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ABSTRACT

This booklet is a catalog of error patterns found in basic arithmetic and algebra courses. It is intended to be used as a resource by instructors and tutors teaching these concepts. The material is divided into major concept headings with subheadings. The error patterns are named and given a brief general description followed by a specific example of the error. Major concept headings include the following: (1) Basic Problem Solving Skills; (2) Averages; (3) Whole Numbers; (4) Fractions; (5) Decimals; (6) Percents; (7) Integers; (8) Exponents; (9) Simple Equations; (10) Ratios and Proportions; (11) Geometry; and (12) Graphing. (RH)

Working Draft

CATALOGUE OF ERROR PATTERNS OBSERVED IN COURSES ON BASIC MATHEMATICS

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1985

Based on observations made by Ronald Narode, Lynn Benander, John Clement, and Jack Lochhead, as well as by Donna Avery, Tom Bassarear, Carol Brossi, Sarah Cleveland, Cliff Konold, Nadim Khoury, Nina Koch, Kwan-Min Lee, Tom Murray, Peter Rosnick, Deborah Schifter, and Larry Weithers. Thanks to Cliff Konold and Bob Markey for their help in editing this manuscript

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TABLE OF CONTENTS

ERROR PATTERNS

Observed in Basic Arithmetic and Algebra Courses

INTRODUCTION

This booklet is a catalog of error patterns found in basic arithmetic and algebra courses. It is intended to be used as a resource by instructors and tutors teaching these concepts.

The material is broken down into major concept headings with subheadings. The error patterns are then named and given a brief, general description, followed by a specific example of the error.

> MAJOR CONCEPT HEADING
>
> SUBHEADING
>
> NAME FOR THE ERROR PATTERN
> ERROR: A brief description of the error
>
> EXAMPLE: A specific example of the error
> [source code]

The material for this catalog was obtained from three sources. The major source for each entry is noted following each example.

SOURCE CODE

[C] In-Classroom observations of tutors and instructors in remedial level mathematics courses at the University of Massachusetts.

[R] The retrospective reports of tutors and instructors.

[I] Clinical Interviews focusing on students' elementary algebraic concepts and problem solving skills.

BASIC PROBLEM SOLVING SKILLS

OVERALL SKILLS AND ATTITUDES

LACK OF PRECISION IN INFERENCE
ERROR: Students are not able to read a problem and accurately make inferences required to solve the problem.

EXAMPLE: Given the Days problem --

"What day preceeds the day after tomorrow if four days ago was two days after Wednesday?"

Once students decide that two days after Wednesday is Friday, in the next step they count days in the wrong direction from Friday. They count backwards in time four days to reach Monday instead of counting forward to Tuesday. Perhaps the word "ago" triggers the idea of counting backwards in time. This an example of an inference inversion. [I]

AVOIDING CONJECTURES -- NEED FOR AN ESTABLISHED APPROACH
ERROR: Students are unwilling to start representing a problem when they don't see a sure method for solving it. [I]

INABILITY TO KEEP TRACK OF PARTIAL RESULTS
ERROR: Students make one or two inferences in a problem, but then "get lost" because they are unable to keep track of the partial results they have produced.

EXAMPLE: In the Days Problem above one student reread the entire problem and attempted to restart the solution a total of eight times. This indicates a difficulty in keeping track of partial results (such as "today is Tuesday"). The simple use of paper and pencil as an external memory for this purpose eluded her during most of the solution. [I]

LACK OF PRECISION IN INFERENCE WITH NESTED RELATIONSHIPS
ERROR: Some students have difficulty processing nested relationships like the "square of the square of a number", or the "ratio of ratios".

EXAMPLE: Students may say Monday when asked: "What is the day before the day before yesterday if today is Wednesday?" [I]

LACK OF PRECISION IN VERBAL EXPRESSION
ERROR: Students may use vague or imprecise language in making inferences during problem solving, leading to errors. [I]

COMPREHENDING AND REPRESENTING

DIFFICULTY IN FINDING THE GOAL AND THE GIVENS (AND KEEPING TRACK OF THEM)
ERROR: Some students especially have difficulty finding and holding in mind the goal of the problem. They begin making inferences from given information without directing their efforts toward the goal. [I]

FAILURE TO DRAW AND IMPROVE ON DIAGRAMS
ERROR: To many students, making a drawing seems to them "extra work" with no clear payoff. Many do not realize that sometimes several drawings or representations need to be attempted and modified before a good representation for the problem is found. [R]

PLANNING AND SEARCHING

DIFFICULTY BREAKING PROBLEMS INTO PARTS
ERROR: In a multi-step problem students can have difficulty identifying a piece of the problem to work on first. In addition, many students have difficulty "suspending" work on most of the problem while they work on one small part of it at a time. They may try to deal with too many results at once and become discouraged. [I]

DIFFICULTY REMEMBERING THE ORIGIN OF A SUBGOAL
ERROR: When a student does successfully solve a piece of the problem, he or she may forget the original reason for solving that piece (losing track of the process which set up the subgoal). Thus the student finds the answer to a subproblem and asks: "Now what did I need that result for, anyway?" [R]

CONFUSING SOLUTION TO SUBGOAL WITH SOLUTION TO PROBLEM
ERROR: When a student solves a piece of the problem he or she reports the solution as the final solution, loosing track of the larger picture. [R]

EVALUATING THE SOLUTION

NOT CHECKING EACH STEP
ERROR: Many students do not review each step in their solution for accuracy, sometimes because they have no clear record of the steps on paper and cannot recall the steps they took.

