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ABSTRACT

Understanding the way farmers respond to risk is a prerequisite for sound agricultural policymaking. Clarifying established agricultural economic theory to describe how individuals make choices among risky alternatives and providing a more precise set of concepts for studying behavior under risk contributes to such an understanding. A mathematically rigorous approach ensures that concepts are defined unambiguously and results are established decisively, and finds that individuals' preferences over an important class of risky alternatives are independent of preferences among certainties. Economic behavior under this type of risk cannot generally be predicted from behavior under certainty. Farmers' aversion to risk, or lack thereof, cannot invariably be determined from the shape of their utility curve of income. Farmers' profit-maximizing choices of production inputs may differ from those traditionally thought to be dictated by classical theory, even when preferences satisfy the explicit assumptions of that theory. Separate sections examine lotteries; the existence, uniqueness, invariance, continuity, and decomposition of measurable utility functions; aspects of measurable utility on the real line; definitions of risk aversion and its relation to concavity; and economic applications to optimal production levels under price uncertainty and optimal saving rates under uncertainty. (NEC)

CONCEPTUAL FOUNDATIONS OF RISK THEORY. Michael D. Weiss, Commodity Economics Division, Economic Research Service, U.S. Department of Agriculture. Technical Bulletin No. 1731.

ABSTRACT

A mathematically rigorous approach to the subject of risk permits us to develop a more precise and unified set of concepts for analyzing individuals' behavior under risk. The traditional method of measuring aversion to risk is not always warranted. Individuals' behavior under certainty cannot always be used to predict their behavior under risk. Consequently, optimal saving rates and optimal production input levels under risk may differ from those prescribed by traditional theory.

Keywords: Behavior under risk, expected utility, risk aversion

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SUMMARY

This report clarifies the theory traditionally used in agricultural economics to describe how individuals make choices among risky alternatives. It uses a mathematically rigorous approach to ensure that concepts are defined unambiguously and results are established decisively. It reveals a hidden and unnecessarily restrictive assumption in the traditional risk literature. It demonstrates that, when this assumption is removed, the behavior of individuals under certain types of risk is not, as previously thought, entirely determined by their behavior under certainty.

The theory of individual choice under risk begins with an economic agent faced with a set of risky alternatives and endowed with a set of preferences over these alternatives. In the classical "expected utility" theory, individuals are assumed to prefer one risk to another if they judge that, on average, its outcome would be more beneficial. Agricultural economics applies this theory extensively. However, many claimed consequences of expected utility theory are actually derived from another, hidden assumption: that an individual's preferences are "continuous" in the sense that slight changes in alternatives do not lead to sharp changes in expected benefits. Exposing and removing this logically unnecessary assumption enables us to reinterpret the implications of the classical theory.

Individuals' preferences over an important class of risky alternatives are independent of their preferences among certainties. Thus, economic behavior under this type of risk cannot generally be predicted from behavior under certainty. For example, a knowledge of the "utility" (a kind of numerical rating) that farmers implicitly assign to various possible incomes is not generally a logically sufficient basis on which to predict their behavior under risk. Farmers' aversion to risk, or lack thereof, cannot invariably be determined from the shape of their utility curve of income. Moreover, farmers' profit-maximizing choices of production inputs may differ from those traditionally thought to be dictated by the classical theory, even when their preferences satisfy the explicit assumptions of that theory.

Understanding the way farmers respond to risk is a prerequisite for sound agricultural policymaking. This report contributes to such an understanding by clarifying established teaching and by providing a more precise set of concepts for studying behavior under risk.

Conceptual Foundations of Risk Theory

Michael D. Weiss

1 INTRODUCTION

Risk is a pervasive influence in agriculture. Indeed, agriculture is one of the few industries in which a crucial production input, weather, can neither be controlled nor predicted. The economic consequences of this fact, including the effects of the resulting price risk, are far reaching. Understanding the ways in which farmers and other participants in the agricultural economy respond to risk is important to effective agricultural planning, policymaking, and analysis.

For agricultural economists, such an understanding must be founded on a general theory of how individuals make choices under risk. But, risk is a subtle concept, and a theory of choice under risk cannot successfully undergo testing, revision, confirmation, and ultimately empirical application, if it is not thoroughly understood. This difficulty applies as much to the currently used "expected utility theory" as it does to improved theories yet to be developed.

The risk literature, particularly the less advanced literature, has not always been conducive to understanding. Discussions of risk have often relied on improperly defined concepts or concepts identified inappropriately with special cases. Some claimed results have depended on incomplete arguments. Overall, the risk literature has not successfully conveyed a clear, conceptual view of risk theory that is comparable, for example, to the established view of the foundations of consumer demand theory. This report will provide agricultural economists with the foundation for such a conceptual view of risk theory.

The report investigates the concepts that underlie expected utility theory, the theory that describes individuals faced with a choice among risky prospects as attempting to maximize their "expected utility" (a numerical measure rating risks against one another). This theory is based on assumptions of transitivity, completeness, independence (see p. 14), and

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adherence to an "Archimedean" property (*7*, p. 292) for preferences under risk.[1/] Moreover, as the report will demonstrate, most treatments of expected utility theory and its applications have relied on an additional, hidden assumption of continuity of risk preferences. Previous writers have discussed the consequences of altering or omitting various of these assumptions (for example, see *3, 4, 7, 12, 13, 14, 15, 16, 21, 29, 42*).

This study continues that line of inquiry by investigating the consequences, both theoretical and practical, of omitting the assumption of continuity. The results on continuity are part of a broader analysis aimed at clarifying the meaning and logical relationships of a variety of concepts important to expected utility theory.

We first define the notion of a "lottery," the formalization of the idea of a risky prospect, and draw the connection between compound lotteries and convexity. We show how two textbook definitions of "lottery" can be interpreted in terms of our definition.

We then define and examine measurable utility functions, utility functions that represent risk preference orderings and that have a linearity property mimicking the computation of the expected utility of a lottery. We present new results on the intrinsic structure of measurable utility functions; we show that any such function can be uniquely decomposed into a "discrete part," an "absolutely continuous part," and a "singular continuous part." Conversely, a measurable utility function is definable from such parts. These results give rise to a new type of discontinuous preference ordering that, in some economic models, allows preferences among "certainties" to be independent of preferences among those "uncertainties" that are represented by continuous lotteries. Such orderings can represent behavior in which choices among certainties are made within a different "frame of reference" (*1, 43*) than are choices among "continuous uncertainties."

We examine the use of functions on the real line to represent measurable utility functions defined on lottery spaces (and thereby to represent the associated preference orderings of lotteries). We describe conditions under which a measurable utility function μ has a "von Neumann-Morgenstern utility function" u on the real line that allows μ to take the "expected utility" form

$$\mu(L) = \int_{-\infty}^{\infty} u(t)dL(t)$$

for each L in the lottery space. We also present new representation theorems showing that *any* function on the real line, even if discontinuous at *every* point, represents some measurable utility function and thus some "rational" preference ordering of lotteries. These theorems are used to extend the use of discontinuous utility in modeling farmers' disaster outcomes (*30*) to include the case in which there is a riskless asset.

The concept of "risk aversion" is defined in a purely ordinal manner. It is shown that the widely claimed equivalence between risk aversion and concavity

1/ Italicized numbers in parentheses refer to literature cited at the end of this report.

of an underlying utility function of money holds in a weakened form for measurable utility functions when preferences are continuous, but fails in one direction, and "appears" to fail in the other, when preferences are discontinuous. Implications for the empirical identification and modeling of risk aversion are discussed.

We conclude by exploring the behavioral consequences of the new type of discontinuous risk preference ordering within two risk models: one, a model of production with uncertain output price, and the other, a model of saving with uncertain interest rate. We show that, when the marginal utility of money is greater under certainty than under "continuous" uncertainty (in a carefully circumscribed sense that must be made precise), then (under additional routine assumptions) the optimal production level and the optimal saving rate will be lower than the corresponding levels obtained through traditional expected utility maximization. We also indicate how economists can use our analysis to determine the optimal production level when a product price support is introduced, as with agricultural commodities.

Readers who master the concepts presented in this report should be much better equipped to understand newer research such as the seminal work of Machina (*28, 29*). Moreover, they should be better prepared to conduct and evaluate empirical studies involving risk because they will have a precise and orderly intellectual framework against which to test the meaningfulness and validity of empirical arguments.

