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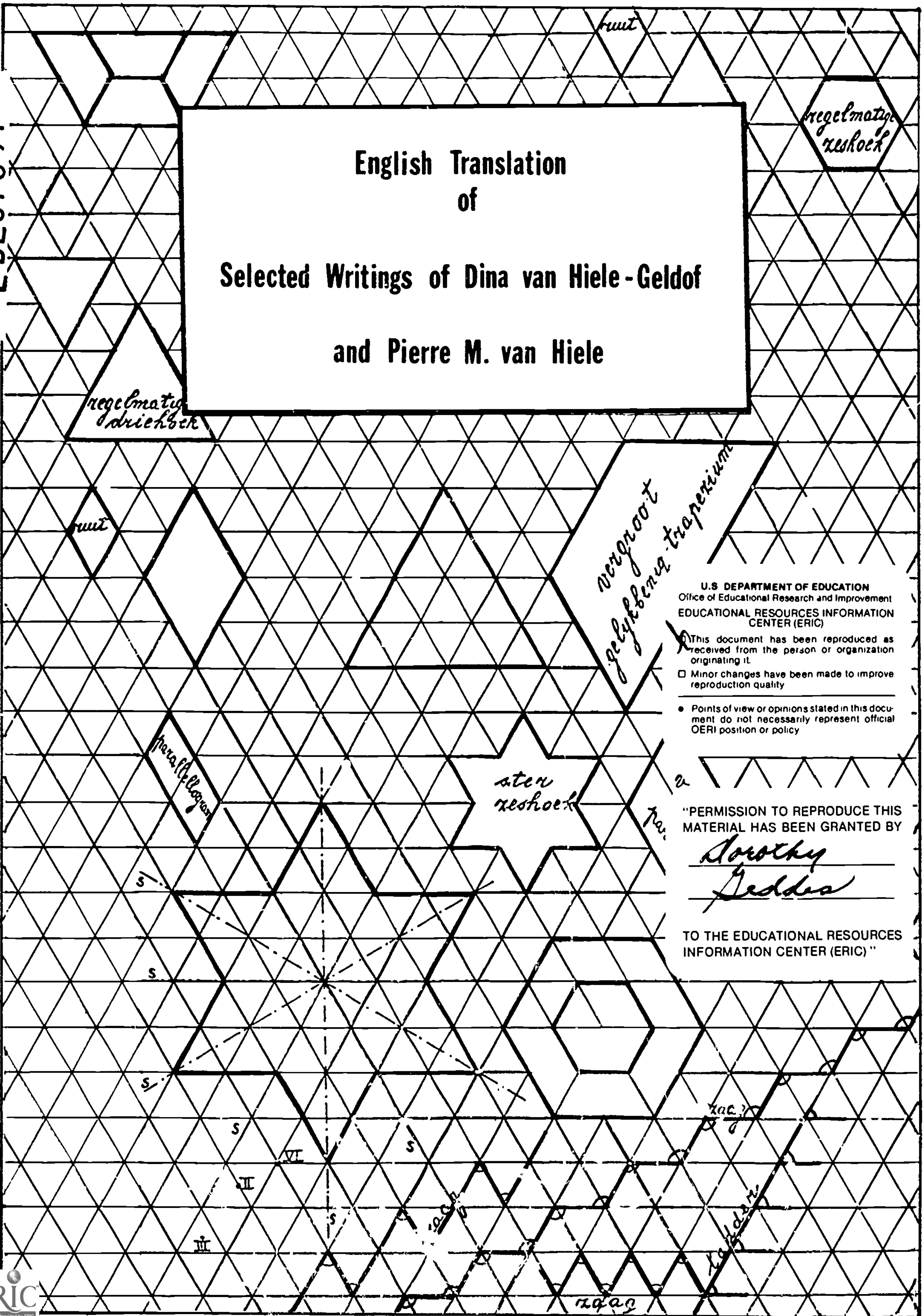
ABSTRACT

After observing secondary school students having great difficulty learning geometry in their classes, Dutch educators Pierre van Hiele and Dina van Hiele-Geldof developed a theoretical model involving five levels of thought development in geometry. It is the purpose of this monograph to present English translations of some significant works of the van Hieles. These translations were done as part of a research project entitled "An Investigation of the van Hiele Model of Thinking in Geometry among Adolescents" which was supported by a grant from the National Science Foundation. Part 1 of the document includes the dissertation of Dina van Hiele-Geldof entitled "The Didactics of Geometry in the Lowest Class of Secondary School" and a summary of the dissertation written by Dina van Hiele-Geldof. Part 2 contains the last article written by Dina van Hiele-Geldof entitled "Didactics of Geometry as Learning Processes for Adults." Part 3 provides a summary of Pierre van Hiele's dissertation entitled "The Problems of Insight in Connection with School Children's Insight into the Subject Matter of Geometry" and an article by the same author about a child's thought and geometry. Included is a bibliography of the writings of both authors, whose doctoral dissertations were presented to the University of Utrecht in 1957. (TW)

English Translation
of
Selected Writings of Dina van Hiele-Geldof
and Pierre M. van Hiele



English Translation

of

SELECTED WRITINGS OF DINA VAN HIELE-GELDOF AND PIERRE M. VAN HIELE

Prepared as part of the Research Project:

AN INVESTIGATION OF THE VAN HIELE MODEL OF THINKING IN GEOMETRY AMONG ADOLESCENTS

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PREFACE

After observing secondary school students having great difficulty learning geometry in their classes, Dutch educators Pierre van Hiele and his wife, Dina van Hiele-Geldof developed a theoretical model involving five levels of thought development in geometry. Their work, which focused on the role of instruction in teaching geometry and the role of instruction in helping students move from one level to the next, was first reported in companion dissertations at the University of Utrecht in 1957. For titles of the dissertations, see Bibliography, pages 255-256. According to the van Hiele model, the learner, assisted by appropriate instructional experiences, passes through these levels beginning with recognition of shapes as a whole (level 0), progressing to discovery of properties of figures and informal reasoning about these figures and their properties (levels 1 and 2), and culminating in a rigorous study of axiomatic geometry (levels 3 and 4).

Since the van Hieles' dissertations and early articles were in Dutch, their findings were not widely disseminated outside The Netherlands. However, a paper, "La Pensée de L'Enfant et La Géométrie," presented in 1957 by Pierre van Hiele to the mathematics education conference - "Pilot Course on the Teaching of Mathematics" - at Sèvres, France and later published, brought the model to the attention of the mathematics education community. The paper was of particular interest to Soviet educators and psychologists who undertook major revisions of their geometry curriculum based on the van Hiele model. In recent years interest in the van Hiele model has been growing in the United States. However research efforts to study the van Hiele model have been hampered since little original source material in English has been available to English-speaking researchers.

It is the purpose of this monograph to present English translations of some significant works of the van Hieles. These translations were done as part of a research project entitled: An Investigation of the van Hiele Model of Thinking in Geometry Among Adolescents, supported under a grant (1980-1983) from the Research in Science Education Program of the National Science Foundation. In particular, the following selections have been included:

From Dutch into English:
- "The Didactics of Geometry in the Lowest Class of Seconday School," Dissertation by Dina van Hiele-Geldof.
- "Didactics of Geometry as Learning Process for Adults" by Dina van Hiele-Geldof.

From French into English:
- "The Child's Thought and Geometry" by Pierre van Hiele.

A brief overview of each of these selections is given below.

Dina van Hiele-Geldof's dissertation, which is presented as Part I of this monograph, is a major source of material on the van Hiele levels. It describes a year-long "didactic experiment" involving two of her own classes of 12 year-olds. The dissertation, consisting of 15 chapters, includes a discussion of some theoretical issues and a detailed description of her teaching experiment. Chapter I contains a discussion of the question, what is a didactic experiment, and research questions are posed. The three main questions investigated in the study are:

1. Is it possible to follow a didactic as a way of presenting material so that the thinking of the child is developed from the lowest level to higher levels in a continuous process?
2. Do twelve year-olds in the first class of secondary school have the potential to reason logically about geometric problems and to what extent can this potential be developed?
3. To what extent is language operative in the transition from one level to the next?

A comparison of other researchers' views of didactics and her own is presented in Chapter II. The Dutch school year consisted of three trimesters. A description of the geometry content and how the students approached some of the problems during the first trimester is given in Chapters III and IV. An account of the teaching method adopted is set forth in Chapter V with the author noting particularly effective activities for developing students' knowledge of space.

In Chapter VI, the author describes how she applies the laws of apperception theory from Gestalt psychology to develop the topic of "Tiles" for the second trimester. A phenomenological analysis of the teaching-learning procedure is given in Chapters VII and VIII along with a discussion of how the author reached her position on didactics. A careful examination of the fundamentals of didactics appears in Chapter IX.

In order to analyze constructive moments (moments of insight and intuition) in the process of learning, the author presents in Chapter X a protocol - a detailed account of class conversations on the topic of "Tiles". This record of her lessons for the teaching experiment and the students' responses are fascinating and provide much insight into the van Hiele model. In Chapters XI and XII, the author analyzes the protocol in terms of the students' transition from undifferentiated thought (level 0) to the formation of visual geometric structures (level 1) and to the development of logical thought (level 2).

Chapter XIII consists of an introspective observation of the author's own process of learning. She describes how her ideas on didactics developed and how one can operate on levels 0, 1 and 2 with respect to "didactics." The author presents in Chapter XIV a synthesis of the didactics resulting from her analyses of the students' process of learning as shown in the protocol. Chapter XV contains a description of the subject matter (congruence of triangles) for the third trimester and procedures for developing the concepts for this unit.

As part of the dissertation requirements at the University of Utrecht, doctoral candidates had to present a summary of their dissertations in a language other than their native language. Dina van Hiele-Geldof chose to write her Summary in English. This Summary appears at the end of Part I along with a set of Tenets she prepared following the defense of her dissertation.

Pierre van Hiele reviewed the Project's first draft of the English translation of Dina van Hiele-Geldof's dissertation. Aside from a few minor suggestions for word changes, he indicated that it was "a very fine translation." Dina van Hiele-Geldof died about a year after completing her dissertation. Dr. van Hiele recommended his wife's last article, written in 1958, in which she gives further clarification of the levels as related to a student's behavior, as an important resource document for researchers. The translation of this article, entitled "Didactics of Geometry as Learning Process for Adults," is presented as Part II of this monograph.

Pierre van Hiele's two major books, Begrip en Inzicht (1973) and Struktuur (1981), focus on the role of insight, intuition, levels of thinking, and structure as they relate to learning. A new compilation of these ideas, soon to be available in English, is currently in press with Academic Press. In order to give some insight into the dissertation of Pierre van Hiele, the English summary which he prepared in 1957 for his thesis, entitled "The Problem of Insight in Connection with School Children's Insight into the Subject Matter of Geometry," is included at the beginning of Part III of this monograph. It is followed by the Project's translation of his article, "La Pensée de L'Enfant et La Géométrie." It was this article which captured the attention of Soviet researchers who, in turn, developed ways of using the van Hiele principles to revise their school geometry curriculum. In the article, Pierre van Hiele describes in detail the levels and phases within levels of his theoretical model for thought development in geometry.

The Project's translation work from Dutch into English was done by Dr. Margriet Verdonck, a native of The Netherlands, living in Brooklyn, New York. Dr. Verdonck translated Dina van Hiele-Geldof's doctoral dissertation in its entirety, the Tenets (see Part I) and her last article (see Part II). The translation from French into English of Pierre van Hiele's article, The Child's Thought and Geometry (see Part III) was done by Rosamond W. Tischler, a member of the Project staff.

A bibliography of the writings of the van Hieles is included at the end of the monograph.

The Project staff hopes that this monograph, containing translations of significant works of the van Hieles, will provide the English-speaking research community with a resource that will shed more light on the van Hiele model. Also available is a companion monograph which reports the Project's research efforts in five areas: (1) clarification and documentation of the van Hiele model in terms of specific student behaviors; (2) development of instructional/assessment materials for students in grades six and nine on properties of shapes, angle sums for polygons, and area of quadrilaterals and triangles; (3) results of a clinical interview study of sixth and ninth grade students using these materials; (4) results of a clinical interview study with preservice and inservice teachers also using the Project-constructed materials; (5) a critical analysis of the geometry content in three textbook series (grades K-8), currently used in the United States, with respect to the van Hiele levels.

PART I

DISSERTATION OF DINA VAN HIELE-GELDOF

entitled:

THE DIDACTICS OF GEOMETRY IN THE LOWEST CLASS OF SECONDARY SCHOOL

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Chapter I

WHAT IS A DIDACTIC EXPERIMENT?

The first question I asked in order to orient myself is: What is an experiment?

If we look at physical, chemical and biological experiments, and at experiments in the natural sciences in general, we can distinguish two categories: the qualitative description of natural phenomena and the quantitative measurements of them. The customary approach is that one isolates the factors that are thought to influence the phenomenon, i.e. that one keeps all but one constant. In an experiment in the domain of learning processes, the situation is somewhat different. As an educator one has the responsibility to make sure that the activities one wants the children to carry out are pedagogically justified.

In "Geistesformung" of Castiello (I), I found a description on pages 54 and 55 of an experiment by Watson on an 11-month old child. This experiment shows that the feelings of fear of a certain object, that one has intentionally instilled in a child, are carried over to similar objects by the child. I would not call this experiment pedagogically justified, because experience has taught us that such feelings of fear can have far reaching consequences. In this respect, there is a <u>limit</u> to the experiment.

Furthermore, it is difficult to obtain reliable results about the subject (i.e. the twelve-year-old pupil) that is being investigated. When pupils found tasks in the investigation uninteresting, or more importantly, when they have had an aversion to them, one cannot come to a correct conclusion about normal children. Therefore one should take into account that the experiment can bring about abnormal states of mind in the children. This could lead to results that cannot be compared with other results.

A second point of difference between this kind of experiment and those that are carried out in the natural sciences, is that in the natural sciences factors can be <u>isolated</u> - undesirable factors can be eliminated - but this is much more difficult here.

The essential difference however resides in the subjectivity of the observer. In the natural sciences, this subjectivity is expressed in the choices of the measuring instruments that register the phenomena. The phenomena are not observed directly: one makes use of indirect methods. Because a didactic experiment involves the observation of psychological phenomena, the subjective attitude of the observer is a factor that is almost impossible to eliminate.

The central problem in didactics is the educational value of the disciplines. It is the opinion of many, that especially for the B-pupils, mathematics is a subject that has an important educational value. The work of Castiello: Geistesformung: Beitrage zur Experimentellen Erforschung der Formalen Bildung (Mind building: Contributions to Experimental Research on Formal Education) is especially significant in the investigation of this educational value. He describes in his book a number of experiments about intellectual development. For the most part these are quick impressions where the subjects did not work under normal circumstances (p. 68). From the answers to a dozen questions (in which it was insisted that all questions be answered in one way or another), Castiello claims to be able to make statements about the logical thinking, the mathematical thinking, the thinking in the natural sciences, etc. of his subjects. I have already pointed to a difference between such an experiment and an experiment in the natural sciences and that the elimination of undesirable factors is much more difficult. This implies that the number of experiments one has to carry out has to be much larger than those Castiello did in order to arrive at reliable conclusions.

Castiello also describes another type of experiment: A number of groups are given different pre-treatments before they are tested. In addition, the groups are compared with another group that did not undergo pre-treatment, i.e. a control group. In this manner, experiments were carried out in the areas of attentiveness, judgment formation, algorithms. The objective was to find out to what extent transfer evolves through practice.

This method of experimenting is also found in the thesis of Mooy (I, p. 60 ff). The objective was to investigate the improvement of learning performance in geometry in children of the first class of the gymnasium (secondary school for those preparing to enter the university) by means of so-called learning conversations. The protocols of the classroom lessons and those of the learning conversations are completely missing. Therefore it is difficult to evaluate whether the classroom lessons that were given to the different groups were indeed the same. Castiello (p. 140) points out in this connection that the way in which the subject matter is transferred is strongly dependent on the teacher (the person in front of the class). In the second place, it was not determined whether the two groups were set up to be as equivalent as possible in so far as their individual aptitudes were concerned. If I have understood it correctly, three classes in one gymnasium were used. Those classes were put together randomly as is customary. A fourth class belonged to another gymnasium. In addition, all those classes were taught by different teachers. Thirdly, there was no fixed control group here, so that the practice factor was not properly taken into account.

Prins (I) carried out investigations in a similar way. He studied the improvement of learning performance in geography in elementary school children. He did take into account the difference in intelligence between his groups. His reports on the learning essays of the children, the learning conversations and the accompanying analyses are extremely informative. However, because this involved the subject of geography and because the test was done on elementary school children I will not dwell on it.

Boermeester (I,p. 58 ff) describes in greater detail an experiment in the area of geometry. The starting point for him was the question: "Would it be possible to improve the results of geometric thinking by means of learning conversations, in such a way that one could talk not of apparent success, but of effective use of the difference between the metastable and stable level within a given maturation period?" His best experiment involves two parallel classes of the second year of a Mulo (advanced elementary school). In order to make the learning conversations as effective as possible, he first analyzes the geometric thinking of pupils in their 1st, 2nd and 3rd years at the Mulo. They were given a difficult problem that ' ' no relationship with the test questions they were to receive later. Not only were they asked to write down the solution, but also to write down all their thoughts.

The next task of the experimenters consisted of thoroughly reviewing the theorems and definitions the pupils had learned up to that time. In order to compare the results from different classes, there should be the same theoretical base.

Then came the first test. This consisted of two geometry problems for the classes 2A and 2B. Then a complete analysis of the mistakes in the work of class 2A was made in order to make the learning conversation as effective as possible. A few days after the first test had been completed, a earning conversation was held in class 2A and a traditional lesson was given to class 2B. Boermeester gives a protocol of the learning conversation. The second test took place one week after the first test. This again consisted of two problems. Here again there was an analysis of the mistakes of class 2A, a learning conversation in that class and a traditional lesson in 2B.

Test III was done during the third week. Again the same treatment for 2A and 2B respectively. Next came test IV in which the first problem differed significantly from the preceding ones.

In Boermeester's work there is a control group. He therefore can arrive at well-organized conclusions.

Even though I value the experiment very highly, I want to make one remark. Boermeester wants to investigate whether geometric thinking can be improved by learning conversations. Therefore, it would have been better if he had first carried out an analysis of the assigned problems starting from the 14 points he gives in Chapter VIII about geometric thinking. He then should have based his judgment of the completed work on this analysis. It is not sufficiently clear from his assessment which functions of mathematical thinking he wanted to test and which problems or parts thereof were to achieve this.

This brings me to the end of my preliminary discussion. The didactic experiments I have discussed so far always had the objective: to investigate the improvement of learning performance by a change in the <u>learning method</u>. The manner of the <u>presentation of the subject matter</u> did not come into question even though Mooy and Boermeester specifically mention psychological objections to it in their introductions. They take particular exception to the abstract way in which the subject matter of geometry is presented in the first year of the secondary school.

This helped to determine the questions for my own didactic experiment. Namely:

1. Is it possible to use didactics as a way of presenting material, so that the visual thinking of a child is developed into abstract thinking in a <u>continuous</u> process? This abstract thinking is requisite for logical thinking in geometry.

2. Is there a need for a child in the first class of the secondary school to reason logically about geometric problems and to what extent can this need be met?

3. What role does language play in the transition from visual to logical thinking?

Each mathematics teacher can repeat the experiment: the protocol (see Chapter X) clearly shows the procedure. In schools where the standards of selection of the pupils are more stringent, the result will be better. In schools where the standards of selection are less stringent, the result will not be as good. Both groups of pupils I worked with were composed randomly: the division took place on the basis of the name roster. From the fact that the class conversations of these groups were practically identical, I think I am able to say that these groups are random samples of twelve-year-olds from the secondary school.

One could imagine the experiment to be as follows: one of the two classes of the first year at the Amersfoort Lyceum would be taught according to the traditional approach and the other class would be taught according to a method where materials are very important. This would be followed by a comparison of the results of both methods. This approach would not be possible for two reasons. First, I no longer feel able to teach according to the traditional methods in the first class. The group which would be taught in the traditional way by me would then not work under normal conditions. Second, the results of the two methods are not really comparable. The groups should be judged according to a standard. This standard is set in relation to one method or the other. Hence, the results seem to be predetermined.

Therefore, the experiment is a protocol of what I am accustomed to doing with the first classes in the geometry course. The report deals with the subject matter covered during the months of January, February and March. The great advantage is that the pupils are tested under practically normal circumstances.

What is new in the experiment is the way in which the subject matter was presented. One may ask oneself whether the experiment remains within the limits I have set in the beginning: "Is this experiment psychologically justified?" Mooy reports in his thesis that it is to be deplored that psychology has not yet established what twelve-year-olds are able to assimilate. Should the didactician then not wait until the psychologist has established how he should organize his didactics?

Langeveld (I, p.11 ff) discusses the following question in great detail:

> We must develop a psychology which is an organic part of and a direct outcome of pedagogy.

He declares:

> In order for psychology to have any significance to an educator, it must be tied to the real-life experiences of the classroom and to the analysis of such real-life situations. Up to now, it is only accidental that an experienced educator could find anything practical in pshchology: it has been too biologically oriented rather than rooted in real-life experiences. It has followed a logical and progressive development - in both the Wurzburger school of psychology and the Wallon and Piagetian school of developmental psychology - which emphasize biological principles. But in fact, for a full understanding of psychology the personal experiences of at least two people are essential.
>
> So it behooves the psychologist who is interested in developing something practical to first adopt methods which would enable him to observe and treat humans in a real-life situation. Secondly, he should be less interested in the scientific application of principles than he is in the understanding of human behavior. Finally, he should formulate his observations and understandings from the people he has worked with and then set out to validate or disprove his impressions - inductively.

If there were a developmental psychology that would satisfy the above-mentioned conditions, the mathematics didactician could possibly find in it the badly needed data on the development of mathematical thinking. Such a psychology would have to derive its data from an analysis of those teaching processes in which this thinking is being developed, because it is there that the psychologist encounters the child in his learning process. The framing of the learning situations, however, is the work of the didactician. Psychologist and didactician will have to deal with the same thing here.

In order to be able to provide child-centered instruction, the didactician will have to have a background in psychology for setting up the learning situation. Conversely, if a developmental psychologist wants to set up a theory that has some value for the teacher, he will have to operate against a background of didactics. Only very close cooperation between didactician, developmental psychologist and teacher will allow for advances in didactics as well as developmental psychology in this area. In this way only will didactics become more than a science merely describing subject-matter and will psychology become a support for the teacher in his daily tasks.

Kohnstamm (I, p.85 ff) cites two circumstances that could be the reason why the work of scholars studying "thinking in children" can possess such serious gaps.

> The first is an incorrect view - incorrect because it is based on a faulty theory of knowledge - a view that one should delineate as clearly as possible the boundaries of the disciplines, even though they deal with the same object. Psychology is not logical. The child psychologist is not responsible for the content of logic: he uncritically accepts its data. He therefore also does not have to investigate what "logical thinking" is, but he keeps to the traditional views in that respect.

A second reason is:

> ......the view that "learning to think" is a biological maturation process, instead of the transfer of historical forms of culture. It is accepted, without closer examination, that "adult-thinking is a uniformly fixed entity to which child-like thinking moves by way of a spontaneous maturation process. As child psychologists we then really have nothing else to do but to investigate what the stages of this maturation process are and to determine at what age those stages are reached. The value of the intentional training, therefore that of the school and teaching, is hereby considered negligible.

That this viewpoint has not become clear for many is frequently apparent from discussions. The saying: "It would be best to have this assessed by a psychologist" has almost become proverbial in teachers' circles.

This shows the necessity for a schooling and training for the teachers in the sense described by A.J.S. van Dam (I):

> The assistant-teacher learns the literature that is important to him in courses - in pedagogy and general didactics, in psychology (general, special and developmental psychology) and he learns didactics of the subject from the subject-didactician.

In this manner, the teacher-to-be becomes acquainted with the relation of these disciplines to each other. This protocol, including analysis, is also intended to be a contribution to the documentation of didactics of initial geometry instruction. It would be a great advancement for the work of the subject-didactician if many experienced teachers were to write down a protocol of their method of procedure.

The psychologists studying thinking make principal use of the mental patterns of behavior of gifted children in order to try to transfer these to less gifted children through mutual contact. Prins and Boermeester tried to promote this mutual contact in their learning conversations in order to arrive at better learning performances among the children. The adoption of good work methods need not be limited to the children however. Perhaps it would also be possible for teachers to learn from each others' work methods. However, in order to get a clear picture of these, protocols of lessons, class conversations, learning conversations and so on, are indispensable. I have been convinced for a long time that we teachers influence each other by talking to each other. This is the most important reason why I attend meetings, as much as possible, of the mathematics work-group of the W.V.O. (Work group on renewal education). I consider this a not-to-be-underestimated contribution to my own development in didactics.

Chapter II

CONSIDERATIONS ABOUT A NEW APPROACH TO GEOMETRY INSTRUCTION

Nieuwenhuis (I) writes the following about "Practical consequences of child-centered instruction:"

> I start again by noting that a large majority of those in teaching circles think that more attention should be paid to the individual child in the course of the learning process. Developments in child psychology have made us especially aware of the shortcomings of our teaching in that respect. This insight gained from child psychology has led many to modify the way in which the subject matter should be presented. This has also led to consideration of the question of what subject matter should be offered at a certain period of time. It can hardly be estimated how many teachers are putting this insight into practice with considerable diligence and energy. All kinds of methods are modified or are replaced by new ones and even outside the methodology, there is much reform taking place in the area of didactics.
>
> However, I cannot help but feel that while doing so it is extremely difficult to withdraw from all kinds of deeply rooted traditions, so that one can hardly talk of a real drastic change. The result is that the often proclaimed and applied renewal of teaching is severely restrained in its course, but even more so in its impact.

Nieuwenhuis further points to:

> the primitive attitude of the didactics that is now being used and which operates according to a certain system of explanations, but which hardly takes into account an analysis of the difficulties that certain pupils encounter. This same didactics has little else to recommend as a solution to these problems except to repeat the same explanation. This is similar to the case of a physician who, in dealing with a patient who does not tolerate a certain diet, does not search for a more appropriate diet, but rather advises the patient to continue as before with the hope that he will get over it.

I totally agree with Nieuwenhuis. I know from my own experience how difficult it is to part with certain habits and tradition. First of all, it is difficult to give up the customary scheme according to which one organizes one's lessons: first to hear the lessons on theorems with their proofs, then to cover the next theorems and finally in the time remaining to give new assignments or to let the pupils do their homework.

Second, it is difficult to abandon the customary subject matter as it is given in the textbooks. This is evidenced, among other things, by the fact that WIMECOS (Union of teachers of mathematics, mechanics and astronomy), even though it creates the opportunity for an informal course of geometry in its new mathematics program, nonetheless refers to extracts of the customary subject-matter, taken from the textbooks, as material for the course.

The above-mentioned teaching method promotes the formation of a large group of "lesson-learners" (see Morrison I, p.57). Those are the pupils who prefer to assimilate the subject matter in the form of recipes. In order to let fewer children develop into this type, I organize my lessons in a totally different way. In the first year there certainly is no fixed pattern of teaching. The instruction is dependent on the objective that has to be met through the "learning unit" which I have established (see Morrison I, p. 42).

Thus I chose plane coverings as the object of study during the months of January , February and March for the school year of 1955-56. They form a clear unit for the children. The teachers at the Montessori Lycea especially try to present the subject matter in a different way as far as form and content are concerned. Through close collaboration one finds the same subject, "Tiles," in varying forms in different schools. I suspect that it had been discussed years ago in the home of Mrs. Ehrenfest.

Given the way I organized this subject, and in view of the objective of my teaching, the class conversation is an indispensable part of the class hour. This is discussed in Chapter VI. The class conversation is precisely directed towards stimulating the children to as lively a thinking activity as possible. The latter is a condition for the pupils to be able to belong to the "transfer-type". (See Morrison I, p.57).

In what way can we activate the thinking? According to A. D. De Groot, the experienced thinker can "switch on" his thinking-energy by a decision of his own will. The beginner, however, here encounters a difficult problem - the problem of motivation (De Groot, I). Whether or not I succeeded with motivation for this subject, I cannot objectively assess. I took care to provide variety. First, a drawing is made according to specific instructions. Next, the children solve a few puzzles which lead to the construction of figures. The children learn to perceive special features in the drawings. They find relations through exploration and finally we try to derive other relations from those relations through a logical process. (See Chapter X).

One can ask oneself why so little has been published by teachers of experimental schools about their experiments. The reason has to be found in the fact that even though certain schools have been designated as experimental schools, adequate staff was not provided in those schools.

Nieuwenhuis writes in the above-mentioned article.

> We absolutely need a staff of people who can devote themselves completely to research and experiment in the area of teaching, who can collaborate with the teachers in the school, who can test, compare and perhaps reconcile different attempts at reforming. Besides those persons, we need staff personnel who can not only take care of diverse special problems, but who can also provide counseling and guidance and who in turn can closely collaborate with the pedagogical and didactic specialists at universities and pedagogical centers.

I have the pleasure of being well-acquainted with a number of these experimental schools. I can hereby corroborate the statement:

> Your work was useful, good - but you have neglected to provide adequate publicity for it and so you left too much room not only for critics, but also especially for persons who criticize in a negative way because they cannot do otherwise. (Biele I)

I would like to add the following however: It is not negligence on the part of teachers at experimental schools. Teachers find it almost impossible to find time to write publications in addition to their particularly difficult daily tasks at school. As far as geometry is concerned, teachers at experimental schools can confirm, from their own experience, what De Groot (I) has stated:

> Geometric comprehension requires a very high level of abstract thinking - all things considered much too high for the average school child of 12 years of age.

The statement of Langeveld (II, p.153) follows the same line:

> A bounded figure is psychologically much more elementary than points and lines.

(and angles - I would like to add). This is in complete agreement with "the law of closure" from Gestalt psychology. If one closely examines the material that is being used in experimental schools, one notices that this is distinctly taken into account.

One only needs to look at the figures in a geometry book in order to know whether or not initial geometry instruction has acquired an elementary character. The omission of axioms is not essential for the transition to the elementary, nor is the elimination of the proofs of a couple of evident (rather "too difficult to prove") theorems. Notwithstanding the reassurances of a few teachers that the teacher does not follow the textbook at all, I kepp doubting. The only protocol I have been able to find about geometry lessons for beginners, points in the other direction. Stellwasy (I, p. 356 ff) gives a description of two lessons to a test class. A short description of the lessons follows.

First the subject matter covered previously is heard. "Which figures are considered in plane geometry?" - one pupil says: "The points." The teacher: "In general: which figures? Where are those figures located?" - Joop: "On the board, on paper, on a sheet, in a notebook." - The teacher repeats his question. When Joop remains silent, another pupil has his turn.

A girl says : "The angles. The angles of a ruler." The ruler had been discussed but in another context. Other pupils now give the correct answer: "in the plane."

The next question is: "How do we picture the plane to ourselves?" The girl whose turn it is says: "Something that has no space nor width." The teacher: "You then have to show me a plane like that!" Another pupil does understand what is meant: a notebook or a ruler. "Do you have such a plane close at hand? Correct, the place where you are resting your elbow! Beautiful."

From the question, with what figures do we start?, we arrive at the point - which has no dimensions. Now the girl can show her knowledge. She gives the answer: "No dimensions."

Geometry is now being constructed in a systematic way:"If we put a few points right next to each other on a piece of paper, what do we get?" The line that thus evolves has a length, but not a width. "Do these lines exist in reality?" "Louise?" - "Yes, they do, Sir," Louise answers.

The difference between a line in the notebook and a line in plane geometry is extensively discussed. Ria tells that such a line "goes from infinity to infinity." The fundamental concepts: point, straight line, plane, have now all gradually become clear. In order to know what is meant by those we introduce definitions in geometry. The teacher now gives an example of such a definition, namely one of a solid that is in space, and therefore is bounded by space on all sides.

Then follows discussion of the boundaries of a solid, etc. Next, line segments, half lines and broken lines are covered and a definition is found for "angle."

The teacher then proceeds with the statement given in the preceding lesson: "A line, and especially a straight line, goes to infinity. As soon as we mark a point on one side, which means an end-point of the line, we call it.... Well, we naturally do not write things down without thinking, so Ida can tell me... or do we write without thinking? You should not do that child, are we going too fast?" Kitty, Jan, Kees: "Two half lines." "No, not two half lines - then we call it: half line. New sentence: If we mark two end-points on that line, we call the segment between the end points....Fokko?"...."line segment." "New sentence: For the build up of geometry, we need a number of basic concepts. Two of those basic concepts are...Kasrel?" "Square." "Paul?" "Point and line." "Yes."

In order to define what we mean by certain concepts we make use of... I would like to know - I just told you - that defining or describing of certain concepts, what do you call that, Ingrid?" "Wim?" "Definitive". "Fokko?" - "Definition." "Another word for definition?" - "Condition." "Correct." Here are a few of these definitions: 1. A solid is a ... I would like Jan to complete this since I have not heard from him very much." "Well, a solid is a thing surrounded by..." - "I will ask someone else." "Something that is surrounded." ..." Yes, not only surrounded, but surrounded on all sides - write down: A solid is a part of space that is bounded on all sides."

The next lesson proceeds in the same way. "Does anybody know which beginning concepts we know?" "The point, the plane." "Yes, but the point is not immediately followed by the plane." - "The line, the straight line." "What is a proposition?" "Does anybody know another word for it?" - "A solid that is surrounded by air on all sides."

From the above conversations it is apparent that geometry has not yet been made elementary by omitting the axioms. There is also the question of whether supplying definitions can effect the <u>formation</u> of concepts in the children. What should the children think when they are given the words "build-up of geometry." What meaning can they attach to the words geometry and build-up? The deductive system of Euclid from which a few things have been omitted cannot produce an elementary geometry. In order to be elementary, one will have to start from the world as perceived and as already partially globally known by the children. The objective should be to analyze these phenomena and to establish a logical relationship. Only through an approach modified in that way can a geometry evolve that may be called elementary according to psychological principles. (See Chapter III).

What should our attitude be, given the fact that we do not have a team of people who can devote themselves to research and experiment in the domain of teaching? Should we wait until that team is provided? Is Mooy correct when he states in his dissertation that there cannot be radical changes in the next few years because first a thorough didactic training of future teachers is needed? I most certainly am a proponent of a thorough didactic training of the future teachers, but I am very skeptical of the statement that this reform can and should stem from these young teachers. De Miranda (at a W.V.O. meeting of the mathematics workgroup) has stated "that young teachers look for stability in their own development; they have not experience - therefore they have to be conservative."

In my view, the renewal of teaching, so strongly desired, will have to come predominately from experienced teachers. The necessity of arriving at a totally different approach to geometry instruction is apparent from the many articles that discuss teaching in general and from the many meetings that are devoted to mathematiucs in particular. The alternative approach should, in the first place, be found in the method of teaching, whereby the pupil more adequately experiences the build-up of the theory. In order to achieve this, the teacher should partition the subject matter into units in advance.

The different units can entail totally different work methods. There are parts that have to be worked through on a more or less individual basis. However, one should keep the following rules in mind:

1. Allow cutting and glueing, only where it is necessary for the build-up of the theory, in order to acquire better insight into the basic concepts of geometry.

2. Reduce to a minimum all computational work. One should not illustrate generally valid rules with concrete numbers too often. If the pupil always needs this method to find the rule, he does not come to an abstraction that rests on understanding of the relations.

3. The fact that the method followed somehow keeps the children happily busy should not be a criterion for its suitability. The saying "the class ought to proceed smoothly" is misleading for young teachers. It would be easy for them to conclude, if everything went smoothly, that the desired self-activity has been achieved and that they can go on working in the same way. Mursell (I,p.116) writes the following about the "principle of individualization:"

   It should be noticed that individualized learning does not necessarily mean individual segregation or a sort of private coaching arrangement. Also it means a great deal more than providing some such device as workbooks that permit a number of children in a room all to go at their own pace while the teacher checks up and offers help. Under such a plan each child may indeed go at his own pace; but it is equally important, within limits, for each to use his own methods and to succeed and even temporarily to fail, each in his own fashion.

The sub-dividing of the subject matter has to be done by the teacher himself; he should define a principal objective around which the rest should be grouped. One should bring unity into the subject matter, organize the subject matter adequately and only teach those methods that are general. Furthermore, the teacher should realize that he is a mathematician, and that his pupils are not. Therefore, he will have to be conscious of the special mathematical orientation of his mind. This special orientation of mind of the mathematician brings with it the following:

1) he wants to build up everything from the bottom;

2) he wants to express himself in symbols and schemes;

3) he aspires to problem-solving.

His pupils, on the contrary, cannot simultaneously have the orientations of the mathematician, the physicist, the chemist, the historian, the artist, etc. This is contrary to the opinion of De Miranda (I, p.59). Therefore, the teacher should be pleased if the pupils are willing to try and understand the analytical approach of a mathematician and if he finds a number of pupils willing to engage themselves in such a process.

The teacher should be careful not to make use of symbols and schemes too soon. He should first be sure that the pupils have insight into problems before stimulating them to give automatic responses. Not all pupils go so far as to aspire to problem-solving.

During the first year, the converse of theorems should not be dealt with, nor should the indirect proof be used. There remain enough problems of a mathematical nature for the children to solve during their first year.

There is one final remark about "self-directed activity." This deals with independent thinking activity. Beth (I) states the following about the principle of this activity:

> It is of the utmost importance to lead the pupil to the learning process in an active relationship, and not in a passive, receptive, so-called sympathetic relationship. It is only in an active attitude that his mental capacities will completely unfold and then only will he be able to benefit most from the teaching. Good teachers have always tried - unconsciously and therefore perhaps not always systematically and effectively - to stimulate their pupils into independent activity. The solving of problems in teaching was introduced with no other goal in mind; originally, the teacher has set himself the task of discussing the theory, and the solving of simple problems has been given to the pupils as an independent exercise. Difficult problems were included in the sets of exercises for the sake of the brightest pupils. After a while, the solving of these difficult problems was required of all pupils with the result that the teacher started to provide the solutions. This finally led to the degeneration of our

> teaching where the teacher no longer placed the emphasis on the theory but instead on methods for solving all kinds of problems. I give this as an example of the fatal consequences of an injudicious application of a principle that itself is excellent and psychologically justified.

The self-directed activity has to be used towards its intended goal. The goal of the self-directed activity is to provide an active relationship between the pupils and the learning process. In order to stimulate twelve-year-old pupils to independent activity it is desirable to apply different work-methods and to observe the pupils. It has been my experience that this diversity has led to more activity. After a period of much individual work, we now have had a period of many class conversations. Whether a certain work-method functions well depends on the teacher and on the group of pupils.

The kind of independent exercises one gives children is very important. If the teacher's view of the learning process in geometry is that it is important to be able to solve all kinds of problems, then he has taken the viewpoint that the children should acquire technical knowledge. In that case he probably will supply the children with many methods of solution. He will analyze all difficulties of thinking that could occur and he will present the problems in a sequence such that "the solving of problems" occurs as smoothly as possible.

The pupil then has no need to think, but only to remember what method to use, which is not difficult because of the sequence of the problems. The teacher then has thought for the pupils and he transmits the result of his thinking to his pupils in the shape of lessons. In this procedure, the method is the same as the one followed in a vocational school, whereas it is not the objective of geometry teaching to provide merely technical knowledge.

If one takes another viewpoint in didactics, the work-method becomes completely different. In order to be able to establish connections with childlike thinking, one starts from an empirical basis and one searches for appropriate material in order to examine and orient this thinking. There is a wealth of material to choose from to practice thinking. The problems have to be set up so that the children can comprehend them and in such a way that it will be possible to guide them toward good "patterns of thinking" from the very beginning. I have worked this out in the following chapters.

Chapter III

A REPORT ON THE FIRST INTRODUCTION OF GROUPS IA AND IB TO GEOMETRY

In order to follow the experiment carefully, it is necessary to give a summary of the geometry I covered with the children from September till Christmas vacation. I will also summarize the way I presented it to the children. Geometry is a subject that can be introduced with many variations, but the end results are always the same.

The first lesson started by showing a cube, made out of colored cardboard. I asked if the pupils had already seen something like that in the past and elicited answers such as: the blocks belonging to a building block set, the tower of progressively smaller cubes, dice. Only in exceptional cases will a child from secondary school not know the name cube.

Next I showed cubes of different sizes. This included showing a cubic centimeter and a skeletal model made with an erector set. They counted how many sticks were needed to construct the cube. In this connection the word edge was mentioned. Together we observed that a cube has 8 vertices. It was always pointed out what was meant by edges, vertices, etc. They counted how many squares are required in order to make a cube out of cardboard. The pupils found out that the six squares can be drawn contiguous with each other, and that this can be done in more than one way. Also, that one could do it in the wrong way. (Excluded was the systematic picking out of the various ways in which to draw those contiguous squares.)

The pupils then constructed the cube. Drawing triangles (i.e. triangles such as those used by a draftsman) were used to draw right angles, rulers were used to obtain the correct scale, and compasses were also used. I explained the procedure and demonstrated slowly while the pupils were drawing along with me. The term perpendicular, used by carpenters, was given concrete meaning by folding a piece of paper that had no straight edges. This folding produced a straight line. A second straight line was obtained by folding the paper such that the second folding line was perpendicular to the first folding line. The angle between the folding lines we called a right angle. Right angles appeared to be present in drawing triangles among other things. With this the first lesson ended.

In the next lesson we continued discussing the cube. With the help of strings we constructed diagonals in the cube made from an erector set. Just like the edges, these diagonals go from vertex to vertex, but they go through the cube. We discovered two kinds of diagonals. They have different names so they can be distinguished from each other: surface diagonals and interior diagonals. They are to be found either on the surface of the cube or the inside. We drew the surface diagonals on the cube which we had made ourselves, and we were able to measure them and to count them. Then we tried to think of ways to measure the interior diagonals as accurately as possible. We had not taped the cover of the cube shut. When I opened my cube, the pupils saw a piece of cardboard that was stuck between opposite edges of the cube. This piece appeared to have the shape not of a square but of a rectangle. The children recognized that if they could make a rectangle that would fit they would have the solution to measuring the interior diagonals. The squares were called lateral faces of the cube. The rectangle was called a diagonal plane because it contains interior diagonals.

The pupils were given instructions on how to draw a rectangle that would fit in the cube and they were asked to check whether it fitted. Doing this they discovered that the rectangle fitted in the cube in several ways. This led us to counting and this counting was arrived at using a system we invented. Counting something twice was also mentioned. Other examples were also discussed.

The difference between the meaning of the words surface diagonals and diagonal plane was again discussed thoroughly.

The pupils constructed a regular tetrahedron. They thereby learned how to construct an equilateral triangle with compasses. This led to the question whether there are still other kinds of triangles. The answer appeared positive since the drawing triangles are different. Together we tried to find the name of such triangles. The characteristic of both drawing triangles is the right angle. This led us to the name of right triangle.

I provided them with the network of a regular octahedron. The result had one diagonal plane already present. This could have been omitted, but then it would be much harder for the students to draw the diagonal planes in their real size. The pupils already had difficulty seeing that there are 3 diagonal planes in a regular octahedron.

As an introduction to basic constructions I paid special attention to symmetrical figures after I had discussed the cube. I drew the left half of a vase (vertical cross section) on the blackboard and I asked them what I had drawn. Indeed, (see figure on p.30), everybody saw what it was. Now came the difficult task to draw the other half. As soon as that had been done, criticism began. We even made use of a mirror to see how wrong the drawing was. We had to find out how to hold the mirror in order to fit the other half on the one drawn on the blackboard.

Through the criticism of the drawing the pupils came up with an analysis of the notion "symmetrical figure". A line can be drawn in the figure such that for each point on the left hand side there is a corresponding point on the right hand side. That corresponding point is obtained by first drawing a line perpendicular to the mirror line and then marking off a point at the right hand side at the same distance as the point on the left hand side.

We pointed out symmetrical objects that surround us, checked for symmetry in coats-of-arms and road signs. We examined the alphabet in connection with symmetrical letters. After that I asked the following questions: "Why does one sometimes refer to one of the two drawing triangles as a semi-square? Would the other drawing triangle perhaps also be something similar?" They flipped the drawing triangle over to get the other side and found out that the other drawing triangle can be called a semi-equilateral triangle. This led us to the geometrical figures: square and equilateral triangle. For these and also for the rectangle and the circle we tried to find the number of axes of symmetry. To do this, they folded self-made paper models in half in several ways. We noticed that a circle possesses innumerable axes of symmetry. Then we drew an equilateral hexagon. This was found to have 6 axes of symmetry.

Through thinking of symmetrical buildings we came to solid mathematical figures such as pyramid, prism, cylinder, cone, cube, sphere. For some figures I showed the pupils paper models, for others they made models. As a practical application of the concept of symmetry, I asked the following question: "I have a round table. How do I go about finding the center of the table?" Their solution was that: first, one has to measure the circumference, divide that in two so that one has obtained 2 points with respect to the center. Then the diameter has to be drawn and divided in two.

The constant ratio of circumference to diameter of a circle was determined experimentally by the pupils. To do this they each measured the circumference and the diameter of a large circular object at home. The tabulation of these observations led to a discussion about the accuracy of measurements, about rounding off numbers, about estimating the result. We drew an equilateral hexagon in the circle and we concluded that the ratio had to be greater than 3. We drew a square around the circle and we concluded from that that the ratio had to be smaller than 4. On the basis of this, a number of the students' measurements were viewed with distrust. The students in question were asked to repeat the experiment.

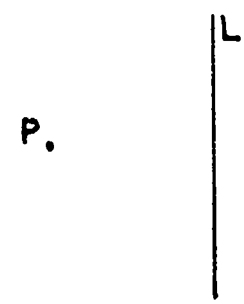
Next, we took four pencils, identical in length. We checked their length by putting them next to each other. With the help of these pencils we tried to construct a quadrangle that had four equal sides, but "that was not a square."

The name diamond appeared familiar to the pupils. Diamond shaped objects were enumerated: nougat blocks, the diamonds in a deck of cards, the diamond in the leaded glass in the front door. Here was an opportunity to demonstrate the necessity for definitions. The word diamond also has another meaning besides the one used in geometry. Therefore, it is desirable to agree upon the meaning of the word diamond (rhombus) in geometry. I showed a couple of diamonds (rhombi) made out of paper. I had them make a flexible model constructed from 4 strips of paper of the same length. The strips were attached to each other with 4 hinges. This showed them that there are different shapes of diamonds (rhombi). We reached the agreement: a rhombus is a quadrangle whose 4 sides are equal. We noticed that the word "four" also can be omitted. The sentence would still be clear. By folding, we saw that a rhombus (diamond) has 2 folding lines (axes of symmetry). We tried to put into words all the properties we had observed by folding:

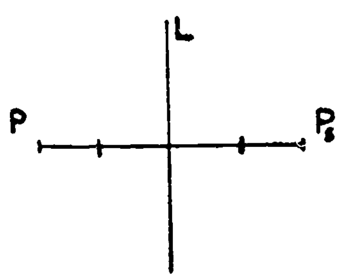
- The diagonals of a rhombus bisect each other.
- The diagonals of a rhombus are perpendicular to each other.
- The opposite angles of a rhombus are equal.
- The diagonals of a rhombus bisect the angles.

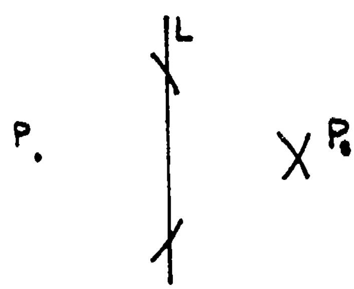
All this was established by showing that the parts fall on top of (match) each other. One can also verify the properties of symmetry by using a mirror. For this we placed the mirror on the folding line and saw the whole rhombus (diamond).

The rhombus became the starting point for many constructions. First measurements were chosen and the rhombus could be drawn anywhere in one's notebook. Shape and size of the rhombus were given, but not the location. After that the rhombus had to be drawn in a certain location. One of the vertices had to coincide with a given point P; two opposite vertices had to be located on a given line L, and the side of the rhombus had to be 4 cm. This determined the location of the rhombus completely. The pupils also recognized, by using the mirror, that the location of the fourth vertex was determined. Then the rhombus was drawn. In the next task more freedom was given in that the side of the rhombus and thus the shape itself could be chosen. We noticed that this did not modify the position of the fourth vertex. This was also confirmed by the mirror - the length of one diagonal was not changed.



I spent a lot of time at these tasks because they form the groundwork for the basic constructions. From the analysis of a symmetrical figure we derived that the mirror point of P with respect to the line L is located on the other side of L at a distance from L equal to the distance of P from L. We saw that this mirror point can also be determined with compasses and that in this construction, the radius of the compasses does not matter.





I drew figures on the blackboard. We called the first method, the drawing and the second, the constructing of the mirror image.

Next, many plane figures were mirrored with respect to a line. The answer to the question was often shown beforehand in the mirror. This was often necessary to provide a concrete meaning to the language we used. The pupils not only heard, but at the same time saw what was required of them. Since the mirror we used had a reflecting surface on both sides, we could also mirror figures that intersect the axis of symmetry. If necessary, the pupils drew along the lines of the given figure with a pencil and they saw the mirror pencil describe the symmetrical figure.

After that I derived 4 important basic constructions from the properties of the rhombus:

I. The bisecting of a line segment
II. The dropping of a perpendicular
III. The erecting of a perpendicular
IV. The bisecting of an angle

A rhombus made out of paper constituted the material used as a concrete basis for the basic constructions. In order to familiarize the pupils thoroughly with these basic constructions, I had them construct precise figures: a regular octagon, a regular dodecagon, a triangle with its medians, a triangle with its angle bisectors, a triangle with its altitudes, a triangle with its perpendicular bisectors and its circumscribed circle. We observed special features of these figures.

The pupils made a regular hexagonal prism and a regular octagonal pyramid out of thin cardboard.

They were required to learn the names median, perpendicular, etc. They had to be able to point out the lines and to indicate how these lines are positioned in the figure. They also had to be able to construct the lines.

As for the construction of perpendiculars, I taught the children a construction method that is based on the kite-figure. We had noticed that the rhombus construction required much space. Therefore it appeared necessary to examine the kite-figure a little further. The students had already come across the kite while reflecting a triangle in one of its sides. A new aspect, however, was the fact that a kite can be seen as a figure consisting of 2 isosceles triangles, but sometimes also as a dividing of a diamond (rhombus). The isosceles triangle was viewed as a semi rhombus. In this way the characteristics of the figures: rhombus, isosceles triangle and kite were experienced as a coherent entity.

The use of the semi-rhombus and the kite are clever means to carry out constructions when there is little room for drawing. A number of students managed to solve difficult problems that were given to them, for example, when they were asked to drop a perpendicular to a line that was drawn completely at the bottom of the page or when they were asked to bisect a line segment that was positioned near the edge of the paper. The students were not required to be able to reproduce these constructions, rather it was treated as a game.

During the last weeks before Christmas vacation we studied angles. The introductory lesson took place as follows: I demonstrated what a straight angle is by stretching my arms sideways and bringing one arm horizontally towards the other. What this arm described we called a straight angle. (A definition was not given). Next, a few children were asked to make a right angle with their arms and to point out through arm movements where the angle in question was.

I asked: "How large is the angle formed by the hands of a clock at 6 o'clock?" Then one of the boys was allowed to form a right angle with his legs. Another drew a right angle on the blackboard with a (drawing) triangle. A semi right angle was constructed on the blackboard with the help of compasses. The names: sides of an angle, and vertex were mentioned. In the connection I talked of the distinction we make between line segment, ray and line according to whether there are two endpoints, one or none. We saw that a protractor is a semi circle that is partitioned into 180 equal parts. Each part is called a degree. If we were to draw the rays, then a straight angle would also be partitioned into 180 equal parts. The small pieces of the arc are called arc degrees, the small angles are called angle degrees. Next, the usual tasks followed: measuring angles using the protractor and drawing angles of a given size with the help of a protractor.

One of the pupils was asked to describe three-quarters of a circle with stretched arms. This was necessary in order to demonstrate that some angles are greater than a straight angle.

The pupils computed the size of the angles formed by the hands of a clock at different hours. I taught them how to mark an angle using three letters, how to copy an angle using compasses, and what the names of the different kinds of angles are.

The names: opposite angles, adjacent angles, supplementary and complementary angles were explained. The pupils were asked to carry out a number of tasks to test their understanding. Through individual contact I made sure that the pupils were able to copy an angle, that they were able to identify an angle using the new notation and that they mastered the concepts: re-entrant (reflex), salient, acute, right and obtuse.

This was the subject matter of geometry in the first stage.

CHAPTER IV

COMMENT ON THE SUBJECT MATTER OF GEOMETRY IN THE FIRST STAGE

As a maxim for the first stage of the initial geometry instruction I would choose: one should allow the children to act thoughtfully with the help of manageable materials.

During the initial stage of geometry instruction, the individual activity of the pupil is central. The question "How shall I make something?" is an important one. This allows the head and the hand to be engaged simultaneously in the learning process. This is the reason for my requesting that pupils make a cube after the cube has been thoroughly examined. When one looks at this exercise from the point of view of geometry, this entails finding a particular ordering of 6 squares. This is an exercise that makes use of the faculty of imagination. It is however supported by the use of material.

A requirement for the material, in general, is that it contains the foundation of the logical development of geometry. In this instance this means that if at a later date one gives a definition for the cube other than "A cube is a polyhedron bounded by 6 squares," one has to demonstrate that any new definition and the original one are equivalent. I want to point out in passing that this definition was used only implicitly during the lessons.

It can be useful to ask how one should join the squares in order to make a cube and also whether it can be done in more than one way. With given pieces of cardboard, the question is: which of those different ways can one use in order to make the cube?

There also is a certain game-element present in the task: "It is possible to make the cube in one piece. Could I do it also?" The interesting thing is that the pupils reach the correct solution by doing. They can correct their mistakes without the help of the teacher. The teacher need not even notice that the pupil did it wrong at first. The material is self correcting.

As one will have noticed, the pupils are asked many questions collectively in the class discussion. They measure and count. Discussion of the properties is not exhaustive. Ample opportunity remains for the pupils to ask themselves questions.

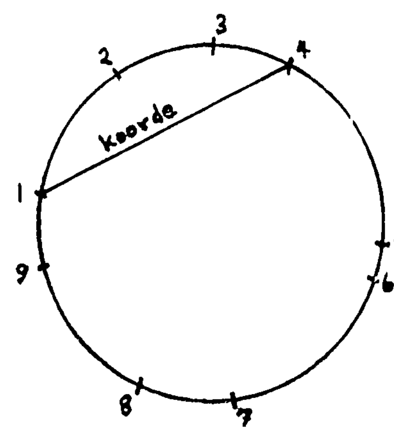
That a diagonal plane of a cube is a rectangle and not a square is not arrived at by reasoning, but, if necessary, is ascertained through measuring. The concepts of square and rectangle are not defined. However, care is taken to see that the pupils know the figures by name, that they can distinguish them from each other and that they can recognize their characteristics. That surface diagonals are of equal length has already been accepted on the basis of an existing structure of observation. This is true also for the equality of the interior diagonals and for the fact that a interior diagonal is greater than a surface diagonal. The pupils are made aware of this structure by means of questions.