NOT ASKING WHETHER THE ANSWER SEEMS REASONABLE
ERROR: Some students will make no attempt to ask whether their answer to a problem is at all reasonable. [R

AVERAGES

WEIGHTED AVERAGES

NOT WEIGHTING WEIGHTED AVERAGES
ERROR: Students compute the average without regard to the weight of each value.

EXAMPLE:
A man walked 3 hours at 4 miles per hour and he took a bus 20 minutes at 30 miles per hour. What was his average speed?

Students may add the two speeds and divide by two rather than computing the weighted average.

[C]

OPERATIONS WITH WHOLE NUMBERS

BASIC OPERATIONS

SYMBOL (AND WORD) ORDER IN DIVISION

ERROR: Students confuse the many ways we order the quotient and divisor when we talk about and write division problems.

> EXAMPLE: $5 \div 3 = ?$
> Students may write $5\overline{)3}$ or 3/5 rather than $3\overline{)5}$ or 5/3. [R]

> EXAMPLE: Five divided by three equals?
> Students say this is the same as five into three or three over five rather than three into five or five over three. [R]

LIMITED INTERPRETATION OF DIVISION CONCEPT

ERROR: There are two main interpretations of division for sets:
1) distributing a set to a given number of equal sized bins (partitioning) or
2) packaging a set into groups of a given size (measuring).

When students interpret division in only one of these ways they are often unable to solve problems requiring a different understanding of division.

> EXAMPLE: $10 \div 2 = ?$
> Different problems require students to think of this number sentence as 10 items distributed evenly into 2 bins with 5 in each bin and as 10 items packaged into groups of 2 with a total of 5 packages. [R]

ORDER OF OPERATIONS

OVERGENERALIZATION OF LEFT TO RIGHT RULE

ERROR: Students work from left to right, ignoring order of operations.

> EXAMPLE: $3 - 2 \times 5 = ?$
> Students may perform subtraction before multiplication arriving at 5 for the answer rather than -7. [R]

COMMUTATIVE PROPERTY

SUBTRACTION AND DIVISION AS COMMUTATIVE
ERROR: Students think all operations are commutative, including subtraction and division. [R]

DISTRIBUTIVE PROPERTY

OVER-DEPENDENCE ON DISTRIBUTIVE RULE
ERROR: Students insist on distributing over parentheses even when it is simpler to add first.

EXAMPLE: 5 (2 + 3) = ?
Students may rewrite this as 10 + 15 first rather than simply multiplying 5 x 5. [R]

USE OF PARENTHESES

OVERGENERALIZATION OF PARENTHESES RULE
ERROR: Students insist on performing all the computations within parenthesees before combining other terms, even when this is not possible.

EXAMPLE: 3 (a + 7) = ?
Students may say this cannot be simplified any further because the a and 7 cannot be combined.
[R]

UNDERSTANDING BASE 10

LACK OF UNDERSTANDING OF BASE 10 SYSTEM
ERROR: Some students are unaware of the patterns which arise from the underlying base 10 structure of our number system. They are not able to take advantage of short cuts or self-checking strategies available to them.

EXAMPLE: The simple algorithm for subtraction with regrouping can be executed without understanding its justification in base 10. Given the problem --

23 - 8 = ?

Students may not think of 23 as equivalent to 10 + 13 when borrowing. They simply follow a mechanical process. [R]

FRACTIONS

NOTATION

CONFUSION WITH BASE 10 NOTATION
ERROR: Students may have trouble shifting from the decimal system and wonder which place the fractional number belongs in. Mixed numbers and fractions containing decimals compound the difficulties. [R]

SWITCHING NUMERATOR AND DENOMINATOR
ERROR: 3/5ths is written as 5/3rds [R]

ASSIGNING NEGATIVE SIGN ONLY TO WHOLE NUMBER IN A MIXED NUMBER
ERROR: Given a negative mixed number students only consider the whole number to be a negative value and the fractional part to be positive.

EXAMPLE: (- 3 1/2) + 5 = ?
Students may think of this as --

(-3) + 1/2 + 5 = 2 1/2

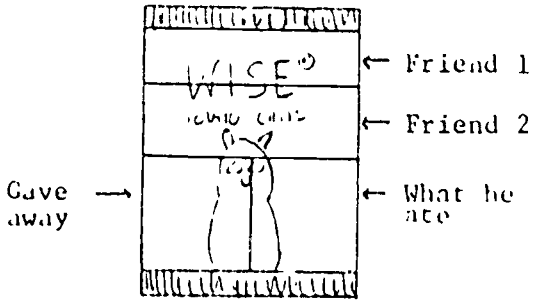
rather than seeing the 1/2 as a negative quantity also. [R]

FRACTIONS AS UNRELATED TO DIVISION
ERROR: Some students have difficulty understanding that 3/4 is another way of describing 3 divided by 4. [C]

USE OF DIAGRAMS

FORMING PARTS OF UNEQUAL SIZES
ERROR: Students draw diagrams in which the sections are not of equal size and arrive at wrong answers.