The theory of choice under risk, dealing as it does with orderings representing individuals' preferences over spaces of cumulative probability distribution functions (see pp. 4-5, 7), is intrinsically a highly mathematical subject. This report, therefore, contains a good deal of mathematics. However, the material should be accessible to many agricultural economics researchers and students who have some knowledge of probability and real analysis, convex sets, and linear algebra. The main prerequisites are a familiarity with basic mathematical notation (especially that of functions and sets) and a willingness to do some hard, rigorous thinking about risk.

The theory of preferences under uncertainty concerns choices that individuals make when confronted with alternative risky prospects. These risky objects of choice are customarily called "lotteries"; this section is concerned with their definition, properties, and types. We will show how lotteries can be defined as cumulative probability distribution functions and will describe the convention for representing compound lotteries that allows this definition to succeed. Several types of lottery are distinguished. We develop the notion of a "lottery space" as a set of lotteries closed under the formation of compound lotteries and show that this closure property amounts to an assumption of convexity. We point out alternative, more general definitions of a lottery as a probability measure or as an element of an abstract "mixture set." We also contrast our approach to the subject with those of two widely used microeconomics textbooks.

2.1 Formalizing the Concept of Lottery

A lottery may intuitively be conceived of as a game of chance in which various prizes occur with preassigned probabilities. These prizes may be money or even other lotteries (that is, the opportunity to play other lotteries and receive *their* prizes). In the latter case, one speaks of a "compound" lottery.

Consider the example of a farmer who faces a probability p of a crop infestation and, hence, a probability 1-p of no infestation. If the first case occurs, he/she faces a spectrum of possible profits depending, for example, on weather and other unpredictable factors. In the second case, there is another (higher) spectrum of possible profits. In effect, with probability p, the farmer receives one profit lottery as a prize, and with probability 1-p, another. This situation has the form of a compound lottery.

The intuitive concept of lottery used in economics is governed by an important convention: two lotteries are considered "equivalent" if they have the same sets of ultimate prizes occurring under the same probability laws, regardless of the processes by which these prizes are achieved. In short, the internal compound structure of a lottery is ignored. The objects of choice are not individual lotteries as one intuitively conceives them, but, rather, are *equivalence classes* of individual lotteries.

It might at first appear that one could define a lottery (or, more precisely, the corresponding equivalence class) mathematically as simply a random variable whose possible values were the various ultimate prizes, those occurring according to the desired probability law. However, the calculation of an overall random variable to represent an empirical compound lottery in terms of its constituent sublotteries would be quite complicated. Thus, random variables are not very convenient as mathematical representations of lotteries. Rather, it turns out that *cumulative probability distribution functions* (*c.d.f.'s*) are more tractable representations.

Recall that, if X is a random variable on a probability space with probability measure P, then the c.d.f. F_X of X is the function $F_X: \mathbb{R} \to [0,1]$ defined by $F_X(r) = P(X \leq r)$ for all $r \in \mathbb{R}$ (where $\mathbb{R}$ is the set of all real numbers). F_X contains all the probabilistic information inherent in X, but in a more convenient format. It can be shown to be:

(1) Nondecreasing on $\mathbb{R}$,
(2) Continuous on the right at each point of $\mathbb{R}$, and to satisfy
(3) $\lim_{r\to-\infty} F_X(r) = 0$ and $\lim_{r\to\infty} F_X(r) = 1$.

Conversely, if $F: \mathbb{R} \to [0,1]$ is any function satisfying conditions (1) - (3), then there exists a random variable of which F is the c.d.f. Thus, the set of all c.d.f.'s is merely the set of all functions satisfying conditions (1) - (3).

Definition. A *lottery* is a c.d.f., that is, a function $F: \mathbb{R} \to [0,1]$ satisfying conditions (1) - (3). The set of all lotteries is denoted $\mathcal{F}$.

Note that, in general, a lottery need not have an expected value.

Bearing in mind the distinction between the empirical concept of lottery and our mathematical representation of it, consider an empirical compound lottery L that offers empirical lotteries L_1 and L_2 as prizes with probabilities p and 1 - p, respectively. Then, if the c.d.f.'s C_1 and C_2 are taken to represent L_1 and L_2, respectively, the c.d.f. $pC_1 + (1-p)C_2$ will represent L. (Note that $pC_1 + (1-p)C_2$ is indeed a c.d.f.; this is readily proved by reference to the defining properties (1) - (3).) *This simple relationship—the representation of compound empirical lotteries by convex combinations of c.d.f.'s—is central to the usefulness of c.d.f.'s as mathematical representations of empirical lotteries.*

Several important types of lottery are now defined.

Definition. For each $r \in \mathbb{R}$, define a lottery F_r by

$$F_r(t) = \begin{cases} 0 & \text{if } t < r \\ 1 & \text{if } t \geq r. \end{cases}$$

Then, F_r is called *degenerate*. The set of all degenerate lotteries is denoted $\mathcal{D}$.

A degenerate lottery is the c.d.f. of a constant random variable. F_r has the prize r with probability 1. In empirical work, degenerate lotteries may be used to represent "certainties."

Definition. A lottery F is called *simple* if it is a convex linear combination of degenerate lotteries, that is, if there exist a positive integer m, numbers $r_1,\ldots,r_m$ (not necessarily distinct), and nonnegative numbers $p_1,\ldots,p_m$ (not necessarily distinct), such that

$$\sum_{i=1}^{m} p_i = 1$$

and

$$F = \sum_{i=1}^{m} p_i F_{r_i}.$$

The set of all simple lotteries is denoted $\mathcal{H}$.

A simple lottery has (with probability 1) finitely many prizes.

Definition. A lottery F is called *discrete* if there exist a sequence $\{r_i\}_{i=1}^{\infty}$ of (not necessarily distinct) numbers and a sequence $\{p_i\}_{i=1}^{\infty}$ of (not necessarily distinct) nonnegative numbers such that

$$\sum_{i=1}^{\infty} p_i = 1$$

and

$$F = \sum_{i=1}^{\infty} p_i F_{r_i}.$$

The set of all discrete lotteries is denoted $\mathcal{H}_\sigma$.

Definition. A lottery F is called *continuous* if it is continuous as a function on $\mathbb{R}$ and *absolutely continuous* if there exists a Lebesgue integrable function $f: \mathbb{R} \to \mathbb{R}$ such that

$$F(r') - F(r) = \int_{r}^{r'} f(t)dt$$

whenever $r, r' \in \mathbb{R}$ and $r < r'$.

(The use of the now-standard, but seemingly unaccountable, term "absolutely continuous" to describe the stated property is apparently a vestige of an earlier period in the development of real analysis, when the definition of absolute continuity for a function F was in the following vein: "For each $\epsilon > 0$, there is a $\delta > 0$ such that

$$a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_n < b_n$$

implies

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon."$$

This property was proved equivalent to the sort of integral condition that nowadays is usually taken as the definition of absolute continuity (see *31*).)

For an absolutely continuous F, the function f is unique and nonnegative almost everywhere and satisfies $\int_{-\infty}^{\infty} f(t)dt = 1$. In short, F is absolutely continuous if and only if it has a probability density function.

Definition. A lottery F is called *singular* if F' equals 0 almost everywhere.

2.2 Lottery Spaces

Different economic problems may involve different types of risk. Thus, economic agents may be confronted with different "choice spaces" of lotteries in different situations. Yet, although the need to consider a variety of choice spaces is well accepted in, for example, consumer demand theory, it has not been given much attention in the risk literature.

What properties should a choice space of lotteries have? Expected utility theory (or the more general "measurable utility theory" pursued in this report) imposes only one condition: that the choice space be "closed under the formation of compound lotteries." Expressed mathematically, and in view of the previous discussion, this requirement is simply: whenever $0 \leq p \leq 1$ and the choice space contains C_1 and C_2, then it must contain $pC_1 + (1-p)C_2$. Observe, however, that the set of all functions from $\mathbb{R}$ into $\mathbb{R}$, endowed with the usual operations of addition/subtraction of functions and multiplication of functions by real numbers, is a vector space over $\mathbb{R}$ containing all lotteries as elements. Thus, the cited requirement can be restated as: the choice space (considered as a subset of this vector space) must be *convex*.

Definition. A *lottery space* is a convex set of lotteries.

One can readily verify that each of the following is a lottery space. the set of all (1) lotteries, (2) simple lotteries, (3) discrete lotteries, (4) continuous lotteries, (5) absolutely continuous lotteries, (6) singular lotteries, (7) lotteries with finite mean, and (8) lotteries that are c.d.f.'s of bounded random variables.