I asked the pupils to make a regular octahedron and a regular tetrahedron in order to show them that there are other regular polyhedra besides the cube. However one positions them, they always look the same. Their shape is attractive. Often the students ask whether there are any more of those regular polyhedra. They don't want to research that however - they only ask out of thirst for knowledge. Even though the latter is satisfied by my answer that there are 5 such polyhedra, there is often the need to show and make the other polyhedra. The construction of these polyhedra stems from an aesthetic need; in this stage the pupils never ask for a proof that there are no more than five.

I gave a test on the cube and the properties discussed in connection with the cube. Except for one item, the subject appeared not to have been too difficult. This item concerned the systematic counting whereby the pupils had to count twice, three times or four times. In general this has not led to insight. Not even 10% of the pupils gave thought to double counting when presented with a new situation. The task was as follows: There are 9 points on a circle. The line segment connecting two of those points is called a chord of a circle. Look at the figure; a chord runs from point 1 to point 4. (The pupils were already familiar with the words circle and line segment. In order that an accumulation of new words would not produce confusion I explained the word chord in detail.)

a. How many chords can be drawn from point 1?

b. How many chords can be drawn in total using only the 9 points?



Koorde means Chord

Question (a) was answered correctly by almost all the pupils. Some pupils thought there were six chords Just as for the diagonals of a polygon, they had subtracted 3.

As for the answer to question (b) most pupils found 9 x 8. A number of pupils reached the correct answer in the following way: 8 diagonals leave from point 1. Only 7 from point 2, because we counted the chord 1-2 already. Six diagonals leave from point 3, etc. In this way they found 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36 chords. These pupils used a more concrete method. They subtracted the already counted diagonals.

A train of thought involving multiple counting appeared to be still too abstract for most pupils at that time. By introducing this problem in another way and at another time, one should be able to provide the pupils with a better insight in this matter.

The introduction to "multiple counting," which I gave, occurred as follows: I said that someone might reason as follows: a square has four vertices, a cube has six squares and as a result a cube has 6 x 4 = 24 vertices. I asked what was wrong with this reasoning and how this error could be corrected. Most pupils answered with the statement: a cube has only 8 vertices, because I see 4 at the bottom and 4 at the top. There were also remarks such as: a vertex is shared by several squares, namely by three squares. But when they divided by three, their answer had to be viewed more as a computation of an answer rather than as an insight that would correct the error. The interest in the paradox, contained in the problem, was present though. However, in order to reach the answer, the abstract working method is not practical - the concrete method is much clearer and shorter. One can conclude from this that one has to present the problem of multiple counting so that the abstract method is more efficient.

For example, one can take a tredecagon (13 sides) and ask how many diagonals there are. Then the pupils can in a class discussion, try to count the diagonals using the concrete method. Ten diagonals leave from the first vertex, ten from the second, 9 from the third, etc. They will then see that the last two vertices are left with none. One gets the sum: 10 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 + 0. In this case the abstract method would be more compact. From each vertex 10 diagonals can be drawn. If one were to say that there are 13 x 10 diagonals one would not have taken into account the diagonals that were counted already. For example, when we check what happened with the diagonal 12-5 we see that it first has been counted with the diagonals from vertex 5. <u>All</u> diagonals will be counted <u>exactly</u> twice because <u>each</u> diagonal has <u>two</u> end points. Instead of computing 10 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 + 0, we can also compute 13 x 10 / 2. The pupils now see two methods: one has the advantage that one clearly sees what one is doing, whereas the other has the advantage that the solution is obtained more quickly. The result obtained by means of an abstract method can be checked via a concrete method. This is reassuring to the pupil.

Indeed, I believe that the problem will be grasped better if treated in this fashion. However, as I already mentioned, it will have to be treated at another point in time. At the time when there is interest in the paradox: "a cube has 6 x 4 vertices," I cannot find enough motivation for the students for the problem: "How many diagonals are there in a polygon?"

So, the assigned problem about the chords was not justified. It assumed a transfer which I had no right to expect. The paradox is suitable for presentation to the children, but it would be wrong to use that type of paradox as a means for testing the children.

By starting with a cube, one complies with the requirement that geometry instruction be initiated through psychologically elementary concepts. The cube is a bounded body which because of its exceptional regularity is easily observable.

Gestalt psychology teaches us the following about observation (Linchaten I, p. 57 ff):

> The world of perception can be described from the aspect of its ordering which makes orientation possible. For the description of the systems or ordering, we can choose different points of view.
>
> -The world of perception is ordered spatially.
>
> -The world of perception is also ordered chronologically.
>
> -The world of perception can be considered as an organized conglomerate of shapes, which together, but also each one individually, stand out against a background. By shape we understand a bounded entity that possesses a certain independence. The context in which the shape is placed we call its field. When other shapes are present in this field they display, in their relationship with one another, an organization which we call field structure. The world of perception also displays an ordering of meaning (that of the meaning, relationship of meanings and meaning structure).

The geometry teacher is initially dealing with the existing spatial ordering. It is important for him to know the field structure that each pupil has of the observed shapes, when these shapes are looked at from a geometric point of view. The point is that the teacher should find the common nucleus of the geometric experiences of all the pupils.

> Some field structures draw attention automatically. Others, on the contrary, only come about in correlation with a certain attitude of the person. Someone who concentrates on a certain task has to adopt a certain attitude. While he is doing this, the field structures itself in its particularities. This would escape a person who did not take on that attitude...
>
> A shape that is seen appears in its totality. It stands before us. The whole and the parts are visible simultaneously, i.e. are given simultaneously. The shape, or the visual object is also always visibly embedded in a context that forms the system of relationships or the background of the shape.

The context in which the cube is taken up in this case is a geometric one (squares, edges, diagonals, vertices). The attention is focused on a particular structuring. For secondary students this only involves making them conscious of this structure. The octahedron and the tetrahedron on the contrary involve an expansion of the world of perception. Many pupils are seeing those polyhedra for the first time. Since these polyhedra are placed in the same context as the cube, the structuring for these (polyhedra) takes place simultaneously.

It has been my opinion that regular and other symmetrical polyhedra are the simplest bounded figures in the plane when they are viewed in a context of symmetry. It appeared from the class discussion (see III, p. 18) that the concept of symmetry is already structured in the field of perception of 12 year-olds.

1. The pupils are able to perceive that the drawing represents one half a vase.

2. The pupils are able to add on the other half.

3. The pupils are able to discover mirror points with respect to a line.

Through asking questions a stronger observational ability develops which brings about the structuring.

By using a mirror with reflecting surfaces on both sides, the pupils were able to determine the axes of symmetry for themselves and they were able to correct mistakes in construction without my help.

The questions; Have you seen symmetrical road signs? capital letters? etc. also stimulate observation with respect to geometric figures. The question: "I have a round table. How can I find the center?" presents the pupil with a simple problem. They follow the discussion attentively: what is being suggested by one pupil is being refuted by another one, and finally they find the answer collectively. First measure the circumference, then divide the circumference by two, this produces 2 symmetrical points with respect to the center, then draw the diameter and divide this in two.

The discussion of the tabulation of the observations: circumference, diameter, ratio can be made interesting for the pupils. The Babylonians used the number three in their computations. How can we show in the figure that we do not use 3?

We first estimate the result before we compute it. How do we go about rounding off? Why do we make so many measurements? All these are important questions.

The symmetry of the rhombus is studied especially because this forms the link with the first basic constructions. The properties of the diagonals of the rhombus can be associated with the construction of the perpendicular bisector of a line segment, a bisector of an angle, a perpendicular through a given point on a given line. The rhombus thereby becomes the concrete basis upon which all basic construction can be patterned. Sometimes one uses a semi rhombus or one uses 2 semi rhombi with the same diagonal (kite).

The pupils never ask why the altitudes, medians, etc. of a triangle intersect in one point. They add this experience to the observations.

An angle is an open figure. This can lead to confusion. The pupil will prefer to see a closed figure. That the lengths of the sides are irrelevant is, in a way, an abstraction. For this reason, the notion of angle has to be introduced with care. I link it with the direction of the (human) legs and I clarify it with arm swings. This also helps to indicate the region belonging to an angle. Re-entrant (reflex) angles and angles greater than 360 degrees can also be demonstrated clearly with this method.

Two circle disks of the same size but made out of different colored cardboard can be slid onto each other after each one has been cut along one radius. This produces a movable model through which the increase in the size of an angle can be visualized.

CHAPTER V

NEW DIDACTICS RELATING TO THE LEARNING OF GEOMETRY IN THE FIRST AND SECOND STAGE

How and in what way do we experience in textbooks new insights relating to the didactics of geometry? Boermeester says in this connection: "All but 3 textbooks begin with the traditional subject matter in the first chapter. Of those three, one starts with that material in the middle of the first chapter and another delays it to the second chapter." I must conclude from this that the mathematician in the teacher cannot yet resist the temptation to build up the geometry subject matter in a logical sequence from the very beginning.

Occasionally one departs from the logical path: if necessary one omits those items that appear to be too difficult for that moment but does not deviate from the sequence of a logical structure. This is most clearly demonstrated by the diagrams. Because school tradition has the figure of 2 parallel lines intersected by a third line right at the beginning of the logical structure, this figure - even though it may be distorted from a logical point of view - must and will be presented to the students at the very beginning of geometry instruction. The fact that pupils see this figure as a complete entity is not considered in their instruction. Pupils most certainly see the equality of alternate-interior angles and of corresponding angles when two parallel lines are intersected by a third line. Sensory perception is correct.

The difficult point, however, is that the teacher wants them to operate using the following theorems: "If two parallel lines are intersected by a third line, the alternate-interior angles are equal" and "If two lines are intersected by a third line, and if the alternate-interior angles are equal, then the lines are parallel." Of course, one can teach the pupils to fill out a scheme corrrectly. However, can one deduce from that that there is comprehension?

In the observation, parallelism is not separate from equality of the angles. They are Siamese twins for the student, as it were. They are there simultaneously. The one is not the result of the other for the pupils.

A similar theorem is: "The base angles of an isosceles triangle are equal" and linked with that: "If in any triangle two angles are equal, the sides opposite those angles are also equal." Should one desire that the pupils view these theorems as separate, this would require preparatory lessons during which the pupils are presented with a number of theorems to be investigated. Their own investigations are used much too infrequently during class lessons. The emphasis is on explaining the subject matter. The tasks where pupils are actively involved are often considered as a check on whether a lesson has been understood and whether the pupil can apply the knowledge. To do this one chooses simple problems that are directly related to the subject matter covered in the lesson.

Would it not be desirable to examine critically the customary method? Is that logical sequence really necessary at the beginning? What psychological data are taken into account by such an introduction to the subject matter? In his "Adventurous Mathematics Instruction," Kruytbosch (I p. 11) says:

> The intuition of the pupil remains idle in front of the unyielding mass of the logically constructed deductive system; his resourcefulness and interest are paralyzed by it. Against this completely finished structure that seems to be almost indestructible, the pupil's initiative, his desire for undertaking, seeking out, trying, discovering do not stand a chance.
>
> The stiff deductive teaching of mathematics is wrong, not only from a didactive perspective, but also from an historic perspective. Many important results have been found through the inductive method. Why then should these be displaced to an a posteriori role by the deductive method?

If one takes the stand that it is more meaningful for a child to prove a theorem that is not self-evident than to prove a self-evident theorem (see De Groot, I), then one should investigate which theorems are evident for the pupils and omit their proofs regardless of whether or not the proofs are difficult.

> There is in the child no need for greater certainty than the graphic evidence.

Geometry instruction that does not follow a logical sequence is often labeled as "elementary geometry". One thinks it undesirable to treat solid geometry in this fashion in the secondary school because one argues that elementary geometry belongs to the elementary school curriculum. By trying to analyze the learning process in geometry, I hope to clarify this idea for the teachers. In this connection I refer to what Langeveld has said (II p. 479 ff):

> In each educational process we are dealing with a substratum which has to be planted. ... We shall have to establish relations with forms of experience, with known objects in order to arrive at a problem setting that is accessible for the child. First of all, one should prepare or uncover a substratum on which something can grow. Initially this substratum should be our point of departure: We should make use of the life experiences which possess reality for the child... He who succeeds in establishing links with situations which have been built by the child

> himself, stands a good chance of speaking about things which can be meaningful to that child.... One can illustrate, organize, focus and help to assimilate the forms of experience in a certain way. In doing this one must take into accoout that it should be possible to go back to the initial experiences, to repeat, to duplicate new aspects, to analyze connections better and in greater depth, etc.
>
> In the course of the teaching process we have three objectives:
>
> 1) to bring forward or clarify certain initial experiences;
>
> 2) to construct relationships in connection with this, all the while "condensing" and returning, where this is required in the light of the objective, to the full concreteness of continuously unfolding new initial experiences;
>
> 3) to provide insight into such "condensation" (= abstraction) methods and to teach how to work with them.

What Langeveld is saying here about didactics in general, one should be able to recognize in special didactics. In my opinion, one does not pay enough attention in geometry to bringing forward and clarifying certain initial experiences. The objects with which the mathematician works are not new for the pupils. They have already made use of this material, either under the direction of other teachers in the school situation, or on their own initiative in a game situation (mosaic box, building block set, erector sets, etc). Therefore, we should take into account what is called autonomy in the learning process by Van Parreren (I, p. 122):

> If a subject is instructed to practice certain activities and if he intends to carry out this assignment, an autonomous learning process appears to be developing.

The geometric figures have already obtained certain meanings. These meanings can lead to inappropriate actions during the initial stages of geometry instruction because the mathematician considers appropriate only those actions that are based on certain logical rules of the game. By starting geometry instruction using the logical structure of thought one really puts the child into an ambiguous learning situation: the meanings which the material possesses for the children do not fit the operations that have to be carried out with the material. This undesirable situation can be avoided by taking care that already existing meanings are utilized as much as possible in the initial learning situations.

The viewpoint "Elementary geometry belongs to the elementary school and logically structured geometry belongs to the secondary school, both courses being independent of each other," is wrong.

The only correct implication in this statement is the insight that elementary geometry is the natural predecessor of geometry and hence pre-eminently suited as a foundation for it. We, as mathematicians, possess the great advantage of being able to establish relations clearly related to the realm of experience. Twelve-year-old children start secondary school with a very diverse knowledge of elementary geometry. This knowledge has been gained, in part, in the elementary school; in part, in everyday life. Although certain regularities have been noticed in the observed stages, the accompanying ordering of the stages has not been expressed in most cases. The language structuring belonging to this ordering is also absent.

During the first two months (called "first period" at the Amersfoort Lyceum) the geometry class-hours were devoted to observation of figures, naming the figures, ordering of the observed figures according to their properties of symmetry and the introduction of the appropriate language structuring. In order to provide the children with the opportunity to exchange their experiences and to practice the accompanying language, they frequently worked in small groups during this period. This method also brought about an increase of the common nucleus of the structure formation in the fields of perception of the children (e.g. the "greatest common divisor" or the cross section of the geometric structure increased). For the pupils, this period is a time of "the making of models" - that is their task. At the same time there is an exploration of the work domain of geometry. Thus there is a relation between the objects with which geometry is dealing and the outside world. This makes it possible to later reinterpret in the world around us that which has been proven in geometry through logical methods.

For example, how often is the word "congruent" only linked to triangles? Has the child experienced that the word "similar" from geometry has something to do with similarity of shape? By doing this, the pupils arrive at a certain abstraction. They start from empirical shapes and proceed towards rigid mathematical drawings and spatial models. In the protocol in Chapter X one can see how a pupil went back to the empirical shape when drawing sidewalk tiles - she also drew sand in the corners of the sidewalk tiles.

The spatial models given to the pupils for examination, the models they make and the figures they draw with the drawing triangles and compasses all help them to make this abstraction.

Since the regularities of the objects around us are most conspicuous, an ordering of geometric figures according to symmetry is usually possible. Cutting out and folding figures help to develop this ordering because this allows the pupils to find the axes of symmetry empirically. By pinning and rotating, the centers of symmetry are determined. A mirror is used to find that one half of a symmetrical figure is the mirror image of the other half. The planes of symmetry, the axes of symmetry or of rotation and the centers of symmetry provide the pupils with appropriate graphic representations of the basic elements of geometry: the plane, the straight line and the point.

The manipulations mentioned above prepare the pupils for constructing figures. This construction requires a new abstraction. Pupils then do not have to visualize an object in order to be able to draw it as accurately as possible with the help of a ruler and compasses. In this fashion they gradually move from sensory perception of figures to graphic representation.

Graphic representation is recorded in precise figures. Figures that are constructed as accurately as possible can, in turn, bolster their representation. Good drawings very often bring one closer to the solution of a problem. Therefore, the objective of the teacher is to provide pupils with a correct view of the graphic representation of geometric figures. He endeavors to do this in the first place by encouraging pupils to increase their technical proficiency at manipulating compasses, drawing triangles, graduated ruler and protractor. This is a skill that provides pupils with much satisfaction. Their task is to construct precise plane figures of diverse shapes.

In the second place the teacher tries to reach his goal by making a clear distinction between the drawing of figures and the constructing of figures. The constructing of right angles (dropping and erecting perpendiculars) and the bisecting of line segments and angles with the help of compasses are based on the graphic structure of the rhombus. This makes it possible to organize this subject matter well. When the material is inviting (see Linschoten I, p. 130), fewer recapitulations are needed to obtai good results. All the constructions are experienced as a con tion of a rhombus (see Fladt I, p. 26). The rhombus is the cen oint - the real basis during the learning of the basic cons. tions. The ultimate objective is reached when all the basic constructions can be carried out independently. Then the end of the first stage has been reached.

Starting from the data gathered through (1) sensory perception, (2) experiences of movements, (3) spontaneously complemented perception, the pupils have arrived at a graphic representation of the objects with which geometry is dealing. They have learned to translate these graphic representations into precise mathematical figures with the help of drawing materials.

I endorse the standpoint of Berghuys: "To recognize graphic figures as being instrumental in discovering theorems and in aiding memory is consistent with the natural feeling of the mathematician." Berghuys (I, p.6) points out:

> It is common practice that when confronted with a geometric problem one immediately picks up a pencil and sketches the figures. This is true for the professional as well as for the amateur mathematician.

From the didactic point of view it would certainly be appropriate to focus more on the visual geometric structure of the field of perception. In particular, one should bring the pupils in close contact with sets of geometric figures around us. Let them really draw plane fillings (tessellations) and not just see them (see Chapter V). One can read in the protocol (Chapter X) what new experiences this can lead to. Here we are clearly dealing with what Langeveld calls under the second point: to have the pupils find relationships in the observed figures. This is necessary in order to bring the pupils to the first level of thinking. The first level is reached if the student makes operational use of known properties in figures with which he is familiar.

Van Hiele points out that the learning process in a child who is studying geometry clearly follows a discontinous course. The teacher then will have to allow for variations in pace in order to bring as many pupils as possible to the second level of thinking. This level is reached when the pupil is able to manipulate geometric relations operationally (see Van Hiele i). The teacher will thus have to make use of a vast quantity of empirical material in order to help a sufficient mumber of pupils attain the second level.

The beginning levels of thinking are easily attained by pupils of the secondary school. Thus it is certainly desirable to exert great patience during the first years and to give children the opportunity to think at a higher level. One should not reduce the subject matter to a lower level for them, nor should one press students to reach a higher level too quickly.

In the second stage it should become clear to pupils what our (mathematical) task is - what our aim is in dealing with geometric objects. Their task is still to draw precise plane fillings (tessellations).

Berghuys (I, p. 75) writes about the empirical origin of mathematics.

> Before we concentrate on the nature of mathematical insight, it is important to examine the process by which it comes into being. It appears then that sensory perception images lead us to form mathematical systems. The world of our senses appears with multiple nuances. It is the task of the human mind to rule and to subdue this world. For that, it is necessary to fit the phenomena into simpler schemes.
>
> The phenomena can be simplified by foregoing certain qualities which at a particular time have no importance for us. The schematizing itself is not mathematics, it is only out of the conscious schematizing, where attention is given to the scheme rather than to what is being schematized, that mathematics is born.

> The practice of mathematics is a consistent development using the acquired schemes, without paying further attention to the original empirical content and not guided any more by the observations, at least not by those observations out of which the original abstractions were derived.

In summary: In the second stage one should start from phenomena that allow spontaneous ordering. When pupils begin noticing how the same scheme can be applied over and over again, the scheme will be recognized as a scheme and geometry can be studied as reasoning. Mathematical modes of thinking are now central. As clarification, it should perhaps be pointed out that it is not yet the appropriate time to build up geometry according to a logical structure.

The important question now becomes: What problems can guide students towards studying geometry according to reasoning so that they can be initiated into the practice of abstracting?

While pupils are dealing with these problems, the teacher should try to discern as much as possible what kinds of difficulties of thinking they could face.

The history of mathematics can probably teach us to find problems appropriate for this stage.

In this context, the following written by Van Der Waerden (I, p. 100) is fitting:

> ... the Egyptians only gave arithmetic propositions without any motivation. For the Greeks these arithmetic propositions did not represent mathematics; these only led them to the question: How does one prove that? If one examines more closely the theorems that are attributed to Thales, one is struck by the fact that these theorems do not belong at the beginning of the discoveries of mathematics, but at the beginning of a systematic logical exposition of mathematics.

For our didactics, I deem important:

1) the fact that Van Der Waerden has searched for a basis for Greek mathematics;

2) his way of reasoning that Greek mathematics can have been derived from Babylonian mathematics;

3) the fact that he discerns different lines of development in the mathematics of the golden century;

4) that there are 4 stages to be found.

First here is a purely empirical stage, completely in keeping with practical applications. Included are computations of surface areas, computations of volume, computations of lengths of line segments in a figure with the help of equal ratios, the theorem of Pythagoras for right triangles with sides that have rational lengths.

Then there is a stage in which the need arises to give proofs, to search for logical relationships and to develop a logical structure. In that stage, however, the expansion of the theorems still continues with empirical phenomena as a base. If we take the historical development as a guide for the instruction of young children, we also arrive at the aforementioned stages.

According to my experience, the first stage will last 4 months. For the second stage one needs the rest of the first year as well as the second year. Suggestions for appropriate subjects for that second stage can be found among other things in T. Ehrenfest-Afanassjewa (I) and in D. J. Kruytbosch (I).

Neither of the two authors had the intention of giving a systematic treatment and classification of the subject matter over the different years. They give examples to awaken interest and to promote thoroughness.

As for the third stage in the process of instruction: to provide insight into abstract methods and to learn how to use them, I refer to what Van Hiele says about the third and fourth level of thinking in Chapter XVI and about the possibility of reaching these levels.

Chapter VI

THE SUBJECT "TILES"

The subject "Tiles" involves a very old problem: only three regular polygons can fill the plane around a point O with their angles (Dijksterhuis I, p. 14). The fact that this cannot be done with regular pentagons, can lead to the much desired question: "Why can this be done with regular triangles, quadrangles and hexagons, but not with regular pentagons, octagons, etc?" The pupils will wonder about the non-fitting rather than about the fitting aspect of things. The fact that six squares can bound a portion of space, or that all right angles are equal (and therefore fit each other), or that a quadrangle can have four right angles, do not impress pupils. The subject lends itself to raising questions about geometric phenomena. This line of development ends in a completely logical structuring of all geometric phenomena (see Chapter X, p.120).

The most important objective of this subject "Tiles" is to present material that can lead to the formation of visual geometric structures in the pupils. This does not imply that the material will automatically do it. It would be more accurate to say: all the activities carried out by pupils using the material according to instructions as well as discussion of the ideas they expressed during these activities can contribute to the formation of visual geometric structures.

Thus class discussion is an essential part in this formation: first, to give instructions pertaining to the material and second, to make sure that the pupils' ideas which have come to my attention during the activities are made common knowledge to the class. In addition, class discussions make it possible to adjust the instructions according to the needs of the group. During this past year I have often been led by ideas of the children. In so doing, I have been able to gather much more information than during the preceding years. I always repeated the question: "What do you see in this figure?" Of course, because I was writing up a protocol, I was intent on obtaining as much information as possible from the children (see Chapter X).

In this introduction of the subject "Tiles", I will first discuss what my original intention was.

In setting up the material, I started from the following laws of the apperception theory of Gestalt psychology:

1. The law of corformity: corresponding figures in transition are often perceived as totalities.

2. The law of proximity: parts that are in each others' proximity are easily perceived as totalities.

3. The law of closure: closed figures are more easily perceived than open figures. In perception one has the tendency to close open figures.

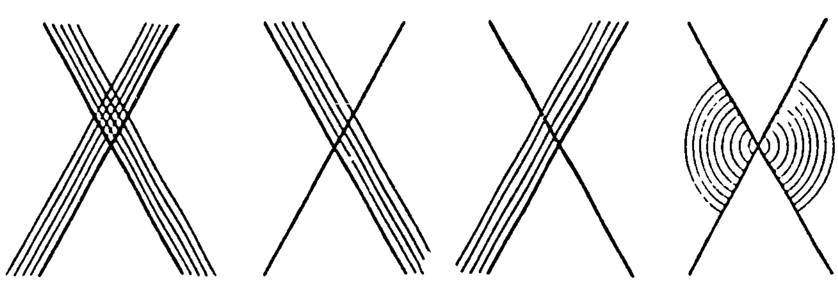
4. The law of exact extension: in perception one has the tendency to extend a figure so that the structure is maintained.

These laws can be concretely traced in geometric figures. Van Hiele (p. 17) discusses these laws in further detail in his thesis. The above-mentioned apperception laws are accepted as laws that are valid for insight - for thinking. Hence we can expect that the introduction of visual geometrical structures, which are based on these laws, will contribute in an important way to the gaining of insight into the nature of geometry.

A.D. de Groot also pointed to the above-mentioned conformity in his lecture mentioned earlier.

> From the Gestalt psychological side, attention is paid to the unique part played by the figure, the illustration, in the development of a geometric concept, and thus in didactics, some problem transformations can be followed up concretely by a "restructuring" of the figure. The elements of the figure undergo function modifications, thus one is able to view the figure in another way.

In the drawings which de Groot provides with the proof of the theorem that opposite angles are equal, he clearly makes use of the above-mentioned law of proximity.



> One has tried to explain lack of geometry (or mathematics) aptitude as a deficiency in the ability to "restructure": the pupil is incapable of arriving at structure and function-modifications which are required in order to understand a proof.
>
> Also when the drawing of auxiliary lines is involved, the ability to modify the structures and to anticipate new structures is especially important.

For this it is necessary that pupils have visual geometric structures at their disposal. These structures are often not clearly imparted to children in the secondary school. (See Chapter XI.)

One will clearly recognize the apperception laws in the following tasks. These form the beginning of the problem of the "Tiles".

1.a. A sidewalk is paved with congruent square tiles. Draw part of a sidewalk. Use a quarter of a sheet of paper. Use your drawing triangle to draw right angles.

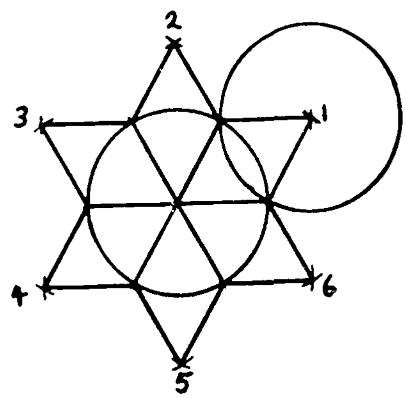
b. It is possible to pave a sidewalk with congruent square tiles in yet another way. Draw a part of that also. (Note: to make the edges one may have to break some tiles in pieces.)

c. Which pattern would you prefer for a bicycle path? Why? (This question is asked to draw attention to the straight lines in the figures 1a and b).

In the second task a stellated hexagon is drawn and pupils are asked to continue this pattern in the same way. One obtains a very regular pattern of rhombi.

2. Draw 6 congruent rhombi, that are contiguous to each other and that meet in one point (see the figure); use compasses and start with the circle. The sides should be 2 cm long. Start another such starshaped hexagon with point 1 as center. Continue this figure also until you have covered half a pe e.

(The pupils see a totally different plane covering being developed in which even a three dimensional image can be seen).



Each student is now given 4 small bags containing regular triangles, pentagons, hexagons and octagons. They are asked:

3.a. Is it possible to pave a floor with regular triangles?

b. Same question with regular quadrangles.

c. Same question with regular pentagons.

d. same question with regular hexagons.

e. Same question with regular octagons.

Write down the results and illustrate with drawings.

(The answers are arrived at by exploration, and the question of why it is not possible to pave a floor with pentagons and octagons remains unanswered temporarily).

The next task is very similar to the preceding one. Each pupil is again given 4 bags. This time, however, they contain irregular triangles, quadrangles, pentagons and hexagons. The pupils are given tasks similar to the one above. They are asked whether straight lines such as in figure 1a. are present. Most students find three arrays of straight lines in the problem of the triangles. In the drawings these lines should be made clearly visible; the drawing has to be done very carefully (a whole page full). The figures drawn in response to the other questions can be made by simply tracing around the shapes that were provided. Half a page is sufficient. The answer again is found through exploration. The pupils find they cannot tile a floor with the hexagon. Again the question of why the pentagons and hexagons cannot be used remains unanswered.

In task 5, we color the smallest angle of each triangle drawn for task 4, with red, the largest one with blue and the third angle is given yet another color. We discover collectively that the three angles of a triangle together form a straight angle. The straight angle is observed in different positions (see Chapter X). Similarly, the pupils find that the four angles of a quadrangle together form a round angle. Then task 6 follows: can the latter proposition be demonstrated with the help of the former? This provides the key to answering the questions that remain unresolved. The regular hexagons stand apart. Only 3 of the 6 angles meet in one point.

The regular hexagons again help us to see that perhaps we could imagine pentagonal tiles with which a floor could be tiled. For this the children are given the following tasks:

1.a. Find pentagonal tiles with which a floor can be covered. Use a sheet of squared paper and cover one quarter or one half with a pattern made up of the same tiles.

b. Repeat the assignment using hexagonal tiles.

c. Color the drawings such that two contiguous tiles have different colors, yet using as few colors as possible.

The logical relationship that has to be discovered in task 6 can be written in the form of a genealogical tree. §See next page.£

The sum of the angles of a triangle equals 180°

The sum of the angles of a quadrangle equals 360°

<u>each</u> quadrangle can be used to fill a plane

The sum of the angles of a pentagon equals 540°

<u>not each</u> pentagon can be used to fill a plane

The sum of the angles of a hexagon equals 720°

<u>not each</u> hexagon can be used to fill a plane

One can see from the protocol written in Chapter X how the same material possesses much additional potential. It is also apparent how during this year we progressed somewhat further, as far as the logical relations are concerned, than during the preceding years. This was possible through the introduction of the figures "saw" and "ladder".

Far more important, however, is the fact that this same material contains many other lines of development which have not yet come up for discussion. Besides the aforementioned lines of development: "the sum of the angles of an n-gon" and "the logical structuring of geometry", the following lines of development can be considered: "equality of areas", "similarity", "kinds of symmetry". These lines of development have not appeared to full advantage in the first seven tasks.

We arrived at the above-mentioned extensions by comparing the drawn plane coverings with each other. One figure is present in the other. This means that the equilateral triangles are present in the rhombi, the rhombi are present in the regular hexagons. By erasing line segments and by adding line segments, the pupils can practice "seeing" one figure in the other.

In this way, the restructuring of a figure is practiced. This is necessary in order to be able to produce a proof and to find the correct auxiliary line (see Chapter X, p. 98). At the same time the method allows for identification of components of the whole structure and completed new structures can be derived (see Chapter X, p. 103). The process of abstracting is practiced in the same way (by abstracting, we mean focussing on a certain aspect of the figure while simultaneously disregarding the rest of the figure). The structuring, restructuring and abstracting are all functions of mathematical thinking. The material (material thought of in a broad sense) contains possibilities for developing logical reasoning and thinking.

The question: "What do you see in the figure?" led to the exploration of enlargements (see Chapter X, p. 104), thus exploring similar figures. In order to obtain an enlargement, all line segments have to be enlarged in the same ratio, but the size of the angles should not change. This material serves as an introduction to the proportionality of line segments (see Chapter XI). The basic figures for the study of proportions, necessary for a logical development of geometry, can be found in the above-mentioned plane fillings (tessellations).

Through comparison of figures of different shapes, it can be ascertained that their areas will be equal, if they can be constructed with the same components. This forms the basis for the study of surface area. This same principle can lead to determing the ratio of the areas of similar figures (only if the ratio of their sides is rational).

Finally one can look at the symmetry of figures (see Chapter X, p.124). Flipping (folding in two, mirroring with respect to a line), rotating (mirroring with respect to a point), shifting (translation) can be demonstrated easily with the material. Axes of symmetry, center of symmetry (with their order) and possible shifts in direction can be determined. This also prevents the pupils from becoming attached to figures in a certain position. It allows for the development of certain dynamics in visual images. This is also helped by letting the pupils look at each others' drawings.

As a climax at the end of the problem of tiles, the following assignment is given: "Color the floor tiled with rhombi. Take as colors for example: white, grey and black, or in any case a very light, a very dark and a regular shade; what do you see?"

An ambivalent figure develops: one sees cubes in two ways: from the top or from the bottom. The perception of the person determines the formation of the structure.

Chapter VII

APPROACH OF THE DIDACTICS OF GEOMETRY BASED ON THE PRACTICE OF TEACHING

In this chapter I will discuss how I reached my present position on didactics. As is the case with most teachers I was prepared for my task only in the area of the didactics of the subject matter. While practicing, one accumulates an array of data. This led me to criticize existing teaching methods and to explore new ways of teaching. My didactics are thus based on (a) experiences gained from practicing, (b) exploration and (c) corrections through the practice. What follows now are purely subjective considerations that evolved by looking back at my own teaching experience and at the way other teachers modified their teaching after they had had some practice in teaching. A new teacher starts his career by using methods that were tried and used by others before him. Very often he uses the method of his former mathematics teacher. He complements these with his own experiences and so arrives at general teaching rules that are important to him.

After further exploration there appear to be exceptions to these rules. In such cases it is necessary to examine other methods and to gain new experiences. So, the teacher is always experimenting. This enables him to establish a progressively more finely structured system of knowledge and of experience with teaching methods. His own methods can finally play an important part in this system. Teachers of schools where more attention is being paid to individual instruction gain more experience, of course.

It has appeared to me and to others that many of the customary methods are not satisfactory. This has led me to develop a didactic method that is strongly based on the experiences gained from practicing teaching. It is of importance to describe these experiences in broad terms.

My first experiences go far back in the past. At the transition from the second to the third class of the secondary school, my schoolmate was given an assignment in geometry. I carried out the assignment, while she played with the chicks. I found that she had absolutely no interest in geometry problems and that I was not able to instill this interest in her. She finally copied the assignment and handed it in. In the third class I used tricks to make her solve simultaneous equations. Her grade on the test was 10, mine was 9. This time she was motivated to concentrate more on her work because promotion to class 4a was approaching.

I want to report another experience. During the school years 1938-39 and 1939-40 I was employed temporarily in different types of schools where I substituted for teachers who were drafted into the army. Before that I had already had three years of teaching experience. I discovered that the pupils were able to solve the problems only if I used the methods, schemes and language of the teacher. Because I experienced this phenomenon in all the schools, I gave a lot of thought to the value of such instruction. My impression gathered from this and later experiences is the following:

When mathematical language is used too early and when the teacher does not use everyday speech as a point of reference, mathematical language is learned without concomitant mathematical insight. The pupils then use a kind of analogy process: they work by a sort of "feel" and they try to guess which answers are expected of them. Their own activities consist of filling out schemes. The thinking activity, necessary to understand the logical background of the schemes, is non-existent. In this connection Kruytbosch (7, p.31) talks about students who prefer to follow thinking schemes because they dare not think. Van Hiele discusses this extensively, among other things, in the Chapter 6, "In what way does insight manifest itself in children?"

The above-mentioned experiences were confirmed by my later experiences. My conclusions are:

1. In general, the children do not know what they are building up.

2. The children do not know what they are building with.

By teaching a few things about a topic, one does not make the subject matter an experience for the children; on the contrary, it remains a story. Initially one can also expect little interest from the children for a mathematical ordering. When is one going to systematize? Only when one wishes to obtain an overall picture of something that presents itself to us more or less chaotically. It is only meaningful to start ordering when one sees that there is something to be ordered. Only then can one start thinking about the way in which this something can be ordered. At this stage then the pupils can search for an ordering scheme under the supervision of the teacher. Twelve-year-old children have not had enough geometry experiences to be able to start ordering them immediately. Even if these experiences were present, they still would have to be brought to a conscious level and the pupils would have to be provided with the appropriate language structure.

It follows from the considerations noted above that the teacher must provide pupils with experiences from which the need for ordering and for logical thinking can arise. Initially, children's thinking is not logical. At least, it is completely different from what one views as logical thinking in mathematics. The statements of children are rather matter-of-fact. So, at the beginning of geometry instruction, one cannot require children to reason in a logical way. Rather, we have to teach them to reason logically. This does not happen by placing a logically structured deductive system in front of them. Kruytbosch (I, p. 21) notes in this context:

> Logic teaches us forms of thinking. The question, however, is whether it teaches us thinking itself.

If the teacher has patience, a logical ordering develops rather spontaneously. Freudenthal (I) points out:

> The most important pedagogical quality is patience. One day the child will ask "Why?". It is useless to start with systematic geometry prior to that day. Stronger yet, it can even be harmful. Geometry instruction should be a means of making the children aware of the power of the human mind - of their own mind. Are we allowed then to rob them of the privilege of making their own discoveries? The secret of geometry is the word "why". He who does not want to be a spoil-sport should be able to keep a secret.

As the child has the natural tendency to grow, he wants to try things out, to discover. It is wrong then for the teacher to present the subject matter as a completely finished entity, to point out what paths to follow or to explain methods which the children can develop for themselves. Such instruction is devoid of the attractiveness of finding things out for oneself. It also lacks the satisfaction that accompanies discovery. The mistake can be made in individual as well as in classroom teaching. It stems from the nature of the tasks which the children are given. This may be the reason why some teachers have not noticed much of that natural curiosity in children.

The natural tendency of the child to develop is not sufficient. In addition the child has to be motivated to develop with the help of the subject matter being presented. In the case of the adept learner, the development has taken place outside the method that is being used. The expression: "I have no talent for mathematics" mostly indicates that motivation is not present. In contrast with the adept learners, one does not see growth in those children.

Thus it is important for the teacher to ask how to develop pupil motivation with the help of the subject matter. Posing this question leads us to the joint domain of pedagogy and psychology. As a way of achieving this I recommend: Start with everyday speech and slowly proceed to mathematical language; refer to situations that are known to the child. In geometry, the latter means referring to observations of figures and relating these observations to a geometric way of viewing them.

<u>This special way of viewing, whereby figures are looked at in a geometric context, has the objective of bringing the children to the first level of thinking in the study of geometry: understanding that the properties of figures characterize them.</u>

The sequence in which the figures are presented should not be that of the elements: point, line, angle, triangle, etc., but it should fit the natural organization of perception. The working method, which follows from it, appears to have its theoretical foundation in Gesalt psychology. In Chapter IX, I will discuss this foundation of my method in more detail.

It should not surprise us that the method used by many mathematics teachers in their instruction shows many similarities to the method of causally-oriented psychology. Their personal intellectual development came about through application of the causal-explanatory method. In conformity with this, one starts from the elements: point - line - angle - triangle, etc. and one builds, in a sequence of axiom - definition - theorem - definition - theorem - etc., a logical system from the bottom up.

The influence of Gesalt psychology on them is noticeable in uncoupling the element "point" and in omitting the first axiom. The whole deductive system however remains. The build-up keeps originating from the elements. It is not easy to shed such "thinking habits" which one has obtained through education.

These teachers follow the association theory in their working method. They build complex ideas out of the elements through synthesis. In doing this they pay special attention to connections between the elements. This is done by considering many special methods for solving problems; one thereby establishes many links. It is evident that this method of working strongly promotes the existence of "lesson learners". It does not lead to an integration of subject matter in most students, neither does it lead to an integration of methods for solving. They remain distinct little methods with the function of solving certain problems. The emphasis is completely on reproducing arguments.

If the question of learning to think is viewed by the teacher as a question of the feasibility of learning methods for solving, he focuses his didactics on the question: How shall I present the subject matter so that the obstacles encountered in thinking can be overcome as easily as possible? He analyzes the difficulties of thinking, provides many methods of solution, increases his pupils' knowledge of the subject. He focuses on the external result: he wants to <u>measure</u> it. This measuring of intensities is a concept which belongs to physics.

On the contrary, if the question of learning to think is viewed by the teacher as providing ways in which thinking can be practiced, the question arises: How should the subject matter be presented in relation to the mind of the child? He teaches the children only general methods. The emphasis is on comprehension through experience and on integration of the subject matter. He leans more towards a humanities point of view.

Whether or not the teacher has found a good solution, in the sense we mentioned above, will become apparent much later. The results of teaching are not immediately measurable. If one thinks one has found the correct way, one should thoroughly investigate whether the method has met its objective. The appropriateness of a didactic method will become apparent when it has been tried over many years.

Measuring the result of education is less important than education itself. The textbooks presently used, with their uniform subject matter, are not good for our method. They do not provide sufficient opportunity for children to find their own methods of solution.

Introductory geometry instruction will have to be built by the teacher himself. However, it would be important for the didactics of introductory geometry to have a pool of problem sets. These should be problem sets which have been tried by experienced teachers over several years and that are accompanied by a protocol and by comments in which the objectives are clearly explained.

Each teacher would be able to draw from this pool in order to arrive at the most fruitful teaching. At the same time, one should make the materials available, e.g. construction kits. One should keep in mind, however, that material that is being used by others without documentation, most probably will be misused as was the case with Ligthart who used wall pictures in his working method. In "Pedagogical Studies" (Jan Ligthart, commemorated by G. Oosterkamp) of April, 1956, one reads:

> The schoolmasters did not completely ignore Ligthart, they hung the wall pictures, which he had recommended as a stimulation for active instruction, on the wall. Then, at a certain hour of a certain day of the week, a wall picture was put in front of the class and commented on.
>
> In this way, everything was brought into the sphere of intellectualism and it dried cut there. Of the children's activity little or nothing was left.

I can imagine that teachers will use construction kits because of the pressure of the times; psychology certainly has its influence. It is expected that children should be kept happily busy during a portion of the class period. This frequently happens by having children listen to some stories by the techer. If during a class period, it doesn't go further than having children merely involved in cutting and gluing materials, one could do better by giving the pupils a construction kit of a plane or a boat or a building.

The objective of using a construction kit, as mentioned above, is to provide each child with his own geometry materials. The child can get so much out of this material if the teacher knows how to place that material in the right context. The pupils - and initially also the teacher - start making discoveries; they explore. If a teacher does not want to conform his didactics to the needs of the child, it does not make sense to force him to change his didactics. Perceiving the needs of the child is not everything. One will have to convince the teacher of the need for certain training (re-training is not correct here; supplemental training is probably better) in the area of pedagogy (psychology).

Chapter VIII

APPROACH OF THE DIDACTICS OF GEOMETRY INSTRUCTION FROM THE THEORY

"There really is no generally recognized didactics for mathematics. Actually the didactics is not elevated above the level of 'I do it this way' and 'I do it that way'." This is written by Streefkerk (I), one of the writers of Euclid, Journal for the Didactics of the Exact Sciences.

I personally object to the frequently used statement: "I have been doing it like that for (several) years and it works very well." As long as subjectivity predominates in this way at the meetings of mathematics teachers, there cannot be a collective scientific development of didactics. The methods of mathematics instruction are still too much determined by tradition and intuition and too little by purposeful reflection. I should point out here that the didactics of mathematics is still a budding science. It is therefore appropriate to cite from Kohler's (I, p.40) second chapter of his Gestalt-psychology. This chapter is entitled: Psychology as a young science.

> Even in our day, Rontgen did not at once make measurements when he discovered X-rays. First of all he had to analyze their properties in qualitative experimentation. Later, of course, his rays could become a means of measuring constants of crystals. Much too easily do we forget the fact that, at their start but also when more particular new fields come into sight, the natural sciences rely almost completely upon qualitative observation. ...wherever we have a good quantitative problem in psychology and correspondingly accurate method of measuring, we can immediately apply procedures which are comparable to those now used in physics. ...for the majority of psychological problems this is not the case. Where in psychology have we that knowledge of important functional relationships on which indirect and exact measurements could be based? It does not exist. Therefore, if the development of more exact methods presupposes the existence of such knowledge, the gathering of it must be our first task. For the most part, our preliminary advance in this direction will have to be crude. People who protest in the name of exactness do not understand our situation in psychology. They realize neither the nature nor the historical background of indirect quantitative methods. If we wish to imitate the physical sciences, we must not imitate them in their highly developed contemporary form. Rather, we must imitate them in their historical youth, when their state of development was comparable to our own at the present time. Otherwise we should behave like boys who try to copy the imposing manners of full-grown men without understanding their

> raison d'etre, also without seeing that intermediate phases of development cannot be skipped. In this respect, a survey of the history of physics is most illuminating. If we are to emulate the natural sciences, let us do so intelligently.

Since I am convinced of the validity of this statement, I have not carried out measurements as are done in the natural sciences. The tests a teacher usually administers at the end of a learning process serve as a means of classifying the pupils, e.g. good, average, poor (or a more detailed classification). These tests are usually not analyzed in the context of psychology of thought. These tests do not provide decisive answers as to <u>the way in which</u> learning evolved. In "Introduction to Psychology" with Langeveld (III,p.399) as principal author, mention is made of an experiment by Gertrud Bauer (one of the students of Selx).

> She let children - 9 to 11 years old - define a number of concepts, e.g. letter carrier, house, thirst. ...She first put the children to work without giving any directions. Then she gave some instructions concerning the correct method of defining by critically discussing a number of definitions. It appeared that the results improved dramatically.

Such investigations do not give us any clues about the didactics to be followed. They only show that there is a significant difference between the presence and absence of instruction. The method that was followed and the way in which the instruction was provided were subordinate. It was precisely because this kind of experiment dealt with determining the ability to learn methods of solution by using <u>quantitative measurements</u>, that the didactics - the way in which learning evolves - was not considered.

If we think, however, that Van Parreren (I) has demonstrated the existence of independent processes in learning, then the didactic approach that is being followed should be taken as an important factor in the experiments designed to give quantitative estimates of intellectual performance. I would like to illustrate this with an example that is known to us all.

The operation "dividing by a fraction" is eventually applied faultlessly by the children of the elementary school. This operation can be taught to them in the following way:

> "The dividing fraction first stands on its head before it is multiplied. (Kurt Strunz I, p.85).

If one should follow such a didactic approach, it would be aimed at a structure of thinking that has not developed in a rational way, i.e. not by discovering a pattern. Therefore, this structure of thinking has not evolved as a result of acquiring insight. This procedure then leads to an independent thinking process that has been established by associations acquired through the material.

In a didactic experiment we will have to ask ourselves what measurements actually make an acceptable substitution possible. When measuring intellectual performance, the didactics that is being followed should not be left out of consideration. Even though the teacher may try to bring the pupils to a structure of thinking developed in a rational way, he has no certainty that he will succeed. Associations can evolve irrespective of the intention of the teacher. I will illustrate this with an example:

The basic constructions of geometry can be learned in totally different ways. The teacher can teach children the manipulations they need in order to arrive at the desired constructions with the help of compasses and ruler. He can also teach them to see the constructions as constructions of a rhombus. In both cases the children will acquire associations. Through these associations independent learning is uncovered. The associations appear to be present because children can carry out the constructions without thinking. They do them rather automatically.

The teacher who has tried to establish a rational base, i.e. the learning of the constructions on the basis of the properties of a rhombus, does not know, however, how the associations evolved. They could have evolved as a result of the pupil observing and imitating the manipulations carried out with compasses by the teacher. This can take place without the acquisition of a gometric concept. The attitudes among the teachers can be very different. It is still possible that the results among the pupils will be the same.

When one introduces subject matter too early, it is expecially true that associations without sufficient rational base will be formed. The ultimate result: actions, answers, solutions can still be correct even though the correct concept may not be present.

Teachers are responsible for the development of this kind of association to a greater or lesser extent. When pupils help each other, it usually consists, especially in the earlier years, of helping in the formation of associations outside a rational base. The schemes given by mathematics teachers to solve certain problems can also cause the formation of associations outside a rational base. I refer here to the known schemes that are customary for computations involving logarithms.

In order to prevent incorrect formation of associations, one should be cautious about providing schemes (e.g. explicit outlines, plans,...) at the beginning of geometry instruction.

It would be different if the solving of problems were the goal of instruction. In that case, the manner in which independent operations came into existence would not play a role. No matter how the associations evolved, the ultimate way of operating is exactly the same. One can no longer object to the upside down position of the fraction which is the divisor: the computations using the trick are carried out faultlessly. Whether or not one agrees with a fraction standing upside down, completely depends on one's objective of teaching.

Such a didactic approach most probably will not lead to integration of the subject matter or of the available methods of solution. Therefore, many other associations will have to evolve before the concept becomes operational (Van Hiele, p.00). An example of a subject that frequently is tackled the wrong way at the modern secondary school is descriptive geometry. There the subject matter is mostly drilled into the pupils' head in the form of recipes. It would be much better if this subject would be taken as an auxiliary subject to stereometry.

Didactics is closely related to pedagogy and psychology on the one hand and methodology on the other hand. The articles in the Euclid Journal cover the area of didactics that relates to methodology. Didactics then is subordinate to methodology and has to orient itself according to the method that is being followed. This is seen in the didactic experiments of Mooy and Boermeester. For them, the method is paramount. The class conversation is held in order to improve the results of the method that is being followed. The method also effects some organizational changes in the school. One or more hours per week are added to the school curriculum during which the pupil can consult with the teacher.

In the article by Beth, referred to above, one can read that Jaensch also wants to complement the existing system in that manner.

> Perhaps it would be possible in teaching mathematics to introduce into the curriculum a systematic logical build-up one hour a week - even that is not necessary - in which free productive construction play is to be used to stimulate understanding.

One tries to maintain the logical systematic build-up of mathematics by means of didactic tools.

Turkstra[1] says that we still are in an impasse as far as initial geometry instruction is concerned. We are imprisoned between two poles: the formal-logical treatment and the graphic-psychological.

> On the one hand, each mathematician with some feelings for didactics accepts that the strictly logical method for beginners is no longer suitable, but on the other hand, one is afraid to grapple with the consequences of a radically new system that displaces the previous one.
>
> The textbooks are a clear reflection of this. Aside from a few that definitely have gone in a new direction, most books have kept the old structure, even though they have adjusted as far as style and objectives are concerned, some even as far as didactic objectives are concerned. However,

> and this is most important, we have not yet found the correct way that would give general satisifaction for our <u>initial</u> geometry instruction.

There has been a tendency in recent years to use the results of cognitive psychologists. For example, Selz has written about the process of thinking in order to account for the didactics that is being followed and to maintain the method of a logically built deductive system. However, these cognitive psychologists do not provide answers about learning. When we interpret their theory in relation to didactics, we will have to do so with the greatest possible care (Van Hiele, p.30).

There still is a large number of teachers for whom the objective of geometry teaching is learning, from the very beginning, a logically built system of theorems. They, therefore, follow the formal logical method of operation; they proceed according to a deductive method and they assume that the analysis of visual geometry has already taken place, or they do not see the need for such an analysis. Klein (I, p. 172) states the following about this:

> Pure logic cannot provide the foundation of mathematics. Deduction can only begin when the first part of the problem is solved: when one understands a system of simple fundamental ideas and assumptions - the so-called axioms which state the most basic self-evident facts.

If one assumes that analysis of the visual has already taken place, one has dealt insufficiently with the question which Klein raises:

> An important question for pedagogy is: how, in an individual, is spatial visualization developed to the degree of precision we need in mathematics?

In reality, teachers take the viewpoint of the university teacher. They consider it their task to merely transfer culture which automatically should include the learning of precise thinking. In this case also (a deductive system built on insufficient foundations), the concept will only become operational when many associations have been established.

It should be pointed out that didactics is just as old as teaching itself, hence very old. From the many modifications it has undergone in the course of time, it appears, just as for the whole realm of teaching, to possesses a dynamic character. The study of didactics, however, failed to develop which explains why it has not become a science.

One cause was a too hasty transition to quantitative measurements. The outcomes of these measurements were judged to be more important than the didactics which should have contributed to the performance of the children. The learning process was not analyzed; therefore, the way in which pupils learned and achieved was not sufficiently investigated. Didactics cannot be judged exclusively by means of tests administered at the end of the learning process.

Didactics is a practical science that has not been sufficiently objectified. Ways of comparing individual teaching performances have not been found nor has a language been created to make statements about this science. Since one is unable to deal with the heart of the problem, one keeps looking for a solution in the organization of subject matter to be taught.

It will only be possible to investigate what positive results several didactic methods have had when the means has been found to compare these methods objectively. A next step will be to investigate how these results have been arrived at. One will subsequently have to find out what factors govern the learning situations and to what extent these factors can be reconciled with the didactic method to be followed.

It seemed appropriate to me to first provide a description of my own experiences, gained from practice, and then to arrive at a foundation of my own didactics while taking into account the results of psychology and pedagogy. This foundation simultaneously provides the motivation for the experiment and determines its objective.

Footnote:

1. Turkstra en Geursen. Psych. did. inl. bij het werkschrift, Groningen 1949. [See p. 55.]

Chapter IX

FOUNDATION OF THE METHOD I USED DURING THE INITIAL GEOMETRY INSTRUCTION

It appears from the preceding chapters that didactics always should be viewed in connection with the objective of teaching As Freudenthal points out, it is the general consensus that learning geometry is a means of letting children experience the power of the human mind, i.e. of their own mind. The goal of teaching geometry is focused upon exact thinking. The degree of rigor is very closely linked to the concept "exact" in mathematics.

What degree of rigor is needed for logical thinking? Heything (I) formulated the following propositions in an article entitled "Mathematical rigor in science and school:"

1. Mathematical rigor is not a fixed concept but has undergone an evolution in the course of time.

2. For the school, only a form of rigor that has been approved by science can be used.

3. Not the degree of rigor attained, but aiming for and developing exact thinking are important for the formative value of mathematics instruction.

4. Aiming for rigor should not take place at the expense of the graphic content of the subject matter.

I would like to add here a fifth rule, expecially for initial geometry instruction:

> In the first years one should not impose the same standards on all students as far as rigor is concerned. The latter should be allowed to vary from case to case.

The teacher may require more from one pupil than from another.

The theory of layers of consciousness of the School of Cologne can perhaps show us how to develop exact thinking. Willwoll (I, p.162 ff) presents the theory as follows:

> The lowest layer contains images of the observed object, while a knowledge of the state of affairs remains in the background. Above it are layers where this knowledge is more extensive and the concrete is less visible (and becomes schematic).
>
> On the contrary, systematic joining of concepts occurs in an abstract uppermost level which spreads over the visual layers and guarantees clarity of concepts and results of

the thinking.

> Although the experience of formulating concepts includes processing of visual material, it is not totally visual. On the contrary, the visual can occur apart from the conceptual, producing only design. The visual is not the essence of thought, but only its garment, its resemblance, its means of expression.

