Jaeger, Kearney, & Wiley, 1985; Raizen & Jones, 1985; Shavelson, Oakes, & Carey, 1986; Shavelson, forthcoming). The major contributions of the Romberg paper lie in its enumeration of the directions for change to be monitored in mathematics education and in its definition of some of the components that should be included in a monitoring system. The requirement that the model specify causal relations, however, is misleading on methodological, historical, and policy grounds.

The purposes of this paper are first, to show why the notion of a causal indicator model of change in mathematics education is misleading and, second, to evaluate the specification of model components against alternative specifications for national indicator systems for monitoring the "health of education." We conclude that (a) if the notion of a causal model is replaced by the notion of a conceptual or logical model of change, we are no longer in disagreement; (b) an important criterion for including components has been omitted in Romberg's analysis that, if included, would lead to a more complete specification of his model; and (c) by comparing Romberg's proposed indicator model with others, a more complete specification of the mathematics indicator model can be attained.

Causal Models and Education Systems

To infer causality from nonexperimental data requires a very strong theory of the causal relations among components of the nation's education system, a theory we do not now possess and one we are unlikely to achieve because of the heterogeneous nature of American education that results from our philosophy of local control of schools. Indeed, one of the most important lessons learned from past attempts to develop social indicator systems underscores the difficulty in arriving at causal inferences from an indicator system of the breadth of coverage required to monitor the "health of society" (Shavelson, forthcoming). Consequently, causal claims for a national indicator system grossly mislead the policy community about what it is possible to achieve with a monitoring system and about the strength of the information on which their policy deliberations will be based.

Methodological Considerations

Causal claims from nonexperimental research must, of necessity, rest on strong theoretical grounds to rule out plausible counterinterpretations to the proposed causal interpretation. A strong theory, one that is logically consistent and empirically justifiable, specifies the components of a causal system and their causal ordering, as Romberg points out. In the absence of strong theory, we run the risk of inaccurately specifying the causal model by omitting components that are required to rule out counterinterpretations or by incorrectly specifying the existence and/or direction of causality. The consequence of weak theory is that we may erroneously infer causal relations where they do not exist or where the causal flow is in the opposite direction.

To be sure, by the careful design of correlational studies, especially by including longitudinal data and admitting to reciprocal causal relations, some of the plausible counterinterpretations can be ruled out (but see Ellett & Ericsson, 1986, on backward causality). Romberg stated that "one can assert that 'X is a possible cause of Y' but not that 'X is <u>the</u> cause of Y.' A suggested relationship alone does not prove causation [sic.; causality cannot be proved inferentially]. Causal claims must rest on other, persuasive evidence about <u>X</u> and <u>Y</u> or on appropriately controlled experiments. However, when experiments are impossible, such as in a study of schools' response to pressures for change, causal modeling allows researchers to develop causal propositions supported by data and logic" (p. 66).

However, we do not have an adequate theory of the nation's education system on which to base causal interpretations. Moreover, a national monitoring system, of necessity, must cast such a wide net to reflect the "health" of mathematics education that it cannot possibly include in its specification the level of detail that would permit causal inferences (Hauser, 1975; Sheldon & Freeman, 1970; Sheldon & Parke, 1975).

The inadequacy of our knowledge about causal relations in an education system is reflected in Romberg's discussion of the relations among components of a mathematics education monitoring system. For example, are we to believe that curricular guidelines cause changes in course requirements, tests, and textbooks as indicated by Romberg's figure 3? On what grounds is this causal assertion made? The literature on implementation (e.g., Berman & McLaughlin, 1975-1979; Crandall and others, 1982-1983; Fullan, 1982, Goodlad, 1975; Sarason, 1982) certainly leads us to question this causal ordering; it is simply too linear to account for what happens in schools and classrooms. Or, are we to believe that by increasing teachers' subject-matter or pedagogical knowledge, we will cause a change in teachers' professional responsibilities, Romberg's figure 4? Or are we to believe that changes in attitudes toward mathematics will cause changes in students' ability to apply their mathematical knowledge to problems? Might not an equally strong argument be made that by improving students' mathematical problem solving ability, we may change their attitudes toward mathematics? On what theoretical and empirical grounds are we to rule out the latter interpretation in favor of the former (cf. Sirotnik & Oakes, 1986)?

The problem of specifying causal relationships in a model of the nation's education system is complicated by the fact that states, local education agencies, schools, and teachers simultaneously--but not in consert--introduce changes into education, any one or any combination of which might "cause" changes in outcomes. To collect sufficient information to rule out counterinterpretations due to multiple possible causes would be prohibitively expensive, in both dollar and respondent burden terms, if it were not virtually impossible to do so.

We conclude that an adequate specification of the nation's education system is beyond our means for both conceptual and cost reasons. Indeed, to portray a national indicator system for mathematics education as a causal model oversimplifies the complexity of arriving at causal interpretations from such a system and misleads the policy community.

Lessons from History

Over the past 100 years, educational and social indicators have, repeatedly, been heralded as instruments of reform. As the first three chapters show, Romberg shares this optimism with his predecessors. But, the excitement and promise quickly gave way to realism (Shavelson, forthcoming). Promises of policy applications were overly optimistic. Indicator systems were, for example, unable to provide sufficiently detailed and accurate information for evaluating government programs. Moreover, indicator data bases, often lacking essential theoretical prerequisites, fell short of expectation for research applications (Sheldon & Parke, 1975; Warren, 1974). These events gave rise to realistic assessments of what indicators can and cannot do.

The literature on social indicators appears to have reached consensus on what indicators cannot do (e.g., Hauser, 1975; Sheldon & Freeman, 1970; Sheldon & Parke, 1975) and provides a reality test for what we can expect to do with education indicators:

1. Set goals and priorities. The very process of developing social indicators is value laden. Those indicators that show startling changes if lodged in one system of

measurement might be regarded as of modest interest if placed in a different system. Decisions about priorities are based on more than just data. Indicators are inputs to the policymaking mosaic.

2. Evaluate programs. Social indicators cannot substitute for carefully crafted evaluation of social programs. They do not permit the necessary level of control or detail.

3. Develop a balance sheet. Social indicators cannot match economic indicators. Evoking an economic analogy and proposing a parallel development of social indicators is misleading, because education cannot put each of its constructs on a common dollar metric as can be done to obtain GNP.

The expectations for social indicators have become more modest: to describe, state problems more clearly, identify new problems more quickly, obtain clues about promising educational programs, and the like. In the end, we should not expect causal ascriptions to guide educators and policymakers in mathematics education reform. Rather, realistically, history suggests that the fundamental role of a mathematics monitoring system may be to describe the conditions of mathematics education, ask better questions, find clues to success, and contribute to and perhaps change policymakers' "cognition," their ways of thinking about education reform (cf. Sheldon & Parke, 1975; Kaagan & Smith, 1985; James & Tyack, 1983).

We conclude that to aspire to a national indicator system that provides causal information about the relationships among the components of the nation's education system flies in the face of past experience with social and educational indicators. The educational system is simply too complex and the goals of monitoring too broad to realize causal interpretations.

Policy Considerations

One of the major reasons that the mammoth social indicator movement of the late 1960s slowly ground to a halt was that the needs of the policy community, the community most needed for financial support, were not adequately dealt with. Rather, some policymakers perceived the social indicator movement as a movement of, by, and for social science research.

The causal model of mathematics education may be open to similar criticism. The monitoring system has been grounded on an analysis of expected changes in the goals of school mathematics, primarily from the mathematics-education community's perspective with the consequence that: "For monitoring purposes, three questions must be raised: Have these anticipated changes actually occurred, and, if so, to what degree? What is the effect of these changes on students? Do these changes improve the health of school mathematics?" (p. 64). What is monitored, then, may not jibe with what federal, state, and local policymakers are most interested in during the current period of reform.

Moreover, the discussion of the components of the monitoring system and their "causal" relations suggests that, even with its relatively narrow focus on particular changes in mathematics education, the level of detail required by the monitoring system comes with a very sizeable budget.

We conclude that, by focusing on a causal model of mathematics education, the monitoring system might better serve the needs of the mathematics education and education policy research communities than that of policymakers. We are not convinced by the unsubstantiated assertion that "having such data would be useful to the [National Science] Foundation, other federal government agencies, state education agencies, local school districts . . . to evaluate efforts to improve practice, to formulate plans, and to identify effective school programs" (p. 77).

Conclusions

Many of the issues raised here would vanish if the model of mathematics education were conceived as conceptual, or logical, or functional in the sense of indexing relations (associations) among its components. Causal claims would be avoided, and a more realistic tone set for what is reasonable to expect from a national indicator system for mathematics education.

The analysis of change in society and mathematics and the implications drawn for mathematics education are important contributions of a strategic nature. Such a model would be likely to push mathematics education in the direction of much needed reform. However, we believe that greater weight should be given to ascertaining the tactical needs of the policymaking community in the identification of central components of the educational system, indicators, and their relations. This modest shift in emphasis might help insure that the information provided by the monitoring system would reflect both anticipated and unanticipated changes in mathematics education and provide data about concomitant changes in other components. A model so grounded would meet policymakers' more general need for information about changes of all types, while still keeping the long-term reform agenda in front of them.

Specification of the Components of a National Indicator System for Mathematics Education

Of paramount concern in designing a monitoring system is that the conceptual model on which it is grounded (a) does not omit important components, and (b) contains the best (reliable, valid) indicators of its components. Romberg is clearly aware of this and addresses these two concerns carefully. But there remains the possibility that something has been omitted from Romberg's model, or something has been included that could be improved. Because of the particular focus of the causal model, it is useful to bring the perspectives of other indicator models to bear in the attempt to uncover possible omissions and detect possible "commissions." In that spirit, we, first, examine his criteria for including components and indicators into the mathematics education model and, second, compare his model with those of the National Academy of Science, the Rand indicator project, and one proposed for the Office of Educational Research and Improvement, U.S. Department of Education. A detailed analysis is beyond the scope of this paper (and probably the reader's interest and patience). Rather, our goal is to point up potential shortcomings to be considered in possible modifications of the current model.

Criteria for Selecting Indicators

The major criterion used by Romberg for selecting indicators is that they reflect a particular set of goals for mathematics education and the changes expected in the reform in mathematics education toward those goals: (1) content and structure of courses; (2) course requirements; (3) sequencing and segmenting of mathematical topics; (4) use of technology; (5) methods of assessment; (6) knowledge and professional responsibility of teachers; (7) the way mathematics is taught; and (8) the policy environment within communities (p. 64). A second criterion is the inclusion of variables that intervene between policy and outcomes. "They are not directly manipulable by policymakers, but they significantly affect outcomes" (p. 73). And a third criterion is the inclusion of "prior variables" because "the teaching and learning of mathematics does not occur in a vacuum" (p. 74).

As a heuristic for testing the adequacy of these criteria, we enumerate without discussion the criteria used by Rand in developing alternative versions of a national indicator system for monitoring mathematics and science education (Shavelson, forthcoming; Shavelson, Oakes, & Carey, 1986):

1. Predicts important outcomes such as student achievement, participation in mathematics and science courses, or dropouts, or is itself an important outcome.

2. Mediates the relation between an input and/or process indicator, and an outcome indicator.

3. Reflects important policies or policy changes in education.

4. Might reflect potential problems or point toward possible actions to solve them.

5. Can be readily interpreted by policymakers.

6. Includes information to describe central features of the system that are essential to understanding how the system works.

There is considerable overlap in the two lists, both in content and philosophy. The major difference is that Romberg's criteria are motivated strongly by his notion of the direction of change mathematics education should take. This is a strategic position; it admits to a reformist perspective. The Rand criteria are broader and tactical in nature, admitting to a more pragmatic policy perspective.

Both types of criteria are needed in developing a monitoring system. Strategic criteria are needed to provide indicators that will reflect anticipated changes in policies and their effects. One possible problem with this approach, however, is that, if change is not in the direction anticipated by the model, the monitoring system may be unable to adapt quickly enough to reflect that change.

Tactical criteria are likely to create a monitoring system that is responsive to policy needs, since it tracks central features of schooling that are likely to be sensitive to policy changes, whatever their type or direction. One possible benefit of this approach is that the monitoring system keeps tabs on changes in federal, state, and local policy arenas and adapts to reflect those changes. The limitation of the tactical approach is the strength of the strategic approach; it fails to anticipate or foster long-term change.

The Wisconsin model of mathematics education may benefit from an application of the Rand criteria. This would serve to add an immediate policy relevance to the model, thereby enabling it to serve the needs of and not just reflect changes in the various policy communities it serves.

Comparisons of Monitoring System Models

By comparing alternative monitoring systems with the Wisconsin model, we attempt to expose weaknesses in the latter.[1] Specifically, the Wisconsin model is compared with the National Academy of Sciences' preliminary model (Raizen & Jones, 1985), Rand's model, and a model proposed by Hall et al. (1985) for the U.S. Department of Education. The last three, compared to the Wisconsin model, include more curricular areas and are vague about

1. This is not to say that the other projects do not share these particular blindspots, or do not have their own difficulties. In fact, our approach emphasizes the somewhat unique strengths of each, without suggesting that the Wisconsin model is the only one that might consider including new components.

the specific purposes they intend to serve. That all four models might have been developed at cross purposes need not concern us here, because we can use them to learn what might be added to or omitted from the Wisconsin model with consequent improvement in the monitoring of mathematics education in the U.S. We first compare the Wisconsin model with the NAS model because, historically, it preceded the others and because it is the least complex. We then present the comparison with the Rand model, followed by a comparison with the Hall et al. model.

The NAS Preliminary Model

The National Academy of Sciences' preliminary model (Raizen & Jones, 1985; Figure 1) focused on mathematics and science

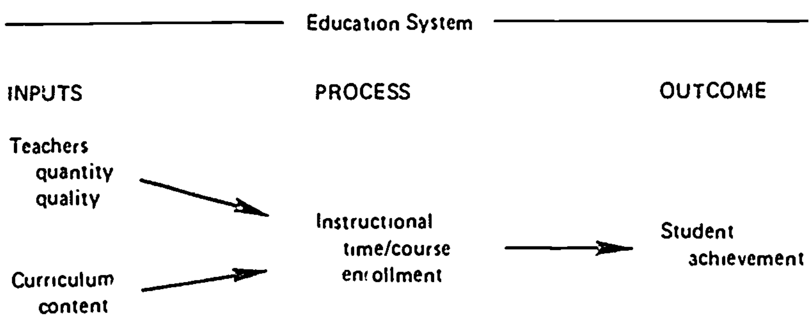

Fig. 1 -- Areas of science and mathematics education to be monitored.

education. The model divides the educational system into bare-bones sets of schooling inputs, processes, and outcomes and includes only four components: teachers, curriculum content, instructional time/course enrollment, and student achievement.[2]

2. The NAS committee responsible for developing this system understood that a more elaborate model was required for developing indicators when it stated, "Even at their best, these indicators are not sufficient to provide an adequate portrayal of the state of science and mathematics education in the nation's schools. There is a need to search for more imaginative and less conventional indicators to guide educational policy, including new indicators that have the potential to take account of likely changes in the function and structure of education" (Raizen & Jones, 1985, pp. 11-12). Thus, the NAS is continuing to develop indicators, and their newer, unreleased models might be considerably more complex than the simple one first proposed.

The NAS model pinpoints one component that is de-emphasized in the Wisconsin model: instructional time. Although instructional time could be considered part of the Wisconsin model's "course requirements," the Wisconsin model apparently downplays instructional time because overemphasizing this component could result in a conception of the teacher's role as purely managerial or procedural. We agree that the teacher's role is much more than managerial, but we still consider the omission of instructional time as a shortcoming of the Wisconsin model. [Ed. note: It is considered under pedagogical decisions variable.] The impressive evidence that instructional time is a major factor in student learning, especially in elementary grades (e.g., Berliner, 1979; Brophy & Good, 1986), suggests that this factor should be considered in any full picture of the education system. Furthermore, it is likely to take even more instructional time to be able to teach in the professional, thoughtful manner the Wisconsin model sees as desirable. Ignoring the issues of whether teachers have adequate instructional time to teach in this manner, or whether they use time wisely, would be to overlook an important feature of the education system. Lastly, instructional time must be considered as a model component because it is both manipulable by policymakers and capable of being squandered or enhanced at the classroom level.

The Rand Model

The Rand model, like the NAS model, is simpler than the Wisconsin model in the division of the educational system into only three domains: inputs, processes, and outcomes (Figure 2). Unlike the Wisconsin (or NAS) model, the Rand model places these domains within the policy context that interacts with and influences each of them.

The Rand model includes two components that were omitted from the Wisconsin model. Specifically, it (1) considers the school as a component and (2) depicts the policy context as a multiple-level, pervasive factor underlying the entire education system.

The Wisconsin model's neglect of the school as a component of mathematics education constitutes a serious omission. School-level decisions transform state and district resources and policies into specific programs for children. They set the conditions under which mathematics teaching and learning occur. School decisions can influence the access students have to higher-level mathematics courses, and the school "culture" can influence the degree to which students are pressed to take challenging courses. Variables at the school level such as course offerings, curriculum differentiation practices, the extent to which school resources are directed toward mathematics, and staff and student attitudes toward mathematics achievement should be explicitly considered in the Wisconsin model. Although each of these variables is somewhat influenced by external (district, state) policy, analyzing them at the school level can provide information about how principals' school policies and

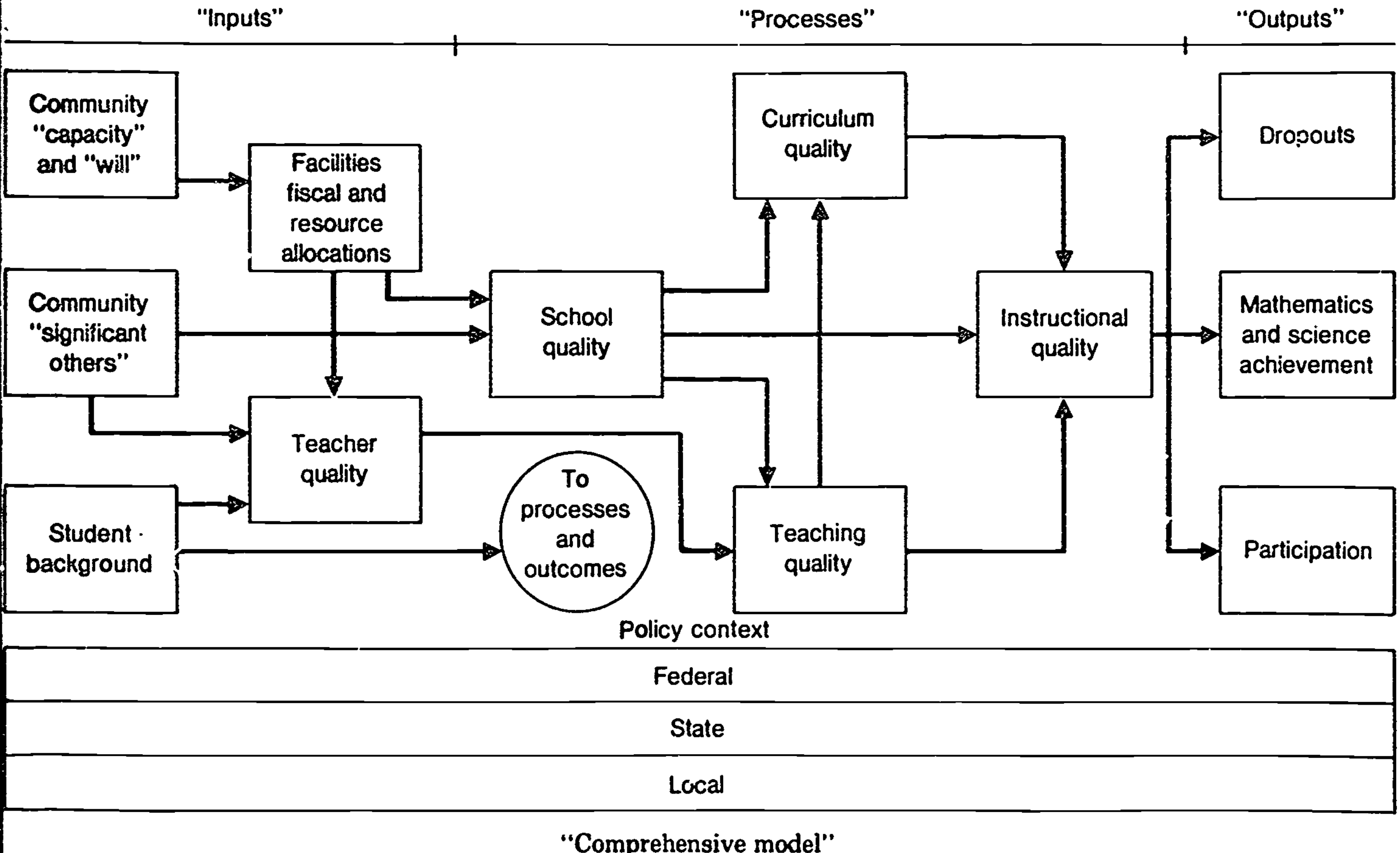

Fig. 2 -- The RAND Model

teachers' individual initiatives may mediate their effects and make a significant difference in educational quality. Including school-level data may also permit analyses of whether principals and teachers perceive the results of policy initiatives to be what policymakers had intended.[3]

Comparison with the Rand model shows that the Wisconsin model's depiction of policy environment is too simple to fully illustrate the variety of ways that policy influences the educational system. The Wisconsin model emphasizes state regulations, but alternative policy levers such as provision of resources, offers of incentives, and use of assessment systems should also be included. A further complication ignored in the model is that policymaking occurs at multiple levels--the policy environment includes mutually interacting federal, state, and local levels. Each level tends to influence somewhat different aspects of schooling. The Wisconsin model's omission of these intricacies of the policymaking environment makes causal modeling seem more appropriate than it actually is, by ignoring the difficulty of disentangling the many, sometimes conflicting, pressures caused by the initiatives of different parts of government.

The Hall, Jaeger, Kearney, and Wiley Model

The Hall, Jaeger, Kearney, and Wiley (1985) model (Figure 3) was developed to help guide the U.S. Department of Education's Center for Statistics' assessments of alternatives for a national data system on all of elementary and secondary education, so their model is not specific to mathematics and science. Similar to the NAS and Rand models, the Hall et al. model uses a simpler, tripartite division among "background," "schooling," and "outcomes." Despite its simplicity, their model illustrates three aspects that might be usefully incorporated into the Wisconsin model: (1) a fuller depiction of the multiple, complex relationships among model components; (2) specific attention to students' educative difficulties; and (3) inclusion of the influence of the environment, society, and culture on goals and resources of schools.

In contrast to Wisconsin's one-way causal flow, one major contribution of the Hall et al. model is to graphically illustrate interactive, mutual influences among educational components. For example, the background variables "educative difficulties" are shown to influence and be influenced by the environment; resources

3. By including "intervening" variables, which are not directly manipulable by policymakers but which significantly affect outcomes, the Wisconsin model acknowledges that individual initiatives can be important. However, we are arguing that the model as it stands (emphasizing planning, classroom events, and pupil pursuits) neglects many school level variables that should also be considered.

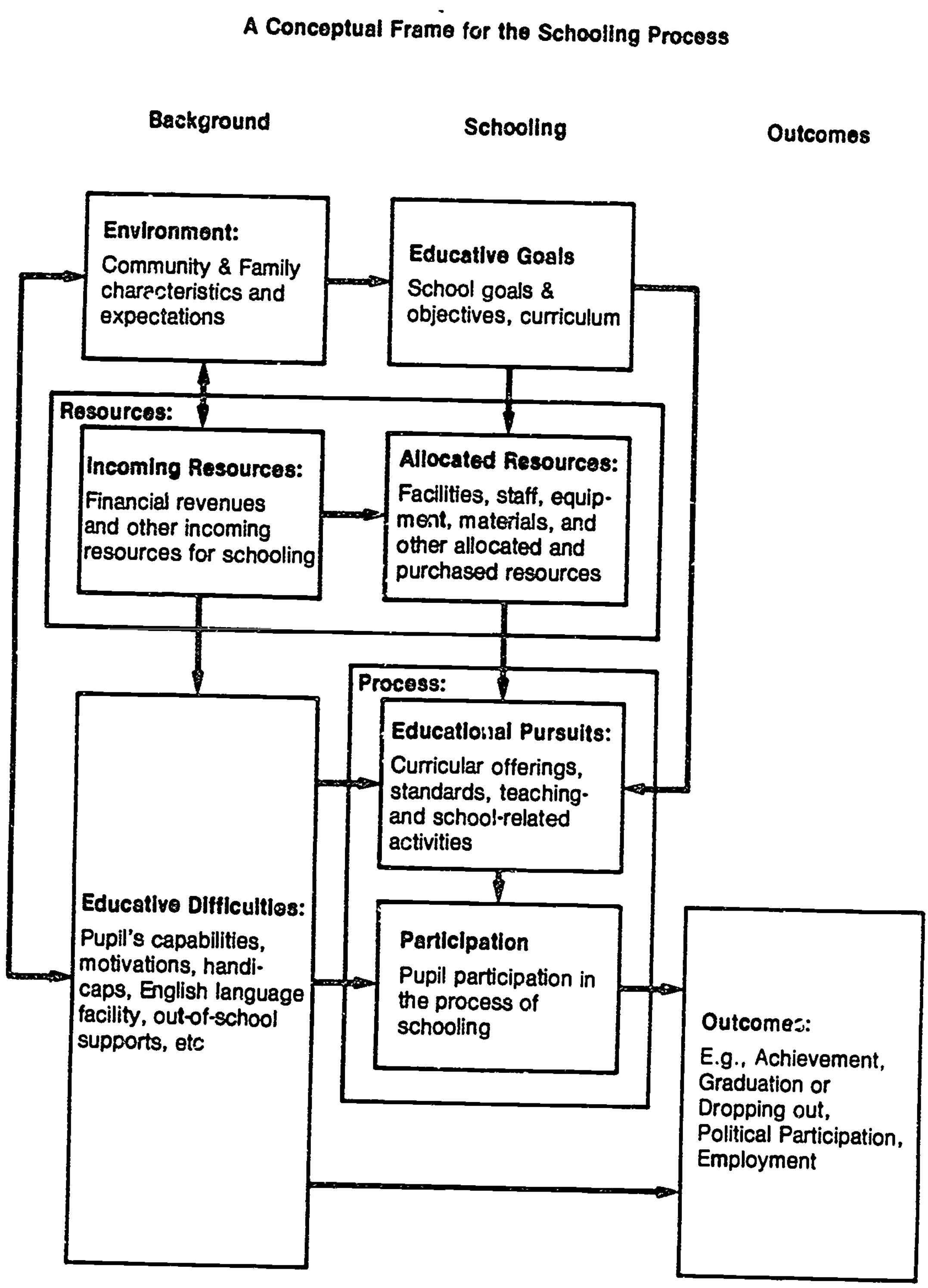

Fig. 3—The Hall, Jaeger, Kearney, and Wiley Model

and environment are also shown as mutually interactive. This more complex scheme captures more of what occurs in schools (e.g., pupil difficulties do influence parent expectations which in turn influence pupil motivation) and presents a more complete picture of environmental influences (on pupils, and resources) than any of the other models considered in this paper.

A second contribution of the Hall et al. model is to suggest that educative difficulties of students should be specifically considered. The Wisconsin model presently makes no provision for the fact some students and schools could face quite different types of obstacles in meeting educational goals and successfully implementing policy initiatives, many of which have been geared toward raising requirements for the mainstream. Given the evidence that students' sometimes idiosyncratic misunderstandings of mathematics pose serious challenges to teachers (e.g., Erlwanger, 1975), these difficulties might be more fully explored in the Wisconsin model.

Lastly, the Hall et al. model sketches the environment's effect on both the goals and resources of the school. By doing this, Hall et al.'s model suggests an important implicit, but not fully realized, possibility for the Wisconsin model: Consideration of whether improvements in student outcomes are due to policy initiatives at all, or whether they are the result of cultural and societal influences. For example, higher mathematics achievement <u>could</u> be the result of changes in family structure (Zajonc, 1986), student perceptions of the job market, or parental expectations, rather than changes in course requirements and content.

Strengths of the Wisconsin Model

Despite the weaknesses exposed by comparing the Wisconsin model with the others, this comparison also demonstrates several areas where the Wisconsin model has contributed unique insights worthy of consideration by other indicator projects. First, the Wisconsin model's distinction among independent and intervening variables appears useful in underlining the fact that policymakers cannot reasonably expect to manipulate all factors relevant to schooling--classrooms are to some extent "black box's" to which policymakers can merely supply certain inputs, such as quality materials, content requirements, and quality instructors.