In view of the importance of normal distributions to the subject of risk, it is interesting that the set of all normal lotteries (that is, normal c.d.f.'s) is *not* a lottery space. To establish this fact by means of a counterexample, let F be the N(0,1) c.d.f. and f the N(0,1) probability density function. Then, there is an $x_0 \in \mathbb{R}$ such that $f(x_0) < 1/2(2\pi)^{1/2}$. Let G be the $N(2x_0,1)$ c.d.f. and g the $N(2x_0,1)$ probability density function. Now, if the c.d.f. $(1/2)F + (1/2)G$ were normal, then its derivative, $h = (1/2)f + (1/2)g$, would be a normal probability density function. But this is impossible, since the inequalities

$$\begin{aligned} h(0) &= (1/2)f(0) + (1/2)g(0) \\ &> (1/2)f(0) \\ &= 1/2(2\pi)^{1/2}, \end{aligned}$$

$$h(x_0) = (1/2)f(x_0) + (1/2)g(x_0)$$

$$< [1/4(2\pi)^{1/2}] + [1/4(2\pi)^{1/2}]$$

$$= 1/2(2\pi)^{1/2},$$

and

$$h(2x_0) = (1/2)f(2x_0) + (1/2)g(2x_0)$$

$$> (1/2)g(2x_0)$$

$$= 1/2(2\pi)^{1/2}$$

show that $h(x_0)$ is smaller than both $h(0)$ and $h(2x_0)$, although x_0 lies between 0 and $2x_0$. This argument establishes that the set of all normal lotteries is not convex. (This fact is certainly known in other contexts (*10*), but it has not been expressed clearly in the risk literature of economics.) However, any set of lotteries has a *convex hull*. Since the convex hull of a set S equals the set of all finite convex combinations of elements of S (that is,

$$\{p_1s_1 +\ldots+ p_ns_n \mid n \in \mathbb{N},\ p_1,\ldots,p_n \in [0,1],\ \sum_{i=1}^{n} p_i = 1,\ \text{and } s_1,\ldots,s_n \in S\},$$

where $\mathbb{N}$ is the set of all positive integers), a lottery space, Co(S), may be constructed from *any* set S of lotteries (such as the set of all normal lotteries) by forming all finite convex combinations of lotteries in S, and Co(S) is the "smallest" lottery space containing S.

2.3 Relationship to Other Approaches

Our definition of lotteries as c.d.f.'s contains the implicit assumption that a lottery's ultimate prizes can be represented by real numbers. Indeed, if F is a lottery and t is a real number, we are interpreting F(t) as the probability that the lottery will provide an ultimate prize in the interval $(-\infty,t]$. Thus, in our approach, the real line represents the set of possible prizes, and these prizes are presumed to be quantities of money or the like.

It is possible, however, to define lotteries so that quite general types of objects are permissible as ultimate prizes. One such definition characterizes a lottery as a probability measure defined on a measurable space (*20*) of prizes. To understand how this approach relates to our own, recall that there is a natural one-to-one correspondence between the set $\mathcal{F}$ of all c.d.f.'s F and the set of all Borel probability measures m on $\mathbb{R}$, given by

$$m((-\infty,t]) = F(t) \quad (F \in \mathcal{F},\ t \in \mathbb{R})$$

(see *8*). For any c.d.f. F, this formula determines a unique Borel probability measure m on the real line (that is, in effect, on our set of prizes) that contains the same probabilistic information as F, but in a different format. Clearly, the correspondence maps convex combinations of c.d.f.'s to convex combinations of the corresponding measures; thus, Borel probability measures on $\mathbb{R}$ share the ability of c.d.f.'s to represent empirical compound lotteries conveniently in terms of their sublotteries. Although the use of point functions such as c.d.f.'s does offer computational advantages, there would have been no *conceptual* barrier to our originally defining lotteries to be Borel probability measures on $\mathbb{R}$. Similarly, given a measurable space consisting of *any* objects considered prizes, one could define a lottery to be a probability measure on that space (see *18*).

An even more general definition of lottery is implicit in Herstein and Milnor (*23*). There a "mixture set" is defined as any set of objects that are capable of being combined with one another, and with weights in [0,1], to form analogs of convex combinations. Convex sets of c.d.f.'s and convex sets of probability measures are subsumed as special cases. (See pp. 11-14 for a fuller discussion of this important paper.)

Because many economists learn the fundamentals of the theory of behavior under uncertainty from microeconomics textbooks, it is instructive to examine how such works typically approach the subject of lotteries. Let us consider two widely used texts, Varian (*44*) and Henderson and Quandt (*22*).

Varian does not define lotteries as specific mathematical entities, but instead characterizes them through several axioms. All lotteries are assumed to have only two prizes (themselves possibly lotteries); a lottery with prizes x and y that occur with probabilities p and 1-p, respectively, is denoted

$$p \circ x + (1-p) \circ y.$$

The axioms are:

$$1 \circ x + (1-1) \circ y = x \qquad \text{(L1)}$$

$$p \circ x + (1-p) \circ y = (1-p) \circ y + p \circ x \qquad \text{(L2)}$$

$$q \circ (p \circ x + (1-p) \circ y) + (1-q) \circ y = (qp) \circ x + (1-qp) \circ y. \qquad \text{(L3)}$$

The intended interpretation of axiom (L1) is that "getting a prize with probability one is the same as getting the prize for certain." (L2) signifies that the order in which a lottery's prizes are specified is of no consequence. (L3), the "compound lottery axiom," requires that the lottery whose prizes are $p \circ x + (1-p) \circ y$ and y (attained with probabilities q and $1 - q$, respectively) be considered the same as the lottery whose prizes are x and y (attained with probabilities qp and $1 - qp$, respectively). In effect, this axiom stipulates that the internal structure of a lottery is immaterial; only the ultimate prizes and their probabilities of being attained are significant. Thus, (L3) is an axiomatic statement of the convention on representing lotteries by c.d.f.'s adopted earlier (p. 4).

C.d.f.'s clearly satisfy the preceding characterization of lotteries. In fact, if we interpret $p \circ x + (1-p) \circ y$ as the convex combination $px + (1-p)y$ whenever x and y are c.d.f.'s and $0 \leq p \leq 1$, then any c.d.f. x can be expressed in the "two-prize" notation as $1 \circ x + 0 \circ x$, and equations (L1) through (L3) reduce to trivially true statements concerning the algebra of functions, namely:

$$1x + 0y = x \tag{L1'}$$

$$px + (1-p)y = (1-p)y + px \tag{L2'}$$

$$q(px + (1-p)y) + (1-q)y = qpx + (1-qp)y. \tag{L3'}$$

(Similarly, probability measures defined on measurable sets of prizes satisfy equations (L1) through (L3).) Thus, there is no logical inconsistency between our definition of lottery and the more general characterization given in Varian (44). Nevertheless, using c.d.f.'s has some advantages. We shall return to this point shortly.

In Henderson and Quandt (22), as in Varian, lotteries are not defined as a known type of mathematical object, but are characterized axiomatically. Only one axiom, an analog of the compound lottery axiom described above, is given. Attention again centers on lotteries having only two prizes; a lottery with prizes A, B (and corresponding probabilities p, 1-p) is denoted

$$(p,A,B).$$

Just as Varian's $p \circ x + q \circ y$ may be interpreted as the convex combination $px + (1-p)y$ of the c.d.f.'s x and y, Henderson and Quandt's (p,A,B) may be interpreted as the convex combination $pA + (1-p)B$ whenever A and B are c.d.f.'s and $0 \leq p \leq 1$. (In particular, when A and B are money prizes (and thus represented in our approach by degenerate c.d.f.'s), (p,A,B) may be interpreted as the c.d.f. $pF_A + (1-p)F_B$, which (under the assumption that $A \leq B$) is a step function F given by

$$F(t) = \begin{cases} 0 & \text{if } t < A \\ p & \text{if } A \leq t < B \\ 1 & \text{if } B \leq t. \end{cases}$$

A similar remark holds for Varian's $p \circ x + q \circ y$.)

Economists generally believe that axiom systems for behavior under risk based on abstract characterizations of lotteries (*22, 23, 44*) imply such standard results as the propositions that individuals act to maximize their expected utility (of, say, income) or that risk aversion is equivalent to concavity of the utility function of (say) income. We will show, however, that these results require an additional assumption. The assumption that preferences are continuous, imposed on a choice space of c.d.f.'s., will prove sufficient (see also *18*). This assumption is meaningful when lotteries are defined as c.d.f.'s, since the set of all c.d.f.'s has a natural topology, the "topology of weak convergence." In contrast, the abstract characterizations of lotteries do not provide for any topological structure on the choice space. Under these characterizations, an assumption of continuous preferences is inexpressible, and the standard results may fail.