EXAMPLE: Three friends equally shared a bag of potato chips. One of them gave away half of his chips and ate the rest. How much of a bag did he eat? Students who do not draw equal parts may not see that the answer is 1/6 of a bag.



This diagram shows that he ate 1/4 of a bag. [R]

UNDERSTANDING PART TO WHOLE RELATIONSHIPS

FRACTION OF REMAINDER = FRACTION OF WHOLE
ERROR: Students equate a fraction of a remainder with a fraction of the whole.

EXAMPLE: Given the problem --

"A man's aunt died leaving half of her one million dollar estate to a charity and one third of the remainder to him. How much did he get?"

Students may say that he received one third of a million dollars rather than seeing that he only received one sixth of the estate. [C]

UNDERSTANDING HOW A FRACTION IS USED

Fractions are used in many contexts. Students who are not able to differentiate between uses often are not able to successfully solve the problem.

CONFUSING FRACTIONS AS OPERATORS WITH FRACTIONS AS SETS
ERROR: Students fail to recognize that a fraction is used in a word problem to describe an operation on a set.

EXAMPLE: Given the problem:

"One day a man earns $20. The next day he earns half of that. How much did he earn on both days?"

Students may say he earned $20 + 1/2 or 20 1/2 dollars rather than seeing that he earned $20 + (1/2 x $20) = $30. [R]

OPERATIONS WITH FRACTIONS

OVERGENERALIZATION OF LCD ALGORITHM

ERROR: Students find a lowest common denominator before they divide or multiply two fractions.

> EXAMPLE: 3/4 x 2/3 = ?
> Students will convert both fractions into 12ths before multiplying, 9/12 x 8/12 = 72/144 = 2, rather than simply multiplying numerators and denominators. [R]

OVERGENERALIZATION OF DIVISION ALGORITHM

ERROR: Students invert before multiplying with multiplication problems.

> EXAMPLE: 1/2 x 1/4 = ?
> Students may rewrite this as 1/2 x 4 = 2 rather than 1/2 x 1/4 = 1/8. [R]

OVERGENERALIZATION OF CROSS-MULTIPLICATION ALGORITHM

ERROR: Students cross-multiply whenever they see a fractional expression of the form a/b x c/d.

> EXAMPLE: 3a/4 x 2/3 = 7
> Students may cross-multiply and rewrite this as 9a = 8, rather than multiplying to find 6a/12 = 7 then reducing to find a = 14. [R]

INABILITY TO USE MEASURING MODEL OF DIVISION

ERROR: Students who are only comfortable with the partitioning or measuring definition of division are sometimes unable to understand the meaning of division of fractions. How many times 1/8 will go into 1/2 is easy to think about (measuring), but distributing 1/2 into several groups where the number of groups is 1/8 (partitioning) is difficult to conceptualize.

> EXAMPLE: If you have a half ball of twine and each kite needs an eigth of a ball of twine, how many kites can you fly. Students may choose to multiply 1/2 x 1/8 rather than recognizing this as a division problem. [R]

SEEING FRACTIONS AND WHOLE NUMBERS AS UNLIKE TERMS

ERROR: Students don't recognize fractions and whole numbers as like terms.

> EXAMPLE: x + 1/3 = 7
> Students fail to see that the 1/3 can be subtracted from the 7 and think that there is not enough information to solve this equation. [R]

A/A = 0

ERROR: Students conclude that the answer is zero if everything cancels out.

> EXAMPLE: 6/7 x 2/3 x 7/4 = ?
> Students may say this is equal to 0 because everything cancels out, rather than seeing it is equal to one. [R]

DIRECTLY SUBTRACTING DENOMINATORS

ERROR: Students subtract the denominators of two fractions as if they were simply whole numbers.

> EXAMPLE: 3x/4 - 1/4 = ?
> Students may say this equals 3x - 1 because 4 - 4 = 0. [C]

MULTIPLICATION OF FRACTIONS AS NON-COMMUTATIVE

ERROR: Students see 1/2 of 2/3 as not necessarily being equal to 2/3 of 1/2. [C]

The following surprises students:

> a/b x c = ac / b = c/b x a
> a/3 = 1/3 x a (Some think 1/3 x a = 1/3a) [R]

a/b/c makes no sense to some. [R]

a/2 doesn't make sense to students who think of a as a label for something. 2/a makes even less sense. They only want to deal with coefficients in front of the label. [R]

Students don't see why you can't have 0 in the denominator if you can have it in the numerator. [R]

WORD PROBLEMS

RELUCTANCE TO REDUCE A FRACTION IN A WORD PROBLEM

ERROR: Students will not reduce a fraction in a word problem because they feel tnat when you attach a physical meaning to a fraction, reducing that fraction changes its value. As long as only abstract computation is required these students are able to reduce fractions.