A second important contribution of the Wisconsin model is its subject-matter based approach, focusing on an ideal toward which mathematics education should strive, rather than what is presently implemented in classrooms. Systems based on present-day knowledge (such as the comparatively recent appreciation of the role of academic learning time) to the exclusion of alternative visions (such as the need for more teaching of "higher-order" thinking skills) could miss much of what we should monitor.

Third, the Wisconsin system's distinction between dependent and consequent variables is useful in illustrating that outcomes of schooling might be quite different, depending on whether they are measured proximally (at the end of a given school year) or distally (months or even years after the course has ended).

Summary and Recommendations

Despite our recognition of the contributions of the Wisconsin model, we have argued both on methodological and practical grounds that a causal model of education can be misleading. We do not have an adequate theory of the nation's education system on which to base causal interpretations, nor do policy changes occur in the controlled manner necessary to infer cause in the absence of adequate theory. Furthermore, attempts to collect enough information to support causal conclusions could result in such a massive amount of data that policymakers would find it unusable and confusing.

Instead of a causal model, we recommend a conceptual or logical model which indexes relations among components. Our comparisons of the Wisconsin model with the National Academy of Science's preliminary model, Rand's model, and a model proposed by Hall et al. (1985) reveal several components that should be considered for inclusion into the Wisconsin model. Specifically, we recommend that the Wisconsin model should more fully consider the following:

- Instructional time
- The school as a separate level of analysis
- The multiple levels of policymakers influencing educational practice
- The multiple directions of causal, conceptual, or logical relationships among components of the educational system
- Student educative difficulties
- Cultural and environmental influences on schooling

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POSTSCRIPT

It is essential to the development of new ideas that they be put into perspective by constructive contrast; Stanic's commentary and the reactions of Shavelson, Oakes, and Carey did precisely that. Both papers considered implications of a new world view and the possibility of school reform.

Stanic raised pivotal questions about the low-skill future possible in a high tech world and the compatibility of Western scientific, technical, and fiscal supremacy with global community. However, it is precisely because of the likely consequences of a low-skill future of a high tech society that a new world view is essential. The claim was not a high-skill future for all but the need for adaptability and innovation. One approach to innovation and creativity is to continue the elitist, dichotomous, high literacy/low literacy tradition common in Western countries (Resnick & Resnick, 1977). This would relegate the majority of the population to low-skill jobs while a super-educated elite took care of innovation; computers would perform many of the functions now employing middle-level management and professional groups. There are already indications that the middle class is shrinking. However, societies characterized by wealthy educated elites, weak or absent middle classes, and a majority of poor are typically oligarchic, communistic, or theocratic dictatorships. Our choice is a matter of values. Democracy evolved through hundreds of years of striving and is intimately associated with the capitalistic system and the rise of the middle class. To sustain democracy, upward social progress must be possible, and that requires commitment to continuity in the social spectrum supported by high general levels of literacy.

Shavelson, Oakes, and Carey simultaneously characterized the notion of organizing monitoring around a new world view as a strategic contribution yet downplayed it by claiming it exclusive to the mathematics-education community. They argued that, with organization around a specific viewpoint, what is monitored "may not jibe with what federal, state, and local administrators are most interested in during the current period of reform." The truth is that a strong consensus about the direction of change and its implications for education has emerged in most of the advanced industrial nations, including the Soviet Union and Japan. It is best articulated in this country by such calls for change as those of the Carnegie Forum on Education and the report of the Holmes Group.

More serious is the implication that current approaches to data collection will suffice. Our position is that policymakers will not be well served by atheoretical data about features of schooling, which are likely to be insensitive to the effects of the reforms. In fact, this monitoring project is part of a deliberate reform movement. Its purpose is to collect information which will support policymakers' decisions about the reform. This requires a framework for data collection which supports present and future purposes and not those of the past.

Shavelson, Oakes, and Carey also addressed the vexed and crucially important question of causality. They argued that simultaneous, but unconcerted actions introduce changes, any one of which might cause a change in outcomes. Therefore, because the cause cannot be isolated, a conceptual nomenclature is more appropriate. The point is well taken, especially with regard to time. For example, while there is an association between homework and achievement, it is unclear whether homework causes achievement, achievement causes homework, or whether there is no causal relationship but each is "caused" by a combination of socioeconomic and motivational factors.

In essence, the question of causality rests on the sequence of correlation; does a higher achievement pattern result in taking more time on homework, or does spending more time on homework result in higher achievement. If achievement precedes homework, does familial pressure (with socioeconomic undertones) contribute, or does motivation? If homework precedes achievement, perhaps motivation, rather than mandate, causes children to spend more time. On the other hand, irrationality would suggest denial of all notions of causality.

While we know a lot, we simply do not know enough. Nevertheless, an analytic scheme for policymakers

> must . . . strive to bring philosophical clarity and system to subject matter, define criteria for the deployment of basic descriptive terms, and explain what sorts of evidence or argument bear on the justification of relevant claims. In working towards these goals, a scheme in progress will, typically, not only incorporate results as given but will criticize, redefine, and reformulate such results from a systematic point of view. [Furthermore,] an analytical scheme will be judged as of higher merit to the degree it does not merely reflect past investigations but points out new directions for inquiry and suggest new questions for reflection. (Scheffler, 1985, pp. 68-69)

We conclude that a pragmatically causal model will provide the policy community with the best information for making judgements about schooling in general and school mathematics in particular.

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WHAT MATHEMATICS SHOULD BE IN THE SCHOOL CURRICULUM?

Considering possible answers to this question is central to the current reform movement. The argument is that all students should learn "more and somewhat different mathematics" than is in the current curriculum. This claim is based on a variety of concerns, four of which are particularly important.

The principal concern is related to the new world view as described in chapter 2. Today's students will be working in the Information Age of tomorrow. Although the real demands of that era are unknown, it would be safe to guess that mathematics--in particular mathematical modeling, higher order thinking, and creativity--will be highly valued. The other concerns are derivatives from this perspective. The second concern is about how mathematics is perceived. Even young children should not view mathematics as a static collection of concepts and skills to be mastered. A major theme in the following chapters is that students should have a different perception of the discipline. Mathematics is dynamic, growing and changing. In the past it was considered sufficient for students to master a few concepts and become proficient at basic computational skills. That is no longer adequate.

The third reason for changing what is taught is based on the technology of calculators and computers that makes paper and pencil calculation less important. As others have noted, we should not be trying to teach students to compete with the $4.95 calculator. While the calculator and computer have made proficiency in calculation less important, they have made understanding of calculation procedures even more important.

The fourth reason for change in content is the expanded use of mathematics. Mathematics has always been a primary tool in the sciences and engineering, but in the past quarter century its application to economics, the social sciences, medicine, law, and other fields has made it an even more important part of the background and training of all students. One consequence is that we should not refer to mathematics as a single discipline. We should refer to the mathematical sciences, indicating that schooling should include areas such as statistics, computer science, econometrics, and biometrics.

It should be apparent that we need answers to the question, What mathematics should be in the school curriculum? The answers, however, are not obvious. The five chapters that follow reflect the positions now taken by a number of persons and groups.

In chapter 6 the summary statement prepared by Henry Pollak for the Conference Board of the Mathematical Sciences is reprinted.

The view presented is a mathematician's view of both what is now considered fundamental and what is not. In chapter 7 the Brazilian mathematician Ubiratan D'Ambrosio, founder of ethnomathematics, presents the argument for change from a sociological perspective. In chapters 8 and 9 two other recent papers, also written by mathematicians on this topic, are reprinted. Peter Hilton presents a cogent argument about what mathematics is likely to be important in the decades ahead. Stephen Maurer outlines activities at the collegiate level which should affect secondary mathematics. Then in chapter 10 an earlier paper by Thomas Romberg is reprinted. This paper presents a mathematics education perspective about what should be included in the curriculum. This section concludes with an invited commentary from the noted mathematician Herbert Greenberg.

Other answers to the question of what mathematics should be taught in schools will soon be forthcoming from a variety of projects including the 2061 project of American Association for the Advancement of Science, the curriculum goals project of the Council of Chief State School Officers, and the standards project of the National Council of Teachers of Mathematics. In addition, several new curricular materials are being developed such as the Transition Mathematics materials from the University of Chicago School of Mathematics Project, the Quantitative Literacy materials produced by the American Statistical Association, and a variety of curriculum development projects recently funded by the National Science Foundation.

There needs to be continued discussion about school mathematics. Given the dynamic nature of the discipline, curriculum content will never be set for all time. However, it is clear that many aspects of current mathematics programs need to be changed.

CHAPTER 6

THE MATHEMATICAL SCIENCES CURRICULUM K-12: WHAT IS STILL FUNDAMENTAL AND WHAT IS NOT

Report from The Conference Board of the Mathematical Sciences
Prepared by Henry O. Pollak

Executive Summary

Our charge from the NSB Commission was to identify what parts of mathematics must be considered fundamental for education in the primary and secondary schools. We concluded that the widespread availability of calculators and computers and the increasing reliance of our economy on information processing and transfer are significantly changing the ways in which mathematics is used in our society. To meet these changes we must alter the K-12 curriculum by increasing emphases on topics that are fundamental for these new modes of thought.

This report contains our recommendations on needed changes--additions, deletions, and increased or decreased emphases--in the elementary and middle school mathematics curricula and a statement of more general concerns about the secondary school mathematics curriculum.

With regard to elementary and middle school mathematics, in summary, we recommend:

- That calculators and computers be introduced into the mathematics classroom at the earliest grade practicable. Calculators and computers should be used to enhance the understanding of arithmetic and geometry as well as the learning of problem-solving.

- That substantially more emphasis be placed on the development of skills in mental arithmetic, estimation, and approximation and that substantially less be placed on paper and pencil execution of the arithmetic operations.

- That direct experience with the collection and analysis of data be provided for in the curriculum to insure that every student becomes familiar with these important processes.

1. This chapter originally appeared as a report from The Conference Board of the Mathematical Sciences (Washington, DC: National Science Foundation, 1983) and is reprinted with permission.

We urge widespread public discussion of the implications of the changing roles of mathematics in society, support of efforts to develop new materials for students and teachers which reflect these changes, and continued and expanded experimentation within the schools.

With regard to the secondary school curriculum, in summary, we recommend:

- That the traditional component of the secondary school curriculum be streamlined to make room for important new topics. The content, emphases, and approaches of courses in algebra, geometry, precalculus, and trigonometry need to be reexamined in light of new computer technologies.

- That discrete mathematics, statistics and probability, and computer science now be regarded as "fundamental" and that appropriate topics and techniques from these subjects be introduced into the curriculum. Computer programming should be included, at least for college-bound students.

Modern computer technology clearly has vast potential for enriching and enlivening the secondary school curriculum. However, we are not now in a position to make firm recommendations. There is need for research on the effects of incorporating technology into the traditional secondary school curriculum. We urge federal support for investigations into this question, including development of experimental materials and prototypes of actual school curricula.

Although we are generally optimistic about the future role of computers, we feel we must highlight one point that worries us even though it is not directly within our charge. The disparity of access between children who have a computer at home and children who do not threatens to widen the educational gap that already exists between different economic strata. It is urgent that programs be designed to address this problem.

We clearly recognize that the most immediate problem is not the mathematics curriculum, but the need for more, and better qualified, mathematics teachers. One section of this report is devoted to recommendations on attracting and training prospective teachers, better using the talents of inservice teachers, and retraining teachers who are inadequately prepared for teaching mathematics. We feel that the coming changes in subject matter and emphasis not only will bring a new sense of vitality to K-12 mathematics, but also will encourage teachers actively to seek and participate in programs of professional development.

The Conference Board of the Mathematical Sciences stands ready to assist efforts to develop immediate strategies for addressing the teacher shortage and to develop long-term strategies for bringing about the curricular changes envisioned in this report.

I. THE NSF/CBMS Meeting

In response to suggestions made at the July 1982 meeting of the NSB Commission on Precollege Education in Mathematics, Science and Technology and, specifically, to a request made by the Educators Panel of the Commission, The Conference Board of the Mathematical Sciences (CBMS) held a special meeting to address the topic of this paper. The meeting was held on September 25-26, 1982, at the headquarters of the Mathematical Association of America in Washington, DC.

Participants in the meeting included the presidents of the American Mathematical Society, National Council of Teachers of Mathematics, Mathematical Association of America, American Mathematical Association of Two-Year Colleges, and Society for Industrial and Applied Mathematics. Two members of the Commission, Frederick Mosteller and Katherine Layton, and two members of the Commission staff, Ray Hannapel and Mary Kohlerman, also participated in the meeting. The other participants were representatives of the CBMS constituent organizations and the CBMS officers.

The initial portion of the meeting was devoted to discussion of six position papers on the fundamentals in the mathematics curriculum written expressly for this conference. Following this, participants joined working groups to address the question of what is still fundamental and what is not in K-8 and in secondary school mathematics. A general discussion of the written reports of the working groups was held during the last hours of the Saturday session.

On Sunday, new working group assignments were made to discuss the implications of changes in the K-12 mathematics curriculum. The reports from these groups were discussed in the closing session of the conference.

II. Recommendations to the Commission

Introduction

In the limited time available during the conference, it was not possible to establish full consensus on every detail of the working group reports. However, there clearly was broad consensus on the need to incorporate calculators and computers, as well as additional data analysis, into the K-12 curriculum and to make the necessary adjustments in the mathematical topics and modes of thought traditionally taught at these grade levels.

Some detailed recommendations on the fundamentals in the K-8 curriculum, what should be emphasized more and what should be emphasized less, are given in the working group report on elementary and middle school mathematics. The corresponding adjustments needed in the secondary school curriculum, where the impact of technology is even greater, are described in more general terms in the two reports on traditional and nontraditional secondary school mathematics. In this area much more investigation and experimentation are required before a firm consensus can be reached.

Recommendations on the challenge of providing children with access to, and understanding of, computers and calculators pervade this report. They are dealt with specifically in the report on the role of technology." A statement of the relationship between the mathematics curriculum and what is, or can now be, taught in other disciplines is included in the report on relations to other disciplines. The report on teacher supply, education, and re-education, contains a variety of recommendations on attracting and retaining well-qualified mathematics teachers.

The question of time in the school curriculum which should be devoted to the study of the mathematical sciences was not addressed in any detail at the meetings. The general feeling was that, at the primary level, there appears to be an approximate balance between topics needing more emphasis and those needing less. At the secondary level, it is not yet clear how much time, in addition to the time that can be gained by streamlining the traditional mathematics curriculum, will be needed for discrete mathematics, probability and statistics, and computer science. This can only become clear after detailed examination of model mathematical sciences curricula and careful consideration of the many competing demands for time in the overall school curriculum.

There was general agreement at the conference that the most pressing immediate problem is the need for more, and better qualified, teachers. No curriculum, no matter how well-founded, can possibly succeed without dedicated and competent teachers to teach it. However, many participants felt that appropriate changes in the curriculum could bring a new sense of vitality to K-12 mathematics and could serve to encourage teachers to actively seek and participate in programs of professional development.

Participants in the conference also were in agreement that their suggestions, even if influential in full, cannot be expected to constitute a "cure-all" for all the shortcomings of K-12 mathematics. In fact, a fundamental improvement in K-12 mathematics can be hoped for only within the framework of a general improvement of the total school environment. Remedies for the difficulties facing the teaching community (low teachers' salaries, low prestige, lack of support by society, lack of classroom discipline, irregular attendance, etc.) are societal in nature and fall outside both the mandate and the competence of this group.

Some Additional Recommendations

In addition to the concerns and recommendations in the working group reports, a few points were emphasized in the general discussions that are of vital importance in the implementation of any curricular changes.

Textbooks. Textbooks play a key role in the mathematical sciences curriculum at all levels. Any major changes in the curricula at the elementary, middle, or high school levels must be accompanied by corresponding changes in textbooks. For this to happen, the groups responsible for preparing textbook series and for adopting textbooks must have substantial subject matter competence and have available to them direct evidence of textbook effectiveness.

Testing. To a large extent, all teachers are under strong pressure to train their pupils to maximize their chances of doing well on standardized tests. As long as these tests stress computations, the pupils are bound to be drilled in computations, regardless of any other guidelines the teachers may have received, and even contrary to the sounder convictions the teachers themselves may have.

We call the attention of the Commission to the power and influence of standardized tests. Properly modified, these can have considerable effect in hastening the hoped-for improvements in the teaching of mathematics in grades K-12.

Articulation. The entrance requirements and course prerequisites of the nation's colleges and universities are major factors in determining the topics in the secondary school curriculum, as well as the amount of time devoted to them. Efforts to change the curriculum at the secondary level must be carried out in a cooperative effort with the colleges and universities.

Equal Access. The disparity of access to computers between children who have a computer at home and children who do not threatens to widen the educational gap that already exists between different economic strata. This disparity is exacerbated by the differences in resources available to different school systems. It is urgent to design programs to address this problem.

Women and Minorities. The conference noted with satisfaction the improvement during recent years in the participation of women in upper secondary mathematics. The many efforts that have led to this improvement must continue to be supported. We look forward to corresponding success with minority and handicapped students and continued improvement in the preparation of women.

Working Group Report: Elementary and Middle School Mathematics

Arithmetic and, more generally, quantitative thought and understanding continue to become more important for more people, but the importance of various aspects of arithmetic has changed and will continue to change as computers and calculators become pervasive in society. The suggestions below are designed to better equip students for life and effective functioning in the developing age of technology. We believe implementing these suggestions into the K-8 curriculum will make students more adaptive to future change, better equipped to use modern technology, better grounded in the mathematical bases for other sciences, and better grounded for further school mathematics.

A principal theme of K-8 mathematics should be the development of number sense, including the effective use and understanding of numbers in applications as well as in other mathematical contexts.

The changes we propose are fairly substantial, but are primarily in emphasis rather than in overall content. We believe they are consistent with, and are natural outgrowths of, recommendations relative to K-8 education of the earlier valuable documents, Basic Mathematical Skills by NCSM and An Agenda for Action by NCTM.

When implemented, the desired changes at the K-3 level lead to even more significant improvements at the 4-6 and 7-8 grade levels. They essentially replace excess drill in formal paper-and-pencil computations with various procedures to develop better number sense on the part of the student.

Special Concerns

1) Thorough understanding of and facility with one-digit number facts are as important as ever.

2) The selective use by students of calculators and computers should be encouraged, both to help develop concepts and to do many of the tedious computations that previously had to be done using paper and pencil.

3) Informal mental arithmetic should be emphasized at all levels, first aimed at exact answers and later at approximate ones. Such activity is necessary if students are to be able to decide whether computer or calculator printouts or displays are reasonable and/or make sense. Informal mental arithmetic involves finding easy, not formal algorithmic, ways of looking at number relationships.

4) There should be heavy and continuing emphasis on estimation and approximation, not only in formal round-off procedures, but in developing a feel for numbers. Students need experience in estimating real world quantities as well as in estimating numerical quantities which appear in complicated form. Methods requiring explicit (right or wrong) answers should be used where possible to help develop estimating procedures. For example, many exercises on comparing complicated fractions with easy ones (e.g., 12/25 with 1/2, and 103/299 with 1/3) can be used to get students to think of complicated fractions as close to, but less than (or more than), easy fractions.

5) There should be a heavy and continuing emphasis on problem-solving, including the use of calculators or computers. Trial and error methods, guessing, and guestimating in solving word problems should be actively encouraged at all levels to help students understand both the problems and the use of numbers. Naturally, examples and illustrations should be appropriate to the students' age, interest, and experience.

6) Elementary data analysis, statistics, and probability should be introduced, or expanded in use, including histograms, pie-charts, and scatter diagrams. The understanding and use of data analysis is becoming a vital component of modern life. The collection and analysis of data should include personal data of meaning to students, e.g., number of siblings; students' ages, heights and weights; data culled from newspapers, almanacs, and magazines; random data such as that produced by urn schemes; and data from experiments in other school subjects.

7) Place value, decimals, percent, and scientific notation become more important. Intuitive understanding of the relative sizes of numbers that arise in the everyday world of applications becomes even more vital.

8) More emphasis on the relationship of numbers to geometry including, for example, number lines and plotting, should lead to better understanding of the concepts of arithmetic and of geometry.

9) Understanding of fractions as numbers, comparison of fractions, and conversions to decimals should have more emphasis while drill on addition, subtraction, and division of numerical fractions with large denominators should have less.

10) Drill on the arithmetic operations on three-digit (and larger) numbers should be de-emphasized. Such computations can and should be done by calculators and computers.

11) Intuitive geometric understanding and use of the mensuration formulas for standard two- and three-dimensional figures should be emphasized. More stress is needed on why the formulas make sense.

12) Function concepts, including dynamic models of increasing or decreasing phenomena, should be taught. (For more details, see 4 in "Traditional Secondary School Mathematics.")

13) The concept of sets and some of the language of sets are naturally useful in various mathematical settings and should be used where appropriate. However, sets and set language are useful tools, not end goals, and it is inappropriate to start every year's program with a chapter on sets.

14) Based on motivation from arithmetic, algebraic symbolism and techniques should be encouraged, particularly in grades 7 and 8.

15) More extensive use of mathematics and computers in many other subjects--including business, languages, social science, and science courses--should be actively pursued. We encourage the consideration of this matter by experts in these fields and welcome opportunities to collaborate on further work in this area.

A discussion of possible computer programming or computer literacy courses is left to other groups for further study.

We call the Commission's attention to the fuller discussions and comments related to the K-8 curriculum in various position papers prepared for the conference.

Implementation Concerns

1) We hope the Commission will encourage widespread public discussion of the implications for K-8 mathematics of the changing roles of arithmetic in society. As an early step, we suggest discussions and conferences between teachers, supervisors, mathematics educators, mathematicians, and editors of textbook series concerning this report and others on the same general topic. Such conferences could be quite inexpensive if most participants are local.

2) We hope the Commission will seek ways to encourage the development and use of textbooks and of teacher-training materials in the spirit of the suggestions made above.

3) We hope the Commission will seek ways to encourage changes in standardized tests toward number sense and problem-solving and away from single-operation computational skills.

4) We hope the Commission will encourage school systems to reassign interested teachers at the 4-6 grade level to become specialists at teaching mathematics or other disciplines. One mode might be a simple trade of classes between teachers, with each teacher concentrating in areas of particular interest and competence. The needed changes in subject-matter emphasis will be much easier to effect if those actually teaching any subject are selected for their special interests and aptitudes. Special inservice training programs should be developed for all such semispecialized teachers, whatever their subject.

5) We hope the Commission will seek ways to improve the status of teachers and the conditions under which teachers attempt to do the important and difficult job of educating future citizens.

6) We believe that the needed changes can be brought about somewhat gradually and with general support of those concerned. There already is discussion in teacher and supervisor groups concerning many of the ideas put forth here. The proposed changes generally involve modifications in the way mathematics is introduced and used in schools rather than addition of new subject matter. The changes should permeate texts and not just be add-ons that can be ignored. There appears to be an approximate balance in time between topics needing more emphasis and those needing less. With the exception of computer use and the possible exception of parts of data analysis, the topics needing added emphasis have been taught and learned in American schools at various times and places in the past. The diminished role of paper-and-pencil computation is perhaps the topic which will provoke most concern and disagreement.

Working Group Report: Traditional Secondary School Mathematics

Current secondary school mathematics curricula are organized into separate year-long courses covering algebra, geometry, and precalculus topics. There are proposals that challenge this traditional division of school mathematics and the position of calculus as the primary goal for able college-bound students. Thus, the following analysis used conventional course headings for discussion of proposed changes in traditional topics, not as endorsement of the status quo.

1) Overall Recommendation. The traditional component in the secondary curriculum can be streamlined, leaving room for important new topics. However, since breakthroughs in technology which allow this streamlining are so recent and the conceivable implications so revolutionary, it is not yet entirely clear what specific changes are appropriate.

2) Algebra. The basic thrust in Algebra I and II has been to give students moderate technical facility. When given a problem situation, they should recognize what basic algebraic forms they have and know how to transform them into other forms which might yield more information. In the future, students (and adults) may not have to do much algebraic manipulation--software like mu-Math will do it for them--but they still will need to recognize which forms they have and which they want. They also will need to understand something about why algebraic manipulation works, the logic behind it. In the past, such recognition skills and conceptual understanding have been learned as a by-product of manipulative drill, if learned at all. The challenge now is to teach skills and understanding even better while using the power of machines to avoid large time allotments to tedious drill. Some blocks of traditional drill can surely be curtailed, e.g., numerical calculations using look-up and interpolation from logarithm and trigonometry tables.

3) Geometry. A primary goal of the traditional Euclidean geometry course is to develop logical thinking abilities. But not every fact need be given a rigorous proof to pursue this goal. Nor need this be the only goal of geometry, nor geometry the only means towards this goal.

We recommend that classes work through short sequences of rigorously developed material, playing down column proofs, which mathematicians do not use. These proof sequences should be preceded by some study of logic itself. Important theorems not proved can still be explained and given plausibility arguments, and problems involving them can be assigned. The time which becomes available because proofs are de-emphasized can be devoted to study of algebraic methods in geometry, analytic geometry and vector algebra, especially in three dimensions. Work in three dimensions is essential if one is to develop any pictorial sense of relations between many variables, and handling many variables is essential if one is to model phenomena realistically.

There is much room for computer use in geometry. The power of graphics packages makes it much easier for students to get a visual sense of geometric concepts and transformations. The need to use algebraic descriptions of geometric objects when writing graphics programs reinforces analytic geometry. Finally, the algorithmic thinking needed to write programs bears much resemblance to the thinking required to devise proofs.

4) Precalculus. What often happens in this course is that students see the same topics yet another time, with more drill but with little new perspective. For better students there may not be a need for a precalculus course if drill is no longer so important and if algebra and geometry are done "right," with the concepts made clear. For instance, one justification for the precalculus course is the perceived need to develop the idea of functions; functions appear in Algebra I and earlier, but current teaching may give too static an understanding. With computers, the concept of function can be made central earlier and more clearly. The computer supports qualitative analysis of the graphs of functions in a dynamic mode of display and also allows detailed analysis of zeros, rates of change, maxima minima, etc.

5) Algorithmics. Computers and programming have made the creative human talents and skills involved in developing and analyzing algorithms extremely important. These talents and skills, emphasized by the group on non-traditional topics, can be exercised quite naturally through traditional topics as well. Much of high school algebra consists of systematic methods for handling certain problems, e.g., factoring polynomials. Such methods are algorithms. Instead of making the student carry out such methods with paper and pencil a boring number of times, have the student do it just a few times and then program a computer to do it. The understanding gained should be at least as great.

6) The Average Student. For the many students in secondary school who are not specially talented in mathematics and not headed for careers in science or technology, current programs are a source of discouragement, anxiety and repetition in a dull "basic skills" program which serves them poorly. We cannot ignore the needs of this large and important group. Computers, as mathematical tools and media of instruction, offer a fresh window into mathematics for them.

7) Cautions. We have suggested that technology provides an opportunity to devote less time to traditional techniques while boosting understanding and allowing more time for more complex, realistic problemsolving. However, there are several cautions. First, there are widespread and deep reservations about how much traditional goals should give way to technology. Second, there is little research data on the feasibility of such changes, and there are almost no prototype school curricula embodying the new priorities. Experimental programs, and research on the results, must be given major support. Third, changes in secondary programs must be carefully articulated with the expectations of colleges and employers, who often have conservative views about curricula. Finally, the syllabi of an

extensive range of standardized tests play a very influential role in setting curricula and the actual classroom emphases of teachers. If curricula are to change, the tests must be changed. Clearly, strong national leadership and cooperation are necessary, from teachers, mathematicians and public policymakers, to meet these challenges and implement significant change.

Working Group Report: Nontraditional Secondary School Mathematics

On two basic principles the panel was unanimous:

- o There is need for substantial change in both the subject of and the approach to teaching in secondary school mathematics.

- o If changes are to be made in secondary school mathematics, we must make haste slowly, taking care at all times to insure full consultation with and support from the secondary school mathematics teaching community.

We have five specific recommendations in the areas of subject matter, approach to teaching, the use of new technology, and teacher training and implementation.

1) Subject Matter. Careful study is needed of what is and what is not fundamental in the current curriculum. Our belief is that a number of topics should be introduced into the secondary school curriculum and that all of these are more important than, say, what is now taught in trigonometry beyond the definition of the trigonometric functions themselves. These topics include discrete mathematics (e.g., basic combinatorics, graph theory, and discrete probability), elementary statistics (e.g., data analysis, interpretation of tables, graphs, surveys, sampling) and computer science (e.g., programming, introduction to algorithms, iteration).