Although expected utility theory may be considered to have begun with Cramer (*9*) and Bernoulli (*6*), it was not until the appearance of von Neumann and Morgenstern's pathbreaking study (*45*) that economists succeeded in their quest to find some rational basis for the intuitively appealing principle that individuals faced with a choice among risky prospects attempt to maximize their "expected utility." Taking as given a preference ordering of lotteries satisfying several plausible axioms, von Neumann and Morgenstern showed that one can define a numerical-valued function μ of lotteries so that:
(1) $\mu(L_1) > \mu(L_2)$ if and only if L_1 is strictly preferred to L_2, and
(2) $\mu(pL_1 + (1-p)L_2) = p\mu(L_1) + (1-p)\mu(L_2)$ whenever L_1 and L_2 are in the choice space and $0 \leq p \leq 1$. Condition (2) is a linearity property reminiscent of taking expected values. Functions satisfying (1) and (2) are called *measurable utility functions*.

Following the simplified and generalized approach of Herstein and Milnor (*23*), we now present basic definitions and sketch the proof of existence of measurable utility functions. We establish the properties of uniqueness and invariance and relate these properties to the historical controversy over whether measurable utility is an ordinal or cardinal measure. We define continuity (including continuity of preference orderings) and use a result of Grandmont (*18*) to establish that any risk preference ordering representable by a discontinuous measurable utility function must itself be discontinuous. Finally, we present new results on the decomposition of measurable utility functions.

3.1 Existence

The Herstein-Milnor proof of the existence of measurable utility functions is based on the abstract concept of a "mixing operation," a mathematical device reminiscent of the process of constructing a compound lottery out of two lotteries and a probability. Recall that, if x and y are lotteries and $0 \leq p \leq 1$, then $px + (1-p)y$ is often termed a "probability mixture" of x and y. In this context, a function that maps each p, x, and y to $px + (1-p)y$ is the prototype of a mixing operation. Although this particular type of mixing operation is of greatest concern to us, Herstein and Milnor actually prove the existence of a measurable utility function for a wide class of mixing operations. Indeed, the objects being mixed do not even have to be lotteries, although lotteries are a particular (and important) case. Thus, the Herstein-Milnor approach reveals that von Neumann and Morgenstern's "measurable utility theorem" is but a special case of a very general result.

We now present the formal definition of a mixture space:

Definition. Let S be a set and $M: [0,1] \times S \times S \rightarrow S$ a function such that the following properties hold for all $a,b \in S$ and all $\kappa,\lambda \in [0,1]$:

$$M(1,a,b) = a \tag{1'}$$

$$M(\lambda,a,b) = M(1-\lambda,b,a) \tag{2'}$$

$$M(\kappa,M(\lambda,a,b),b) = M(\kappa\lambda,a,b). \tag{3'}$$

Then, (S,M) is called a *mixture space*, M a *mixing operation on* S, and S a *mixture set with mixing operation* M.

This general concept of a mixture space becomes more familiar under the following notational convention: given a mixing operation M on S, any $a,b \in S$, and any $\lambda \in [0,1]$, use the symbol "$\lambda a + (1-\lambda)b$" to denote $M(\lambda,a,b)$. In this notation, the symbols "λa" and "$(1-\lambda)b$" need not themselves be assigned any algebraic meaning; that is, these composite symbols are not necessarily intended to denote the result of any mathematical operation. Similarly, the symbol "+" should not necessarily be interpreted as having any meaning in itself, such as summation. Rather, given λ, a, and b, it is only the undivided symbol "$\lambda a + (1-\lambda)b$" to which meaning is here being attached—namely, as an alternate means of denoting the function value $M(\lambda,a,b)$. Under this convention, properties (1') – (3') reappear as

$$1a + (1-1)b = a \tag{1"}$$

$$\lambda a + (1-\lambda)b = (1-\lambda)b + \lambda a \tag{2"}$$

$$\kappa[\lambda a + (1-\lambda)b] + (1-\kappa)b = (\kappa\lambda)a + (1-\kappa\lambda)b. \tag{3"}$$

The suggestiveness of the notation "$\lambda a + (1-\lambda)b$" and of the properties (1") – (3") (which read like the properties (L1') – (L3') of lotteries (p. 10)) is, of course, no accident, for an important example of a mixture space is furnished by any lottery space S paired with the probability mixing operation that maps $\lambda \in [0,1]$ and lotteries $a,b \in S$ to the lottery $\lambda a + (1-\lambda)b \in S$. Thus, although Herstein and Milnor prove their general results without ascribing any algebraic meaning to the notation "$\lambda a + (1-\lambda)b$", this notation *does* coincide with standard algebraic notation, and *can* be interpreted algebraically, when the mixture space consists of a lottery space with probability mixing. Henceforth, whenever we treat a lottery space as a mixture set, the use of probability mixing will be implicitly assumed. (For other applications of the concept of a mixing operation, such as to color vision, see *19*.)

Definition. Let S be a set on which is defined a complete weak preference ordering (that is, a complete transitive relation), $\succsim$. The corresponding strong preference ordering, $\succ$, is defined by:

$$a \succ b \text{ if and only if } a \succsim b \text{ and not } b \succsim a.$$

The corresponding indifference relation, $\sim$, is defined by:

$$a \sim b \text{ if and only if } a \succsim b \text{ and } b \succsim a.$$

We call $(S,\succsim)$ a *preference space*.

Definition. Let $(S,\succsim)$ be a preference space. A function $\mu: S \to \mathbb{R}$ is called *order-preserving (with respect to* $\succsim$*)* if, for any $a,b \in S$, one has $\mu(a) > \mu(b)$ if and only if $a \succ b$ (or, equivalently, $\mu(a) \geq \mu(b)$ if and only if $a \succsim b$).

Definition. Let (S,M) be a mixture space. A function $\nu: S \to \mathbb{R}$ is called *linear* if, for all $a,b \in S$ and all $\lambda \in [0,1]$,

$$\nu(M(\lambda,a,b)) = M(\lambda,\nu(a),\nu(b)),$$

that is,

$$\nu(\lambda a + (1-\lambda)b) = \lambda\nu(a) + (1-\lambda)\nu(b).$$

The preceding notion of linearity should not be confused with linearity for a vector-space mapping, although the two notions are closely related. The notion used here may be motivated by the fact that a real-valued function defined on an interval of the real line satisfies the second formula given in the definition if and only if it is both concave and convex, and the latter condition holds if and only if the graph of the function is a segment of a straight line. (Such a function is linear in the vector-space sense if and only if its domain is the entire real line and its graph contains the origin.)

Definition. Let S be a set with a complete weak preference ordering $\succsim$ and a mixing operation M. We call $(S,\succsim,M)$ a *preference mixture space*. A linear, order-preserving function on S is called a *measurable utility function*.

The central question addressed by Herstein and Milnor is that of the *existence* of a measurable utility function on a set S. In the situation of most concern to us, S is a set of lotteries and M is the probability mixing operation on S. Herstein and Milnor's main result (*23*) is:

Theorem 1. Suppose $(S,\succsim,M)$ is a preference mixture space for which the following assumptions hold:

(1) For any $a,b,c \in S$, the sets $\{\alpha \in [0,1] \mid \alpha a + (1-\alpha)b \succsim c\}$ and $\{\alpha \in [0,1] \mid c \succsim \alpha a + (1-\alpha)b\}$ are closed; and

(2) For any $a,a',b \in S$, if $a \sim a'$, then

$$(1/2)a + (1/2)b \sim (1/2)a' + (1/2)b.$$

Then, there exists a measurable utility function on S.

For a detailed proof, see Herstein and Milnor (*23*). However, the intuition underlying the construction of a measurable utility function on S is as follows: given $a \succ b$, consider the "interval"

$$S_{ab} = \{x \in S \mid a \succsim x \succsim b\}.$$

It is first proved that, for each $x \in S_{ab}$, there is a unique element $\mu_{ab}(x)$ of $[0,1]$ such that

$$x \sim \mu_{ab}(x)a + (1-\mu_{ab}(x))b.$$

(This result is analogous to the simple fact that, if a, x, and b are real numbers for which $x \in [b,a]$, then there is a unique $\mu_{ab}(x) \in [0,1]$ such that

$$x = \mu_{ab}(x)a + (1-\mu_{ab}(x))b.)$$

Of course, $\mu_{ab}(x)$ is determined not only by x, but by a and b as well. To arrive at a method of assigning a "utility value" to x alone, one now selects, and henceforth "holds fixed," two elements r_0, r_1 of S satisfying $r_1 > r_0$. Then, given any $x \in S$, one chooses any $a,b \in S$ for which $x, r_0, r_1 \in S_{ab}$ and defines

$$M_{ab}(x) = [\mu_{ab}(x)-u_{ab}(r_0)]/[\mu_{ab}(r_1)-\mu_{ab}(r_0)].$$

It can be shown that, if also $a',b' \in S$ are such that $x, r_0, r_1 \in S_{a'b'}$, then

$$M_{a'b'}(x) = M_{ab}(x).$$

Thus, $M_{ab}(x)$ depends only on x and not on which particular a,b are chosen, and it can be denoted $\mu(x)$. The function $\mu: S \to \mathbb{R}$ thus defined can be shown to be linear and order-preserving, hence a measurable utility function on S.