EXAMPLE: If a word problem asks how many people bought red bikes, and a student set up a correct equation and arrived at an answer of 3/45ths of the people bought red bikes, they would not reduce 3/45ths because 3/45ths of a group of people seems significantly different than 1/15th of that group. [R]

DECIMALS

NOTATION

INVERTING THE DECIMAL SYSTEM
ERROR: Students may think hundreths are bigger than tenths because hundreds are bigger than tens.

EXAMPLE: Students may say .03 is greater than .3 because hundreths are larger than tenths. [R]

EXAMPLE: Students may say .12 is greater than .125 because the 5 is out in the thousanths place which is smaller than the hundreths place. [R]

FAILURE TO RECOGNIZE EQUIVALENT DECIMALS
ERROR: Students think that adding zeros onto the end of a decimal number changes the value of the number.

EXAMPLE: Students may say .2 is greater than .20 because tenths are bigger than hundreths. [R]

OPERATIONS WITH DECIMALS

MISPLACING DECIMAL POINT IN WHOLE NUMBERS
ERROR: Given a list of decimal and whole numbers, students will place the decimal point in front of a whole number.

EXAMPLE: Given " .03 + 4.6 + 12 = ? "
Students will convert 12 to .12 before adding. [R]

FAILURE TO RECOGNIZE DECIMALS AND WHOLE NUMBERS AS LIKE TERMS
ERROR: Given an equation containing a decimal and whole number students will not see that they are able to combine the terms.

EXAMPLE: Given " 3 + x = .8 "
Students will not recognize that they can subtract 3 from the 8 tenths. [R]

DIVISION BY A DECIMAL LESS THAN ONE
ERROR: Students are not able to see how this produces a number greater than both the quotient and divisor. [R]

UNDERSTANDING PART TO WHOLE RELATIONSHIPS

MEASUREMENT CONVERSION ERRORS
ERROR: Students equate one tenth of a unit of measure with one smaller unit of measure.

EXAMPLE: 2.5 feet = ?
Students may convert this to 2 feet and 5 inches rather than recognizing that .5 of a foot is 6 inches (or 2.5 cups = 2 cups 5 ounces). [R]

PERCENTS

NOTATION

DIRECT TRANSLATION TO WHOLE NUMBER
ERROR: Students may interpret a percentage such as 33% to be the number 33. [R]

DIRECT TRANSLATION TO DECIMAL NUMBER
ERROR: Students may interpret a percentage such as 65% to be the number .65 regardless of the set the 65% is operating on. [R]

ASSUME PERCENTAGES ONLY OF ONE WHOLE OR 100 PARTS
ERROR: Students assume that a percentage can be taken of only one whole or of 100 parts.

> EXAMPLE: 20% of $1.50 is interpreted to be the same as 20% of $1.00 or $.20. [R]

NON-EXISTANCE OF PERCENTS LESS THAN ONE
ERROR: A decimal followed by a percent sign is uninterpretable.

> EXAMPLE: .94% is considered an invalid number. [C]

PART TO WHOLE RELATIONSHIPS

A PERCENTAGE NOT SEEN AS A PART OF A WHOLE
ERROR: Some students do not see how 20% of x and 80% of x add up to all of x. [R]

OPERATIONS WITH PERCENTAGES

CONFUSION WITH ROLE OF PERCENT SIGN
ERROR: Students may not know what to do with the percent sign once they've completed the computation to determine the percentage.

> EXAMPLE: 10 x 50% = ?
> Students may find that 50 percent of 10 is 5 but they're not sure if the answer is 5% or 5. [R]

CONFUSING PERCENTAGES AS OPERATORS WITH PERCENTAGES AS GIVEN SETS OF ONE WHOLE
ERROR: Students assume that 30% of a is the same as 30% of one whole. [R]

WORD PROBLEMS

20% OF SOMETHING IS 20% OF ANYTHING
ERROR: Occasionally a student will think that the value for 20% of one quantity is the same as 20% of another quantity. [C]

PERCENT OPERATION REVERSAL
ERROR: When given that the result of a percent operation is a, the student will perform the percent operation on a to find the answer rather than working backwards from a to find the original amount.

EXAMPLE: Given the example --

> The price of the meal including a 15% tip was $5.75. What was the cost of the meal without the tip?

Students may find 15% of $5.75 -- $.86 -- then subtract $5.75 - $.86 to get $4.89 rather than solving this equation --

$$A + (15\% \times A) = \$5.75$$

to find that the meal was $5.00. [C]

"PERCENTAGE INCREASE" VS. "PERCENTAGE OF"
ERROR: Students confuse a percentage increase with a percentage of a given quantity.

EXAMPLE: Given a problem --

> Due to a major gas shortage, gas prices increased 150% above the former price of $1.00 per gallon. What was the price of gas after the shortage?

Students may think of this as a one step problem, finding 150% of $1.00 and say the price of gas was $1.50 after the shortage.