According to this theory, the human mind is partitioned in different layers. Certain layers are never present by themselves. The lowest layer, the layer of graphic representations, is directly connected with sensory perception. In the higher layers, thinking is more prominent, graphic information is being ordered and mutual relationships are being ascertained. In the highest layer, abstract thinking occurs (i.e. thinking in schemes, non-graphic thinking in categories) through which the world of concepts is being ordered in a surveyable way and through which the goal - directedness of thinking - is brought about.

> The visual foundations of thought do not surpass thought itself. They are governed and shaped by it.

This lowest layer provides for actual experience.

> Personal motivation is more related to the layers of visual representations than to the more abstract highest layer.

Should one ignore this layer of individual representations in didactics, one risks that either the instruction will degenerate into verbiage because it has no links with the reality of the child, or individual representations will crop up at the most unexpected moments. These individual representations are not brought under the control of thinking precisely because of this negation in didactics. They have not been incorporated into the structuring and they impede thinking.

> Thinking should be concerned with restructuring visual material only after it has aimed at understanding the real subject matter. Conceptualization is not visual in its nature; likewise a blurred outline remains a visual representation of things. We find that it supports thought in the formation of concepts, structuring intricate visual perceptions as it does so.

The above-mentioned statements from psychology show the need in didactics for establishing links with the lowest graphic layer in order to be able to arrive at fruitful instruction.

The special didactics of geometry therefore starts with perception of figures. Perception of figures is an object of study of Gestalt psychologists. Kohler gives extended reports about perception. Experiments have shown that the joining of parts into a group takes place spontaneously during perception. Kohler (I) devotes a chapter to "spontaneous organization and group formation in the fields of perception."

> This ordering cannot be the result of complex learning events but must occur through the senses. The ability to work analytically grows with age.
>
> There is an interrelationship between the visualization of time and the visualization of space.

This same organization that brings ordering into space appears to be present also in the ordering of time and, according to some cognitive psychologists, in thinking as well. This is not the appropriate place to discuss more thoroughly the physiological exposition of Kohler on this meaningful organization. The four laws of apperception theory of Gestalt psychology have been mentioned in Chapter VI. Van Hiele (p. 18) has discussed these laws in his dissertation in relation to insight and he further refers us to the work of Van Parreren. This author provides a necessary addition to Gestalt psychology by his distinction between intention and autonomy in the learning process. Van Parreren (I, p. 140) writes:

> Apart from intentional and rational <u>guidance</u> that consciousness can provide to action, conscious processes can also be the <u>result</u> of autonomous acting and they can reinforce the latter through their emphases.

According to Gestalt psychology, one should aim in the first place at letting pupils experience the subject matter as an integrated entity. One thereby maintains the same organization as the one that exists in perception. It is therefore desirable to partition the subject matter into "units" that form surveyable entities for pupils.

We will also have to pay attention to the age and mental level of the child, because the capacity to analyze increases with age. Should one present subject matter that conceptually belongs to too high a level, then the pupils can only arrive at results through independent actions based on associations acquired from the material. These associations are formed independently of the intention of the teacher, and this will happen even though the subject matter may form a beautiful "unit."

One can also conclude, on the basis of the theory of Van Parreren, that pupils must be given enough time to arrive at the correct formation of associations. One therefore should not start with a new unit until the previous one has been thoroughly assimilated. In this connection, Van Hiele (p. 41) points out the importance of the plateaus in the learning curve. Streefkerk (I) writes:

> There are as many didactics as there are authors; or perhaps not. Are the didactics that form the basis for all

> those books except for one, (I am thinking, for example, of the books of Dr. Van Hiele) not really all the same, such that there is only one (perhaps completely wrong) didactics?

My answer to this is clear: from a psychological point of view they are all incorrect indeed. In all these books the method that is being followed is primitive: a completely logical system is built up from the elements.

Starting from the building elements rather than from the total building plan can no longer be accepted, because it is completely founded on association theory in its most primitive form. By holding on to this theory, didactics has remained far behind the progress of psychology. Strunz (I, p. 24) points out in this respect:

> The teacher of mathematics - for whom psychology was an unimportant subject to be dispensed with during his study at the university - lacks the requisite psychogical insight needed to question himself about his teaching, not only about didactics and the psychology of performance but also about experience, structure, and aspects of developmental psychology.

In the first stage I therefore started from the field of perception that was already structured more or less in a geometric sense. This geometric structuring has to be the center of attention at the beginning of geometry instruction. The figures, long known from observation, are placed in a geometric context. I have done that by means of models of a cube. First a few questions were asked about it.

The question: "How should the squares be drawn so that they can form a cube upon folding?" changes the receptive attitude of the pupils into an actively organizing attitude. This can be ascertained very easily among twelve-year-olds: the expressions of joy are spontaneous. This is the best attitude in order to arrive at the structuring of the field of perception.

A cube is a regular bounded figure. Kohler (I, p. 145) writes:

> Simple and regular wholes, also closed areas, are formed more readily and more generally than irregular and open wholes. The order of sensory fields, in this sense, shows a strong predilection for particular kinds of organizations...

In Chapter III I expanded on the observation lesson I gave about symmetry. The results of it can be found in Chapter IV. These agree with the above-mentioned statement by Kohler.

During this stage the pupils make models, draw figures, find geometric properties that characterize the figures and finally they construct the figures with compasses and ruler. The pupils have oriented themselves; they have been brought in touch with the geometric objects, their names and their characteristics.

At the end of this stage, the basic constructions can be carried out independently. The properties of figures need not yet be known as associations. They have only partially reached the autonomous layer.

Since I am dealing here with the foundation of the method, I refer to Chapters III, IV and V for an overview of the subject matter covered during the first stage. The pupils were to work through the first five chapters of our workbook for practice.

In the second stage, it has to become clear to the pupils what we are going to do with these geometric objects. We have to give them the awareness that the human mind is capable of logical ordering, of exact thinking. For the didactics of geometry, this means that one should provide appropriate preparation and that one should search for favorable conditions that will allow the pupils to come to an active structuring of the field of thinking.

We again present integrated entities. These however now consist of sets of figures. For this e choose figures that are already present in the visual layer of thinking. We place these in a geometric context whereby the pupils will structure these complex figures in a geometric sense. Since this involves a particular structuring of the field of percep on, the guidance of the teacher is necessary. The latter should not assume that the structuring will take place spontaneously. The pupils, however, spontaneously contribute significantly to this structuring.

The pupils should gradually proceed from visual representations to ideal geometric representations of the figures. The latter are the objects of geometry.

The transition from concrete to abstract can be found in the Chapter "Tiles". There the sidewalks with tiles gradually change into plane coverings with geometric objects. The visual representation is incorporated in the structuring. The children no longer see a picture of something that was perceived visually (a sidewalk) but a picture of something that only exists in thought.

The plane coverings are subsequently placed in a geometric context through the ordering according to geometric characteristics, such as: congruence (which includes equality of line segments and equality of angles), symmetry, similarity, etc. In so doing we always return to the graphic representation; parallel lines, enlargements, saws and ladders have to be incorporated in the structuring (i.e. they also have to be taken up in the geometric context). Through this, the pupils are made aware of the geometric properties of those figures.

The material plays an important role in this. Congruence of plane figures is experienced by means of cardboard models that fit on each other. Symmetry is experienced by means of a mirror that has a reflecting surface on both sides; similarity by means of enlargements. The magnifying glass and the distorting mirror can provide help in detecting the correct characteristics of similar figures.

Parallelism of lines can be linked to direction, distance and slope in the graphic representation. These lead to the geometric characteristics: lines having the same direction, lines which are equally distant, lines with equal slopes. The axiom of parallelism initially does not play a role at all. Through this method of observation, geometric relations and geometric orderings evolve during the second stage. However, one only talks about mathematical thinking when the pupils operate with the relations and orderings, when relations between relations are being found, or when a scheme is recognized as a scheme.

How can we activate this productive thinking in didactics? The productive thinking has been studied in particular by Selz and by Duncker. They study productive thinking in adults and build their theory on protocols acquired through introspection of the test persons. We didacticians, however, are dealing with the genesis of productive thinking. If we wish to draw conclusions from the theory of cognitive psychologists, we will have to test these conclusions in the context of school practice.

Some conclusions of the cognitive psychologists on the thinking process now follow briefly.

Selz talks, in "On the laws of the ordered thinking process", about an operation of a complex. As I mentioned earlier, the same organization that exists for the sensory perception, also appears to guide the thinking process. Selz (I, p. 94, p. 119) writes:

> It has been well established that complex patterns can be reproduced in their totality. It is essential that such patterns be recalled in such a way that they cannot be broken down into their elements.
>
> Research has shown many times over that the will to <u>recall</u> causes <u>certain intellectual operations</u> necessary for such recollection to begin to work. The choice of the will to reproduce a complex pattern produces an intellectual operation which is essentially a <u>reproduction process</u>. On the other hand, a recollection of a schema of the complex pattern could facilitate the process of abstraction or of combination.
>
> The choice of the will to reproduce a complex pattern sets into motion the intellectual operations necessary to generate such a pattern.

Selz distinguishes the following thinking operations:

> Reproduction of a complex pattern, abstraction, combination and completion of a complex figure.

For the completion of a complex figure he gives (p. 128) three laws:

1. A given part of a complex figure which is part of a whole tends to cause the reproduction of the whole figure.

2. A schema, anticipating all the characteristics of a complex figure, tends to cause the reproduction of the whole figure.

3. A firmness of purpose, focused on completing a schema of the anticipated complex figure, tends to cause the reproduction of the whole figure.

Duncker (I, p. 100), however, gives the following critique:

> I believe Selz has gone too far when he considers even involuntary reproductions as determined: that is, as exactly anticipated.

Similarly, Van Parreren (II) argues against this determination: thinking does not always proceed according to rational actions, but there are also independent actions on the basis of associations that have been formed.

Even though Duncker uses a slightly different terminology from Selz, both theories have many points in common. Therefore, it is not necessary to expand on this theory here.

The way a person thinks is not only determined by natural aptitudes and by biological maturation, but education plays a very important role as well. We therefore cannot trace the thinking of twelve-year-olds from the theory of Selz. Langeveld (VI, p. 40) argues that it is incorrect to view the child as a scaled down edition of a future adult.

> It is exactly because there is such latitude in human development that man can be educated.

He views "the principle of exploration" as being of utmost importance; man continously breaks through the concrete surrounding phenomena; behind each discovery is another one.

One often does not sufficiently realize what the influence of teaching is on mental development. This is apparent from certain statements, e.g. bright pupils will always succeed. This sounds as if the kind of teaching does not matter for them. Through better didactics, where the inclination for exploration is more consciously taken into account, a shift in the type of pupil could evolve from the algorithm type to the more structuring type (van Hiele, p. 74).

Prins and Van Gelder (I) concur on the necessity of a return to the phenomenon of learning after a reflection upon the psychology of thinking and learning and on the basis of investigations during recent years. They argue that more attention should be paid to receptive-structuring moments of thinking and that the phenomenon of learning should not be reduced to a cognitive process - even though differentiating may have already taken place.

The words "structure" and "structured" are suitable for use in discussing the foundations of the didactic approach that is being followed. I will, in my explanation, establish connections with the graphic layer of thinking of the reader, in so doing I adhere to my own didactices. Even though a parquet floor is not totally visible, because the furniture covers parts of it, we spontaneously complete the structure in our thinking. In this we can recognize the laws of apperception theory. For the law of the correct continuation implies that we think the pattern is continued under and behind the furniture, exactly in the same way as we perceive it in the visible part of the parquet floor. The parquet is judged differently by a dancing couple than by an interior decorator. The latter will view the floor in relation to the attributes of the whole room, whereas the dancing couple judges the floor by its degree of slipperiness. The mathematician views it in yet another way. He perceives the geometric particularities of the floor such as parallelism, congruence, etc. The context varies according to the attitude of the observer. The parquet floor is structured differently by the above-mentioned persons. The establishment of the context in itself already elicits a spontaneous structuring.

Duncker (I, p. 91) gives the following example:

> The reader may try the following experiment: given the suggestion that he search out everything "red" in his environment, he looks in his room or on the street for everything of that color. There will be an astonishing change - perhaps a familiar one - in the "form" of his environment. It will be structured in a "red" way: everything that appears red will spring forth in an unseemly way. Previously unnoticed objects (like billboards, book bindings, or neckties) will become the chief spokesmen of the surroundings and establish a relationship among themselves. It doesn't happen gradually, but "leaps into the eye" and dominates the color structure of the environment. The state of recognition limps after it. It is also possible to give the suggestion that one search out everything "round" in the environment. Suddenly one discovers totally different forms; the presence of red seems to dissolve.

> Similar results occur in the field of perception whenever one is instructed to attend to a so-called "suggestion."

A particular structuring evolves as a result of the attitude. A mathematician requires a different attitude from a physicist who aims at a completely different structuring. A mathematician, for example, will pay attention to the equality of shape or area when comparing a marble and a wooden windowsill. A physicist on the contrary will look at the equality of temperature.

The context in which figures are put by a mathematician always has a varying geometric character: the figures can be examined for their symmetry, similarity, possibility for stacking, etc.

We therefore are dealing with two forms of spontaneous perception. The first spontaneous perception is independent, the second evolves under the influence of the context. Therefore, when the teacher provides an appropriate context, the children can make discoveries.

A frequently encountered mosaic for a parquet floor is the herringbone pattern. At first one sees lanes in such a parquet floor. After closer examination these lanes seem to be made up from rectangular boards that meet each other in a certain way. Thus structures can sometimes be redifferentiated.

Another frequently used mosaic consists of large squares where each square is made up of four rectangular boards. Here too one first perceives the squares that form the understructure and only after that does one perceive the higher structure of the rectangular boards. This mosaic consists of the same elements as the preceding one.

In the above-mentioned example the thinking operations can be clearly identified because they are linked to perception. The nongraphic thinking operations, however, take place in the same way. This is one of the basic principles of Gestalt psychology. I will therefore consider thinking and perceptin from the same viewpoint: as analogous mental functions that are governed by the same operations. There exists an interaction between both functions, such that, especially in goemetry, it cannot always be determined which function provides the solution and to what extent one function supports the other.

Berghuys (I, p. 81) describes mathematical insight as follows:

> Mathematics in itself is <u>not empirical knowledge</u>. Mathematical activity consists of a schematizing of empirical data, but mathematics, as a science, is a study of this schematization. The knowledge of the schematizing is not the knowledge of the empirical data. Therefore, mathematical insight is something different from empirical intuition.

Nor can insight be characterized as intellectual knowledge of being. For the scheme that is being studied is the form of the empirical data which have given rise to that schematizing. Therefore, it does not provide a knowledge that belongs to the domain of the mind, completely above the phenomenon; the domain of the mind is precisely elevated above ach scheme because of its self-possession.

Mathematical insight however involves both: the empirical material that is being dealt with in the scheme, as well as the mind that holds the material in its grasp in order to master it. For the human mind, with its will and intellect, is known for its ability to subjugate empiricism. At the same time, empiricism is known for its property of letting itself be manipulated by the mind.

In this way mathematics is a knowledge of the mind as well as of the world; not a knowledge of a mere a priori kind, but also of our empirical surroundings.

Perhaps one could talk more readily of a point of tangency between intellectual and sensory knowledge. For the intellectual part is not known in itself, but in its relationship to empiricism; and the empirical part is only known in its relationship to the intellect.

However, this implies that typically mathematical subjects of thought can never be completely pure.

A purely intellectual consideration in which some object would be transparent for us in its whole nature is not within reach for us humans.

The theory of thinking (which henceforth will also include perception) will be summarized briefly in order to make it easily available for the didactics.

As a result of an assignment given by the teacher, goal-oriented thinking evolves in the pupil. This thinking is supported by associations that have already been acquired. The thinking directs itself to a structure. The operations of thinking bring about:

1) a more refined structuring of the perceived structure;

2) a viewing of the perceived structure as a component structure of another structure;

3) the ability to expand the perceived structure;

4) the ability to recognize the isomorphism of the perceived structure with an already known structure.

It is not difficult to recognize here the work of Selz.

As a follow-up, we then have the following rules for didactics. The teacher should aim for the following:

1) that his pupils can build perception structures in a geometric sense;

2) that they will diagnose these structures as component structures of more complex ones;

3) that they will expand these structures in so far as the context permits;

4) that they learn to recognize corresponding elements in isomorphic structures.

It is very important that children have a correct concept of the context which the teacher has established, otherwise a spontaneous structuring cannot take place.

The teacher tries to bring his pupils to a higher level of thinking by means of the four above-mentioned operations. In this way, productive thinking can be practiced. Apart from that, the teacher should establish which subject matter should be learned independently. The transition to a higher level of thinking can only take place when a sufficient number of associations, of the subject matter covered, have been formed.

The teacher will gradually modify the context in a more mathematical direction; he will move further away from empiricism. Then the difference from a physical or technical context becomes clear. A conclusion that has been arrived at in a mathematical way is correct and cannot be refuted by observations. For example, when it has been proven that the sum of the angles of a triangle is 180 degrees, the correctness of this theorem is not affected by an experiment that would produce a result of 179 degree. In the technical and physical sciences one must allow for the possibility that it may be necessary to establish corrections by doing new experiments.

The development of exact thinking has its origin in conscious perception. First of all, facts of experience are ordered. The space surrounding the observer is structured in a geometric sense. The objective of "educating to exact thinking", in fact, implies that during the first years the pupils are taught to better understand and master the properties of space in which the whole of life takes place. One could talk here about an intermediate goal: learning to know and understand space. Van Hiele (p. 144) elaborates on this and names some fifteen geometric aspects that could come up for study during the structuring of space.

Van Hiele (p. 151) further points out that it is far from certain whether it is meaningful for the pupils, who should be counted among the future "consumers" of mathematics, to be able to prove theorems. Therefore, an intermediate goal should be considered at least as important, if not more so than the goal with which I started this chapter. For the "producers" as well as the "consumers" of geometry will have to know and to understand space.

It appears justified to me not to apply the same rigid standards to all pupils of the first years because structuring ability increases with age (see IX, p. 60). One should rather consider these years as an observation period in which to prepare the pupils for different options offered by study programs.

I would like to return to the concrete starting point in the didactics of my initial geometry instruction: the cube. The objective of teaching is: to know and understand space better. This knowing and understanding is viewed differently by the mathematician and the physicist, the biologist, the artist, etc. Therefore, the pupil will first have to be informed about the objects with which these subjects deal and about the background against which these objects have to be perceived. The objects and the context will appear most clearly to the pupils when these can be directly connected to already acquired experiences. For the initial geometry instruction, there are experiences which all pupils have had, that is, in relation to the block building set and the mosaic building set. The pupils have even structured space in some sense, for they have built houses, churches, etc.

By starting with the cube, the pupils rapidly learn to know the names of a number of geometric objects. Next, regularity (symmetry) can be chosen as a context. On the one hand, geometric objects thereby acquire characteristic properties; on the other hand, space thereby becomes structured in a certain sense. In this way, properties of geometric objects are recognized in objects of the real world.

If one chooses as a context the stacking of geometric objects, as is done in the problem set "Rhombododecahedron", one obtains a better foundation for the geometric concepts of area and volume on the one hand; and on the other hand space becomes structured in such a way that the objects of the real world can be thought of as being built from mathematical objects. A house then becomes a rectangular parallelopiped on which a tri-faced prism or a four-faced pyramid is placed. A hurch tower then consists of, for example, a regular prism on which a regular pyramid has been placed. A water tower is then viewed as a large cylinder, etc. A tree must be strongly idealized before one could recognize a cylinder in it. In the e way, one would have to idealize the leaves in order to be able to show their symmetry. For that reason, leaves are less suitable for arriving at an analysis of symmetry; many objects and tools lend themselves better for learning symmetry. Many of these objects and tools are more clearly symmetrical and they can be more clearly associated with geometric objects. (See IV, p. 30.)

It is possible for space to become more finely structured by pupils by varying the geometric context. However, it appears <u>necessary</u> to me that one ask oneself in what way space is already structured for the twelve-year-old child and on which level of thinking the geometric context for 12-13 year-olds can be considered to be. For this, we have to know how far the child has proceeded in his development in that respect during the elementary school period. The progress of development of intellectual functions of the child has to be borne in mind especially.

Oberer (I, p. 356) concludes the following on the basis of his investigation of school children 7 to 13 years old:

> As P. Vogel has shown, we can establish that in a thirteen-year-old, not a single category of thought process can be found which had not already been in evidence by the age of seven.

In connection with this he made use of a classification according to sixteen "forms of thinking" among which are the following:

> classification, function, property, whole and part, means-goal, cause-effect, genesis and development.

He comes to the conclusion that the child is capable of a synthetic train of thought, but that during the whole elementary school period, analysis type thinking predominates. The most important form of thinking according to him is "whole-part" and he postulates that the other forms of thinking can be reduced to that one. The form of thinking "whole-part" should be viewed as a "consciousness of belonging together" for the seven-year-old child. The child names the components of a whole without the connection. In older children however a "consciousness of relationships" develops.

> In the main, there are objective learnings which can be formulated as expressions.

Therefore, in didactics we should certainly not forget to choose a clearly visible goal for the child. The child would like to know why he is doing something: the teacher should not lose sight of the motivation. In my special didactics during the first two months, that means for the pupils: What can we make with the help of compasses, drawing triangles, protractor, cardboard? The teacher hopes to realize his objective: a geometric structuring of space by the children. Space was already classified into parts by the children. These parts in themselves form more or less restructured wholes.

I wish to start from these wholes in order to establish the correct connections in the thinking of children. I cannot imagine that 12-year-old children have already acquired enough experiences so that they are able to make a classification of the whole of reality. Just as we are convinced that the total view of an infant does not cover the entire reality, it is also probable that the total view of a twelve-year-old child will considerably differ from that of adults. This involves the scope as well as the structuring of the image. If the didactician, thinking that this would be desirable according to holistic psychology, wishes to start with the entire reality in order to let twelve-year-olds classify it, he will have raised his expectations too high. One cannot structure space geometrically (i.e. one cannot abstract to geometric solids) if one does not know what that implies. Therefore we have to start by letting the children experience with known objects what geometric structuring means - or stated differently - we have to make the children clearly aware of the context in which the perceived objects are placed. If this is accomplished, then the children themselves are capable of structuring reality around them in a geometric sense.

Likewise, the physics teacher will have to make clear to the pupils, with the help of actual experiences, what a physical context entails. After that, the children are capable of structuring the world around them in a physical sense. They then have experienced what it means when one perceives something in a physical context.

If one proceeds differently, one incurs the risk of having to establish too many associations because the desired structuring then cannot be built on top of an already existing structuring. Both extremes: starting from point, line, etc. and starting from the entire reality are not suitable for the child; for both approaches, too large an abstraction is required. For both starting points, a philosophical attitude is needed that does not suit a twelve-year-old. Concrete entities, that can be easily manipulated, are the best possibilities with which to start. From that point one can turn to a thorough examination: the parts of the perceived entity can be investigated, and conversely, the perceived object can be considered as part of a whole. In this way, analytical as well as synthetic thinking can be practiced by means of graphic representations.

Selz (II, p. 8 ff) draws attention to the fact that Gestalt psychologists as well as association psychologists start from dynamic principles.

> Gestalt psychology differentiates itself fundamentally from associational psychology in that it does not employ synthetic structures to categorize phenomenological entities made up of physical elements. It uses just the opposite strategy and starts from the origins of perception, from the wholeness of the source, and searches for a phenomenological structure based upon real experience; it does not use artificially created categories to justify its inner workings, but rather uses the dynamic principles of phenomenological integrity: wholeness, size, order, connection and organization, principles which underlie the nature of this psychology and which account for the diversity of its views. It has little to do with the elegant simplicity of construction as it exists in the physical world.

In contrast with this, Selz says:

> These dynamic principles of Gestalt psychology do not fully explain the structure of phenomenology, but at least establish the foundations of psychological principles of perception which is the basis of its origin. Among such psychological principles is the recognition that the strength of any theory is not acutally self experienced and thus cannot be considered phenomenological evidence. The structure of phenomenology, particularly when using principles of phenomenological development, cannot solely be explained by psychological theory. The appearance of an explanation from psychological theory occurs by combining the principles of phenomenological integrity: wholeness, size, order, connection and organization when these dynamic principles of wholeness, size, order, connection and organization are entirely different. The structural principles of world phenomena are moreover a vast manifold and not a small wonderful simplex like the dynamic principles of the building of a physical world.
>
> The question of constructing phenomenological questions, consequently, is the cornerstone for a new holistic (man-made) psychology.

Selz then gives the following build-up:

> In order to achieve a new holistic (man-made) psychology we need more than a foundation of truth as evidenced in the quality of physical sensation, something the old psychology relied heavily on. We need to be able to understand and combine the aspects of phenomenology.
>
> Just as there are two qualities of this psychology, understanding and combining aspects of phenomenology, there are similarly two qualities for combining aspects of phenomenology to achieve the new holistic psychology: the qualities of comparison and repetition.

He then discusses the system "the sounds", "the quality of colors", "the layers of time" and "the layers of space" and finally he talks about the phenomeno-logical system: "the wholes and shapes". However, a thorough discussion of these topics lies beyond the scope of this book.

Thus there is a similarity between the way in which a structuring evolves that is based on hearing and, for example, a structuring that is based on sight. As we know, there already is a wide range in the extent of structuring in the area of music among twelve-year-olds. In the first place, the environment (the milieu) of the child will have had a strong influence. Natural aptitude will also be an important factor. In general, school will not have taken much part in this structuring. The same, however, can also be said of the extent of geometric structuring of space in the child and of the extent of physical and biological structuring. Therefore, the range of the extent of geometric structuring of space in twelve-year-olds should also be very wide. In the case of spatial structuring, the environment of the child is also the most important factor, the school having only little influence. In connection with the subject of arithmetic, undesirable (faulty) associations can even have been acquired. I especially refer to the geometric concepts of area, volume and proportion.

We teachers at the secondary school, therefore, are dealing with a group of pupils who, because of different natural aptitudes and different environments, are very diverse in the extent of their geometric structuring of the world around them. They even think with the help of a number of associations which have a restraining influence on the geometric structuring. The latter problem could be alleviated through a closer contact between elementary school and secondary school. The Arithmetic Committee, that has already made a start on this, hopefully will be able to carry out its operations in the actual didactic domain. It has already outlined what subjects should be covered in elementary school.[1]

Kohnstamm (I, p. 91) similarly had high expectations of improved didactics:

> Practical conclusions concerning didactics can only be drawn when the logical functions, which apparently can be developed from the first year of the elementary school on, are incorporated into the subject matter and into the assignments which can inspire the child and which also are of real importance for his future. The work of Montessori and Decroly point in that direction, but we are only at the beginning of an era where didactics will be built on the results of the new child psychology. This much seems certain already, that we will have to drastically modify current opinions on the illogical character of the mind of the child.

For the time being teachers at the secondary school cannot count on that and they will have to devote the first period (approximately two months) to finding out how far each child has proceeded with geometric structuring. Through conversations (individual and class conversations) the teacher can find a common base for this geometric structuring. This will be different for each group. By using better didactic methods in the elementary school, it should be possible to obtain more homogeneous groups in the future. Then the formative value of teaching could become an important factor in addition to the development resulting from natural aptitude and environment.

Interest is extremely important for this formation. Kohnstamm calls it "the direction-giving and the problem-conquering moment of intelligence". It is one of the most difficult tasks of didactics to decide how to take this factor into account as much as possible. I gave a few problem sets in Chapter XV that can arouse interest in children. The majority of the problem sets evolved in connection with questions by pupils who could not cope with the customary class system and who completed their education at the Utrecht Lyceum.

Since, after reaching the first level of thinking, a period of active (productive) thinking is expected from the children, and since it is especially this thinking that is the center of attention of the geometry-didactician, I carried out my experiment during the months of January, February and March. During those months, the subject matter that is being presented is strongly geared towards the transition to the next level of thinking.

The purpose of the experiment was to give a clear description of the learning situation in which I put the children and to set up an analysis of the ensuing learning processes in the children. Especially important for the didactician is the investigation of the normal course of the learning process. There could also be investigations of the course of those learning processes that lie outside the normal range. This type of study, however, should be done in collaboration with a school psychologist. These investigations in turn could provide the didactician with indications that could complement and refine his own didactics. Given the number of children in the first classes, it would be almost impossible for the teacher to devote sufficient attention to such special situations.

Because of the large size of the groups, it has been impossible for me to organize a sufficient number of individual question-and-answer sessions between teacher and pupil in order to be able to make reliable statements about the genesis of thinking in the individual. There is not sufficient quiet and time in school to do this. The school psychologist here has a vast potential field for work. One advantage the psychologist has is that, unlike laboratory work, he can observe children under normal circumstances during lessons by the teacher and that he can complement these observations by individual interviews outside the lessons (e.g. during his free work hours). In this way, the development of a number of pupils under rather normal circumstances could be followed.

I am forced to limit myself in my analysis and to investigate to what extent the structuring of perception, the structuring of thinking and the language structuring - the third accompanying the first two - are being developed by children in a group as a result of the learning process that has its origin in the learning situation presented to them.

Just as it was an important question for psychology as to what method should be used, it is also an important one for didactics. If didactics wants to be elevated above the unscientific level "I do it this way", one will first of all have to reflect on the way in which to exchange ideas about the didactic method that is being followed.

If the subject matter that is being covered in the method is different from the customary subject matter, a description of the material used should be provided.

One should indicate which didactic, psychological and pedagogical principles formed the starting point. Likewise, the intermediate educational goals of the subject matter should be indicated, in order to be able to examine later whether these goals have indeed been reached. Next, a protocol should follow, in which the contents of the lessons should be described as accurately as possible, so that the person who practices teaching can find out how the material has been presented. Finally, one should give an interpretation of the entire study.

Only then is a discussion on a scientific level possible:

1) about the learning situation in which the children have been placed;
2) about the progress of the learning process;
3) about the correctness of the interpretations.

We will have to arrive at such a method in order to avoid acquiring a chaotic array of data. This also prevents talking past each other (i.e. not communicating).

Ruttmann (I, p. 53) expresses it as follows:

> A scholarly explication of any subject having to do with one's intrinsic experiences means something quite different from following absolute and logical principles of human thought. These principles simply repeat in their formulation and application rigidly exact methods which produce so-called scholarly outcomes.
>
> Since this particular method is required for scholarly results, it determines how a person will initiate any inquiry. Furthermore, it is in this manner that standards are established.

I would like to advocate the presentation of a protocol of the lessons as a method for the stud of didactics, especially given the stage in which the latter still remains. This is the best way to provide a picture of learning situations and of t'e learning process. Some examples are: the protocol given by Boermeester (I, p. 94 ff), Brandenburg (I) and the protocol mentioned by Stellway (I, p. 356).

Before one proceeds to analyzing the way in which the solution to a geometric problem evolves, it is necessary to analyze the didactic method that is being used. For the way in which the children solve geometric problems could be dependent on the particular didactic method followed. In our own careers, it has happened several times that a modified didactic approach resulted in completely different methods of solution. The pupils then came up with solutions which we ourselves, not being raised with this didactic approach, did not know and which appeared to originate from the didactic method followed.

The following problem serves as an example:

> Construct a circle such that it touches a given circle in a given point, and in addition, such that it touches another given circle.

In Van Thijn[2] the following hint is provided with this problem: "by applying no. 53". No. 53 is as follows: "When a circle touches two other circles, then the line that connects the two points of tangency goes through one of the points of conformity of the last-named circles."

One pupil analyzed the problem as follows: "If the circles were the same size, I could solve the problem, because then the center of the circle to be constructed is located on the perpendicular bisector of the line segment MN connecting the centers of the given circles and on the line MP. The point of intersection S of these lines is the center and SP is the radius of the circle to be found. This circle has a common tangent at P with the given circle M. Then came a moment of insight: I can replace the given circle M by a circle with the same radius as the given circle N and which has the same line of tangency at P as the given circle M. There are two of those circles, and therefore there are generally two solutions."

This solution, where the pupil first goes back to a special case, namely the <u>symmetrical</u> case is a consequence of the didactic method followed. For in our didactics, symmetry from observation is being incorporated into the geometric structuring. See Chapter IV, p. 30.

Whereas the hint provided in the textbook leads to the examination of the set of circles that touch two given circles externally, the analysis given by the pupil leads to the examination of a much simpler set of circles, namely those that touch each other in the same point P.

Van Thijn gives the following problem on p. 82: "Draw a circle that touches a given straight line and a given circle, where the point of tangency P on the straight line is also given." (no. 34).

This problem also can be solved by drawing an auxiliary circle of the same size as the given circle and that touches the given line in P. The supplement of Van Thijn however gives the following hint:

> Let M be the center and R the radius of the given circle; the center X of the circle to be found is located on the perpendicular one can draw at P on the given line; if we

assume, first of all, that the tangency between the two circles is external, one has MX - PX = R; the problem is then reduced to the construction of a triangle MPX with data as in no. 53 of section 6; should the tangency between the circles be internal, then one has to construct a triangle MPX of which PX - MX = R, compare with no. 51 section 6.

The problem to which the former problems have been reduced, in turn are again artificial problems where the correct auxiliary lines have to be found.

For this problem also one sometimes provides a hint. First construct a circle that is concentric with the circle to be found and where the radius is equal to the sum or the difference of the radii of the given circle and the circle to be found. In this manner the problem is being reduced to the construction of a circle that goes through a given point and that touches a given line at a given point.

One should ask oneself whether it is meaningful to submit problems such as the ones described above to all pupils. This depends on the function one assigns to the problems. Van Hiele (p. 146 ff) describes the functions the problem can carry out in geometry. In addition, one should ask oneself whether it is meaningful, in connection with the learning process, to provide hints with problems. With problems that have the goal of inducing productive thinking, hints can strongly interfere with the progress of the phases of the thinking process (De Groot II, p. 53). For problems such as the ones mentioned above, the statement by De Groot about a set-up in a chess problem is valid: "In the first phase, the subject allows the set-up to work on him: the problem takes on shape for him". (p.76).

When the hints provided with the problem do not fit with the associations that are activated in the pupil by the assigned problem, then these will disturb the first phase - named the phase of problem formation by De Groot. Even if the hints are provided in a supplement, they can disturb the different phases during which consecutive ideas for arriving at the solution are worked out. For how can the pupil know at what moment it is useful to look at the hint? Even then the hint can disturb his thinking. The most important question however is: Which experiments have shown that the hints that are being provided are justified from a didactic perspective?

The method used by the cognitive psychologists is an experimental introspective method. This is less suitable for children, because a command of language is essential for that method. There is a question whether results acquired from introspection of adults are also valid for pupils of the secondary school.

If one assumes that it is desirable to give hints accompanying problems, one would do better to indicate more than one method of solution and include some that are found by the pupils themselves. For then there is greater probability that the pupil will receive a hint that fits into his scheme of thinking. The psychology of thought that has been delineated for adults, cannot give rise to a psychology of learning for the secondary school. The thinking of those young people will first have to be studied more thoroughly. This can be done through a collaboration between didactician and school psychologist by placing the child in the most appropriate learning situations. One should choose learning situations so that an experimental exploratory method of investigation is possible. In those learning situations it is possible to observe the pupils under normal circumstances.

The individual learning process is influenced by many factors of a psychological nature, the knowledge of which is valuable for the teacher as a pedagogue, but that are beyond the scope of this didactic experiment, therefore also beyond the scope of this dissertation.

The above-mentioned example shows that there is a close relationship between the methods of solution used by the pupils and the didactic approach one follows. If one emphasizes, in the didactic experiment, the improvement of learning performances through transfer of methods of solution from good pupils to other pupils, then there is no change in the presentation of the subject matter - it is only being refined in certain aspects.

The following points should be investigated carefully:

1. The study of which learning situations most appropriately connect with the thinking of children. This not only implies a study of the difficulties of thinking the subject matter can provide, but also a study of the manner in which that subject matter can be presented.

2. A systematic analysis of the learning process as it takes place among children on the basis of protocols of the lessons.

3. An adaptation of the scope and arrangement of the subject matter to the mind of the child.

Only after some light has been shed on these points, can one proceed with the question: What should we measure and which measurements are representative of actual progress in the learning process?

In setting up learning situations, one can be guided by statements of psychologists in order to try to establish the best possible connections between those learning situations and the psyche of the child.

Footnotes:

1. Richtlijnen voor een nieuw leerplan rek. op de L.S. Purmerend 1956. [See p. 73.]

2. Zie Van Thijn, Kobus, Wasscher. Verz. van plan. vraagstukken met supplement. p. 87. no 53, 54. 18e druk Groningen 1952. [See page 76.]

Chapter X

PROTOCOL OF THE WORK-PIECE "TILES"

[Note: Figures 1-12 referred to in this chapter can be found on pages 143-151.]

In order to report as accurately as possible on the class conversation, I drew a seating chart of the classroom for each lesson. Each compartment contained only the name and the number of the pupil so that much space remained to write down notes.

While the children were working out an assignment, I wrote down in shorthand, under the names, details of the conversation and at the bottom of the sheet I wrote down general remarks. I also made notes, after I went around in the class, of reactions to the tasks. At the end of the class hour, I added more remarks when needed. The notes were always transcribed during the same afternoon or evening. Even though I aimed at giving a precise reproduction of the class conversations, one should not assume that the protocol literally reproduces what the children said.

Since the names of the pupils are not mentioned in the report, I labelled the answer of a pupil with Pp. Sometimes I use Pps to indicate that many pupils gave that answer. I mention the number of the Pp, in those cases where the answer can be of importance, in order to make an analysis of the performances of pupils individually. (My remarks are labelled Tr.)

I told the children at the outset that I would work somewhat differently with them. I told them: "Geometry is to be worked on only at school. You will write in a notebook that has blank sheets of paper, ruled paper and squared paper. This notebook is to be handed in after each lesson. I am going to ask all kinds of questions. We are going to do different things. We are going to draw and make puzzles. Since everyone should have an opportunity to think, questions should not be answered immediately. We will agree on a sign for you to show that you know the answer. Do not lift your hand high up in the air. Do not look at what your neighbor is doing. Most often there are several correct answers. Even when you are solving a puzzle, you should not look at the result of someone who has already finished. This would take all the fun out of solving the puzzle. Make sure you have all the necessary materials for each geometry lesson: drawing triangles, compasses, protractor, ruler, eraser, 2 colored pencils. Fortunately, I have been able to give all of you a satisfactory grade on your Christmas report card. This grade will remain so, on the condition that you cooperate and that your behavior is good. If you contribute interesting solutions and ideas, your grade can be increased " In this manner I hoped to let the children work in reasonably normal circumstances.

The first class conversation with group Ia now follows. (This group consists of 24 pupils, of which 6 were repeating the class. In order to obtain as clear a picture as possible, these 6 pupils did not participate in the experiment. At the Amersfoort Lyceum it is possible to let those pupils work at their own level in a "workroom".)

Tr. On the label write your name and under your name write the title of the work-piece: "Tiles". We are going to study different floors that can be paved with tiles. I have written a sentence on the blackboard. There are two difficult words in that sentence: "A sidewalk is tiled with congruent square tiles." Do we know the word sidewalk?

Pps. ....(* Translator's Note: There are several words to indicate a sidewalk in Dutch. The one used in the sentence written on the blackboard is more formal, hence not as likely to be known by the pupils as the word that is commonly used. The pupils answer by giving the commonly used name.)

Tr. Do we know the word congruent? (Nobody appeared to have heard this word before, as I expected. I tried to approach the word in the following way.)

Tr. When I look at the chairs on which you sit, I notice that they are congruent. (Several pupils thought they knew it now: the same tiles, equal tiles.)

Pp. 17 When you stack two tiles, they have to fit. (We then started looking at the phrase "the same".)

Tr. If I say: Tomorrow you will sit on the same chair again, I do not mean the chair of your neighbor. So, the phrase "the same" is not clear enough.

Tr. Now the word equal: What is equal?

Pps. The area.

Tr. Let us try. Area is measured in .....?

Pps. Square centimeters.

Tr. Imagine now two tiles each of whose area is 12 square centimeters. They are thus equal. Should those tiles really be congruent?

Pp. 8 No, because they can differ in their length and their width.

Tr. Could you show that on the blackboard?

Pp. 8 Draws two rectangles 2 x 6 and 3 x 4 square centimeters.

Tr. Indeed they are not congruent - the word equal is also not such a good choice.

Pp. 17 They do not fit on each other either. (We conclude that Pp. 17 has given the best answer so far.)

Tr. When you sat down at the breakfast table this morning, were there any congruent objects on it?

Pps. Yes, plates, knives, forks, mugs.

Pp. 11 At our house the mugs are not congruent. (My conclusion that she came from a large family was correct).

Pp. 11 Each one of us has a different mug. Then you know which one is yours.

Tr. But in the afternoon when you have company do you then drink out of congruent teacups at home?

Pp. 11 Yes, but then we have different teaspoons.

Tr. When I look carefully at your chairs, they are not congruent (The pupils understood immediately. I made reference to the little labels with a number at the back of each chair.)

Tr. What are those numbers for?

Pp. 10 To tell them apart.

Tr. Correct, in order to be able to make a distinction, to be able to distinguish them. The teaspoons of Pp. 11 serve the same purpose. You can thereby distinguish congruent teacups. Who can now finish the following sentence. Congruent objects are.....

Pp. 17 They are objects that fit in each other.

Tr. I believe that congruent teacups do not entirely fit in each other, the ear (handle) is in the way. With tiles it works beautifully

Pp. Objects that have the same volume.

Pp. 8 No, because then it can still be high or low.

Pp. 2 Congruent objects are objects that cannot be distinguished from each other. (This was accepted with general agreement - it was stated well. Then the pupils divided the first blank sheet into four parts by a vertical and a horizontal line. I did this simultaneously on the blackboard.)

Tr. Task: Now draw, on a quarter of a sheet, a sidewalk paved with congruent square tiles - make the side 2 cm. Make the right angles neatly with the drawing triangles. Think of the sidewalk or of a tiled floor at home. Draw what you see before you in your mind. It can perhaps be done in several ways. (As they made this drawing, I was struck by the fact that everybody immediately drew continous lines). (Figure 1 and 2 [see pages 142-143] were both drawn without clear preference; one figure was drawn as often as the other. Almost all pupils found the two possibilities by themselves.)

Tr. How are the tiles most often positioned?

Pps. As in figure 2.

Tr. Why are they positioned like that?

Pp. Because it makes it much stronger.

Tr. Can you feel at the handlebar of your bicycle whether the tiles are in the wrong position?

Pp. Yes, then the wheel gets caught in a groove, just as in a rail.

Tr. In which direction do you have to ride your bicycle?

Pps. In that one. (They indicate the correct direction with their fingers.)

Pp. 5 (Was just thinking of the direction perpendicular to the one indicated. When we drew the direction in the figure on the blackboard, they saw it clearly. Then I remarked that ten days before I had been in a city where the tiles were positioned in yet another way.)

Pp. 10 Yes, on its edge.

Tr. Whoever understands what Pp. 10 means by that can try to draw it. (Again I saw many pupils first draw parallel lines, but now in an oblique direction. A number of pupils did not understand at all. They said: Then you obtain exactly the same. In this they were also right. Therefore I said: do not rotate it too far.)

Pps. But then it stands on it's vertex. (We also observed that the word edge is not clear enough: by edge is meant a side. The word "vertex" is better here.)

Pps. May I also r tate my notebook? Then I have it already, (Next, we looked at the three figures.)

Tr. What do I see in figure 1?

Pps. Lines.

Tr. Can you say something about those lines?

Pps. Yes, parallel.

Pp. 15 They are at the same distance from each other. (I pointed to the horizontal lines and called this an array of parallel lines.)

Tr. Does the figure have another such array of parallel lines?

Pps. Yes, so (vertical hand movement).

Tr. Does figure 2 also have such an array of parallel lines?

Pps. Yes, only so (horizontal hand movement).

Tr. Does figure 3 (see page 144) also have an array of parallel lines?

Pps. Yes, I see them.

Pps. I do not. (One of the other pupils then held a ruler along that set and moved it across two arrays.)

Pp. 3 So they do not have to be vertical or horizontal. (We have demonstrated this clearly once again with the ruler. There are many directions.)

Tr. Should the tiles be of exactly the same size in order to be able to tile a floor with them? Imagine for a minute that you have two kinds of tiles, large and small squares (I allowed for thought, then came.....)

Pp. 11 They should not be, if the small ones together precisely form a large one.

First class conversation in Ib.

The group Ib consists of 27 pupils of which 6 were repeating the class. The numbering of the pupils now begins with 19 and ends with 39. Of course the class conversation with this group differs in a few points - I want to mention these differences. Three pupils had already heard the word congruent.

Pp. 33 It means the same.

Tr. Where had you seen that word?

Pp. 33 I had this in the seventh class in elementary school.

Pp. 32 Yes, it means equal. I did not have it in school, but I don't remember where I have seen or heard it.

Pp. 25 I saw it in the back of the geometry book. It is mentioned there: first case, second case.

Tr. Do you also know what it means?

Pp. 25 It says: Congruent figures are figures that can cover each other completely.

Tr. Now, this is beautifully said. I will see whether the others would perhaps also guess it. Those three chairs are congruent. (I pointed towards three unoccupied chairs.)

Pps. It will probably mean the same.

Pps. Yes, it has to be equal.

Tr. To cover each other, as Pp. 25 has said, and also completely, is difficult to realize. It could be done with tiles. If I say to Pp. 33 and Pp. 34: You are sitting on the same chair....... Why do you laugh now?

Pp. 39 Then they would sit on one chair.

Tr. No, I mean from the same factory and of the same make and so on. Thus we see that in any case the phrase "the same" can lead to confusion. Now the word "equal". What is then equal?

Pps. The area.

Pp. 30 The shape and ..... the measurements.

Tr. Yes, that is said very well. The measurements are perhaps somewhat difficult to obtain, but it is correct. What is area really? (Pp. 36 and Pp. 39 and probably some more children thought that area is length times width).

Tr. What do you compare the area with? In what units do you measure it? What do you express it in?

Pp. 37 In square centimeters, or square decimeters or square meters.

Tr. Now take some tiles of 12 square centimeters. Are those equal?

Pps. Yes.

Tr. Do they then also have to be congruent, or can they be different?

Pp. 32 It can still be 2 X 6 and 3 X 4. (This was drawn on the blackboard.)

Tr. The word equal also leads to confusion. Look carefully at the back of the chairs. (It was clear: they had different numbers.)

Tr. What is that for?

Pp. 24 In order to know to which table it belongs.

Tr. Good, who can say it somewhat differently?

Pp. 30 In order to be able to distinguish them.

Tr. Let us try again. Congruent objects are......(I poi t to Pp. 19).

Pp. 19 O jects that are equal. (Protests from all sides. I again mention that we just saw that the word equal was not a good choice because equal tiles n ed not be congruent.)

Pp. 21 They are objects that cannot be distinguished from each other.

Pp. 38 The teacups of our set at home are not congruent.

Tr. Then they most certainly have little cracks by which you can recognize them.

Pp. 38 Yes, that is what I meant.

Pp. 37 In fact, objects are never congruent, because the fingerprints on the objects are different.

Tr. Let us imagine that we have a very special microscope with which you could observe those fingerprints. Then the objects can be distinguished by those fingerprints and then they are indeed not congruent. (While drawing, some students constructed square after square and they had difficulty seeing the existence of more possibilities.)

Second class conversation Ia.

Tr. Last time we talked about a set of parallel lines. Another word for parallel is ....? One of you even observed that the lines in figures 1, 2 and 3 are at the same distance from each other. (With the help of two large drawing triangles I demonstrated on the blackboard how one can draw a set of parallel lines. I drew a set where the distance between the lines was the same and a set where that was not the case. The direction was completely random.)

Tr. Instead of using one of the triangles, you can also use a ruler. (This also was demonstrated.) Now we are going to draw a tiled floor that is very common. It involves rhombi where six acute angles repeatedly meet in one point. Look carefully how I construct it. (I constructed a star built on a hexagon figure. See page 41.) Now you are going to construct this tiled floor for yourself in your notebook. First divide a circle in 6 equal parts. Then construct the star based on a hexagon. Then construct another such star at the vertices of the other acute angles of the rhombi that have been drawn already. Cover half a page (see figure 4, page 145). It took the pupils the remainder of the hour to do this. I watched how they were doing it and I occasionally helped one or another.

Pp. 3 Whispered to me that he saw cubes in the figure. This gave me the idea to inquire of the others what they saw in the figure. I received as answers: hexagon, everywhere those stars, rhombi, zig-zag lines that are interrupted but that continue further on. but nobody else noticed cubes.

Second class conversation with Ib.

Tr. First I will review what we did last time. We first talked about the word congruent. We saw that the phrase "the same" does not really reflect what one means by congruent. Covering each other completely, as Pp. 25 had stated, was appropriate for flat figures, for example, tiles. The word "equal" where we all immediately thought of area was confusing, because tiles that have the same area are not necessarily congruent. (Agreement in the class: Yes, 2 X 6 and 3 X 4.)

Tr. Pp. 30 brought up something interesting yesterday. Does someone remember it? Maybe you yourself remember it?

Pp. 30 No, I only said equal area. (Then she started to hesitate.) No, I said equal shape and something else.

Pp. 32 (Completed and said) equal measurements.

Tr. Congruent figures are figures of equal shape and with the same measurements. After that, we observed that the numbers on the chairs, the teaspoons in the teacups, etc. allow us to distinguish congruent objects. If we were able to see fingerprints on the cups (as Pp. 37 mentioned yesterday), then these cups would not be congruent any more. For then they would be distinguishable from each other. Then you drew a sidewalk that was covered with congruent square tiles. Most of you have already found two different possibilities. I have already drawn three on the blackboard. (A few students had also found the third case.)

Tr. How are the tiles mostly positioned on the bicycle path?

Pps. (Without hesitating this time): So (hand movement).

Tr. Why, so?

Pp. Otherwise you could get caught in a groove.

Tr. Do you notice something when you compare figure 1 and figure 3?

Pp. They are all the same squares.

Tr. Yes, who can say it in a different way?

Pp. 29 It is exactly the same figure, but rotated. (This elicited agreement from different sides..)

Tr. What do you see in figure 1 ?

Pp. Squares.

Pp. Everywhere right angles.

Pp. Lines.

Tr. Can you say something about those lines?

Pp. Parallel.

Pp. .... .(*Translator's Note: Pupil gives the Dutch synonym for parallel.)

Tr. Correct. I see a whole lot of them.

Pps. Yes (they make a horizontal movement with their hands).

T What else do you see?

Pp. Still more parallel lines, vertical (this was again accompanied by a hand movement). There are equal pieces in between.

Tr. Is there a set of parallel lines in figure 2 also?

Pos. Yes, like that (horizontal movement).

Tr. No more?

Tr. Yes, if you draw it through.

Tr. They are not there - you can think of them as there. And in figure 3 ? (The answer was yes from all sides without hesitation. Two sets. Pp. 19 pointed them out by putting a ruler on the figure and by shifting it across the figure. Then I asked if they had ever drawn parallel lines by moving two triangles along each other.)

Tr. Try to obtain a set of parallel lines by shifting your ruler on the table. (Some found them after searching for awhile.)

Tr. Why is it more difficult for me to do it on the blackboard then for you on the table?

Pp. It is difficult to hold. (They watch carefully to see if I will succeed. I drew a line, another one 10 cm lower, again one 10 cm under the second one, and so on. They are all in a oblique position.)

Tr. What did I make equal?

Pp. 33 A decimeter.

Tr. Yes, 10 cm is a decimeter. But that is not what I asked.

Pp. 28 The distance.

Tr. Correct, I made the distances between all the lines 10 cm. Now I will add a few other ones at different distances. Do we still remember that we made many nice figures before Christmas vacation? We made regular polygons among other things. (I picked up a rhombus made out of paper and asked: Was this one of them?)

Pp. 27 (Enters. She lives outside of Amersfoort and always arrives somewhat late. She is included in the conversation at once.) No, because two vertices have acute angles and the two other vertices have obtuse angles.

Tr. Is that not allowed?

Pp. 27 No, they have to be equal.

Tr. Can't an acute angle and an obtuse angle be equal?

Pp. 27 No.

Tr. Tell me why not. (Silence)

Tr. You see how difficult it is to make clear that which is so simple. We will first have to ask ourselves what an acute angle is and what an obtuse angle is, because that is what really matters. (Upon this, the tongues were loosed.)

Pps. Below 90 degrees and the other above 90 degrees then they cannot be equal. (At this point I took a rectangular notebook.)

Tr. Is this rectangle then a regular polygon?

Pp. 32 (Wanted to tell it quickly and started): That point (by this she meant the point of intersection of the diagonals, because her fingers pointed toward those lines) is not equidistant from the angle..... (here she stopped; it was visible from her face that, at the same moment she saw that the point of intersection of the diagonals was equidistant from the vertices).....not equidistant from those lines. (Here she pointed towards the sides of the rectangle.)

Tr. Correct, that is good. The point of intersection of diagonals is equidistant from the vertices, but not equidistant from the sides of the rectangle. (I simultaneously pointed towards the parts I talked about.)

Tr. Can somebody phrase it somewhat simpler?

Pp. 37 The sides have to be equal. (Agreement.)

Tr. Who could tell me again what a regular polygon is? (I point towards Pp. 38.)

Pp. 38 It is a figure with equal sides.

Tr. (I again pick up the rhombus): And this rhombus then? It has equal sides and you just rejected it.

Pp. 38 They have to have equal sides and equal angles.

Tr. Which regular polygons did we construct?

Pps. Hexagons, octagors, dodecagons.

Tr. Good. Now smaller. There is also a regular pentagon. Which one has the smallest number of sides?

Pps. Three.

Tr. Could there not be two?

Pps. No. (A few boys arrived at the figure drawn here, Pp. 35 and Pp. 36)

Tr. If there are curves, where is the angle then? And shall we also call those curves, sides? (We did not dwell on it. Next, I gave each of the pupils a little bag containing regular triangles, one with regular pentagons, one with regular hexagons and one with regular octagons. I wrote the following questions on the blackboard:

Can one tile a floor with:
1) congruent regular triangles?
2) congruent regular quadrangles?
3) congruent regular pentagons?
4) congruent regular hexagons?
5) congruent regular octagons?

The outcome of each puzzle had to be written in the notebook. The children immediately noticed that I had not given them a bag containing regular quadrangles.) Yes, why did I not give one like that?

Third class conversation Ia.

(I entered with a box full of transparent little bags which I had just collected from the pupils in class Ib. I showed them a few.)

Tr. Do you remember what these could be?

Pps. Regular polygons.

Tr. What do you think my question will be?

Pp. 16 Whether a floor can be tiled with them. (I again wrote the questions on the blackboard. Here the pupils also noticed that the bag containing the quadrangles was missing.)

Tr. Answer the questions right after you finish making the puzzles. Then draw neat tiled floors in your notebook. You may omit edges where tiles have to be broken. Pp. 3 stacked the regular pentagons and pointed out to me that they were not completely congruent. Indeed, some were quite different. I whispered to him that they really were intended to be congruent. He accepted this. Pp 2 quickly finished and he put the shapes back into the bag. He could draw it without using the shapes. In passing I also asked some others whether they would need the shapes while drawing.

Pp. 16 Thought he would, in the cases where it did not work.

Pp. 5 Did not know yet.

Pp. 15 Said he did not need them, but drew right isosceles triangles instead of equilateral ones. He noticed that it was wrong only when I put the shape in front of him.