Examining the above reasoning in the context of real numbers may provide additional insight. For real numbers $a > b$ with $x \in [b,a]$, one easily calculates that

$$\mu_{ab}(x) = (x-b)/(a-b).$$

Given $r_1 > r_0$ and assuming $x, r_0, r_1 \in [b,a]$, one finds that

$$M_{ab}(x) = [(x-b)/(a-b) - (r_0-b)/(a-b)]/[(r_1-b)/(a-b) - (r_0-b)/(a-b)],$$

which does not depend on a or b. Thus, in this case,

$$\mu(x) = (x-r_0)/(r_1-r_0).$$

Note that μ is order-preserving with respect to the relation $\geq$ on $\mathbb{R}$. Furthermore, μ is linear on $\mathbb{R}$ with respect to the mixing operation M' defined by the ordinary algebraic formula

$$M'(\alpha',a',b') = \alpha'a' + (1-\alpha')b'.$$

In fact,

$$\begin{aligned}\mu(\alpha'a' + (1-\alpha')b') &= [(\alpha'a' + (1-\alpha')b') - r_0]/[r_1-r_0] \\ &= \alpha'(a'-r_0)/(r_1-r_0) + (1-\alpha')(b'-r_0)/(r_1-r_0) \\ &= \alpha'\mu(a') + (1-\alpha')\mu(b').\end{aligned}$$

When S is a lottery space, condition (2) of Theorem 1 is known in various forms as the "independence axiom." Its interpretation is that, if a, a', and b are any lotteries and an individual is indifferent between a and a', then he/she is indifferent between the compound lottery offering prizes a and b, each with a 50-percent probability, and the compound lottery offering prizes a' and b, each with a 50-percent probability.

3.2 Uniqueness and Invariance

Although Herstein and Milnor (23) address the question of the *existence* of a measurable utility function, they do not explicitly consider the *uniqueness* of such a function. We now address this issue, following Varian's approach (44), but filling in some gaps in his treatment.

We begin with the following result, which, although basically well-known, is apparently often confused with the "invariance property" (see p. 17):

Proposition 1. Let μ and ν be order-preserving functions on a preference space $(S,\succsim)$. Then, there exists a unique function

$$f: \text{Range }(\nu) \rightarrow \text{Range }(\mu)$$

such that

$$\mu = f \circ \nu.$$

Moreover, f is increasing. (*Note:* Here and in the remainder of this report, "$\circ$" has its usual meaning of composition of functions.)

Proof. Given any $r \in \text{Range }(\nu)$, choose any $a \in \nu^{-1}(r)$, and put

$$f(r) = \mu(a).$$

Note that f is a well-defined function, since, if $a,a' \in \nu^{-1}(r)$, then $\nu(a) = r = \nu(a')$, from which it follows that $a \sim a'$ and $\mu(a) = \mu(a')$.

Furthermore, $\mu = f \circ \nu$, since, for any $a \in S$, we have $a \in \nu^{-1}(\nu(a))$, so that $\mu(a) = f(\nu(a))$. Clearly, f is unique, for if also $\mu = g \circ \nu$, then $0 = (f-g) \circ \nu$, so that f and g agree on Range (ν). Finally, f is increasing; for, suppose $r,r' \in \text{Range }(\nu)$ and $r > r'$. Then, there exist $a,a' \in S$ such that $\nu(a) = r$ and $\nu(a') = r'$. Necessarily, $a \succ a'$, so that $f(r) = \mu(a) > \mu(a') = f(r')$.

Q.E.D.

For measurable utility functions, the range assumes a particularly simple form:

Proposition 2. Let μ be a linear function on a mixture space (S,M). Then, Range (μ) is an interval.

Proof. Consider any $r,r' \in \text{Range }(\mu)$, $\lambda \in [0,1]$. We have $r = \mu(a)$, $r' = \mu(a')$ for some $a,a' \in S$. Then $\lambda a + (1-\lambda)a' \in S$ and

$$\begin{aligned}\mu(\lambda a + (1-\lambda)a') &= \lambda\mu(a) + (1-\lambda)\mu(a') \\ &= \lambda r + (1-\lambda)r' \\ &\in \text{Range }(\mu).\end{aligned}$$

Q.E.D.

To characterize the function f of Proposition 1 when μ and ν are measurable utility functions, we shall need the following:

Definition. Let S be a subset of $\mathbb{R}$. A function A: $S \to \mathbb{R}$ is called *affine* on S if there exist $a,b \in \mathbb{R}$ such that, for all $x \in S$,

$$A(x) = ax + b.$$

If $S = \mathbb{R}$, such a function is called simply *affine*.

Lemma 1. Let S be an interval in $\mathbb{R}$. Then, a function A: $S \to \mathbb{R}$ is affine on S if and only if, for any $x,y \in S$, $p \in [0,1]$,

$$A(px + (1-p)y) = pA(x) + (1-p)A(y).$$

Proof. If A is affine on S, a simple calculation shows that the above formula holds. To prove the converse, assume that this formula holds and (without loss of generality) that S contains more than one point. Consider any $c,d \in S$ with $c < d$. For any $w \in [c,d]$, it is easy to show that

$$w = t_w c + (1-t_w)d,$$

where $t_w = (w-d)/(c-d)$. Since $t_w \in [0,1]$, it follows that

$$\begin{aligned} A(w) &= t_w A(c) + (1-t_w)A(d) \\ &= t_w(A(c)-A(d)) + A(d) \\ &= w(A(c)-A(d))/(c-d) + (cA(d)-dA(c))/(c-d), \end{aligned}$$

so that the restricted function $A\big|[c,d]$ is affine on $[c,d]$. However, it is easily shown (for example, by differentiation) that the coefficients of $A\big|[c,d]$ are the same over all intervals $[c,d] \subset S$. It follows that A is affine on S.

Q.E.D.

We can now prove the following:

Theorem 2. Suppose μ and ν are measurable utility functions on a preference mixture space $(S,\succsim,M)$. Then, there exists an increasing affine transformation A such that

$$\mu = A \circ \nu.$$

(That is, "a measurable utility function on $(S,\succsim,M)$ is unique up to an increasing affine transformation".)

Proof. By Proposition 1, there is an increasing function

$$f\colon \text{Range}(\nu) \to \text{Range}(\mu)$$

such that $\mu = f \circ \nu$. Consider any $c,d \in \text{Range}(\nu)$, $p \in [0,1]$. We have $c = \nu(x)$ and $d = \nu(y)$ for some $x,y \in S$. Thus

$$f(pc + (1-p)d) = f(p\nu(x) + (1-p)\nu(y))$$

$$= f(\nu(px + (1-p)y))$$

$$= \mu(px + (1-p)y)$$

$$= p\mu(x) + (1-p)\mu(y)$$

$$= pf(\nu(x)) + (1-p)f(\nu(y))$$

$$= pf(c) + (1-p)f(d).$$

It follows by Lemma 1 that f is affine on Range (ν). Since f can clearly be extended to an affine transformation, the theorem is proved.

Q.E.D.

Corollary. Suppose μ and ν are measurable utility functions on $(S,\succsim,M)$. Then, whenever $x_1,x_2,y_1,y_2 \in S$ and $y_1 \not\sim y_2$,

$$[\mu(x_2)-\mu(x_1)]/[\mu(y_2)-\mu(y_1)] = [\nu(x_2)-\nu(x_1)]/[\nu(y_2)-\nu(y_1)].$$

(Intuitively, the ratio of utility differences depends only on $(S,\succsim,M)$ and the lotteries chosen, not on the choice of measurable utility function.)

Proof. Obvious.

In addition to being unique up to an increasing affine transformation, measurable utility functions are also *invariant* under increasing affine transformations:

Theorem 3. If μ is a measurable utility function on $(S,\succsim,M)$ and A is an increasing affine transformation, then $A \circ \mu$ is a measurable utility function on $(S,\succsim,M)$.