This problem is actually a two step problem. First the increase must be calculated, then the increase must be added onto the original price. [R]

50% INCREASE VS. DOUBLING

ERROR: Students interpret a 50% increase as doubling.

EXAMPLE: Given the problem --

> A store owner increases the price of his merchandise by 50% of the price he pays to determine the sale price of each item. If he buys a rug for $50.00, how much does he sell it for?

students may translate a 50% increase as doubling and say that the rug's sale price is $100.00 rather than solving the following equation --

$$\$50.00 + (50\% \times \$50.00) = \$75.00$$

to find that the rug would sell for $75.00. [R]

INTEGERS

BASIC OPERATIONS

FAILING TO DISTRIBUTE -1
ERROR: Students do not distribute the negative sign in front of an expression in parenthesees.

EXAMPLE: - (a + b) = ?
Students may say this is equal to - a + b rather than - a - b. [R]

OVERGENERALIZATION OF NEGATIVE x NEGATIVE = POSITIVE
ERROR: Students misapply this rule to addition.

EXAMPLE: (-3) + (-7) = ?
The incorrect answer would be 10. [R]

RESISTANCE TO NEGATIVE x NEGATIVE = POSITIVE RULE
ERROR: Students refuse to believe this rule is true.

EXAMPLE: (-5) x (-2) = ?
The incorrect answer given would be -10. [R]

USE OF DIAGRAMS

INABILITY TO INTERPRET NUMBER LINE OPERATIONS
ERROR: Students are unable to produce a number line explanation for operations with negative numbers. [R]

EXPONENTS

NOTATION

CONFUSING EXPONENT WITH MULTIPLICATION

ERROR: Students may want to multiply by the exponent rather than raising the base to the given power.

EXAMPLE: Given " $2^3 = ?$ "

Students may say this is the same as 2 x 3.

[R]

IGNORING NEGATIVE SIGN IN NEGATIVE EXPONENT

ERROR: Students answer a problem containing a negative exponent as if the exponent were positive.

EXAMPLE: Given " $5^{-2} = ?$ "
Students answer 25. [C]

NEGATIVE EXPONENT IMPLIES NEGATIVE WHOLE NUMBER

ERROR: Students solve a problem containing a negative exponent as if the exponent were positive, then multiply their answer by negative 1.

EXAMPLE: Given " $5^{-2} = ?$ "
Students answer -25. [C]

NEGATIVE EXPONENT IMPLIES NEGATIVE FRACTION

ERROR: Students find the correct fractional value but then multiply by -1.

EXAMPLE: Given " $5^{-2} = ?$ "
Students answer $-(1/5)^2 = -1/25$. [C]

NEGATIVE EXPONENT IMPLIES A NEGATIVE FRACTION RAISED TO A POWER

ERROR: Students invert the base, take its negative and raise it to the given power.

EXAMPLE: Given " $5^{-3} = ?$ "
Students answer $(-1/5)^3 = -1/125$. [C]

ZERO EXPONENT CONFUSION
ERROR: Students have a difficult time accepting that any number to the zeroth power is one. It makes more sense to them for it to equal zero. [R]

OPERATIONS WITH EXPONENTS

MISSING TERMS WHEN SQUARING A SUM
ERROR: Students may apply the distributive property incorrectly and simply move the exponent inside a set of parentheses.

EXAMPLE: Given $(a + b)^2 = ?$
Students may say this equals $a^2 + b^2$
[R]

COMBINING LIKE BASES WITH DIFFERENT EXPONENTS IN ADDITION
ERROR: Students will attempt to combine two numbers with the same base even though they are raised to different powers.

EXAMPLE: Given " $a^3 + a^5$ "
Students may say that these can be combined to equal a^8. [R]

MULTIPLYING EXPONENTS IN MULTIPLICATION
ERROR: When asked to multiply two numbers with the same base raised to different powers, a student may multiply rather than add the powers.

EXAMPLE: Given " $a^3 \times a^5 = ?$ "
Students may answer a^{15} rather than a^8. [R]

WORD PROBLEMS

GEOMETRIC VS. MULTIPLICATIVE INCREASE

ERROR: Students represent a geometric increase as a multiplicative increase.

EXAMPLE: Given the problem --

> "A woman invested $1,000 in a high risk business venture. Every year for the next six years her money doubled. How much was her investment worth at the end of six years?"

Students may reply that she then had --

$1,000 x 6 = $6,000

instead of recognizing this as a geometric increase. (P)

SIMPLE EQUATIONS

UNDERSTANDING VARIABLES

Variables are used in a wide variety of contexts. They can represent unknowns, indeterminants, independent variables, dependent variables, constants, parameters, etc. When students begin working with letter variables they are only familiar with using letters to represent units of measure, labels for a group of objects (apples), names of lines and angles in geometric figures, and as abbreviations for words which describe an object or group of objects (w for width). Understanding the use of a letter in the context of a number sentence demands a clear understanding of the fact that a letter variable stands for a number, not an object or unit.

LETTERS AS LABELS

ERROR: Students assume that a letter variable stands for an object, not a number.