2) Approach to Subject Matter. The development of computer science as well as computer technology suggests new approaches to the teaching of all mathematics which should emphasize:

- o algorithmic thinking as an essential part of problem solving, and

- o student data gathering and the investigation of mathematical ideas to facilitate learning mathematics by discovery.

3) Technology. New computer technology allows not only the introduction of pertinent new material into the curriculum and new ways to teach traditional mathematics, but it also casts doubt on the importance of some of the traditional curriculum, just as the hand calculator casts similar doubts about instruction in arithmetic. Particularly noteworthy in this context at the secondary level are:

Symbolic manipulation systems which even now, but certainly far more in the near future, will allow students to do symbolic algebra (and calculus) at a far more sophisticated level than they can be expected to do with pencil and paper.

Computer graphics and the coming videodisc systems which will enable the presentation and manipulation of geometric and numerical objects in ways which should be usable to enhance the presentation of much secondary school mathematical material.

One caveat which we would stress is that this technology and related software packages must be used not to enable students to avoid understanding of the essential mathematics but rather to enhance such understanding and to allow creative experimentation and discovery by students as well as to reduce the need for tedious computation and manipulation.

4) Teacher Training. There are two aspects of this, both dealing with secondary school mathematics, on which we wish to comment:

a) Retraining of current teachers in the new topics, approaches, and technology.
One possible new approach to this might be the use of college students to aid and instruct secondary school personnel as part-time employees, perhaps using such incentives as forgiveness of student loans.

b) Education of new teachers.
Crucial to long-term solution of the secondary school mathematics education problem is that the requirements for degrees in mathematics education be, as necessary, changed to incorporate modern content and approaches. In particular, we believe that all prospective teachers of secondary school mathematics should be required to take at least one year of discrete mathematics in addition to traditional calculus requirements, one semester or one year of statistics (with focus on statistical methods rather than mathematical statistics), and one year of computer science.

5) Implementation. We recognize that the kinds of changes proposed here not only require much more study than has been possible by our panel but that also they will never be implemented unless there is dedicated cooperation among concerned people and groups: secondary school teachers of mathematics and their professional organizations; college curriculum people in schools of education and in mathematics departments, including their organizations; and state and local education authorities and their organizations.

A conference bringing together these groups to discuss the relevant problems and to plan future action might be the most fruitful next step to provide some momentum for the changes we believe are necessary.

Working Group Report: The Role of Technology

Computers and related electronic technology are now fundamental features of all learning and working environments. Students should be exposed to and use this technology in all aspects of school experience where these devices can play a significant role. More specific recommendations follow.

1) The potential of technology for enhancing the teaching of mathematics and many other subjects is vast. Development of such resources should be supported at a national level. Specific examples include computer-generated graphics, simulations, and video-disc courseware materials. There should be efforts to create a network providing easy access to such banks of material.

2) While computing technology promises to enhance learning, differential access to the benefits of that technology could widen the gaps in educational opportunity that already separate groups in our society. It is imperative that every effort be made to provide access to computers and their educational potential to all sectors of society.

3) As a general principle, each mathematics classroom should have computers and other related electronic technological devices available to facilitate the computing and instruction required for mathematics learning and competency. Such availability of computers and other electronic technological devices in the mathematics classroom is as important as the availability of laboratory equipment for science instruction.

4) Hand calculators should be available in mathematics classrooms (both in elementary and secondary schools) for students on the same basis that textbooks are now provided.

5) Support should be given for broad developments in software that may be useful in the schools. School districts shoul. encourage their teachers and students to engage in cooperative development activities and to find ways to recognize and disseminate the products of those efforts.

6) Computer literacy involves not only the use of computers to accomplish a great spectrum of tasks but also a general understanding of the capabilities and limitations of computers and their significance for the structure of our society. Development and implementation of appropriate programs to teach these more general concepts should be supported.

7) Possible curricular changes emanating from technological changes will require careful study and deliberation over a long period of time. This activity must be encouraged and supported from a national level. The exploratory projects should bring together teachers, curriculum developers, mathematicians, and affected interested parties from business and industry. The new programs developed should be tested extensively in a variety of settings to

insure that they work with real students and schools before extensive implementation is attempted.

8) The interplay between word-processing, computers, data bases, and data analysis methods assists in breaking down barriers between disciplines, thus offering an opportunity for schools to provide a range of holistic problem-solving experiences not typical in school today. Using the technology as an aid, students can plan and conduct data collection, analysis, and report writing that is realistic, attractive, and far beyond normal expectations in today's schools.

9) The availability of well-trained, highly qualified teachers of mathematics is a must in a technological society. Support should be given to organizing programs for inservice training and retraining of current teachers of mathematics (elementary and secondary) who are inadequately prepared to teach a technologically oriented curriculum, but have the capacity to profit from such programs to strengthen their mathematical preparation and teaching skills.

10) While technology provides opportunities, it also makes demands. The world becomes a more complex place in which to live. If we are to insure that a broad spectrum of society can function and participate actively in the business/industrial community and decision making of the country, it is imperative that students become adept in the precise, systematic, logical thinking that mathematics requires.

Working Group Report: Relations to Other Disciplines

Along with the effects of computational technology on the mathematics curriculum, it is also necessary to consider how this technology and the proposed curricular changes affect the relationships between the curricula in other disciplines and the curriculum in mathematics. We have interpreted the phrase "other disciplines" rather broadly.

First, using a narrow view, we must look at the effects these curricular changes will have on science education. There has always been a necessary interaction and coordination between the science and mathematics curricula, particularly with the physical sciences. At a minimum, this revised curriculum, which encourages student use of calculators and computers and emphasizes a good sense of estimation, provides an opportunity for elementary and high school education to be more realistic and eliminates the use of specialized problems with "easy numbers." If we raise our sights a bit, there is an opportunity for a better coordinated and integrated total science education. Furthermore, the introduction of statistical ideas, data handling procedures, and discrete mathematics provides an opportunity for a more mathematical discussion of social science problems at the elementary and high school levels. Similarly, changes in currently available tools will undoubtedly affect courses in business and commercial programs.

Related questions arise on the other side. What do the school programs and the college programs in natural sciences, social sciences, and business require in the mathematical preparation of entering students? We believe the suggested curriculum can only be an improvement, but discussions with leaders of those disciplines is required.

Taking the broad view, we also believe that this modified curriculum, which provides students with the same (or greater) ability to use mathematics as well as an ability to use and appreciate the technology, will provide for a wiser citizenry. The graduates of such a program should be better equipped to deal with "poll results" and statistical data references to the economy and sociological problems.

We believe there is one serious area in which the nation needs more data for the development of an appropriate mathematics curriculum. Namely, what are the needs, in terms of mathematical skills, of the students who seek technical vocational employment without going on to further schooling? Furthermore, what are the mathematical needs of students going on to technical or vocational schools? Although we do not know the answer, we believe the new curriculum will do at least as good a job as the existing one. A conference or meeting to explore this area would be an excellent idea and would complement our work.

Working Group Report: Teacher Supply, Education, and Re-education

Efforts to improve and update the mathematics curriculum and to increase the mathematics, science, and technology literacy of all citizens require the support of qualified mathematics teachers at all levels. At present there is a serious and well-documented shortage of qualified teachers of mathematics at the elementary and secondary school levels in most areas of the country. Economic, employment, and social conditions forecast that the current short supply may indeed be a long-term problem. Furthermore, even in geographic locations where adequate supplies exist, the frequent turnover of mathematics teachers tends to impede learning.

The following recommendations address the need to increase the supply of mathematics teachers as well as to improve the qualifications of the teacher and, thereby, the learning of mathematics:

1) While state and local efforts by industry, business, and academe to deal with the teacher shortage are laudable and should continue, the magnitude of the problem is national in scope. An articulated national commitment with federal leadership and support is needed for its resolution. The public should be made aware of the problem through more effective publicity.

2) Incentives of all types need to be studied to <u>attract</u> and <u>retain</u> qualified teachers of mathematics. Financial incentives should

be given special attention with priority assigned to those which do not create undue inequities and tensions among colleagues, in order to avoid being counterproductive.

Examples of possible incentives and support systems include the following:

a) Forgiveness of student loans and/or interest on loans for those who enter the teaching field.

b) High entry-level salaries for special expertise (e.g., computer training).

c) Reduced teaching loads to allow teachers to pursue graduate study or other advanced training in the mathematical sciences and applied areas.

d) Financial support of graduate study or other advanced training in the mathematical sciences and applied areas.

e) Salary differentials by discipline.

f) Summer positions and other cooperative arrangements with business and industry to supplement a teacher's income (with the obvious caveat that the short supply of teachers is largely due to the fact that higher industrial salaries lure teachers away; industry would have to be discouraged from using this arrangement for recruitment purposes).

3) In an era when content and technology are changing so rapidly, incentives are needed to keep qualified teachers in the field abreast of current trends in the mathematical sciences. Inservice workshops, NSF-type institutes, retraining courses, industrial experiences, and other forms of continuing education can serve to refresh the faculty and renew its commitment to teaching.

4) In some parts of the country, teachers from other disciplines are being assigned to teach mathematics classes. These teachers need considerable subject-matter training and assistance in developing appropriate teaching strategies to reach a level of preparation close to that of regular mathematics teachers and to succeed in their new assignments.

5) Encouraging colleges and universities to loan their faculty, and business and industry to loan their mathematically oriented employees to teach courses in the secondary schools could be mutually beneficial. Qualified retirees or near-retirees also might be recruited to enter the teaching field. (Of course, the issues of appropriate teacher training and certification need to be addressed.)

6) In states where this is not the norm, it is recommended that teacher certification requirements be stated in terms of the specific topics to be covered in the subject area rather than in terms of just total number of credits.

7) Recommendations regarding the mathematical fundamentals to be covered in educating qualified teachers of mathematics include the following:

Elementary Level. It is strongly suggested that mathematics at the elementary school level be taught by teachers who specialize in mathematics. Whether the teacher specializing in mathematics should be assigned to all grades or just to grades 4-6 (or 4-8) requires further study. An alternative approach would be to identify those teachers in a given school who most enjoy teaching mathematics. Those teachers could be assigned to teach all mathematics courses across a grade level, while other teachers do similarly in reading and writing.

The following recommendations pertain to both the regular elementary school teacher and the teacher specializing in mathematics:

For entry into the mathematics education program for elementary school teachers, at least three years of college-track mathematics in high school is recommended. College mathematics courses should provide a sufficient background to understand the relationships between algebra and geometry, functions, elementary probability and statistics, instruction in the use of the hand-held calculator, and some exposure to computers. Creative approaches to problem-solving should also be included in the curriculum. Training should be at least one level above what is being taught. This background is particularly important in light of children's awareness of the world around them through television, other media, computers, and so on.

Secondary Level. Secondary school mathematics teachers should have course work in mathematics equivalent to a major in mathematics. Requirements for those who will teach mathematics should include the equivalent of a two-year calculus and linear algebra sequence, discrete mathematics, probability and statistics, and appropriate computer training. These courses should develop in the student a sense of "mathematical maturity" in the approach to problem-solving.

Note that college and university curricula for educating mathematics teachers should be re-examined and revised in accordance with the above guidelines and goals. Contingency plans should be developed in case separate departments of mathematics and computer science are established at the secondary level in the future.

Conclusion

The recommendations cited here require careful planning and implementation. With high technology a mainstay of our present and future society, it is imperative that we recognize and promote mathematics as a powerful, useful, and enjoyable component of our lives.

CHAPTER 7

NEW FUNDAMENTALS OF MATHEMATICS FOR SCHOOLS

Ubiratan D'Ambrosio

If we look at the educational system as a whole, mathematics is a dominating subject in schools. Together with reading and writing, it constitutes the spine of a system aimed at providing equal opportunity for all. At the same time, it helps to prepare our young for the future advancement and betterment of the socioeconomic and political framework of society. The three _R_'s have dominated school scenery for decades. Is this to be maintained?

The emergence of computers surely will affect the scenery and, in predicting education of the 1990s, an important role should be reserved for information-processing equipment. Although influential in teaching all the three _R_'s, the use of computers will, by its very nature, directly affect mathematics education. Indeed, it must prompt new investigations of the nature of mathematics itself. In addition, pedagogical action, as conceptualized by D'Ambrosio (1981, 1985a), will be deeply affected, and the curriculum, seen as the strategy for pedagogical action, will call for new components.

Although the computer issue is relevant to this paper, the essential concern here is to identify a few indicators of mathematics' contribution to societal goals and thus to set up the appropriate framework for establishing a monitoring system for mathematics education. Obviously, I am thinking about long-range effects and broad, global, societal goals. As has been stressed in earlier works (D'Ambrosio, 1979), mathematics appears to be a strategy to attain overall societal goals. It is not easy to define such long-range goals, as they are immersed in the concepts of progress and development. But a few values are permanent in any model of global policy. The American model is dominated by the democratic ethos within a welfare state. A growing, equally prevailing force is the ecological ethos, which is closely related to concerns about the primacy of our species and "the internalization of holistic thinking in science and culture" (Falk, 1986, p. 68). This ecological ethos calls for imaginative new models of social, political and economic organization as described in the Declaration of Venice: "The challenge of our time--the risk of destruction of our species, the impact of data processing, the implications of genetics, etc.--throw new light into the social responsibilities of the scientific community, both in the initiation and application of research" (UNESCO, 1986).

Our responsibilities as educators in a democracy go beyond reproducing past and current models. We are driven to create a future that shall be, in many ways, better than the present. It is here that one must ask the question, how much does mathematics education have to do with it? The answer is unequivocal: everything!

A CRITICAL APPROACH TO MATHEMATICS EDUCATION

Mathematics is deeply rooted in our cultural systems and thus, is loaded with values. Although not sufficiently studied as yet, the analysis of ideological components in mathematical thought reveals a strong connection with a certain socioeconomic model. These mathematical ideologies parallel the ideological components of education in general, as stressed by Apple (1979), Giroux (1981) and the proponents of critical theory. Together with some eminently conservative practices, such as medicine when dealing with normality and law for hierarchy, mathematics promotes a model of power through knowledge. I could paraphrase Duncan Kennedy (1983) by saying that mathematics teachers indoctrinate students to believe that people and institutions arrange themselves in hierarchies of power according to their mathematical ability. The "superiority" of high achievers in mathematics is recognized by all; mathematics ability is the mark of the genius. Taken together, critical approaches to cognition, to social structure, and to states' interdependencies, i.e., to the global world arrangement, urge us to examine the role of mathematics in our educational system from a fresh perspective. Issues such as environmental decay, individual privacy and security, widespread hunger and disease, and the threat of nuclear war are new to the exercise of thinking about the future.

Undeniably, the future is impregnated by science and technology--for good or evil--and mathematics is at their root. A few years ago, "The Economist," a London weekly, published a lengthy article entitled "You cannot be a 20th century man without maths" (1979). The responsibility of mathematics educators toward the future is a focal one, and we need to understand our role in this very complex net of shared responsibilities. This is the framework from which we should discuss a system to monitor the health and progress of mathematics education, which is the objective of this paper.

We cannot avoid reflecting briefly upon the way policymakers will use information provided by a monitoring system. In this respect, we clearly need to educate policymakers. As expressed by Israel Scheffler (1984), we will need to design a curriculum for policymakers rather than to merely provide them data. Such an approach recognizes that policymakers must understand the learning process and become aware of the position of mathematics in everyday life, with its complexity of human activities, experiences, purposes and needs, and its consequential tensions and creativity. This calls for a broader understanding of the nature of our discipline and its position in the full range of human knowledge. As Scheffler (1984) puts it, "The policymaker needs to be multilingual, to learn to speak and learn various disciplines' dialects, and to employ them conjointly in understanding the problems" (p. 154). Is this less true for the mathematics educator?

THE AUDIENCE FOR A MONITORING SYSTEM

Several issues are to be considered in planning a monitoring system. The most fundamental is its audience. Here I prefer to expand the issue to the more general question of school system accountability. Of course, a monitoring system will be used by state and local policymakers responsible for managing the educational system. In addition, however, the fact that these policymakers respond to public demand is the foundation of a representative democracy. This is clearly evidenced by the tax system which prevails in financing American education. Hence, although primarily designed to be available to state and local policymakers, the monitoring system must be accessible to the entire population and must address the issues which are day-to-day concerns of parents and pupils as well. In discussing the improvement of science teaching in a special 1983 issue of Daedalus devoted to scientific literacy, J. Myron Atkin wrote that "most people want something practical, or at the very least, recognizable" (p. 178). Hence, we must insure the accessibility of our data to a very broad audience. These data must be available to all of those responsible for decisions, i.e., policymakers and their constituents. In addition, they must carry a pedagogical component in the sense that they must be somewhat instructionally designed. Explanation, interpretation, and a critique of the results must accompany the information provided by the monitoring system. While this is sure to be biased by the monitoring system's set of values, there is no way to avoid this. After all, mathematics education is impregnated with values, as discussed in earlier works (D'Ambrosio, 1985a).

Also, I cannot disregard the internal composition of the school itself, i.e., the relationships among teachers, teachers and principals, principals and supervisors, and so on. Summing up, one must consider all the forces playing a role in the school system, a highly complex net of influences that shapes mathematics education. This was discussed by Paulus M. Gerdes at ICME-5, as it pertains to the highly traditional society of Mozambique, and it has been evidenced in classical literature. Examples include the judiciary process, which moved against Gustave Flaubert following the publication of "Madame Bovary", and, more recently, Giovanni Lada's "Padre Padrone." An interesting approach to the expectations surrounding education can be found in the work of Teresa Amabile (1983). These expectations, and the interrelationships among the various actors on the educational stage, are fundamental components which must be considered when monitoring the entire play.

Thus, an efficient monitoring system must take into account the expectations of all those involved. These expectations range from effectiveness and the enhancement of creativity to pure utilitarism and promotion of skills. Of course, these are not dichotomous; both respond to a particular view of society as a whole. This kind of dichotomy is discussed clearly by Plato in Book VII of The Republic. Also, Marrou's comments (1982) are rewarding (p. 73). It is undeniable that more or less emphasis on one aspect or the other is a political decision, closely related to overall societal goals. Is The Paideia Proposal (Adler, 1984) more akin to American ideals than the "back to basics" movement? Or do both aim at the same model of society?

If The Paideia Proposal is deemed more representative of social goals, the attitude toward mathematics education will reflect the remark that "all students study mathematics until the twelve years of basic schooling. . . . Mathematics is central to the manipulation and the innovation of information. Mathematics illiterates will be left behind. In addition, reasoning is one of the most human things that human beings do" (Adler, 1984, p. 84). How well can one monitor such a Comenian approach, as compared with the "answer-oriented" mathematics education which prevails nowadays, and which Garth Boomer (1986), with evidence drawn from Romberg (1984), has properly called "catechistic" teaching?

A CREATIVITY-ORIENTED PROGRAM

Let me discuss the basic issue, which remains open. Should mathematics education move into a creativity-oriented, hence basically open, curriculum, which is very difficult to evaluate in the short term? Or should mathematics education stick to the performance-oriented traditional model? Several examples of open, creativity-oriented programs have been proposed throughout history, but short-term assessment of them is practically impossible. Impact evaluation, as it sometimes has been called, is in an unsatisfactory stage as yet. Affective components may be the only indicators upon which one can rely. Monitoring must then be directed to small group behavior, using qualitative rather than quantitative instruments. This will have immediate implications for the curriculum. Clearly, monitoring systems act in a dual-target mode. Although aimed at policymakers, their reflection on curriculum and classroom management is unavoidable. Every teacher who is aware of the evaluation scheme will be deeply affected by it in his or her practice. There is no way, and no reason, to keep the monitoring system "secret." As a result, the entire educational system is affected by the theoretical framework on which the monitoring system rests, and the monitoring system itself deeply affects the behavior of teachers and influences the educational system.

WHY TEACH MATHEMATICS?

Let me concentrate on the central questions I want to address concerning the monitoring system. These questions, which will indicate the health of the educational system, can be grouped according to what may be looked on as reasons to teach mathematics with such intensity in the school system.

Several reasons to teach mathematics have been identified throughout the history of mathematics. I consider fou : to be essential:

1. Utilitarian
2. Formative
3. Cultural
4. Aesthetic

I do not hesitate to say that all four reasons are equally valid, but there has been a growing imbalance in the last 100 years resulting in a utilitarian overemphasis. This has been a mistake, and the mistaken character of strongly utilitarian-oriented mathematics education is reinforced by the widespread use of calculators and computers. The traditional, skills-oriented mathematics curriculum is obsolete and inefficient. On the other hand, utilitarianism pays only lip service to a new emphasis on applications to real world problems; an authentic approach must go in a different direction. There is no authenticity in the so-called "problem-solving situations" stressed in the beginning of this decade. Even in its broader conception, as described in NCTM's Agenda for Action (1980), specific problems are emphasized and presented in a formulated, already codified, mode. "Real situations" were indeed simulated situations, and although there has been and continues to be an appeal to deal with "really real" situations, they cannot get into the classroom unless the attitudes towards mathematics changes.

This is, more than anything else, the result of an epistemological barrier. Curricular systems are not present in the classroom. Instead, mathematics curriculum is designed conservatively, relying on topics in their final forms, or, in Kuhnian terminology, theories that have attained the stage of "normality." This is superbly described by Philip Kitcher (1983) in the context of his argument against mathematical apriorism: "experts demonstrate their expertise by producing verifiable solutions to problems which baffle us, that they produce plausible arguments against our contentions (arguments whose plans are too well hidden for us to detect), and that they offer convincing psychological explanations of our mistake" (p. 54). The underlying epistemology in mathematics education practices is aprioristic, while a Bachelardian approach has been absolutely ignored in education overall, and particularly in mathematics education. Clearly, when Bachelard (1981) says that "L'etat logique est un etat simple et meme simpliste" (p. 27), and that this state can not serve as proof in the case of a psychological reality, he opens a new direction for an approach centered in the psychoemotional complexity of the student, rather than in the transmissable techniques that a teacher tries to convey to his pupil. Indeed, he refers to William James; regrettably, James became marginal in math education--as did Bachelard's epistemology. The current and dominant approaches to the philosophy of mathematics tend to mask the fact that mathematics is closely related to reality and to the individual's perception of it. Reality informs the individual through a mechanism which we have insisted upon calling sensual rather than sensorial (D'Ambrosio, 1981), precisely to stress the importance of the psycho-emotional component. The key issue in problem solving--which appears basic when one addresses a question such as, How well have the students in our state learned to solve complex problems?--may, indeed, be the result of a misleading question. Consequently, in order to monitor it, we may have to distort the entire attitude in the classroom. Complex problems are related to a new consciousness state, as William James recognized in his observation that the state of consciousness in which we recognize an object is a new one as compared with the state of consciousness in which we have known the object. This may be the reason we insist on calling the impact of reality upon the individual sensual rather than sensorial. Research being conducted by Regina L. De Buriasco in Rio Claro, Brazil, attempts to identify the emotional aspects of children's perception of such mathematical notions as "half" and "more or less." Regrettably, mathematics education has tended to suppress the emotional aspects of the individual perception of reality.

ON EVALUATION

The alternative approach to problem solving calls for the effective immersion of children in global practices. Evaluation and the concept of examination take on new dimensions. Problem solving is viewed in a much broader way, which combines modeling processes and creativity-enhancing programs. Evaluation becomes a qualitative rather than a quantitative issue, an affective-oriented search rather than a performance-oriented one. Thus, the monitoring system must take into account some new indicators.

The issue of which indicators may be used in a qualitative, affective-oriented evaluation system is a fundamental one when we shift from traditional problem solving to a modeling approach. A very imaginative proposal is implicit in an analysis of detectives' behavior by Umberto Eco and Thomas A. Sebeok (1983) when they add abductive reasoning to general considerations of reasoning processes. While discussions about problem solving focus on inductive-deductive modes of thinking, abduction, which may be conceptualized as a conjecture about reality which needs to be validated through testing, seems to be a basic component of an effort to deal with a real situation. According to Charles S. Pierce, abduction, together with induction and deduction, is an essential mode of the cognitive process. Although much progress has been made in our understanding of the human mind since the times of James and Pierce, their approach to reasoning seems to be quite suitable to our understanding of the mind-body processes.

Particularly appealing for renewed mathematical education is evidence gathered by sociologists, who propose a new vision of the cultural phenomenon. Charles J. Lumsden and Edward O. Wilson (1981) try to understand culture through a sequence of components they call learning, imitation, teaching, and reification. All but reification appear in several species. Reification, i.e., "the mental activity in which hazily perceived and relatively intangible phenomena, such as complex arrays of objects or activities, are given a fictitiously concrete form, simplified, and labeled with words or other symbols" (p. 381) is solely characteristic of human beings.

Putting this together, we see that Lumsden and Wilson, and before them Pierce, recognized codification processes acquired through psychological mechanisms that run contrary to the linear structure which characterizes and underlies mathematics education. The codes are acquired through a reificative process and then "stored" for further use in different situations. Among these codes is mathematics. In fact, both suggest that it would be best to immerse children in an environment where mathematical challenge comes naturally. Similarly, the work of Amabile (1983), the psycho-pedagogical framework implicit in the LOGO proposal (Papert, 1980), the message of The Education of Henry Adams, and the roots of Dewey's pedagogical thought agree with this approach.

In bringing these considerations to bear on the practice of mathematics in schools, emphasis should be shifted to "really real" situations. Projects of a global nature, such as building a cabin, mapping a town or assessing the water consumption of a community, provide situations which will require modeling and problem posing. Problem solving occurs as a consequence; it acquires meaning, and its solution makes sense. A methodology which can be traced back to Eliot Wigginton's experiment, as particularly seen in Foxfire 6, takes into account the child's own environment and starts with "fact finding," which is gathering information about a situation. The method proceeds through modeling and finally ends with "realization," the transformation of the result into action or objects. This is based on a cycle of reality - individual - action - reality, discussed in earlier works (D'Ambrosio, 1985a).

This is unmistakenly an open, activities-oriented approach to mathematics education, which draws on the environment, thus relying on previous knowledge. This leads to what I have labeled ethnomathematics, which restores mathematics as a natural, somewhat spontaneous, practice. Although research on the influence of previous ideas on the experimental approach to science education is frequently cited, mainly by the Piagetian school (see for example, Marmeche, Meheut, Sere & Weil-Barais, 1985), efforts in mathematics education to identify ethnomathematical practices and to recognize them as valuable background are relatively recent. The advantages of building up on the transition from an ethnomathematical practice to a formal approach is as yet unexplored and remains largely ignored.

From these considerations, I pass to another set of issues that refers to differences in exposure to mathematics by race, by social class, and by sex, and investigates how these differences are reflected in the level of performance, attitudes, enrollment, and use of mathematics.

DIFFERENTIATED CURRICULA

These issues have been the subject of much emphasis during the last two decades. Indeed, evidence has been gathered on the relatively low mathematics achievement of girls, of blacks, of Native Americans and other groups. Some explanations have hinted at a gender or racial inability to perform well mathematically; such a conclusion has been rejected by all sectors involved. Other explanations point to a social structure intentionally aimed at depriving women and certain ethnic and cultural groups of a full mathematics education. Of course, this theory fits into a social model of male domination in a culture in which power through knowledge limits social leadership to those with a better mathematical background. This is the prevailing position in every school system, and as a consequence schools aim to provide the same mathematics education to all students, assuming first that all students will be able to absorb equally well this form of knowledge, which seems to be correct to the best of our understanding of learning and teaching dynamics. Second, this approach assumes that mathematics fits into the mind structure of the human species, which has mathematics blueprinted in it. This is Kantian apriorism, which has prevailed in the philosophy of mathematics and has its reflection in the totality of mathematics education.

New approaches to the nature of mathematical knowledge, as, for example, in Kitcher (1983), and the growing attention being given to ethnomathematics open a new and broad area of research in the anthropological approach to mathematics, which is loaded with constructions deriving from psycho-emotional and cultural issues. Consequently, some topics of mathematics which draw upon psycho-emotional and cultural motivation will naturally be met with differing levels of enthusiasm by woman, blacks, and the poor independently of the level of exposure. For example, certain mathematics chapters have more appeal to women than do others; some are more attractive to the middle class than are others; some appeal to white, Protestant teenage boys, and so on. What is undesirable, and should be avoided, is the valuation in the school systems of one kind of mathematics over others. Of course, this is a very touchy issue, but research such as that carried on by Ethington and Wolfe (1986) cannot be disregarded. The new scholarship on women very clearly leads to a more creative approach, which does not ignore differences in interest or in psychological development. As Maher and Rathbone (1986) put it, "The essential error has not been in seeing female behavior as different but in judging them as inferior" (p. 222).