Proof. Obvious.

Though apparently often confused with one another, the concepts of "uniqueness" and "invariance" for utility functions are, in a certain precise sense, exact opposites. To exhibit this relationship in the case of measurable utility functions, let $(S,\succsim,M)$ be a preference mixture space, and let U be the set of all measurable utility functions on $(S,\succsim,M)$. Define a set-valued function, T, on U as follows: for each $\mu \in U$, $T(\mu)$ is the set of all transforms of μ by increasing affine transformations; that is,

$$T(\mu) = \{A \circ \mu \mid A \text{ is increasing and affine}\}.$$

Then, invariance means:

$$\text{for each } \mu \in U,\ T(\mu) \subset U,$$

while uniqueness means:

$$\text{for each } \mu \in U,\ U \subset T(\mu).$$

Another source of much confusion in the literature has been the question of whether "measurable utility" is an ordinal or cardinal measure (see *22*, p. 52; *5*, *17*). A complete characterization of measurable utility as a "measurement device" is provided by Theorems 2 and 3, which imply that, in the language of modern measurement theory, measurable utility defines an "interval scale" (see *36* for the requisite background in measurement theory).

3.3 Continuity

The question of when a measurable utility function is continuous presupposes that the meaning of "continuous," as applied to a function whose domain is a set of lotteries, is understood. A full consideration of this topic would involve general topology. However, a less abstract approach using the notion of a "metric space" will suffice for our purposes.

The concept of the continuity of a function f at a point x_0 may be expressed informally and heuristically by the requirement that, whenever x approaches x_0, f(x) approaches $f(x_0)$. When the context admits of some appropriate notion of distance, this characterization may be re-expressed as: whenever the distance between x and x_0 approaches 0, f(x) approaches $f(x_0)$. We now introduce formally the notion of a "distance function" or "metric."

Definition. Let S be a set and $d: S \times S \to [0,\infty)$ a function satisfying:

(1) For all $a,b \in S$, $d(a,b) = 0$ if and only if $a = b$;

(2) For all $a,b \in S$, $d(a,b) = d(b,a)$ ("symmetry"); and

(3) For all $a,b,c \in S$, $d(a,c) \leq d(a,b) + d(b,c)$ (the "triangle inequality").

Then, d is called a *metric on* S and (S,d) is called a *metric space*.

As an example, a metric d_1 on $\mathbb{R}^2$ can be defined by

$$d_1\left[(r_1,r_2),(s_1,s_2)\right] = [(r_1-s_1)^2 + (r_2-s_2)^2]^{1/2}$$

for any $(r_1,r_2),(s_1,s_2) \in \mathbb{R}^2$. This metric corresponds to the usual notion of the distance between two points in the plane.

Now, there is perhaps no immediately obvious notion of the "distance" between two lotteries. However, it can be shown that there exists a metric, d, on the set F of all lotteries, having the property that, for any lottery L and any sequence $\{L_n\}_{n=1}^{\infty}$ of lotteries, one has $d(L_n,L) \to 0$ as $n \to \infty$ if and only if $\{L_n\}_{n=1}^{\infty}$ *converges weakly* to L (*11*, p. 285). (Recall that weak convergence of $\{L_n\}_{n=1}^{\infty}$ to L means that, for each point t of continuity of L, $\lim_{n\to\infty} L_n(t) = L(t)$. Weak convergence defines a natural topology on F.) This correspondence between weak convergence and a notion of distance motivates the following:

Definition. Suppose S is a set of lotteries, $\mu: S \to \mathbb{R}$ is a function, and $L_0 \in S$. Then, μ is *continuous at* L_0 if, whenever $\{L_n\}_{n=1}^{\infty}$ is any sequence of elements of S that converges weakly to L_0, one has $\mu(L_n) \to \mu(L_0)$ as $n \to \infty$. If μ is continuous at L for each $L \in S$, μ is called *continuous*. Moreover, a weak preference ordering $\succsim$ on S is called *continuous* (and we may speak of "continuous preferences") if there exists a continuous order-preserving function on S representing $\succsim$.

Definition. If S is a lottery space and $(S,\succsim)$ is a preference space, we call $(S,\succsim)$ a *lottery preference space*.

Theorem 4 (18). Let $(S,\succsim)$ be a lottery preference space. Then, there exists a continuous order-preserving function on $(S,\succsim)$ if and only if

(1) For any $L_0 \in S$, the sets $\{L \in S \mid L \succsim L_0\}$ and $\{L \in S \mid L_0 \succsim L\}$ are closed in S. (For the set $\{L \in S \mid L \succsim L_0\}$, for example, this means that, for any sequence of lotteries L_n satisfying $L_n \succsim L_0$ for all n and converging weakly to a lottery L, one has $L \succsim L_0$.)

Moreover, there exists a continuous measurable utility function on $(S,\succsim)$ if and only if, in addition to condition (1), the following condition holds:

(2) For any $L_1, L_2, L_3 \in S$ and any $t \in [0,1]$, if $L_1 \sim L_2$, then

$$tL_1 + (1-t)L_3 \sim tL_2 + (1-t)L_3.$$

For the proof, see Grandmont (*18*). (Note that Grandmont uses the more general definition of a lottery as a probability measure.)

At various places in this report, we will want to be able to conclude that a preference ordering represented by a discontinuous measurable utility function is itself discontinuous. What will justify this assertion? After all, in general, the mere existence of a discontinuous order-preserving function certainly does not imply that the corresponding preference ordering is discontinuous. (Indeed, given any *continuous* preference ordering, one can always construct a *discontinuous* order-preserving function for it by composing one of its continuous order-preserving functions with a discontinuous increasing function from $\mathbb{R}$ into $\mathbb{R}$.) However, measurable utility functions enjoy the following distinctive property: *a preference ordering having a discontinuous measurable utility function must be discontinuous.* In fact, *a preference ordering represented by a measurable utility function is continuous if and only if each of its measurable utility functions is continuous.*

To prove these claims, recall that, by Theorem 2, all measurable utility functions on a lottery preference space $(S,\succsim)$ are transforms of one another by increasing affine transformations. Thus, whenever *one* is continuous, *all* are continuous. Suppose there exists a discontinuous measurable utility function on $(S,\succsim)$. Then, all measurable utility functions on $(S,\succsim)$ must be discontinuous, and it follows from the second half of Theorem 4 that either condition (1) or condition (2) of that theorem must fail. But condition (2) holds, since $(S,\succsim)$ has a measurable utility function. Thus, condition (1) must fail, and it follows from the first half of Theorem 4 that $(S,\succsim)$ has no continuous order-preserving function; that is, $\succsim$ is discontinuous.

3.4 Decomposition

We will now prove a structural decomposition theorem for measurable utility functions defined on a lottery space. We will see how such a function may be broken down into a "discrete part," an "absolutely continuous part," and a "singular continuous part." This result will further our understanding in three respects. First, it will reveal explicitly the separateness of an economic agent's behavior toward discrete lotteries (and thus toward degenerate lotteries, which represent certainties) and that agent's behavior toward continuous lotteries ("continuous uncertainties") when this behavior is represented by a measurable utility function. It will thereby provide--for some situations--a theoretical rationale for the related conjectures: (1) that individuals use a different frame of reference (*1, 43*) when choosing under certainty than when choosing under certain forms of uncertainty and (2) that individuals' risk preferences are discontinuous. Second, this result will show how a measurable utility function relates to known classes of lotteries that may lend themselves to econometric and statistical applications. Third, it will suggest a new, canonical method of constructing measurable utility functions (a method that we shall use, implicitly, in sections 4-6 of this study). This new method will reveal the existence of an entirely new class of discontinuous preference orderings under uncertainty; previously, the property of discontinuity had apparently been associated only with lexicographic orderings (*16, 21, 28,* 42), and then rather tenuously.

Although measurable utility functions remain our basic concern, we will state and prove the decomposition theorem for *linear* functions, as the decomposition depends only on the algebraic structure of the function. However, any increase in generality is only apparent, because a linear function μ on a lottery space S is automatically a measurable utility function on $(S,\succsim_\mu)$, where $\succsim_\mu$ is a complete transitive relation on S defined by

$$L_1 \succsim_\mu L_2 \text{ if and only if } \mu(L_1) \geq \mu(L_2) \quad (L_1, L_2 \in S).$$

The decomposition of linear functions on lottery spaces will be seen to be rooted in the decomposition of the lotteries themselves. Thus, we begin with some remarks about lotteries and their decompositions. Details and further background may be found in (*8, 31*).