EXAMPLE: If b = the number of books and r = the number of records, then "r" can never equal "b" because books are different from records. [C]

ASSUMED ONE-TO-ONE MAPPING BETWEEN LETTERS AND NUMBERS

ERROR: Students assume that if a = 7 then b cannot equal 7 also. [R]

SINGLE LETTER FIXATION

ERROR: Some students may feel comfortable working with only one letter variable and be confused if any other letter is introduced. [R]

NEGATIVE VARIABLE MUST HAVE NEGATIVE VALUE

ERROR: Students may assume that -a has a negative value. -a could not equal a positive number. [R]

VARIABLE ALWAYS EQUALS UNKNOWN

ERROR: Some students feel that the letter should always represent the sought after quantity in a problem even when it would be more convenient to treat another quantity as the unknown. [R]

ALL EQUATIONS HAVE ONLY ONE CORRECT SOLUTION
ERROR: Some students think that equations can have only one correct solution. If they find a solution which works, they assume they have completed the problem.

EXAMPLE: Given the equation " S + B = 100 " When some students find a solution S = 20 and B = 80 which satisfies the equation, then they assume that S and B can assume no other values. [C]

UNSTABLE VARIABLES
ERROR: Some students lose track of which letters stand for which variables. They assign a letter to one variable in a word problem and later use the same letter for a different variable. (This may also indicate that the student is not differentiating between concepts.) [I]

EXAMPLE: The student uses C for both the number of cars and the length of a car.

MISSING VARIABLE
ERROR: Students may fail to include a relevant variable that has been described in the problem. [I]

EXAMPLE: The student includes the number of items but not the price in calculating the cost.

REDUCING MULTIVARIABLE TO SINGLE VARIABLE
ERROR: Some students convert multivariable problems to single variable problems by successively modifying a partial result. While this strategy can be successful, it is very difficult to check and often leads to incorrect answers. [I]

EXAMPLE: Given the problem --

If a car went 50 miles per hour for 3 hours and gets 30 miles to a gallon of gas, how many gallons of gas did it use?

Rather than introducing two variables to solve this problem (one for the distance traveled and one for the number of gallons of gas used) a student may only assign a letter to represent the number of gallons of gas used.

G = (50 mph x 3 hrs) x 30 mpg = 4500 gallons

Here the student multiplied rather than divided by 30 mpg but with the nesting of partial results it is very difficult to locate the error.

IRRELEVANT VARIABLE
ERROR: A student introduces an unneeded variable in an equation. [I]

EXAMPLE: When asked to compute the distance a car traveled given the number of gallons of gas used and the number of miles per gallon the car uses a student may introduce a variable for the speed of the car.

OVERUSED VARIABLE
ERROR: A student uses the same variable more than once, inappropriately, in an equation.

EXAMPLE: d/r = rt [I]

EXPRESSIONS VS. EQUATIONS

An expression such as " x + 2 " has an entirely different meaning from the equation " x = 2 ". Students are not always able to appreciate this difference.

OPERATING ON EXPRESSIONS AS EQUATIONS
ERROR: Students begin operating on an expression as if it were an equation. For example, given the expression --

a/2 + 3/b

students may begin to perform the cross-multiplication algorithm to conclude ab = 6, as if this expression were the equation --

a/2 = 3/b

and not follow through with the remainder of the problem. [R]

USE OF EQUAL SIGN

The equal sign is used in a variety of ways --

to give a definition: let x = the number of students
to state a fact: $x = 7$
to give a constraint: $3x + y = 12$
to give the results of computation: $x = 7$

Students often become confused with these different uses and are unable to determine how to solve a given problem.

EQUAL SIGN AS STEP MARKER
ERROR: Students use the equals sign to mean "and the next step is."

EXAMPLE: $3x = 7$
$= x = 7/3$ [R]

TRANSLATING WORD PROBLEMS INTO EQUATIONS

REVERSAL ERROR
ERROR: Students invert the multiplicative or additive relationship between two variables. There are several possible sources for this error:

1) They think of the letters as labels rather than as numerical values. (6S stands for 6 students rather than 6 times the number of students.)

2) They automatically write algebraic symbols in the same order as key words in the problem statement, even when inappropriate.

3) They place the larger number with the letter which represents the larger group to show that it's the larger group.

The correct approach which shows the greatest understanding is termed the Operative Approach. Here the student thinks about multiplying the number in one group by a constant in order to make it the same size as the other group. The "sing song" proportion approach which uses the linguistic "A is to B as C is to D" format can also be used effectively in some situations, but may indicate a lower level of understanding.

EXAMPLE: Given the problem --

> There are six times as many students as professors. Write a number sentence to illustrate this relationship.

Students may say --

$6 \times s = p$

rather than seeing that --

$6 \times p = s$ [I]

SEQUENTIAL MAPPING OF WORDS TO NUMBERS
ERROR: Students create number sentences from a word problem by copying each of the numbers in the order it appears and inserting operation signs between each number. [R]

NEED TO UTILIZE ALL NUMBERS IN A WORD PROBLEM
ERROR: Students may assume that all numbers given in a word problem ought to be in their equation. Extraneous information confuses them. [R]

USE OF SYMBOL MANIPULATION SCHEMES
ERROR: The student generates a preliminary equation for a word problem that he feels is not yet correct. The student then attempts to modify the preliminary expression using symbol manipulation rules algebraically to try to reach a valid equation. [I]

PSEUDOEQUATIONS
ERROR: Students may write an equation which incorrectly expresses the relation of correspondence between specific values of two variables.