This is how ethnomathematics comes into the picture. In this context, the problem implicit when I ask about some children receiving more or different exposure to content than others as a consequence of race, of social class, of gender is a false problem. The issue instead resides in valuing one kind of mathematics over others. Explicitly, by bringing into the classroom mathematics that is closely related to activities more appealing to young girls, the performance of these girls should improve relative to their performance on problems related to typical boys' activities. The same happens when drawing on cultural issues, as with some aspects of mathematics which touch on the racial or religious backgrounds of some children, for example. Much research is needed in understanding different reactions of children to these issues. Research is scarce because of the mistaken trend towards a single mathematics program for all students that has prevailed in the last decades. Most probably, the observed differences in performance by girls or blacks are due to sociocultural background, although research has not eliminated the issue of a genotypical influence or differentiated neuronal assemblies. As "taboo" as this subject may be, research such as that carried on by Benbow and Stanley (1983), or reported in the Dahlem conference (Conference, 1985), must be critically examined.

In any case, the key issue is to provide a multiplicity of directions and diversified curricula to best suit the different psycho-emotional and cultural patterns of children. Very much in line with Howard Gardner's theory of multiple intelligences (1983), the multiplicity and diversity of mathematical experiences lead to curricula based on situations; this agrees with the general approach implicit in the work of Amabile (1983) and in creativity-enhancing projects. Again, I return to the modeling of real situations as the most adequate method to deal with such diversities.

OTHER THAN UTILITARIAN VALUES IN MATH EDUCATION

Naturally, the discussion in the previous section implies the recognition of values in mathematics education that are on equal standing with its utilitarian value. Indeed, in many cases, the cultural and esthetic values, which imply how mathematical ideas are formed, are even more important. The utilitarian value, which has become prevalent in the last hundred years, had been left throughout history to other domains of education. As I have discussed (D'Ambrosio 1979, 1980), utilitarian views of mathematics existed simultaneously with academic mathematics, until the epoch of the big changes which resulted from the great social and political new directions of the last century. It is clear that the imbalance between utilitarianism and other values that has occurred in the last hundred years has caused a dehumanization of science, technology, and society as a whole. It is time we restore the humanistic focus to general education, to education for all, and hence to mathematics for all.

Consequently, the quality of the mathematics curriculum, considered with respect to sex, race, social class or in any international comparison with the curriculum in other countries, must be confronted in a different way. Quality is assessed not merely by performance, attitudes, or enrollment--and even less by analysis of the content of curriculum on the three levels usually considered: intended, implemented, attained. Not only does curriculum analysis lead to false evaluation of the system, but it masks components of social injustice and discrimination by sex, by race, and by social class.

In advocating recognition of the ethnomathematical focus on the curriculum, I am implicitly recognizing mathematics as a system of codification that allows describing, dealing, understanding, and managing reality. This is attached to a broad concept of the definition of knowledge vis-a-vis reality (D'Ambrosio 1985b, 1986). These codes go through two distinct processes: one derived from family and peer groups, and one whose institutionalization is loose and as yet not clearly understood. Both belong to the domain of anthropologists and have recently entered into the consideration of school systems. As yet there has been a resistance to look into these issues in American school systems. These ethnomathematics considerations--of course there are numerous ethnomathematics--are close to what Basil Bernstein (1971) calls the "restricted code," as contrasting with the "elaborated code," when dealing with language, or what Ivan Illich (1982) calls "vernacular" language or universe.

A few characteristics of ethnomathematics should be stressed:

1. It is limited in technique since it draws on narrow resources. On the other hand, its creative component is strong, since it is not bound to formal rules obeying criteria unrelated to the situation.

2. Although it is broader than ad hoc knowledge, it is context bound, and therefore particularistic contrary to the universalistic character of mathematics, which claims, and ideally aims, to be context-free.

3. It operates through metaphors and systems symbols which are psycho-emotionally related, while mathematics operates with symbols that are condensed in a rational way.

Of course, this leads to a hierarchization of transmission of knowledge and to the fundamental issue of legitimation knowledge. While ethnomathematics draws much of its validity from how it works in a given situation or whether it fits an individual's overall view of the world, mathematics draws its authority from a sequential hierarchization; starting with authority of the teacher, reaching finally the authority of rational thought through the authority of the written and printed word. This rationalistic goal leaves in its wake values that are rooted in the cultural context to which ethnomathematics is a natural codification. To face mathematics education in such a way that it embodies the child's value and culture, i.e., his/her ethnomathematics, seems to be a desirable road to a more humanistic version of rationalism.

The step from ethnomathematics to mathematics can be seen as similar to the step from oral to written language. Written language (reading and writing) builds on the knowledge of oral expression already possessed by the child, and the introduction of written language does not suppress oral language.

To understand and respect ethnomathematical practices opens up a vast potential for a sense of inquiring, a recognition of specific parameters, and a feeling for the global equilibrium of nature. Yet, in the school system, in all levels of scholarly and professional life, ethnomathematical practices are devalued, and in most cases considered irrelevant to mathematical knowledge.

I have referred to scholarly and professional life. Let me clarify this by extending the concept of ethnomathematics to higher levels of knowledge such as physics, engineering, biology, and so on. People like Paul Dirac, in introducing the "delta function," can be identified as an ethnomathematician, and the calculus practiced by most engineers, physicists and biologists fits into what we might call ethnocalculus. Sylvanus Thompson, when writing his Calculus Made Easy in 1919, was indeed putting ethnomathematics into print. The examples at this level could be multiplied. Regretably, at the children's level this form of knowledge does not have enough strength to be noticeable to the point of publication. The school systems, which are essentially centered in curriculum and in performance to achieve it, eliminate ethnomathematics. Besides the published works of Lacey (1983), Lave (1981), and Saxe[1], recent work being conducted in several countries is beginning to appear in print.

CONCLUSION

Let me finally address the difficulty of establishing a monitoring system which will be able to measure the health of the mathematics education system in its cultural, esthetical and formative values, and for which the utilitarian value is defined as the ability to deal with "really real" situations. Cultural, esthetic and formative aspects of education need special tools for assessment. A respectful and dignifying vision of one's own being

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[1] Reference in Newsletter of the International Study Group on Ethnomathematics, Patrick J. Scott, Editor, University of New Mexico, College of Education, Albuquerque, New Mexico.

(a major issue in cultural aspects) cannot be measured directly, and formative or esthetic values are hardly weighed. But surely instruments to assess the effectiveness of such values and cultural aspects during the educational experience must be developed.

Parameters that relate to enrollment figures and possibly to classroom affective attitude will easily be developed in the monitoring system. With respect to the utilitarian values, which I see best achieved through global projects, new schemes assessing participation, involvement, and reporting must be devised. Reporting, participation and involvement seem to provide a good strategy; again, we see Foxfire (Wigginton, 1980) as a model.

The description in this paper of the desirable components of a monitoring system departs substantially from current evaluation practices and reflects an inclination toward the elimination of traditional exams, tests, and similar practices in the school systems. These forms of assessment should be replaced by others focused on individual, less competitive, growth.

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CHAPTER 8

CURRENT TRENDS IN MATHEMATICS AND FUTURE TRENDS IN MATHEMATICS EDUCATION

Peter Hilton

My intention in this talk is to study, grosso modo, the dominant trends in present-day mathematics, and to draw from this study principles that should govern the choice of content and style in the teaching of mathematics at the secondary and elementary levels. Sr . of these principles will be time-independent, in the sense that they should always have been applied to the teaching of mathematics; others will be of special application to the needs of today's and tomorrow's students and will be, in that sense, n' ,. The principles will be illustrated by examples in order to avoid the sort of frustrating vagueness that often accompanies even the most respectable recommendations (thus, "problem solving must be the focus of school mathematics in the 1980s," NCTM, 1980, p. 2).

However, before embarking on a talk intended as a contribution to the discussion of how to achieve a successful mathematical education, it would be as well to make plain what are our criteria of success. Indeed, it would be as well to be clear what we understand by successful education, since we would then be able to derive the indicated criteria by specialization.

Let us begin by agreeing that a successful education is one that conduces to a successful life. However, there is a popular, persistent and paltry view of the successful life which we must immediately repudiate. This is the view that success in life is measured by affluence and is manifested by power and influence over others. It is very relevant to my theme to recall that, when Queen Elizabeth was recently the guest of President and Mrs. Reagan in California, the "successes" who were gathered together to greet her were not Nobel prizewinners, of which California may boast remarkably many, but stars of screen and television. As the London Times described the occasion, "Queen dines with celluloid royalty." It was apparently assumed that the company of Frank Sinatra, embodying the concept of success against which I am inveighing, would be obviously preferable to that of, say, Linus Pauling.

1. The text of a talk to the Canadian Mathematics Education Study Group at the University of British Columbia in June 1983. Reprinted from For the Learning of Mathematics 4, 1 (February 1984), pp. 2-8. FLM Publishing Association, Montreal, Quebec, Canada.

The Reaganist-Sinatrist view of success contributes a real threat to the integrity of education; for education should certainly never be expected to conduce to that kind of success. At worst, this view leads to a complete distortion of the educational process; at the very least, it allies education far too closely to specific career objectives, an alliance which unfortunately has the support of many parents naturally anxious for their children's success.

We would replace the view we are rejecting by one that emphasizes the kind of activity in which an individual indulges, and the motivation for so indulging, rather than his, or her, accomplishment in that activity. The realization of the individual's potential is surely a mark of success in life. Contrasting our view with that which we are attacking, we should seek power over ourselves, not over other people; we should seek the knowledge and understanding to give us power and control over things, not people. We should want to be rich but in spiritual rather than material resources. We should want to influence people, but by the persuasive force of our argument and example, and not by the pressure we can exert by our control of their lives and, even more sinisterly, of their thoughts.

It is absolutely obvious that education can, and should, lead to a successful life, so defined. Moreover, mathematical education is a particularly significant component of such an education. This is true for two reasons. On the one hand, I would state dogmatically that mathematics is one of the human activities, like art, literature, music, or the making of good shoes, which is intrinsically worthwhile. On the other hand, mathematics is a key element in science and technology and thus vital to the understanding, control, and development of the resources of the world around us. These two aspects of mathematics, often referred to as pure mathematics and applied mathematics, should both be present in a well-balanced, successful mathematics education.

Let me end these introductory remarks by referring to a particular aspect of the understanding and control to which mathematics can contribute so much. Through our education we hope to gain knowledge. We can only be said to really know something if we know that we know it. A sound education should enable us to distinguish between what we know and what we do not know; and it i a deplorable fact that so many people today, including large numbers of pseudosuccesses but also, let us admit, many members of our own academic community, seem not to be able to make the distinction. It is of the essence of genuine mathematical education that it leads to understanding <u>and</u> skill; short cuts to the acquisition of skill, without understanding, are often favored by self-confident pundits of mathematical education, and the results of taking such short cuts are singularly unfortunate for the young traveller. The victims, even if "successful," are left precisely in the position of not knowing mathematics and not knowing they know no mathematics. For most, however, the skill evaporates or, if it does not, it becomes outdated. No real ability to apply quantitative reasoning to a changing world has been learned, and the most frequent and natural result is the behavior pattern known as "mathematics avoidance." Thus does it transpire that so many prominent citizens exhibit both mathematics avoidance and unawareness of ignorance.

This then is my case for the vital role of a sound mathematical education, and from these speculations I derive my criteria of success.

Trends in Mathematics Today

The three principal broad trends in mathematics today I would characterize as (i) variety of applications, (ii) a new unity in the mathematical sciences, and (iii) the ubiquitous presence of the computer. Of course, these are not independent phenomena, indeed they are strongly interrelated, but it is easiest to discuss them individually.

The increased variety of application shows itself in two ways. On the one hand, areas of science, hitherto remote from or even immune to mathematics, have become "infected." This is conspicuously true of the social sciences, but is also a feature of present-day theoretical biology. It is noteworthy that it is not only statistics and probability which are now applied to the social sciences and biology; we are seeing the application of fairly sophisticated areas of real analysis, linear algebra and combinatorics, to name but three parts of mathematics involved in this process.

But another contributing factor to the increased variety of applications is the conspicuous fact that areas of mathematics, hitherto regarded as impregnably pure, are now being applied. Algebraic geometry is being applied to control theory and the study of large-scale systems; combinatorics and graph theory are applied to economics; the theory of fibre bundles is applied to physics; algebraic invariant theory is applied to the study of error-correcting codes. Thus the distinction between pure and applied mathematics is seen now not to be based on content but on the attitude and motivation of the mathematician. No longer can it be argued that certain mathematical topics can safely be neglected by the student contemplating a career applying mathematics. I would go further and argue that there should not be a sharp distinction between the methods of pure and applied mathematics. Certainly such a distinction should not consist of a greater attention to rigor in the pure community, for the applied mathematician needs to understand very well the domain of validity of the methods being employed, and to be able to analyze how stable the results are and the extent to which the methods may be modified to suit new situations.

These last points gain further significance if one looks more carefully at what one means by "applying mathematics." Nobody would seriously suggest that a piece of mathematics be stigmatized as inapplicable just because it happens not yet to have been applied. Thus a fairer distinction than that between "pure" and "applied" mathematics would seem to be one between "inapplicable" and "applicable" mathematics, and our earlier remarks suggest we should take the experimental view that the intersection of inapplicable mathematics and good mathematics is probably empty. However, this view comes close to being a subjective certainty if one understands that applying mathematics is very often not a single-stage process. We wish to study a "real world" problem; we form a scientific model of the problem and then construct a mathematical model to reason about the scientific or conceptual model (Hilton & Young, 1982). However, to reason within the mathematical model, we may well feel compelled to construct a new mathematical model which embeds our original model in a more abstract conceptual context; for example, we may study a particular partial differential equation by bringing to bear a general theory of elliptic differential operators. Now the process of modeling a mathematical situation is a "purely" mathematical process, but it is apparently not confined to pure mathematics! Indeed, it may well be empirically true that it is more often found in the study of applied problems than in research in pure mathematics. Thus we see, first, that the concept of applicable mathematics needs to be broad enough to include parts of mathematics applicable to some area of mathematics which has already been applied; and, second, that the methods of pure and applied mathematics have much more in common than would be supposed by anyone listening to some of their more vociferous advocates. For our purposes now, the lessons for mathematics education to be drawn from looking at this trend in mathematics are twofold: first, the distinction between pure and applied mathematics should not be emphasized in the teaching of mathematics, and second, opportunities to present applications should be taken wherever appropriate within the mathematics curriculum.

The second trend we have identified is that of a new unification of mathematics. This is discussed at some length by the National Research Council (1970), so we will not go into detail here. We would only wish to add to the discussion by the NRC the remark that this new unification is clearly discernible within mathematical research itself. Up to ten years ago the most characteristic feature of this research was the "vertical" development of autonomous disciplines, some of which were of very recent origin. Thus the community of mathematicians was partitioned into subcommunities united by a common and rather exclusive interest in a fairly narrow area of mathematics (algebraic geometry, algebraic topology, homological algebra, category theory, commutative ring theory, real analysis, complex analysis, summability theory, set theory, etc., etc.). Indeed, some would argue that no real community of mathematicians existed, since specialists in distinct fields were barely able to communicate with each other. I do not impute any fault to the system which prevailed in this period of remarkably vigorous mathematical growth--indeed, I believe it was historically inevitable and thus "correct"--but it does appear that these autonomous disciplines are now being linked together in such a way that mathematics is being reunified. We may think of this development as "horizontal," as opposed to "vertical" growth. Examples are the use of commutative ring theory in combinatorics, the use of cohomology theory in abstract algebra, algebraic geometry, functional analysis and partial differential equations, and the use of Lie group theory in many mathematical disciplines, in relativity theory and in invariant gauge theory.

I believe that the appropriate education of a contemporary mathematician must be broad as well as deep, and that the lesson to be drawn from the trend toward a new unification of mathematics must involve a similar principle. We may so formulate it: we must break down artificial barriers between mathematical topics throughout the student's mathematical education.

The third trend to which I have drawn attention is that of the general availability of the computer and its role in actually changing the face of mathematics. The computer may eventually take over our lives; this would be a disaster. Let us assume this disaster can be avoided; in fact, let us assume further, for the purposes of this discussion at any rate, that the computer plays an entirely constructive role in our lives and in the evolution of our mathematics. What will then be the effects?

The computer is changing mathematics by bringing certain topics into greater prominence--it is even causing mathematicians to create new areas of mathematics (the theory of computational complexity, the theory of automata, mathematical cryptology). At the same time it is relieving us of certain tedious aspects of traditional mathematical activity which it executes faster and more accurately than we can. It makes it possible rapidly and painlessly to carry out numerical work, so that we may accompany our analysis of a given problem with the actual calculation of numerical examples. However, when we use the computer, we must be aware of certain risks to the validity of the solution obtained due to such features as structural instability and round-off error. The computer is especially adept at solving problems involving iterated procedures, so that the method of successive approximations (iteration theory) takes on a new prominence. On the other hand, the computer renders obsolete certain mathematical techniques which have hitherto been prominent in the curriculum--a sufficient example is furnished by the study of techniques of integration.

There is a great debate raging about the impact which the computer should have on the curriculum (Ralston & Young, 1983). Although I do not take sides in this debate, it is plain that there should be a noticeable impact, and that every topic must be examined to determine its likely usefulness in a computer age. It is also plain that no curriculum today can be regarded as complete unless it prepares the student to use the computer and to understand its mode of operation. We should include in this understanding a realization of its scope and its limitations; and we should abandon the fatuous idea, today so prevalent in educational theory and practice, that the principal purpose of mathematical education is to enable the child to become an effective computer even if deprived of all mechanical aids!

Let me elaborate this point with the following table of comparisons. On the left I list human attributes and on the right I list the contrasting attributes of a computer when used as a calculating engine. I stress this point because I must emphasize that I am not here thinking of the computer as a research tool in the study of artificial intelligence. I should also add that I am talking of contemporary human beings and contemporary computers. Computers evolve very much faster than human beings so that their characteristics may well undergo dramatic change in the span of a human lifetime. With these caveats, let me display the table.

HUMANS	COMPUTERS
Compute slowly and inaccurately.	Computer fast and accurately.
Get distracted.	Are remorseless, relentless and dedicated.
Are interested in many things at the same time.	Always concentrate and cannot be diverted.
Sometimes give up.	Are incurably stubborn.
Are often intelligent and understanding.	Are usually pedantic and rather stupid.
Have ideas and imagination, make inspired guesses, think.	Can execute "IF...ELSE" instructions.

It is an irony that we seem to teach mathematics as if our objective were to replace each human attribute in the child by the corresponding computer attribute--and this is a society nominally dedicated to the development of each human being's individual capacities. Let us agree to leave to the computer what the computer does best and to design the teaching of mathematics as a generally human activity. This apparently obvious principle has remarkably significant consequences for the design of the curriculum, the topic to which we now turn.

The Secondary School Curriculum

Let us organize this discussion around the "In and Out" principle. That is, we will list the topics which should be "In" or strongly emphasized, and the topics which should be "Out" or very much underplayed. We will also be concerned to recommend or castigate, as the case may be, certain teaching strategies and styles. We do not claim that all our recommendations are strictly contemporary, in the sense that they are responses to the current prevailing changes in mathematics and its uses; some, in particular those devoted to questions of teaching practice, are of a lasting nature and should, in my judgment, have been adopted long since.

We will present a list of "In" and "Out" items, followed by commentary. We begin with the "Out" category, since this is more likely to claim general attention; and within the "Out" category we first consider pedagogical techniques.

OUT (Secondary Level)

1. Teaching Strategies
   - Authoritarianism
   - Orthodoxy
   - Pointlessness
   - Pie-in-the-sky motivation

2. Topics
   - Tedious hand calculations
   - Complicated trigonometry
   - Learning geometrical proofs
   - Artificial "simplifications"
   - Logarithms as calculating devices.

Commentary

There should be no need to say anything further about the evils of authoritarianism and pointlessness in presenting mathematics. They disfigure so many teaching situations and are responsible for the common negative attitudes towards mathematics which regard it as unpleasant and useless. By orthodoxy we intend the magisterial attitude which regards one "answer" as correct and all others as (equally) wrong. Such an attitude has been particularly harmful in the teaching of geometry. Instead of being a wonderful source of ideas and of questions, geometry must appear to the student required to set down a proof according to rigid and immutable rules as a strange sort of theology, with prescribed responses to virtually meaningless propositions.

Pointlessness means unmotivated mathematical process. By "pie-in-the sky" motivation we refer to a form of pseudomotivation in which the student is assured that, at some unspecified future date, it will become clear why the current piece of mathematics warrants learning. Thus we find much algebra done because it will be useful in the future in studying the differential and integral calculus—just as much strange arithmetic done at the elementary level can only be justified by the student's subsequent exposure to algebra. One might perhaps also include here the habit of presenting to the student applications of the mathematics being learned which could only interest the student at a later level of maturity; obviously, if an application is to motivate a student's study of a mathematical topic, the application must be interesting.

With regard to the expendable topics, tedious hand calculations have obviously been rendered obsolete by the availability of calculators and minicomputers. To retain these appalling travesties of mathematics in the curriculum can be explained only by inertia or sadism on the part of the teacher and curriculum planner. It is important to retain the trigonometric functions (especially as functions of real variables) and their basic identities, but complicated identities should be eliminated and tedious calculations reduced to a minimum. Understanding geometric proofs is very important; inventing one's own is a splendid experience for the student; but memorizing proofs is a suitable occupation only for one contemplating a monastic life of extreme asceticism. Much time is currently taken up with the student processing a mathematical expression which came from nowhere, involving a combination of parentheses, negatives, and fractions, and reducing the expression to one more socially acceptable. This is absurd; but, of course, the student must learn how to substitute numerical values for the variables appearing in a natural mathematical expression.

Let us now turn to the positive side. Since, as our first recommendation below indicates, we are proposing an integrated approach to the curriculum, the topics we list are rather of the form of modules than full-blown courses.

IN (Secondary Level)

1. Teaching Strategies
   - An integrated approach to the curriculum, stressing the interdependence of the various parts of mathematics
   - Simple application
   - Historical references
   - Flexibility
   - Exploitation of computing availability

2. Topics
   - Geometry and algebra (e.g., linear and quadratic functions, equations and inequalities)
   - Probability and statistics
   - Approximation and estimation, scientific notation
   - Iterative procedures, successive approximation
   - Rational numbers, ratios and rates
   - Arithmetic mean and geometric mean (and harmonic mean)
   - Elementary number theory
   - Paradoxes

Commentary

With respect to teaching strategies, our most significant recommendation is the first. (I do not say it is the most important, but it is the most characteristic of the whole tenor of this article.) Mathematics is a unity, albeit a remarkably subtle one, and we must teach mathematics to stress this. It is not true, as some claim, that all good mathematics--or even all applicable mathematics--has arisen in response to the stimulus of problems coming from outside mathematics; but it is true that all good mathematics has arisen from the then existing mathematics, frequently, of course, under the impulse of a "real world" problem. Thus mathematics is an interrelated and highly articulated discipline, and we do violence to its true nature by separating it--for teaching or research purposes--into artificial watertight compartments. In particular, geometry plays a special role in the history of human thought. It represents man's (and woman's!) primary attempt to reduce the complexity of our three-dimensional ambience to one-dimensional language. It thus reflects our natural interest in the world around us, and its very existence testifies to our curiosity and our search for patterns and order in apparent chaos. We conclude that geometry is a natural conceptual framework for the formulation of questions and the presentation of results. It is not, however, in itself a method of answering questions and achieving results. This role is preeminently played by algebra. If geometry is a source of questions and algebra a means of answering them, it is plainly ridiculous to separate them. How many students have suffered through algebra courses, learning methods of solution of problems coming from nowhere? The result of such compartmentalized instruction is, frequently and reasonably, a sense of futility and of the pointlessness of mathematics itself.

The good sense of including applications and, where appropriate, references to the history of mathematics is surely self-evident. Both these recommendations could be included in a broader interpretation of the thrust toward an integrated curriculum. The qualification that the applications should be simple is intended to convey both that the applications should not involve sophisticated scientific ideas not available to the students--this is a frequent defect of traditional "applied mathematics"--and that the applications should be of actual interest to the student, and not merely important. The notion of flexibility with regard to the curriculum is inherent in an integrated approach; it is obviously inherent in the concept of good teaching. Let us admit, however, that it can only be achieved if the teacher is confident in his, or her, mastery of the mathematical content. Finally, we stress as a teaching strategy the use of the calculator, the minicomputer and, where appropriate, the computer, not only to avoid tedious calculations but also in very positive ways. Certainly we include the opportunity thus provided for doing actual numerical examples with real-life data, and the need to re-examine the emphasis we give to various topics in the light of computing availability. We mention here the matter of computer-aided instruction, but we believe that the advantages of this use of the computer depend very much on local circumstances, and are more likely to arise at the elementary level.

With regard to topics, we have already spoken about the link between geometry and algebra, a topic quite large enough to merit a separate article. The next two items must be in the curriculum simply because no member of a modern industrialized society can afford to be ignorant of these subjects, which constitute our principal day-to-day means of bringing quantitative reasoning to bear on the world around us. We point out, in addition, that approximation and estimation techniques are essential for checking and interpreting machine calculations.

It is my belief that much less attention should be paid to general results on the convergence of sequences and series, and much more on questions related to the rapidity of convergence and the stability of the limit. This applies even more to the tertiary level. However, at the secondary level, we should be emphasizing iterative procedures since these are so well adapted to computer programming. Perhaps the most important result--full of interesting applications--is that a sequence $\{x_n\}$, satisfying $x_{n+1} = ax_n + b$, converges to $b/(1 - a)$ if $|a| < 1$ and diverges if $|a| > 1$. (For one application, see Hilton & Pedersen, 1983a.) It is probable that the whole notion of proof and definition by induction should be recast in "machine" language for today's student.

The next recommendation is integrative in nature, yet it refers to a change which is long overdue. Fractions start life as parts of wholes and, at a certain stage, come to represent amounts or measurements and therefore numbers. However, they are not themselves numbers: the numbers they represent are rational numbers. Of course, one comes to speak of them as numbers, but this should only happen when one has earned the right to be sloppy by understanding the precise nature of fractions (Hilton, 1983). If rational numbers are explicitly introduced, then it becomes unnecessary to treat ratios as new and distinct quantities. Rates also may then be understood in the context of ratios and dimensional analysis. However, there is a further aspect of the notion of rate which it is important to include at the secondary level. I refer to average rate of change and, in particular, average speed. The principles of grammatical construction suggest that, in order to understand the composite term "average speed" one must understand the constituent terms "average" and "speed". This is quite false; the term "average speed" is much more elementary than either of the terms "average" or "speed" and is not, in fact, their composite. A discussion of the abstractions "average" and "speed" at the secondary level would be valuable in itself and an excellent preparation for the differential and integral calculus.

Related to the notion of average is, of course, that of arithmetic mean. I strongly urge that there be, at the secondary level, a very full discussion of the arithmetic, geometric and harmonic means and of the relations between them. The fact that the arithmetic mean of the non-negative quantities $a_1, a_2, \ldots, a_n$ is never less than their geometric mean and that equality occurs precisely when $a_1 = a_2 = \ldots = a_n$, may be used to obtain many maximum or minimum results which are traditionally treated as applications of the differential calculus of several variables.

Traditionally, Euclidean geometry has been held to justify its place in the secondary curriculum on the grounds that it teaches the student logical reasoning. This may have been true in some Platonic academy. What we can observe empirically today is that it survives in our curriculum in virtually total isolation from the rest of mathematics; that it is not pursued at the university; and that it instills, in all but the very few, not a flair for logical reasoning but distaste for geometry, a feeling of pointlessness, and a familiarity with failure. Again, it would take a separate article (at the very least) to do justice to the intricate question of the role of synthetic geometry in the curriculum. Here, I wish to propose that its hypothetical role can be assumed by a study of elementary number theory, where the axiomatic system is so much less complex than that of plane Euclidean geometry. Moreover, the integers are very "real" to the student and, potentially, fascinating. Results can be obtained by disciplined thought, in a few lines, that no high-speed computer could obtain, without the benefit of human analysis, in the student's lifetime.

$$\text{e.g., } (7^{10^6})^{12} \equiv 1 \bmod 13$$

Of course, logical reasoning should also enter into other parts of the curriculum; of course, too, synthetic proofs of geometrical propositions should continue to play a part in the teaching of geometry, but not at the expense of the principal role of geometry as a source of intuition and inspiration and as a means of interpreting and understanding algebraic expressions.