Pp. 15 The sides have to be equal, and I did not do it correctly. (Pp. 11 thought it was easier to have the shapes around. A moment later I saw that she traced around the shape with her pencil. We looked at the drawing and imagined that if she constructed the polygons with compasses it would be much neater. She did that very easily. A couple of pupils who could not construct the center of the second hexagon were helped when I suggested using a mirror. Those were pupils who understood that the center had to be constructed. It appeared later that some other children had not constructed the centers. Many pupils immediately drew the horizontal lines through. The other sets of lines they noticed later. They used those then as a check (see figure 5, page 146). Pp. 12 draws a large triangle first and then divides it into small ones. Pp. 9 observes that the floor tiled with squares (regular quadrangles) should not be drawn any more, and that the floor tiled with triangles can be obtained from the figure tiled with rhombi, drawn during the preceding lesson. The latter can be done by drawing in the diagonals. It is also noticeable from the figures of the other pupils that they have been thinking about the stellated hexagon. Only Pp. 2 makes use of the method whereby one drawing triangle is moved along the side of another one to draw parallel lines. He is also fastest in finishing everything.)

Pp. 2 I cannot construct a regular pentagon.

Tr. You constructed the regular hexagon by dividing the circle in six equal parts. You did that with the compasses. You can also divide a circle in five equal parts, but then you have to use a protractor. (He solved the problem after a few minutes. I saw him divide 360 degrees by 5 and he made the central angle 72 degrees.)

Third class conversation Ib.

Tr. For this lesson I brought six rhombi with which you can make a stellated hexagon. I also brought eight rhombi with which you can make a stellated octagon. (One of the girls made it.)

Tr. If I want to construct this stellated hexagon on the blackboard, how do I start?

Pp. First we draw a circle.

Tr. I do not see a circle.

Pp. We imagine it to be there (points to a circle that runs through the outermost vertices).

Tr. If I tell you that the sides of the rhombi have to be 2 cm, can I then draw that circle? Is its radius then perhaps 4 cm?

Pp. No.

Tr. What should I do? (It appeared diffuclt to forget about that first circle.)

Pp. The innermost vertices are also located on a circle. We can start with that.

Tr. I will construct a stellated hexagon on the blackboard and you do it in your notebook. Whoever is ready, construct other stellated hexagons at the vertices of the acute angles of the rhombus. Pay attention! Do it correctly. (In this class there also was only one pupil who saw a spatial figure in the rhombi. Pp. 34 saw stairs.)

(Then came the plane coverings with the regular polygons. Here also, the pupils paid attention to the other sets. Even though, in this group, the construction of the stellated hexagon had immediately preceded (during the same class hour) the drawing of the plane coverings, I did not find that the pupils were inclined to use that construction to help them draw the regular triangles and regular hexagons (see fig. 6, page 147). Pp. 32 observed that all line segments of the stellated hexagons are 2 cm, including the line segments that were not there - she pointed to the diagonals and she thus saw other figures in the drawing.)

Fourth class conversation Ia.

This lesson again started with the distribution of bags.

Tr. These little bags now contain irregular triangles, quadrangles, pentagons and hexagons. The assignment is again: can a floor be tiled with these? (It was surprising to see that it sometimes took less time to cover the plane with quadrangles (see figure 10, page 149) than with triangles. It may have been related to the fact that the quadrangles can be made to fit in only one way, whereas there is more than one possibility with the triangles.)

Tr. I see that you all made the same figures. Are there straight lines in those figures?

Pps. Yes, three sets (see figure 9, page 1'3).

Tr. Now draw a whole pageful of those triangles. You may draw the first triangle by tracing around the shape with your pencil - the other triangles should be drawn such that the three sets of parallel lines appear clearly. How do we do that?

Pps. With the drawing triangles. (I had to help a few pupils individually so they could do it.)

Tr. You may draw the other figures by simply tracing around the model with your pencil.

Fourth class conversation Ib.

Pp. 32 What about those regular bi-angles that were mentioned sometime ago. Do they exist? Was what Pp. 35 drew on the blackboard correct?

Tr. Do you feel like including this one among the others?

Pp. 32 No, it looks so totally different.

Tr. What is different?

Pp. 32 It has those funny arcs. The others do not have that.

Tr. Let us agree that the polygons we are dealing with can only have straight lires. From now on we assume that polygons are only bounded by straight sides. Then the smallest possible number of sides is three. (Just as in the other group I then handed out the bags and the pupils started to make puzzles. Pp. 30 first makes a star, then turns half of the triangles around and finally arrives, with a little help, at a plane covering with kites. Also in this group many pupils need to be helped in order to draw parallel lines using two drawing triangles.)

Fifth class conversation Ia.

(We first compared the pupils' drawings with each other.)

Tr. Here I have one where first a square was constructed and next to it another square, etc. How did this one do it? (I held another notebook up.)

Pps. First drew horizontal lines.

Tr. And then?

Pps. Then marked 2 cm off everywhere.

Tr. Yes, at many places. Is that necessary? (I again held up another notebook.)

Pps. No. He measured less often.

Tr. So, we have to use judgment when we draw. What about the bicycle path? Do I not have to measure much more often in that case?

Pps. No. The other pieces are exactly in the middle.

Tr. Yes, in fact you can draw them simultaneously. (I showed a few more drawings.) Are these squares large enough? What has been made 2 cm here?

Pps. The diagonal.

Tr. These (I held up another notebook) are somewhat askew. You can position them <u>exactly</u> on their vertex. (The pupil had drawn sides of the squres parallel to a diagonal of the rectangular sheet of paper). How did you do that?

Pp. I first drew a large square along the edge of the paper and then I drew a diagonal inside. I repeatedly marked off 2 cm on that diagonal.

Tr. Now we will look at the drawings of the triangles. We all thought about the horizontal lines. But now look at another set of lines. On the floor tiles with rhombi there also are small line segments that belong to the same line.

Pps. It is very difficult to draw it correctly.

Tr. Now I am looking at the floor tiles with regular hexagons. What is the difference between the two drawings? Those who sit in the front can see it well.

Pps. In the one figure there are small arcs in the middle of the hexagon, but not in the other.

Tr. Those are arcs of a circle that have been drawn with compasses in order to correctly construct the centers of the hexagons. It has been done accurately in the first drawing. Can we also tile a floor with regular pentagons?

Pps. No.

Tr. I also want to see a drawing of that. We cannot yet construct the pentagon with the compasses, but we can do it neatly with the help of a protractor. How shall I divide the circle I am drawing on the blackboard into five equal parts? (A number of children know it.)

Tr. I first have to do a computation. (How many more children know it?)

Pps. Divide 360 degrees by 5, that is 72 degrees.

Tr. Tell me what I have to do (see figure 7, page 147).

Pp. Position the protractor near the center.

Tr. Do you mind that the protractor is smaller then the circle I drew?

Pps. No. (That is the general opinion. We mark off an arc of 72 degrees five times on the circumference of the circle.)

Tr. It does not look so neat on the blackboard. That is because of the chalk. Make sure the point of your pencil is nice and sharp. How can I succeed in placing the second pentagon neatly against the first one? (We look in the mirror and see where it has to be placed.)

Pps. Circle around with the compasses. (I demonstrate it on the blackboard.)

Tr. What now?

Pps. First the circle.

Tr. And then?

Pps. Mark off five times again. (Forgot to make the compasses wider and thus marked off the side of the regular hexagon.)

Tr. You did not do it correctly. You first have to make the compasses wider. (At the end of the hour, many children asked whether they could draw all the figures once more. I gave them all a new blank sheet. At the next class hour the chalk for the compasses for use on the blackboard was nicely sharpened.)

Fifth class conversation Ib, was practically the same as that of Ia.

Sixth class conversation Ia.

During that day a lot of the work had to be drawn. Just that day I had almost lost my voice. The pupils remade various drawings. Pp. 16 and Pp. 18 made the puzzles first because they had been sick the previous day.

Sixth class conversation Ib.

Tr. Place the first page with figures 1, 2, 3 and 5 in front of you. We have many more parallel lines in figure 1 then in figure 2. By positioning the tiles differently, a whole set of parallel lines has disappeared. In figure 5 - the figure with all those triangles - we have three sets of parallel lines. Could I also get rid of a large number of grooves (continous straight lines) here by positioning them differently?

Pp. By moving the second row over a distance of half a tile.

Tr. Does it <u>have</u> to be over a half a tile?

Pp. No. (One of the pupils draws on the blackboard what is being discussed.)

Tr. Could I get rid of the parallel lines in figure 9? They also are all triangles?

Pps. Yes, of course, in the same way.

Tr. How is it that you all, except for one, made the same drawing? We see that there are many possibilities of doing it.

Pps. We thought that the sides had to fit.

Tr. Yes, but the assignment only involved tiling the floor. It can be done in many different ways. What about the quadrangles of figure 10?

Pp. You cannot move the quadrangles because there are no parallel lines.

Tr. On the second page you drew a floor with rhombi and a floor with regular hexagons: figure 4 and figure 6. Do we see something when we compare those figures with each other?

Pps. There are hexagons in figure 4.

Tr. In order to check whether you all see those in the figure I will ask you to erase a few line segments in figure 4 so that I can find the hexagon you have seen. (This appeared not to be difficult.)

Pp. 38 There also are still greater hexagons in the figure. (The other pupils also see that now.)

Tr. Which figure contains more line segments, figure 4 or figure 6?

Pps. Figure 4.

Tr. I thus have to add line segments to figure 6 in order to see figure 4 in it. (This appears to be a difficult task: a lot of help is needed.)

Tr. Can we also derive figure 5 from figure 4?

Pps. Divide the rhombi in 2 triangles.

Tr. And from figure 6?

Pps. Yes, by drawing the radii.

Tr. Can we also see the rhombi and the hexagons in figure 5? Is there also a stellated hexagon in it? (This went very easily.)

Tr. Now take figure 9. Each triangle has three angles. Pick out a triangle and color the smallest angle of that triangle with red. (All chose the correct angles.)

Tr. How do I go about helping somebody who does not see which angle is the smallest one?

Pps. Use the protractor to measure the angle. (There was no other opinion.)

Tr. Now color with red all angles of the same size. (I had to warn many students not to forget any. The warning itself was sufficient.)

Tr. Now color the largest angle of the triangle blue and do the same for all angles of that size. Finally give the third angle another color. (I made the same drawing on the blackboard. See figure 9.)

Tr. We will now ask ourselves what we see at such a vertex.

Pp. Opposite angles that are equal.

Pp. I see three different angles.

Pp. The angles of the triangle are there, each twice.

Pp. All angles together are 360 degrees.

Tr. Yes, good; one more thing.

Pp. Three of them make a straight angle.

Tr. Good, we indicate the straight angle with an arc. Do I also see that straight angle in the other points?

Pp. Yes, of course, exactly the same.

Tr. Can I see that straight angle in yet another way?

Pp. 25 Yes, on that other line.

Pps. Yes, also in that other direction.

Tr. That straight angle can be positioned in three different ways in the figure. We indicate each straight angle with a semi-circle. Now go over to figure 10 - the floor tiles with irregular quadrangles. Everybody has a quadrangle made out of cardboard. In order to find out where the equal angles are, we number the angles of the cardboard model 1, 2, 3 and 4. You then color equal angles in the figure with the same color. Thus we use four different colors. (Very soon, most children colored the angles without using the quadrangle made out of cardboard.)

Tr. What do we now notice in such a point?

Pp. There is exactly one of each angle, because there are four different colors.

Pp. The four angles together are 360 degrees.

Tr. Good, we will call that a round angle. Which angles were those? (I hold the cardboard quadrangle up.)

Pp. The four angles of the quadrangle.

Tr. The four angles of a quadrangle form a round angle when we fit them together. Now watch out: I know that the angles of a triangle form a straight angle when fitted together. I claim that I can predict that the angles of a quadrangle have to form a round angle when I fit them together. Who thinks that he also can predict that?

Pp. 30 It is true for a regular quadrangle because four right angles together form a round angle. For a non regular quadrangle, it is also true, because it is only parceled out differently.

Pp. I thought so too. You take away something from one angle and you add it to the other.

Pp. 37 Four angles always form a round angle. (He could not add anything to it.)

Pp. 38 I divide the quadrangle in two halves - so - (movement along the diagonal) one triangle forms a straight angle, so two form a round angle.

Pp. 34 I also had the idea of dividing the quadrangle in two triangles.

Tr. Who explained it most clearly? (All pointed towards Pp. 38.) (I color the three angles of one triangle and then the three angles of the other triangle with chalk, in order to make clear for everybody that one then has obtained exactly the four angles of the quadrangle.)

Tr. You have all found that a floor cannot be tiled with the pentagons. Pp. 30 has tried to do it stubbornly - she did not want to give up. Perhaps, had she been able to say beforehand that it would not work, she would not have spent so much time trying to do it.

Pp. 30 It cannot be done because then you would get more than 360 degrees. (Agreement from all sides.)

Tr. Can you say it exactly? (The bell rings. I hear 3 X 180 degrees from many sides, one girl say 2 1/2 times.)

Seventh class conversation Ia.

Tr. Pp. 17 does not draw squares in figure 3, but.....

Pp. 17 Rhombi. (He is given material consisting of a board with nails[1] and loose rubber bands in order to make a plane covering with squares set <u>on their vertex</u>.) (Moving the tiles with the intention of getting rid of parallel lines here also did not present difficulties. Pp. 9 had already drawn the equilateral triangles in that way and the figure had already been looked at with that in mind. Here also restructuring required much effort. It appears to be difficult to redraw (in figure 6) those line segments that have been erased in figure 4. When coloring the smallest angle of the irregular triangle of figure 9, Pp. 15 took an angle with the shortest side. Therefore, I demonstrated using the material described in Chapter IV (last paragraph) how an angle can grow. Without my saying anything, he told me that he had precisely taken the largest, the obtuse angle, because he thought that the smallest angle should be adjacent to the shortest side. After we had colored the angles, we started making observations about the angles which meet at a point.)

Pp. 15 In a triangle there is one of each, and at such a point there are two of each.

Pp. 9 If I take a mirror, I can s e that angle in the mirror.

Tr. Where do you place the mirror and what do you see? (It was demonstrated on paper and on the blackboard that it is not possible to make the three colors of the mirror image coincide with the colors of the real object.)

Pps. Folding. (The pupils again saw that this also did not work and that in fact it came to the same thing as mirroring,.)

Pp. 8 Had the idea to fold twice, whereby the first folding occurred along the bisector.

Tr. Yes, you can arrive at it by doing it that way, but I can see it immediately. (The word .....* reminds them of something.)
(*Translator's Note: The Dutch expression translated by the word "immediately" contains the word "turn". So the pupils are reminded of the word "turn" and this leads them to the correct answer. It is a play on words that is not meaningful in English since there is no equivalent translation for that expression which was translated by "immediately".)

Pp. 16 Half a turn.

Pp. 4 Rotate half a turn.

Tr. Put your pencil on a point and rotate the notebook half a turn. Pay attention to the colors.

Pp. 3 Three together form a straight angle.

Tr. Do we all see that a red, a blue and a yellow angle together form a straight angle? (It appears to be so because they now also find straight angles on the other lines. Next, the pupils colored the four angles of the quadrangle. They found that the four different angles of the quadrangle meet in one point and there they form a round angle.)

Tr. When I know that the angles of a triangle together form a straight angle, how can I then predict that the angles of a quadrangle together form a round angle?

Pp. 16 You divide a circle in four quadrants and you say that it is parceled out somewhat differently in the case of the other quadrangles.

Pp. 3 (Wants to know first, whether a quadrangle can be circumscribed by a circle. The answer came from all sides of the class: a triangle can always be circumscribed by a circle but not a quadrangle. I ask a couple of pupils, but they have no idea.)

Pp. 3 (hesitating) Yes, if only that diagonal divided it into two halves, then .....

Pp. 5 The diagonal divides the quadrangle into two triangles. One of the triangles is a straight angle and so is the other one.

Tr. Who has now said it most clearly?

Pp. Pp. 5

Tr. But who gave the idea to Pp. 5?

Pps. Pp. 3 (Pp. 3 now also sees that the two parts need not be halves. The three angles of the first triangle and the three angles of the second triangle are now marked with chalk on the blackboard and it is verified that those together are the four angles of the quadrangle).

Tr. The floor can be tiled with triangles and also with quadrangles, but not with pentagons, hexagons, etc. One of you spent a lot of time trying to do it. We would not have to search so long, if we could predict that it is not possible.

Pp. 9 You then get more than 360 degrees.

Pp. 15 Shows that it is 3 X 180 degrees, because he divides the pentagon on the blackboard into three triangles. We count 9 angles that together exactly form the five angles of the pentagon.

Tr. Does this reasoning also hold for hexagons? for heptagons?

Pps. Yes, because then you get still many more.

Tr. We have already made a plane covering with hexagons though. Where?

Pps. Yes, the regular hexagons. (We look at figure 6 again. The bell rings. I hear a few pupils say: here there are three in one point.)

Seventh class conversation Ib.

Tr. Last time we saw that figure 6 can be obtained from figure 4 by erasing a few line segments. Also from figure 5. And figure 4 could also be derived from figure 5 by erasing line segments. Conversely, we had to draw line segments in order to obtain figure 6 back from figure 4 or 5. Would this procedure perhaps produce something new in the other figure? Take for instance figure 9, the plane covering with the nonregular triangles. Erase a line segment. Does anybody know the new figure?

Pps. Yes, a parallelogram.

Tr. With three letters 'L'. Can I obtain other ones?

Pps. Yes, erase another line.

Tr. Perhaps still more.

Pp. 37 Yes, exactly three, because there are three different sides.

Tr. Can we say something about the parallelogram?

Pp. 25 Yes, opposite angles are equal.

Pp. 36 Opposite sides are equal.

Tr. Are the parallelograms congruent?

Pps. No.

Tr. How do you know that?

Pp. 21 They do not fit on each other, the measurements are different.

Tr. I can say something about those parallelograms though, when I compare them with each other.

Pp. 32 Yes, the areas are equal.

Tr. How does she know that the areas are equal?

Pp. 19 Each one consists of two of the same triangles. (I drew the erased lines back on the blackboard such that they could again see that the triangles are simply attached to each other in different ways.)

Tr. Are there any more parallelograms in the figure, if I allow more line segments to be erased?

Pps. Yes, a whole lot.

Tr. Draw one with a colored pencil. (This did not produce any difficulties.)

Tr. Now look at one of the first parallelograms and try to find a parallelogram that is an enlargement of that one. Draw the outline. (Most children take a parallelogram that is enlarged three times. Some students, among others Pp. 25, did not succeed immediately.)

Tr. A photographer sometimes makes an enlargement of a photograph. A person on the second photograph is then an enlarged image of the first photograph. Have you ever looked in a distorting mirror? Is that also an enlarged image?

Pp. No, it makes you very tall, or very short and fat.

Pp. You are stretched in one direction.

Tr. Correct, you are enlarged in only one direction. That long and narrow parallelogram of yours has looked in the distorting mirror - it is also enlarged in only one direction.

Pp. Now I see it - I need that one.

Tr. How many of the smallest ones can fit in the enlargement? How many do fit in it?

Pps. Four.

Tr. Pp. 28 has drawn a still larger parallelogram. It has been enlarged three times. How many parallelogram can fit in it?

Pps. Six.

Pps. Nine.

Pp. There are three rows, each with three parallelograms.

Tr. By erasing a line segment in figure 9 we obtained a parallelogram. Now erase two line segments. Who knows the figure which we now have?

Pps. A trapezoid.

Tr. Can you find more than one?

Pps. Yes, they look different.

Tr. What is the difference between a parallelogram and a trapezoid? What is the distinction?

Pp. The opposite angles of a trapezoid are not equal.

Pp. The opposite sides are not equal.

Pp. 38 A trapezoid has one pair of parallel sides and a parallelogram has two.

Tr. What did we find last time at the end of the class period?

Pp. 26 The angles of a triangle together form a straight angle, 180 degrees.

Pp. 19 The angles of a quadrangle together form a round angle, 360 degrees.

Tr. And then?

Pp. 37 Draw the diagonal, then you have two triangles, 2 X 180 degrees = 360 degrees.

Tr. Yes, we could predict that the angles of a quadrangle together are 360 degrees when we know that the angles of a triangle together are 180 degrees. We then asked whether we could cover a plane with pentagons. The answer is no. Can we also predict that?

Pp. Yes, because there are three triangles, it is more than 360 degrees.

Tr. How much exactly?

Pp. 3 X 180 degrees = 540 degrees.

Pp. 34 Draws the three triangles in the pentagon.

Tr. How much is that for a hexagon?

Pp. 4 X 180 degree = 720 degrees.

Tr. Is it thus not possible with hexagons?

Pp. No.

Tr. Is it never possible with hexagons?

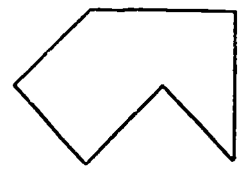
Pp. Yes, we have drawn it in figure 6.

Tr. Why is it possible with the regular hexagons?

Pp. 24 If the six angles are 720 degrees, then three angles are 360 degrees.

Tr. Do we all see that? Only three angles meet in one point of the figure. Would there be any other hexagons, except for the regular one, with which we can cover a plane?

Pps. We thought so. (I drew the hexagonal tile shown here on the blackboard and I gave each student a sheet of squared paper.)



Tr. Try at home whether you can cover a plane with this tile. Use a quarter or a half of a page for each drawing. You may also try and find other hexagons with which it is possible to cover a plane. We call this paper, squared paper. Are those geometrical rhombi.....? [*Translator's Note: The Dutch word for squared paper is ruitjespapier. Ruit, however is also the word for rhombus. That is the reason for the above question by the teacher.]

Pps. No, squares.

Eighth class conversation Ia.

I gave the same introduction as in class Ib. The pupils erased a line segment in figure 9.

Tr. Which figure did we obtain?

Pp. 12 A rhombus.

Pp. 11 No, because the sides are not equal.

Pp. 4 It is a rectangle.

Pp. 11 No, because the angles are not equal.

Pp. It is a parallelogram.

Tr. You have not all erased the same line. How many different parallelograms are there?

Pp. 11 There have to be three different parallelograms because there are three different sides.

Tr. Make all three. Are they not congruent?

Pps. No.

Pp. 1 (hesitates.)

Tr. How could I check that?

Pp. 1 They do not fit on each other when you cut them out and you stack them.

Tr. What do we see when we look at the parallelograms?

Pp. 1 Each time two parallelograms have a common side.

Pp. 9 The lengths and widths of a parallelogram are equal.

Tr. That is the right idea but we prefer to say side instead of length and width.

Pp. 9 The sides are equal.

Tr. Which ones? All of them?

Pp. 9 The opposite sides are equal.

Pp. Opposite angles are equal.

Tr. Look carefully at how the three parallelograms are made. What do you notice about them?

Pp. 12 Equal area.

Tr. Who can explain why the areas of the three parallelograms are equal?

Pps. They are the same triangles. (Next we searched for an enlargement. All pupils made a two-fold enlargement. The story of the distorting mirror appeared to be necessary here too. The name trapezoid appeared to be known.)

Tr. What is the distinction between parallelogram and a trapezoid?

Pp. The sides are no longer equal.

Pp. The angles are no longer equal.

Pp. 4 The areas are not equal, because the one is made out of two and the other is made out of three triangles.

Pp. 8 But it can be equal though. You certainly can draw a trapezoid that has the same area as a parallelogram. (Nobody doubts that, not even Pp. 4.)

Pp. 7 A trapezoid has only one pair of parallel sides and a parallelogram has two pair.

Pp. 16 I wanted to say the same. (Then we reviewed what we found at the end of the preceding lesson.)

Tr. Why is it possible to tile with regular hexagons?

Pp. Because all sides are equal.

Tr. The regular octagon also has equal sides, and yet with regular octagons it is not possible. It cannot thus be a matter of equality of sides.

Pp. Because all angles are equal.

Tr. We are getting closer, but that was also the case with the regular octagon and pentagon.

Pp. Yes, but only three angles meet.

Tr. Yes, let us do some computations. How many degrees are the six angles when taken together?

Pp. 720 degrees.

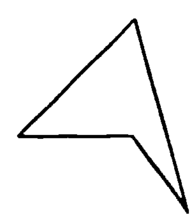
Tr. And then?

Pp. The three angles together are half of that, that is 360 degrees.

Tr. Good, and that is not the case with the non-regular hexagon. It does not work with the non-regular hexagon. It does work with the regular hexagon. Would there be any other hexagons with which it would be possible?

Pps. I think so. (These pupils were also given a model of a hexagonal tile and they were asked to try and find hexagonal and pentagonal tiles for themselves at home.)

During the next lesson the pupils worked on a problem. Each pupil was given a quadrangle made out of cardboard and of a shape as shown here. The question was: Would it be possible to pave a floor with tiles of that shape? Do not only answer yes or not, but write down all your thoughts on this problem. When you are ready, draw a line under your work and turn the page so I can see that you have finished. After 20 minutes I handed out bags with the same tiles. Task: Check whether the answer you gave is correct. You may add comments on your sheet under the line you drew. (See further discussion in Chapter XII.)



Eighth class conversation Ib.

First I drew some interesting findings of the children on the blackboard - pentagonal and hexagonal tiles. Then I gave the task of coloring those tiles such that no two of the same color were adjacent to each other and such that the number of colors was as small as possible. The same colors were allowed to meet in a vertex though. The pupils with less imagination could use the given examples.

Tr. Last time you investigated whether it was possible to pave a floor with quadrangular tiles of this form. (Here I drew the quadrangle with the re-entrant angle on the board.) A number of you had decided beforehand that that was not possible. Then it appeared that it was possible after all. How could I have predicted that it would be possible?

Pps. This quadrangle can be divided in two triangles and hence the angles together form a round angle.

Tr. I wrote down the following on the blackboard. You will give me the figures and tell me each time how you obtained them.

The sum of the angles of a triangle is 180°

of a quadrangle 360° | of a pentagon 540° | of a hexagon 720°

Tr. What would those arrows mean? You sometimes see them in a history book or in a family novel.

Pp. 28 The three items below follow from the item above.

Pp. 37 A genealogical tree.

Tr. Exactly, now I claim that from "the sum of the angles of a quadrangle is 360 degrees" something else follows. What?

Pp. With each quadrangle a floor can be tiled.

Tr. Good, we will write that down:

Each quadrangle can be used to cover a plane.

What can I write under: The sum of the angles of a pentagon is 540 degrees?

Pp. 19 A floor cannot be tiled with pentagons.

Tr. But what has been drawn on the blackboard here then? Those are pentagonal tiles with which it was possible.

Pp. 32 A floor cannot be tiled with regular pentagons.

Tr. Then I do not know much about the other pentagons. What about those?

Pp. 34 Not every pentagon can be used to cover a plane.

Tr. Do we agree with that? And what do I put under: The sum of the angles of a hexagon is 720 degrees?

Pps. Exactly the same: Not every hexagon can be used to cover a plane.

Tr. Do we remember what a parallelogram is?

Pps. Yes.

Tr. There was another kind of figure also.

Pps. Trapezoid.

Tr. Do we remember what the distinction is between those two figures?

Pp. 39 The parallelogram has equal sides, "so and so" (hand motions) and the trapezoid does not have that.

Tr. Do I know then what a trapezoid is? The sides "so and so" not equal. (I drew a quadrangle with unequal sides.)

Pp. In the trapezoid one pair of sides is parallel and in the parallelogram two pairs.

Tr. Next time you are all to bring a sheet, covered with parallelograms; you may choose the measurements yourselves. Would it also work with trapezoids.?

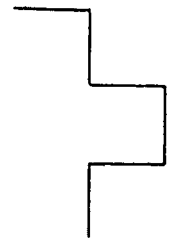
Pps. Yes, because they are quadrangles.

Tr. Then you will also draw a half pageful of trapezoids at home. (They wrote that down on their schedule.)

Tr. We will look again at the first page of our figures. In those figures, we noticed many straight lines. In figures 4 and 6, on the second page, we did not see straight lines. While you were drawing I asked you: What do you see in the figure? (Pp. 34 answered then: No straight lines, but zig-zag lines.) Do you see one? Color it then.

Pps. I see many more than one.

Tr. If you see more than one, then color two different ones. Again start with figure 1. (A number of children had the tendency to color the broken line shown here when they worked on figure 2. While they were doing it they started to doubt. They asked me whether that line was also a zig-zag line. I then drew the correct figure on the blackboard. Back and forth, back and forth, etc.



Tr. What does this figure remind you of?

Pp. Flashes of lightning.

Pp. 37 You can make one like that with a folding pocket-rule and then you can give it the shape you want.

Tr. Since we are talking about the carpenter, what does he use? (Before I finished the sentence the words came from all sides.)

Ninth class conversation Ia.

First I drew on the blackboard the pentagonal and hexagonal tiles the children had found and I gave them the task of coloring them by using as few different colors as possible.

Tr. The quadrangle you looked at last time had a strange form. Some of you therefore thought that it would not be possible to pave a floor with it. Afterwards it appeared that it was possible after all, when you tried it out with a large number of cardboard models which you were given. How could I have predicted it?

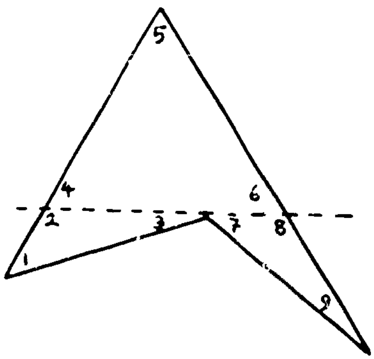
Pps. The quadrangle can be divided into two triangles and then the four angles together form a round angle.

Pp. 2 You could also divide the quadrangle into three triangles.

Tr. Yes, certainly - let us now add up the angles.

Pp. Then you take too much.

Tr. Let us look at it carefully. Angles 1, 2 and 3 together 180 degrees. Angles 4, 5 and 6 together 180 degrees. That is together 3 X 180 degrees. What did I take too much?



Pp. A straight angle there (pointing) and again one at the other side.

Tr. Thus take away 2 X 180 degrees. I am left with 1 X 180 degree.

Pp. A Straight angle has been forgotten at the point (pointing).

Tr. Do we all see it? The re-entrant angle has not been counted completely. (Indicated it with an arc). We thus get....?

Pp. Yet again 360 degrees.

Tr. Even though we can compute the sum of the angles in different ways, we always obtain 360 degree..

Pp. Yes, of course, it has to be like that. (We then wrote down the genealogical tree, see eighth class conversation Ib.)

Tr. What follows from: The sum of the angles of a pentagon is 540 degrees?

Pp. 15 With some pentagons it works.

Tr. You mean: a floor can be tiled. Yes, from what does that follow?

Pp. 15 From the tiles drawn on the blackboard.

Tr. Yes, but what I really asked was: What follows from: The sum of the angles of a pentagon is 540 degrees?

Pp. 2 It is only possible if, for example, three angles together form a complete angle.

Pp. 7 It is possible with irregular ones.

Tr. How do you know that?

Pp. 7 I can see it from those tiles. (Points towards the drawings on the blackboard.)

Pp. 5 It does not work with all pentagons.

Tr. Thus in summary we can say: Not every pentagon can be used to cover a plane. Pp. 2 has indicated how it can be and we have seen from the figures that it is possible indeed.

Tr. And how about the hexagons?

Pps. Not every hexagon can be used to cover a plane.

Tr. Last time we found a couple of new quadrangles. What are their names again?

Pps. Parallelogram and trapezoid.

Tr. Can somebody tell me what the difference is between those two?

Pp. 4 Yes, for a parallelogram one line segment had been erased, and two for a trapezoid.

Tr. Good, we arrived at the figures that way. But Pp. 10 was not here last time. Who can tell her exactly what the distinction is between the two figures?

Pp. 4 A trapezoid has one pair of parallel sides, and a parallelogram has two. (Many wanted to say the same thing.)

Tr. For the next lesson you will make a floor, tiled with parallelograms whose measurements you may choose for yourselves. Would it also be possible using congruent trapezoids? (See figures 11 and 12, page 150.)

Pps. Yes, because it is a quadrangle.

Tr. Good. It has to work. Make half a page full for each figure. (Next we looked at figure 6 and we found zig-zag lines, see class conversation Ib. When asked what they saw in the figure, they answered:)

Pps. The letter Z, the letter N, the letter W, the letter M.

Pp. 10 A flash of lightning. (Some pupils did not think that was true.)

Pp. 15 A saw.

Tr. We will call this figure a saw. From what can we recognize a saw in such a figure?

Pp. 3 You can see that from the angles.

Tr. Can you recognize it from something else? (This took awhile).

Pp. 12 From the parallelism of the lines.

Tr. I call a saw an open figure. Why?

Pp. Because you can continue on and on - there really is no end.

Tr. Who can name other open figures?

Pp. 2 Straight lines.

Pp. 17 An angle and a point.

Tr. There are ladders in the figures. The uprights can run towards each other as in figure 5, or they can be parallel as in figure 1 . Try and find semi-ladders in the figures - one upright with rungs.

Pps. You can only look for ladders in the figures where there are straight lines. Figure 6 cannot be taken into consideration, neither can figure 10.

Ninth class conversation Ib.

Tr. What did we look for in the figures yesterday?

Pps. Saws.

Tr. I called the saw an open figure. Is a triangle also an open figure?

Pps. No.

Tr. Name another open figure.

Pp. 37 A cylinder, you can look through it.

Pp. 35 A straight angle.

Pps. Acute angles. Obtuse angles.

Tr. I can see in figure 5 a ladder with two uprights that run towards each other and many rungs. (I drew an upright with rungs on the blackboard.) This is also an open figure. Look for ladders, one upright with rungs in figures 1 through 12 (pages 143-151).

Pp. There cannot be any in figure 6, because there is no upright.

Tr. Last time we looked for saws in the figures. From what can I recognize a saw?

Pp. 34 From the angles that have to be equal.

Tr. Can I recognize the saw from something else? From the lines perhaps?

Pps. Yes, they have to be parallel, one set 'so' (hand movement) and one set 'so' (hand movement).

Tr. From what do I recognize a ladder?

Pps. From the parallel rungs.

Tr. Can I also see it from the angles?

Pp. 1 The rungs form the same angles with that line (by this is meant the upright) on the upper side.

Tr. And on the underside?

Pp. Nct equal to the upper ones. All lower angles are equal to each other.

Tr. So as not to get confused we call all upper angles a set of corresponding angles and all lower angles another set of corresponding angles. When I see that all lower angles are equal, is the figure then a ladder?

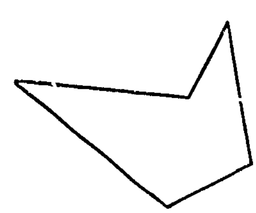
Pps. Yes.

Tr. We now write in the notebook:

> I recognize the saw from its equal angles, or from its parallel lines (two sets).
>
> I recog ize the ladder from its parallel lines, or from its equal correspondin angles (one set is sufficient).

(Next each child was given a pentagon of the shape drawn here).

Tr. There is a re-entrant angle in the pentagon you received. Try to find out how many re-entrant angles a pentagon can have at most. Write down all your thoughts on this. Next, try to draw such a pentagon on the other side of your sheet. (They were given ten minutes to find the answer. See further discussion in Chapter XII, p.162.)



Tenth class conversation Ia.

Tr. Tell me again how the saw can be recognized in the figures.

Pp. From the equal angles.

Pp. Also from the parallelism of the lines.

Tr. How do I recognize the ladder?

Pps. From the parallel rungs.

Tr. Is there also another way to recognize a ladder?

Pp. 2 The rungs each time form the same slope with respect to the upright. (I drew half a ladder with sloping uprights.)

Tr. Do you understand what Pp. 2 means?

Pps. Yes, those angles are equal. (One pupil shows it on the blackboard.)

Tr. Are the other ones also equal?

Pp. No, only when it is a straight ladder.

Tr. Good. Are the lower angles not equal to each other here then?

Pp. Yes, equal to each other, but not equal to the upper ones.

Tr. We call all upper angles a set of corresponding angles - they correspond to each other - they are positioned like that in the figure. The other set of corresponding angles is always on the underside. (Next, we write down how we recognize the saw and also the ladder, see 9th class conversation Ib.)

Tr. Figure 9 has many saws and many ladders. Have you found those? (I check that). In that same figure we have seen a parallelogram and we have drawn an enlargement of it. We are going to look now at the other pages and try and find figures that also have been enlarged. Which figure do we see in the first tile floor?

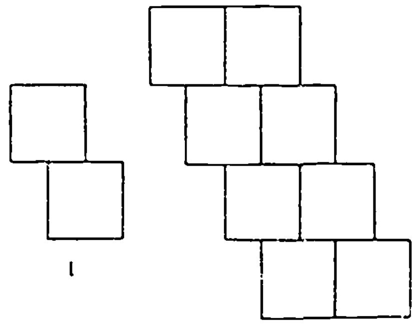
Pps. A square.

Tr. Are there also enlargements in the figure?

Pps. Yes, two times, three times.

Pp. In mine even four times.

Tr. Draw their outline. I will come around and look at what you are finding. (One does not find an enlargement in figure 2. Pp. 6 hesitates. Would the figures drawn here be enlargements of each other?)


I

II

Tr. Is the shape still the same? When I look at figure I with a magnifying glass, do I get figure II if the glass magnifies exactly twice?

Pp. 6 No, there is only one indentation.

Pp. I see other figures in addition to triangles in figure 5.

Tr. Then you should check whether there also are enlargements of those other figures. If you find any, color them. Perhaps you will find other figures that are enlarged in the preceding drawings as well. (Rectangles are found in figure 1. In figure 4, only hexagons and enlargemnts. In figure 6, no enlargement. In figure 3, triangles, rhombi, trapezoids, parallelograms, hexagons, stellated hexagons, etc..) (The last ten minutes are spent on the same problem about the pentagon as in Ib.)

Tenth class conversation Ib.

Tr. Take the sheet covered with parallelograms in front of you (see figure 11). What figures are numerous in that drawing?

Pps. Ladders.

Pps. Saws.

Tr. Color one of the angles in the middle of the sheet with red. Look for a ladder or a saw of which this angle forms a part. Then color the same angles of the ladder or the saw with red. Then you will look for another ladder or saw that has a red angle. Color the angles that are equal to that red angle with red also. Go on in that fashion. Count the number of ladders and saws that you have used. Record your result.

Pp. There are many more ladders then you make use of. Should I count those also?

Tr. No, you only count those you have used. (I had to help here and there to find the saws. Almost all the children chose ladders. Many had to be made aware of the fact that there were more angles of the same size as the first angle.)

Tr. Now check whether you have used the saw at least once.

Tr. Some time ago we made a genealogical tree. Who can remember what it looked like? What was at the top?

Pps. The sum of the angles of a triangle is 180 degrees.

Tr. What followed from that?

Pps. The sum of the angles of a quadrangle is 360 degrees.
The sum of the angles of a pentagon is 540 degrees.
The sum of the angles of a hexagon is 720 degrees.

Tr. How did that follow from it?

Pps. You divide the quadrangle into two triangles and then the angles together are 2 X 180 degrees = 360 degrees. The pentagon is divided into three triangles and the hexagon into four.

Tr. I thus use "the sum of the angles of a triangle is 180 degrees" successively twice, three times and four times. I indicate that with two arrows in the following manner:

The sum of the angles of a triangle is 180°.

The sum of the angles of a quadrangle is 360°.

The sum of the angles of a pentagon is 540°.

You have already written down this genealogical tree in your notebook. Now put in the correct number of arrows.

Tr. I now draw a simple parllelogram on the blackboard and I color one angle with red. Is there another angle that has the same size as this one?

Pps. Yes, the opposite angle.

Tr. I do not color this angle with red yet, but I write down the following rule: The opposite angles in a parallelogram are equal. I would like to know whether this rule also has forefathers. (I drew two arrows above it.)

Pps. Yes, I know it: the ladder.

Tr. Where do you see a ladder?

Pp. Extend the line.

Tr. Come and draw the ladder in the figure (blue). Good. Which angle has the same size as this red angle? Color that one also with red. What now? Is that it?

Pps. Now a saw.

Tr. Good. You come and color this one (green).

Pp. Then the angle colored last is again equal to that one. (At the same time he colored the opposite angle of the parallelogram with red.)

Tr. Are we ready?

Pps. Yes.

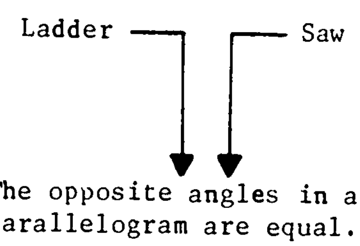
Tr. Who are the forefathers?

Pps. The ladder and the saw.

Tr. How many arrows?

Pps. One from each.

Tr. Write this short tree in your notebook.



Pp. 21 Can't you also do it in another way?

Tr. How would you do it?

Pp. 21 Fold the triangle.

Tr. Look carefully. You can also use a mirror.

Pp. 21 No, I can see it already. Then it would become a kite.

Tr. Pp. 23 has a genealogical tree, even though she may never have checked who her forefathers were. Pp. 21 also has one. Perhaps they descend from the same great-great-great-grandmother because both their names are M. Then their genealogical trees are linked somewhere. Instead of two genealogical trees, there would be only one in fact. The short tree we just found may perhaps belong to that other larger genealogical tree. Only I do not know where the link is situated. We have already found many truths in the figures. Practicing geometry is, among other things, looking for relationships that exist between truths. It is interesting to try and join everything in one large genealogical tree. It may take us as much as three years. We ultimately want to know what they are descended from and that, of course, is the most difficult.

Pp. 38 I know it: it is the point. A line originates out of a point, an angle out of a line and next a triangle.

Tr. We will see whether you are correct. In the meantime I am very doubtful about that. Let us first try and see whether we can find a forefather of: "The sum of the angles of a triangle is 180 degrees". (I drew a triangle on the blackboard).

Tr. Does anyone perhaps see a forefather? I first have to add something to the drawing.

Pp. 32 Through that point a line parallel 'so'. (She indicates a line through the top, parallel to the base).

Tr. Do you understand it? 'So'. (I drew the line).

Pps. (Enthusiastically) I see it - I know it already. A saw and another one.

Tr. Now calm down. First Pp. 28 will color a saw and color the equal angles. Next Pp. 30 will color a saw and color the equal angles. Is that it?

Pps. Yes, they form a straight angle.

Tr. Who is then the forefather?

Pps. The saw.

Tr. How many times has it been used? How shall I write it then?

Pp. Saw and then two arrows underneath and then...

$$\text{Saw} \;\Downarrow\!\Downarrow$$

The sum of the angles of a triangle is 180°.

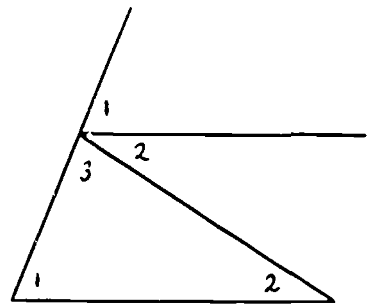
Tr. Where can you write this in your notebook? Is there any room left?

Pps. Above the other one. (The pupils write it above).

Pp. 36 It can also be done in another way.

Tr. Yes, tell us about it

Pp. 36 I make a ladder. I extend that line. And I add another rung. (He draws the figure shown here on the blackboard).



Tr. Which forefathers are involved here? And how many times?

Pps. The ladder once and the saw once.

Tr. This is outstanding. We thus see that it can be demonstrated in more than one way. Can it be done using the other vertices?

Pps. Yes.

Eleventh class conversation Ia.

(This class conversation was practically the same as the preceding one until:

Ladder $\downarrow$ Saw $\downarrow$

The opposite angles in a parallelogram are equal.

Then it is asked:)

Pp. 4 Could it be done differently? You can also divide the parallelogram into two triangles which are the same.

Tr. Yes, it can be done like that also. We will do that some other time. Do you understand the way we have done it now?

Pp. 4 Yes, that was not difficult.

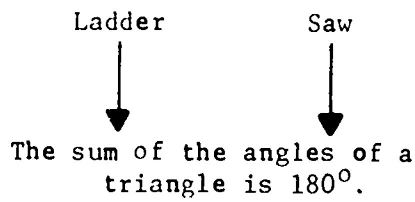
Tr. Pp. 4 and Pp. 7 perhaps have the same great-great-great-grandfather when you would look at their genealogical trees, because both their names are P. Then their genealogical trees are linked together there. Let us see whether we can investigate of what the rule:

The sum of the angles of a triangle is 180 degrees, is a result. (I drew a triangle on the board.)

Pp. 11 I can see how it should be done (here she looks in her notebook.) You extend that line and draw a parallel line, and then you have a ladder. (See above figure.)

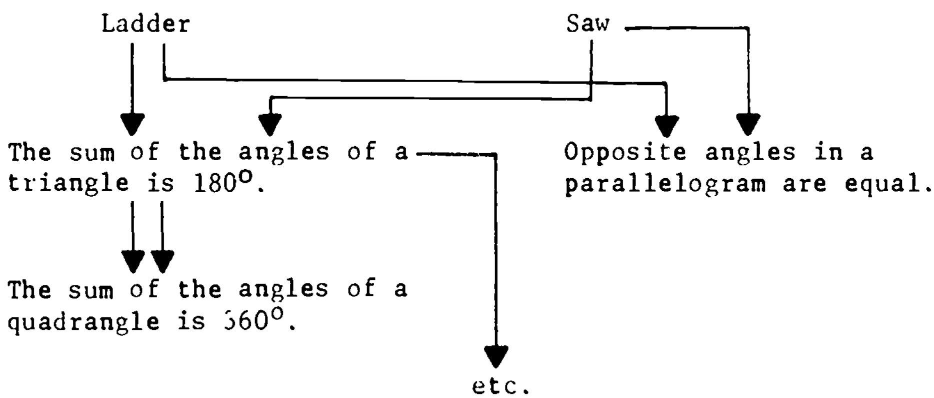
Tr. Can we all see what Pp. 11 means? (I drew the lines in the figure. One of the pupils colored the ladder and gave the corresponding angles the same color. Another pupil colored the saw and gave the corresponding angles the same (different) color, so that the straight angle was made obvious.)

Tr. Where can I place the tree?



Pps. Above the other one.

Tr. Write it there in your notebook. That same ladder and saw already were the forefathers of another tree. We can thus link them in a tree.



You now have seen an example of what we frequently do in geometry. We already know a great deal about geometric figures. We are now going to try and find certain relationships, and finally we will try to join everything we have found into one genealogical tree.

Tr. In the other group, the children drew this line. (I drew a line through a vertex parallel to the opposite side.)

Pps. Yes, I see it. With saws. (Here too, a couple of children colored the equal angles with the same color and again the straight angle was made obvious.)

Eleventh class conversation 1b.

Tr. Place figure 9 in front of you. You found two parallelograms in this figure. One parallelogram was an enlargement of the other. Most of you had enlarged the parallelogram twice. Pp. 28 had enlarged it three times. We now start with figure 1. Do we see a figure in this drawing that is also present in an enlarged form?

Pps. A square. Twice. Also three times. Perhaps four times.

Tr. You will draw those figures and their enlargements on a new sheet of paper. Draw their real size. Start right now because I believe you have a lot to do. I will walk around. Feel free to ask anything.

Pps. There is not anything in figure 2 is there?

Pps. We find the same in figure 3 as in figure 1.

Tr. Such figures you need not draw twice, of course. Write under the drawings which figures they were taken from.

Pps. I see hexagons and enlarged hexagons in figure 4.

Pps. There are many figures in figure 5.

Tr. Yes, there are many figures in figure 5, If there is an enlargement of such a figure it should be included, of course. Perhaps you will also find another figure in the first tiled floor.

Pps. A rectangle.

Tr. Whoever did not finish, may continue at home.

Twelfth class conversation Ia.

Tr. We are going to look at the tiled floors in a completely different way. Imagine the floor extended so that you do not see the ends. Could I then find axes of symmetry in the first tiled floor? (I had already drawn that tiled floor on the blackboard.)

Pp. Vertically, along that line that is there already.

Tr. Good. (I mark it with colored chalk. May I take this one also vertically? (I point to another one.)

Pps. Yes.

Tr. We will draw only one of each kind. Do you see any other kinds?

Pps. Horizontally.

Tr. Good. (I drew a horizontal axis of symmetry.) any more?

Pp. Obliquely, along the diagonal.

Pps. And of course also along the other diagonal. (I drew both of them on the figure.)

Tr. Whoever finds it difficult should use a mirror. Are there any more?

Pp. 16 Divide the tiles exactly in half. Vertically.

Pps. And horizontally as well. (I drew those on the board.)

Tr. Does the figure have centers of symmetry? That means: Can I find a point such that when I rotate the figure around that point, I get the same tiled floor?

Pp. Yes, I see one. (He comes and indicates a point of intersection in the middle of the drawing.)

Pps. Yes, all those points - vertices.

Tr. Of what order is that center?

Pp. It is of the fourth order because you rotate the figure a quarter of a turn.

Tr. Are there, except for those points of intersection, any more centers of symmetry?

Pp. The centers of the squares.

Tr. What is the order of these?

Pps. They are also of the fourth order. (I draw such a point and write down IV next to it.)

Tr. Are there any more centers of symmetry?

Pps. No.

Tr. They are not so easy to find. I will help a little.

There are more centers of symmetry, but not of the fourth order. It is the same case as with the rhombus. (I drew a rhombus.) This figure has a center, but not of the fourth order.

Pp. Of the second order because you have to rotate half a turn.

Tr. Yes, can you now find one like that in the tiled floor?

Pp. In the middle. (He comes and indicates the correct point on the blackboard.) I write down II next to it.

Tr. Last week you searched for enlargements and you drew them. I now give you the following assignments:

I. Draw the figures and the accompanying enlargements side by side on a new sheet of paper.

II. Draw axes of symmetry in the tiled floors.

III. Indicate centers of symmetry, as I have demonstrated on the blackboard and write down next to them what the order of each one is. First you work on figure 1, then figure 2, etc.

(The class conversation then proceeded similarly to the eleventh class conversation Ib.)

Twelfth class conversation Ib.

Tr. On the board I drew the figures and enlargements which could be obtained out of the first tiled floor. What did you find?

Pps. A square. Enlarged two, three and four times. A rectangle. Enlarged two and three times.

Tr. Did anybody find something in the second tiled floor?

Pps. No.

Tr. And in the third tiled floor?

Pps. The same figures as in the first one.

Tr. And in the fourth tiled floor?

Pps. A hexagon and a hexagon that is enlarged twice.

Tr. And in the fifth tiled floor?

Pps. Triangles. Enlarged once, twice, three times and four times. The same hexagons as in the fourth tiled floor. Trapezoid, enlarged twice and three times. Rhombi, enlarged once, twice and three times.

Tr. The whole blackboard is covered with figures. Let us first look at them. The side of the square is doubled - next taken three times and then even four times. Has the area also been doubled?

Pps. No, it has become four times as large.

Tr. How do you know that?

Pps. You can divide that large square into four small squares. (I draw the division in the large square.)

Tr. And when I enlarge the side three times?

Pps. Then it is nine times.

Tr. And four times enlarged?

Pps. Then it becomes sixteen times.

Tr. Now the rectangles.

Pps. It is the same.

Tr. How do you know that?

Pps. The large rectangle can be divided into four rectangles and the next one into nine.

Tr. Now with the triangles.

Pps. The second one can be divided into four triangles.

Pp. We have seen that in the algebra class. Then we have 9, 16, 25, etc.

Tr. Now the regular hexagon.

Pp.36 It does not fit in it.

Tr. Would the area really be four times as large?

Pp.36 Yes, four times.

Tr. How can I see that?

Pp.37 Divide.

Tr. What does Pp. 37 mean?

Pps. Divide the hexagon into triangles and then count them.

Tr. How many triangles in the first one?

Pps. Six. (Next, together we count the number of triangles in the second hexagon that first has been divided into six large triangles.)

Pps. 24. Hence, four times as large.

Tr. And if I were to take a hexagon that has been enlarged three times?

Pps. Then it becomes nine times as large as the first one.

Tr. Now the trapezoid.

Pps. Again, it does not fit. Also divide into triangles.

Tr. Now you make the divisions for yourselves in your notebook. What do you find?

Pps. 3, 12, 27.

Tr. Would it always come out?

Pps. Yes.

Tr. Let us then write this down in your notebook. When I look at the first row of figures, what has remained the same? When I look at the second row of figures, what has remained the same?

Pps. The shape.

Tr. Upon enlargement, the shape of the figure remains the same. Write down:

> The shape of the figures remains the same upon enlargement. We therefore call the figures similar.
>
> When the measurements are enlarged 2 times, the area becomes 4 times as large.
>
> When the measurements are enlarged 3 times, the area becomes 9 times as large.
>
> When the measurements are enlarged 4 times, the area becomes 16 times as large.

(Next, we looked for the axes of symmetry and the centers of symmetry. The lesson further proceeded as in the twelfth class conversation Ia.)

Thirteenth class conversation Ia.

(For the beginning of this lesson see twelfth class conversation Ib.)

Tr. If I enlarge the side of the square 3 times, how many times is the area then enlarged?

Pp. 14 Six times.

Tr. How do you know that?

Pp. 14 It is nine times. I can fit the small square nine times into the large one, three rows of three. (I draw the division on the blackboard.)

Tr. And if I enlarge the square four times?

Pps. Then the area becomes 16 times as large.

Tr. What about the second figure: the rectangle?

Pps. Again 4, 9, 16.

Tr. And the small triangles?

Pps. Then it fits three times.

Pp. 15 No, four times.

Pp. 2 Yes, four times.

Tr. How do we find out who is correct?

Pps. Divide into small triangles. (One of the pupils does that on the blackboard.)

Tr. We find four times. And what about the next triangle?

Pps. Nine little triangles.

Tr. Now we turn to the areas of the hexagons. If the side has been enlarged twice, what do you think the area of the enlargement will be?

Pps. It does not fit in it.

Tr. What do you think the answer should be?

Pps. Four times also.

Tr. How shall I demonstrate that?

Pp. 10 Divide into small triangles.

Pp. 7 First compute the area.

Tr. Good, how do you compute that?

Pp. 7 I do not know.

Pp. 6 I had thought of dividing the hexagons into rhombi.

Tr. Good, come and demonstrate it on the blackboard. We find 3 rhombi and 12 rhombi. It fits. Hence, again four times.

Pp. 15 Can it not be done in another way?

Tr. Yes. How did you want to do it? Come and draw it on the blackboard.

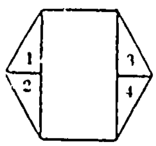


fig. 1

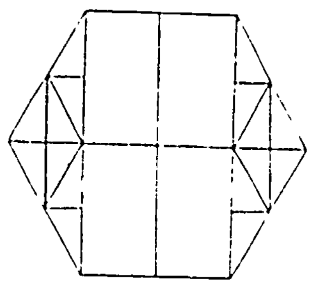


fig. 2

(He draws two parallel lines and after some hesitation, he divides the isosceles triangle into two congruent triangles. See figure 1 above. Is it finished?

Pp. 15 No. (He draws figure 1 and counts four times as many triangles in figure 2 above as in figure 1.)

Tr. Is it necessary to still further divide the triangles in figure 1?

Pp. 15 No, but I did not see how to do it then.