> EXAMPLE: 20 cm = 300 grams may be written to describe the weight and length of a rod.
[I]

TOTALS (WHOLE-PART) CONFUSION

ERROR: The student erroneously tries to include the ratio relationship between two groups in an equation representing an additive total.

EXAMPLE: Given the problem --

> There are six times as many students as professors. Write a number sentence to illustrate this relationship.

Students will write 6S + P = T or 6P + S = T.

[I]

VERTICAL RULE SYMBOLIZATION

ERROR: The student symbolizes the vertical patterns in a data table, typically to predict the next values in the sequence from the immediately preceding values.

EXAMPLE: Given the table --

S	W
3	100
6	200
9	300

Students may say
$S + 3 = W + 100$

[I]

SQUARE FUNCTION REVERSAL

ERROR: The student places the exponent on the wrong side of the equation written from a data table for a word problem.

EXAMPLE: Given the table --

x	y
2	4
4	16
5	25

Students may write
$x = y^2$

[I]

SQUARING THE CONSTANT

ERROR: Students may introduce a constant and use an exponent operator on the constant instead of on a variable.

EXAMPLE: Given the problem --

> Energy E varies directly with the square of the voltage V; when voltage is multiplied by 3, E goes up by a factor of 9.

Students may respond with -- $E = K^2V$ [I]

ADDITIVE VS. MULTIPLICATIVE VS. GEOMETRIC RELATIONSHIPS
ERROR: Students often confuse additive, multiplicative and geometric relationships with each other.

EXAMPLE: " 9 times more women " may be translated into " 9 + w " rather than " 9w ". [I]

RATE VS. VALUE
ERROR: Students often have trouble differentiating between information which describes the rate of change of a variable and that which describes the value of the variable. [R]

RATE VS. CHANGE IN RATE
ERROR: Students often have trouble differentiating between information which describes the the rate of change of a quantity and that which describes the change in the rate of change of a quantity. [R]

SOLVING EQUATIONS

FAILURE TO RECOGNIZE UNWRITTEN COEFFICIENTS OF ONE
ERROR: Students treat "a" and "1(a)" as different quantities.

EXAMPLE: Given the problem --

$$a + 3a = ?$$

students may say that this expression cannot be simplified rather than seeing that this expression is the same as --

$$1a + 3a = 4a$$ [R]

X = X CONFUSION
ERROR: When everything cancels out and students are left with $x = x$ they may either conclude that then $x = 0$ or that $x = 1$. [R]

NOT RECOGNIZING LIKE TERMS

ERROR: Some students have difficulty determining which terms of an equation can be combined.

EXAMPLE: Given " 2 + x = 2x " some students are not able to pick out the like terms. [C]

SWITCHING INVERSE OPERATIONS

ERROR: When simplifying an equation a student may subtract when they need to divide or divide when they need to subtract.

EXAMPLE: Given " 2 + x = 4 " some students will divide both sides of the equation by 2 -- 1 + x/2 = 2. They then become confused because their equation has become more complicated. [C]

EXAMPLE: Given " 2x = 5 " students may subtract 2 from both sides and conclude x = 3. [C]

RATIOS AND PROPORTIONS

NOTATION

CONFUSING PART TO PART AND PART TO WHOLE RATIOS
ERROR: Students do not understand how a ratio which describes a part to part relationship --

2 shaded parts to 3 unshaded parts

relates to a ratio which describes a part to whole relationship --

2 shaded parts out of 5 parts [R]

ORDERING REQUIREMENTS FOR RATIOS
ERROR: Students think the order of the ratio is very important.

EXAMPLE: The ratio 3 men to 5 women is seen as very different from 5 women to 3 men. [R]

ORDERING REQUIREMENTS FOR PROPORTIONS
ERROR: Students think there is only one correct way to set up a proportion and are confused when they discover other ways which seem to work.

EXAMPLE: Given a 3" by 4" rectangle and its enlargement who first side is 9", students may attempt to set up the proportion to find the missing side but become confused when --
3/4 = 9/x, 3/9 = 4/x, 4/3 = x/9, and 9/3 = x/4
all seem to work. [R]

RATIOS VS. PROPORTIONS
ERROR: Students do not differentiate between ratios and proportions. Often proportions are not viewed as equations at all. [R]

OPERATIONS WITH RATIOS AND PROPORTIONS

OVERGENERALIZATION OF CROSS-MULTIPLICATION ERROR
ERROR: Students see two ratios and immediately apply the cross-multiplication algorithm. [R]

RATIOS AND PROPORTIONS WITH DECIMALS AND FRACTIONS
ERROR: Students who are successfully working with ratios and proportions with whole numbers will sometimes become confused when ratios and proportions with decimals and fractions are introduced. [R]

WORD PROBLEMS

SETTING UP NON-PARALLEL PROPORTIONS
ERROR: Students may construct a proportion from a word problem without regard to the quantities represented -- arriving at a proportion which does not express the ratio relationship indicated. They may put a part with a part and equate this ratio with another part to a whole.