My final recommendation is also directed to the need for providing stimulus for thought. Here I understand, by a paradox, a result which conflicts with conventional thinking, not a result which is self-contradictory. A consequence of an effective mathematical education should be the inculcation of a healthy scepticism which protects the individual against the blandishments of self-serving propagandists, be they purveyors of perfumes, toothpastes, or politics. In this sense a consideration of paradoxes fully deserves to be classified as applicable mathematics! An example of a paradox would be the following: Students A and B must submit to twenty tests during the school term. Up to half term, student A had submitted to twelve tests and passed three, while student B had submitted to six tests and passed one. Thus, for the first half of the term, A's average was superior to B's. In the second half of the term, A passed all the remaining eight tests, while B passed twelve of the remaining fourteen. Thus, for the second half of the term, A's average was also superior to B's. Over the whole term, A passed eleven tests out of twenty, while B passed thirteen tests out of twenty, giving B a substantially better average than A.

The Elementary School Curriculum

This article (like the talk itself!) is already inordinately long. Thus I will permit myself to be much briefer with my commentary than in the discussion of the secondary curriculum, believing that the rationale for my recommendations will be clear in the light of the preceding discussion and the reader's own experience. I will again organize the discussion on the basis of the "In" and "Out" format, beginning with the "Out" list.

OUT (Elementary Level)

1. Teaching Strategies
   Just as for the secondary level
   Emphasis on accuracy

2. Topics
   Emphasis on hand algorithms
   Emphasis on addition, subtraction, division and the order relation with fractions
   Improper work with decimals

Commentary

The remarks about teaching strategies are, if anything, even more important at the elementary level than the secondary level. For the damage done by the adoption of objectionable teaching strategies at the elementary level is usually ineradicable, and creates the mass phenomenon of "math avoidance" so conspicuous in present-day society. On the other hand, one might optimistically hope that the student who has received an enlightened elementary mathematical education and has an understanding and an experience of what mathematics can and should be like may be better able to survive the rigors of a traditional secondary instruction if unfortunate enough to be called upon to do so, and realize that it is not the bizarre nature of mathematics itself which is responsible for his, or her, alienation from the subject as taught.

With regard to the topics, I draw attention to the primacy of multiplication as the fundamental arithmetical operation with fractions. For the notion of fractions is embedded in our language and thus leads naturally to that of a fraction of a fraction. The arithmetical operation which we perform to calculate, say, 3/5 of 1/4 we <u>define</u> to be the product of the fractions concerned. Some work should be done with the addition of elementary fractions, but only with the beginning of a fairly systematic study of elementary probability theory should addition be given much prominence. Incidentally, it is worth remarking that in the latter context, we generally have to add fractions which have the same denominator--unless we have been conditioned by prior training mindlessly to reduce any fraction which comes into our hands.

Improper work with decimals is of two kinds. First, I deplore problems of the kind 13.7 + 6.83, which invite error by misalignment. Decimals represent measurements: if two measurements are to be added, they must be in the same units, and the two measurements would have been made to the same degree of accuracy. Thus the proper problem would have been 13.70 + 6.83, and no difficulty would have been encountered. Second, I deplore problems of the kind 16.1 x 3.7, where the intended answer is 59.57. In no reasonable circumstances can an answer to two places of decimals be justified; indeed all one can say is that the answer should be between 58.58 and 60.56. Such spurious accuracy is misleading and counterproductive. It is probably encouraged by the usual algorithm given for multiplying decimals (in particular, for locating the decimal point by counting digits to the right of the decimal point); it would be far better to place the decimal point by estimation.

Again, we turn to the positive side.

IN (Elementary Level)

1. Teaching Strategies
   - As for the secondary level
   - Employment of confident, capable and enthusiastic teachers

2. Topics
   - Numbers for counting <u>and</u> measurement--the two arithmetics
   - Division as a mathematical model in various contexts.
   - Approximation and estimation
   - Averages and statistics
   - Practical, informal geometry
   - Geometry and mensuration; geometry and probability (Monte Carlo method)
   - Geometry and simple equations and inequalities
   - Negative numbers in measurement, vector addition
   - Fractions and elementary probability theory
   - Notion of finite algorithm and recursive definition (informal)

<u>Commentary</u>

Some may object to our inclusion of the teacher requirement among the "teaching strategies"--others may perhaps object to its omission at the secondary level! We find it appropriate, indeed necessary, to include this desideratum, not only to stress how absolutely essential the good teacher is to success at the elementary level, but also to indicate our disagreement with the proposition, often propounded today, that it is possible, e.g. with computer-aided instruction to design a "teacher-proof" curriculum. The good, capable teacher can never be replaced; unfortunately, certain certification procedures in the United States do not reflect the prime importance of mathematical competence in the armory of the good elementary teacher.

We close with a few brief remarks on the topics listed. It is an extraordinary triumph of human thought that the same system can be used for counting and measurement--but the two arithmetics diverge in essential respects--of course, in many problems both arithmetics are involved. Measurements are inherently imprecise, so that the arithmetic of measurement is the arithmetic of approximation. Yes, 2 + 2 = 4 in counting arithmetic; but 2 + 2 = 4 with a probability of 3/4 if we are dealing with measurement.[2]

The separation of division from its context is an appalling feature of traditional drill arithmetic. This topic has been discussed elsewhere (Hilton & Pedersen, 1983b); here let it suffice that the solution to the division problem 1000 ÷ 12 should depend on the context of the problem and not the grade of the student.

Geometry should be a thread running through the student's entire mathematical education--we have stressed this at the secondary level. Here we show how geometry and graphing can and should be linked with key parts of elementary mathematics. We recommend plenty of experience with actual materials (e.g., folding strips of paper to make regular polygons and polyhedra), but very little in the way of geometric proof. Hence we recommend practical, informal geometry, within an integrated curriculum.

We claim it is easy and natural to introduce negative numbers, and to teach the addition and subtraction of integers--motivation abounds. The multiplication of negative numbers (like the addition of fractions) can and should be postponed.

As we have said, multiplication is the primary arithmetical operation on fractions. The other operations should be dealt with in context--and probability theory provides an excellent context for the addition of fractions. It is however, not legitimate to drag a context in to give apparent justification for the inclusion, already decided on, of a given topic.

The idea of a finite algorithm, and that of a recursive definition, are central to computer programming. Such ideas will need to be clarified in the mathematics classroom, since nowhere else in the school will responsibility be taken. However, it is reasonable to hope that today's students will have become familiar with the conceptual aspects of the computer in their daily lives--unless commercial interests succeed in presenting the microcomputer as primarily the source of arcade games.

2. If AB = 2 ins., and BC = 2 ins., each to the nearest inch, then AC = 4 ins. to the nearest inch with a probability of 3/4.

But this is just one aspect of the general malaise of our contemporary society, and deserves a much more thorough treatment than we can give it here. It is time to rest my case.

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CHAPTER 9

THE EFFECTS OF A NEW COLLEGE MATHEMATICS CURRICULUM ON HIGH SCHOOL MATHEMATICS

Stephen B. Maurer

The title is too narrow by half. It suggests the influence goes all one way, with changes at the college level filtering down to the secondary school. Perhaps, in the long run, that will be true, but in the short run, to the extent that the interface between the two levels is now carefully synchronized, what is presently taught at one level constrains what changes can be made at the other. Even in the long run, if changes at the college level demand major changes for which secondary teachers are not ready because of training and philosophy, then changes at the secondary level will be very slow.

To summarize the arguments below: High schools are now teaching very little of the discrete mathematics material this conference proposes students should learn. In that sense, colleges have a free hand to teach this material in the first two undergraduate years and need not worry about what is done with this material in high schools. Every college professor has at some time thought, "I wish the high schools didn't teach calculus; the little bit the students learn just messes them up." Whether or not such thoughts are fair, there will be no cause for similar thoughts about discrete mathematics, at least in the short run. Professors can--indeed will have to--start teaching discrete mathematics from scratch. On the other hand, if the current calculus sequence is replaced by an integrated calculus and discrete sequence, the whole current advanced placement program will be thrown into disarray, and both high school students and high school teachers will be very unhappy. If calculus were replaced entirely as a subject in the first two undergraduate years, the consternation and disarray would be all the greater--but this is very unlikely.

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However, secondary teachers are aware that new sorts of mathematics and new applications have become important, and they are already hoping to include some of this in their curricula in the 1980s. In particular, computing and simple applications of elementary mathematics to social and management sciences are already widespread. This provides an opportunity for discrete mathematics to come into the secondary curriculum. To some extent, what is needed to lay the groundwork in high school for the later study of discrete mathematics (so that professors need not start entirely from scratch) is for schools to do a better job with precalculus topics; students and teachers often rush through this material to get to calculus. However, it is also vital that these topics be taught with an algorithmic viewpoint and with somewhat revised emphasis; the need for this change--even the nature of this change--is barely perceived at the secondary school level.

In the short run, then, we must be very careful about how we introduce a new college curriculum, lest we rend asunder the ties between high school and college mathematics. In the long run, however, there are excellent opportunities for making the two levels fit together at least as they do now, if changes are carefully thought out and energetically pursued.

The Short Run: How the High School Curriculum Constrains Collegiate Mathematics Change

What mathematics is taught in the high schools now? Pretty much the same things as were taught 20 years ago, when I went to high school, and 10 years ago, when I taught high school. To be sure, there have been many innovations since, but most have fallen by the wayside. The standard program today (but hardly universal) for students who continue mathematics in college is:

- 9th grade - algebra I, which goes through quadratic equations.
- 10th grade - geometry, a fairly traditional Euclidean version, mostly in two dimensions and mostly with two-column proofs.
- 11th grade - algebra II.
- 12th grade - precalculus.

What is in this precalculus course? It varies a lot, but according to Martha Zelinka of the Weston, Massachusetts, schools, it contains "everything the students should have learned before but didn't, and everything the teachers should have taught before but didn't." In addition to a lot of review, the course covers coordinate geometry, especially of conics, functions and relations; exponential, logarithmic and circular functions; perhaps vector geometry; permutations, combinations and elementary probability; sequences and series; and elementary theory of equations.

Able students generally take these courses during their first three years of high school, and take a calculus course in the senior year, usually equivalent to half a year of college calculus, but sometimes equivalent to a full year.

It is important to point out what is not included above. Algebra still consists mainly of learning certain methods of simplification which solve certain traditional problems. Little is taught about proof in algebra. In particular, the viewpoint is not emphasized enough that algebra is the means of manipulating expressions into whatever form (simpler or more complex) is needed for the desired conclusion. There is little taught about algebraic structures or mathematical induction--and if it is taught, it is taught with the same narrow set of examples used in the past. Students continue to think that induction is a special method for proving summation formulas. They are not taught that it is the fundamental method for proving just about everything in discrete mathematics.

When combinatorics is taught, it consists of traditional counting problems solvable with permutations and combinations; recursive methods, e.g., difference equations and algebraic method of generating functions, are not taught. Not surprisingly, even the notation needed to talk about general combinatorial problems easily--set notation, Sigma and Pi notation (especially indexed over sets), iterated Sigma notation, etc.--is also not taught. This may seem like a small point; indeed, most discrete mathematics textbooks seem to assume, when they start using such notation, that students can pick it up immediately or have always known it! In my experience, this is emphatically not the case. In fact, one of the greatest stumbling blocks students encounter is the inability to read and write useful notation easily; these are skills that many students do not pick up quickly. In addition probability problems are still those which can be solved by dividing one counting problem by another. Even simple stochastic models are generally not discussed in high schools. Matrix algebra is not discussed. Nor are such useful and easily begun subjects as graph theory.

To be sure, there is an alternate course to calculus which is sometimes offered in 12th grade and which includes some of the things listed above. At one time, there were high hopes for this course in terms of how it would affect the interface between high school and college, and in terms of what a mathematically educated person would know, but today it is usually billed as a course for weaker accelerated students. Consequently, it doesn't have much depth or much of the hoped-for effect.

The one thing I have left out, and which is very different from 20 years ago, is computers. Most schools have some equipment, if only a single microcomputer or a single time-sharing terminal. Many have much much more. Most able students have had time with the computer (at home if not at school) and have written, no doubt, some nice programs. At many schools, there is some effort to integrate the computer into mathematics courses, using it to compute areas under curves (in grade 9 as well as in calculus), to approximate roots of equations, and to do statistical analyses of actual, and perhaps simulated, data. More often, however, computers are used simply to provide graphics and play games, only some of which are educational. At any rate, as far as programming is concerned, computers are used mostly for numerical algorithms; they are regarded simply as big calculators. (The latter have appeared in the schools, too, but with some debate.) The more general concept of computers as universal symbol manipulation machines, and all the interesting mathematical questions arising from this viewpoint, are not touched on. This fact is abetted by the language most often used in schools, BASIC. This is a wonderful language for learning how to run your first simple program without much fuss, and for doing small to medium numerical computations, but it's poor for fostering good, structured, algorithmic thinking, or for giving one access to the most powerful programming techniques and data structures. (Note: The latest version of BASIC, BASIC 7, is a structured, recursive language, and thus overcomes most of these objections. However, BASIC 7 is currently available only at the home base, Dartmouth, and it is not clear whether it ever will spread broadly. Even if it were instantly available in the schools, it is not clear that teachers would or could avail themselves of the new features and change the way they teach programming.) In short, computers in the schools provide a teaching tool but not a mode of thought and not an object of mathematical study.

As indicated before, both students and teachers see college-preparatory high school mathematics as heading toward, and for able students culminating in, calculus. Indeed, the main source of pride in many high school mathematics departments is the number of students who yearly take and succeed in calculus. There is a standard measure of this success: performance on the College Board's Calculus Advanced Placement examinations. There are two well-defined courses, Calculus AB and Calculus BC, and corresponding three-hour AB and BC examinations, given each May. The former course is usually considered the equivalent of one college semester, the latter two. These calculus examinations, first offered in the late 1950s, were taken in 1981 by about 33,000 students, which is estimated to be greater than 60% of all high school students taking an AP (i.e., college level) calculus course, but perhaps only 30% of all those taking some sort of calculus course at the secondary level. High schools take AP courses and examinations very seriously. Most high schools with adequately trained staff pattern their calculus offering(s) after the AP curriculum. (More students might take the AP examinations except for the cost and the fact that many colleges give credit on the basis of their own placement exams during orientation, or simply on the basis of high school transcripts; also, students who don't plan to take more mathematics in college often don't participate.)

That calculus should be the culmination of high school mathematics was a fine idea when the AP program began; the mathematical world at large thought so, too. This conference is premised on the idea that calculus is no longer the sole keystone of advanced mathematics. This premise has gained wide support at colleges and universities, and there is growing enthusiasm there for curricular change. However, at the high school level, the old idea still holds sway and is reinforced by the current Advanced Placement program. This is going to make the transition to a pluralistic view much more difficult for both the colleges and the schools.

It is not necessarily high school teachers who will balk at removing calculus from its pedestal. Students--having heard from their parents or older siblings that calculus is the "real math"--may be even more conservative. Here at Swarthmore, when we introduced an experimental freshman one-semester discrete mathematics course, we first offered it in the fall. But we could lure hardly any freshmen away from calculus; mostly upperclassmen signed up. The second year, we offered the course in the spring, giving students a semester to get "tired" of calculus and giving us more time to advertise. The enrollment doubled--from 9 to 18--still not much of a draw when 250 students take calculus yearly.

We can conclude that high school calculus is not going to go away over night. Even if the Advanced Placement examinations in calculus were terminated suddenly, this would only raise a big howl. Besides, they aren't going to be terminated, and shouldn't be. After all, in the physical sciences, calculus is still the keystone.

Therefore, even if discrete mathematics is introduced at the freshman-sophomore college level, even as part of an integrated sequence, in the short run many of the best students will continue to enter college having taken calculus in high school. These students will require at least second and third semester courses of regular calculus at college. If these courses are not available, we penalize our best students by making them repeat. And if these students finish calculus separately from discrete mathematics, then they must take discrete mathematics separately from calculus. Finally, if a sequence of parallel continuous and discrete courses must be given for these students, why also offer an integrated sequence, except perhaps as an experiment in a few schools to measure possible value for the long run?

The Long Run

One can hope that, in the long run, secondary education will respond to changes at the collegiate level rather than constraining such changes. Is this a reasonable hope in the case of discrete mathematics? I think so. First of all, secondary schools are always sensitive to what colleges want, or at least to what they require. Second, there is already the realization at the secondary level that changes of the sort this conference has in mind may be in order. There have been strong calls for action by secondary mathematics leaders. On the other hand, none of these calls speaks precisely to the philosophy of the current conference.

Specifically, the study of algorithms is not seen as the central glue of the called-for changes. Disseminating this philosophy of algorithmic centrality to the secondary level is the major additional step which must take place if the secondary curriculum and the proposed new collegiate curriculum are to dovetail well.

Current Proposals for Secondary Change. First, the National Council of Teachers of Mathematics has issued An Agenda for Action (1980), a report announcing recommendations for the 1980s. Of its eight summary recommendations, three bear on the goals of this conference.

1. "Problem solving be the focus of school mathematics in the 1980s."

Here, problem solving refers to much more than the traditional textbook problems. In particular, more realistic applications, involving a broader variety of methods, and applying to additional fields such as social sciences and business, are intended. So are problems which require computing devices as aids for solutions.

3. "Mathematics programs take full advantage of the power of calculators and computers at all grade levels."

The sort of uses the report mentions specifically are analysis of data, simulations, and use as an interactive aid in the exploration for patterns. It is also suggested that all students become computer-literate citizens and that computer-aided instruction can be helpful but cannot replace student-teacher and student-student interactions.

6. "More mathematics study be required for all students and a flexible curriculum with a greater range of options be designed to accommodate the diverse needs of the student population."

One of the subrecommendations to this is 6.3: "Mathematics educators and college mathematicians should reevaluate the role of calculus in the differentiated mathematics program." Specifically, the report refers to the similar line of thinking stated in the MAA's PRIME 80 conference report (1978). Recommendation II.1 of the MAA report says: "The MAA should undertake to describe and make recommendations on an alternative to the traditional algebra-calculus sequence as the starting point for college mathematics."

It should be noted that NCTM has also done a survey of mathematics teachers and educational administrators, the PRISM Project (1981), to see what current and proposed mathematical activities are supported by these groups. In general, there is very strong support for the recommendations listed above. That is, it is not simply that leaders wish these changes; teachers and local school systems seem receptive. (On the other hand, declining enrollment will make number 6 above, at least, hard to implement.)

Second, the College Board, through its in-progress Project EQuality (1982), is attempting to bring about in college preparatory secondary education a rededication to high standards in six central areas, mathematics being one. Both E and Q are capitalized to emphasize that both equality of opportunity and quality of offerings are project goals. For mathematics specifically, an internal College Board report (currently undergoing extensive review) calls for an increase in both the minimum and the desired amount of mathematics students learn in high school. Students should learn about "applications and problem solving," the "language, notation and logical structure" of mathematics, and "computers and statistics," as well as most of the traditional "algebra, geometry and functions." Under logical structure is listed "appropriate experiences in pattern recognition, algorithm development, and inductive reasoning." Under computers are listed both computer literacy and computer programming ability.

Third, the College Board (n.d.) started a new Advanced Placement Examination in Computer Science given for the first time in May 1984. This decision was made after several years of discussion about what sort of new mathematics-related AP, if any, to give. The proposed syllabus for the associated course is quite ambitious. It will cover an "honest" full-year introductory course in computer science, as presently given at many universities, rather than the courses or experiences in computing which most high schools now make available to their students, and which used to be what students learned in introductory university computer courses. The course will include nonnumeric as well as numeric subject matter and will emphasize ways to think and write well algorithmically, using block-structured programs and appropriate data structures. Also, the course will require Pascal, a sufficiently sophisticated language.

These are the relevant current recommendations with which I am familiar. It should be noted that some high school teachers have very strong objections to Project EQuality and to the proposed AP Computer Science course. Many question how they are to fit in all the old material and all the new material. (The report does suggest that mathematics is important enough that perhaps students should be asked to take more than one mathematics course at a time at certain points. But what will teachers of other subjects say to that? A more plausible way to make time, though perhaps one no more likely to happen, is for the earlier topics in high school mathematics to be taught in grades 5-8, where currently not much at all is happening mathematically.) As for the new AP, practically no high school gives such a course now. In fact, few teach Pascal, and many will have difficulty making it available with their existing equipment. Moreover, most high school teachers do not know Pascal, and they certainly do not know much about the difference between a computing and a computer science course, and why the change has been made at the collegiate level.

The high school reactions to the new AP course should be monitored closely. Although the teachers I have spoken to are somewhat taken aback by the proposed curriculum for the new course, they are taking it seriously because it is an AP program and are eager for their schools to prepare themselves to give such a course. (The College Board intends to provide assistance in establishing summer teacher training institutes to help prepare high school faculty to teach the course.) It will be several years before we know much about how things turn out, but a possible moral for the advocates of the discrete is: If you want to bring about a major change in what schools teach, arrange for a new AP on the material you want!

Let me summarize what I think will happen at the high school level, relative to discrete mathematics, on the basis of the broad support for the recommendations above. That is, I think these things will happen even if there are no further changes toward discrete mathematics at the college level.

At least some additional topics from discrete mathematics--including statistics, finite probability, and the language and use of computer algorithms--will slowly filter into the high school curriculum or get additional attention there. As algorithms are used more often, mathematical issues related to algorithms will seem increasingly natural. That is, the infiltration of further topics from discrete mathematics will become easier.

What Else is Needed. As stated earlier, there is a caveat to this optimism. For all the planned changes at the high school level, none are quite what this conference has in mind. What is missing is an emphasis, let alone elevation to status of central theme, on the "algorithmic way of life." While the envisioned revised secondary curriculum involves algorithms, it does not involve consciously applying an algorithmic viewpoint to one's entire mathematical training, or applying mathematical analysis to the study of algorithms. For example, I have not seen suggestions that teachers

1. describe traditional mathematics topics in algorithmic language;
2. present algorithms as a proof method, e.g., if you come up with an algorithm which stops only if it finds what you want, and you can prove it stops, then you have simultaneously proved the existence of the object and shown how to construct it;
3. discuss the correctness and efficiency of algorithms, i.e., show how to apply logic, induction and counting methods to verify algorithms and determine their complexity; or
4. present the modern precise idea of an algorithm, and some of its particular techniques such as recursion, as among the great ideas in human intellectual history.

In short, algorithmics is not seen as a major mode of thought around which to tie much of the mathematics one will teach or learn. Rather, it is seen merely as a powerful but ancillary computational tool.

If the collegiate curriculum does change to give algorithmics this central role, even the changes which will inevitably occur at the secondary level will result in a rather large gap between secondary school and college mathematics, unless the spirit of this change reaches the high schools, too. Right now, I see no signs that this spirit is even beginning to reach the schools; an awareness of such a spirit at the college level does not exist at the secondary level.

It is important to understand why there is such a gap in perceptions. The centrality of algorithmics to mathematics is still quite a new and minority viewpoint at the college and university levels. After all, most mathematicians alive today (including this writer) were brought up to believe that the quintessence of mathematical method and beauty is the existential proof. Now it is suggested we should change our esthetics: proofs involving algorithms are equally, if not more, beautiful and central. One can reject this suggestion by insisting that the aesthetic of mathematics should not, or by definition cannot, change; that is, this difference in attitude towards proof is precisely the difference in philosophy that separates mathematicians from computer scientists. I don't hold this view--I think it confines mathematics rather than liberating it--but even if one does hold it, it is irrelevant at the high school level. High schools, like most small colleges, do not have computer science departments. It is mathematics teachers who will teach computer science, statistics, operations research, etc., to whatever extent it is decided these subjects are appropriate at the secondary level. Therefore, high school teachers must be imbued with this algorithmic viewpoint, regardless of whatever regrettable steps university mathematicians may take to shield themselves from it.

In addition, topics covered in traditional secondary mathematics are not, at least on first analysis, very suitable for illustrating the algorithmic point of view. Too many questions can be answered by formulas. For instance, if secondary mathematics were less concerned with solving quadratic equations, for which there is a formula, and more concerned with finding shortest paths through networks, for which there is not, then it would be natural to introduce the idea that a problem is solved when it has a provably correct algorithm, and well solved when it has a correct algorithm of low computational complexity, and go on to study the mathematics of algorithm verification and analysis. But the shortest-path problem is not as central as the quadratic equation. Neither is any of the many other examples I can think of with algorithmic but not formulaic solutions. One still has to become facile at elementary algebra before one can do much else, and for most students absorbing even elementary algebra takes <u>lots</u> of time.

The reasons for this gap make it clear that the retraining problem for high school teachers is of paramount importance. If this were a period when a flood of new, young teachers were entering the profession, already versed in algorithmics from their just-completed college or university training, the problem might not be severe. But, of course, exactly the opposite is the case. Fortunately for me, this thorny issue of retraining is the topic of another paper for this conference.

The comment above that students need time to absorb algebra illustrates another thorny issue already mentioned earlier: there isn't much slack time to play with in the secondary curriculum. If calculus were removed as the finale of high school mathematics, that would still leave open only the senior year, and only for the honors students. If teaching were somehow improved (and students also improved!) so that all precalculus material was learned in the two years of algebra and one of geometry (recall the quote early in the paper), that would leave another year, but I see little hope of such an "improvement." Consequently, the only way to get the algorithmic view into the high school curriculum is by very careful planning to interlace this new viewpoint with the old material.

Answers to questions schools haven't yet asked. (1) What might a new advanced placement mathematics course include whether it is offered instead of, or in addition to, calculus? (2) Given that algorithmic mathematics seems rather foreign to current secondary mathematics, but that planned changes may lead to a change of perception, in what ways might parts of discrete mathematics and "prediscrete" mathematics be put into the earlier years of school mathematics?

I regard the second question as more important, because if collegiate mathematics changes, schools will derive their own answers to the first question in any event. But interweaving preparatory discrete material into the earlier years is harder, and no significant changes along these lines will occur unless strong guidance is given.

As for the second question, the most important point is that the viewpoint expressed earlier, i.e., algorithms aren't needed for secondary mathematics because questions on this level are solved by formulas, is misleading. An explicit algorithmic approach has not been used at that level, but it could be. Take that quadratic formula. If you want a computer to evaluate it for you, you've got to write a program with a few branches, at least if you want to distinguish between single and multiple root cases, and if you have a machine which balks (like most) on being asked to take square roots of negative numbers. Granted, the complexity of this program is not such as to make computer scientists salivate, but we are talking about ninth graders without much computer experience. More important, even if you don't have your computer solve the quadratic for you, but prefer to use your hand calculator or even hand calculations, you can still think about the formula as a summary for an algorithm. Indeed, the same can be said of many procedures in secondary mathematics: solving triangles, graphing equations, solving simultaneous linear equations in two or three variables.

Again, an explicitly algorithmic approach to such topics in vacuo is unnecessary and maybe unnatural, but an algorithmic approach is necessary later on, in problems where formulas are not available. Why not start looking at problems this way sooner? At present, the change in approach a student confronts when he first reaches problems without formulas is more of a shock than it need be.

Here are a few more examples of topics which could easily be approached algorithmically. The best is Horner's method for evaluating polynomials; it is still taught in some places, if only in its equivalent form of synthetic division. There isn't any general formula for this in ordinary mathematical language (without using dot-dot-dot in a confusing way); all the books explain it by example. But it has a very simple formulation as an algorithm. Also, it's clear that it works faster than "direct" evaluation, but it is not immediately obvious why its output is the value of the function. In short, this is one place in secondary mathematics where the issue of verification and analysis of complexity seem natural and simple. Indeed, the whole subject of polynomials cries out for an algorithmic approach--to division, factoring, graphing, finding rational roots, approximating real roots, even approximating complex roots (an active research subject, but some basic algorithms could be studied by bright high school students).

Of course, in one sense, all of high school algebra is the study of solution "algorithms." I use quotes because the methods we use to solve harder algebra problems are not all that well understood. We are just now getting "smart" symbolic manipulation software packages which seem to have taken algebra II and gotten at least a B. Getting students to formalize their algorithms for more than simple parts of algebra is too hard. But formalizing easier parts is well worth doing. It might even be worthwhile to go back and program the arithmetic algorithms of elementary school, e.g., multiplication and division of numbers in base 10 representation. The fact that such algorithms are taken for granted makes them no less brilliant or illustrative.

Another good topic for an algorithmic approach is sequences and series, which at present means arithmetic and geometric series. Viewed algorithmically with recursive definitions, it is obvious that these are but special cases of linear recursions, and one might get much farther in the study of sequences in high school if one takes this recursive approach.

Of course, to the extent that computing is made an adjunct to the regular mathematics courses, there are lots of activities new to high school mathematics classes which can be introduced and which use algorithmic thinking. For instance, one can attack previously too difficult probability problems by number crunching or simulation, or look for patterns in elementary number theory, or give consumer math some content at last by doing realistically complicated financial programs. But I am concentrating here on how standard topics can be viewed differently, not on how new applications can be added. Besides, examples of new applications with computers have been discussed for some time; see for instance CBMS (1972).