Tr. I erased two line segments in the first figure. Now you erase the correct line segments in the second figure. (He did this correctly.)

Tr. However we divide the figure, we always find that if the side has been enlarged twice, the area becomes four times as large. And if the side is enlarged three times?

Pps. Then the area becomes nine times as large.

Tr. Now we will turn to the trapezoids. Does the same apply here too?

Pps. Yes.

Tr. Make a division for yourselves so that this is apparent. (All divide the trapezoids into triangles.)

Tr. When we look at such a row of figures, for example, all rectangles or all triangles, what has remained the same?

Pps. The shape.

Tr. Figures that have the same shape, we call similar. Now write down in your notebook: When a figure is enlarged, the shape remains the same. Therefore, the figures are called similar. When the measurements are enlarged twice, the area becomes 4 times as large, etc.

Pp. 3 Those numbers are the square numbers. Can't we summarize those sentences?

Tr. Yes. Certainly. Try it.

Pp. 3 The areas of the figures are the square numbers.

Tr. Is that so? Look once again at the row of the triangles. Is the area of the second triangle equal to four?

Pp. 3 No, it is not correct.

Tr. Time is up now. You will see later that you are right, that those three sentences can be summarized into one sentence.

Thirteenth class conversation Ib and twelfth class conversation Ia.

Tr. Place figure 5 in front of you: the plane covering with equilateral triangles. You found the centers of symmetry in that figure. A center of symmetry of the sixth order - what does that mean?

Pp. After rotating six times it stands right again.

Tr. And after a sixth of a turn? Think of the plane as being extended far away.

Pp. Then it fits also. After rotating six times it went around completely.

Tr. After rotating a sixth of a turn, the figure covers itself again. It looks again as before. By rotating, the figure can cover itself. This can be done in another way. How?

Pps. Through mirroring.

Pps. Through folding.

Tr. Yes, we also sometimes talk about flipping over. Along which line shall I flip?

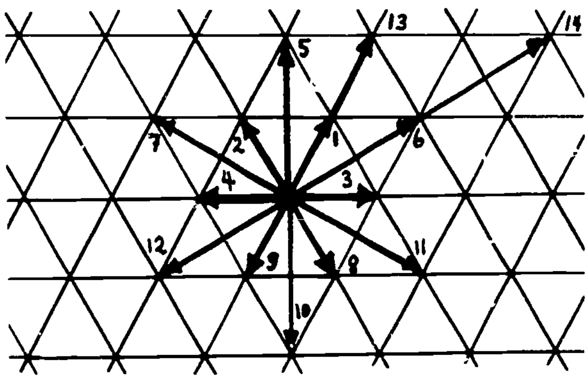
Pps. Around that one, or that one.

Tr. What are they called?

Pps. Axes of symmetry.

Tr. The tiled floor can also cover itself again through displacement (translation). We still think of the tiled floor as being extended far away.

Pp. Yes, up obliquely. (I indicate this with an arrow, No. 1.)



Tr. Are there any other solutions?

Pp. Yes, up obliquely on the other side (I draw arrow No. 2.)

Pp. Horizontal. (Arrow No. 3.)

Pp. Also to the left. (Arrow No. 4.)

Tr. Any more?

Pp. Up vertically, (Arrow No. 5.)

Tr. Any more?

Pp. Yes, from that point to that point.

Tr. Come and show it to us. (Arrow No. 6.)

Pp. Then also to the left. (Arrow No. 7.)

Pp. May it also go downwards?

Tr. Yes, why not?

Pps. Then a whole lot downwards. (Arrows No. 8, 9, 10, 11, 12,)

Tr. Any more? Could I also slide it over two tiles?

Pps. Yes, of course. (I draw arrow No. 13.)

Pps. I can do the same with all those arrows.

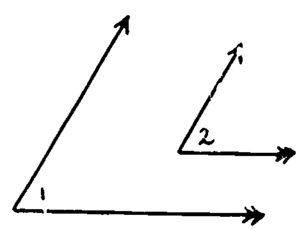
Tr. Good, any more?

Pp. Yes, I see some more. (He comes and indicates which one he means. I draw arrow No. 14.)

Pp. It can go on and on.

Tr. There are many directions in which one can displace (slide) a plane covering such that it will cover itself again. (In both classes the pupils followed the same sequence. First to the right, then upwards, then to the left, then horizontally to the right and to the left, then vertically, then downwards.) Next the pupils were given the following task.

Tr. draw an angle, and then a second angle whose legs are parallel to the legs of the first angle. Can you say something about those angles?



Pps. They are equal.

Tr. I color the first angle with red. Now you will try and find out what this equality follows from. I wrote, with two arrows above it:

Two angles, whose legs are parallel, are equal.

You write down all your thoughts.

Fifteenth class conversation Ib and fourteenth class conversation Ia.

I started this lesson by writing the following problem on the blackboard:

1. An entrance hall of 17 by 20 dm is being paved with square tiles whose sides are 1 dm. Draw this tiled floor. Scale 1:10.

Tr. Do all of you know what that means?

Pps. 1 cm on the drawing is 10 cm.

Tr. What measurements do we have to use for the drawing?

Pps. 17 cm and 20 cm.

Tr. 2. Draw the same entrance hall, this time paved with regular triangles whose side is 1 dm. Scale 1:10.

3. Draw a hexagonal market place (regular hexagon) with a side of 10 meters. Scale 1:10. Pave this also with square tiles whose sides are 1 m.

4. Draw the same market place, but now covered with triangular tiles, whose side is 1 m. Scale 1:100.

(The pupils who had received a satisfactory grade for a certain algebra test were allowed to start drawing immediately. The others spent the rest of the hour discussing with me the errors they made on the algebra test. They had to make the drawings at home.)

Sixteenth class conversation Ib and fifteenth class conversation Ia.

Tr. Let us first look at the drawings of the entrance hall. If the square tiles cost F 1.60 and the triangular ones F 0.70, which pavement is the most expensive?

Pps. This can be answered immediately. The square tiles.

Tr. Why?

Pps. Because half of F 1.60 is equal to F 0.80. The square tiles are more expensive.

Tr. But you need less of them?

Pps. Yes, but you have twice as many triangles as squares.

Tr. Who else found that? (Six pupils in both classes.) Look carefully at your drawings. Have you drawn regular triangles whose side is 1 cm? (They had drawn semi squares.)

Pps. No, one side is longer than 1 cm.

Pp. 11 The triangle is one half of the square though.

Tr. Let us draw it precisely.

Pp. 11 (Draws a square with an equilateral triangle in it with the compasses.)

Tr. Is it half of it? Less than half or more than half?

Pp. 11 It is half if it goes all the way to the top. (She had not drawn it incorrectly in her notebook, but she had not seen that there were more than 17 rows.)

Tr. Good. (I drew it on the blackboard.) How do you krow that?

Pp. 11 The two outmost pieces are then together exactly as large as the triangles.

Tr. Which tiled floor is more expensive?

Pps. Then you first have to count the tiles. (I point to one of the boys who writes the computation on the blackboard.) There are 17 x 20 = 340 square tiles needed. This costs 340 x F 1.60 = F 544.

Tr. Now we will compute the other one. (They counted 20 rows, each of 40 tiles.)

Pp. There are 20 x 40 = 800 triangular tiles needed. This costs 800 x F 0.70 = F 560.

Pp. 10 That is more expensive.

Tr. But the tiles that are now in the entrance hall are expecially nice. We will now examine the regular hexagon. (I drew the hexagon and I asked how the pupils had done it. All but one pupil had drawn the figure.)

Pps. Mark the circumference with points.

Tr. How many triangular tiles do we need and how many squares? (Many pupils first wanted to count them.)

Pp. 25 In one triangle there are 100, hence there are 600. (100 had been counted.)

Pp. 36 There are 100 in the large triangle. We learned that in algeb.. class.

Pp. 22 I first took half of it.

Tr. This trapezoid?

Pp. 22 Yes, and then counted.

Tr. How?

Pp. 22 I just counted, the odd numbers: 21 + 23 + 25 + 27 + 29 + 31 + 33 + 35 + 37 + 39. And then multiplied the answer by two.

Pp. 19 I did the same thing, but I have added the first and the last and multiplied that by 5.

Pp. 5 I summed up 1 + 3 + 5 + ...+ 19, that is 5 x 20. I multiplied that by six.

Tr. Good, you divided the hexagon into six large triangles. Has anybody done it in a different way?

Pp. 7 Yes, but it does not come out right. I took the lower half. First a triangle 10 x 21. Then two are added to each subsequent row. This adds 9 x 2 = 18 to it.

Tr. Let us look carefully at what you did. (In the meantime I have drawn the figure on the blackboard.)

Pp. 7 At first 2 are added. But then 4 are added. 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 = 90. And 210 + 90 = 300. The top half is also 300. That comes out.

Pp. 9 I took 20 x 10, that is much easier.

Tr. Where are those 20 x 10?

Pp. 9 That rhombus, then you have 10 rows of 20 tiles. And you take that three times, because the hexagon has three rhombi. (we recognize that this is the quickest method.)

Tr. Now the number of square tiles. How do we determine that? (The drawings were good.)

Pp. 36 Then we first have to compute the area. (His voice gets lost.)

Pp. 17 First that triangle, that is 17 x 10. (She was allowed to round off the length at 17. I had already mentioned that last time during the drawing.)

Pps. You can place the triangles next to each other. This makes a rectangle of 17 by 15.

Tr. Hence 17 x 15 = 255 tiles are needed. (A couple of students had to think carefully that one cannot just put together the triangles, indicated by 1 and 3 on the figure of page 128, one triangle first has to be flipped over.

Tr. Now another question. What is the area of the market place? (It takes awhile, then I hear one pupil chuckle and finally they all understand.)

Pps. That is the same.

Tr. Say it then.

Pps. 255 square meters.

Tr. And the area of the hexagon on the blackboard.

Pps. 255 square centimeters.

Tr. Thus, the area of a figure is the number of square centimeters that are needed to cover the figure. Remember this. You cover the figure with square tiles of 1 square centimeter.

Seventeenth class conversation Ib.

Tr. We now know many figures. I will write down which quadrangles we have seen. Whoever I point to should name one. (In this way we formed the row: square, rectangle, trapezoid, parallelogram, rhombus, kite, irregular quadrangle.) We have also drawn the enlargements which we observed in the plane coverings. Now we will try it without those plane coverings. Everyone take a blank sheet. Draw a triangle and then that triangle enlarged twice. (All but one pupil, carried out the construction according to case SSS. Only Pp. 5 copied an angle and used the case SAS.) Now a square. Take the side to be 2 cm. Enlarge it 1 1/2 times. (Some pupils made the side of the new square 5 cm. When I asked them: "What did you multiply by?" They were able to correct the error immediately. Pp. 33 made a rectangle. I asked him: "Have you enlarged the square now?")

Pp. 33 No, the other one has to be 3 too.

Tr. If you want to clarify for somebody what a trapezoid is, you can do that by making a drawing saying: That is it. You can also try to define it - say it in words.

(I point to Pp. 35 for him to try it.)

Pp. 35 I was sick during that lesson when you discussed the trapezoid.

Tr. That does not matter. You do know what it looks like, don't you? Come and draw it first. (Pp. 35 draws a trapezoid.)

Tr. Good, now try and say it in words.

Pp. 35 The upper and underside have to be parallel.

Tr. Can this not be then? (I drew a trapezoid where the parallel sides are not horizontal.)

Pp. 35 Yes. The upper and underside or the left and the right side have to be parallel.

Pp. 28 You could say it shorter: One pair of sides has to be parallel.

Tr. Is it something like this? (I drew a hexagon with one pair of parallel lines.)

Pp. 35 No, it has to be a quadrangle.

Tr. You have not said that yet. Say it again now.

Pp. 35 A trapezoid is a quadrangle in which two sides are parallel.

Tr. When is the trapezoid isosceles?

Pps. When the legs are equal. When the two other sides are equal.

Tr. Draw an isosceles trapezoid and then draw that trapezoid twice enlarged. (The pupils found many solutions. There was one who did not dare tackle it.)

Pps. I extended the legs, then I obtain a triangle - I enlarge that one first and then I take the legs of the trapezoid twice.

Pps. I first take the lower side twice, and then the altitude and then the legs.

Pps. I copy two angles with the compasses.

Pps. I divide the trapezoid into a rectangle and two triangles and then I enlarge the rectangle first.

Pp. 29 (Made use of the axes of symmetry and took the altitude and the halves of the parallel sides twice.)

Pp. 38 (Divided the trapezoid in two triangles with a diagonal, and enlarged all sides and the diagonal twice.)

Tr. You all have found many solutions. Who can say in words what a parallelogram is?

Pp. 31 A parallelogram is a quadrilateral with two sets of parallel lines.

Tr. Everyone draw one. I now ask you to draw a parallelogram whose sides are twice as large but one that is not similar to the one you first drew. (I wrote down this problem on the blackboard.) (Pp. 19 took the altitude twice as large. He thought that the parallelograms had to be similar. Pp. 35 took the same altitude and thereby knew that they could not be similar. Most pupils estimated the angles by eye. Therefore I had to ask several times: "How do you know that the parallelograms are not similar?")

Pps. Because the angles are not equal?

Tr. How can I be sure of that?

Pps. By measuring the angle with a protractor. (Many then wrote down the degrees.)

Pps. I did it before with compasses.

Tr. Could I ask the same question for a rectangle? (I pointed to the problem on the blackboard.) It says: Draw a rectangle, and next a rectangle whose sides are twice as large, but one that is not similar to the rectangle you first drew.

Pps. No.

Tr. Why not?

Pps. The angles remain equal.

Tr. If I were to change the word to parallelogram, would it then be possible?

Pps. No, also. The angles are also right.

Tr. And if I were to change it to rhombus? (Two pupils, Pp. 25 and Pp. 26 first thought it could not.)

Tr. What is a rhombus?

Pp. 25 Then the sides are equal and the diagonals bisect each other, and ......

Tr. When I ask you to draw a rhombus, what do you do?

Pp. 25 I make the four sides equal.

Tr. Is it a rhombus then?

Pp. 25 Yes.

Tr. You then should not make such a long sentence about the diagonals.

Pp. 25 A rhombus is a quadrangle with four equal sides.

Tr. Could we draw a rhombus that is not similar to another?

Pps. Yes, you can change the angle of the rhombus.

Tr. Good, now draw it. (Most pupils draw a diagonal in the rhombus. This was enlarged more than twice then. The sides are farther apart from each other then, they say.)

(Then I draw two circles on the blackboard.)

Tr. Are those circles similar?

Pps. Yes.

Tr. The enlargement has to be indicated. How large is it? How could I determine this? (I point towards Pp. 20.)

Pp. 20 You have to measure the radius then. (I measure the radius of the smallest circle. It is 12 cm.)

Pp. 20 And then the other one. (I measure 27 cm.)

Pp. 20 Then I have to divide 27 by 12. The enlargement is 2 1/4.

Tr. I now draw two very simple figures. (I draw two unequal line segments.) What are those figures called?

Pps. Line segments.

Tr. Are they similar?

Pps. Yes, of course.

Tr. How do I find the enlargements?

Pps. Measure the line segments.

Sixteenth class conversation Ia.

The differences with the seventeenth class conversation Ib were: in the case of the isosceles trapezoid the solution involving the axis of symmetry and the solution involving the use of the diagonal were not found. All pupils thought that one can draw a rhombus whose sides are twice as large as those of a given rhombus and that is not similar to that rhombus. I therefore asked them to draw a rhombus that was similar to that given rhombus and one that was not similar. This time the diagonal was used. Pp. 16 enlarged both diagonals, drew them perpendicular to each other and made them bisect each other. This produced a similar rhombus. The non-similar rhombus was obtained by changing the length of the diagonals, but by making sure that the side remained the same through the use of compasses. Pp. 3 thought that the circles were not similar. The others thought they were.

Tr. How can we convince Pp. 3 that this is indeed so?

Pp. 3 I can see it already, it has been enlarged twice.

Tr. How do you know that?

Pp. 3 I estimate that.

Tr. Could you find it exactly?

Pp. 3 Yes, by measuring the radii. (I measure 10 cm and 22 cm.)

Pps. The enlargement is 2 1/5.

Tr. I now draw two line segments. Are they similar?

Pps. Yes, just as for the radii, first measure. (I measure 17 cm and 39 cm.) the enlargement is 2 5/17.

Tr. Thus circles are always similar. One can always be considered as an enlargement of the other. So are line segments. Are there perhaps any other figures that are always similar?

Pp. 3 Squares.

Pps. Equilateral triangles.

Pp. 16 Yes, hexagons. Those hexagons which we drew with a circle. (I drew regular hexagons in the circles on the blackboard.)

Tr. How large is the enlargement?

Pps. Also 2 1/5.

Pp. Also those other regular polygons.

Tr. Can you tell me again what a regular polygon is?

Pps. All sides have to be equal and also all angles.

Tr. Can I choose the angles of a regular hexagon?

Pps. No.

Tr. Can I say what their size is?

Pp. 60 degrees.

Tr. What kind of angle is it? (I point towards the angles of the hexagon on the blackboard.)

Pps. Obtuse angles.

Pp. 17 It will be 120 degrees.

Pp. 16 That is twice 60 degrees.

Tr. How do you divide the hexagons.

Pps. Into six triangles. (I draw them in the hexagon.)

Tr. Now, start at the center.

Pps. One sixth of 360 degrees is 60 degrees. All those other angles are also 60 degrees because they are all equal.

Tr. I can also compute it in another way. (I drew a triangle, a quadrangle, a pentagon and a hexagon on the blackboard.) What do I know about the sums of the angles?

Pps. For the triangle 180 degrees, for the quadrangle 360 degrees, for the pentagon 540 degrees, for the hexagon 720 degrees.

Pp. 17 Then you divide by 6, that is 120 degrees.

Tr. Can you now compute the angles of the regular pentagon? (I pointed towards the figure on the wall.)

Pp. 16 540 degrees - 5 = 108 degrees. During the next two hours the pupils were given a test on tiles. They were given a sheet of paper completely covered with isosceles triangles (base 2 cm and legs 4 cm.) They were asked:

I. Which bounded figures can you find in the drawing? Draw their outline, indicate them by writing down a number in them and fill out the chart below.

No.	a Name	b Is there also an enlargement? Yes, no, 2x, 3x, many	c How many different shapes are there? One, two, many.
1	isosceles triangle	Yes, 2x, 3x, many	one —— ——
2			
3			
4			
..			
..			

II. Have you also found hexagons? Is there a regular one? If so, draw the outline with red. Do not forget to write down the answers and to explain.

III. Are there any figures that do not have the same shape, but that have the same area? Give name, number or enlargement.

IV. Are there any open figures? Color them. Indicate them with their name.

V. Draw the axes of symmetry. Indicate them with the letter S.

VI. Construct one enlargement of the figures. (Draw the enlargement on the sheet.)

Time: 2 x 45 minutes.

Footnote:

1. Geo-plan van Gattegno (I p. 172). [See p. 100.]

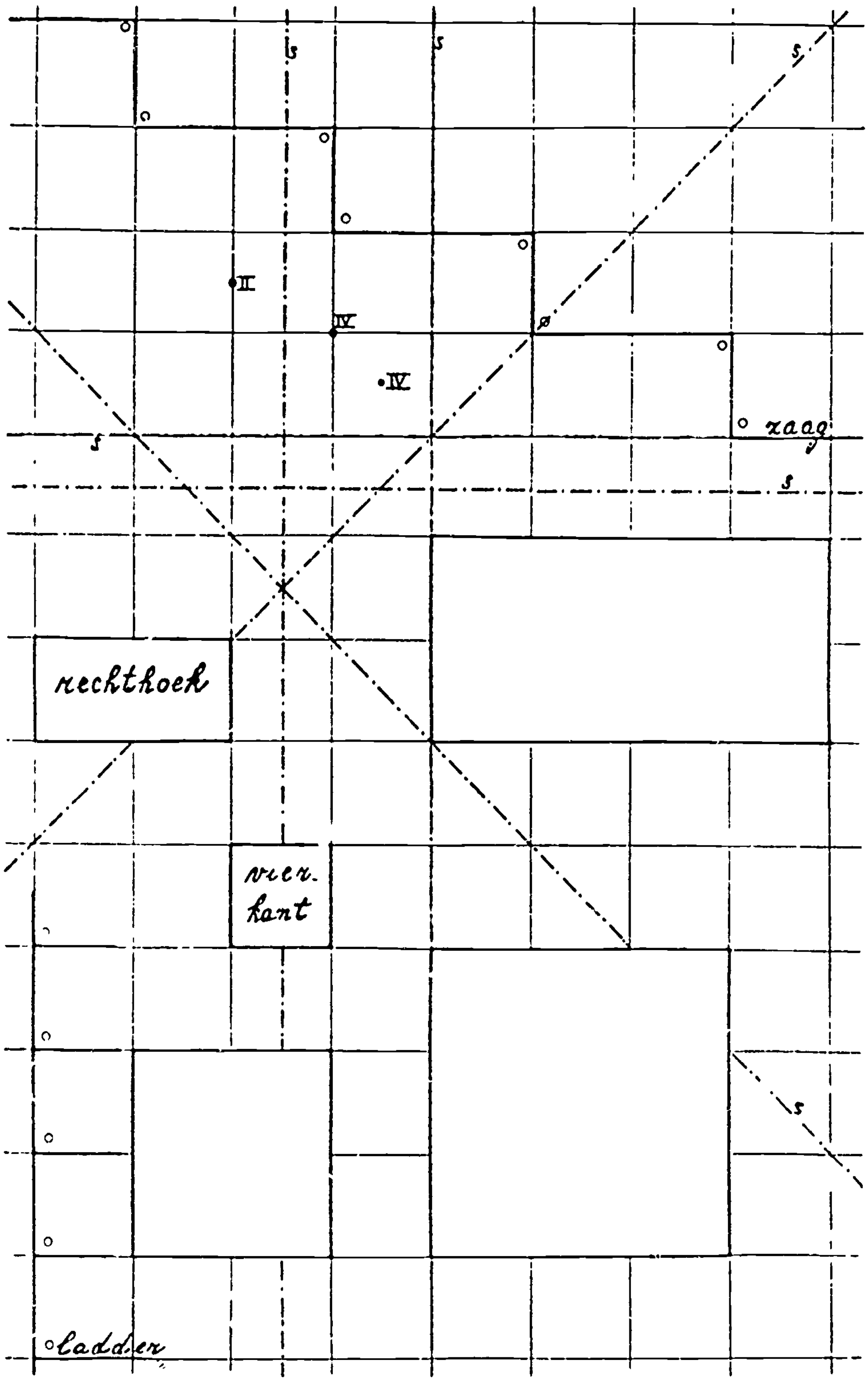


Fig. 1

Glossary:

rechthoek	means	rectangle
vierkant	means	square
zaag	means	saw

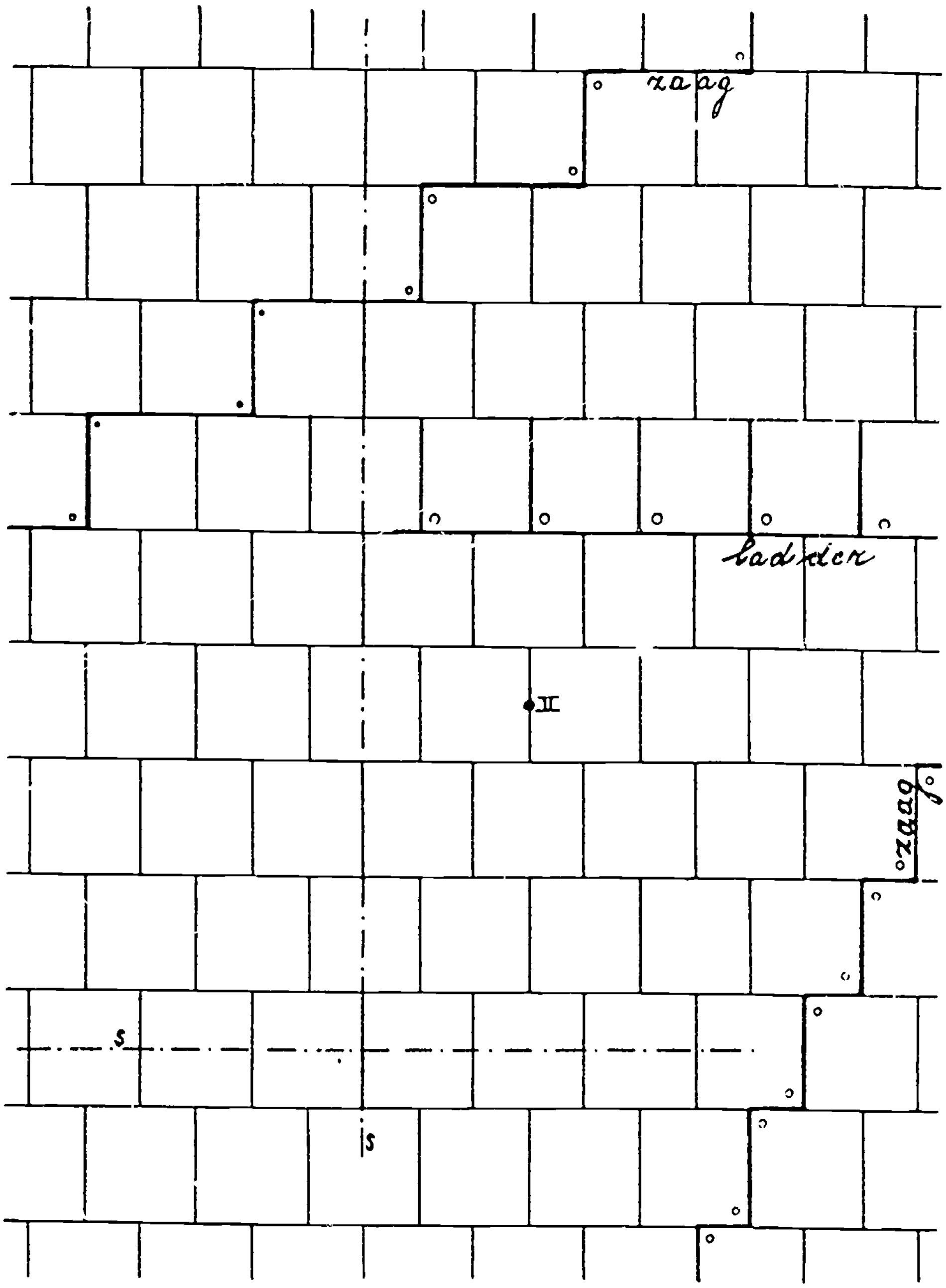


Fig. 2

Glossary: zaag means saw

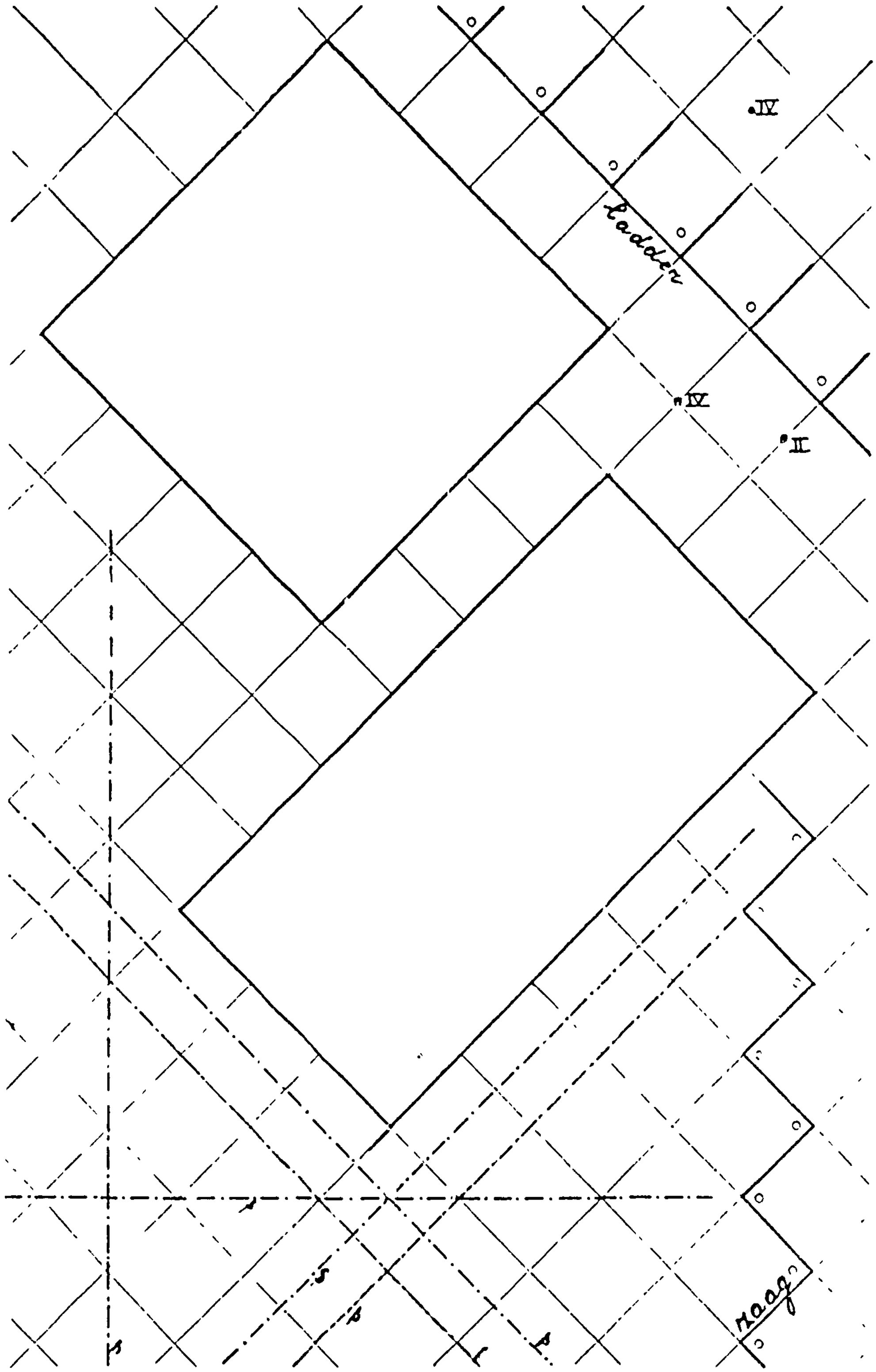


Fig. 3

Glossary: zaag means saw

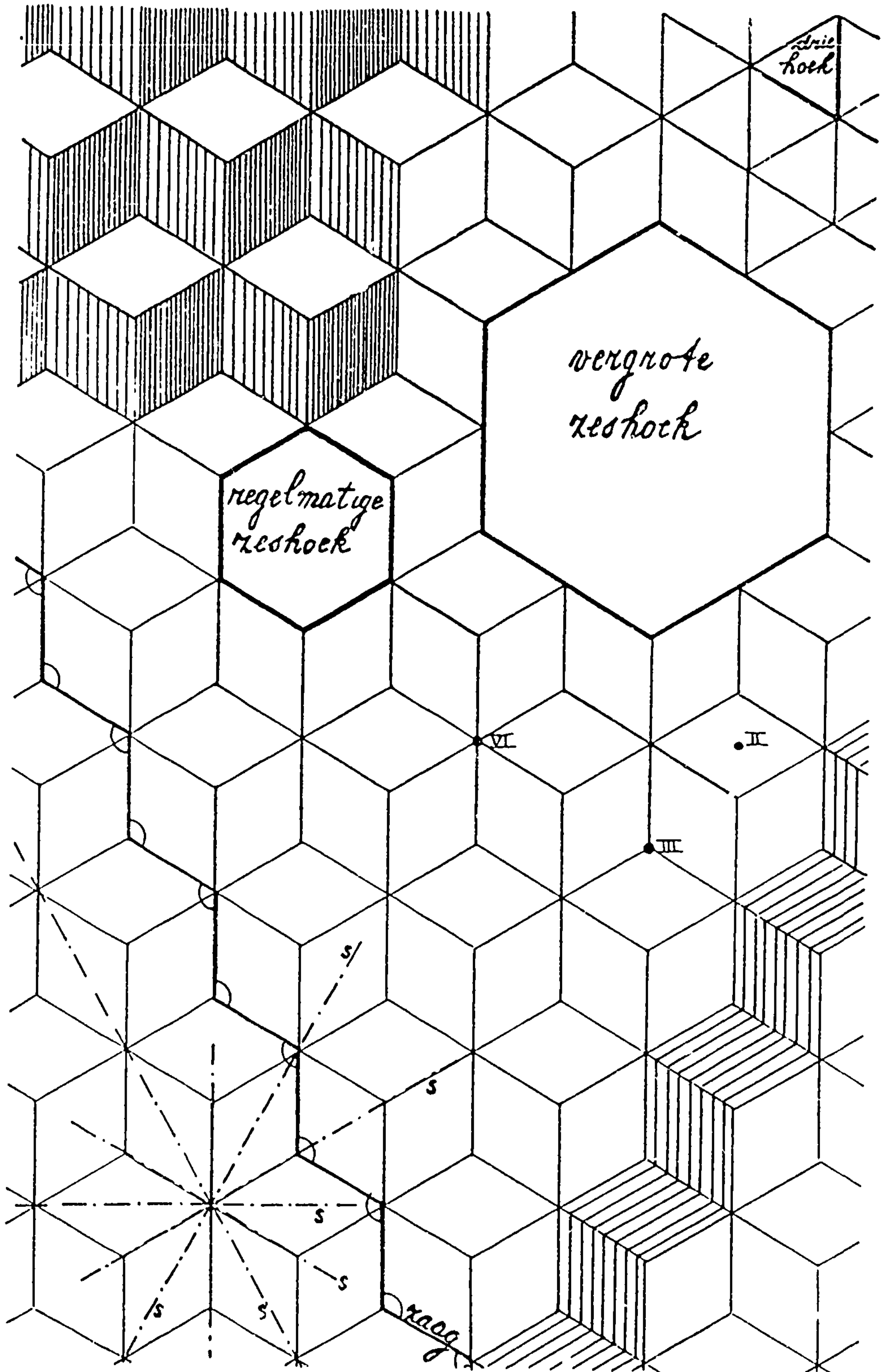


Fig. 4

Glossary:

driehoek	means	triangle
regelmatige	means	regular
zaag	means	saw
zeshoek	means	hexagon

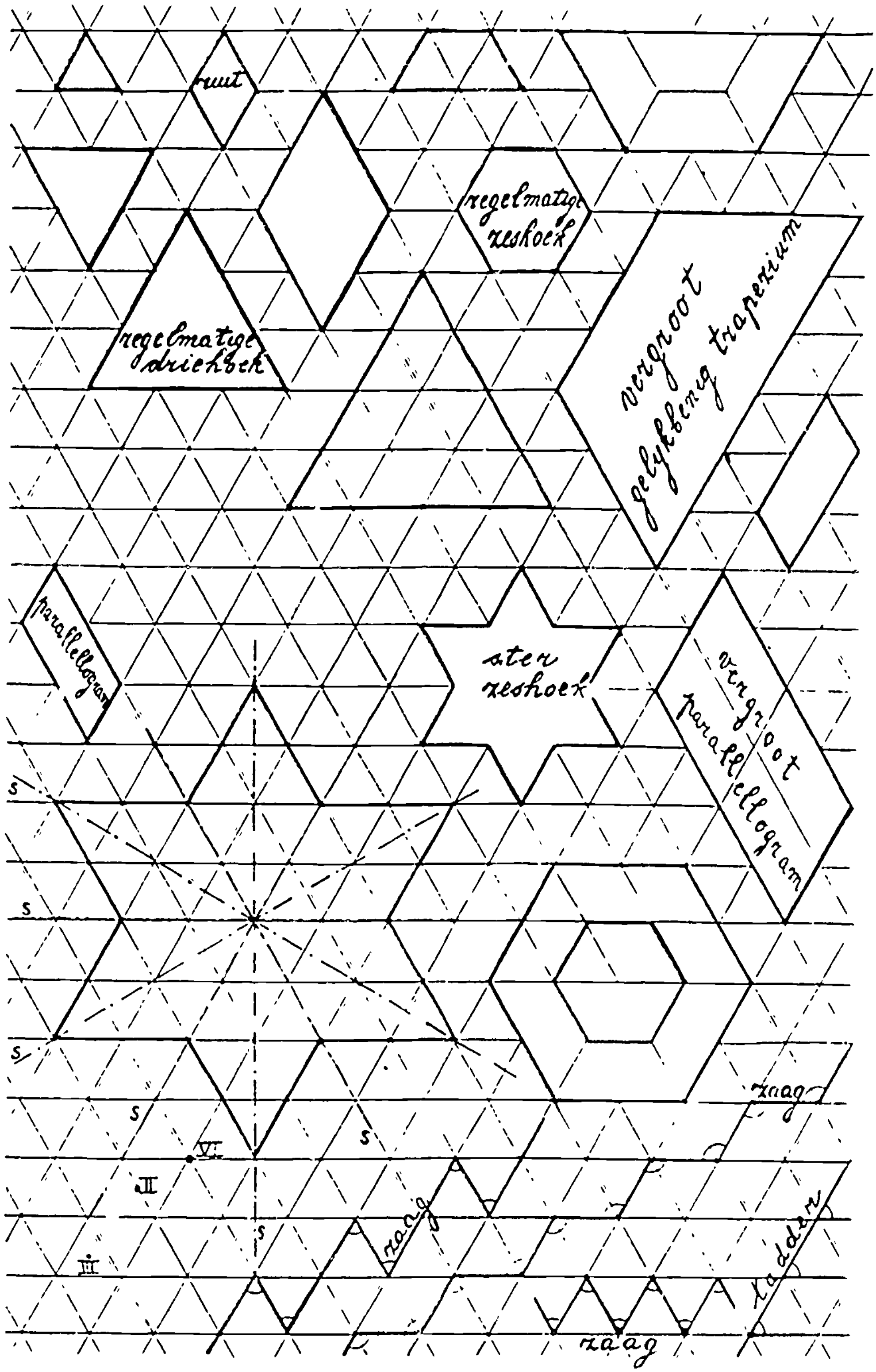


Fig. 5

Glossary:

driehoek	means	triangle
gelijkbenig	means	isosceles
trapezium	means	trapezoid
regelmatige	means	regular
ruit	means	rhombus
ster	means	star
vergroot	means	enlarged
zeshoek	means	hexagon
zaag	means	saw

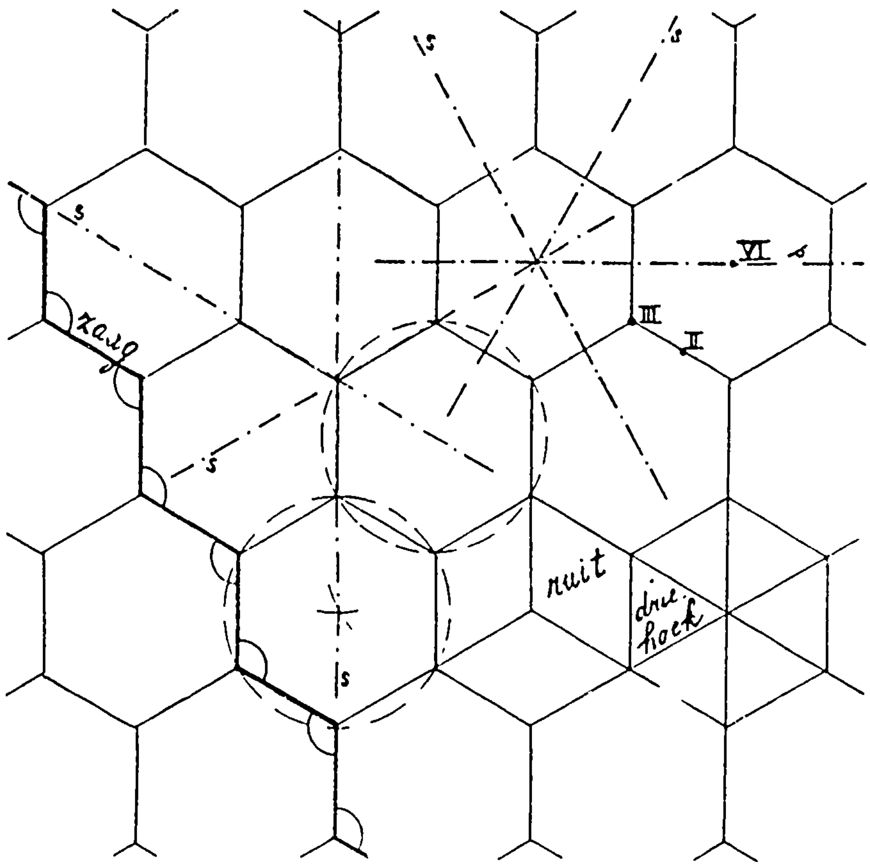


Fig. 6

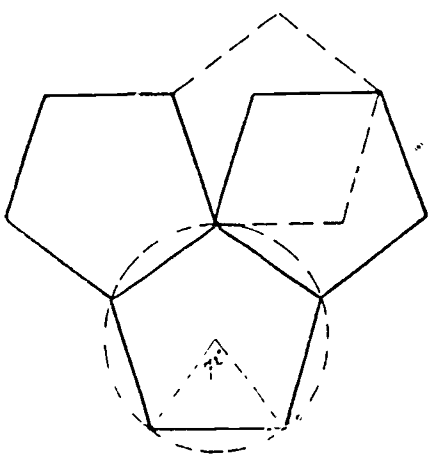


Fig. 7

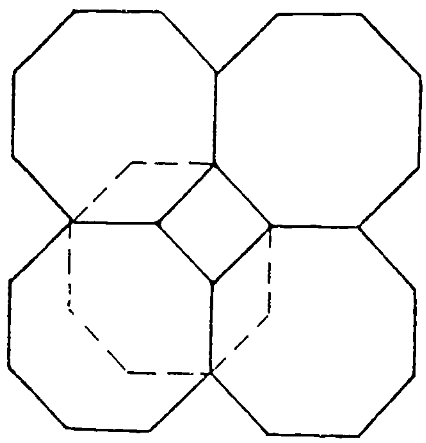


Fig. 8

Glossary:

driehoek	means	triangle
ruit	means	rhombus
zaag	means	saw

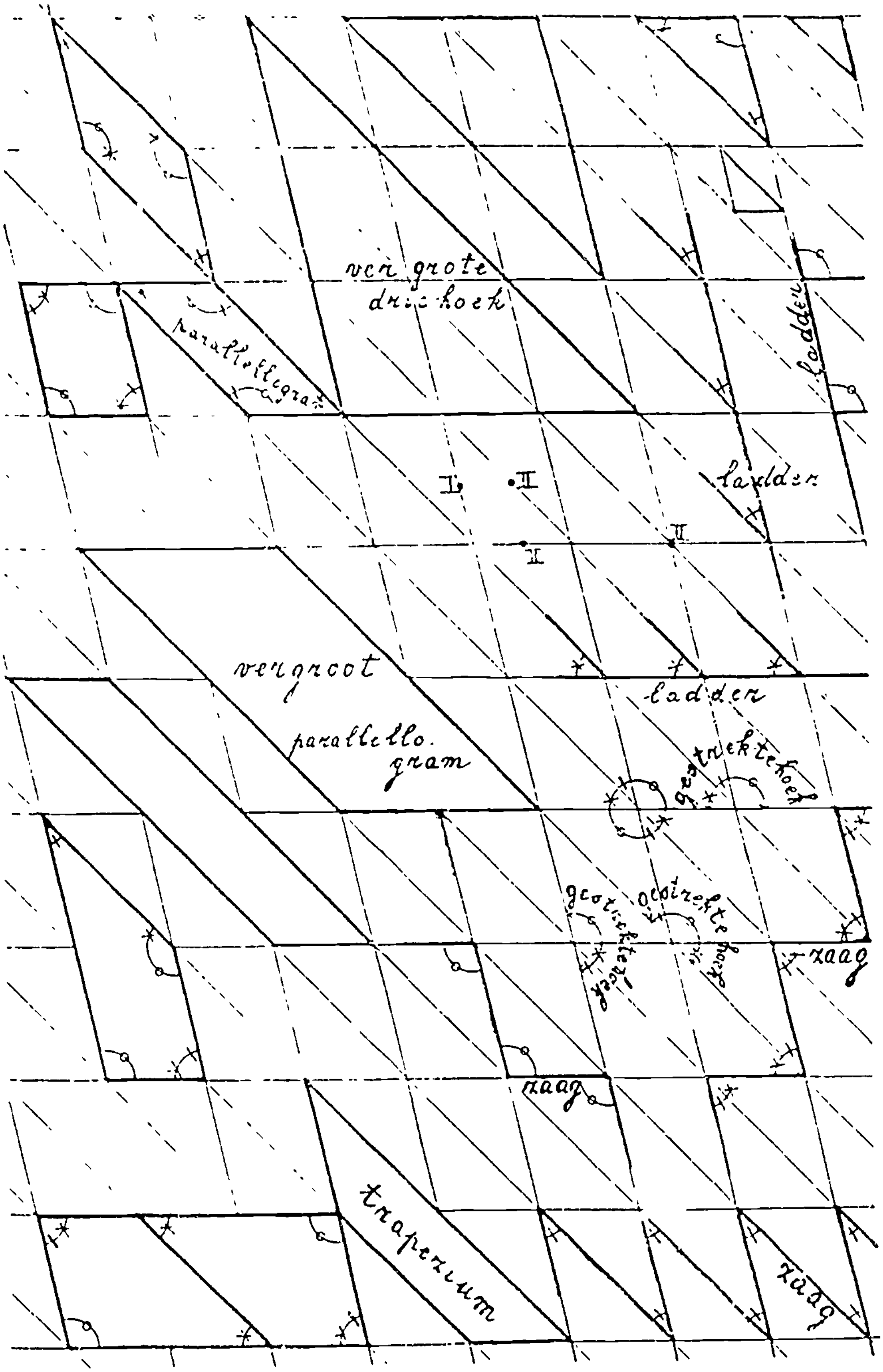


Fig. 9

Glossary:

driehoek	means	triangle
gestvektehoek	means	straight angle
trapezium	means	trapezoid
vergroot	means	enlarged
zaag	means	saw

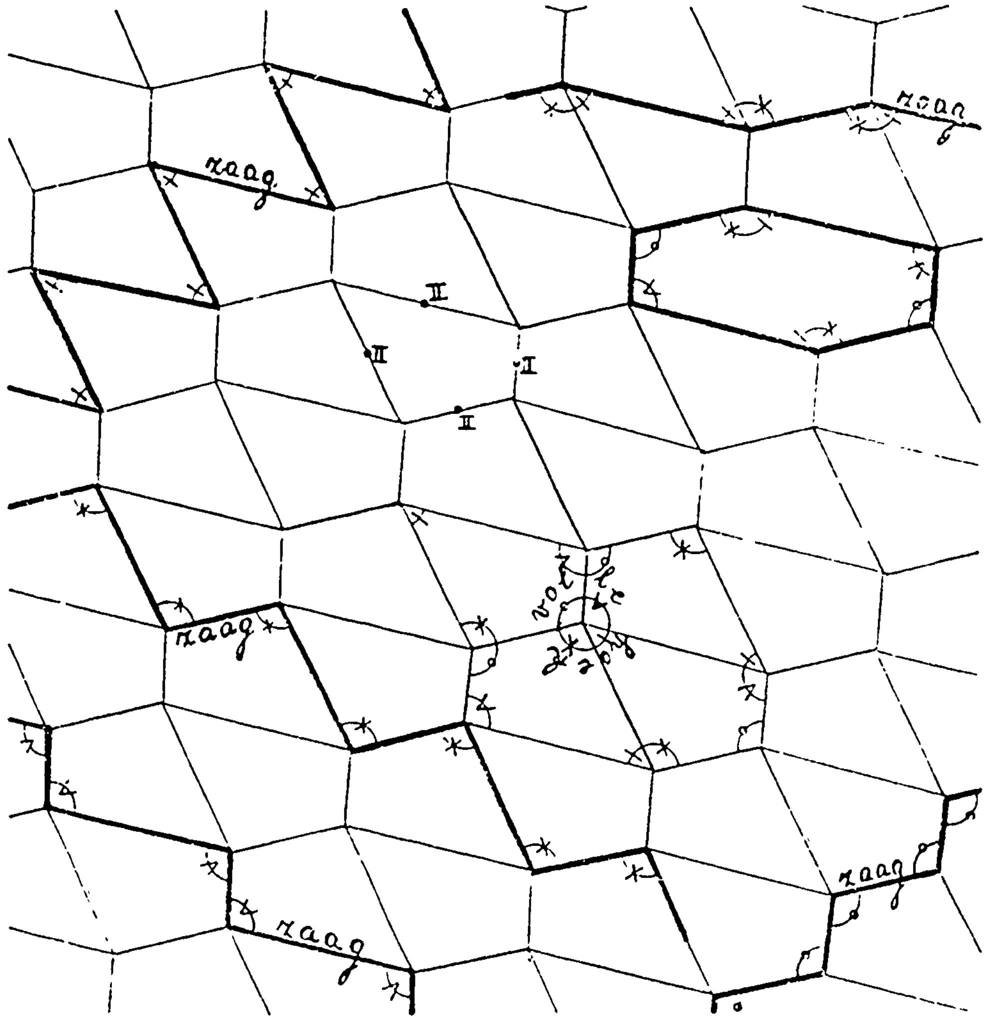


Fig. 10

Glossary:	volle hoek	means	round angle
	zaag	means	saw

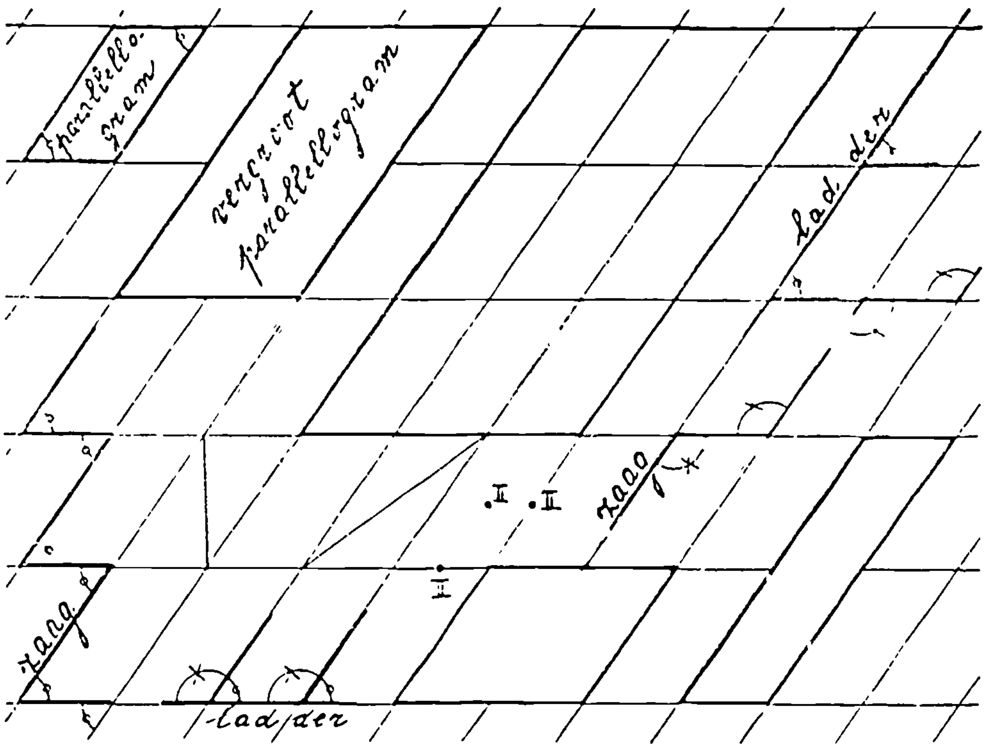


Fig. 11

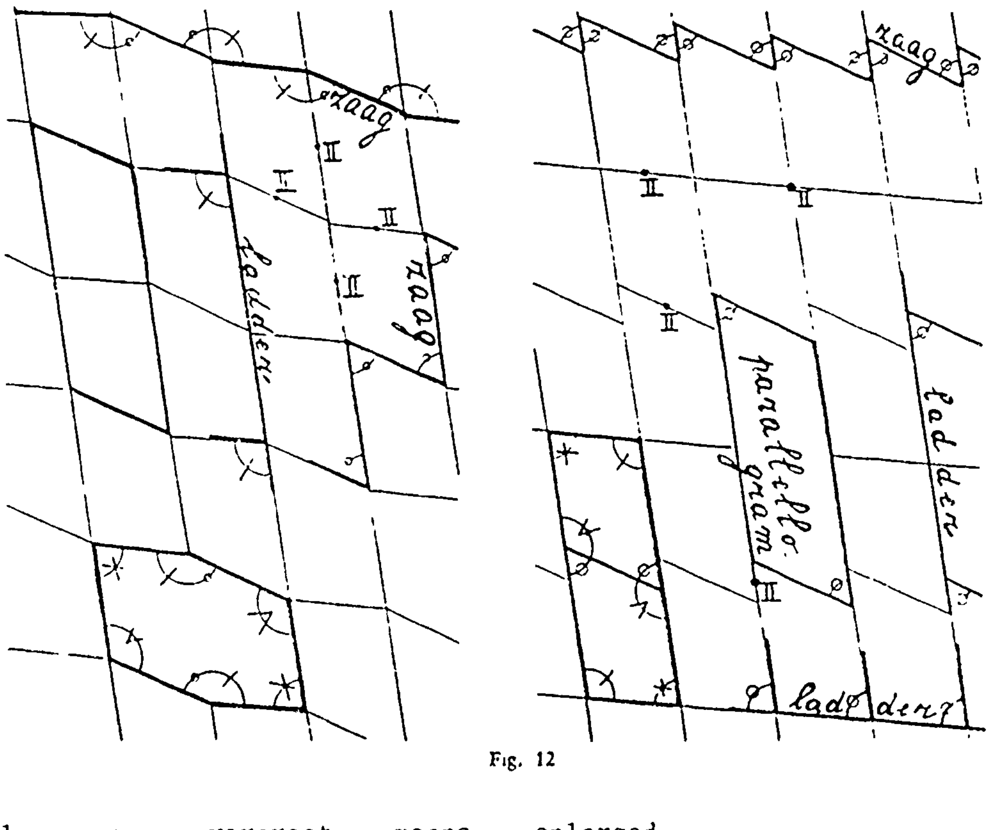


Fig. 12

Glossary:	vergroot	means	enlarged
	zaag	means	saw

Chapter XI

ANALYSIS OF THE PROTOCOL IN RELATION TO THE VISUAL GEOMETRIC STRUCTURING OF THE ALREADY GLOBALLY STRUCTURED FIELD OF PERCEPTION.

I first will give some comments on the didactics followed in this connection.

During the first trimester (September until Christmas vacation) the pupils have become acquainted with geometric solids and figures, i.e. they have observed and globally structured the figures and their components that were presented to them. The accompanying language structuring consisted of giving names to the objects. A geometric structuring had to be established over this global structuring. To achieve this I placed those figures, among other things, in the context of "symmetry". In doing so, one should make sure that this concept of symmetry, that is present in a globally structured way in the pupils, has been analyzed by them first (see IV, p. 30), so that they will be able to arrive at a geometric structuring of the field of perception.