EXAMPLE: Given this problem --

Researchers would like to determine the number of fish in a pond. They remove a sample of 50 fish from the pond and tag them. Several days later they return to the pond and remove 40 fish. 10 of them are tagged. Approximately how many fish are there in the pond?

Students may construct the following proportion --

10 tagged fish	50 fish	
30 untagged fish	x fish	[C]

OVERUSE OF "SING-SONG" RATIOS
ERROR: Students use the "sing song" formula -- "A is to B as C is to D" inappropriately.

EXAMPLE: Given the problem --

Mr. A owns 2 cars and drives one of them to work every day. His first car gets 12 mpg and uses 3 gallons for this trip. His second car gets 18 mpg and uses how many gallons for this trip?

Students may set up the proportion

$12 / 3 = 18 / x$

and not see that this implies that the car with the better gas mileage uses more gas for the trip. [R]

GEOMETRY

PERIMETER, CIRCUMFERENCE, AREA, VOLUME

THERE'S A FORMULA FOR EVERYTHING
ERROR: Students use any available formula to answer a geometry problem. [R]

CONFUSING LENGTH AND AREA
ERROR: Students have trouble differentiating between units of linear and area measure. They will give a correct answer with the wrong units. They will give the area when asked for the perimeter and vice versa. [C]

CIRCULAR UNITS
ERROR: Some students are uncomfortable expressing the area of a circle in terms of square units. [R]

BASKETBALL PERIMETER
ERROR: In basketball, the perimeter of the basketball court is the area just outside of the court. Students may erroneously apply this concept to the definition of perimeter in mathematics. [R]

THE 12 SQUARE INCH SQUARE FOOT
ERROR: Students do not see square units as a squaring operation.

> EXAMPLE: A square foot has 12 square inches, a square yard has 3 square feet, etc. [R]

OVERGENERALIZATION OF RECTANGLE AREA FORMULA
ERROR: Students determine the area of any quadrilateral by multiplying any two sides. [R]

MIXING CIRCLE'S AREA AND CIRCUMFERENCE FORMULAS
ERROR: Students think " 2 pi r " and " pi r^2 " describe the same quantity. [R]

TRIANGLES AND THE PYTHAGOREAN THEOREM

OVERGENERALIZATION OF FORMULA FOR AREA OF TRIANGLE
ERROR: Students apply the formula --

A = 1/2 base x side

to all triangles, not only right triangles. [R]

OVERGENERALIZATION OF PYTHAGOREAN THEOREM
ERROR: Students apply the pythagorean theorem to triangles without a right angle. [R]

ROTATION OF THE PYTHAGOREAN THEOREM
ERROR: Students name the missing side c when it is the leg, not the hypothenuse, of the triangle and substitute into the formula --

$$a^2 + b^2 = c^2$$

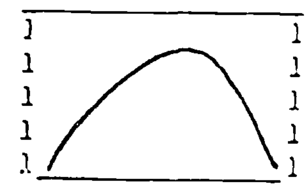
EXAMPLE: To find the length of missing leg "x" of a triangle with hypothenuse 26 and leg 10, a student may substitute into the pythagorean theorem -- $10^2 + 26^2 = x^2$ [C]

QUALITATIVE GRAPHING

GRAPH AS PICTURE : GLOBAL CORRESPONDENCE

ERROR: Students draw the graph to represent the contour of an object in the problem.

EXAMPLE: When asked to draw a graph representing the speed of a person walking over a hill with respect to time, a student might draw the shape of the hill.

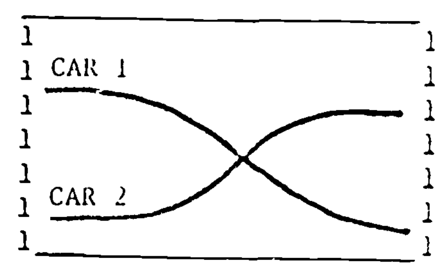


[R]

GRAPHS AS PICTURE : FEATURE CORRESPONDENCE

ERROR: Students may draw the graph so that a particular feature corresponds with a pictorial feature of the situation.

EXAMPLE: When asked to draw a graph representing the speed of two cars on a track with respect to time, students will make the graphs of each car cross where one car passed the other one.



[R]

ALL GRAPHS ARE ALLOWABLE

ERROR: Students may say that all graphs are possible - even graphs which place one object in two places at one time. [R]

ALL GRAPHS BEGIN AT THE ORIGIN

ERROR: Students may begin all of their graphs at the origin rather than making the graph reflect the positions indicated by the values on each axis. [R]

SLOPE HEIGHT CONFUSION

ERROR: Students may use height instead of slope to represent a rate.

EXAMPLE: Asked to identify the point of maximum increase in temperature, the student chooses the point of maximum temperature on a graph of temperature vs. time. [R]