To the extent that the training in computer use in schools becomes substantial, both in terms of methodological sophistication and size of problems handled, the mathematics of algorithms will become more natural. For instance, if recursive programming is introduced, it will not be obvious how many steps the programs take. Difference equations can come to the rescue and, incidentally, show that a recursive program is often longer to run than a corresponding longer-to-write iterative program. Another example: If students are told to alphabetize a long list of names using the computer, they may discover as they wait for their output that the usual first method, bubble sort, is not such a good idea; after that wait they may well be eager to see a mathematical analysis of the running time. Finally, if one learns algorithms for which it is unclear whether they work at all, verification becomes an issue. Fairly simple algorithms which come to mind in this category are: Euclid's algorithm; certain base change algorithms, especially for numbers with decimal parts; algorithms for certain solitaire games; the Gale-Shapley marriage algorithm; various graph-search algorithms; and algorithms for finding random combinatorial objects, say a permutation. But these rather quickly go far afield from concerns in school mathematics.

Finally, there are topics already taught in high school, but in light of their increased importance in algorithmics, they are not taught enough. Induction and counting stand out. They could be given increased emphasis, with examples drawn from algorithmic problems.

If the sorts of changes described above were made in school mathematics, then a college level course in discrete mathematics could proceed much more quickly than at present. My personal experience in teaching an algorithmic discrete course at the freshman-sophomore level is that I must spend a lot of time <u>introducing</u> induction, counting methods and algorithmic thinking, not to mention basic notation.

Now for the first question about advanced placement. If discrete mathematics at the introductory undergraduate level is introduced and maintained as a separate course from calculus, schools could offer both as AP courses. Given the NCTM recommendation that the school curriculum become broader and more flexible, schools would probably want to offer both. But if their resources are limited and they must choose one, as many will, which should we recommend?

Until recently, I would have said discrete mathematics. Like most mathematicians, I have felt intuitively that the discrete is intrinsically simpler than the continuous. Such a view, carried to its logical conclusion, underlies the ambitious Cambridge Report proposals of 1963 (Educational Services Incorporated) on changing secondary mathematics. It was proposed therein, for example, that students should be introduced to difference equations in junior high school, and that by the end of high school they should have covered more or less a complete course in difference equations paralleling in structure and sophistication the standard college course in differential equations. Such sophistication would be attainable, presumably, because the subject matter is finite.

But if discrete mathematics is simpler, then why did the kind of algorithmic discrete material we are talking about develop so much later than calculus in mathematical history? And why do discrete mathematics students find the subject more difficult than calculus?

The answer, I think, is this: Because algorithmic discrete mathematics does not have many formulas, or even many systematic methods for solving problems, students have no choice but to resort to "pure" mathematical reasoning. In short, the mathematics of algorithms requires considerable ingenuity and mathematical maturity. (If this is true, we have conversely that tackling such mathematics may be a good, if somewhat trying, way to develop such maturity.)

It is not that doing algorithms is especially hard. Indeed, high school students seem to enjoy this a lot, and many are very good at it. But the mathematics of algorithms is in parts quite difficult. This should not be surprising. The "mathematics of calculus," i.e., the theory of functions of a real variable, is a lot harder than "calculus itself," i.e., learning the sorts of problems calculus can solve, learning to solve them, and developing an informal sense of why the methods are valid. We don't try to teach the mathematics of calculus in calculus courses anymore, but leave it to the upperclass years. This is probably wise. The mathematics of algorithms is easier than the mathematics of calculus, but it may still be too hard for high school. Yet, if that mathematics is left out of a discrete mathematics course, it seems we are left merely with computing exercises and the jumble of watered-down topics in the old finite mathematics course.

In short, if a school is to give just one AP mathematics course, I recommend calculus. This circumvents a lot of technical problems, too, such as how high school teachers are to be retrained to teach a complete discrete mathematics course (perhaps an even more difficult retraining issue than how to prepare them to teach the algorithmic viewpoint throughout their courses).

What if the integrated two-year college sequence comes into being? Should we develop a new AP course that is half calculus, half discrete mathematics? If the colleges can agree on a fixed order in which to cover topics during the two-year course, so that schools could give a course covering the first semester, and an honors course covering the first year, then schools should do so; anything which allows able students to gain time at college, without skipping some things they need and covering other things twice, is to be applauded. Whether such a fixed order can be determined remains to be seen, especially since there doesn't seem to be any natural order in which to sequence things. This seems to me yet another reason to keep the continuous and the discrete in separate courses, at both the secondary and undergraduate levels.

Additions After the Conference

I saw no need to delete anything from this article on the basis of the conference discussions, but I felt that a few additions were in order.

The algorithmic way of life. Although many other papers referred to more or less the same list of discrete mathematics topics as necessary additions to the curriculum, there was, with the exception of Wilf's (1983) paper, very little written support for the assumption I made that an algorithmic frame of mind should become pervasive. There was very little mention that just doing algorithms isn't enough, but that thinking mathematically in terms of algorithms should be the glue. Also, there should be a new esthetic in which algorithmic proofs are better than existential proofs, and the general circle of ideas surrounding induction and recursion are among the great ideas an educated person should know.

However, my viewpoint was very definitely supported verbally at the conference. It is also stated explictly in the reports of the curriculum workshops which took place on the last day. These reports speak not only of topics but of overall themes, and framing one's mathematical thinking algorithmically is one of these themes.

Retraining. As emphasized before, taking such a pervasive algorithmic viewpoint in a new curriculum will make the retraining issue much thornier--even at the collegiate level. To be sure, mathematics professors already know the discrete mathematics topics proposed for inclusion, or could easily learn them, but to learn a new attitude is much harder. And for almost all mathematicians it will be a new attitude.

Moreover, if the retraining is going to be hard at the collegiate level, imagine the difficulty at the secondary level! This was brought home to me clearly when several of the very fine secondary teachers to whom I sent this paper told me that, when I got to talking about the algorithmic point of view, they really didn't know what I was talking about. In fact (as Dick Anderson pointed out at the conference), to many teachers the phrase "emphasizing algorithms" means "back to basics"--drilling in the classical numerical and algebraic manipulative skills so that students can do them without thinking, which usually results in students doing them mechanically rather than with understanding. Also not clear to my high school readers was the difference between a computing and a computer science course. For instance, they were shocked to hear that computer scientists find something lacking in BASIC. Also, some of them asked me what Bubble Sort is; since sorting is an archtype problem for computer science issues, lack of familiarity with the Bubble algorithm (at least by that name) suggests lack of familiarity with these issues.

To be sure, I was writing for college and university mathematicians rather than secondary teachers, and I was aware that more detail would be necessary for the latter group. For instance, I didn't include any specific examples of what I mean by using algorithms to prove theorems, figuring my audience could supply several. (Here are the bare bones of one example. Consider the Euler circuit theorem, that a connected graph with an even number of edges incident at each vertex can have all its edges traversed in one continuous circuit. Avoid the standard proof by contradiction, which assumes the conclusion is false and considers a minimum counterexample G, and in which one shows that G would contain a cycle C and that G-C would be a smaller counterexample. Rather, give a precise statement of the "follow your nose" algorithm which finds a cycle in any graph G meeting the hypothesis, prove that it finds a cycle, and prove that after repeating this algorithm several times, the set of edges left in G must be empty. Finally, state an algorithm for merging these cycles, and show that the merger is a continuous circuit, as desired.) Even if I had stated several of my examples in detail, they are not about theorems well known to high school teachers. Obviously even to introduce the issues of this new curriculum clearly to high school teachers will require a very different article than this one.

I had though that the issue of retraining high school teachers would be discussed in the Weissglass (1983) paper, but all I knew then about the paper was the title, and I misinterpreted it. His paper was about how a new curriculum would be a boon in perspective to <u>future</u> teachers, whatever specific topics they ended up teaching, not about how to retrain <u>current</u> teachers to teach a new perspective. So let me say a little about this issue now.

First, it really is a retraining issue. We can't expect there will be many new secondary teachers in the next several years. Second, there is an optimistic viewpoint that suggests current teachers will revise their perspectives anyway, but I don't believe it. This optimistic viewpoint argues that because computers are already in the schools and will continue to play a larger and larger role there, the algorithmic point of view will naturally insinuate itself in time. Doing a lot of algorithms eventually will cause teachers to think about, and then to teach, the mathematics of algorithms. I don't believe it. If this were a natural and easy development, why hasn't it happened already, pervasively, on the university level, where computers and large-scale problems which must be solved algorithmically have been around longer and where faculty, in order to do research, keep abreast of changes? No, I am a pessimist on this: A lot of retraining of teachers will be necessary, and some of professors, too.

How will this retraining be funded, in light of the current retrenchment? Fortunately, school systems and even teachers themselves sometimes are willing to pay part of the cost of, say, summer institutes. The teachers are willing to pay because their salaries increase if they take further courses or get advanced degrees. Usually the college or university running the institute is responsible for part of the costs, but such institutions can only meet such costs from outside grants. There has been a lot of discussion lately, at educators' conferences and in the press, about the dire state of math and science education in the United States. We can only hope that through concerted efforts such discussions will be translated into action at the highest levels, i.e., new government and foundation funding.

Is there room in the high school curriculum for changes? In the body of this paper I cautioned about thinking there was much room. I argued that a changed perspective on mathematics could be taught by carefully changing examples, but I warned against thinking lots of new topics could be added. Yet it has been suggested at this conference and elsewhere (see the Lucas, 1983, paper) that there can't really be significant change in mathematics education until room is made for lots more material at the high school and junior high school levels.

Wilf (1983) argues that the $19.95 symbolic manipulation hand calculator now available will allow us to eliminate most of the drill from calculus, and thus considerably condense that course. In my paper, I do mention in the high school context the currently available, moderately powerful microcomputers which can do similar manipulations, but I don't discuss whether the calculator-to-come could have a similar condensing role at that level. Much of high school mathematics deals with mastering symbolic manipulation; surely the ratio of skills to concepts is higher in high school algebra than in calculus. With that $19.95 machine, couldn't we condense high school mathematics drastically? What about with the $139.95 machine!

While I do think that the advent of such calculators will brighten the time-squeeze picture, I still caution against expecting too much. First of all, while there was considerable agreement that we should already do away with most arithmetic drill and allow students to use today's calculators instead (see Anderson's, 1983, paper). in contrast there was considerable worry about where eliminating algebraic drill would lead. Few think that skill in arithmetic is closely tied to general mathematical ability, but the relation between algebraic skill and mathematics is less clear. It may just be that skill and training in symbolic manipulation are closely tied to success as a mathematician, or scientist, or engineer, or even to being an astutely analytic businessman; if we don't "screen" students for other subjects by continuing to test whether they have become skilled at traditional algebraic manipulation, we may not pick out those who are good at other sorts of symbolic manipulation, either. Clearly, participants agreed that much more needs to be known about this issue.

Conference participants agreed that we know what arithmetic skills we still want people to have in the computer age. They should know at least how to estimate effectively. That is, they still need good number sense. But what is the equivalent "good algebraic sense"? We don't know. Possible partial answers which were tendered include the ability to sense how many solutions a system of equation should have, and the ability to know what form an algebraic expression would best be worked into in order to draw from it easily whatever information is needed. But until we have a better idea of what algebraic sense people should have, we should be careful about throwing out the algebraic training we give now.

Let me say a bit more about this matter of converting algebraic expressions from one form to another. In the body of my paper, I say that schools spend too much time solving equations and not enough in more general manipulations from one form to another. It is the former skill, not the latter, which we can expect the Wilf calculators to perform. True, there is no reason to suppose that calculators won't eventually be able to do most of the algebraic rewriting that people have ever found useful, but it seems to me this is a lot farther off. Furthermore, the user still will have to tell the calculator what sort of rewriting he or she wants--at least if the calculator has not produced an appropriate rewrite on its own. In short, truly effective algebraic manipulation with calculators will have to be interactive.

This conclusion emphasizes for me the importance of continuing to give students thorough algebraic training. Without having had much training in arithmetic calculation, one can still know what sort of beast one wants when one has an arithmetic problem. But one can't know what sort of algebraic beast one wants without considerable training in algebraic calculation. Furthermore, as I point out in the paper, learning any mathematical skill well takes most students, even bright ones, a long time.

The advent of the Wilf calculator does, I think present a wonderful opportunity nonetheless. It will naturally guide schools to spending more time on algebraic manipulation between forms, rather than in just getting the form x = some number.

One more point related to this issue of algebraic manipulation. It has been suggested to me that in this paper I continue the outworn tradition of giving the quadratic equation too much prominence. But I was quite careful. I did not say the quadratic equation was of crucial importance. I only said it was more important than the shortest path problem--a modest claim, I think. But let me be more specific about how I think the quadratic equation continues to be important.

I agree that it is wrong today to make students spend lots of time solving the canonical quadratic over and over again--by factoring, by completing the square, and by the formula. But in fact, most teachers I know of don't make their students do a lot of this. The students spend much more time reducing some equation or equations in other forms to canonical quadratics, or taking a word problem and finding it is a quadratic, or maybe even doing some more amorphous modeling problems which lead to quadratics. Now, of these three activities, the Wilf calculator may well do most of the first (in addition, of course, to solving the quadratic), but it will not do the last two. Yet they are important, especially the modeling with quadratics (which admittedly, like modeling in general, is not done enough in the schools). Unless secondary students get experience with linear and quadratic modeling, the only modeling experience they will get in school is modeling by computer simulation. One might argue that this is good, that almost all sufficiently accurate models are too complicated to handle analytically, and so the sooner students switch to computer modeling, the better; in particular, one might claim that linear and quadratic models are almost always too crude to be useful.

Even if Newtonian physics didn't exist, I would reject this argument. With computer modeling only, one is apt to miss the forest for the trees. Because one is not forced to think through to conclusions with computer models, one is likely to miss patterns even when they are there.

This was recently brought home to me in the course of work I am doing with an economist colleague. He brought me a certain model of economic dynamics. He had already successfully translated it from concepts into a set of differential equations. However, he had been unable to solve these analytically, either explicitly or qualitatively, and so he had already run many computer simulations. In particular, he wanted to know if a certain variable always changed monotonically. He believed it would, and it did in all his simulations, but he wanted to know if I could prove it.

After thinking about it for some time, it seemed to me (without yet a rigorous argument) that his belief was wrong, that if the initial values were related in a certain way, the variable of concern would not behave monotonically. I told my colleague this. He doubted me but was willing to run his program again with the initial conditions I proposed. The next day he phoned me to say that, upon looking at his previous reams of data again, he found he already had a case with that sort of initial conditions, and sure enough, the variable already had been non-monotonic--he had just not noticed it previously!

The story has a happy ending for my colleague. I went on to prove analytically that monotonicity wouldn't always occur, but in so doing I determined when it would occur, and this happened to include all the situations in which he was really interested. But it's what happened in the middle that is important here. One cannot live by computer modeling alone--at least not without living dangerously! And if we don't teach some analytic modeling in high schools, most students will never get this message. Analytic modeling at that level will necessarily involve linear and quadratic models. Key properties of such models often hinge on the roots of a quadratic. Ergo, the quadratic equation is still needed!

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CHAPTER 10

A COMMON CURRICULUM FOR MATHEMATICS

Thomas A. Romberg

Deliberate teaching requires choices as to what to teach.
Herbert Kliebard
Curriculum Inquiry 6 (1977)

Some mathematics should be taught to students, but an adequate presentation of a "common curriculum" for mathematics cannot consist of a list of topics to be covered, however extensive and carefully prepared. I use the word "curriculum" as a course of study, its contents, and its organization, and my task in this chapter is to consider four questions which shape an outline for a common curriculum for mathematics. The questions to be examined are:

1. What does it mean to know mathematics?
2. Who decides on the mathematical tasks for students and for what reasons?
3. What should be the principles from which a common curriculum can be built?

WHAT DOES IT MEAN TO KNOW MATHEMATICS?

This question is not easily answered. When nonmathematicians, such as sociologists, psychologists, and even curriculum developers look at mathematics, what they often see is a static and bounded discipline. This is perhaps a reflection of the mathematics they studied in school or college rather than a sure insight into the discipline itself. John Dewey's (1916) distinction between "knowledge" and "the record of knowledge" may clarify this point. For many, "to know" means to identify the artifacts of a discipline

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(its record). For me and many others, "to know" mathematics is "to do" mathematics.

Mathematics viewed as a "record of knowledge" has grown to be a stupendous amount of subject matter. The largest branch builds on what collectively is called the real number system, which includes the ordinary whole numbers, fractions, and the irrational numbers. Arithmetic, algebra, elementary functions, the calculus, differential equations, and other subjects that follow the calculus are all developments of the real number system. Similarly, projective geometry and the several non-Euclidean geometries are branches of mathematics, as are various other arithmetics and their algebras. Unfortunately, this massive "record of knowledge, independent of its place as an outcome of inquiry and a resource in further inquiry, is taken to be knowledge" (Dewey, 1916, pp. 186-187).

The distinction between knowledge and the record of knowledge is crucial. A person gathers, discovers, or creates knowledge in the course of some activity having a purpose: this active process is not the same as the absorption of the record of knowledge--the fruits of past activities. When the record of knowledge is mistakenly taken to be knowledge, the acquisition of information becomes an end in itself, and the student spends his time absorbing what other people have done, rather than having experiences of his own. The student is treated as a "piece of registering apparatus," which stores up information isolated from action and purpose (Dewey, 1916, p. 147). I do not assert that informational knowledge has no value. Information has value indeed to the extent that it is needed in the course of some activity having a purpose, and to the extent that it furthers the course of the activity. "Informational knowledge" is material that can be fallen back on as given, settled, established, assured in a doubtful situation. Clearly, the concepts and processes from some branches of mathematics should be known by all students. The emphasis of instruction, however, should be on "knowing how" rather than "knowing what," even though in my description of a common curriculum for mathematics I shall refer to some of the concepts and procedures (the "what") of mathematics.

To appreciate what it means "to do" mathematics, one must recognize that mathematicians argue among themselves about what mathematics is acceptable, what methods of proof are to be countenanced, and so forth. Doing mathematics cannot be viewed as a mechanical performance, or an activity that individuals engage in by solely following predetermined rules. In this light, mathematical activity can be seen more as embodying the elements of an art or craft than as a purely technical discipline. This is not to say that mathematicians are free to do anything that comes to mind. As in all crafts, there will be agreement, in a broad sense, about what procedures are to be followed and what is to be countenanced as acceptable work. These agreements arise from the day-to-day intercourse among mathematicians. Thus, a mathematician engages in mathematics as a member of a learned community that creates the context in which the individual mathematician works. The members of that community have a shared way of "seeing" mathematical activity. Their mutual discourse will reinforce preferred forms and a sense of appropriateness, of elegance, of acceptable conceptual structures (King & Brownell, 1966). Furthermore, the community promotes and reinforces its own standards of acceptable work, and, as Hagstrom (1965) suggests, a major characteristic of a mathematical/scientific community is the continued evolution of its standards. Not only does the range of acceptable methods vary, but in mathematics especially the standards of rigor have themselves been subject to continued modification and refinement, a point well illustrated by Bell:

> How did the master analysts of the eighteenth century--the Bernoullies, Euler, Lagrange, Laplace--contrive to get consistently right results in by far the greater part of their work in both pure and applied mathematics? What these great mathematicians mistook for valid reasoning at the very beginning of the calculus is now universally regarded as unsound. (1945, p. 153)

Nor did Bell have the last word, for during the 1970s, mathematical logicians such as Robinson (1974) and Keisler (1971) found a way to make rigorous the intuitively attractive infinitesimal calculus that was developed by Newton and Leibniz and extended by those master analysts to whom Bell refers.

Given this perspective--to know mathematics is to do mathematics within a craft--what are its essential activities? Even with a superficial knowledge about mathematics, it is easy to recognize four related activities common to all of mathematics: abstracting, inventing, proving, and applying.

Abstracting

The abstractness of mathematics is easy to see. We operate with abstract numbers without worrying about how to relate them in each case to concrete objects. In school, we study the abstract multiplication table--a table for multiplying one abstract number by another, not a number of boys by the number of apples each has, or a number of apples by the price of an apple. Similarly, in geometry we consider, for example, straight lines and not stretched threads--the concept of a geometric line being obtained by abstraction from all the properties of actual objects except their spatial form and dimensions. Thus, the basic concepts of the elementary branches of mathematics are abstractions from experience. Whole numbers and fractions were certainly suggested originally by obvious physical counterparts. But many concepts have been invented that are not closely tied to experience. Irrational numbers such as the square root of 2 were invented to represent lengths occurring in Euclidean geometry--for example, the length of the hypotenuse of a right triangle whose arms are both one unit long. The notion of a negative number, though perhaps suggested by the need to distinguish debits from credits, was nevertheless not wholly derived from experience. Mathematicians had to create an entirely new type of number to which operations such as addition, multiplication, and the like could be applied. The notion of a variable to represent the quantitative values of some changing physical phenomenon, such as temperature or time, goes beyond the mere observation of change. The farther one proceeds with the mathematics, the more remote from experience are the concepts introduced and the larger is the creative role played by mathematicians.

This process of abstracting is characteristic of each branch of mathematics. The concept of a whole number and of a geometric figure are only two of the earliest and most elementary concepts of mathematics. They have been followed by a mass of others, too numerous to describe, extending to such abstractions as complex numbers, functions, integrals, differentials, functionals, n-dimensional spaces, infinite-dimensional spaces, and so forth. These abstractions, piled as it were on one another, have reached such a degree of generalization that they have apparently lost all connection with daily life, and the "ordinary mortal" understands nothing about them beyond the mere fact that "all this is incomprehensible." In reality, of course, such is not at all the case. Although the concept of n-dimensional space is no doubt extremely abstract, it does have a completely real content, which is not difficult to understand.

Some mathematical abstractions have become so important that their absorption by students is taken as evidence of knowing mathematics. To illustrate, consider two types of abstractions: procedures and concepts. Procedural knowledge involves acquiring solution routines for a series of problems in a specific domain (for example, adding whole numbers, solving linear equations). Conceptual knowledge involves learning the labels used to name objects, relationships, procedures, and so forth (for example, "six" for the numerosity of a particular set, "parallel" for certain lines or planes). Some procedures and concepts from the record of mathematical knowledge should be learned by all students. However, they should acquire the knowledge through activities that give it meaning. The concepts and procedures should be formed under conditions where thought is necessary, rather than simply by means of routine and repetition.

But a traction is not the exclusive property of mathematics; it is characteristic of every science, even of all mental activity in general. Consequently, the abstractness of mathematical concepts does not in itself give a complete description of the peculiar character of mathematics. The abstractions of mathematics are distinguished by three features. In the first place, they deal above all else with quantitative relations and spatial forms, abstracting them from all other properties of objects. Second, they occur in a sequence of increasing degrees of abstraction, going very much further in this direction than the abstractions of other sciences. In fact, it is common for branches of mathematics to feed on each other, yielding ever more abstract notions. Finally, mathematics as such moves almost wholly in the field of abstract concepts and their interrelations. While the natural scientist turns constantly to experiment for proof of his assertions, the mathematician employs only argument.

Inventing

I have chosen "inventing" rather than "discovering" to describe this aspect of what mathematicians do even though for this chapter the distinction between the terms is not important. Discovery involves a law or relationship that already exists, but has not been perceived. Inventing involves creating a law or relationship. There are two aspects to all mathematical inventions: the conjecture (or guess) about a relationship, followed by the demonstration of the logical validity of that assertion. All mathematical ideas--even new abstractions--are inventions (like irrational numbers). Also, to assist them in the invention of their abstractions, mathematicians make constant use of theorems, mathematical models, methods, and physical analogues, and they have recourse to various completely concrete examples. These examples often serve as the actual source of the invention.

However, for students who are learning mathematics, "discovering" relationships which lead to abstractions, theorems, models, and so forth, known to the mathematical community but not to the student, can serve the same purpose. In this regard, instructional activities that require "problem solving" can give students an opportunity to experience inventing. I am hesitant to use the term "problem solving" since it has become a popular catchword in mathematics education with many meanings (Kilpatrick, 1981, p. 2). I use it here to describe instructional activities that have three implied parts: (a) a complex task is to be solved whose solution is not intended to be obvious; (b) the concepts and procedures needed to solve the task are known by the student; and (c) the "problem" is to find a strategy (or heuristic) that can be used to connect the known ideas with the unknown. Such problem-solving activities are important, for only by this means can the variety of strategies (heuristics) common to the craft of mathematics be learned.[2]

Proving

No proposition is considered as a mathematical product until it has been rigorously proved by a logical argument. If a

2. Learning to invent or to discover is not simply explaining a set of rules to be followed. It truly is an art. Readers interested in this topic should see Hadamard (1954), Polya (1957), or Wickelgren (1974).

geometer, reporting a newly invented theorem, were to demonstrate it by means of models and to confine himself to such a demonstration, no mathematician would admit that the theorem had been proved. The demand for a proof of a theorem is well known in high school geometry, but it pervades the whole of mathematics. We could measure the angles at the base of a thousand isosceles triangles with extreme accuracy, but such a procedure would never provide us with a mathematical proof of the theorem that the base angles of an isosceles triangle are congruent. Mathematics demands that this result be deduced from the fundamental concepts of geometry, which are precisely formulated in the axioms. And so it is in every case. To prove a theorem means for the mathematician to deduce it by a logical argument from the fundamental properties of the concepts related to that theorem. In this way, not only the concepts but also the methods of mathematics are abstract and theoretical.

The results of mathematics are distinguished by a high degree of logical rigor, and a mathematical argument is conducted with such scrupulousness as to make it incontestable and completely convincing to anyone who understands it. Mathematical truths are, in fact, the prototype of the completely incontestable. Not for nothing do people say "as clear as two and two are four." Here the relation "two and two are four" is introduced as the very image of the irrefutable and incontestable. But the rigor of mathematics is not absolute; it is in a process of continual development. The principles of mathematics have not congealed once and for all, but have a life of their own and may even be the subject of scientific quarrels. Furthermore, proving should not be seen as being independent of invention. As Lakatos (1976, p. 5) has argued:

> Mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations.

For example, non-Euclidean geometries were invented as a result of attempting to prove Euclid's fifth postulate.

Unfortunately, there is one problem associated with proofs in mathematics. The writings of mathematicians (recorded mathematics) often give a misleading view of their work. The process of invention is ignored in most published articles since only the proof of an assertion is usually presented.

Applying

In the final analysis, the importance of mathematics arises from the fact that its abstractions and theorems, for all their abstractness, originate in the actual world and find widely varied applications in the other sciences, in engineering, and in all the practical affairs of daily life; to realize this is a most important prerequisite for understanding mathematics. The exceptional breadth of its applications is another characteristic feature of mathematics. In the first place, we make constant use, almost every hour, in industry and in private and social life, of the most varied concepts and results of mathematics, without thinking about them at all. For example, we use arithmetic to compute our expenses or geometry to describe the floor plan of an apartment. Of course, the procedures or concepts here are very simple, but we should remember that in some period of antiquity, they represented the most advanced mathematical achievements of the age. Second, modern technology would be impossible without mathematics. Scarcely any technical process could be carried through without building an abstract mathematical model as a basis for carrying out a sequence of more or less complicated calculations; and mathematics plays a very important role in the development of new branches of technology. Finally, it is true that every science, to a greater or lesser degree, makes essential use of mathematics. The "exact sciences"--mechanics, astronomy, physics, and to a greater extent chemistry--express their laws by means of abstract mathematical formulations and make extensive use of mathematical apparatus in developing their theories. The progress of these sciences would have been completely impossible without mathematics. For this reason, the requirements of mechanics, astronomy, and phvsics have always exercised a direct and decisive influence on the development of mathematics. In other sciences, mathematics plays a smaller role, but here too, it finds important applications. Of course, in the study of such complicated phenomena as occur in biology and sociology, the mathematical method cannot play the same role as, let us say, in physics. In all cases, but especially where the phenomena are most complicated, one must bear in mind, if one is not to lose the way in meaningless play with symbols, that the application of mathematics is significant only if the concrete phenomena have already been made the subject of a profound theory. In one way or another, abstract mathematics is applied in almost every science, from mechanics to political economy.

An Example: Distance

To illustrate these four aspects of "doing" mathematics, let us examine the idea of distance (Shreider, 1974). By considering different examples of distance, it is possible to formulate concepts and procedures to solve various problems concerning the "shortest" path between two points on a surface, the geometric properties of multidimensional spaces, methods of "noise" reduction in the coding of information, and so forth.