Should one skip the analysis of the concept of symmetry, then one cannot expect that the pupils will rise above the already existing global structuring, because the context does not allow for an extension of the structure. The analysis of the figures examined for their symmetry brings to the foreground many characteristics of equality in those figures. The properties are thus found empirically and not by way of reasoning. The accompanying language structuring consists of verbalizing these characteristics in geometric terminology.

This makes it possible to establish a visual geometric structure for the objects.

By global structure I mean the structure that is present in the things around us, before an analysis in a geometrical sense has taken place. (See Van Hiele, p. 192). This global structure need not be the same for everyone.

Above this global structure, another structure, which I want to call the visual geometric structure, is established through the analysis of objects in a geometrical context on the basis of empirical truths.

Above this visual geometric structure, the more abstract structurings of thinking are established.

During the second trimester (January, February, March) the geometric objects remain visual. Figures, drawn as accurately as possible, then represent geometric objects for the pupils.

Above the visual geometric structuring, a structuring of thought has to be established by the pupils. This includes: arriving at properties of figures by way of reasoning.

My didactics were directed towards discovering and helping to develop those visual geometric structurings that lend themselves especially well to extension of the structuring of thought. This extension can take place when the visual structures are put in the correct context by the teacher. The correct context here, for example, is the logical connection of the properties.

The plane coverings appeared to me to be the most appropriate material for that. It was presented to the children under the title "Tiles". Van Albada had used this subject for his pupils at the Rotterdams Montessori Lyceum. It then involved no more than the drawing of tiled floors. Hence, a global structure was established. It was observed that only some tiles could be used to pave a floor, and others could not be used to pave a floor. I have expanded this subject for the pupils of the Utrechts Lyceum in order to promote the establishment of a visual geometric structure in the pupils (description see Chapter VI.)

I will now examine whether it is apparent from the protocol that the pupils have arrived at a visual geometric structure of the plane coverings and how far this reaches. As a starting point of the treatment, I have taken the global structuring of plane coverings insofar as this is present in the pupils on the basis of their acquaintance with sidewalks and tiled floors.

It appears from the first lessons that the groups arrive at a clear distinction between the concepts "equal" (equality of area), "congruent" (all properties equal except for the location) and "the same" (identically equal) through their analysis. The difficulties that certain pupils experience are to be found in the language structurings (see XIII, p. 174).

I have clear proof that the concept "congruent" was well understood by the assignments which the pupils completed on congruent triangles during the month of May. Many pupils wrote $\overline{AC} \cong \overline{BC}$ and angle A ≅ angle B, even though the equality sign was always used in the examples in the book. In this connection one should compare Tarski (I, p. 67). I had not experienced this before, but I also had not held an extended class discussion on this subject during previous years.

The analysis of parallelism of lines is necessary in order to obtain a geometrical structuring. Through class conversations, children learn what their collective knowledge about this is. In this way they arrive at an extension of the geometric structuring of the field of perception. The characteristic of equal distance between parallel lines appeared to be generally known. This appears from the answers: They are the same distance from each other (Pp. 15, X, p. 84); there are equal line segments between the lines (X, p. 89). A small number of pupils do not recognize the parallelism of the lines when they are oblique (X, p. 84). The remark of Pp. 29 (X, p. 88) that fig. 3 is the same as fig. 1 and the question: May I also rotate my notebook, then I have it (fig. 3) already?, are proof that the properties "same direction" and "equal distance" are experienced as belonging together. By drawing parallel lines using displacement of a triangle along a ruler, the geometric characteristic "the same slope with respect to a third line" is implicitly established.

This experience was also verbalized later. Pp. 2 (X, p. 116): The rungs each time hav the same slope with respect to the upright. Pp. (X, p. 115): The rungs form the same angle on the upper side with the upright.

Nobody named non-intersecting as a characteristic. This is also obvious because it would be an observation of something that is not there. In order to arrive at an analysis in this sense, one has to start from another structure that is no longer global.

It is exactly this characteristic that is being used in the customary build-up of the definition of parallelism. From a didactic point of view, this entails many difficulties, among which is the need for indirect proof in the initial geometry in order to show that when two parallel lines are intersected by a third line, the corresponding angles are equal. Similarly, the proof of the theorem: "Two lines are parallel if, when cut by a third line, corresponding angles are equal", using the concept of central symmetry, is based on geometric structurings that have not yet been established. Furthermore, these geometric structurings require a language structuring (XIV, p. 189) that has not yet been acquired by the pupils. It is for that reason that I introduce the parallelism of lines and the equality of corresponding angles and of alternate interior angles as Siamese twins: they are always present simultaneously.

The name "ladder" is used to evoke the relation between parallelism and equality of corresponding angles; the name "saw" to evoke the relation between parallelism and the equality of alternate interior angles. The teacher can try and see whether the pupils are able to separate the Siamese twins only when the pupils themselves have established logical orderings in a system of theorems and when they have experienced that the ordering is not fixed a priori and that it is thus possible to establish different orderings in a system of theorems. It is my experience that this structuring is established with great difficulty even in the second year of the secondary school - for this, it is necessary that the pupils be on the second level of thinking.

The geometric characteristics of parallelism in the visual field of perception were understood well. This was apparent from greater dexterity in handling drawing triangles and ruler when pupils were drawing figures 1, 2, 3 and 5 again, where general use was made of the equal distance concept in order to draw parallel lines faster.

The property of equality of angles became apparent by coloring equal angles, when the task was given that an angle could be colored with red only after a ladder or a saw had been found that already contained a colored angle (X, p. 117). In contrast with this, the task (X, p. 98): "color with red all angles in fig. 3 that are as large as the angle that is already colored with red" assumes only a global structure.

Since I could not predict that all children would possess a global structure of a plane covering with equilateral triangles, regular hexagons, ..., I gave each child little bags filled with regular triangles, pentagons, hexagons and octagons. Indeed, when drawing, it appeared that only a few pupils had at their disposal a sufficiently fine structure of the large equilateral triangle divided into small ones. This figure had been discussed during the algebra class, in connection with a sequence of odd numbers and a sequence of square numbers.

The context, which was an arithmetic one, namely "counting", has brought about associations with respect to convenient counting. This became apparent when the pupils were asked to determine the number of triangular tiles that were needed to pave a hexagonal marketplace (see X, p. 134): Pp. 36: "There are 100 in the large triangle, we have seen that in the algebra class." The answer of Pp. 19 stems from a formed structure; he divides the hexagon in two trapezoids and then conveniently counts: 21 + 23 + 25 +......+ 37 + 39 and finds 5 X 60. Pp. 9 also shows us that a geometric structure of the hexagon which leads to a very appropriate action. She divides the hexagon into three rhombi. Each rhombus contains 10 rows of 20 tiles. The above named pupils combine the arithmetic structure acquired earlier with the given geometric structure and refine the latter with the former.

In order to let the pupils again observe the global structure of the plane design with congruent rhombi, I used the rhombi from a mosaic box with which a stellated hexagon was first made (X, p.91). In the protocol one can read how, starting from the stellated hexagon, the pupils constructed a plane figure built with rhombi positioned in a regular fashion. Here and there, the plane covering threatened not to become regular, because not enough attention was paid to the requirement that six acute angles must always meet in one point. This meant that quite a few corrections had to be made.

Seeing the straight lines and the parallel lines in the figures 1, 2, 3, etc., is a consequence of the abstracting ability that each person possesses to a greater or lesser extent.

Some children spontaneously perceive regular hexagons in the tiled floor constructed from rhombi; others perceive zig-zag lines, cubes, a step, etc. (X, pp. 88-93). Others perceive figures only when their attention is drawn to them through words. Duncker call this a structuring on the basis of so-called "Einstellungen" (see examples in Chapter IX, p. 65).

Using the terminology of the rules for the didactics I have given in Chapter IX, abstracting can be defined as follows: Abstracting is the recognition of structures or partial structures of a more complex structure. The questions: "Do we see something when we compare figures 4 and 6?"; "Can we also derive fig. 5 from fig. 4 or from fig. 6?" are to be considered as practice for abstracting and restructuring. The children clearly expressed an active interest here.

For those who have difficulty with abstracting, one can use as didactic help: "Erase a few of the lines". Conversely, when finer structuring is involved, one can say: "Add a few lines." In this way new structures evolved out of fig. 9 (fig. 11 and 12, p. 151).

The organizing principles emerged through my repeatedly asking the question, "What do you see in the figure?" The zig-zag lines (p. 111) were the starting point for the structuring of the geometrical figures. Saws and ladders were looked for and drawn in the figures (p. 113). The remark by Pp. 38 (p. 98): "There are still larger hexagons in it", led to structuring the figures geometrically in the sense that enlargements were looked for (p.125). The saws and the ladders were also structured geometrically. This appears in the protocol (p. 115), because the pupils were able to name the characteristics.

That enlargement also obtained a geometric structure appeared first of all in the large variety of construcutions that were made when an isosceles trapezoid had to be enlarged two-fold (p. 138). Second, it appeared while constructing non-similar parallelograms, etc. (p. 139), where "the inequality of the angles" was used as the characteristic. Third, it was apparent during the computing of the multiplication factor of circles and line segments (p. 140). Fourth, it appeared during the search for figures that are always similar (p. 141).

The notion of equality of area precedes the finding of area because the notion of equality of area is precisely necessary for measurement of area. The pupils appear to be able to decide on the equality of area independent of size. The equality of the areas of three non-congruent parallelograms was accepted on the basis of divisions into mutually congruent parts (pp. 103-108). Pp. 8 even remarkded that one certainly can draw a trapezoid and a parallelogram whose areas are equal (p. 108).

Then follows the comparison of areas (p. 126). It appears not to be redundant to repeat the division into congruent parts, because Pp. 14 (p. 128) thought that when the side of the square is enlarged three-fold, that area is enlarged six-fold. This mistake was not made in the other group. This favorable result can be attributed to the fact that the same mistake had been made earlier in that group. Several pupils thought in a previous class conversation that a parallelogram that is enlarged three-fold contains 6 small parallelograms (p. 105).

By choosing a unit of area one finally arrives at the concept of size of a surface. Paving a floor or a marketplace with tiles provided the experience needed to show that the counting of the number of square tiles that cover a marketplace is the same as determining the number of square meters in the area of that marketplace; this means determining the size and the area of the marketplace (p. 135).

By analyzing tiled floors in the arithmetic context "counting", structures develop that can be called visual geometric, as soon as it is established that the determination of the area of a figure is the same as counting the number of squares (the chosen unit) with which the figure can be covered.

Only after this new visual geometric structure has been formed can one proceed to a higher structure where the area of the figure is linked to a formula.

I had the impression that Pp. 36 and Pp. 39 (p. 84), and probably a few more pupils, linked the formula "length X width" to the word area, without a formed geometric structure being present. Their answers were based on associations that had developed in elementary school. Van Hiele (p. 202) exresses this as follows: "The pupils, starting from the analyzed structure of the teacher, have formed out of that, a global structure for themselves. This global structure has little or nothing to do with real objects. Therefore, the pupils are not able to make use of their global structure in a concrete stituation."

Finally, the movements that can be made to cover a tiled floor (translations, flipping over, rotations) were looked at.

The translations did not entail any difficulties. One can read on page 130 and following how the orientation of a line that ws present in the figure as a side was first found as the direction for a translation. This line ran upwards in an oblique direction. Only afterwards was the horizontal direction found, first to the right and then to the left. The vertical direction (up) was observed even later, probably because this direction was not present as a side in the figure. Later other totally different oblique directions upwards, and finally also downwards, were found. Similarly, when moving the tiles in fig. 9 in order to let the grooves disappear, it appears that the translation is easily seen.

Flip-overs and rotations are easily perceived when the tile is a regular polygon. It is less easy when the tile is not regular. After the inaccuracy of the statement of Pp. 9: "When I take a mirror, I can see that angle in the mirror (for fig. 9)", was demonstrated with the help of a mirror, a number of pupils wanted to demonstrate the equality of the angles by folding.

Pp. 8 felt that the equality could be shown with the help of two flip-overs (p. 101).

It was also hard for them to see that the angles can cover each other by rotating around the vertex a half turn. Many rotated their notebook as I had showed them in order to see it better: hence, their visual representatio of symmetry was still insufficient.

A test, the questions of which can be found on p. 201, was given to the pupils at the end of the second trimester. The purpose of the test was to determine to what extent the pupils had been able to form associations of visual geometric structures.

The results of the 38 pupils (Pp. 30 had left the country in the meantime) are given in the table on the next page.

The first questio is divided in three parts; the fifth question in two parts. Problem VIb was added by the pupils during the test. This question was as follows: Draw in the figure VI (= VIa) the axes of symmetry and the centers of symmetry.

It was accepted that part (a) of I was correct when in column (a) the names of the figure found by the pupils were correct and if there were not too few of them. Later I discussed the test individually with each student. This was necessary because questions I, III and IV had not been posed in such great detail that one could make judgments on the basis of the written work only. Indeed, even though we tried very hard to present the problems clearly, we had no guarantee that they really were. Similarly, the rotations that were used in the third and the fourth columns, appeared to present difficulties, even though they had been explained verbally with an examp

Part (b) of I was deemed correct when the enlargements had been seen: VIa was only considered correct when it was clear that use had been made of the characteristics of enlarging when they were drawing. The answers to this sequence of questions indicate for whom enlarging has become a visual geometric structure. A plus sign in the table indicates that the figures were correct. A vertical line in the table indicates that a small mistake had been made that could be immediately corrected during the individual conversation or that the ruler had been used more than the compasses. A minus sign in the table indicates that I was not convinced that the associations had been properly formed. For 5 pupils out of 38 I did not know with certainty whether they could make use of visual geometric structures in the context of similarity.

In problem Ia a vertical line was used when the only existing language structure could not be used without mistakes. For instance, Pp. 9, 33 and 39 used the word 'square' for the rhombus whose side was 4 cm. In addition, Pp. 39 used the word 'round' for circle. However, on inquiry, the mistake appeared not to reside in the visual geometric structure. That same rhombus was classified as a parallelogram by many pupils. This was evidently caused by the rhombus being positioned on one of its sides. For that same reason, an isosceles triangle that is positioned on one of its legs is very often perceived as a scalene triangle. On inquiry, this mistake was immediately corrected. I have indicated this in table Ic by a pl s sign iollowed by a dot. More than half of the pupils did not immediat ly perceive the parallelogram as a rhombus.

Pp. 11 and Pp. 13 appeared to have difficulties with the notation. The bad result of Pp. 27 can be explained by her having missed 20 minutes of each class hour. She appears not to be on the first level of thinking.

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Results of the .est

Many pupils reacted as fo..ows when they learned what mistakes they had made in II: "How I have been caught!" The vertical lines in the list of the results indicated that they corrected themselves clearly upon inquiry. The criteria for such correction were that the pupils not only had to tell that the angles were not equal in the hexagon colored with red, but also had to tell how they could see that those angles could not be equal. Many characteristics were given by them. For example, "it cannot be circumscribed by a circle," which had to be further explained. Or, "there are not six equilateral triangles," which also had to be demonstrated. One pupil pointed towards the lines connecting the center of the opposite sides of the hexagons and remarked that in one case the line was perpendicular on the sides, and that in both other cases it was not. A pupil rarely thought of looking more closely at the size of the angles. Only when I asked them to compare the size of the angles of the hexagon was it observed that the inequality of these angles is based on the fact that the upper angle of the composing isosceles triangles is smaller than a base angle. For 5 pupils I was not certain whether a good structure had been formed.

It is remarkable that the 9 pupils who had found on their own the rhombus with side of 4 cm of Ic, also correctly answered the question concerning the hexagon. In total, 18 pupils out of the 38 found on their own that the figure did not contain regular hexagons. This larger number can be explained by the more detailed questioning which directed attention towards that which had to be perceived.

Question III was generally answered correctly. A few students followed their own notation. Many students found, upon inquiry, many more figures of equal area than they had written down. During this conversation many pupils discovered that one-fold enlargements of figures that have the same area produce new figures that again have the same area. Only Pp. 20 gave as an answer to question IV: "No, there are no open figures." He could not explain well what his thoughts were. He appeared to be able to draw saws and ladders.

The questions Va, Vb and VIb deal with symmetry. They were not allowed to use a mirror to determine the axes. Of the 38 pupils, 8 could determine the axes of symmetry and the centers of symmetry without any mistakes. They found them for the plane coverings as well as for the simple figures. Eighteen pupils were <u>not</u> able to determine the axes and the centers without any mistakes by simply looking at the figures. They needed a mirror. In order to see that the isosceles triangle does not have any center of symmetry, they had to make use of a sheet of transparent paper on which the same plane covering had been drawn. During the test, I also noticed that a few pupils stealthily used the drawing triangles as mirroring surfaces.

When we look at the columns of the table, we can draw conclusions for the performances of each child individually. This however is not the objective of the experiment.

The horizontal rows given a decisive answer to the question whether the hypothesis of the didactics we followed is correct as far as the first part is concerned: "Can visual geometric structures be established in 12 year-old pupils The answer is very clear from the table: <u>The pupils can form these structures</u>

for themselves, when the organizing principle is: enlargement, parallelism or regularity. For the organizing principle of symmetry, the following restriction has to be made: during the first year one can make use of symmetry as an organizing principle only when one works with scissors and glue, i.e. when one starts from the special empirical action of folding, rotating and mirroring of figures. Here one thus can see my above-mentioned maxim confirmed: "One should let the children act thoughtfully with manageable material as a help." The ability for visual representation in twelve year-olds is generally insufficient for the observation of symmetries. They identify straight and oblique symmetry. They have not separated these notions from each other into abst ct thinking. This can be explained as follows. The children have experienced during the first trimester that an axis (vertical) of symmetry divides the figures into two congruent parts through a one-to-one correspondence of the points (see observation lesson in IV, p. 30). Since they have not yet reached the second level of thinking, and hence are generally not able to discern whether or not a relation is symmetrical, they have the tendency to identify a line that divides the figure into two congruent parts as an axis of symmetry. Indeed, mistakes were made if the axes (oblique) of symmetry divided the figure irto two congruent parts. Here again we have an example of "Siamese twins."

Chapter XII

ANALYSIS OF THE PROTOCOL IN RELATION TO THE STRUCTURING OF THOUGHT

My field of investigation encompasses both experiences the pupils gain as a group with the help of the material presented to them and individual and class conversations as described in Chapter X. Statements about the group in general do not imply that these are applicable to all pupils. Here also the well-known rule applies: the whole contains more than the sum of its parts.

From the protocol it is apparent that the group is on the first level of thinking in plane geometry. Corresponding with what was mentioned above, this implies that a conversation on that level of thinking is possible with the group. It does not mean that all pupils should be on that level. For those pupils in whom insight is on the verge of materializing, these class conversations can be very illuminating; similarly, for the pupils that are not yet on the level.

The levels of thinking have been discussed by Van Hiele (I). The first level of thinking only implies that the pupil sees that geometric figures are characterized by certain properties, that he is able to apply known properties in an operational way to a known figure. This does not imply that the pupil has already arrived at a classification of geometric figures that are known to him. The formation of associations is not yet complete: the properties do not yet represent factual knowledge. The teacher should not yet expect that the pupil knows exactly which properties belong to a given figure and which figure is determined by certain given properties. The attainment of the level only means that pupils see that a geometric object possesses certain determining properties and so are able to search for the characteristics. A classification made by the pupils is to be considered by the teacher as proof that the subject matter has been assimilated, that associations have been formed, that the subject matter can be handled independently. That a conversation on the first level of thinking is possible is apparent from the protocol on p. 89 where the pupils are reasoning that a rhombus and a rectangle are not regular polygons, and on p. 105 from the argument given why the figure obtained cannot be a rhombus nor a rectangle. It is also apparent from pages 110 and 113, where the pupils construct the new figures parallelogram and trapezoid using their properties. It is also apparent from the responses to the questions on how one can recognize a ladder and saw. At the same time the concept of parallelism seems to be well understood there.

On p. 101 one can read in the protocol how the pupils tackle the following task: "I know that the angles of a triangle, when juxtaposed, together form a straight angle; I assert that I can predict that the angles of a quadrangle form a round angle when I fit them together such that they touch each other. Does anybody think he or she also can predict that?"

The protocol clearly indicates that the pupils first look outside the logical context for a satisfying explanation that will reflect on what grounds they can accept the correctness for themselves. Pp. 30 starts with a regular quadrangle and thus first takes a special case in which the relation is recognized. Pp. 16 (p. 101) also thinks of a square - she has already juxtaposed the four angles, because she starts with a circle with four quadrants. Pp. 3 apparently does not understand what Pp. 16 means, because he asks whether a circle can circumscribe a quadrangle. The special case is subsequently generalized: with another quadrangle one obtains a different division of the round angle, at one place a little is removed, at another a little is added; the pupils reason on the grounds of reasonableness. The working method described above: the generalization of a principle that has been ascertained for a particular case should not be rejected in itself. For in physics, many important laws have been established in this way. The pupils are told by the teacher that this method is not legitimate in mathematics; a proof should follow.

Thus my conclusion is that the pupils do not make use of premises that are available to them (see Ehrenfest II). This result should not surprise us. For reasoning of this kind, insight on the second level of thinking is required; a new relation has to be found with the help of a known geometric relation.

From the protocol it also appears that both groups did arrive at a logical reasoning stage. It shows especially in the class conversation on p. 101 how this was possible, how the pupils finally reached the goals with each others' constructive help. Pp. 3 first saw the special case: When the two triangles, obtained by drawing the diagonal in a quadrangle, are congruent (he talks about the two <u>halves</u>), then it follows from the theorem - the sum of the angles of a triangle is 180 degrees, that the sum of the angles of a quadrangle is 360 degrees. Because he divided the quadrangle into two triangles, Pp. 5 was able to give the general proof.

Because geometric insight was involved here, I did not want to draw the attention of the pupils to the language they used: One triangle is a straight angle. I only drew attention to the angles of the triangles by indicating them with chalk. I had mentioned that we were talking about the angles of the triangles and that each three-some forms a straight angle when the angles are put together. The teacher should separate the language and the thought expressed by means of that language. Here the important thing was the idea and this was approved because it was understood by fellow pupils. That the pupils were receptive to a logical train of thought appears from the answer to my question of who had expressed it most clearly. In one group Pp. 38 was pointed to without exception and in the other group, Pp. 5 aided by Pp. 3.

That all pupils participated intensively is apparent from the fact that all methods of solution were immediately transferable. The group we now able to demonstrate directly in an exact way that a floor cannot generally be tiled with congruent pentagons and hexagons. Also, the group was able to answer the question why it can be done with regular hexagons. Pp. 2 is farthest along in the analysis of the problem, for he remarks that it is possible to tile a floor with those pentagons and hexagons for which three angles together form a round angle (X, p. 112).

I gave the question about the quadrangle with the re-entrant angle (p. 108) in order to see if a logical train of thought had already become possible. Of the 38 pupils who participated, 16 immediately gave a correct description. Of the other pupils, only three arrived at a correction of their train of thought after seeing the proof. The remaining pupils gave a report in which they attributed their lack of success to an insufficiency in the graphic structuring.

Here are a few of their statements:

Pp. 4: You can divide the figure into two triangles, so it is possible.

Pp. 6: It is not possible because the sides are not equal. The opposite angles are not equal either.

Correction: It is possible because one can divide this figure into 2 triangles and the angles together form a round angle.

Pp. 7: Yes, because one can divide it into two triangles. Each triangle has three vertices that together form a straight angle. It has two straight angles - together a round angle.

Pp. 8: Yes, it is possible, one can divide it into two triangles that together possess angles that form 360 degrees. One point, where they meet, is a round angle.

Pp. 10: Yes, it is possible because all those figures fit in each other, with sides, with angles.

Pp. 11: When one puts the cardboard model down, one asks oneself whether two points would fit in that recess (indentation) but that does not work and also no other points can fit in it because something is left or it is too small.

Correction: When one puts a cardboard model down and right next to it one with the same side against it, three other points can fit in the recess. When you go on like that it fits well.

Pp. 13: It is not possible. Because the figure can be divided into three triangles each of 180 degrees, they then do not form a round angle because a round angle is 360 degrees and this is 3 X 180 degrees = 540 degrees.

Correction: It is possible. Four different angles merge in one point. For the angles together form one round angle.

Pp. 14: It is not possible because they do not fit in each other.

Correction: It is possible after all, but I had thought at first that two had to fit in each other.

Pp. 16: Yes, I think that you can make a tiled floor here because first of all you can divide it into two triangles. The figure really looks a little like a half step and if there is one half, another half has to fit in it, and in that one still another and so on. Therefore all those figures together have to form a floor.

Pp. 19: One cannot make a figure with that triangle because when one places 2 points of the other triangles in the tail, there remains a hole.

Correction: One can make a figure with that triangle by putting the small triangles around and around and then one can place another row on top of it.

Pp. 22: I think it <u>cannot</u> be done, because two equal salient angles fit in the re-entrant angle of the figure and one of those salient angles is opposite the re-entrant angle. In this way a circle is formed, but the center remains open. Because the two sides that come out of the salient angle are not equal, one cannot fill the middle of the circle.

Correction: I have now seen that it <u>is</u> possible after all. If one puts three different angles in the re-entrant angle, namely the largest angle of the figure, the smallest angle of another figure and the middle angle of yet another figure then it fits precisely. If one places against the longest side of the first figure the longest side of another one and one proceeds in that fashion, one can cover the whole floor.

Pp. 25: It is possible I think because it is a quadrangle and the four angles of a quadrangle together form a round angle when they meet in one point; but now all of a sudden I see that one angle a re-entrant angle in the quadrangle and therefore I do not know it anymore. I think it is not possible because of that re-entrant angle.

Pp. 26: Those little figures fit in each other, because when you divide that little figure in the middle you get two triangles and one triangle contains 180 degrees and two triangles 360 degrees thus it could become a round angle and there remains no hole it it then.

Pp. 29: You cannot cover a floor with this figure, because the acute angle does not fit in the obtuse angle. The angles do not fit into each other.

Correction: It is possible with this figure, when you place the three acute angles in the obtuse angle, it fits precisely so you can go on like that.

Pp. 30: In order to pave a floor with tiles, those tiles not only should be congruent, but they also have to fit in each other. I believe that you cannot have a floor with these tiles because they do not fit in each other. The upper angle is too small for the lower angle and two of those upper angles together are too large. I could make one strip though.

Correction: One can tile a floor with those tiles and this by means of a strip. One has to make that strip twice and then put strip 2 against strip 1.

Pp. 39: It is not possible because if now the points were equal and the points were as large as the recess, then it would be possible, then the sides would also be equal.

Correction: It is possible if one first puts them next to each other, one with the recess upwards and next to it one with the point upwards, then one places the small point in the recess, then next to that one, one with the point in the recess and next to that one, one puts one with the long point in the recess, etc. This way it forms a closed entity.

Let me first investigate what the possibilities are of arriving at a solution. I have chosen this particular quadrangle because most certainly there is no global structure of it present in the pupils. This assumption is confirmed by the reports of the children: there is only one exception. Only Pp. 10 succeeds in forming for herself a global structure of this plane covering and to establish a visual geometric structure. For she observes that it fits with the sides and with the angles. She told me later also, that she had pictured to herself the plane covering with the cardboard models. Because I give a quadrangle, the global structure of which is not present in the pupils and which cannot be found easily outside empirical situations, the other possibility of arriving at a solution is given a better chance.

This other solution is based on an extension, in an abstract sense, of the already existent visual geometric structure of a plane covering with convex quadrangles. The context allows for this extension of structure, because there has been abstract reasoning in a previous class conversation, during which logical reasoning took place in arriving at the conclusion that the angles of a quadrangle together form a round angle. This was derived from the fact that the angles of a triangle form a straight angle. The quadrangle chosen for that was convex.

A number of pupils appeared to be able to establish an abstract structure of thought. They noticed that the quadrangle can be divided into two triangles. Some gave an additional comment that hence the angles of the quadrangle form a round angle. The analysis of Pp. 25 attests to the insight that a theorem that has been proven for the case of a convex quadrangle cannot simply be transferred to the case of a quadrangle with a re-entrant angle. However, he did not arrive at a proof that the theorem also holds for that kind of quadrangle.

Pp. 13 got stranded because he held the following viewpoint: Because the quadrangle can be divided into three triangles, the sum of the angles of the quadrangle is 3 X 180 degrees. He did not find the mistake in this reasoning, neither did he later. This led me to come back to that topic in a class conversation. There the paradox was quickly solved (X, p. 112).

From the report of Pp. 16, it is apparent how great the need of the children is to have an explanation based on global structure, besides abstract reasoning. The idea of a step could be clearly found back in her drawing. She thus had looked for a global structure upon which she could base a visual geometric structure.

It did not occur to the other pupils to look for a solution according to a way other than the visual method. They did not go beyond the global structure that empiricism provided them.

There were two pupils who were unable to make a plane covering with the cardboard models in the time that was given to them. Those were Pp. 15 and Pp. 18. The latter had been absent frequently.

I conclude from the reports <u>that the pupils are able to set up an abstract structure of thinking, but that they prefer an explanation based on a visual goemetric structure</u>.

If one presents the problem to others, for example, mathematics teachers, they generally do not solve the problem without visual geometric structuring. They also start from a global structure, and establish a visual geometric structure next, but they then much more rapidly set up the logical structure above that. They have, as it were, obtained an association for a structuring through, i.e. from the visual geometric structure to the abstract structure.

The above-mentioned preference of the children is also apparent from their way of tackling the following problem; "How many re-entrant angles can a pentagon have at most?" (X, p. 115). This problem can also be solved in two different ways.

One possibility is that one tries to arrive at the solution through drawing; here one forms a geometric structure through "trial and error."

The other possibility is that one establishes a logical structuring above an existing geometric structure - "relations among the angles of a polygon." The latter method was chosen by only 2 of the 36 pupils.

Pp. 21: I think two angles, because a pentagon is 540 degrees. A re-entrant angle is more than 180 degree. Therefore there cannot be three re-entrant angles, because 3 X 180 degrees = 540 degrees, but the re-entrant angle was more than 180 degrees.

Pp. 38 expresses himself with much greater difficulty and doubted his own reasoning, because when he did not succeed in drawing the figure he changed the number 2 back to 1. He wrote:

> This pentagon can at most have (2) 1 re-entrant because it was said that a re-entrant angle is more than 180 degrees. Because the angles together are 540 degrees.

("Said" here referred to earlier lessons, because I had not given any hints. I did not want to focus attention on the number 180.)

Sixteen pupils gave the correct answer by means of visual geometric structuring. For the other nineteen this structuring is not fine enough to arrive at a good solution.

According to the main idea of the didactics I am proposing, it is necessary that the pupils get a global structure of a system of theorems.

In the protocol one can find (X, p. 110 ff) how I made use of a well-known structure, the genealogical tree, to introduce that idea. We are dealing here with an isomorphism that is only based on a metaphor (Van Hiele, p. 195). The resemblance is only global. The objects with which one is now working are the geometry theorems (relations) themselves. These objects came into being during the visual geometric structuring. They are the laws (relations) that have been supplied through empiricism and that have been translated into geometric language. It is the special empiricism (folding, rotating, etc.) that was used for the visual geometric structuring of the geometric objects.

On the basis of experience in teaching I came to the following conclusions in VII, p. 47:

1. The children generally do not know <u>what</u> they are building up.

2. The children do not know what they are building <u>with</u>.

We can now understand these conclusions in the context of the didactics. For two kinds of objects are being studied without discrimination, and one does not stress that enough, i.e. in what is given and in what had to be proved. The pupil thinks that the figure illustrating what is given and what has to be proved is the geometric object. During the proof, however, he should become aware that proven theorems are his objects. As soon as the theorem concerning the geometric object (= figure) has been proved, a new object of another building structure comes into being. Prior to that, this new building block was not allowed to be viewed as a block.

By treating the subject matter in this way, the pupils cannot come to a satisfactory structure formation. The essential element is <u>not</u> whether or not the first page should start with the axioms, whether or not to start with an introductory course, whether or not to work with notebooks, but is the <u>way</u> of working. The way of working, whatever method one follows, has to be such that the children can come to the formation of new structures through analysis of structures that are known to them and that have been placed in a geometric context by the teacher.

During the first trimester I did this by starting from the already globally structured field of perception. Because the children participate actively in the analyses of the geometric objects (the figure and the models) in a geometric context (symmetry), the globally structured field of perception is structured by them in a geometric sense. The structures formed in this way I have called visual geometric structures.

<u>The pupil who has visual geometric structures at his disposal is on the first level of thinking</u>. In him, associations have developed for the geometric characteristics of the figures. He knows, on the basis of empiricism, the relations among the elements of a geometric figure. This is necessary in order that the pupils can understand what one is building with in geometry.

The pupils now still have to experience how it is being built. The structure of the building-up of the theorems is isomorphic with the structure of an imaginary web in which the knots represent the geometric relations. These knots are connected in a certain way and in certain directions. These connections and directions are determined by the organizing principles. The threds of the web are the logical relations. In each knot many arrows meet and many leave from there. There are points out of which arrows only leave. These particular points represent the axioms.

No object in the field of perception possesses this structure. The genealogical tree is a structure that is known to the children and that can be modified most easily into the desired structure. That this modification is possible, appears from the answers in the protocol on p. 110. For it was accepted without further explanation that a relation can follow from a single other relation. It was also accepted (p. 121) that one has a certain amount of freedom in the choice of the relations from which a certain relation can follow. That the resemblance between the structures only consists of the global was clear for both groups of pupils.

The pupils have understood how it is being built. This appears from the protocol on p. 122, where they readily established the connection between two genealogical trees.

I was surprised to see how quickly the pupils produced the proof of the theorem: "The sum of the angles of a triangle is 180 degrees" (X. p. 120 ff). In fact they did not look for auxiliary lines; these were present automatically. This is the result of having the visual geometric structure of fig. 9 at one's disposal. The given triangle is lifted out of (abstracted from) this figure with the necessary auxiliary lines. This was most clearly demonstrated by Pp. 11; she placed fig. 9 in front of her and told how the figure of the given triangle has to be extended (p. 121).

The formation of the new structure "the ordering of the relations" involved great enthusiasm.

In order to check whether the drawing of the correct auxiliary line(s) is indeed related to having a visual geometric structure at one's disposal, I assigned the following difficult problem after ten days: "Show that two angles are equal when their legs are parallel and have the same orientation." One can read on p. 132 how I introduced that problem.

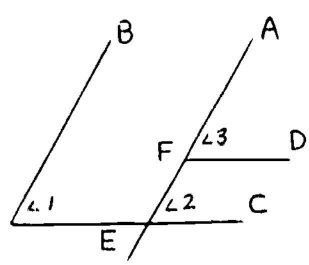
Of the 36 pupils who were given this problem, 3 just drew the given figure, 3 pupils drew auxiliary lines that could not have led them to the objective and they got stuck, 4 pupils tried to work with saws and ladders but did not establish a clear ordering in the figure nor in the accompanying reasoning.

One pupil did not draw auxiliary lines, but wrote: "These angles are equal because when one displaces a line it will be exactly as straight and as oblique as the first one, on the paper. This holds thus for both the lines." After that, she concluded the angles were equal. So she worked with a translation - a visual geometric structure.

Still four other pupils worked with a visual geometric structure, but they did draw extensions. Two of them worked with a translation and two with a rotation. The latter mentioned that opposite angles in a parallelogram are equal. Their work did not show a clear ordering.

Twenty-one pupils set up a more-or-less clear logical ordering. Since the lowest number of links that is necessary is two, I labelled each solution that consists of two links as adequate. Of those, there were 13; three among those did not have a completely correct accompanying description. Here are some of the solutions.

Pp. 1 draws the connecting line between the vertices and uses two ladders to arrive at the solution.



Four pupils extend one leg of one

of the angles. The description of Pp. 5 is given here as an example. It shows a good language structure.

> "The angles are equal because when you extend line A it forms a ladder with line B and line C. Angle 1 is found again at E and angle 2 is found again at F. Hence, angle 1 and angle 3 are equal. C is a ladder with A and B. A is a ladder with C and D. The lower or upper angles of a ladder are always equal. Angle 1 and angle 2 are both lower angles of ladder A-B-C. Angle 2 and angle 3 are both upper angles of A-C-D. Angle 2 is therefore equal to angle 1 and angle 3. Angle 1 and angle 3 are therefore equal.
>
> Ladder
>
> ↓↓
>
> Two angles whose legs are parallel are equal."

Four pupils extend both legs, but use only one.

Pp. 26 wrote:

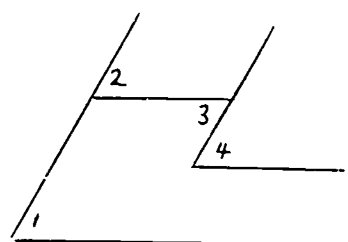
> "When you extend the lines then you get a ladder for each angle and a line from one angle forms a rung of the ladder of the other angle, therefore those angles are equal. Ladder has corresponding angles."

Here only the most necessary was written down which shows that she has a good thinking structure at her disposal.

Four pupils drew an auxiliary line that could still be called appropriate when one thinks that a saw has many equal angles.

Pp. 22 wrote:

> " Saw ↓ Ladder ↓
>
> Angles whose legs are parallel are equal.



> If one wishes to show that these angles are equal, one should make use of one saw and one ladder, because in a

ladder the corresponding angles are equal, and in a saw all angles are equal. One first should go up one rung on the ladder (2) and with the saw two angles towards the angle where we have to arrive at (4) (see drawing)."

This answer also points to a good language structure. The saw is clearly seen as one figure.

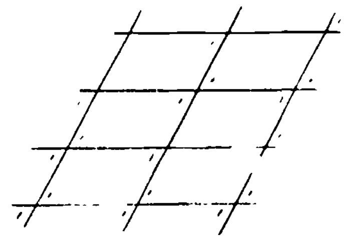
The remaining 8 pupils drew a different number of auxiliary lines, one, two and as much as a complete plane covering with parallelograms. Five of them used no more than three links and represented their way of thinking better in the figure than in their words. Pp. 17 drew the following figure.

The angles 1 were all colored with red. Underneath was written:

"The ladder and the saw

↓ ↓

In a parallelogram the opposite angles are equal; hence also the adjoining angles."



The reference to adjoining angles probably meant the opposite angles.

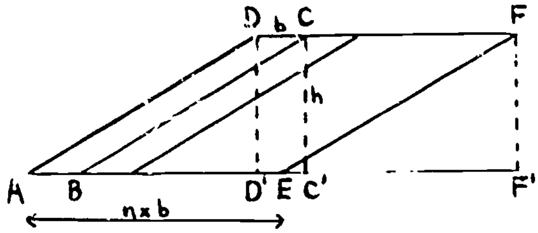
This test cannot be considered as an examination in its usual meaning. With such a test one wants to check on which students have finished the learning process. This test was purposely given during the period of structure formation, i.e. at a moment when one could expect that the learning process was finished in none of the pupils.

Before one administers tests in the normal instructional process, the teacher first consciously proceeds to the formation of associations. This has been omitted here completely. Therefore, the result of this work does not reveal a picture of the abilities of the pupils: the three pupils who handed in blank sheets, had beautiful grades on their reports, as well as for mathematics.

From the result of the test, it is apparent that having at one's disposal visual geometric structures facilitates the transition to logical structures of thinking and that the finding of auxiliary lines should not present difficulties herewith.

The visual geometric structure of figure 11 led me to finding the solution to the problem presented by De Groot (I):

Prove that the area of a parallelogram is equal to the base times the altitude for the case that the projection of the raised side on the base is greater then the base.



The proof rests on the fact that one has to define a number n such that n x b is greater then the projection of the raised side on the base. For the rest, the figure speaks for itself:

Area ABCD $= \frac{1}{n}$ area AEFD $= \frac{1}{n}$ area D'F'FD = area D'C'CD.

Chapter XIII

THE LEARNING OF TECHNICAL LANGUAGE. ANALYSIS OF THE TEACHING PROCESS (PROTOCOL) IN RELATION TO LANGUAGE STRUCTURING.

The learning of a new subject is always accompanied by the learning of technical language.

The language problems which the children experience in geometry can be compared with those I underwent during the study of my didactics. I thereby arrived at the following working method in order to study language problems during the learning process of geometry:

First: a global analysis of my own learning process by means of introspective self-observation with the objective of arriving at organizing principles.

I will first describe the way in which the theoretical foundations of the didactics I follow have been found. Here attention had to be focused on two formations: concept and language of the subject "didactics." The formation of language structures that belong to the subject follows the formation of concepts. The formation of concepts has taken place previously.

Chapter I of this study is an orientation: What is a didactic experiment? The mental correlate (see Mannoury, p. 137) of this word - the meaning I attached to this word for myself - did not concur with the meaning others had attached to this word. I thought that a didactic experiment was set up in order to investigate the phenomenon that is called didactics, in this case the didactics of geometry. Instead I found as examples of such experiments: learning conversations that had the objective of improving learning performances. In addition, some appeared o think that didactics should be derived from psychology.

Langeveld makes a statement about the relationship between pedagogy and psychology that casts doubts on the above-mentioned relationship between didactics and psychology. This statement seems to be in accord with my experience and it made me assume that the mental correlate of the expression "didactic experiment" does not yet contain a clearly defined common element among teachers and didacticians. It is therefore incorrect to consider the expression "didactic experiment' as a technical term.

In order to approach the phenomenon, I investigated in chapter II how didactics appears to others and I compared this with the outcome of my own impressions. After seeing a protocol from Stellwag, it became clear to me that there are big differences in views on the didactics of mathematics. It further appeared to me that there are still two words for which there exists no clear common mental correlate, namely "elementary" and "self-active". To me, geometry is elementary in the secondary school if it starts from the global structures that have reference to the world around us. Pupils are self-active when they actually participate in the analysis of the structures.This can take place, for example, by their "acting thoughtfully" or by their participating in a collective discussion.

Upon further consideration of the work of mathematics teachers at experimental schools, I came to the following conclusion: one cannot and should not expect publications from experimental schools about the didactics to be followed. The work of the pioneers is still practically at the stage of exploration (See Koning I).

In Chapters III and IV, I consider my own didactics in relation to the subject matter and the learning situations of the first trimester. At the beginning of this work that was completely new to me, I started from a global structure that I already possessed. The language I use in these chapters should not be labeled as "technical language;" it belongs to the linguistic style I had at my disposal at that moment.

In this global analysis of didactics, I thus arrive at the following conclusion as far as language is concerned: there are a number of words for which the mental correlate differs strongly among teachers and this is apparently caused by their starting from their own linguistic style. Up to now insufficient coordination has taken place through mutual exchange of ideas. The above-mentioned words are technical terms: "didactic", "elementary", "self-active." If one wants these words to function effectively in the technical language, then didacticians of the discipline should aim for less diversity of the mental correlate among those who use the words.

Through the study I subsequently made, in order to show that the learning situations I created conform to the conditions the teacher poses for each educational process, I arrived at an extension of my linguistic style. This primarily consisted of delineating more precisely the mental correlate of the technical terms I had learned from the practice. Use was made of this language structure in Chapter V in which the learning of geometry was analyzed according to the principles Langeveld has established for the teaching process. The subject didactics thereby took on shape. Through study and testing, I acquired for myself a clearer mental correlate of technical terms such as: structure, context, intention, autonomy, association, plateau, level of thinking, habit formation. Associations were also acquired for the technical terms.

In Chapter VIII I pointed out that teachers, in schools where more attenton is being paid to individual instruction, naturally gain many more experiences than those in schools where instruction is given according to the traditional method. In the former schools the pupils often have the opportunity to ask their teacher for clarifications during one or more hours per day. The practical action of the teacher is then directed towards the individual learning process, with the incidental objective of helping the pupil move forward. The thinking of the teacher during this intuitive action is executive-practical (see Langeveld II, p. 315). Thus he acquires global structures of the didactics, much in the same way as parents form these for themselves to a greater or lesser extent when they help their children with their homework.

The example Langeveld gives of this form of thinking, "I become skillful at handling the hammer by using it" can perhaps be used as a metaphor for the above-mentioned situation.

If we compare the case mentioned above with the case where instruction takes place in a classroom situation where fewer questions are asked by the pupils, the form of thinking of the teacher will be more directive-practical. For the teacher has now prepared himself for teaching his lesson. Langeveld here makes a further distinction between practico-practical and theoretico-practical. In the first alternative, the action is the object of study: "how do I handle the hammer?". The teacher has then reached the first level of thinking: "the aspect of the didactics" and is at the stage where he says, "I do it this way and I do it that way and it works very well."

The compulsory use of the executive-practical form of thinking during individual class hours increases in value when the teacher afterwards chooses the action as the object of his thinking. One can ask oneself: "How do I handle the hammer?" but also: "How do I want to handle the hammer?" Langeveld calls the latter theoretico-practical; it can be oriented towards the action. The didactic experiments of Mooy, Boermeester and Bunt[1] are all at this level. Hence the teacher does not dwell on learning itself, but on the result of teaching. This result is measured at the end of the learning process. The didactics has a means-goal relationship with respect to the learnability of intellectual performances and thus stems from an approach of the natural sciences (see Langeveld V, p.231).

If one wants to penetrate into the essence of didactics, one should choose as the object of thinking not only the action of the teacher but also the teaching situation. Here the subject is co-determining the instructional process and its result.

Theoretico-practical thinking is then directed towards the action as well as towards the thinking: "What should I know in my thinking when I ask myself how I want to handle the hammer?" It is exactly this orientation towards the thinking that provides the opportunity for analysis through which a higher level can be attained.

The opinion that the problem set "Tiles" can contribute to the formation of visual geometric structures in the pupils and that this formation is desirable, thus stems from the practice of teaching, the latter by means of the theoretico-practical form of thinking.

In order to be able to arrive at insight into the didactics one follows, a more thorough analysis of the nature of the didactics followed is needed. This is necessary because I was not able to analyze the protocol. I did not see clearly enough the organizing principles along which I could analyze the didactics, and language failed me completely. Therefore it was necessary to approach the didactics more closely (see Chapters VII and VIII). I was successful at this by making comparisons between several didactic approaches. This enabled me to define my own didactics better.

Thinking is now directed more explicitly (see Langveld II, p. 315) towards finding out how the didactics that is being followed really works. It is now also possible to arrive at a theory of didactics and its foundations.

From the analysis it appears that I have objections to a certain didactic approach if: (1) it advocates a building up of geometry by way of the elements, (2) it stresses the supplying of methods of solution too much, and therefore (3) it promotes too strongly the learning (memorizing) of lessons. In contrast with this I propose that my didactic approach: (1) promotes a building up of geometry by way of structure, (2) first directs the thinking activity of the pupils to the analysis of structure prior to the formation of associations and (3) concurrent with that, provides an opportunity for the pupil to develop thinking focused on structuring.

Starting from the fact that the important element in geometry is to develop logical thinking, my further analysis in chapter IX brought me to the works of the cognitive psychologists, Selz and Duncker. Cognitive psychologists analyze thinking but they do not analyze the gradual development of the thinking. The dissertation of De Groot (II) is an experimental investigation that analyzes the thinking of the chess player but it does not study the process by which chess is learned If one wants to analyze learning, one has to focus on learning processes and learning situations. However the disciplines are closely related by the nature of things. Findings from the psychology of learning can inspire the cognitive psychologist while he is setting up his theory. Conversely, the results obtained by the cognitive psychologist, through analysis of his thinking processes, can be of help to the educational psychologist in his explorations. There is thus a certain interaction between the two disciplines, similar to the interaction we know between mathematics, physics and chemistry. One discipline is not an auxiliary discipline of the other, nor does one follow from the other.

The learning processes that study the learning to think may not be viewed as thinking processes, nor should the teaching processes be confused with the learning processes. During a learning situation in which the teacher places his pupils, not only the children but also the teacher find themselves in a learning process. The learning process of the teacher forms a part of the teaching process. This learning process the teacher goes through belongs to the study of the didactician. Initially, when the didactics has yet to be analyzed, the teacher himself is this didactician and the only way to obtain data is through self-observation.

Since the teacher has learned the subject matter during his training, but has not learned how to teach this subject matter, the above-mentioned learning process starts when he enters school. One can then rightly say that the children are guinea pigs This statement also holds for the pupils of experimental schools. The pupils are no longer guinea pigs when the teacher has analyzed the didactics he uses and when this didactics has acquired a theoretical basis.

If one views the teaching process not only as a process during which transfer of culture takes place, but also as an educational process, it is necessary to bring the theoretical foundations into agreement with the requirements of pedagogy. The teacher's instructional method should not be an imitation of instructional methods of former teachers. The increase in scientific insight into the disciplines of pedagogy and psychology llows for a much better approach to teaching problems than was possible 30 years ago. In addition, the change in the "notion of education" requires a totally different relationship between teacher and pupil.

In order to delve into the foundations, an analysis of the didactics is necessary first of all. This entails:

1) setting up hypotheses on which the didactics is based;

2) analyzing the teaching process in order to test these hypotheses;

3) setting up a theory of the didactics on the basis of the laws thus obtained.

I hope to have sufficiently shown with this that the didactics does not simply follow from pedagogy, nor from the psychology of learning or child psychology and that the didactics of mathematics certainly does not follow from cognitive psychology.

The mathematics didactician investigates what the possibilities are for a teacher to teach mathematics, i.e. how he can bring the pupil to exact thinking. He cannot investigate this by taking snapshots before and after the learning process. Such an action is based on an over-simplification of the problem to be studied. In reality, the psychology of thought, psychology of learning, pedagogy, general didactics, child psychology and the discipline of mathematics itself all meet in the domain of the didactics of mathematics. The mathematics didactician should know all these disciplines and in addition he should know semantics.

In my teaching I started from the customary objective: the learning of exact thinking. According to Gestalt psychology, thinking has already begun in conscious perception. Honigswald (I,

p. 206) points out:

> The "sense" of the perception is the Gestalt; and the designa.ion of the sense of the Gestalt is determined by the perceptions. Gestalten are "produced" objects.

Therefore the situations into which I brought the children were focused on the objective: learning to know and to understand space. From these learning situations, one can proceed to the learning situations which are adapted to the learning of exact thinking. As far as the didactics of the subject matter is concerned, there is a question whether one can create such learning situations that fit with the objective of mathematics teaching by using a method that is founded o: association theory.

If one takes the viewpoint of child-centered instruction, 'he goal: learning to know and to understand space in a geometric context is then not an intermediate, but a temporary goal in the sense indicated by Langeveld (IV, p. 63):

> The temporary goals are rest points along the way to the general and ultimate goals... They are given in stages along the developmental path of the child.

After studying the work of Selz it appeared possible to me to define the organizing principles of my didactics. The language to express this properly was still absent however. I have acquired this technical language by reworking the language structures Selz and Duncker used with the help of the word "structure." Therefore the following rules for the didactics evolved: The teacher should aim at the following:

1) that his pupils will build up the global structures in a geometric sense;

2) that they will recognize these structures as component structures of higher ones;

3) that they will extend these structures, in so far as the context allows it;

4) that they will learn to recognize corresponding elements in isomorphic structures.

With the help of this language, I established the ordering principles of my didactics. The latter thereby became a discipline. By using this language in Chapters XI and XII, I acquired the <u>structure</u> of a technical language.

The course of events sketched above clearly shows that first a formation of concepts took place in myself and that only afterwards could I proceed to the formation of a technical language. We see demonstrated here what Brouwer (I) calls "the primate of the idea above the language".

We can now draw a parallel for the formation of the language structures in the child during geometry instruction. I present the analysis of this formation in the following scheme: At the start of the instruction each child possesses his own linguistic style.

In the course of the teaching process the teacher will:

1) try to find the common mental correlate of words and concepts which are present in the child and which he will need;

2) clarify and complement this mental correlate so that it coincides with the meaning of the word in the discipline;

3) expand the language style in so far as necessary.

When in the course of the learning process, the aspect of geometry has become clear, i.e., when the pupils have acquired associations for the visual geometric structures, the teacher proceeds to the pre aration of abstract geometric structures (structures of thinking). During the structuring period that now follows in the course of the learning process, the children use their linguistic style and not the technical language. The establishment of the concomitant language structure takes place only after concepts have been formed.

Prins and Van Gelder (I) distinguish logical-structuring moments and receptive-structuring moments in the learning process. These two moments can be clearly found in the analysis of the learning process I went through myself. The first-named moments occur during the concept formation I sketched; the second moments take place just prior to the formation of the language structure. One has to keep in mind however that both moments can be further divided. Logical-structuring consists of intentional analyzing, immediately followed by structures of thinking that are formed independently. Receptive-structuring consists of intentionally taking on a receptive attitude, whereby associations are formed independently (for example, language) and as a result of that, structures are formed independently.

Children make frequent use of other means for understanding than language. The spoken language is complemented by looks and body movements. During the discussions, many head, shoulder and hand movements are added to express approval or disapproval, doubt or certainty. The children have yet to experience that language is the means of supporting the analysis - between the speaker and the listeners - and that for this, an orderly sequence of spoken or written sentences is necessary. Many pupils in the first class are not yet able to make use of language in that way. This is especially apparent from the reports (Chapter XII, p. 164 ff) in which the pupils attempt to write down the thoughts they have while they are solving a problem. The purpose of the discussions is to listen to each other and to speak to each other. In this way a more correct use of language is acquired along with more precise thinking.

During the first class conversation, the mental correlate of the word "congruent" is explored and clarified. Next, the meaning of that word as a technical term is more precisely established. It appears in the protocol how we arrive at: Congruent means "indistinguishable with respect to shape and measurements" (Pp 2., p. 83 and Pp. 30, p. 85). If we wish to distinguish congruent objects, we apply distinctive signs. So one uses different teaspoons with congruent teacups (Pp. 11, p. 82). For the same reason we affix a number to congruent chairs (Pp. 10, p. 82 and Pp. 30, p. 86). At the same time an action is linked to congruence: they fit on each other (Pp. 17, p. 81 and Pp. 3, p. 90). The concept of congruence has to be detached from area and volume: figures with the same area need not be congruent (Pp. 8, p. 82). In order to observe congruence, the children look at the shape and the measurements of objects and figures. They experience that all characteristics have to correspond, except for the location.

During the second class conversation, the concept of parallelism had to be linked to equality of direction. Observing that the grooves in the tiled floor (fig. 2) are parallel does not prove that this connection has been established. The direction in which one should ride a bicycle on a tiled path should not coincide with the direction of the grooves (Pp. 5, p. 83). In this fashion attention is drawn towards the word direction. Direction can be linked with movement. The pupils rotate fig. 2 in order to obtain fig. 3: the parallel grooves then take on another direction. It appears from the class discussions that Pp. 3 (p. 84) cannot yet adequately distinguish between the notions parallel, horizontal and vertical. The displacement of the ruler along the grooves of tiled floor no. 3 was necessary in order to clarify that in a global geometric structure the concept parallel is linked to equality of direction (Pps., p 84).

The word edge appeared to possess several meanings. Since this seems to involve a mental correlate (Mannoury I, p. 38), the difficulty could be solved immediately by pointing out to the children that mathematics has two other words at its disposal, two technical terms, "side" and "vertex."

I also want to analyze here to what extent the children have a need for defining. We see this need especially in the process of exploring a concept.