Any discrete lottery is singular (*8*, p. 12). We shall be interested, however, only in those singular lotteries that are *continuous*, of which a classic example is the Cantor distribution (*11*, p. 141). The sets of all singular continuous, absolutely continuous, and discrete lotteries, respectively, are convex and pairwise disjoint.

Now, it is well-known that, for any lottery L, there exist $p_1, p_2, p_3 \in [0,1]$ and lotteries L_1, L_2, L_3 that are, respectively, discrete, absolutely continuous, and singular continuous, such that

$$p_1 + p_2 + p_3 = 1$$

and

$$L = p_1L_1 + p_2L_2 + p_3L_3.$$

The numbers p_1, p_2, p_3 are unique. Although L_1, L_2, L_3 are not generally unique (for example, if L is discrete, then $p_2 = p_3 = 0$, and L_2 and L_3 may be arbitrary absolutely continuous and singular continuous lotteries, respectively), the decomposition is unique in the sense that the product functions p_1L_1, p_2L_2, and p_3L_3 are unique. These functions may be viewed naturally as the (uniquely determined) discrete, absolutely continuous, and singular continuous parts of L.

However, unless $p_i = 1$, p_iL_i will not be a c.d.f.; thus, a linear function μ that may be defined at L will not generally be defined at p_iL_i. To achieve a convenient decomposition of μ, we will need to extend the domain of μ to contain all the above parts p_iL_i of each lottery L at which μ is defined. To facilitate this process of extension, we assume that, whenever μ is defined at L, it is also defined at each L_i for which $p_i \neq 0$. (Note that $p_i \neq 0$ if and only if L_i is unique in the obvious sense. In fact, if $p_iL_i = q_iM_i$ and $p_i \neq 0$, then, since

$$p_i(\lim_{t\to\infty} L_i(t)) = q_i(\lim_{t\to\infty} M_i(t)),$$

we obtain $p_i = q_i \neq 0$ and, therefore, $L_i = M_i$. The converse is obvious.)

We will now make these ideas more precise.

Definition. A function $K: \mathbb{R} \to [0,1]$ is called a *subdistribution function* (*s.d.f.*) if:

(1) it is nondecreasing;

(2) $\lim_{t\to-\infty} K(t) = 0$; and

(3) it is right-continuous at each $t \in \mathbb{R}$.

One can easily verify that K is an s.d.f. if and only if there exist a lottery F and a $p \in [0,1]$ such that $K = pF$. The number p is unique (in fact, $p = \lim_{t\to\infty} K(t)$). Moreover, F is unique unless K is the zero function.

Definition. An s.d.f. K is called *discrete* (respectively, *absolutely continuous*, *singular continuous*) if there is a $p \in [0,1]$ and a discrete (respectively, absolutely continuous, singular continuous) lottery F such that $K = pF$.

The sets of all nonzero s.d.f.'s that are, respectively: (1) discrete, (2) absolutely continuous, (3) singular continuous, are pairwise disjoint (where "nonzero" means "not the zero function"). Note that it follows from our definitions that the zero function is a discrete, absolutely continuous, *and* singular continuous s.d.f. Allowing this "degenerate" case will simplify our work.

We can now reformulate our description of the decomposition of a lottery as follows:

Proposition 3. If K is any s.d.f., there exist unique s.d.f.'s K_1, K_2, K_3 such that K_1 is discrete, K_2 is absolutely continuous, K_3 is singular continuous, and

$$K = K_1 + K_2 + K_3.$$

Notation. If S is any set of lotteries, we define S_* as the convex hull (within the vector space over $\mathbb{R}$ of all functions from $\mathbb{R}$ into $\mathbb{R}$) of $S \cup \{0\}$ (where "0" denotes the zero function on $\mathbb{R}$).

Observe that a set S of lotteries is empty if and only if

$$S_* = \{0\}$$

and is nonempty if and only if

$$S_* = \{pL \mid p \in [0,1] \text{ and } L \in S\}.$$

The sets of all (1) s.d.f.'s, (2) discrete s.d.f.'s, (3) absolutely continuous s.d.f.'s, and (4) singular continuous s.d.f.'s are of the latter form.

Notation. For simplicity, we denote by S_1, S_2, S_3 the sets of all lotteries that are, respectively: (1) discrete (so that $S_1 = H_\sigma$), (2) absolutely continuous, (3) singular continuous. Accordingly, S_{1_*}, S_{2_*}, S_{3_*} are the sets of all s.d.f.'s of th respective types.

Definition. Let S' be a convex set of s.d.f.'s. A function μ': $S' \to \mathbb{R}$ is called *linear* if, for any $p \in [0,1]$, $K_1, K_2 \in S'$, we have

$$\mu'(pK_1 + (1-p)K_2) = p\mu'(K_1) + (1-p)\mu'(K_2).$$

Theorem 5. Let μ be a linear function on a lottery space S. Then, μ has a unique linear extension μ_* on S_* satisfying

$$\mu_*(0) = 0;$$

that is, there exists a unique linear function μ_* on S_* such that

$$\mu_*(0) = 0$$

and

$$\mu_*\Big| S = \mu.$$

Proof. If S is empty, then μ is necessarily the empty function. In this case, $S_* = \{0\}$, and the zero function on S_* is the unique linear extension of μ taking zero to zero. Suppose, then, that S is not empty. Relying on the characterization of S_* described earlier, define

$$\mu_*(pL) = p\mu(L)$$

whenever $p \in [0,1]$, $L \in S$. Note that μ_* is well-defined on S_*, since either $pL = 0$, in which case $p = 0$ and $\mu_*(pL) = 0$, or $pL \neq 0$, in which case p and L are uniquely determined by pL. Furthermore, μ_* is linear. To establish this point, consider any $K_1, K_2 \in S_*$ and $\alpha \in [0,1]$, and put

$$Z = \alpha K_1 + (1-\alpha)K_2.$$

Of course, $K_1 = p_1L_1$ and $K_2 = p_2L_2$ for some $p_1, p_2 \in [0,1]$, $L_1, L_2 \in S$. Now, if $Z = 0$, then $\alpha p_1 = (1-\alpha)p_2 = 0$, so that

$$\begin{aligned}\mu_*(Z) &= 0\\ &= \alpha p_1\mu(L_1) + (1-\alpha)p_2\mu(L_2)\\ &= \alpha\mu_*(K_1) + (1-\alpha)\mu_*(K_2).\end{aligned}$$

Suppose, then, that $Z \neq 0$, and put

$$Z_\infty = \alpha p_1 + (1-\alpha)p_2.$$

Since $Z_\infty \in (0,1]$, we have

$$\begin{aligned}\mu_*(Z) &= \mu_*\Big[Z_\infty[(\alpha p_1/Z_\infty)L_1 + ((1-\alpha)p_2/Z_\infty)L_2]\Big]\\ &= Z_\infty\mu\Big[(\alpha p_1/Z_\infty)L_1 + ((1-\alpha)p_2/Z_\infty)L_2\Big]\\ &= Z_\infty\Big[(\alpha p_1/Z_\infty)\mu(L_1) + ((1-\alpha)p_2/Z_\infty)\mu(L_2)\Big]\\ &= \alpha\mu_*(K_1) + (1-\alpha)\mu_*(K_2).\end{aligned}$$

Finally, we show that μ_* is unique. Suppose that μ^* is any linear function on S_* such that $\mu^*(0) = 0$ and $\mu^*\big|S = \mu$. Then, for any $p \in [0,1]$, $L \in S$, we have

$$\begin{aligned}\mu^*(pL) &= \mu^*\Big[(1-p)0 + pL\Big] \\ &= (1-p)\mu^*(0) + p\mu^*(L) \\ &= 0 + p\mu(L) \\ &= \mu_*(pL).\end{aligned}$$

Thus,

$$\mu^* = \mu_*.$$

Q.E.D.

Notation. Given any linear function μ on a lottery space S, we will continue to use the "star symbol" "μ_*" to denote the unique linear extension to S_*.

Theorem 5 amounts to the assertion that the mapping that takes ν on S_* to its restriction $\nu\big|S$ is a one-to-one correspondence from the set of all linear functions ν on S_* satisfying $\nu(0) = 0$, *onto* the set of all linear functions on S. In fact, the *existence* of an extension for each linear μ on S amounts to the fact that the correspondence is *onto*, while the *uniqueness* amounts to the fact that the correspondence is *one-to-one*. This correspondence allows us, intuitively speaking, to view μ_* as merely another form of μ, and, where convenient, to study μ_* instead of μ. In particular, μ_* has a more naturally described decomposition into discrete, absolutely continuous, and singular continuous parts than does μ.