Every child is familiar with problems of "how far" apart two or more sites happen to be. For instance, how far it is from home to school, or from home to a grandparent's house? Answers to such "how far apart?" questions inevitably vary--three blocks, two hours by car, and so forth, depending on the context. Also, there may be more than one answer to any one problem. For example, the distance by car between two points (home and grandmother's) may differ from the distance by train. Despite the differences, it is evident that all meanings taken on by the word distance have something in common. The first task for the mathematician is to abstract from the spatial-temporal facts about the world the fundamental properties of each of the different meanings for distance. There are three basic properties: (a) there exist two (or more) fixed points in space; (b) there is at least one "path" joining two points that is interesting (such as: shortest, requires the least effort); and (c) a measure of "how far apart" the two points are can be found.

The next problem for the mathematician is to invent a measuring procedure for some of the more interesting paths. One usually begins such investigations with the easiest cases and then goes to more complex cases. The simplest case for distance is between two points (say, M and N) on a plane surface. The simplest interesting path is the shortest path, which can be represented by a straight line, and the simplest measure is length. In fact, in practice then one uses an instrument (for example, a ruler) to estimate the length. However, in many situations, using a ruler is impractical, as in finding the height of a tall tree or the distance across a lake.[3] Also, since different lengths require different instruments, which have varying degrees of precision (but always involve some error), a more general procedure is needed.[4]

The next step for mathematicians is to define distance operationally in terms of a rule for the set of all points on a plane. If we characterize each point by an ordered pair of coordinates, say $M = (m_1,m_2)$ and $N = (n_1,n_2)$ then from the Pythagorean theorem we can develop the closed algebraic formula for the shortest distance (d) between M and N:

$$d(M,N) = [(m_1 - n_1)^2 + (m_2 - n_2)^2]^{\frac{1}{2}}$$

This formula then can be used as a definition of distance between any two points on a plane.

In the same manner, we can characterize the distance between two points P and Q in three dimensions in terms of an ordered triple of coordinates, $P = (p_1,p_2,p_3)$ and $Q = (q_1,q_2,q_3)$, so that:

$$d(P,Q) = [(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2]^{\frac{1}{2}}$$

3. For these cases, properties of similar triangles and angles (trigonometry) can be used to find the distances.

4. This is one distinguishing characteristic between mathematics and its applications, particularly engineering. Mathematicians are not concerned with the instruments for making the estimates; engineers are.

Although one can readily understand the properties of two or three dimensional distance, mathematicians do not hesitate to abstract beyond our temporal space to consider by analogy the distance between two points in an n-dimensional space. Thus, two points, R and S, in n-dimensions can each be represented by an ordered n-tuple of coordinates, $R = (r_1, r_2, \ldots, r_n)$ and $S = (s_1, s_2, \ldots s_n)$ so that

$$d(R,S) = [r_1 - s_1)^2 + (r_2 - s_2)^2 + \ldots + (r_n - s_n)^2]^{\frac{1}{2}}$$

Finally, let us examine distance on a different surface; for example, the shortest distance on the earth's surface between two points, such as from Chicago to London. If we consider the earth to be a sphere of radius r, then we can define the distance between two points M and N on the surface of the sphere to be the length of the smaller arc of the great circle passing through the points M and N.

Although one could continue to examine different notions of distance, at some point, mathematicians attempt to build a more abstract definition that preserves the general properties of each of the cases. Clearly, abstracting the key features from several exemplars of an idea is an important aspect of doing mathematics. For distance, a more general definition is that of a metric space.

I have tried to illustrate in this example how mathematicians build abstract systems on common notions in the real world. They develop new concepts (such as metric space), utilize other ideas (the Pythagorean theorem or the great circle of a sphere), and define new operational procedures in terms of quantitative and spatial concepts and procedures.

Inventing is illustrated in this example by the Euclidean n-space formulation of distance by analogy to the two- and three-space formulations. But mathematicians are more creative than this. For example, many interesting metric spaces on the plane arise out of a consideration of differently defined distances. One interesting class of metric spaces is obtained when one defines a metric d_p on the plane by the formula:

$$d_p(M,N) = [\,|x - x_1|^p + |y - y_1|^p]^{1/p}$$

The spaces so obtained are called Minkowski spaces. What is interesting is to change the value of p in the defining equation. For example, if p = 1, the set of points N for which $d_p(M,N) = r$ is a square with center M and diagonals of length 2r parallel to the coordinate axis. Also, if p approaches infinity, the set of points is also a square, but with sides of length 2r parallel to the axes (see figure 1).

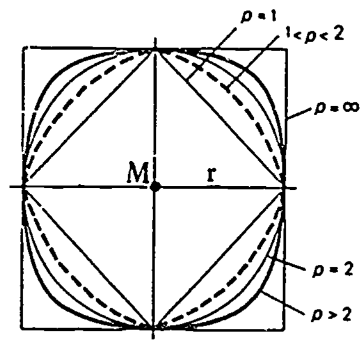

Figure 1. Minkowski spaces for different values of p. (Source: Reprinted from What is Distance, p. 23, by Yu. A. Shreider by permission of The University of Chicago Press, © 1974 The University of Chicago.)

These are only a few of the many examples of metric spaces mathematicians have invented. Yet no invention is accepted without also demonstrating the truth of its properties via a deductive argument. In the above examples, I have not proved that each metric space described fulfills the four basic properties. However, mathematicians would not accept that these spaces are metric spaces without proof. Finally, the applications of the mathematical ideas about distance should be apparent, for using different definitions, scientists and engineers have developed means of measuring distance between subatomic particles and between galaxies (whether they are conceived to exist in a Euclidean space or some other). Applications of the idea range from space technology to plotting air traffic routes to save fuel.

In summary, I believe that to know mathematics is to do mathematics: abstracting, inventing, proving, and applying are its basic activities. The challenge to teachers is to organize a course of study which provides students the opportunity to experience these activities and thus "to know" mathematics and not just to "know the record" of past mathematical activities.

WHO DECIDES ON THE MATHEMATICAL TASKS FOR STUDENTS AND FOR WHAT REASONS?

I raise this question because what is taught in schools requires deliberate choice. In the last section, I argued that the instructional activities chosen should give students experiences in abstracting, inventing, proving, and applying. But what is now taught in schools (or was taught when I was in school) bears little resemblance to this view of mathematics.

The decisions about mathematical activities in school have not been made by mathematicians. Thus, who makes the decisions? In

American schools, one could argue that which tasks are assigned to students, how much time is spent with what emphasis, what is to be judged and rewarded (or punished), and so forth, are curricular decisions individual teachers make. But, it would be naive to conclude that teachers alone make the decisions about what mathematics is taught. The fact that mathematics is taught by a teacher to a group of students leads one to expect that a teacher's pedagogical principles and practices, soundly based or not, and the constraints of a particular learning situation will shape the kind of mathematics children learn and how. Thus, while teachers may be in a position to make content choices, real or imagined constraints limit such decisions. For example, Stephens (1982) has shown that elementary teachers see themselves making decisions about "my children" and not about "your mathematics." Furthermore, one should not assume that mathematical inquiry will fit comfortably into the time slots of the conventional classroom, or that it will escape distortion altogether when adapted to fit the exigencies of a subject-based curriculum. The school has a wider social mandate than simply to teach mathematics, or any other subject: through what it teaches, the school helps to define and to legitimate what is to count as work by teachers and students, and what kinds of knowledge are to be valued above others. For many teachers, making content decisions is not seen as part of their job.

The actual decisions about what mathematics is taught and how it is interpreted are influenced by curriculum developers, school boards and administrators, publishers, and others interested in what is taught in schools. Given that there are over 16,000 separate school districts, each with a board of administration, over 500 publishers of educational materials, and so forth, there is no real answer to who influences curriculum decision making. However, the combined influences of these and other interest groups tend to perpetuate existing traditions about schooling, which in turn act to limit the choices available to teachers. As Popper (1949) has argued, the role of tradition in society is twofold: first, traditions create a certain social structure, and second, traditions are something which we can criticize and change. Curricular traditions, such as teaching a year of geometry to fifteen-year-old students, provide regularities in the social structure of schools. The mere existence of these regularities is more important than their merits or demerits. They bring order and rational predictability into the social world of schools. To illustrate curricular traditions, I will discuss three perspectives.

The Discipline Perspective

Attention in schools is fixed on subject matter. Mathematics is separated from science, grammar, and other subjects. Within each subject, ideas are selected, separated, and reformulated into a rational order. For mathematics, this selection and organization has not been made by mathematicians. Seldom do they have a voice in school mathematics. Curriculum development starts by subdividing each subject into topics, each topic into studies, each study into lessons, and each lesson into specific facts. This curricular tradition is so ingrained that most educators simply take it for granted. In fact, the modern mathematics movement of the past twenty-five years really challenged only the content aims and how content was subdivided--not the discipline tradition per se--and this chapter still reflects that tradition. Mathematics is so universally accepted as a part of school curricula that it is quite easy to see why little thought is given to its overall justification.

However, assuming the place of mathematics in school programs is justified, there are two serious problems within this discipline perspective. First, as was argued earlier, mathematics is too often viewed as "a record of knowledge." Second, many parents expect their children to have the same curricular experiences they had. Since they had to master a set of computational skills, they expect schools to teach their children the same things in the same way. However, as satirically narrated by Peddiwell (1939) in his classic The Saber-Tooth Curriculum, traditional courses such as one on "saber-tooth-tiger-scaring-with-fire" sometimes outlive their usefulness (that is, "scaring-with-fire" continues to be taught even after saber-tooth tigers have become extinct). Today's typical mathematics program is crowded with lots of "scaring-with-fire" topics. Usiskin (1980) recently did the profession a service by listing five traditional topics in algebra and geometry that should be omitted. Many other topics should also be considered for omission. There is no question that the computer (and the calculator) have made obsolete the slide rule, logarithmic approximations, statistical approximation procedures, and so forth.

The adherence to curricular stability--teaching this generation of students the concepts, procedures, and values taught to previous generations--is clearly reflected in the "back-to-basics" movement, the banning of calculators from classrooms by school administrators and teachers, and so forth. However, such a position fails to appreciate that today's students will not be working in today's world but in the twenty-first century. Some of the skills needed by productive citizens then will be quite different from those emphasized in today's school. In particular, the current technological revolution brought on by the "chip" has created a whole new set of skills all should learn.

The need to challenge instructional traditions based on this technological revolution is evident with respect to the teaching of the arithmetic of whole numbers, algebraic routines, right triangle trigonometry, and even many procedures in calculus. For example, while the concepts and procedures of arithmetic are and will continue to be central to the learning of all mathematics, the tradition has been to emphasize getting students to become proficient at a set of procedural skills--finding sums of long columns of figures, doing long division with large numbers, finding square roots, and so forth. There is no question that these skills were essential in the post-Renaissance growth of business and industry. One can almost see behind the child doing sums in today's classroom a Victorian clerk, poring over a ledger, complete with flickering candle and quill pen. Today, small machines available to anyone can do all the calculations expected of any clerk faster and more accurately (and do a lot more as well). We need to teach students how to tell tomorrow's machines what to do. This does not mean that we no longer need to teach computational skills. Understanding the concepts and procedural skills is still needed, since computers only do what they are told to do, but extensive drill on computational skills is obsolete. In summary, the discipline tradition and how mathematics has been characterized in school mathematics need to be challenged.

The Psychological Engineering Tradition

As a result of the vast research on individual differences, human development, and human learning, educational psychologists believe that their knowledge should influence how curricula are developed. Although it cannot be argued that current curricula are based on sound psychological principles, it is commonly assumed by many educators that they should be. Many psychologists believe that the teaching of concepts, meanings, or skills is usually done by a teacher or textbook writer in intuitive, unanalytic ways; thus, an improvement would certainly be made if psychological principles were used as a basis for curriculum development. For example, whether or not one believes in the details of Piagetian research, it is now commonly accepted by most educators that young children think differently from older children and adults and that learning proceeds from concrete experiences to abstractions that go beyond such experiences.

Nevertheless, the actual necessary connection between current psychological knowledge and classroom instruction is not clear for two reasons. First, most psychologists have related their theories to a very limited view of mathematics. Too often they have operationally defined mathematics in terms of performance on a standardized test, or have addressed a limited set of routine concepts or skills. Thus, they have focused only on getting right answers. Second, too often they have assumed that information derived in laboratory settings generalizes to classrooms. Unfortunately, learning in a classroom is not simply the sum of individual learning experiences. Classrooms are social groupings where the structure of many activities is dictated by a need to manage or control the group.

In summary, it has become commonplace to justify new programs in terms of psychological principles (or more likely to use the name of a noted psychologist--Piaget, Bruner, Gagne, Bloom, and so forth). The actual connection between the psychological notions and classroom instruction is in doubt.

Critical Sociology

This third perspective on curriculum views curricular knowledge as a mechanism of socioeconomic selection and control. The question of selection of content is seen as a form of the larger distribution of goods and services in society. One poses political questions such as:

> Whose knowledge is it? Why is it being taught to this particular group, in this particular way? What are its real and latent functions in the complex connections between cultural power and the control of modes of production and distribution of goods and services in an advanced industrial economy like our own? (Apple & Wexler, 1978, p. 35)

As a result, the study of educational knowledge becomes a study in ideology that seeks to investigate what is considered legitimate knowledge by specific social groups and classes, in specific institutions, and at specific historical moments.

Although most people are likely to associate the knowledge distributed by schools primarily with the knowledge incorporated into textbooks, the sociologists of school knowledge have recognized that textbooks constitute but one of the many vehicles through which information of various kinds is disseminated. In particular, social and economic control is effectuated in schools both through the forms of discipline schools have (that is, the rules and routines that ensure order, the "hidden curriculum" that reinforces norms of obedience and punctuality, and so forth) and through the forms of meaning the school distributes. For example, scheduling arithmetic after recess to quiet the students is clearly a form of social control.

Many recent studies illustrate the manner in which curriculum content relates to the interests and ideology of some particular groups, as opposed to that of others. For example, Anyon (1980) examined children's work in long division in four types of schools that were using the same mathematics text series. The schools differed in terms of social class. In the working-class school, the children's work was to follow the steps of a mechanical procedure involving rote behavior and very little decision making or choice. The teachers rarely explained why the work was being assigned, how it might connect to other assignments, or what idea lay behind the procedure to give it coherence and perhaps meaning or significance. In the middle-class school, Anyon found that the children's work was to get the right answer. If one accumulated enough right answers, one got a good grade. The child had to follow the directions in order to get the right answers, but the directions often called for some figuring, some choice, and some decision making. In the affluent professional school, Anyon found that the children's work was creative activity carried out independently. The students were continually asked to express and apply ideas and concepts. This work involved individual thought and expressiveness, the expansion and illustration of ideas, and a choice of appropriate method and material. Thus, division (finding averages) became a procedure one uses to solve problems. And finally, in the executive elite school, work was developing one's analytical, intellectual powers. The children were continually asked to reason through a problem to produce intellectual products that would be both logically sound and of top academic quality. What is important in this example from the sociology of knowledge is the importance of establishing the ideological linkages between a curriculum and the system of meanings and values of the effective, dominant culture (Williams, 1961). Working schools train workers, and executive elite schools train executives. In fact, one of the traditions which needs to be examined and challenged is how "knowing mathematics" is operationally defined in the classroom.

In summary, while teachers in fact decide on what is taught, they are influenced by a set of curricular traditions. Traditions may be explicit and recognized (like content being organized in disciplines) and others implicit and unvoiced (like the influence of social class). Either way, such traditions undoubtedly have considerable influence on what is actually taught. And, in proposing a common curriculum in mathematics, I am both aware of such traditions and challenging some of them directly.

WHAT SHOULD BE THE PRINCIPLES FROM WHICH A COMMON CURRICULUM CAN BE BUILT?

To consider this question, first recall that the work for students is defined by the instructional activities given them. Students are expected to listen, do assignments, complete homework, work alone (or in groups), take tests, and so forth. Tyler (1959) has argued that to build an effectively organized group of such activities, three major criteria should be met: continuity, sequence, and integration. To these I would add a fourth: content integrity. By this I mean that the activities given to students should give them experiences in abstracting, inventing, proving, and applying mathematics, and at the same time, the concepts and skills they learn should form the basis for a much wider range of mathematical activities.

Obviously, no student can recreate all of mathematics. Thus, only some exemplars from various branches of mathematics should be selected. Nor should the student be exposed to the multitude of mathematical concepts and procedures in a willy-nilly fashion. They should be chosen and organized in an evolutionary manner so that new concepts and procedures can build on (evolve from) other concepts and procedures. Thus, to engineer a common curriculum for mathematics meeting these four criteria, three principles should be followed.

Principle 1. Instructional Activities Should Emphasize Processes

To know mathematics as a craft means that instructional activities should require students to actively "do" something. More specifically, at least four basic sets of processes used in mathematics can be identified: relation, representation, symbolic-procedure, and validation (Romberg, 1975).

Relation processes. These are used to relate objects according to common attributes. Some important relation processes are the following. Describing is the process of characterizing an object, set, event, or representation in terms of its attributes. Classifying is the process of sorting objects, sets, or representations into equivalent classes on the basis of one or more attributes. Classifying is basic to mathematics, for it requires the student to look at how things are alike; if common attributes of things are identified, then generalizations about the class can be made. Comparing is the process of determining whether two objects, sets, events, or their representations are the same or different on specified attributes. When comparing, the student focuses on an attribute to decide whether two things are the same or different on that attribute. Ordering is the process of determining whether one of two objects, sets, events, or their representations is greater than (>), equal to (=), or less than (<) the other on a specified attribute. The process of ordering gives a background for developing the natural order of numbers. Joining is the process of putting together two objects, sets, or representations that have an attribute in common to form a single object, set, or representation with that attribute. In the process of joining, one begins with at least two objects or sets and puts them together to make one object or set. This is most often represented with a sentence such as $5 + 7 = \square$, where the unknown is the sum. However, situations may be posed where one of the two objects or sets is unknown; these situations are represented by sentences such as $5 + \square = 12$ and $\square + 7 = 12$. Separating is the process of taking apart an object, set, or representation whose parts have an attribute in common to make two objects, sets, or representations each with that attribute. Separating, as well as joining, enables the children to solve problems that they will later solve symbolically with addition and subtraction. Grouping is the process of arranging a set of objects into equal groups of a specified size with the possibility of one additional group for any leftovers. Partitioning is the process of arranging a set of objects into a specified number of equal groups with the possibility of one additional group for any leftovers. Grouping and partitioning are closely related processes. Both allow students to consider problems that will be solved by multiplication or division. In grouping, one knows how many objects are in each group, but does not know how many groups there are. In partitioning, the student knows the number of groups, so he or she deals out the objects one by one, giving each group the same amount, and then counts the number in each group. When the action has been completed, in either a grouping or partitioning situation, it is impossible to tell how it was done. Both processes are represented symbolically the same way. Both grouping and partitioning are used to convert from one unit to another within a system of measurement. For example, in changing 4 meters to centimeters, the student thinks of 4 groups of 100 centimeters or 400 centimeters. Changing 20 quarts to gallons, the students think of the problem as how many groups of 4 does it take to make 20, or $\square$ (4) = 20. Grouping is also the basis of place value. Students may group a set of objects by tens and represent it, for example as 3(10) + 7. This notation is given a special name: expanded notation. The student goes from this notation to the usual notation or compact notation, 37. Grouping is also used in connection with the addition and subtraction algorithms. Partitioning is often used to build an understanding of fractions. If a set can be partitioned without any leftovers, then each group is a fractional part of the whole set. For example, if 12 cookies are distributed to 3 people, each receives one-third of the cookies.

<u>Representation process</u>. These allow the student to progress from solving problems directly to solving them abstractly. Students begin by solving problems directly with the objects or sets involved in the problem. In solving many problems, they gradually learn to use physical representations, then pictorial representations, and finally, symbolic representations to help them. Representing not only includes going from the concrete to the abstract, but also includes going in the other direction. For example, students can represent the symbol 6 with six objects.

<u>Symbolic-procedure processes</u>. These are a great deal of what is commonly considered as mathematics. These are the procedures one uses to transform symbolic statements into equivalent statements. An <u>algorithm</u> is a finite sequence of steps one uses to close an open mathematical sentence. The common algorithms children learn (addition, subtraction, multiplication, and division of whole numbers) are examples of such procedural processes. A <u>sentential transformation</u> is a finite sequence of steps one uses to change an open mathematical sentence to an equivalent open sentence. For example: when 314 + $\square$ = 843 was changed to 843 - 314 = $\square$, a sentential transformation was made. Such transformations are efficient in problem solving since they provide the student, as in the example above, a way of changing an unworkable problem to one in which an algorithm can be used. A <u>structural transformation</u> is a finite sequence of steps one uses to change a symbolic phrase to an equivalent phrase. For example, the phrase 3(5+4) can be changed to 3 x 5 + 3 x 4 because of the structural characteristic of the distributive property for multiplication over addition with whole numbers.

<u>Validation processes</u>. These are the processes used to determine whether a proposed proportion is true. There are three basic ways of validating: authority, empiricism, and deduction. <u>Authority validation</u> is the process of determining validity by relying on some authority. For example, if a child checks his

answer to a problem by comparing it to an answer book or to the teacher's answer, he is relying on authority. Empirical validation involves representing a proposition with objects, pictures, or other symbols to assist in determining its validity. For example, suppose for the problem 9 $\square$ 6, a student puts > in the box. To determine if 9 > 6 is a valid sentence, the student should represent 9 and 6 with cubes and visually show that the 9 cubes are more than the 6 cubes. Similarly, for the problem 6 + 3 = 10, the student could represent 6 and 10 with pictures and clearly show that 3 is an invalid solution. Logical deductive validation is the process of determining validity by a deductive argument based on agreed upon common notions, definitions, axioms, and rules of logic. This process is at the heart of mathematics.

In summary, these four basic sets of processes should be considered illustrative (certainly not exhaustive) of what it means to "do" mathematics. Each specific instructional activity should expect students to use one or more mathematical processes. In addition to processes or procedural routines such as these, there is also a set of executive routines or heuristics which should be learned. Thus, while the semantics of many activities should direct a student to use a particular process, some activities need to be provided that require the student to decide on a strategy or choose between processes. For example, to learn the strategy suggested by Polya (1957), "If you cannot solve the proposed problems, try to solve first some related problem" (p. 31), students must have exposure to problems where that strategy is appropriate.

Principle 2: Instructional Activities Should Be Grouped into Curriculum Units

Tyler's (1959) notion of integration "refers to the horizontal relationship of curriculum experiences" (p. 55). One interpretation is that a set of activities should be related to each other to give meaning to the set. Unfortunately, during the past two decades, with "individualized" programs, this concern was not heeded. We were able to break mathematical learning into hundreds of specific behavioral objectives, but the problem was how to put them back together again so that students had an integrated knowledge of mathematics.

The problem is not new. Dewey (1902) argued for activities related to experience in his classic, The Child and the Curriculum, and Brownell (1947) demonstrated the efficacy of meaningful instruction nearly half a century ago. It is widely accepted that meaningful learning is better than rote learning. The difficulty lies in engineering a "meaningful" mathematics program. It is deceivingly easy to specify objectives, lay out or create a hierarchy, and engineer a rote learning program based on that framework (Gagné, 1965). The danger, as Erlwanger (1975) has shown, is that mastery of a set of objectives is no guarantee that the student can do mathematics.

One answer that is emerging from current work in several areas is that activities should be grouped into curriculum units that take two to three weeks to teach. The activities should be related to a "story shell" which provides a reason for doing each activity and makes the learning clear and meaningful.

Story shell. This is not the place to go into a lengthy discussion of the research on "story shell curriculum units." But let me trace some of the argument, beginning with cognitive psychologists who have been interested in what people remember from texts they have read (that is, the relationship between reading comprehension and memory storage). Several scholars in the 1970s (Lakoff, 1972; Mandler & Johnson, 1977; Rumelhart, 1975; Stern & Glenn, 1979; Thorndyke, 1977; van Dijk, 1977) proposed "story grammars" (sets of rewrite rules) with two objectives: first, they are grammars for real stories, and second, they are theories about the representation of stories in memory.

This line of research, although demonstrably inadequate in representing real stories, has turned out to be extremely valuable as a characterization of how to utilize planning knowledge in reading stories. In particular, Black and Bower (1980) have proposed a two-part critical path rule that predicts memory accuracy: (a) the best remembered part of a story is the critical path that provides the transition from the beginning state to the ending state of the story; (b) if the story describes the critical path at various levels of detail, then the higher (that is, the more general, less detailed) the level of statement, the better remembered it will be. What is becoming clear is that the "critical path" (or "solution episode" or "story shell") is what is remembered best.

Next, relying in part on this memory research, Sternberg (1981) is currently studying learning of new vocabulary words in context (where the new words are included in a familiar story line). Such learning is clearly more successful and meaningful than the rote learning of new vocabulary words. Finally, instructional research by Good and Grouws (1981) lends tangential evidence to this argument. They asked teachers to spend more time explaining the content of each mathematics lesson and found that the students learned considerably more. The implication is that in explaining content, teachers put the ideas in context. Thus, a curriculum unit should have a story to tell posed in an episodic, problem, or "detective" format; and the story should fit into a larger tale (like a chapter in a good novel).[5]

5. There are several examples of Instructional Units that have story lines. Martin Covington and colleagues studied general creative thinking and problem solving using mystery problems following a story line. See Covington, 1968.

Also, several experimental mathematics programs created

Clear and meaningful activities. From personal experience in developing an elementary mathematics program, I am convinced that the creation or selection of activities within the story shell framework is also critical (Romberg, 1982). One could take an existing text, add a story line, expect students to read the story, and still have students answer questions on work sheets. That is not what is being proposed; instead I suggest that the following four characteristics be considered:

1. The activities should be related to how children process information. Most traditional programs fail in this regard for four reasons. First, the bulk of the developmental literature, as reviewed by Lovell (1972), suggests that curriculum units be organized around a set of activities in which (a) "the pupils work in small groups or individually at tasks which have been provided"; (b) "opportunity is provided for pupils to act on physical materials or to use games"; and (c) "social intercourse using verbal language is encouraged since it is an important influence in the development of concrete operational thought" (p. 176). Most programs are not organized around activities like these.

Second, if one takes seriously the notions of curricular spiralling, then one must review prior concepts and skills and get ready for others to be learned later. Most programs do not both "review" and "prepare." Third, the activities to teach concepts should differ from those to teach algorithms, problem-solving heuristics, and so forth. In most programs, one page looks like most other pages. Fourth, sequenced activities that require students to assimilate new information will differ from those that require accommodation. Again, in most programs, little attention is paid to students' methods of processing information in creating activities.

2. The reasons for each activity within the unit should be clear to each student. Too often the only reason a student sees an activity is that "it is the next one in the book" or "my teacher assigned it."

3. Ideally, every unit should include activities that expect students to abstract, invent, prove, and apply. Again, too much emphasis is placed on learning abstract concepts and procedures with the only rationale being "someday you will need this skill." Each story line should include problem-solving activities.

excellent topics with these characteristics, such as Stretchers and Shrinkers developed by University of Illinois Committee on School Mathematics (Hoffman, 1970), Minnemast Units developed by the Minnesota Mathematics and Science Teaching Project (Werntz, 1968), or some topics from Developing Mathematical Processes developed at the Wisconsin Research and Development Center (Romberg, Harvey, Moser, & Montgomery, 1974, 1975, 1976).

Students should invent and argue about solutions to problems and their validity. Furthermore, the problem-solving activities should not be the word problems often seen in texts (Usiskin, 1980).

4. The curriculum unit should be objective referenced, with tests (and observations) related to those objectives. Since part of the job of teaching is to judge the words and actions of students, the use of some behavioral objectives for evaluation purposes has proven to be helpful.

In summary, I believe it is possible to create curriculum units that are a collection of activities integrated around a story line that can give meaning to the concepts and procedures.

Principle 3. Curriculum Units Should Be Related Via Conceptual Strands

To develop continuity, curriculum units need to be related to each other as part of a larger story. For mathematics, one way to develop the larger story is to let history be our guide. All of elementary mathematics has been created in response to problems. Yet, few adults are aware of the story of mathematics. "No subject, when separated from its history, loses more than mathematics" (Wolfe, 1945, p. vi). The story of how elementary mathematics has developed can be told via seven strands: whole number arithmetic, spatial relations, measurement, fractions, coordinate geometry, algebra, and statistics.[6]

By strand, I mean a cognitive subdivision of mathematics that has a rich history (or story line). I have chosen to emphasize strands because, although a familiarity with number combinations and operations can develop from manual and visual experience, this of itself will take the student little beyond the level of paleolithic man. Our numeral system is the product of centuries of

6. I suspect few mathematics educators would disagree with these seven strands; many might have expected some others. Let me comment on two, problem solving and computer literacy, which I did not include. Problem solving was not included because problem-solving activities should be included in most curriculum units. Dealing with it separately would falsely separate strategies from concepts and procedures where they can be used. Problem solving should be learned from problem situations embedded in the context of the "story shell" of each instructional unit.