Pp. 32 (p. 94) thus wants certainty about "regular bi-angles" that had been talked about, in passing, a few sessions before. After her analysis that the sides are funny arcs, she only wanted the statement that such figures are not counted among the regular polygons. She had no need for a complete definition yet. When reasoning that a rectangle is not a regular polygon, one looks for a characteristic of regularity to which the rectangle does not conform. This can be, for example: "the point of intersection of the diagonals is not equidistant from the sides" (Pp. 32, p. 90).

When the pupils were asked to indicate the zig-zag lines in a given figure, they only asked me for approval of their judgment in cases of doubt (p. 110). Therefore, no definition was given for the term "zig-zag line." The pupils discover while drawing which lines do not belong to them.

The procedure was similar in the drawing of enlargements. I asked a question when the drawing threatened to go wrong (e.g. Pp. 6, p. 117).

It is simple for the students to ascertain that a rhombus is not a regular polygon, namely, because the angles are not equal (Pp. 27, p. 89). They thereby have a clear concept of a square in mind. It is much more difficult for them to say what a rhombus is. (Pp. 25, p. 139). The pupil will then tell all he knows about a rhombus. The pupils have the tendency to incorporate into a definition all the properties of a figure.

As soon as the pupils know how to construct a rhombus in one way or another, they can come to a correct formulation of a definition (Pp. 25, p. 139).

It is difficult for the pupils to make the correct distinction between trapezoid and parallelogram (p. 137). This precedes the defining of both figures. In order to be able to arrive at a good definition of trapezoid, a critical attitude is required and the group does not possess that yet. Therefore the definition evolved only because I always produced counter-examples.

Upon reading the test, given on p. 142, I encountered a couple of seemingly contradictory answers. Question II read: Have you also found hexagons? Is there a regular one among them? If yes, draw the outline with red. Do not forget to write down the answers and to explain.

Pp. 22 gave the following answer to this question: "Among the hexagons I found there is also one regular one, because all six sides are equal. The angles are not equal."

Upon inquiry, it appeared to me that she really did not make a distinction between regular and semi-regular. This phenomenon can be explained as follows:

For a pupil, a complex of properties he has not yet ordered belong to a figure. A similar complex of not yet ordered properties also belong to the word regular. There now are two possible modes of connection. The first is that one calls a figure regular when it possesses the whole complex of properties that are linked to regularity. One can only make use of this mode of connection here to ascertain that a given figure is not regular. For, in order to show that a figure is regular, the totality of the properties of regularity has to be established first. Indeed, many pupils have acted according to the above-mentioned mode in order to show that a given figure is not regular: they named a property that regular hexagons ought to possess and that the given hexagon did not possess (see p. 160). There is, however, still another possible mode of connection between figure and regular. One can link them also when the complexes of properties that belong to them possess one or more common elements. The pupils very often still have to experience what the custom (usage) is. The rule of the game became clear to many pupils only during the oral conversation. Their reaction "I have been caught" indicates that they had been tested on rules of the game which had not been sufficiently established for them.

Before one can proceed to the defining of particular figures in the teaching process, it is necessary first to establish a totality of properties of such a figure and second to order these properties according to known principles. Only after this has been completed, can one arrive at definitions in which "necessary and sufficient" characteristics can play a role.

The ordering of geometric figures according to geometric characteristics belongs to the first level of thinking.

The finding of interdependence and the ordering of relations takes place on the second level of thinking.

The ordering in a system of theorems is established when the pupils are on the third level. Then insight into the theory of geometry has been acquired.

I will explain this further with an example. It follows from empiricism that a regular pentagon has: (I) equal sides, (II) equal angles, and (III) equal diagonals. On the second level of thinking, one can make use of the ordering principle congruence, i.e. one can deduce the congruence of triangles on the basis of certain characteristics. The pupils then can establish the following ordering:

> III follows from I and II according to SAS and also II follows from I and III according to SSS.

It is only possible on the third level to conclude from this that there are (at least) two definitions for a regular pentagon.

For an analysis of the language structure in another sense, it is desirable to have a shorthand report at one's disposal.

Footnote:

1. Dr. L.N.H. Bunt, Geschiedenis van de wiskunde als onderwerp voor het gymm. A. Groningen, 1954. (See p. 176.)

Chapter XIV

FURTHER ANALYSIS AND FOUNDATION OF THE DIDACTICS

It appears necessary to me to summarize my conclusions about the didactics in the preceding chapters. In the initial stage of geometry instruction I start from the global structure the children have acquired. Before the study of geometry can start, it is important to build these structures geometrically. For this I choose objects from the surrounding world that lend themselves especially well to bringing about the geometric structures: the extended building block set.

Since the pupils have to be able to understand the structure of the analysis, it is not desirable to start from a total picture of the world. A pupil proves he possesses the structure of the analysis when he shows that he can manipulate the organizing principles. One of these organizing principles is "symmetry." This rests on the act of thinking named by Delacroix (I, p. 240): the identification of component parts. If the teacher draws half of a vase, then each pupil can add the other half to it. For he has a concept of a whole vase at his disposal. The existing global structure of the symmetry concept becomes structured geometrically through an analysis of the deviations the completed vase has with respect to the concept of a vase. For this analysis I make use of an object for which the pupils have a clear concept. It has become evident to me that the head of a cat is much less suitable; the activity of the pupils is then focused more on an aesthetic rather than on a geometric structuring. The teacher will have to arrive at the most effective material by trial.

During the first trimester the organizing principle of symmetry is prominent because this connects best with the natural organization of perception. Empiricism provides diversity; the intellect identifies and orders (Delacroix, p. 209). The pupil makes drawings and models, searches for and <u>uses</u> planes of symmetry, axes of symmetry and centers of symmetry; he is actively involved with <u>hand</u> and <u>head</u>, he identifies and orders, in short, he explores.

In a synthetic build up one ought to reflect on the objects with which one is working. To do this we call on the developmental psychologists who assume that "to develop" means: to establish a structure for oneself. In Piaget (I, p. 15) for example, one reads the following about this:

> Intelligence, in effect, was showing up as a coordination of actions. These actions are first of all, simply material or sensory-motor (that is, without the use of the symbolic function or representation), but, already, they organized themselves in schemes which carried with them certain structures of totality.

There are no indications that abstract thinking among twelve-year-olds has developed to a point that its relations with the material world and action have been disconnected. This is the reason for my maxim that the didactics should start from a "thinking activity with manageable material as a help." Since the making of models, etc. is not a goal, but a help, scissors and glue are only of temporary use. By "elementary geometry" for the secondary school, I therefore understand a geometry that starts from the global structures of thinking that we find in children. The requirement of establishing connections with the existing global structures during the formation of new mental structures remains valid for the whole geometry course.

One will have to accept that the child who studies geometry initially will have totally different structures of thinking at his disposal than the teacher had expected to establish. A beginning pupil, for example, does not see a reversible relation divided in its two parts: in this area the act of thinking "identification - distinction" has not yet been operative. The separation of the Siamese twins appears not to be possible yet. During the first year, therefore, one should avoid any symmetrical relations; or better, one should never make use of them.

The twelve-year-old child has acquired structures through his life experiences. We call these structures global structures. These all came into being as a result of the act of thinking: identification - distinction. One can obtain totally different global structures of the same object depending on the context in which one views the object. The example of the parquet floor already points in that direction. The global structure of a dancing couple is mainly based on it being more or less slippery. For the interior decorator the structure of the parquet floor is linked to that of the whole room. The organizing principle has an aesthetic character. For the very young child the parquet floor takes on a global structure in which the organizing principles are linked to the possibilities for play which it provides.

Some children discover rather early the geometric structure in the parquet floor. We have observed this as follows: our youngest daughter (6 years, 10 months) was able to reproduce the herring-bone pattern of such a floor with small rectangular tiles without looking at the parquet floor. She could also carry that out faultlessly with tiles in which the sides containing the right angles have totally different ratios. This showed that she had a structure of an organizing principle at her disposal. Her one and a half year older sister (8 years, 3 months) was unable to reproduce the herring-bone pattern with tiles, even though she had repeatedly looked at the parquet floor. The geometric structure of the parquet floor had not yet been sufficiently formed in her. She could see the lanes, but not the herringbone. She had not yet seen through its structure.

The ordering principle can be pointed out by observing the regularity of the figures. When these structures are sufficiently fine, one can speak of the existence of visual geometric structures. Initially they still have a global character. The existence of such structures is a necessary condition in order to be able to start geometry instruction.

The structures that have to serve as starting-points for the child at the beginning of geometry instruction can be divided into two categories: the perception structures that are especially studied by the Gestalt psychologists and the structures of the organizing principles that have been defined by Van Heile (p. 143).

Briefly, one can imagine the didactics of geometry as being built up in the following manner.

One starts from the global structures in a geometric context. These can be grouped into perception structures and organizing structures.

The organizing structures can come into being in various ways, by putting into practice the relation "to fit in" (puzzles) and "to fit on" (sticker-puzzles). By letting pupils, for example, cut out figures and investigate the different ways these will fit in the holes, one provides geometric structuring material for them.

When the pupil is able to structure geometrically, he reaches the first level of thinking: he is able to apply known properties to a known figure in an operational way; he acquires visual geometric structures of the figures, but he cannot yet incorporate links in his reasoning.

It is only after this period of active formation of visual geometric structures of geometric objects (figures) that it becomes meaningful to insert a period during which associations are formed. I then let the pupils do a lot of constructing during the lesssons; through the use of compasses and ruler, the characteristic properties become attached to the figure. If one introduces this period too early, an association formation will most probably be established, but the pupil is not on the level of thinking - he therefore misses the opportunity to make operational use of his knowledge. In this procedure one takes into account the plateaus in the learning curve. (See Van Heile, p. 41.) If everything proceeds well, the pupil has ascended to the first level of thinking from level zero (the global structuring). One could say that the concept of "geometry" came into being because the theorems are the objects of geometry proper.

In the learning process we now have a repetition: first a period of structure formation followed by a period of appropriate association formation. As far as the structure formation is concerned, however, there is a difference: the transition from level zero to the first level is, in essence, different from the transition from the first to the second level. On level zero everything is global from a geometric point of view. Through the analysis in a global geometric context, geometry is born, as it were. The transition from the first to the second level only brings about a structuring-through within the geometric context.

There can be a transition to the second level of thinking precisely because of the opportunity that has been created to make use of the organizing principles of the context. These principles provide geometric characteristics rather than a global aspect.

This level of thinking implies that the pupil sees that an ordering can be established in the relations, that one relation can follow from other relations. He can possibly look for new relations himself by starting from given relations, or he can order a number of given relations. He thus can make operational use of the relations.

I will illustrate this with an example: I take parallelism as the organizing principle. Globally this means sameness of direction. Visually, this means acquaintance with the geometric characteristics of parallelism - among other things, equality of angles. In order to give this characteristic a signal character, I introduced the names saw and ladder.

If one now orders figure 9 (which evolved through ordering of a large number of congruent triangles) according to the parallel lines that one can observe in it, one *finds* the relation: "The sum of the angles of a triangle is 180 degrees". (See X, p.98). If however one orders figure 9 according to the ladders and saws that are present in it, a structuring-through towards the following scheme is possible:

Ladder ↓ Saw ↓

Sum of the angles of a triangle is 180°;

Hence, a structuring-through towards an *ordering* in the relations is possible. (See X, p. 119 ff). By this structuring-through, a new structure has been formed. Having this new structure at his disposal, the pupil proves that he has attained the second level of thinking. He then has grasped the concept of what geometry is all about.

The first level of thinking implies comprehension of the aspect of geometry and the second level implies comprehension of the essence of geometry.

The didactics aimed to prepare for the second level of thinking is similar to the one aimed to prepare for the first level of thinking. The learning process again has a period of structure formation, followed by a period of association formation. These periods, however, are much longer. This is especially due to the fact that the concepts by which we have ordered still have to be structured-through. The visual geometric concept "congruent" implies: "all elements equal." After structuring through, it implies: "to be able to operate with the cases of congruence." The visual geometric concept enlarging (the corresponding angles equal, the corresponding line segments proportional) has to be structured-through to the cases of similarity. The visual geometric concept "saw" and "ladder" has to be structured-through to the known theorems about parallelism in connection with alternate-interior angles and with corresponding angles. I have indicated this earlier with the words: the separation of the Siamese twins. Therefore, the second level of thinking is a necessary condition here; this subject ought not be presented to beginners. There is even a question whether the attainment of the second level of thinking is sufficient, whether perhaps the third level of thinking wouldn't be necessary (see Van Hiele, p. 174). The relation between parallelism and equality of angles is symmetrical and the symmetry is essentially used. When the pupil has directed his attention towards this symmetry of the relation, one can then say, for the first time that this relation is a logical relation for him. The pupil is then on the third level of thinking.

In order to let the pupils better experience the essence of geometry I use a known global structure, "the genealogical tree." The genealogical tree clearly provides the opportunity for extension at both ends: the antecedents and the descendents. The former involves searching for a set of mutually independent axioms, the latter involves the finding of new theorems from the theorems that are already known.

In order to promote the formation of new structures by the children, the teacher should aim at the following:

1) that his pupils will build up the global structures in a geometric sense;

2) that they will recognize these structures as component structures of higher ones;

3) that they will expand these structures, in so far as the context allows it;

4) that they learn to recognize corresponding elements in isomorphic structures.

In this connection I start in the second trimester from the visual geometric structures: the plane coverings (see fig. 1 through 12). Since these may not yet be structured in a visual geometric way, I start from the global structures of sidewalks, tiled floors and mosaics which the pupils made with congruent cardboard figures. These global structures of perception are structured in a geometric sense by having the pupils make drawings of these mosaics as accurately as possible. These visual geometric structures are compared with each other; the one is viewed as a component structure of the other. They are refined by applying ladders, saws, enlargments, etc. This preparation is necessary in order to make it possible for the pupils to acquire those new structures belonging to the higher level of thinking. These contain he same elements as the visual geometric structures. The novel aspect is the ordering that is established among the relations. Subsequently, it is important that the pupils have at their disposal the structures of the organizing principles that are used in the analysis.

For congruence of triangles, the latter means: to be able to conclude the equality of six elements from the equality of three elements. The customary terminology for this is as follows: congruence follows from the given characteristics and the equality of the remaining elements follows from this congruence.

The empiricism - the correspondence of the construction - implies five characteristics for the congruence of triangles.

The same procedure is used for the theory of parallelism: the saw and the ladder can be recognized either from parallelism of lines, or from equality of angles. From each of these figures two characteristics are valid. The accompanying terminology here is: Parallelism of lines implies the presence of the saw and this implies equality of angles. Equality of angles implies the existence of the saw and this in turn implies parallelism of lines.

A pupil who is on the third level of thinking and who has the structure of organizing principles at his disposal can reason as follows:

> If two triangles have two sides and the included angle equal (S.A.S.), then the remaining elements are also equal (A.S.A.). If two triangles have one side and both the adjacent angles equal (A.S.A.), then the remaining elements are also equal (S.A.S.).
>
> If two lines are parallel, then the alternate interior angles are equal. If alternate interior angles are equal, then the lines are parallel.

Many teachers make use in their didactics of tools such as ladder and saw, but they use a terminology that can only be understood by pupils that are on the third level of thinking. They also sometimes use schemes that do not belong to the pupils' present level of thinking.

Similarly, the proof of the above-mentioned symmetrical relations requires a level of thinking that is too high for the first year, because the ordering of a system of theorems forms the base for these proofs.

One can easily follow the analysis of the children in the protocol.

When I ask for the forefathers of the theorem: "the sum of the angles of a triangle is 180 degrees," the children have to do the following:

a) extend the figure of the single triangle until it is incorporated into a pattern of triangles;

b) abstract, i.e. choose that part of this pattern that is necessary for reasoning about the theorem. This amounts to choosing the correct auxiliary lines.

c) establish the ordering of thinking that leads to the conclusion that the sum of the angles of the triangle is 180 degrees.

This is an example of an inductive thinking process. If one derives from the above-mentioned theorem that the sum of the angles of a quadrangle is 360 degrees, one is dealing with deductive reasoning that was started with an inductive thinking process.

The deductive as well as the inductive thinking processes are intentional. The teaching of exact thinking has to be directed, in the first place, towards increasing the versatility of the pupils' thinking. This versatility consists of having the pupils learn to switch smoothly from inductive to deductive reasoning.

The pupils thus have to experience that when there is a relation between A and B, there is a possibility that A is the premise and B the conclusion, but also the converse. They will also have to see that the reasoning which establishes such a relation can have variations. One can have A - P - B, but also A - Q - B.

As soon as the pupils are able to draw correct conclusions in this way and can examine the conclusions of others, they have reached the third level of thinking. I therefore would like to indicate this third level of thinking by "insight into the theory of the subject geometry". Only when the pupil has gained this insight, is he able to following an axiomatic build-up of geometry and to participate in the actual construction of a system of theorems. From that moement on, instruction can be given by means of a deductive system. This is a very appropriate means of establishing results, although one shall have to keep in mind that one arrives at these results by way of a thinking process in which induction and deduction continously alternate.

My conclusion therefore is as follows: A system of teaching that has the tendency to be based exclusively on deduction or on induction is wrong. In the initial instruction neither purely deductive nor purely inductive reasoning is appropriate. What is important here is that the pupils establish relations between empirically obtained results, and that they gain insight into the empiricism. The first goal is not yet the building up of a system of theorems, but the development of thinking. The class as a unit analyzes at a certain level, and the pupils learn to analyze by participating in discussions. The teacher only guides discussions and provides favorable learning situations. Parts of a system of theorems are found, and only after insight has been gained into the structure of this system is it appropriate to try and build a system from a certain given sets of axioms by using a deductive approach.

One should not expect that the pupils themselves would be able to arrive at a certain system of axioms from the parts, found by them, of the system of theorems by using an inductive approach. In order to establish such an ordering, it is necessary that one looks at geometry from a higher viewpoint, and for this it is necessary that one has reached the fourth level of thinking. (Van Hiele, p.175.)

I now will give an overview of the didactics of geometry as a whole in relation to learning situations.

In the first learning situations one starts from existing global structures of perception and structures of ordering. The objects are figures. A number of figures and concepts are placed in a geometric context by the choice of organizing principles. Initially this is a known ordering (fitting, piling, etc.) The figures thus become geometric symbols (see Meyerson I, p. 574); they are carriers of geometric relations that we have found by means of empiricism. These geometric relations become the objects of study of geometry. The goal of the first learning situations is to let the pupils experience the aspect of geometry in an empirical way. The pupils acquire visual geometric structures. The structure of the organizing principle takes on a geometric symbol through the concepts of congruence and parallelism (ladder and saw).

The structuring-through makes it possible to place pupils in learning situations where they can experience the dependency of relations. By letting pupils analyze at their level, an ordering of certain relations evolves. Known relations can be a consequence of other known relations and new relations can be discovered from known relations. The goal of these learning situations is to bring pupils from empiricism to the essence of geometry. Through this analysis it becomes possible for pupils to expand their visual geometric structures into structures that belong to the second level of thinking. For this transition to the second level, it is necessary that they have acquired (although still implicitly) the structures of the organizing principles. What has been going on in the pupils' thinking can be indicated as follows: The organizing principles (e.g. congruent and parallel) have taken on a signal character and can be recognized. Ladders and saws each possess two characteristics; triangles have five. In the latter instance, the signal character lies in the word congruent.

After the structure of the organizing principles has been made explicit, the pupils are capable of reasoning such as: the parallelism of two or more lines implies the existence of a ladder and this in turn implies the equality of corresponding angles.

The equality of two sides and the included angle implies the congruence of triangles and this implies, for example, the equality of the third side.

If one now takes the geometric relations of a figure itself as objects, then a separation in the relations, namely, into premises and conclusions, evolves after the structure of thinking on the second level has been made explicit. While making the accompanying structures explicit, the following scheme evolves: given - to be proved - proof.

Only when the pupils have these structures at their disposal can the teacher introduce learning situations in which the pupils experience how an ordering in the relations evolves. Through analyzing the theorems for their correctness, the pupils ascend to the third level of thinking. The nature of the signal character of the figures is being analyzed (see, for example, Chapter XIII, p.183). In these learning situations, the thinking activity is also directed towards finding the converse of theorems and drawing up definitions. The schemes of the ordering of the relations becomes the object of study in order to obtain insight into the theory. Pupils who possess this insight into geometry have reached the third level of thinking.

Such pupils are then capable of the following arguments:

> If two lines are intersected by a third line and the corresponding angles are equal, then the first two lines are parallel; and conversely: if two parallel lines are intersected by a third line, then the corresponding angles are equal.
>
> If in two triangles two sides and the included angle are equal, then the third side and its adjacent angles are also equal, and conversely.

The teacher can place those pupils in learning situations in which they can actually participate in the building up of a system of theorems.

Only the pupils who have reached the scientific insight (fourth level) can study the foundations of the theory. They are able to help with the building up of a deductive system from the foundations. Among them, and only them, one finds persons who can compare different theories, who can seek out missing axioms in other geometries and who can establish the foundations of a new theory and build a deductive system on it.

Since the learning process of the pupil who is studying geometry is founded on the special didactics of geometry, I will compare this learning process once again with that of the teacher who is learning didactics.

It can be useful for general didactics to proceed with my analysis of the last named learning process which I started in the preceding chapter. I will thereby make use of the already mentioned terminology of Langeveld (V, p.228) and I will develop my own terminology on the basis of his.

The initial objects in the learning process of the teacher are, among others, the performances of the pupils. These have an undifferentiated shape for the beginning teacher. The teacher possesses certain global structures of his objects. In this learning process, one can also expect that different teachers will have very different global structures. For there are differences in training, experiences, intuition and environment. During the explanations, the shape becomes differentiated. The teacher acquires structures of his objects. These structures belong to the aspect of teaching. Explaining is a purposeful action and therefore encompasses a moment of thinking (executive-practical). When the teacher reflects on the explaining, his actions become more differentiated. He acquires structures of the <u>explaining</u>. These belong to the aspect of teaching (practico-practical form of thinking).

We see the following points of correspondence between both learning processes: one starts from observations; the action is initially acting-thinking and thinking-about-the-acting (technical thinking). Then a change takes place in the object of study. The reflection upon the teaching process is the condition to arrive at a higher level, i.e. to gain comprehension of the essence of teaching.

If the theoretico-practical thinking is only directed towards the action of the teacher, one only takes into account a single aspect of didactics, namely, the transfer of the subject matter. If, however, one takes the teaching situation as a reciprocal action situation, then the theoretico-practical thinking leads to a formation of structures that belong to the essence of teaching. Then didactics is being placed in line with the humanities. However, there are also factors that influence the teaching process and that lie outside the science of the subject to be taught. One of these factors is the <u>thinking</u> of the pupils. In order to understand the nature of this thinking, the teacher will first have to perceive the aspect of it. There he will also have to start with the stage where he makes use of the executive-practical and practico-practical forms of thinking. He can ascertain through observation that the thinking of twelve-year-olds differs remarkably from the thinking of adults. This aspect of teaching is well-known to most experienced teachers. In order to reach the essence of it he will have to follow the genesis of the thinking with the help of the practico-practical and theoretico-practical thinking process. He will have to observe, theorize, experiment.

When the factors that influence the teaching process have been diagnosed and have been perceived and understood in the context "educating," when in addition they have led to meaningful integration, then the teacher has gained insight into the didactics: he then sees the organizing principles on which the didactics rests. If he reaches the point where he has at his disposal the structures that belong to this level, he is then able to formulate provisional goals in relation to the incomplete goal that is partially determined by the subject. This theoretical thinking is now explicitly directed towards the didactics. Langeveld (II, p. 315) talks about subject-oriented theoretical and reflexive theoretical.

The similarity between both learning processes is that each new factor, that brings a change in the learning process, has to be diagnosed in the original objects and has to be perceived in the correct context. Gaining of insight into the essence of that factor is possible by directing the thinking onto the process itself. For the learning process of the pupil in geometry, those factors are the new organizing principles that are added, e.g. "parallelism". This organizing principle is added after "congruence" is first diagnosed in the learning situations "sidewalks." It is subsequently perceived and understood in a geometric context by means of ladders and saws. Because the thinking is directed towards the characteristics of parallelism, the pupils gain insight into the essence of parallelism.

For the learning process of teaching, the thinking of twelve-year-olds is a factor that initially is insufficiently taken into account. This factor first has to be diagnosed in the teaching situation, has to be perceived and understood in the context "educating." It can only become an organizing principle in the theory of didactics when the thinking of the teacher has been directed towards the characteristics of the genesis of the thinking and towards the levels of thinking. In this way, the teacher touches upon the essence of an important organizing principle: the ordering of the subject matter according to the levels of thinking.

Chapter XV

THE LEARNING PROCESS IN THE LAST TRIMESTER

The subject matter to be covered during the third trimester was determined by structures of thinking that were present in the children at the beginning of that trimester. These partly evolved in the learning process of the second trimester.

By manipulating the organizing principles: "to fit, to pave, to enlarge," the pupils had acquired visual geometric structures. Plane figures that possess a certain regularity were now geometrically structured for them by being linked to geometric characteristics. Sets of congruent figures took on a geometric structuring for the pupils because they learned to see ladders, saws and enlargements in them.

During the use of the first-named organizing principle, attention was paid to equality of the elements of figures; for the second, to equality of orientation of lines; for the third, to equality of shape of figures. The theories of congruence, parallelism and similarity are based on these principles.

Before the transition to the second level of thinking is possible, these organizing principles themselves have to be geometrically structured. Ladders, saws and enlargements are geometrically structured; they already have geometric characteristics.

By building on the above considerations, the learning process can be developed in the following directions:

1. The organizing principle "congruence" can be structured through. This was already used in a global geometric way, such as "to fit in" and "to fit on."

2. One can allow associations to be formed that are connected with parallelism (ladders and saws), in order to let the pupils have the principle at their disposal.

3. One can expand the number of visual geometric structures by starting from new known global geometric structures and determining their organizing principles.

The bringing together of figures to make a whole is brought about not only through the organizing principle of "paving" but also through "stacking." We encounter a very special form of "stacking" among the building kits. There, a number of plane figures are put together until they form a spatial entity. These spatial entities in turn can be chosen as objects for stacking.

The making of networks is thus one of the expansions of visual geometric structures mentioned under point 3. In the context "stacking," one can prepare for the concept "volume" in the same way as paving was preparation for the concept area. (p. 136 ff).

In the last trimester I first introduced learning situations that had the goal of bringing about a structuring through of the organizing principle "congruence." It was a period with much individual work. Under my guidance, a large number of constructions of triangles were carried out. Since many teachers introduce the congruence of triangles in this way, it is not necessary to give a detailed description of this teaching process.

The results were collectively established in a class conversation. The characteristics SSS, SAS, ASA were derived from the corresponding constructions. It had already been experienced in the course of the constructions that one is not entirely free in the choice of angles and sides. For instance, the sum of two sides of a triangle has to be greater than the third side. Since the property of the sum of the angles of a triangle was known, the characteristic AAS could also be found. It appears from the construction that SSA is not a complete congruence characteristic. It is one, however, when the angle is a right angle. Strips and protractors were made available to the pupils who liked working with these materials.

Next, a number of constructions of triangles were made where altitudes, medians, and bisectors were involved in the instructions given. Because it now involved the formation of associations for five characteristics, the precise measurements were given. In a free choice of the data, one runs a risk of encountering new problems, namely that of dependency and contradiction of data. (During the constructions of triangles from their elements, this had surfaced, but in a very restricted form.) Attention was being paid to the possibility of multiple solutions, however, because this would reinforce the correct interpretation of the case SSA.

As for learning to know space in a geometric sense, I judge the making of models to be very important. Honigswald (I, p. 220) says:

> It is not spatial figures per se which are defined by a system of geometric knowledge. Rather it is certain determining criteria which chartacterize the spatial figures in a concrete way. Indeed, the fact that the criteria are related to figures - and only to figures - gives them a singular characteristic: it marks them as moments of true perception.

I therefore ended the course with an activity about the "making and drawing" of solids. This contributes to the formation of global stereometric (three dimensional) structures. I will reproduce this activity in shortened question-and-answer form. This most clearly shows what the pupils are able to achieve by themselves in this area.

First lesson

Our starting-point was again the cube. The model I used for demonstration had a loose upper base and a loose diagonal plane. This partition divided the cube into two pieces for which the name "prism" was mentioned. The first task was as follows: "Just as we succeeded earlier in making a cube out of one single piece of paper, we can make the prism in a similar way. Try it.

Most pupils circled around the edge (4 cm) twice in order to obtain the upper base. They thereby observed that in doing this one does not easily make mistakes. Only one pupil needed help in making the upper base.

Next I showed a regular four-faced pyramid the base of which fitted on the upper base of the cube. I therefore could place this pyramid in the cube. I did this and mentioned that the altitude of the pyramid was exactly equal to half the altitude of the cube. I then asked: "How many of those pyramids are needed in order to build a whole cube?" One pupil said, "two." I also received answers such as "four" and "six." Nobody bid higher. Once the number 6 had been mentioned, more and more pupils agreed with this number. One of the pupils was asked to explain why 6 was the correct answer.

Next I asked: "Can we also make a network of the pyramid?" Pps: "A square with an isosceles triangle on each side." Some thought that it had to be an equilateral triangle. Tr: "How can we find out who is correct?" Pps: "By measuring." In this way it was found that the triangle has to be isosceles. Tr: "How long should the legs be? The model I have is larger than the model you have to make - and nobody has the pyramid." I showed two cubes side by side; in one there was the diagonal partition and in the other the pyramid. Then a couple of pupils saw it: the leg of the isosceles triangle is half the diagonal of the cube. The pupils explained it to each other: "When you extend a leg you arrive exactly at the opposite vertex."

Then came the second task: "Draw the network of this pyramid." A few pupils did not take half of the diagonal of the rectangle, but half of the diagonal of the square. They discovered their mistake because the network did not produce a pyramid. The pupils went home with the assignment to make a prism and three pyramids for the following lesson.

Second lesson

Tr: "What is the volume of the prism?" Pps: "32 cc because two prisms together form a cube." Tr: "What is the volume of the pyramid?" Pps: (after some hesitation) "10 2/3 cc." Tr: "Show me that 6 pyramids together have the same volume as two prisms." All hands were needed to hold the pyramids.

Now came the first task: "Glue the prisms together so that they form a cube and glue a pyramid to each face of the cube. Tr: "How many lateral faces does this solid have and what do they look like?" Pps: "Triangles. Rhombi." Finally, the answer "rhombi" was predominant. Tr: "How do you know that they are rhombi?" Pps: "Two triangles together form a rhombus." We felt with the palm of the hand that the triangles were positioned in one plane. Tr: "How many rhombi are there?" Pps: "12" Tr: "The name of this solid is a rhombododecahedron."

The solid can be made by drawing twelve rhombi and gluing them together. Tr: "Which measurements of the rhombus are already known?" Pps: "The diagonal is 4 cm and the sides are the legs of the isosceles triangle, thus 3.5 cm." (A few pupils obtain 3.4 cm.) I showed a model of a rhombododecahedron and asked one of the pupils to find the vertices of the cube and to point them out. Fortunately I had a few models to distribute among them. It then became a race to find the vertices of the cube. This assignment was necessary in order to arrive at an ordering of the vertices of the rhombododecahedron. We found: There are 6 vertices where each time 4 acute angles of the rhombus meet and 8 vertices where each time 3 obtuse angles meet. Then came the task: "Make this rhombododecahedron out of one piece." During the work I gave hints: "Number the vertices where 3 obtuse angles meet 1, 2, ...,8 and the vertices where 4 acute angles meet I, II, III, ... VI. Connect the sides that have to be glued together with an arc."

More than half of the pupils succeeded right away in putting together the rhombi in the correct way. The figure was finished in one hour of free work.

<u>Third lesson</u>.

I started by asking the volume of the rhombododecahedron. Pps: "Twice the volume of the cube of course, 128 cc." The pupils asked me to give similar additional assignments. I then drew the following figures: six contiguous regular triangles that met in one point; four contiguous squares that met in one point; three regular hexagons that met in one point. I asked whether they recognized these figures. Pps: "We saw those when we had the 'tiles.' It did not work with regular pentagons, because a small space was left and it did not work with regular octagons either."

Tr: "Draw those three regular pentagons again on a piece of cardboard. The protractor can only be used once. The rest has to be carried out with compasses and rulers." I simultaneously drew it on the blackboard and drew an arc in the open space. Tr: "Why do I draw an arc there?" Pps: "It can be glued together." Tr: "Exactly. If I want to, I can make a tri-faced top hat. It is possible to make a solid that is bounded with only regular pentagons. You need twelve of those, in each point three of them meet. Try it."

Whoever was finished with this assignment was allowed to start with the following investigation:

Are there solids that are bounded by:

a. regular hexagons?
b. regular quadrangles?
c. regular triangles?
d. regular octagons?

Fourth lesson

I redrew on the blackboard the three figures that had been discussed at the beginning of the preceding lesson. They were named figure 1, figure 2, and figure 3 (they are the basic patterns of the tiled floors 5, 1 and 6 respectively). Tr: "Has anybody succeeded in making a solid that is bounded by regular hexagons?" Pps: "No, it does not work. There is no open space." Tr: "And squares?" Pps: "Yes, that is the cube." Tr: "There is no open space in figure 2 though." Pps: "That one square is in the wrong position. It has to be drawn somewhere else." Tr: "Correct, I have to erase that then. Then I have an open space and I can glue the two open sides together. I drew an arc in the open space. Could I not also do that with the hexagons: erase one, such that there is some open space?" There was agreement until a few saw that this also would not work. Th hexagons then fall on each other.

Tr: "Is it possible with regular octagons?" Pps: "No, then you also have only two."

Tr: "Now the triangles. Who has found a solid that is bounded by regular triangles?" Most pupils had found the tetrahedron. A few also found the icosahedron. Tr: "How many triangles meet at each point?" Pp: "In the icosahedron, five." Tr: "In which figure on the blackbord should I erase something?" Pps: "In figure 1." I then erased a line. Tr: "What else can I do?" Pps: "Remove two segments or three segments." Tr: "You certainly all found what the end result is. That is the tetrahedron. There apparently has to be another solid where four triangles meet in one point." One of the pupils proudly showed a model. It was bounded by 10 regular triangles. When we looked at it more closely it appeared that five triangles met in two vertices and in the other vertices only four met.

The task now was: "Make a solid that is bounded by regular triangles where four meet in one point." It was not given how many triangles would be needed in all. It was a surprise when the known octahedron made its appearance. Many pupils had alredy drawn too many triangles when it was whispered: "There are eight of them."

In this manner we came empirically yet systematically to the existence of five regular polyhedra.

Fifth lesson

Tr: "There are also solids that are bounded by a number of squares and a number of equilateral triangles." Each child was given the assignment of making 8 equilateral triangles and 8 squares, each with a side of 4 cm. They then were asked to investigate the following:

a. There is a solid that is bounded by two triangles and three squares. Make it and give it a name.

b. There is a solid that is bounded by 4 triangles and one square. Make it and give it a name.

c. There is a solid that is bounded by 8 triangles and two squares, such that at each vertex one square and three triangles meet. Make the solid out of one piece.

The assignments a. and b. were given in order to allow the names prism and pyramid to obtain a broader meaning and to emphasize the characteristic features of these solids.

One pupil solved question c by combining into one solid the not yet glued pyramids (of question b.) made by his neighbor and himself.

Sixth lesson

Two other assignments were given:

d. There is a solid that is bounded by 8 triangles and 6 squares, such that at each vertex 2 squares and 2 triangles meet. The triangles are opposite each other at a vertex. Make such a solid out of one piece.

e. There is a solid that is bounded by 8 triangles and 18 squares. At each vertex 3 squares and one triangle meet. Make this solid out of one piece.

Since now the number of faces became too large to be held together, only the systematic drawing method was left. The information concerning the figures that meet at one point provides the key to the solution. A global ordering is established and the stereometric figures that this evolves are structured in a visual stereometric way.

The last solid appeared to be too difficult for many pupils. I therefore showed the solid. After the pupils had observed the edges of 8 squares that appears on the solid, the solution no longer seemed difficult.

In this way over a period of three weeks, a whole set of solids was gathered that were worth studying more closely. In order to let the pupils focus better on the geometric characteristics, I had them draw a number of these solids on blank paper. This took place as follows: The solid that had to be drawn was placed in a certain way. We imagined it was intersected by a certain vertical plane (frontal plane). The intersection was drawn in actual size. The edges that belonged to the intersection were drawn thickly; the remaining lines of the intersection were thin. These lines were merely auxiliary lines. We then looked for the lines that were perpendicular to the frontal plane. Sometimes these were edges, sometimes diagonals or altitudes. These lines were drawn obliquely at an angle of 30 degrees with the horizontal. They were drawn half their real size. The figure was finished by showing the edges thickly. The invisible edges were drawn in a thick dotted line.

I had pre-drawn the cube in two different positions on the blackboard. First with a lateral face positioned frontally and later with a diagonal plane in the frontal position. In both cases the cube rested on one of its faces. Using this fixed procedure the pupils drew solids that had been discussed earlier. They did this during four class hours. Among these solids there was also the solid of assignment e. which we called a hall lantern. A few pupils could even draw this solid with a little help. The dodecahedron and the icosahedron were not part of this assignment.

The children often had difficulty finding the points where the perpendiculars met the frontal plane. For the most part they had not sufficiently realized that the finding of these points was important.

The last test of the first year had the objective: to investigate whether sufficient associations had been formed for the visual geometric structures. To do so, a regular dodecagon was drawn collectively. The vertices were labeled with the letters A through L and the center with the letter M. Next the pupils had to carry out the following assignments:

1. Write down two triangles that are congruent to triangle MAD and which have E as a vertex.

2. Write down three triangles that are similar to triangle MAD and which have E as a vertex.

3. Write down five triangles that are congruent with triangle LBD and which have E as a vertex.

4. Write down two triangles that are similar to triangle AEI and which have D as a vertex.

5. Fill in: BFHL is a ..... (commented on orally).

6. Give the names of as many different figures as possible that have the points A through M as vertices.

The pupils were advised in an oral comment to draw as few figures as possible.

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ENGLISH SUMMARY

by

Dina van Hiele-Geldof

of

THE DIDACTICS OF GEOMETRY IN THE LOWEST CLASS OF SECONDARY SCHOOL

This thesis is an inquiry into the didactic possibilities of geometrical instruction in a class where the child is given concrete material in a systematic way so as to unfold visual thinking and to transform it in the abstract way of thought which the logical system of geometry demands.

The author is mathematics teacher at a secondary school in Amersfoort (Holland). Teaching of geometry in Holland starts in the lowest form of the secondary school, i.e. at the age of 12. The descriptions in Chapter III refer to the teaching of mathematics to children at this school during the months of Sept.-Dec. 1955. The children had been divided into two groups in alphabetical order, so they may be considered as having been chosen at random. Chapter V gives an account of the method adopted.

When initiating a child into geometry we take into account it's previous experience. To do so, we start from known geometrical shapes, such as: cube, square, rectangle, rhombus, etc. The primary perception of these objects leads to undifferentiated structures that are analyzed under the guidance of the teacher. By a phenomenal analysis she will draw her pupils attention to the geometrical qualities of the shapes in order to clarify the context of the subject-matter. Paper folding, cutting out and construction of models are important tools in order to develop the children's knowledge of space, especially that of the symmetry of geometrical shapes. By these activities they enrich their store of visual structures.

The relations found should be settled by a joint effort of teacher and pupils, for the latter are not yet familiar with the technical terminology and must learn it by practice. The structures that finally emerge from this analysis can be considered as symbols of the subject-matter of geometry. The word 'symbol' should here be interpreted as meaning 'a mental substitute for a complex of undifferentiated relations that is subsequently elaborated in the pupil's mind.' The rhomb, for instance, is a symbol of the following characteristics: it has four equal sides, equal opposite angles, diagonals that bisect the angles and are perpendicular to each other.

The fact of working at and designing geometric models and, in particular, of making the necessary drawings and constructions, leads to the acquisition of a system of signals for these symbols. Finally the symbols are recognized even if only part of their characteristics can be seen. Anticipation of a symbol then becomes possible.

To encourage the pupils' personal exploration we facilitate the orientation in undifferentiated structures by means of a number of natural organizing principles that are already present in the pupils' minds in an undifferentiated form (i.e. not geometrically analysed), such as: division of plane and space, piling of objects, symmetry with respect to planes, lines or points. These organizing principles are the factors that permit a schematic anticipation of the symbols. At this stage we say that the first level of thinking - the aspect of geometry - has been reached.

This implies for example that a pupil who knows the properties of the rhombus and can name them, will also have a basic understanding of the isosceles triangle = semirhombus.

In this teaching situation the pupil will first acquire symbols, then the signals matching these symbols and finally he will learn how these symbols and signals may be used. This first stage, the period of the pupils' orientation by 'practical thinking', takes approximately 20 lessons.

Pupils at the first level of thinking have acquired special structures that have taken the place of the original undifferentiated structures; we call them visual geometrical structures. At this level a geometrical shape is still interpreted as the totality of its geometrical properties. The pupils are not yet capable of differentiating them into definitions and propositions. These two will have to wait until the children have reached a higher level of thinking for they belong to the deductive system of propositions. Logical relations are not yet a fit study-object for pupils who are at the first level of thinking.

For the transition to the second level of thinking the attention of the pupils should be deflected from the primary subjects to the relations between the properties of geometrical patterns. We shall have to organize the aggregate of geometrical properties. Furthermore, the 'implication' must acquire its special meaning within the context of geometry.

The forming of signals at the second level of thinking demands a preliminary orientation, now in the field of visual geometrical structures. The pupils will never understand certain essential geometrical concepts, such as parallelism, unless the visual structures have acquired a more subtle differentiation.

For the next stage in the hierarchy of structures, on the second level of thinking, general class discussions are undispensable. During the months of January, February and March of the Sept. 1955 - Sept. 1956 course I took notes during the lessons and worked them out immediately after. The syllabi have been described in Chapter VI. They form an introduction to: the logical connexion of relations, the theory of parrallelism, the theory of congruence, the theory of superficies and the theory of translation. Chapter X gives a detailed account of discussions on the subject of 'tiles'. With the help of these discussions we can analyse constructive moments in the process of learning. Chapter XI contains an analysis of the transition from the undifferentiated to the visual geometrical perception field whereas Chapter XII analyses the development of logical thought. We paid special attention to this particular period as it gives an insight into the pupils' way of proceeding from concrete situations to abstract thought patterns.

As a basis we took the child's undifferentiated structure of a sidewalk. The Dutch tiled sidewalk is a typical illustrative object of the concept of parallelism. The pupils were given the task of building up a tiled floor from congruent tiles of card board, such as triangles, squares, etc. In this way the concrete basis will not be lost. The pupils were instructed, carefuly to copy the patterns in their exercise-books, and were told to compare the results and to elaborate them by drawing in 'ladders', 'saws', enlargements, etc. (see figs. 1 to 12 inclusive).

In so doing we made use of the following didactical principles:

1. The pupils will structure the perception structures in a geometrical sense.

2. These structures are subsquently refined, i.e. differentiated in a more subtle way.

3. Structures will be visualized as components of a hierarchically higher structure.

4. Ismorphic structures will be identified.

The orientation is effected by means of the observation of parallels, congruent angles, similar shapes, of relations such as: the three angles of a triangle together form a straight angle (see fig. 9).

In this way organizing principles such as 'parallel', 'congruent' and 'similar' acquire the meaning of symbols.

The organizing principle 'congruent' becomes the symbol of the property: 'indistinguishable with respect to form and measures'.

The 'ladder' becomes the geometrical symbol for the property: 'to have two sets of parallel lines and equal angles'.

The enlargement grows into a geometrical symbol for the properties of equivalent patterns: 'corresponding angles are equal, the ratios of corresponding linear segments are the same, the ratio of their areas equals the square of the ratios of two corresponding sides'.

The use of these symbols finally leads to their recognition by the pupils. Ladder and saw then become signals: each of these patterns is recognized, either by the parallelism of their lines or by the equality of their angles.

The introduction of the 'implication' into the geometrical context opens the way to the exploration of the essence of geometry. We appealed to the pupils' knowledge of the (undifferentiated) structure of the genealogical tree.

The 10th class discussion shows how the pupils found the ancestors of the property: 'the sum of the angles of a triangle is a straight angle'. The plane filling with congruent irregualr triangles apparently functioned as mother-structure. Once the pupils had accepted the above-mentioned geometrical structures in their functions of symbol and signal, the auxiliary lines, needed to prove the relation were, as it were, incorporated into this mother-structure.

As soon as the pupils begin to understand the signal character of the symbols of ladder and saw, they are able to understand more advanced thought structures, such as: 'the parallelism of the lines implies (according to their signal character) the presence of a saw, and therefore (according to their symbolic character) equality of the alternate-interior angles.

Thus a new structure has appeared, the elements of which are the organizing principles of the original structures. The second level of thinking is reached: the essence of geometry. The didactical principle that guides the pupils towards this higher level of thinking seems to be: 'Make sure the pupils have at their disposal a number of visual geometrical structures that contain the same elements as the abstract structures you wish to confer.'

During the structuring period of the learning process the children are not expected to express themselves in technical terms but are allowed to use their own pattern of speech. The acquisition of the concept should precede that of the describing linguistic structure (see Chapter XIII).

To clarify the new context we inserted - during the last term of the first year - various practical tasks that lead to the intermediate goal of the subject matter: the five criteria of the congruence of triangles. The subject matter and procedure of this particular term are described in Chapter XV.

The pupils comes to accept the criteria on the ground of the uniqueness of construction. They work with protractor and with strips, and also with a pair of compasses and a ruler. The criteria are found empirically.

Once the pupils have accepted the organizing principle 'congruence' as a symbol and a signal, they are able to advance towards the next thinking structure: 'one of the criteria implies the presence of congruent triangles, and therefore, because of its character of a symbol, the equality of the other element of these triangles'.

However, to get this far we shall need a total of appr. 50 lessons. In our shcool we are then at the beginning of the second year.

Having given a phenomenological analysis of the teaching-learning procedure in Chapters VII and VIII I have tried to arrive at a closer scrutiny into the fundamentals of didactics in Chapter IX. To this purpose I studied Gestalt-psychology (Kohler), psychology of thinking (Selz and Duncker), psychology of learning (Van Parreren), pedagogy and development psychology (Langeveld), didactics (Strunz) and semiotics (Mannoury).

A careful scrutiny into and analysis of the children's process of learning enabled me to arrive at a synthesis of didactics (Chapter XIV) - partly owing to an introspective observation of my own process of learning (Chapter XIII).

During the second year the rhomb becomes a geometrical symbol of an organized aggregate of relations between its elements - the elements of the rhomb. Pupils who have these symbols and their signals at their disposal can understand what is meant by 'proof' in geometry. They have arrived at the second level of thinking.

Once the pupils have mastered sufficient second-level thinking structures, we can start changing the context. Now the logical relations themselves move into the foreground and we can start investigating the nature of these relations. This leads to an even higher level of abstraction, such as: 'parallelism of the lines implies equality of the corresponding angles and vice versa.'

On reaching this third level of thinking, which we call insight into the theory of geometry, we can start studying a deductive system of propositions, i.e. the way in which the interdependency of relations is effected. Definitions and propositions now come within the pupils' intellectual horizon.

Finally we can choose as a subject-matter the system of propositions itself. A comparative study of the various deductive systems within the field of geometrical relations is a subject reserved for those, who have reached the fourth level of thinking in geometry. Of these we can say, that they have acquired a scientific insight into geometry.

Tenets

I

In order to be able to arrive at an efficient study of a certain subject, it is desirable to investigate

1. whether more than one level of thinking is involved in the study of the subject;

2. which levels should be attained in order to reach the goal of the study;

3. how the attainment of the levels of thinking can be supported didactically.

II

It is important for the formation of concepts not to use schemes that belong to the unformed structure of thinking during the structuring period in the learning process.

III

It is meaningful to compare didactic methods scientifically only when the foundations and the theory of these didactic methods have been established and tested.

IV

Just as the analysis of learning processes in an experimental situation allows the investigation to penetrate more deeply into a real thinking process itself, the didactician will be able to penetrate more deeply into the genesis of thinking upon analyzing the learning processes in the school.

V

The question "what do you see?" is meaningful only after one has made sure that the subject knows the context in which this question has to be answered.

VI

The problem of selecting a school for a child could be kept within certain limits if it were possible to give child-centered instruction during the first two years of the secondary school. This would allow for natural selection to take place.

VII

For the most part the organizing principles of general didactics will have to be obtained through a mutual comparison with special didactics.

VIII

The choice of subject matter for geometry for the introductory course will have to be based, according to the newer insights of pedagogy and psychology, on figures that form a geometric structure and not on their elements.

IX

The introduction of axes and centers of symmetry acquires special meaning only if one starts from observations which are reinforced by manipulation such as folding, rotating, fitting on each other, etc.

X

The formation of visual geometric structures in the pupils is aided most efficiently by allowing them to use appropriate material, such as building kits.

XI

Since different subjects often start from the same structures or from the same objects, it would be desirable, at least in the beginning, to coordinate the instruction of these subjects.

XII

It is desirable to analyze the methodologies used in geometry instruction in mathematical, psychological and didactic perspectives so as to be able to determine to what extent they differ didactically.

XIII

It is possible and desirable that pupils, who appear to possess sufficient aptitude for it, recognize that the theorems of geometry they know can be organized in a deductive system that possesses similarities with a "genealogical tree" and that this genealogical tree can be chosen at will from a network of available relations.

XIV

One could interpret elementary geometry instruction as the formation of visual geometric structures through analyses of the global structures that are provided by empiricism. In connection with the introductory course in geometry proposed by WIMECOS, it is desirable to investigate the psychological correlate of the expression "elementary geometry."

XV

For the natural sciences one will have to look for subject units for the initial instruction which meet the following conditions:

1. they have to make clear what the context of the course in question is;

2. they have to belong to the immediate realm of interest of the children;

3. they have to provide enough subject matter for discussion without there being the need to go to a higher level of abstraction.

XVI

Even though concrete courses (with an empirical approach) should be the starting-point as much as possible in mathematics teaching, one should not say that mathematics instruction should be based on the teaching of the natural sciences. For mathematics, as well as the natural sciences, can develop from the same basic empiricism. They differentiate themselves from each other at a rather high level of abstraction.

XVII

More than before, one will have to collect material in order to allow the pupils to orient themselves in the topics of physics. For the development of technology is such that a spontaneous orientation cannot easily take place.

PART II

DIDACTICS OF GEOMETRY AS LEARNING PROCESS FOR ADULTS

(Last article written by Dina van Hiele-Geldof)

DIDACTICS OF GEOMETRY AS LEARNING PROCESS FOR ADULTS

The essence of didactics is the encounter of three elements: the pupil, the subject matter and the teacher. This is true for general didactics as well as for didactics of special subjects. A methodology may be seen as a particular way to bring about this encounter, principally by ordering the subject matter.

Didactics should not be viewed as applied pedagogy, or as applied psychology, or as an outcome of the science of the subject to which one wishes to introduce the pupils.

The didactical approach that developed as a result of interaction with practice and that is able to combine the pedagogical aspect, the psychological aspect and the scientific aspect of the discipline into one encompassing structure, will potentially be most fruitful for practice.

In this article I will not consider the pedagogical aspect - the relationship between pupil and teacher, nor will I discuss the sociological aspect - the relationship among the pupils. I wish to limit myself to the learning process - the relationship between pupil and subject matter - in order to focus attention on the particular structure of didactics. For adults too can still learn.

I will start from the practice; I will tell what I do and I will explain what my objectives are. I do not stress what I do, but why I do it. The function of the material, of the classroom conversations, etc. is central.

The subdividing of the ultimate goal of teaching into intermediate goals is important in order to be able to arrive at principles of the didactic method particular to the discipline. The method intuitively rests on these principles.

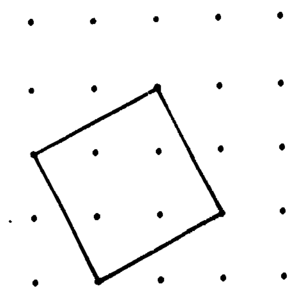
The goal of teaching, as it has been formulated, namely, that we teach children exact thinking, or we teach children the method of mathematicians (i.e. deduction), or we teach them Euclidian geometry, does not state anything fundamental about the didactic problem: how this learning takes place.

The ultimate goal of teaching will have to be clarified by intermediate and temporary goals so that one will be able to understand the structure of geometry learning.

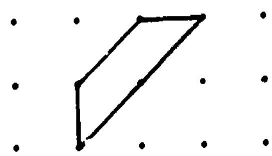
Information (Inquiry)

My first geometry lesson at the secondary school is information for me. It is a fact that man is able to perceive structure in almost any material however unordered it may be, and that this structure can be perceived in the same way by different people. This allows man to discover the intrinsic ordering in the material that is presented to him. For example, the knowledge of shapes is developed through manipulation of material objects.

Young children are already able to distinguish geometric figures. For example, they can make a square with a rubber band on the geoboard of Gattegno. They can make a square, positioned obliquely (see figure), when one indicates two consecutive points of this square on the geoboard. This geoboard is provided with nails at the location of the grid points.



If we make an isosceles trapezoid, they can make a trapezoid that has completely different measurements and that is in another position. Figures are recognized by their shape as a whole.



Reflection upon the manipulation of material objects, by taking the relations between those shapes as an object of study, can lead to geometry. This is the idea from which I start and I wish to know which shapes have already been differentiated in a geometric sense by the twelve-year-old pupils.

This becomes apparent during the first class conversations. I ask them, for instance, to tell me what regularities they perceive in a cube. I show them several cubes of different sizes. The equality of the plane figures as well as of the line segments and the right angles are always spontaneously mentioned. The parallelism and the similarity are not mentioned spontaneously.

In addition, I ask the pupils why they are so convinced that this equality exists. It appears that the pupils are proposing as a method, the action of "fitting." The equality that is being observed is not a result of measuring with the ruler, but of experiences gained previously through the stacking of building blocks, the making of puzzles, the drawing of the outlines of figures, etc.

The regularity that has been discovered in the shapes during the period preceding the secondary school has already established a global structure (i.e. not yet analyzed in a geometric sense) in the shapes such that it is possible to start with geometry lessons in the secondary school. The pupils themselves make the relation of equality explicit. For congruence is the means by which we order geometrically. Goal and means are clearly brought forward.

I know from the inquiry (first geometry lesson) that the pupils are capable of purposeful action. We can call these first class conversations the informative phase in the learning process of the pupils. They discover which aspect, out of the multitude of experiences they have already had, we are dealing with. The action of <u>fitting</u> allows the observation of <u>equalities</u> of parts in figures.

Directed orientation

The pupils are brought into a new phase of learning through these conversations. They now look at figures in a certain way. Figures are not only being folded, but in the process of doing so, one purposefully looks at what equalities of parts of the figure are being revealed.

During the preschool years children learn manipulations, e.g. folding, etc. These manipulations which we need have been sufficiently mastered by the pupils. They are now accompanied by a <u>more conscious perception in a geometric sense</u>. The pupils are actively engaged in cutting out figures and in subsequently checking in what way those figures fit in the openings; networks of known spatial figures are being made and these are checked by actually making the figure; it is also investigated in how many different ways plane figures can be folded in two through actual folding.

This period during which the manipulation is prominent and is being required of the pupils, we call directed orientation. The empirical experiences are broadened through manipulations. There is continual investigation of how one part of a figure can take the place of another part.

Explicitation

The results of the manipulation of material objects are now expressed in words. Equalities that have been observed are enumerated. Each child need not find out everything for himself. Subjective experiences are exchanged. In this way the figures acquire geometric properties. The theorems are expressed by the pupils. The role of the teacher here consists of introducing the necessary technical terms.

One could call this the objectification of the subjective experiences. It conforms to the customary "geometry" terms from the native language, in which the experiences are expressed. During the information-conversations, the pupils speak their own language. What is crucial in these conversations is to get to know what knowledge the pupils bring with them.

The goal of explicitation is to establish properties of figures. As a result, the shape as a whole becomes less important and the figure becomes a conglomerate of properties.

A rhombus then is a figure whose four sides are equal, whose opposite angles are equal, whose diagonals bisect the angles, whose diagonals bisect each other and form right angles with each other.

The properties are being read from a particular object, namely, a rhombus made out of paper that could be folded in two along each of its diagonals.

Materials

One could ask oneself whether it is necessary to start with the cube.

Once the teacher has thoroughly examined the goal of his first lessons, once he knows the function of the class conversations, he can vary his method.

A completely different presentation can be obtained by means of cut-outs. The pupils cut out the plane outline of spatial figures and glue these to make a solid. Then follow class conversations about these solids. The teacher remains in the background and listens, and hears what knowledge the pupils possess.

In each case one starts from whole entities in which something can be seen. For example, in the figure of two intersecting lines, initially no angles are perceived. It is nothing more than two lines. Geometric spectacles are needed in order to perceive elements in this figure. The material that can serve for a phenomenological analysis should possess a shape that is already differentiated in a geometric sense by the pupils. We call this didactic principle the experiencing of the context.

The function of the very first material can be described as follows: it has to be representative in the sense that it allows the context to become clear.

During directed orientation, which involves expanding empirical experiences, the teacher introduces new material. The function of the material then is such that it should be able to contribute to the discoveries of the pupils. Given the fact that the objective now is that the pupil purposefully searches for results of "fitting", we are not dealing with merely materials, but with material geared towards the task. The manipulations that have to be carried out with the material, place the concepts of congruence and of symmetry within reach of the child. During this phase of the learning process, the pupil is dependent on the ability of the teacher to find the appropriate tasks. It would be wrong to assume, as is sometimes done, that the adult has to be prepared to be a child among the children if he wants to provide adequate guidance to the learning process of the child during this phase.

The world of experience of the adult is completely different from that of a twelve-year-old pupil. The teacher sees and knows the objects in a way different from that of the pupil. The exploration of the teacher should concern the didactic approach -

by generating tasks, by creating favorable learning situations - but not the exploring of the material.

If he were to do so, he would run the risk of missing his goal. He would let the children discover all kinds of things which in reality would only be an imitation of what he himself discovered. Whether the child is learning depends on the wllingness of the child to carry out tasks with the material. When the teacher allows his pupils to use scissors and glue in the classroom, he will find that twelve-year-olds certainly have the inclination to explore.

A figure undergoes a metamorphosis as a result of the manipulations followed by a phenomenological analysis and an expliciting of its properties: it becomes what we call a geometric symbol. An abstraction has taken place.

The first intermediate goal during the initial geometry instruction is the formation of geometric symbols. They develop in the course of expliciting the results of fitting with insight. Fitting with insight is possible when it appears from the information-lessons that the pupil is capable of purposeful action.

Free orientation

By comparing symbols with one another, by searching for similarities and differences, the pupils orient themselves in the domain of symbols. For example, now they will begin perceiving a square as a rhombus because the square possesses all the properties of the rhombus. It is a special rhombus: a rhombus whose angles are right, or a rhombus whose diagonals are equal. At the same time, these two properties are experienced as inseparable from the rhombus "being square." The right angles of the rhombus necessarily imply the equality of the diagonals and conversely.

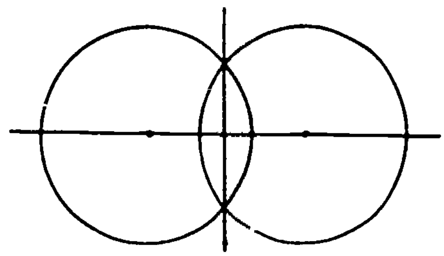
By working with figures, the pupils finally will recognize the figures by some of their properties. The symbols acquire signals, so that it becomes possible to build known figures, where a concept has been formed through analysis, out of their elements. During this phase there is not yet a real problem setting. It is rather an ordering of the manipulations that have to be carried out: I first have to do this, than that, in order to obtain the intended result. A new type of manipulation now develops: the drawing of figures of which elements are given or chosen. During free orientation the teacher appeals to the inventive ability of his pupils.

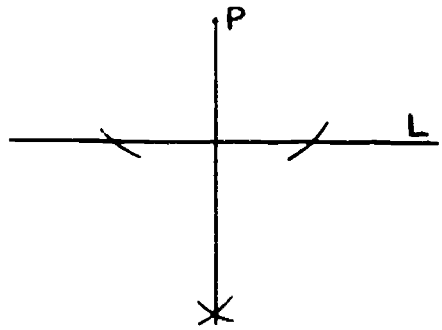
Empirical experiences can be expanded still more by joining figures that are already known, for example, by mirroring a triangle along one of its sides, or by rotating a triangle half a turn around the midpoint of one side.

Integration

Free orientation finally leads to being oriented in the domain of the symbols. The symbols then possess field characteristics. The operation "to fit" is reversible and associative. One knows beforehand whether it will fit without actually fitting it. The manipulations are understood, there is insight into the operation. The concept congruent has acquired a geometric context. Congruent triangles can be recognized and can be constructed. The concepts which one has formed of the figures play an important role.

Whether integration has taken place can be seen from the operation with the figure as a totality of properties. Example: How can you demonstrate that the lines connecting the points of intersection of two circles with equal diameters is perpendicular to the lines connecting the centers of both circles and also, how can you demonstrate that both connecting lines bisect each other? The answer is: because a rhombus can be seen in the figure.





Another example: How do you know that the construction (see figure) to drop a perpendicular from a point P to a line L is correct? The answer is: because a kite can be seen in the figure. This action already implicitly contains the deduction. The deduction however is not yet made explicit in this phase.

When integration has taken place, we say that the pupil has reached the first level of thinking. He then knows the domain as a totality of relations. Study is shifted from the shape as a whole to the network of relations that is now, as it were, stretched over the form of appearance. We can call these new figures "ideal" figures. They came about by taking the manipulations at level zero, the fitting, as the object of study. Because of that, this operation becomes reversible. The ideal figures, in turn, now determine the form of appearance. A clear example of this is the square: at level zero, a square is not perceived as a rhombus; at the first level of thinking, it is self-evident that a square is a rhombus. An analysis and a free orientation were necessary in order to arrive at this generalization from the particular object. The new opportunity to be able to operate with a figure as a totality of properties in fact means that the geometric symbols have lost their abstract character. The ideal figures of the first level of thinking are as concrete as the shapes as a whole are for someone at the zero level.

The second intermediate goal of the initial geometry instruction can then be formulated as follows: to learn to understand those symbols of the discipline that belong to the first level of thinking, namely, the aspect of geometry, and that allow geometric thinking about space.

Summary

The transition upward from the zero level to the first level of thinking is characterized by 5 phases:

1) information by means of representative material gathered from the existing substratum of empirical experiences in order to bring the pupils to purposeful action and perception;

2) directed orientation which is possible when the child demonstrates a disposition towards exploration and is willing to carry out the assigned operations;

3) explicitation through which subjective experiences are objectified and geometric symbols are formed;

4) free orientation which is the willful activity to choose one's own acitons as the object of study in order to explore the domain of abstract symbols;

5) integration which can be recognized as being oriented in the domain, as being able to operate with the figures as a totality of properties.

Substratum

Using this same scheme, we can also follow the exploration of the child, which is essential in order to provide the child with insight into the change of shapes, through which he is able to distinguish shapes.

1. Information here coincides with the discovery that the objects are movable.

2. In his orientation the child is restricted to his environment; he can only manipulate the objects that are within his reach.

3. A change in the manipulation takes place when the child becomes aware of the fact that certain situations recur as a result of his manipulations. He establishes a connection between his own manipulations and the result of these manipulations.

4. Then follows the willful activity: he desires certain situations to recur; his actions become goal-oriented.

5. After repeated trials and failures, a mastering of the

manipulation finally follows: the child is able to stack, to fold, to fit, etc. Figures can be recognized in various positions; they have a shape, i.e. they are not tied to a location, and they are distinguishable from each other.

This creates the possibilitity of starting to study geometry at the age of 12. Furthermore, at this age the native language is sufficiently developed to bring this implicit knowledge into the open in a class conversation.

Broadening the context through the introduction of new aspects.

The pupils have reached this first level of thinking after some 25 lessons. How does the learning process proceed from there on? There are two options: either to broaden the context by letting pupils experience other aspects drawn from the substratum, namely, parallelism of lines and similarity of figures, or to modify the context and to take the ideal figures as object of study, i.e. the network of relations itself. We first have to expand the context because one single aspect is too limiting for future problem settings. One single aspect leads to a certain structure which is experienced througl exploration. During that stage, a child is not surprised at whether or not figures fit each other. It is accepted as fact. The relations are discovered through manipulations and they determine the structure. Similarly, the falling of objects does not create astonishment. This can only become a problem when it is considered from a certain point of view. For this, the pupil will first have to have experienced the physical aspect.

We encounter this same phenomenon in didactics. The teacher who bases his instruction on one single aspect and who does not see other aspects, does not have a real didactic problem. His explorations lead him to the practico-practical stage; he arrives at a structure about which no doubt exists.

When, in didactics, several different aspects are not considered, this can lead to a structure that ought to be incompatible with one or more aspects. Apart from the aspect relating to the science of the discipline proper (which has always been considered), lately another aspect, one relating to developmental psychology, has come under consideration. In professional teaching circles one is at present looking for the overarching structure that includes both. At the meetings of mathematics and physical science teachers on April 14 of this year at Utrecht, Beth talked about the didactic consequences of the research into the foundations of the exact sciences.

His opinion is that the science of the discipline requires that the secondary school teacher perform a psychic (mental) intervention. This would mean that not only the mathematics teacher but virtually all teachers should perform such a psychic intervention. For why would one assume the physical sciences' thinking or the thinking of the humanities is more accessible than mathematical thinking? Geography and history classes deal with relations between facts rather than with the facts themselves.

If this intervention is valid for most subjects taught at the secondary school, this statement of Beth means nothing more than that all teaching requires a psychic intervention. Nobody will dispute the necessity of providing instruction. It is exactly by means of guided learning processes that the child can grow faster and reach a developmental level he could not attain without instruction. I cannot imagine that Beth would place mathematical thinking so high that only this discipline would require a psychic intervention.

If we were to arrive at conclusions on the basis of considerations of the discipline, that a particular psychic intervention appears necessary for the <u>learning of the subject</u>, then the didactitians, who no longer can ignore the aspect of developmental psychology in their own subject, will have to demonstrate the need for that intervention in the case of mathematics instruction and they will have to indicate how to perform it. Teachers have not learned how to perform this intervention during their study of the discipline, nor can it be logically derived from the science of that discipline.

In my opinion, it is not justified, from the point of view of developmental psychology, to perform a particular psychic intervention as long as the analyses of the learning situations, which would demonstrate the need for such an intervention, have not been carried out. We cannot be cautious enough with our interpretations. These interpretations need not come about in a logical way, but rather in a phenomenological way. Not clearly seeing the distinction between the levels of thinking, which results in one's not perceiving the gap between teacher and subject matter on the one hand and pupil and subject matter on the other hand, can lead to incorrect conclusions.

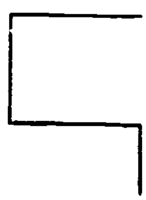
The tenor of this article is precisely to demonstrate how a gradual ascension to mathematical thinking is certainly possible. If instruction itself is already viewed as a psychic intervention, then "geometry-learning" does not need a particular invention once we, the teachers, have grasped the structure of this learning.

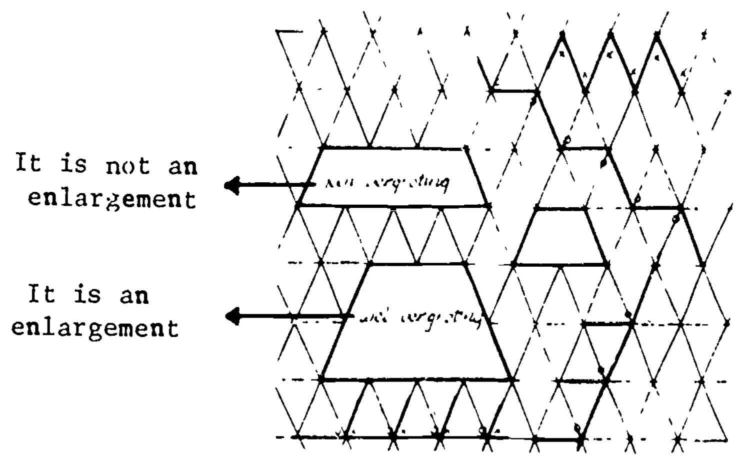
<u>Parallelism and similarity</u>

Let me now return to my method. I ought to make clear how I let the pupils experience the aspects of parallelism and similarity by drawing on the substratum and I also should indicate which empirical experiences have yet to be acquired. The pupils are at the first level of thinking. Because of this, they know that they have to search for relations.

The first task I give is as follows: draw part of a sidewalk paved with congruent square tiles whose side is 2 cm. One could also consider a tiled wall in the kitchen. After the pupils have collectively found three ways of doing that, I ask them what they observe in those figures. They discover, among other things, sets of parallel lines. The empirical experiences are broadened by providing them with bags that contain congruent regular triangles, pentagons, and hexagons respectively. They themselves formulate the tasks: "Attempt to pave a floor with these figures." Neat figures are drawn, in the process of which, close attention is being paid to the parallel lines.

"Ladders", "saws" and "enlargements" are discovered in these plane coverings. No definitions are constructed, but information is gathered. For example: "Is the (accompanying) figure also a saw?"





The answer is: "You can feel that in the movement of your hand. We follow a zig zag pattern, back and forth, always in the same direction."

Information concerning the meaning of an enlargement is gathered with the help of counterexamples the children know from experience. The image projected by a distorting mirror does not enlarge the figure equally in all directions.

During the phase of directed orientation, the children are asked to color angles of equal size in the plane coverings. The function of this coloring is to focus attention on the geometric structure of these plane coverings, to make the relations more visible.

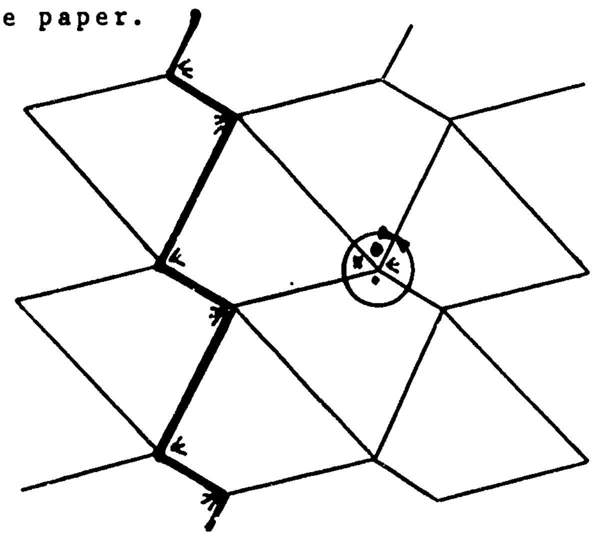
They finally can state explicitly: "Ladders have two sets of equal corresponding angles and one set of parallel lines; saws have two sets of parallel lines and equal angles. In a parallelogram, the opposite sides are equal and parallel, and the opposite angles are equal, etc."

One should avoid letting pupils ascertain equalities by measuring with ruler or protractor. For then the above statements are not generated by the pupil himself. The operation "to measure parts of different figures" is a control operation to test the correctness of hypotheses that are already present.

This phase of the learning process however deals precisely with the development of hypotheses. In addition, in mathematics one will not test hypotheses by measuring, but one will verify them through deduction. If, for example, one allows the pupils to ascertain the equality of the base angles of isosceles triangles by measuring them, we do not provide a good background for deduction to be used later. The folding of an isosceles triangle made out of paper and the fitting into the opening of a triangle that is cut out, and then reversed, provide a better background for the proof which makes use of the congruence concept. The generalization then rests on the idea that each isosceles triangle can be folded in half; the infinite repeatability of the action is accepted.

During the period of free orientation, plane coverings of congruent triangles and of congruent quadrangles are drawn. The plane coverings become structured geometrically by looking for ladders, saws, enlargements, axes of symmetry, etc. The plane coverings are compared with each other. A structuring-through of the figures takes place. The figures rhombus, quadrangle. square now also have parallel sides. The sum of the three angles of a triangle is 180 degrees; the angle sum for a quadrangle is 360 degrees.

Very frequently the children do not immediately discover the saws in the plane coverings with quadrangles. I made them move a pencil along the sides of an angle colored in red while not lifting the pencil from the paper.



During this period, saws acquire signals. They can be recognized either by equal angles, or by parallel lines. Ladder also possesses two signals. Because, at this moment, there is not yet a systematic search for characteristics, the pupils do not discover all the characteristics of a parallelogram. What is important here is that each child himself discovers signals as a result of his manipulations. This is shown by the fact that the children are able to select the data that will allow them to draw the figures.

The same applies to the drawing of enlargements. The cases of similarity have not yet been stated. The children however are able to draw enlargements of any figures. They know that all line segments have to be enlarged equally and that the angles remain the same.

The existence of integration thus is apparent from the fact that the pupil works with the figure as a totality of properties. Figures possess signals that do not all have to be stated explicitly but that appear to be present in the action.

The number of geometric symbols is increased, among other things, through the use of geometrically structured plane coverings. That the pupils know how to use these symbols, that they understand them, is evident when they show good intuition in finding the correct auxiliary line.

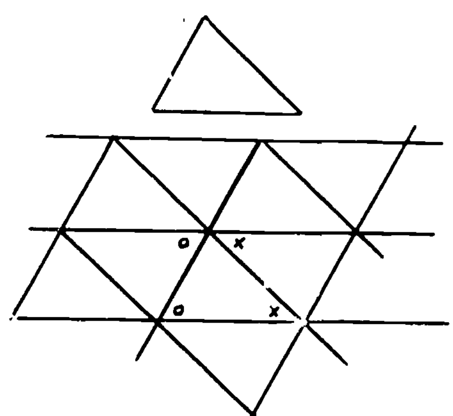
Modification of the context

This finding of the correct auxiliary line is important in the learning process where we modify the context. It then involves verbalizing the connections among the relations, the knowledge of which is implicitly contained in the action. The ordering of relations should now become a purposeful action. By starting from empirical experiences we can make the children become conscious of this purpose. During the informative phase of this learning process, I make use of the familiar structure of a genealogical tree in order to focus pupils' attention on how one relation follows from the other.

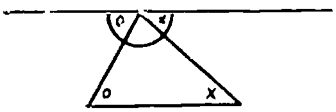
During a class conversations, the pupils collectively find how the statement "the sum of the angles of a quadrangle is 360 degrees" can follow from "the sum of the angles of a triangle is 180 degrees." (A protocol of these class conversations can be found on page of my dissertation.)

Another example. The pupils collectively look for the antecedent of "the sum of the angles of a triangle is 180 degrees."

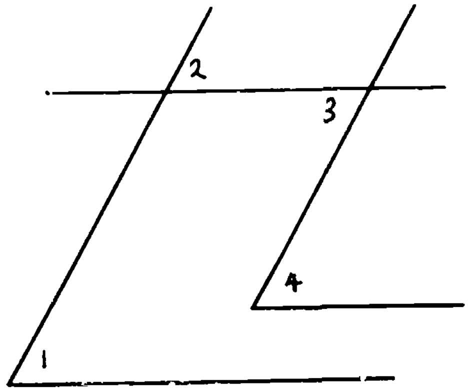
They themselves find the idea of drawing an auxiliary line through the vertex, parallel to the base. The single triangle is extended in their thoughts and viewed as an element in the plane covering consisting of triangles that are all congruent with the given triangle. From this they abstract what is necessary, i.e. one pair of parallel lines and two saws.



The freedom in the choice of the antecedents is also noticed, because the ladders as well as the saws are named by them as possible antecedents. This shows that the genealogical tree is viewed only as a global resemblance, as a metaphor.



When looking for the antecedents of the theorem "angles whose sides are parallel are equal," many pupils produced an auxiliary line which we do not normally draw (see figure). Antecedents are one ladder and one saw. Because a ladder can be seen, angle 1 = angle 2. Because a saw can be seen, angle 2 = angle 3 = angle 4.



During the first year I do not proceed beyond this experiencing context. The pupils further broaden their empirical experiences during the last trimester. They make solids that are bounded by congruent squares and congruent regular triangles, whose sides are equal. They do not get to see the model beforehand. The tasks become gradually more difficult.

The first assignment is: make a solid that is bounded by 1 square and 4 regular triangles. The fourth assignment however is as follows: make a solid that is bounded by 6 squares and 8 triangles. It is added that 2 squares and 2 regular triangles meet at each vertex, such that the triangles are opposite each other. New solids which they have never seen before are constructed.

In order to focus attention on the geometric structure, the pupils draw top, front and side views. In addition, the pupils learn to correctly depict the figures on a plane.

The function of this operation is dual. It aims at:

1) forming visual geometric structures, because there always has to be a purposeful search for parallelism and perpendicularity before drawing can begin;

2) promoting construction skills.

During the second year, a start can be made with the learning process which must lead the pupils to the second level of thinking. The description of this can be found in Euclides XXXIII, Gray 1958.

One can at once start with directed orientation. The pupils learn deduction. The figures become sets of <u>ordered</u> properties.

Levels of thinking

Work at the first level of thinking in geometry allows a continously increasing knowledge of space in a geometric sense. Apart from the knowledge of space in a geometric sense, the children learn to know space in a physical sense, in a biological sense, etc. Even though the objects of study are frequently the same, the difference between the disciplines is expressed during symbol formation.

Furthermore, the modus operandi, the method used to objectify the acquired experiences, is not the same in all disciplines. These differences should already become apparent through activities at the basic level, because they cause a differentiation into geometric thinking, physical thinking, etc.

As a result of sensory experiences that objects are movable and can take each others' place, the pupils have become familiar with the aspect of form. Forms are recognized by their shape and, as a result, name giving is possible. At the basic level of geometry, purposeful action takes place in order to arrive at the essence, the structure, of the forms. Because we introduce manipulation (fold-ups, fitting-in, etc.) in the guided learning process, the children arrive at a discovery of the structure of the forms.

In order to be able to arrive at an exchange of sensory perceptions, we attempt to record the structure in language symbols. This leads to purposeful acting in order to reproduce the results of fitting in the form of geometric properties. The names of the forms now have acquired a content; they have become geometric symbols. Orientation leads to recognition.

We therefore call this first level of thinking, the aspect of geometry. The purposeful fitting hereby has changed into geometric thinking. It is a thinking that is oriented towards the result, the structure, the properties of the forms, but it is also a thinking that is directed towards the form of thinking itself in order to arrive at the structure of thinking.

In order to penetrate into this structure, the essence of geometric thinking, goal-oriented thinking, again first has to be objectified by means of language. We accelerate the learning process, for example, by introducing the geometric symbol "to follow from" by means of a genealogical tree, itself drawn from the substratum. After the structure of geometric thinking has been made explicit, a purposeful deduction can take place. The results are recorded in an ordering of the geometric relations. We call this second level of thinking: the essence of geometry.

The practical results of this second level of thinking are that one possesses insight into the appearance of forms in space. The determination of the structure of the forms now is an application of geometric thinking. The structure no longer need be approached by means of thought-filled action, by means of fitting, because the properties can be determined by means of deduction.

The second level of thinking however has theoretical results as well, because it implies a modification of the context through the appearance of a new aspect. As purposeful deduction finally becomes a thinking habit, the operation has become reversible and associative. Deduction is recognized, takes on shape, becomes a symbol in the new context; the mathematical aspect becomes prominent. In summary, we could characterize geometric thinking at the second level of thinking as follows.
It is the thinking that:

1) unveils the structure of geometry;

2) leads to insight into the forms; (This becomes evident because the thinking habit brings with it concrete action.)

3) brings out the aspect of geometry.

Mathematical thinking now acquires structure because a new way of thinking has become purposeful.

The third level of thinking is analogous:

1. The structure of mathematics is revealed.

2. Insight into geometry evolves because the thinking habit involves concrete action with a geometric form of thinking.

3. It brings out the aspect of logic.

At the fourth level of thinking, "exact thinking" provides:

1) the structure of logic;

2) insight into mathematics because the thinking habit implies concrete action in mathematics.

By making these distinctions, the object "level of thinking" becomes a symbol; it acquires structure for us. It becomes an applicable concept.

If we now focus our attention on this structure, we can conclude that there has to be a "fifth" level of thinking, i.e., insight into the subject logic.

The structure becomes more obvious when I investigate the results for a person who masters the successive levels of thinking. When a form of thinking has been mastered, then the structure of this thinking becomes so familiar that it is replaced by a routine action. One could say, it proceeds almost automatically, associations have been formed. The substratum becomes richer.

In the course of the guided learning process, we can notice that a time comes when the pupil has mastered the symbols of the technical language that belong to geometric operations. Later on, the pupil will have mastered the symbols of the technical language that belong to geometric thinking. The substratum is richer because one can talk of mathematical perception.

Later still, he knows how to think geometrically. A goal-oriented operation or a mathematical method, namely, deduction. When the person has reached integration, the operation takes place almost automatically, the thinking process proceeds quickly. As a result of practice, formation of associations has taken place. The thinking is brought back to observation and to concrete operations. The substratum contains new perception structures and new operations. One sometimes refers to it as a "sixth" sense. The intent can be disparaging: he is a real mathematician (from a sensory point of view), or it can be praising: he is a genius (where one stresses the thinking).

In order to arrive at the fourth level of thinking, three modifications of the context have been necessary, after the first context has been brought into consciousness.

He who is capable of working productively on the fourth level of thinking, can try to pattern geometry according to an axiomatically built system of theorems. He thereby takes an a priori point of view. When he becomes conscious of this viewpoint, he knows that aspect of his own actions that bears on the psychology of thinking.

If the logician remains operating from a mathematical point of view, he can approach the subject along a mathematical train of thought. This, however, does not bring his thinking to a higher level of thinking. An a priori point of view implies that exact thinking is the highest form of thinking.

When the logician omits considering the aspect of the psychology of thinking when investigating the foundations, he will not succeed in stating the essence of logical thinking even though he experiences a mathematical thinker within himself (his substratum is rich). Many mathematicians wrestle with seeming problems because they exclude the psychological aspect, namely, personality.

The question of whether mathematics is only based on a thinking operation is an apparent problem. Thinking and operation can be distinguished, but they cannot be separated. Concept and language can be distinguished, but cannot be separated.

The thinking operation itself first has to be made conscious through language symbols and the language symbols are a consequence of the thinking operation. To be sure, the objects of study of a logician are the thinking operations of a mathematical thinker, but his information has to be acquired by means of sensory perception, i.e. only a mathematical thinker can arrive at such a study.

The background of the analyses, in the course of the investigation of the foundations, will precisely have to be the mind of the human being. Exact thinking will then acquire structure.

The addition of the language symbols "aspect" next to "object" and "structure" next to "system" could provide clarification, if one were to delve beneath the surface to the concepts of these symbols and if one were to record them in a definition. Perhaps this would provide content for the "operational definition" concept. This can be a positive contribution of logic to the test psychologist.

The logician will have to proceed to a new form of thinking. For him, this represents an exploration from his substratum. This is the opposite of the deductive method with which he is so familiar. Induction is required of him!

For those who do not exclude the aspect of the psychology of thinking as a means of ordering, new perspectives open up. For them, it is evident that there are logical and non-logical statements. It is exactly through this that the logical acquires shape. An analysis in this sense opens up the possiblility of reaching the fifth level of thinking.

PART III

1. ENGLISH SUMMARY OF DISSERTATION
   by Pierre van Hiele

2. THE CHILD'S THOUGHT AND GEOMETRY
   by Pierre van Hiele

ENGLISH SUMMARY
by
Pierre Marie van Hiele

of

THE PROBLEM OF INSIGHT IN CONNECTION WITH SCHOOL CHILDREN'S INSIGHT INTO THE SUBJECT MATTER OF GEOMETRY

In this study we have tried to examine the meaning and functions of insight during a process of learning. In view of the subject's extent we had to limit the scope of our study somewhat. For that reason we have confined ourselves to the study of mathematical insight in general and of geometrical insight in particular.

A teacher first meets evidence of his pupil's insight when, as a result of the process of learning, the latter reacts adequately to situations that were not included in this process. This inference, however, is admissable only if the pupil's action has been executed throughout with deliberate intent, i.e. if we can safely assume that no chance element has played a part in it.

The above definition of insight corresponds well with that of the Gestalt theory. 'To act as a result of insight means: to act on the strength of a structural achievement'. This definition links up observation and thought and though initially we may interpret it in a purely figurative sense we shall see later on, when we come to a closer analysis of the concept of thought, that it hides a far deeper meaning too.

An analysis of rational thought will reveal three importnat moments in it:

1. the forming of structures;

2. the forming of valencies [associations], (as when, for example, learning 'by heart');

3. analysis (see Selz's research).

The above-mentioned forms of thought are fundamentally dissimilar and demand an entirely different personality adjustment. Analysis is an action requiring an 'active' adjustment, as it occurs under the continous pressure of an individual's 'intent.' The forming of valencies [associations], on the contrary calls for a 'receptive' adjustment - it is an autonomous process. Structural formation, finally, demands the ability of rapid mental switchovers from a receptive to an active adjustment and vice versa: receptive in its acquiescence to the absorption of the 'spontaneous' structures emanating from the material; active in its concentration on the analysis and modification of these structures, once they have been formed.

We could classify the structural types as follows:

1. Structural expansion.
2. Structural refinement.
3. The construction of superstructures.
4. The transition to isomorphic structures.

Each succeeding structural form demands a 'higher' insight than the preceding one.

The creation of a structure demands two basically distinct acts of thought: the identification of its components, and their classification.

We can distinguish two types of identification: undifferentiated identification and identification following analysis of the object. The former leads to an undifferentiated structure, the latter to a structure based on analysis.

The analysis of an object enables us to abstract and eliminate a certain number of its conceptual moments. The resultant conceptual impoverishment will lead to new forms of identification and thus to new structures.

In reality, of course, the study of geometry concerns itself mainly with the examination of structures 'as such,' after they have been abstracted from the object. Consequently, undifferentiated structures cannot be called truly mathematical, and this applies equally to the type of insight that produces them.

The study of the classifying principles of interrelated structures will sooner or later lead to a building up of the classifying principles themselves. At first these structures will be undifferentiated but they are likely to lose their original form when analysed. The result will be a new 'higher' structure, embodying the classifying principles of the original ones.

This is an entirely new process of thought: we call it the 'transition to a higher level of thinking.' This transition can only be effected if we have accumulated enough symbols leading to this new level (i.e. after so many concepts have condensed into the symbols that we can use the latter to guide us in our study).

In mathematics, and in particular in geometry, it is easy to follow this trend. The presentation of concrete (study) material evokes visual undifferentiated structures. Children become familiar with these structures fairly early in life, long before they reach the level of secondary education.

It is important to define the geometric aspects that crop up when we study a concrete foundation. They are:

a. The perception and recognition of geometric patterns.
b. The division of plane and space.

c. The use and disposition of congruent patterns.
d. Similar patterns.
e. Stacking of patterns.
f. Transformation of patterns.
g. Symmetry with respect to a plane.
h. Symmetry with respect to a line.
i. Symmetry with respect to a point.
j. Surface and content.
k. Movements in space: translation, rotation, screw.
l. Curves.
m. The fact that mirrors need not produce congruent images.
n. The plane projection of spatial patterns.
o. The intersection of patterns.

All these aspects are significant in geometrical practice. It is therefore important to give them proper consideration when we compile our syllabus.

The teaching of geometry could be interpreted as the realization of a two-fold aim:

1. The study of geometry gives the pupils a definite angle of approach to, and understanding of, the characteristics of space and this shows them how to achieve a certain domination over space.

2. Nowhere else will the pupils find a better chance:

   a. to create a logical and coherent scientific system;

   b. to develop their capacity for the acquisition of knowledge, not by practical experience but by the application of pure thought.

If, in his geometry class, the teacher sufficiently stresses the fifteen aspects mentioned above, he will undoubtedly achieve the first aim.

Realization of the second aim, however, largely depends on the amount of attention paid to the various levels in geometrical thinking.

A pupil reaches the first level of thinking as soon as he can manipulate the known characteristics of a pattern that is familiar to him. For instance: if he is able to associate the name 'isosceles triangle' with a specific triangle, knowing that two of its sides are equal, and to draw the subsequent conclusion that the two corresponding angles are equal.

As soon as he learns to manipulate the interrelatedness of the characteristic of geometric patterns he will have reached the second level of thinking, e.g. if, on the strength of general congruence theorems, he is able to deduce the equality of angles or linear segments of specific figures.

He will reach the third level of thinking when he starts manipulating the intrinsic characteristics of relations. For example: if he can distinguish between a proposition and its reverse.

Provided the process of teaching lasts long enough, the symbols used in it will progressively lose their original significance until finall their only functi n will be that of junctions in a network of relations. Usually this network will be coherent in a meaningful way though the relations themselves have been determined mainly by valencies, formed during the process of learning.

This sequence of events is precisely what we need in geometry, as the network of relations is, actually, what we are aiming at in our syllabus. However, as teachers we always ought to remember that this network of relations should emerge during the process of learning and from the concrete situations. Only then we can expect the growth of a reversible structure, i.e. a structure in which the children can find their way back from abstract relations to concrete situations. This is what we need if we are aiming at a productive application of geometrical knowledge.

Unfortunately the teaching of geometry frequently follows other ways: much too often the teacher tries to instill a direct knowledge of the network of relations without having passed through the intermediate stage of concrete situations. At the first glance this way of tackling the subject can be quite successful, and it is true that factual knowledge about the network of relations is often acquired in less time. It has the disadvantage, however, of largely neglecting the important matter of the basic substructures; the analytical link with concrete (visual) structures will be absent, for the pupil will have created his entire network of relations by an imitative process incited by the teacher's structural exposition. As the resultant structures have little mutual cohesion we cannot expect much transfer, neither within the field of geometry itself, nor with respect to other spheres of activity.

By now it is clear that the teacher of geometry faces a double task:

a. he should help his pupils to transform the structures, produced in their visual field of observation, into geometrical structures;

b. he should teach the children the use of the algorithms in several parts of mathematics.

The understanding of algorithms depends on a special kind of insight, which we call algorithmical insight.

In addition to this we find a more general insight, based on structural forms that are able to be developed into algorithms.

And finally there are those levels of thinking that permit insight into completely new principles of thought.

The teacher will not find it easy to ensure an adequate development of these two latter types of insight within the bounds of classroom tuition. For one, the usual test-methods and examinations do not work in this respect: these types of insight cannot be adequately assessed by means of test-papers. If the test problems cover a limited field of acquirements, algorithmic skill will suffice to solve them. It is a fairly simple matter to teach a child a number of manipulative structures that enable him to reduce a high-level problem to a lower level of thinking.

We could avoid this handicap by embodying the whole range of available subjects into our test-papers. As the pupils could hardly have memorized all the algorithms they would be unable to reduce the problems to a lowe. level. But even then it is doubtful whether they would manage to solve them, for though some of the pupils will undoubtedly possess the structural understanding needed to develop algorithms, the average span of time available for a test-paper will rarely cover the process of thought involved in it.

The result is that both examinations and test-papers tend to push the pupil towards algorithmical insight instead of leading him on towards the far more valuable higher forms of insight.

This does not imply that it is impossible to perceive and assay these two higher forms of insight. If the teacher-pupil relationship is based on confidence, then the pupil's reactions will show the teacher how, and to what extent, he is absorbing and digesting the subject matter. Once we know which level the pupil has reached, we can learn by a painstaking analysis of the process of learning, how to bring about a further increase of insight.

It is highly important to know how the child itself experiences insight. The acquisition of insight into the many spheres of matter that lie within the range of the dealings and aptitudes of a human being is one of the basic necessities of life. Moreover, our own inner urge impels us to it: the conciousness of acquired insight is a memorable inner experience and gives us a feeling both of power and of safety. If, from term to term, we see no sign of developing insight, then we may safely assume that the child has no contact with the subject matter. There can be many reasons for such a negative approach. We shall name three only: - maybe we are presenting the subject matter in too small separate units that do not have enough self-evident mutual cohesion; maybe we are operating on a level of thinking that is beyond the pupil's understanding and, thirdly and lastly, the subject matter itself may have no bearing at all or the child's own world. However, if we constantly remind ourselves to base our presentation of the subject matter on the firm foundation of visual material, then there will be little danger of the child's losing contact with it.

THE CHILD'S THOUGHT AND GEOMETRY

by P.M. van Hiele

The art of teaching is a meeting of three elements; teacher, student, and subject matter. Since it is very difficult to keep all of these things in view at the same time, one has a tendency t neglect one of them, which gives an incorrect view of the situation. Because if one neglects subject matter, one only sees the relationship between teacher and student; if one loses sight of the student, then one only sees the structure of the subject matter. Sometimes one does not sufficiently realize that the teacher is there to direct the student's studies.

Nevertheless let us acknowledge for the sake of argument that one should take into account the three aspects mentioned above without omitting any of them. There remains nonetheless a great danger and it appears to me that it has not been sufficiently recognized. The difficulty which arises is that the subject matter as met by the student is of a completely different structure from that known by the teacher.

If we agree that the aim of our teaching is that the student should know how to prove theorems, it is highly improbable that the student's thought aims directly towards this goal. Improbable, because the student will not be able to grasp, in its intrinsic sense, the idea of proving a theorem. In fact, if he had this idea, he would not have the need to learn it. Understanding mathematics comes down to this: knowing the relationships between theorems that one studies. As soon as one understands the meaning of these theorems, one knows their relationships at the same time.

All this is very simple and shows us clearly why mathematics is so difficult for students. The teacher knows the relationships between the theorems, but he knows them in a different way than the student. His explanation of these relationships does not suffice to make them intelligible to the student. What the student must understand in the first place is that there are such things as theorems. This is all that one can expect from a beginning student. The following example will illustrate what I mean.

A teacher wants to teach plane geometry to beginning students. He uses symmetry with respect to a straight line, in order to teach them the relationships between equality of segments or angles, perpendicularity, etc. He teaches them that the points on the axis of symmetry are invariant, that symmetric segments have the same length, that symmetric lines intersect on the axis of symmetry. In order to see if the students have understood what he has taught, he gives them the following problem: "Let ABC be a triangle for which the extensions of the sides meet the line L. Construct the symmetrical triangle with respect to L." The teacher imagines the following solution: "The lines AB and AC meet the axis L in two points that we will call P and Q. These points are invariant under symmetry. Then the distances AP and AQ are invariant, so that one can construct the symmetrical point A'. In the same way one finds the points B' and C'."

All of this reasoning, this whole way of conceiving the material, is the reasoning of a teacher who knows all the relationships. The student is completely incapable of developing a similar process of thought without the teacher's help. The teacher has used the fact that the lengths of symmetrical segments are the same as the basis for his argument. Such a technique is meaningless for the students because they have not yet seen a counterexample; they have not yet seen transformations which change the length of segments.

But there is a more important reason for us to oppose this method of teaching of which we have given an example: it requires students to reason with the help of a system of relations between ideas whose meanings they do not even know. It is a matter of "points," "axis of symmetry," "segments," "to meet," "invariant," "to change length," "triangle," "extension." Obviously, the teacher has explained these expressions, he has shown points and segments, he has demonstrated at the blackboard what is meant by extending a segment. Possibly he has asked the students to formulate the definition of vertical angles. It is even possible that the definition was found to be not quite correct and that as a result he showed this by means of a counterexample. One must realize however that it is the teacher who is giving the counterexample. The students would fail because to be in a position to give a counterexample one must have a system of relations at one's disposal, and they do not have one.

I hope that the thoughts that I have just presented to you will have clearly shown that the teacher reasons by means of a system of relations that he alone possesses. Starting with this system, he explains the mathematical relations that the students end up manipulating by rote. Or else the student learns by rote to operate with these relations that he does not understand, and of which he has not seen the origin.

At first glance things seem to be in order: the students will end up having at their disposal the same system as the teacher. Is this not the proper goal of the teaching of mathematics, namely: the possession of a system of relations identical for all those that use it, appropriate to express arguments, a system in which the relations are linked in a logical and deductive fashion?

Let us not be too optimistic. First of all, a system of relations structured in this way is not based on the sensory experiences of the student. Though it is possible that the system of relationships itself has inspired some experiences on the part of the student, the mathematical experiences that the student has been able to have are based only on the system imposed by the teacher. This system, imposed and not understood, forms the base of his reasoning. As one knows, a system of relations which is not based on prior experience has the potential of being forgotten in a short time.

Therefore the system of relations is an independent construction having no rapport with other experiences of the child. This means that the student knows only what has been taught to him and what has been deduced from it. He has not learned to establish the connections between the system and the sensory world. He will not know how to apply what he has learned in a new situation.

Finally, the student has learned to apply a system of relations that has been offered to him ready-made, he has learned to apply it in certain situations specifically designed for it. But he has not learned how to construct such a system himself in a domain which is still unstructured. If, on the other hand, we were to succeed in ensuring as a result of our teaching that the students are capable of constructing for themselves a deductive relational system in a new domain, we would have produced the optimal mathematical training.

In general, the teacher and the student speak a very different language. We can express this by saying: they think on different levels. Analysis of geometry indicates about five different levels.

At the Base Level (Level 0) of geometry, figures are judged by their appearance. A child recognizes a rectangle by its form and a rectangle seems different to him than a square. When one has shown a six-year-old child what a rhombus is, what a rectangle is, what a square is, what a parallelogram is, he is capable of reproducing these figures without error on a geoboard of Gattagno, even in difficult arrangements. We have used the geoboard in our research so that the child will not be bothered by the difficulties resulting from drawing figures. At the Base Level, a child does not recognize a parallelogram in the shape of a rhombus. At this level, the rhombus is not a parallelogram, the rhombus seems to him a completely different thing.

At the First Level of geometry, the figures are bearers of their properties. That a figure is a rectangle means that it has four right angles, diagonals are equal, and opposite sides are equal. Figures are recognized by their properties. If one tells us that the figure drawn on a blackboard has four right angles, it is a rectangle even if the figure is drawn badly. But at this level properties are not yet ordered, so that a square is not necessarily identified as being a rectangle.

At the Second Level properties are ordered. They are deduced one from another: one property precedes or follows another property. At this level the intrinsic meaning of deduction is not understood by the students. The square is recognized as being a rectangle because at this level definitions of figure come into play.

At the Third Level, thinking is concerned with the meaning of deduction, with the converse of a theorem, with axioms, with necessary and sufficient conditions.

One can probably thus distinguish five levels of thought in geometry. This number is moreover of little importance in understanding what a level of thought is.

These levels - as we have said - are inherent in the elaboration of thought; they are independent of the method of teaching used. It is possible, however, that certain methods of teaching do not permit attainment of the higher levels, so that the methods of thought used at these levels remain inaccessible to the students. The following points can contribute to a specification of levels of thought:

a. At each level there appears in an extrinsic way that which was intrinsic at the preceding level. At the base level, figures were in fact also determined by their properties, but someone thinking at this level is not aware of these properties.

b. Each level has its own linguistic symbols and its own system of relations connecting these signs. A relation which is "correct" at one level can reveal itself to be incorrect at another. Think, for example, of the relation between a rectangle and a square. Numerous linguistic symbols appear at two successive levels; moreover they establish a liaison between the various levels and assume continuity of thought in this discontinuous domain. But their meaning is different: it becomes manifest by other relations among these symbols.

c. Two people who reason at two different levels cannot understand each other. This is what often happens between teacher and student. Neither of them can manage to follow the thought process of the other and their dialogue can only proceed if the teacher tries to form for himself an idea of the students' thinking and to conform to it. Some teachers make a presentation at their own level while asking students to reply to their questions. In fact, it is nothing but a monologue, for the teacher is inclined to consider all the answers which do not belong to his system of relations as stupid or misplaced. A true dialogue must be established at the level of the students. For this to happen, the teacher must often, after class, ask himself about the responses of his students and strive to understand their meaning.

d. The maturation which leads to a higher level happens in a special way. Several stages can be revealed in it (this maturation must be considered above all as a process of apprenticeship and not as a ripening of a biological sort). It is thus possible and desirable that the teacher aids and accelerates it. The aim of the art of teaching is precisely to face the question of knowing how these phases are passed through, and how help can effectively be given to the student.

Let us now examine the phases which, in the process of apprenticeship, lead to a higher level of thought.

The first phase is one of inquiry: the student learns to know the field under investigation by means of the material which is presented to him. This material leads him to discover a certain structure. One could say that the basis of human knowledge consists of this: mankind is characterized by the revelation of structure in any material, however disorganized it may be, and this structure is experienced in the same way by several people, which results in a conversation that they can have about this subject.

In the second phase, that of directed orientation, the student explores the field of investigation by means of the material. He already knows in what direction the study is directed; the material is chosen in such a way that the characteristic structures appear to him gradually.

In the course of the third phase, explicitation takes place. Acquired experience is linked to exact linguistic symbols and the students learn to express their opinions about the structures observed during discussions in class. The teacher takes care that these discussions use the habitual terms. It is during this third phase that the system of relations is partially formed.

The fourth phase is that of free orientation. The field of investigation is for the most part known, but the student must still be able to find his way there rapidly. This is brought about by giving tasks which can be completed in different ways. All sorts of signposts are placed in the field of investigation: they show the path towards symbols.

The fifth phase is that of integration; the student has oriented himself, but he must still acquire an overview of all the methods which are at his disposal. Thus he tries to condense into one whole the domain that his thought has explored. At this point, the teacher can aid this work by furnishing global surveys. It is important that these surveys do not present anything new to the student; they must only be a summary of what the student already knows.

At the close of this fifth phase a new level of thought is attained. The student has at his disposal a system of relations which are related to the whole of the domain explored. This new domain of thought, which has acquired its own intuition, is substituted for the previous domain of thought which had a completely different intuition.

The objectivity of mathematics rests on the fact that new systems of relations are agreed on by different people. The new symbols are linked by the same relations among many people. If one decides that the goal of education should be the uniqueness of the relational system, one could restrict oneself to having that learned. And the student would seem to understand the reasoning perfectly, for it would result in correct conclusions based on his relational system. But that is not to say that he would attach to it the same significance as his questioner. This significance cannot be disentangled solely from the language used, it depends also on the experiences which led to the formation of the relational system, that is, it depends on what happened at a lower level of thought.

If one does not take the content of the symbols into consideration, but only their relations, one could say that from a mathematical point of view, everything is perfect. The student is capable of handling the relational system of deduction without mistakes. But from the pedagogical and didactic point of view, and from the social point of view, one has wronged the student! One has committed a pedagogical error because one has stolen from the student an occasion to realize his creative potential. From the didactic point of view, one has neglected to let the student discover how to explore new domains of thought by himself. Finally, one has wronged society because one has provided the student with a tool which he can handle only in situations which he has studied.

The theory of levels of thought leads to the following important conclusions.

1. One has been able to see that the levels of thought are inherant in thought itself; thus they are not only the concern of those who occupy themselves with didactics. The levels of thought have, for example, a certain importance for mathematics itself. One can only express onself clearly in mathematics when one uses symbols belonging to one's own level. If one manipulates functions, it is of little importance that they are defined by the expression f(x) or by the equation y = f(x). One learns to know the function while using it and out of this activity flows the content of the notion of function. If one asks oneself, at a higher level, the questions of what a function is, of <u>what one has really done</u>, one will arrive at the conclusion that it is a pairing of elements x and of elements f(x). The function is defined neither by f(x), nor by y = f(x), but rather by the symbol for the pairing which one can represent, if one wants, by f. Error results from trying to give a definition at a lower level of thought, from exploiting a structure contained implicitly in an activity before it has become sufficiently familiar. Because this attempt is doomed to failure, one limits oneself to representing either the result of this action, f(x) - or else the action itself, y = f(x). The mistake is not only a didactic one, but also theoretical. (This example is drawn from a conversation with Professor Freudenthal.)

One makes an analogous error when one tries to construct a system of axioms using symbols which belong to a level of thought which is too low. Systems of axioms belong to the fourth level where in fact one no longer asks the question: what are points, lines, surfaces, etc.? At this fourth level, figures are defined only by symbols bound by relations. To find their appropriate content, it is necessary to return to lower levels where the content of these symbols can be perceived. But with this content, these symbols belong to a relational system which cannot be axiomatized because it cannot have direct liaison with logic.

2. Just as a child only learns his native language by applying grammatical rules (which are deduced from current usage), he only learns mathematics by applying mathematical rules. These rules only become firm, that is, become explicit, when one questions oneself about activities displayed at a lower level. It is in this way that all mathematical rules are formed, even the rules of formal logic. The application of rules is important, but the rule of application resides above all in the exploration of new domains bordering those where the rules and laws have been developed.

3. Two or more people can understand each other in a specified area of thought when they use a language in which they experience the same relations between the linguistic signs. The certainty of mathematics is based on the infallible way in which mathematical language can be used. The "mathematician at any price" can be happy with this: LANGUAGE is everything for him and he hardly cares what a symbol represents for others. (Just think of the point-line duality in the projective plane!) There is no problem from the algorithmic point of view. But if one is also concerned with knowing if agreement will still occur when the field of investigation is broadened, it is desirable to examine whether the symbols used by the questioner have a common base. It will not suffice then to learn the linguistic symbols and their liaisons, but it will be necessary to start with the same material at the lower level and to see if one succeeds, starting from there, in developing the same domains of higher level symbols.

Description of a geometry course.

The first part of a geometry course ought to allow the attainment of the first level of thought, which we will call the aspect of geometry. The aim of teaching is as follows: geometric figures such as cubes, squares, rhombuses, rectangles, circles, etc. should become bearers of their properties. A rhombus is no longer recognized by its appearance, but, for example, by the fact that the sides are equal or that the diagonals are perpendicular and bisect each other, or these two properties together.

One uses a collection of concrete geometric figures and materials with which students will themselves make models of the figures. The manipulations which the students perform with this material will be the base of a new relational system in the process of formation.

The second part of the course should allow the attainment of the second level of thought, which we will call the essence of geometry or the aspect of mathematics. The aim of instruction now is to learn the relations which link properties of figures. For example, the sum of angles of a triangle is 180 degrees; the alternate interior angles formed by two parallel lines and a transversal are equal. What is more, one begins, during this period, to order properties of figures logically. The first property mentioned above becomes antecedent to the following: the sum of the angles of a quadrilateral is 360 degrees.

Material could consist of a series of congruent triangles or quadrilaterals with which students could try to construct a paving. Here again, students learn to uncover a structure while manipulating a material. In a paving constructed from congruent triangles, they see systems of parallel lines, parallelograms, trapezoids, hexagons with their centers of symmetry, etc. appear. This material later suggests in a natural way the auxiliary line needed to show that the sum of angles of a triangle is 180 degrees, using the method of alternate-interior angles.

The third part of the course should allow the attainment of the third level, that of discernment in geometry, or the essence of mathematics.

The aim of instruction is now to understand what is meant by logical ordering (what do we mean by: One property "precedes" another property?).

The material is made up of geometric theorems themselves. In the ordering of these theorems certain ideas will become apparent, namely: the link between a theorem and its converse, why axioms and definitions are indispensable, when a condition is necessary and when sufficient. Students can now try to order new domains logically, as for example when they first study the cylinder. Analysis of what they see will teach them that the cylindrical surface contains lines and circumferences. After having stated a definition, they will be able to try to prove the existence of lines and circumferences.

If the course could be continued further (which is generally impossible in general education), the fourth level would be attained, that of discernment in mathematics. The aim of teaching at this level would be to analyze the nature of a mathematician's activity and how it differs from the activity displayed in other disciplines. One cannot attain this fourth level until one is sufficiently familiar with the procedures of mathematicians that one can do them automatically. One must form within oneself associations such that one step induces others. And it is only when these steps can be integrated that one can grasp the structure of mathematical activity.

But a similar integration takes place at the time of transition from one level of thought to a higher level. In the course of passage from the base level to the first level, it is manipulation of figures which gives birth to structure. This nourishes thought at the first level. Thus the figures become new symbols defined by their relations with other symbols.

At the first level the context is different from that of the base level. Action developed in this new context furnishes an integration which makes access to the second level possible, and so forth.

The teacher who deliberately strives to lead his students from one level to another, gets them ready to develop a deductive system by themselves and to uncover faults in a deductive argument. Acting this way, the teacher does not impose domains where thought should be practiced, but helps the students to specify them on their own. This does not mean, as has already been stated above, that he will leave the student the burden of discovering everything, but that he will require from the student some particular activity which in each of the five stages is directed in a different way. Application of these principles will surely not mean a lightening of the task of the teacher. But he will have the satisfaction of knowing what he is doing and of understanding his students' reactions better.

Teaching a deductive system requires patience above all. This system only exists at the third level of thought, and its essence is only perceived at the fourth level. It may appear tempting to build geometry from transformations of the plane, but then one again has the aim of building a deductive system. One cannot confuse this construction with the elaboration of geometric thought. If one takes transformations as a departure point, one is already supposing the existance of a pre-existent domain of thought. Someone who is too hasty reduces his own domain of symbols, with children, to intuitive symbols which belong at a much lower level and which do not have the meaning he gives to them. Under these circumstances, one is forming algorithmic types, that is, minds capable of applying algorithms in a satisfactory way without their knowing their content sufficiently. In this case one is teaching the material without sufficient formative value.

When one directs instruction too rapidly towards a mathematical relational system because one disdains teaching the geometric relational system, one risks losing mathematics forever. What one gets is a verbal relational system in which new operations are impossible. One finds an example of such an error in the teaching of fractions in Holland. In this instruction, a verbal relational system is established. For most of the students, operations with fractions are completely incomprehensible. If in teaching the teachers only recognized that the relational system of the students is more valuable than that of the teachers!

The heart of the idea of levels of thought lies in the statement that in each scientific discipline, it is possible to think and to reason at different levels, and that this reasoning calls for different languages. These languages sometimes use the same linguistic symbols, but these symbols do not have the same meaning in such a case, and are connected in a different way to other linguistic symbols. This situation is an obstacle to the exchange of views which goes on between teacher and student about the subject matter being taught. It can perhaps be considered the fundamental problem of didactics.

Footnote:

1. Dr. P.M. van Hiele, professor at the Lycée de Bilthoven (the Netherlands), is the author of a thesis on the problem of intuition (in particular on the role of intuition in the teaching of geometry). This thesis was defended before the University of Utrecht on July 4, 1957.

   Dr. van Hiele delivered this paper to the conference "pilot course on the teaching of mathematics", organized by O.E.C.E. at Sèvres, November 17 - 27, 1957. We thank him vigorcusly for having authorized us to publish this remarkable study of the problems of initiation.

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of

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