Computer literacy poses a more difficult problem. All students should learn to communicate with and use computers. Such instruction is the responsibility of schools but should not be considered as another part of mathematics. On the other hand, the use of computers in mathematics is important, but it needs to be considered as a tool to solve problems in many of the units within the seven strands.

mathematical exploration and invention. It possesses such deceptive simplicity that it can be mechanically mastered with no reference to its history. And therein lies the danger. Because children are capable of insight and are not disposed to accept arbitrariness unquestioningly, the deadliness of uninformed teaching is immediate. Our decimal system, our rules of calculation, our arithmetical and geometric terminology--all seem natural and inevitable to anyone who has successfully mastered them by mechanical methods. To see them for what they are--a blend of logic, history, and convention--seems not only difficult, but unnecessary; to show them as such to children, almost impossible. Yet it is not only possible, but essential if the foundations of mathematical understanding are to be well and truly laid.

The <u>whole numbers arithmetic strand</u> includes counting, additive structures, and multiplicative structures. "Counting" means the assignment of numbers to sets of objects (finding out how many) and includes learning the terminology of the Hindu-Arabic numeration system. Learning "additive structures" means learning to write addition and subtraction sentences to represent certain concrete numerical situations and learning the procedural rules for addition and subtraction (Vergnaud, 1981). Similarly, learning "multiplicative structures" means learning to write multiplication and division sentences to represent certain concrete situations and learning the procedural rules for multiplication and division.

The story of how the concepts and procedures of arithmetic were developed is central to the history of mankind. The concepts of arithmetic correspond to the quantitative relations of collections of objects. These concepts arose by way of abstraction, as a result of the analysis and generalization of an immense amount of practical experience. They arose gradually. First came numbers connected with concrete objects, then abstract numbers, and finally the concept of number in general. Each of these concepts was made possible by a combination of practical experience and preceding abstract concepts. Similarly, the operations on whole numbers and procedural rules for finding sums, differences, products, and quotients arose as a result of practical experience with joining, separating, grouping, and partitioning sets and looking for shortcuts for counting the sets resulting from those processes.

The importance of this strand cannot be overestimated. The concepts and procedures of arithmetic, which generalize an enormous amount of experience, reflect in abstract form relationships in the actual world that one meets constantly and everywhere. It is possible to count the objects in a room, stars, people, atoms, and so forth.

At the same time, every abstract concept--in particular, the concept of number--is limited in its significance as a result of its very abstractness. In the first place, when applied to any concrete object, it reflects only one aspect of the object and therefore gives only an incomplete picture of it. For example, mere numerical facts often say very little about the essence of a matter. In the second place, abstract concepts cannot be applied everywhere without certain limiting conditions. It is impossible to apply arithmetic to a concrete problem without first convincing ourselves that their application makes some sense in the particular case. If we speak of addition, for example, and merely unite the objects in thought, then naturally no progress has been made with the objects themselves. But suppose we apply addition to the actual uniting of the objects. We put the objects together, for example, by throwing them into a pile or setting them on a table. In this case, not merely abstract addition takes place, but also an actual process. The process is not arithmetical addition, and it may even be impossible to carry out. For example, an object thrown into a pile may break; wild animals, if placed together, may tear one another apart; and materials put together may enter into a chemical reaction: a liter of water and a liter of alcohol poured together produce not two, but 1.9 liters of mixture as a result of partial solution of the liquids.

The spatial relations strand includes the basic concepts of geometry. The story here is similar to arithmetic. Early man took over geometric forms from nature. The circle and the crescent of the moon, the smooth surface of a lake, the straightness of a ray of light or of a well-proportioned tree existed long before man himself and presented themselves constantly to his observation. In nature, our eyes seldom meet with straight lines, equilateral triangles, or squares. Clearly, the chief reason men and women gradually worked out a conception of these figures is that their observation of nature was an active one, in the sense that, to meet their practical needs, they manufactured objects that were more and more regular in shape. They built dwellings, cut stones, enclosed plots of land, stretched bowstrings in their bows, and modeled their clay pottery. In bringing these to perfection, they correspondingly formed the notion that a plot is curved, but a stretched bowstring is straight. In short, they first gave form to their material and only then recognized form as that which is impressed on material and can, therefore, be considered in itself, as an abstraction from material. By recognizing the form of bodies, humans were able to improve their handiwork and thereby to work out still more precisely that abstract notion of form. Thus, practical activity served as a basis for the abstract concepts of geometry.

Geometry operates with "geometric bodies" and figures. But a geometric body is nothing other than an actual body considered solely from the point of view of its spatial form, in abstraction from all its other properties such as density, color, and weight. A geometric figure is a still more general concept, since in this case, it is possible to abstract from spatial extension also. Thus, a surface has only two dimensions; a line, only one dimension; a point, none at all. A point is the abstract concept of the end of a segment, of a position defined to the limit of precision so that it no longer has any parts. Thus, geometry has as its object the spatial forms and relations of actual bodies, removed from their other properties and considered from the purely abstract point of view.

The self-evidence of the basic concepts of geometry, the methods of reasoning, and the certainty of their conclusions has the same source as in arithmetic. The properties of geometric concepts, like the concepts themselves, have been abstracted from the world around us. It was necessary for people to draw straight lines before they could take it as an axiom that through every two points it is possible to draw a straight line. They had to move various bodies about and apply them to one another on countless occasions before they could generalize their experience to the notion of superposition of geometric figures and make use of this notion for the proof of theorems, as is done in the well-known theorems about congruence of triangles.

The measurement strand involves learning to assign numbers to attributes of objects and then using the concepts and procedures from the numbers strand to solve problems of length, weight, or other properties involving measurement. This strand relates the concepts and processes of whole numbers to those of geometry. Whole number arithmetic begins with the notion that each separate object is a unit. A collection of discrete objects is a sum of units, which is, so to speak, the image of pure discreteness, purified of all other properties. Geometry, on the other hand, considers properties of a single homogeneous object, which in itself is not separated into parts, but which may nevertheless be divided in practice into parts as small as desired. Lengths, areas, and volumes have the same property. Although they are continuous in their very essence and are not actually divided into parts, they nevertheless offer the possibility of being divided without limit.

Here we encounter two contrasting kinds of objects: on the one hand, indivisible, separate, discrete objects; and on the other, objects that are completely divisible, not divided into parts, but continuous. We therefore have two properties--discreteness and continuity--and their abstract mathematical images: the whole number and the geometric extension. Measurement involves a blending of these ideas: the continuousness is measured by separate units.

The fractions strand involves learning to name fractional parts for certain situations (Kieren, 1977), learning the conventional symbolism for representing fractions (both common and decimal), and learning the procedural rules for operations on fractions.

Historically, the need for fractions arose out of measurement problems. In the process of measurement, the chosen unit is not ordinarily contained in the measured magnitude an integral number of times, so a simple calculation of the number of units is not sufficient. It becomes necessary to divide the unit of measurement in order to express the magnitude more accurately by parts of the unit; that is, no longer by whole numbers, but by fractions. This was the way fractions actually arose. They arose from the division and comparison of continuous geometric magnitudes. The first magnitudes named lengths, areas of land, and volumes of liquids. In the earliest appearance of fractions, we see the mutual action of arithmetic and geometry. This interaction led to fractions, as an extension of the concept of number from whole numbers to fractional numbers (or as mathematicians say, to rational numbers, expressing a ratio of whole numbers).

The coordinate geometry strand extends the interaction of arithmetic and geometry to a higher level of abstraction by developing a general procedure for the assignment of numbers to points in any space. Concepts and procedures such as naming points on a line starting at any point (the number line), going in either direction on the line (directed numbers), naming coordinates on a plane, and so forth, are to be learned in this strand.

The algebra strand incorporates the notions derived from generalized arithmetic. It deals only with mathematical operations on numbers considered from a formal point of view, in abstraction from given concrete numbers. The abstractions find expression in that magnitudes are denoted by letters on which calculations are carried out according to well-known formal rules.

Algebra, in contrast to generalized arithmetic, retains this basis, but widens it extensively. Algebra considers "magnitudes" of a much more general nature than numbers and studies operations on these "magnitudes," which are to some extent analogous in their formal properties to the ordinary operations of arithmetic: addition, subtraction, multiplication, and division. A simple example is offered by vectors, which may be added by using a parallelogram rule. The degree of generalization in contemporary algebra is such that even the term "magnitude" may lose its meaning, and one speaks more generally of "elements" on which it is possible to perform operations similar to the usual algebraic ones. For example, two motions carried out one after the other are evidently equivalent to a single motion, which is their sum; two algebraic transformations of a formula may be equivalent to a single transformation that produces the same result; and so forth. It is possible to speak of a characteristic "addition" of motions or transformations. This and more is studied in a general abstract form in contemporary algebra.

The statistics strand is a "body of mathematics of obtaining and analyzing data in order to base decisions upon them" (Wallis & Roberts, 1956). In this sense, statistical concepts and procedures are a natural bridge between real problems and mathematics. Data are often gathered to help decide questions of practical action. Statistics help decide what kind of information is needed; how to collect, tabulate and interpret it; and how judgments can be made on the basis of this information.

In summary, the fundamental concepts and procedures in these strands are the building blocks that enable any student to cope with many relevant problems in later life and work situations.

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CHAPTER 11

MATHEMATICS EDUCATION: A REALLY REAL, REAL WORLD PROBLEM
Reactions to Chapters 6-10

Herbert J. Greenberg

D'Ambrosio, in chapter 7, advocates the study of really real situations as the way for children to learn mathematics. I propose to approach the real world problem of mathematics education in the same way and to confront this problem with the solutions recommended by the preceding authors (and vice versa).

The problem admirably suits D'Ambrosio's paradigm: it is undefined, unformulated, and uncodified. Indeed, none of the authors attempt to define or formulate the problem, but all recognize its complexity, and between them, touch on most, if not all, of its aspects: curriculum, pedagogy, teachers, schools, student motivation, individual differences, discipline, publishers and tests, as well as broader sociological, economic, and political questions relating to American society. Moreover, it is not clear whether our task here is to deal with the problem of mathematics education, or with the metaproblem of monitoring mathematics education. However, the presumption seems to be that the two must, or will, go hand in hand. (Personally, I never metaproblem I didn't like.)

We have before us five papers that collectively provide both a general perspective on dominant trends in mathematics and the mathematics that is important for the schools and the specifics of content changes recommended for the next decade. Before discussing these, however, let us reflect on the second of four basic questions on the shaping of a common curriculum posed by Romberg in chapter 10, namely, Who decides on the mathematical tasks for students and for what reasons?

Romberg points out that it is not the teachers who make the decisions on what is taught, but rather traditions embodied in curricular perspectives. He enumerates three: the perspective of the discipline itself, the perspective of educational psychologists, and the perspective of sociologists. To these three, I would respectfully add a fourth: the perspective of mathematics educators.

What we find in Pollak, Hilton, and Maurer--chapters 6, 8, and 9--is the perspective of the discipline; in D'Ambrosio, we find the perspective of the sociologist; in Romberg, we find the perspective of the mathematics educator. The perspective of the educational psychologists, except incidentally, is absent--I am tempted to say, mercifully absent.

Turning to the substance of these papers, those by mathematicians Pollak, Hilton, and Maurer form the basis of Romberg's dictum that "all students should learn more and somewhat different mathematics than is in the current curriculum." The key words here are "all," "more," and "somewhat different," although the very serious implications of the word "all" are largely ignored. There is even general agreement on what topics ought to be added or deleted.

There is no need here to repeat the conclusions and recommendations contained in the preceding papers. The authors have done their jobs thoroughly and well, and the reader has already seen them. However, for purposes of comparison and criticism, it is necessary to pull together some of the central themes.

Pollak's paper is, in all respects, the most comprehensive, reporting as it does on the recommendations contained in six position papers on the fundamentals in the mathematics curriculum, that were written for a conference held by the Conference Board of the Mathematical Sciences. Recommendations made to CBMS for elementary- and middle-school mathematics emphasize the early introduction and use of calculators and computers; the development of skills associated with "number sense" such as mental arithmetic, estimation, and approximation; and the collection and analysis of statistical data. At the secondary level, the recommendations, again in considerable detail, are aimed at streamlining traditional curricular components to make room for fundamental new topics in discrete mathematics, statistics, probability, and computer science, with special emphasis on algorithmic thinking.

Pollak takes care to convey the caveats and concerns of the separate reports when he stresses the urgency of the central noncurricular problems: the need for more and better qualified mathematics teachers, the need for improvement of the total school environment, and the lack of support of the schools. In the same vein, he calls attention to key factors that will affect the implementation of any curricular changes, namely textbooks, testing, articulation, and equity considerations, including the new one of equal access to computers. He reiterates the need to "make haste slowly," and calls attention to the vast potential provided by modern computer technology.

Hilton draws from "the dominant trends in present-day mathematics . . . principles that should govern the choice of content and style in the teaching of mathematics at the secondary and elementary levels" (p. 149). From these principles, he is led to recommendations about which topics should be "in" and which should be "out" at each level.

In his choice of basic topics, Hilton's recommendations, not surprisingly, are substantially the same as those in the reports to CBMS contained in Pollak's paper. However, he adds a number of highly interesting topical ideas. For example, he suggests including elementary number theory, both because, "the integers are very 'real' to the student and, potentially, fascinating" (p. 159), and because the study of elementary number theory might be a better way to teach logical reasoning than the usual theorem proving in Euclidean geometry.

Other novel, noteworthy, and strongly mathematical recommendations put forth by Hilton for the secondary curriculum are concerned with rates and the rational numbers, averages, interactive procedures, and paradoxes to stimulate thought. Clearly, Hilton has in mind both our responsibility to the more mathematically able students, and our future dependence on them. To that, I say, "Bravo!"

In addition to his curricular recommendations, Hilton supplies us with separate recommendations pertaining to teaching styles and strategies that call for, "an integrated approach to the curriculum, stressing the interdependence of the various parts of mathematics" (p. 156).

Hilton, like the others, calls attention to the need for "confident, capable, and enthusiastic teachers," and believes that the computer should have a "noticeable impact" on the curriculum.

Maurer directs his attention to the place of discrete mathematics in the secondary school curriculum. Specifically, he identifies "algorithms is not seen as a major mode of thought around which to tie much of the mathematics one will teach or learn" (pp. 172-173). Maurer deplores the fact that computers in the schools are "simply regarded as big calculators" and that, "all the interesting mathematical questions . . . are not touched on," a situation that he ascribes in part to the teaching and use of BASIC, a language that is, "... poor for fostering good, structured, algorithmic thinking" (p. 168).

Recognizing, however, that even very fine secondary teachers don't know what is meant by <u>algorithmic thinking</u>, let alone how to introduce it in their classes, Maurer provides a considerable service by being very explicit about what the algorithmic approach is, and how it can be introduced and used in the high schools. I found this discussion very valuable.

Equally valuable are the frank and blunt conclusions Maurer reaches on the difficulty of discrete mathematics, the undesirability of integrating discrete and continuous mathematics into a single course, the value of traditional topics and the consequent lack of room for change in the high school curriculum, and the problem of retraining high school teachers. All but the last of these conclusions, to a large extent, contradict conventional wisdom and the usual utterances and are all the more convincing coming from one who is an advocate of change.

My own observations as an instructor at an institution attempting an integrated discrete mathematics and calculus freshman course bear out Maurer's conclusions. Our assumption also was that "the discrete is inherently simpler than the continuous." But my classroom experience, and student reactions and evaluations, clearly contradicted this assumption.

As Maurer puts it, "it is not that doing algorithms is especially hard. . . . But the mathematics of algorithms is in parts quite difficult" (p. 177). One is then forced to conclude that "the mathematics of algorithms is easier than the mathematics of calculus, but it may still be too hard for high school" (p. 177).

On the prospect of condensing or eliminating topics in high school algebra to make room for new topics, Maurer suggests that that which is desirable at the elementary level, vis a vis arithmetic skills and drill, may not be desirable in the teaching of algebra: "We know what arithmetic skills we still want people to have in the computer age. They should know at least how to estimate effectively. That is, they still need good number sense. But what is the equivalent 'good algebraic sense'? We don't know" (p. 181). Maurer speculates that, "it may just be that skill and training in symbolic manipulation are closely tied to success as a mathematician, or scientist, or engineer, or even to being an astutely analytic businessman" (p. 180). Accordingly, Maurer stresses, "the importance of continuing to give students thorough algebraic training" (p. 181), which tells us, when combined with other mathematical requirements and expectations, that "there isn't much slack time to play with in the secondary curriculum."

Leaving now the perspective of the mathematicians, I wish to turn to the perspective of the sociologist in the paper by D'Ambrosio. The questions he raises are truly vital ones, as distinct from disciplinary ones, and serve to remind us of the really real, in our real world problem of mathematics education.

D'Ambrosio says that mathematics education aims to provide "equal opportunity for all. At the same time, it helps to prepare our young for the future advancement and betterment of the socioeconomic and political framework of society" (p. 135).

He states that "mathematics is deeply rooted in our cultural systems and thus, is loaded with values" (p. 136). It is hard to deny his assertion that, "mathematics promotes a model of power through knowledge" and that "mathematics teachers indoctrinate students to believe that people and institutions arrange themselves in hierarchies of power according to their mathematical ability" (p. 136). Think of our fond classroom claims for the "power" of mathematics, and think, too, of the hierarchy implicit in the way we extoll the mathematical achievements of school children in certain countries.

D'Ambrosio reminds us that, in addition to the utilitarian reasons for teaching mathematics, there are formative, cultural, and aesthetic reasons, and that these latter reasons have been set aside and largely forgotten for a long, long time. Worse than that, he points out that the utilitarian has come to mean merely a traditional skills-oriented mathematics that is obsolete and inefficient and pays "lip service to a new emphasis on applications to real world problems" (p. 139).

D'Ambrosio believes that "an authentic approach must go into a different direction," and advocates the study of what he calls "'really real' situations," (p. 139) and the "effective immersion of children in global practices" (p. 140). To accomplish this, D'Ambrosio recommends "an open, activities-oriented approach to mathematics education, which draws on the environment, thus relying on previous knowledge. This leads to what I have labeled ethnomathematics, which restores mathematics as a natural, somewhat spontaneous, practice" (p. 141).

D'Ambrosio raises still another set of issues "that refers to differences in exposure to mathematics by race, by social class, and by sex, and investigates how these differences are reflected in the level of performance, attitudes, enrollment, and use of mathematics" (p. 142). He believes these issues, too, can be addressed by the approach he recommends.

In summary, D'Ambrosio sees immersion in the classroom in "really real" or global problems, proceeding from the "child's value and culture, i.e. his/her enthnomathematics" as a "desirable road to a more humanistic version of rationalism" (p. 145). He sees the "step from enthnomathematics to mathematics," as similar to "the step from oral to written language" (p. 145).

The picture provided by D'Ambrosio, while interesting, stimulating, and in many ways valid, leaves many questions unanswered; issues having to do with teachers, schools, children, curriculum, and the ultimate step from ethnomathematics to mathematics remains unaddressed. D'Ambrosio acknowledges the "difficulty of establishing a monitoring system which will be able to tell about the health of a system facing mathematics education in its cultural, aesthetical, and formative values and for which the utilitarian value is focused as the capability of dealing with 'really real situations.'" His inclination is "toward the elimination of traditional exams, tests, and similar practices in the school systems" (p. 146).

It is hard for me to imagine that the system of mathematics education advocated by D'Ambrosio, ideal though it might be, could ever "prepare our young for the future advancement and betterment of the socio-economic and political framework of society" (p. 135), a goal he identifies as one of the aims of mathematics education.

I see the need for these cadres to know a great deal of the "elaborated code" of mathematics and science, and there is a giant gap to be filled between that code and the "restricted code" of ethnomathematics, as D'Ambrosio clearly recognizes in characterizing ethnomathematics.

Romberg, from the perspective of a mathematics educator and researcher in mathematics education, places his emphasis on pedagogy, specifically the doing of mathematics in the classroom, as contrasted with merely the learning of "the record" of mathematics. By "doing," he tells us he means abstracting, inventing, proving, and applying. Romberg's perspective tells readers and teachers the "how," largely leaving the "what" to the previous authors.

Romberg speaks to curriculum designers, as well as to teachers, telling them that "instructional activities should be grouped into curriculum units," and that "curriculum units should be related via conceptual strands." He provides specific and provocative suggestions about how curriculum units can be grouped and related, making "story shells" the basis of each two- to three-week group of curriculum units, and the larger story of mathematics itself the basis for conceptual strands linking the unit groups.

Romberg's answer to the question of how individual differences can be considered is that "a new common curriculum for all students should be developed. . . . The new program should have at its base a 'core' program and also provide a variety of options."

So much for perspectives and prescriptives. Let us think once again of the really real, real world problem of mathematics education. In addition to the use of inductive and deductive modes of thinking in problem solving, D'Ambrosio calls our attention to the importance of abduction, "which may be conceptualized as a conjecture about reality which needs to be validated through testing [and] seems to be the basic component to deal with a real situation."

Clearly, each of the papers we have before us represents abductive thinking. We are presented with model solutions to the real problem, solutions that remain to be validated. But what are the chances of success of these models, as yet untested against the global situation of teachers, students, schools, and the multitude of other critical factors identified by the authors?

In June, 1984, a meeting of mathematicians and mathematics educators took place at Carleton College to consider what could be done to improve the state of mathematics education. In a short paper, entitled "Only in America," I said, "In all candor we are forced to ask, at least among ourselves, is anything needed other than well-prepared competent teachers, good materials (already available), students who are dedicated to the task, and hard work? Clearly, these old-fashioned virtues have sufficed in the past, and still suffice today throughout those countries of the world with which we must compete." I called this the "doctrine of common sense."

The models proposed here, however, claim that more is needed, the dictum being that "all students should learn 'more and somewhat different mathematics' than is in the current curriculum," and this mathematics, according to the authors, should be differently organized and differently taught.

But how is this to be accomplished? Aren't we talking about the same teachers, the same students, the same schools and the same society that are today failing in mathematics? In short, the same system? If we tinker with the system by moving in the directions indicated for curriculum and pedagogy, will we contribute to the solution of the problem, i.e., improve mathematics education, or are we likely to exacerbate the problem by creating more confusion, fear and loathing of the subject? Unfortunately, the mathematics community has done this before.

To me, it is insufficient to say that this time we will work with the teachers. The reality of the situation as it pertains to our teachers appears to be this: We do not have the quality in the quantity needed, and there is no indication that we ever will. All of the authors agree on the seriousness of this problem.

Under these circumstances, it may very well be that the old-fashioned virtues enumerated in the doctrine of common sense, though necessary, are not sufficient to the task. So where can we turn? Perhaps we must be more willing to experiment with new modes of instruction, just because of the uniqueness and difficulty of our situation, the size and complexity of our population.

The new modes I refer to, recognized in one way or another by all the authors, arise from "the machine." Not, let us hope, the "teaching machine," but rather, teaching by machine. Perspectives on the future of mathematics education in the next decade that do not take into account powerful, compact, and affordable personal machines, might, one day soon, appear to be as unrealistic as perspectives on the future of the movie industry that ignored the existence of the VCR, TV, and three-dimensional holograms.

I would like to distinguish here between two kinds of technology-related recommendations. The first are those that recognize the use of the computer as a computational tool, as a conceptual visual aid, and as the motivating force behind new topics for the curriculum (computer science). These are by and large contained in the preceding papers.

The second kind of recommendations have to do with instruction by machine, i.e., the use of technology to deliver the curriculum, or appropriate parts of the curriculum, correctly, consistently, predictably, efficiently, patiently, tolerantly, individually, motivationally, and interestingly, drawing on vast and as yet uncreated stores of material directed to the eye, the ear, the mind and the imagination.

It is this latter kind of recommendation, having to do with teaching by machine, that is at least implied in the "Working Group Report: The Role of Technology," contained in Pollak's paper. Their very first recommendation states that

> the potential of technology for enhancing the teaching of mathematics and many other subjects is vast. Development of such resources should be supported at a national level. Specific examples include computer-generated graphics, simulations, and video-disc courseware materials. There should be efforts to create a network providing easy access to such banks of material. (p. 129)

This is not the place to present or debate the arguments for and against the desirability and feasibility of teaching by machine in the schools. Let two points suffice: first, that the technological advances of the last few years have greatly improved the chances of success, and second, that machines are now used extensively for teaching in the work place and in the military, both sectors where success is essential to the enterprise.

Can the subject matter of school mathematics be packaged and delivered by technology? Not all of it, but telecommunications, personal computers, networking, and videodiscs can replace much routine and one-way instruction that today occupies class time and, by and large, can do it better.

Despite the positive arguments and claims, however, it is by no means certain that teaching by machine will ever prove successful in the schools. Trials are underway, however, in this country and abroad, and research on the use of computers in education has been funded by NSF under its Applications of Advanced Technologies Program. A significant contribution that the mathematics community can make, in any event, is to see to it that machines are properly used, and that proper software is created for that purpose.

Finally, let me return to the doctrine of common sense. While I subscribe fully to the recommendations of the previous authors calling for the teaching of more and different mathematics, I believe strongly that we as mathematicians and educators must reaffirm the importance of the old-fashioned virtues as necessary prerequisites for students to succeed, and for our system of education to succeed. That is:

(1) Teachers must be competent and well-prepared in their subject.

(2) Materials of instruction must be rigorous and of high quality.

(3) Students must be dedicated to the task, i.e., want to learn and be willing to work hard at it.

(4) Home and school environments must be conducive to learning.

(5) Expectations and standards must be high.

Anything less will not get the job done, no matter what or how we try. To pretend otherwise is to do a great disservice to all. Indeed, the future health and progress of school mathematics in the United States will perhaps be most dependent on the extent to which these common-sense requirements are met.

On this point I am optimistic, for I see encouraging signs, from such diverse sources as governors' conferences and teachers' unions, that the need for these requirements is once again recognized. Common sense has long been a strong point of the American people--this might be labeled the "public perspective"--and, in the long run, this is perhaps the most important perspective of all in a democratic society.

POSTSCRIPT

The debate about what mathematics should be included in the school curriculum is critical if educators are to properly respond to the demands for reform. Four issues raised by Herb Greenberg require further comment.

First, Greenberg noted the absence of implications from educational psychologists. Their perspective is the topic of chapters 12-16 in Volume 2. Although they have little to say about what mathematics should be taught, they have a lot to say about how it should be taught.

Second, his comments about Maurer's attention to topics such as discrete mathematics and algorithms and how they might influence the secondary curriculum reflect a vision of incremental rather than radical change. The distinction between these change strategies is important. Most prefer incrementalism because it minimally disrupts. However, too often it results only in nominal change (Romberg & Price, 1983). Sometimes change that radically affects how teaching and learning are done is warranted. We believe that it is warranted now.

Third, Greenberg was right on target when he identified "all," "more," and "somewhat different" as the key terms to be considered in this section. The importance of "all" was addressed in the first chapters of this monograph and will not be discussed again. "More" implies that there are mathematical topics which are not now taught or emphasized which should be. It seems clear from the papers in this section that topics such as statistics, mathematical modeling, and algorithms need such treatment. However, the term "somewhat different" needs further clarification. It should be seen more as "different emphasis" than as "omission" or "replacement." For example, naive readers may conclude that because of the calculator long division will no longer be taught. This is not so. What should change is what is being emphasized. The difference should be that understanding of the process of division and how it is used should be emphasized rather than speed and proficiency in carrying out the computational algorithms. Overall "somewhat different" should reflect the shift from acquisition of facts and skills to understanding the basis, use, and interrelationships of those facts and skills.

Finally, Greenberg's "doctrine of common sense" recognizes that good teachers in some schools have always emphasized understanding, higher order thinking, and problem solving. However, that has not been the case in most schools. As Resnick and Resnick (1977) have documented, the U.S. historically has had a dual educational system. There are a few "elite" schools and a lot of "common" schools. The shift in emphasis described above would not affect the former but should radically affect the latter.

In conclusion, the belief that "all students should be taught more and somewhat different mathematics" remains true. It should be a consequence of the reform effort.

References

Resnick, D. P., & Resnick, L. B. (1977). The nature of literacy: An historical exploration. Harvard Educational Review, 47, 370-385.

Romberg, T. A., & Price, G. G. (1983). Curriculum implementation and staff development as curriculum change. In G. A. Griffin (Ed.), Staff development: Eighty-second yearbook of the National Society for Study in Education (Part II). Chicago: University of Chicago Press.