Definition. Let S be a set of s.d.f.'s with the property that, whenever $K \in S$ and $K = K_1 + K_2 + K_3$ (where $K_i \in S_{i_*}$, $i = 1,2,3$), then $K_i \in S$ $(i = 1,2,3)$. Then, we call S *decomposable*.

Thus, a decomposable set of s.d.f.'s is simply one that, whenever it contains an s.d.f., also contains its discrete, absolutely continuous, and singular continuous additive parts. For example, if S is the set of all lotteries, or the set of all lotteries with finite mean, then S_* is decomposable.

Definition. Given any decomposable set S of s.d.f.'s, we define the *projection operators*

$$\pi_i : S \to S \cap S_{i_*} \quad (i = 1,2,3)$$

as follows: for any $K \in S$, write $K = K_1 + K_2 + K_3$, where $K_i \in S_{i_*}$

$(i = 1,2,3)$. Then

$$\pi_i(K) = K_i \quad (i = 1,2,3).$$

The π_i's are analogous to the canonical projections associated with a direct-sum decomposition of a vector space (*25*, p. 161). In fact, they satisfy the following properties:

(1) π_i is linear, in the sense that

$$\pi_i\left[pJ + (1-p)K\right] = p\pi_i(J) + (1-p)\pi_i(K) \quad (p \in [0,1];\ J,K \in S);$$

(2) $\pi_i(S) = S \cap S_{i_*}$;

(3) $\pi_i \circ \pi_i = \pi_i$ (π_i is "idempotent");

(4) $\pi_i \circ \pi_j = 0$ if $i \neq j$ (π_i and π_j are "orthogonal"); and

(5) $\pi_1 + \pi_2 + \pi_3 = I_S$ (the identity operator on S).

In essence, the property of being "decomposable" will ensure, in the forthcoming decomposition theorem, that the "additive parts" of μ_* are well-defined. The property ensuring that the parts of μ itself (in a sense that will later become clear) are well-defined is expressed by:

Definition. Let S be a set of s.d.f.'s with the property that, whenever $K \in S$ and $K = K_1 + K_2 + K_3$ (where $K_i \in S_{i_*}$, $i = 1,2,3$), then

$$K_i/\lim_{t\to\infty} K_i(t) \in S$$

for any i for which $K_i \neq 0$. Then, S is called *hereditary*.

In particular, whenever $L = p_1L_1 + p_2L_2 + p_3L_3$ is an element of a hereditary set S of lotteries (where $p_i \geq 0$, $L_i \in S_i$, $i = 1,2,3$), then any L_i whose p_i is positive must itself be in S. Examples of hereditary sets are the set of all lotteries and the set of all lotteries with finite mean.

The notions of "hereditary" and "decomposable" are related in the following way:

Proposition 4. A set S of lotteries is hereditary if and only if S_* is decomposable.

Proof. If S is empty, $S_* = \{0\}$, and the equivalence holds trivially. Assume $S \neq \emptyset$, and suppose first that S is hereditary. If $K \in S_*$ and $K \neq 0$, then

$K = pL$ for some $p > 0$, $L \in S$, so that also $K = pq_1L_1 + pq_2L_2 + pq_3L_3$ for some $q_i \in [0,1]$, $L_i \in S_i$ such that either $L_i \in S$ or $q_i = 0$ $(i = 1,2,3)$. Thus, $pq_iL_i \in S_* \cap S_{i_*}$ for each i. It follows that S_* is decomposable. To prove the other half of the equivalence, suppose S_* is decomposable, and consider any $L \in S$. We have $L = p_1L_1 + p_2L_2 + p_3L_3$ for some $p_i \in [0,1]$, $L_i \in S_i$. Since, necessarily, $L \in S_*$ and each $p_iL_i \in S_{i_*}$, we have $p_iL_i \in S_*$, so that $p_iL_i = q_iM_i$ for some $q_i \in [0,1]$, $M_i \in S$. However, L_i and M_i are lotteries; thus, if $p_i \neq 0$, then $L_i = M_i \in S$. It follows that S is hereditary.

Q.E.D.

The foregoing proposition allows us to define projection operators on S_* whenever S is a hereditary set of lotteries.

With these preparations, we can now give the desired decomposition theorem:

Theorem 6. Let μ be a measurable utility function (in fact, any linear function) on a hereditary lottery space S. Then, μ_* has a unique decomposition

$$\mu_* = \xi_1 + \xi_2 + \xi_3$$

into linear functions ξ_i on S_* such that $\xi_i(0) = 0$ and "$\xi_i(K)$ depends only on the ith part of K" (that is, $\xi_i = \xi_i \circ \pi_i$ $(i = 1,2,3)$, where each π_i is the projection operator from S_* onto $S_* \cap S_{i_*}$).

Proof. To prove the existance of the decomposition, note that, by the properties of the projection operators, we have

$$\begin{aligned} \mu_* &= \mu_* \circ I_{S_*} \\ &= \mu_* \circ (\pi_1 + \pi_2 + \pi_3) \\ &= (\mu_* \circ \pi_1) + (\mu_* \circ \pi_2) + (\mu_* \circ \pi_3). \end{aligned}$$

Clearly, each $\mu_* \circ \pi_i$ is linear and takes the zero function to 0. Moreover, for any $K \in S_*$, $\mu_* \circ \pi_i$ depends only on the ith part, $\pi_i(K)$, of K, since

$$\begin{aligned} [\mu_* \circ \pi_i](K) &= [\mu_* \circ \pi_i \circ \pi_i](K) \\ &= [\mu_* \circ \pi_i](\pi_i(K)). \end{aligned}$$

Thus, existence is proved.

To establish the uniqueness of the decomposition, let ξ_i $(i = 1,2,3)$ be *any* linear functions on S_* such that $\xi_i(0) = 0$, $\xi_i = \xi_i \circ \pi_i$, and

$$\mu_* = \xi_1 + \xi_2 + \xi_3.$$

Then, we need merely observe that, by the properties of the projection operators,

$$\begin{aligned}\mu_* \circ \pi_i &= (\xi_1 \circ \pi_1 + \xi_2 \circ \pi_2 + \xi_3 \circ \pi_3) \circ \pi_i \\ &= 0 + 0 + \xi_i \circ \pi_i \circ \pi_i \\ &= \xi_i\end{aligned}$$

for each i.

Q.E.D.

The linear functions ξ_i $(i = 1,2,3)$ appearing in Theorem 6 may be considered, respectively, the discrete, absolutely continuous, and singular continuous "parts" of μ_*. Thus, the theorem shows how the extension to S_* of a measurable utility function on S is "built up" from its discrete, absolutely continuous, and singular continuous parts. (Actually, Theorem 6 can easily be generalized to encompass situations in which lotteries are uniquely decomposed with respect to classes other than S_1, S_2, and S_3.) Let us examine in more detail how this decomposition of μ_* relates to μ itself. To do so, we need:

Proposition 5. For any sets S and T of lotteries,

$$(S \cap T)_* = S_* \cap T_*.$$

Proof. Since $(S \cap T) \cup \{0\} \subset S \cup \{0\}$, we have $(S \cap T)_* \subset S_*$. Similarly, $(S \cap T)_* \subset T_*$. Thus, $(S \cap T)_* \subset S_* \cap T_*$. To establish the reverse inclusion, note first that it reduces to $\{0\} \supset \{0\}$ if either S or T is empty. Suppose S, T are nonempty, and consider any $Z \in S_* \cap T_*$. Since $0 \in (S \cap T)_*$, we may assume $Z \neq 0$. Now, we have $Z = pL = qM$ for some $p,q \in [0,1]$, $L \in S$, $M \in T$. Necessarily, $p = \lim_{t\to\infty} pL(t) = \lim_{t\to\infty} qM(t) = q$, so that $L = M$, $S \cap T \neq \emptyset$, and $Z \in (S \cap T)_*$. Thus, $(S \cap T)_* \supset S_* \cap T_*$.

Q.E.D.

Proposition 6. Let μ be a linear function on a lottery space containing a lottery space T. Then

$$\left[\mu|T\right]_* = \mu_*|T_*.$$

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Proof. Both functions $\left[\mu\middle|T\right]_*$ and $\mu_*\middle|T_*$ have T_* as their domain and, if $T \neq \emptyset$, both map an arbitrary element pM ($p \in [0,1]$, $M \in T$) of T_* to $p\mu(M)$. If $T = \emptyset$, then $T_* = \{0\}$, and both functions are the zero function on T_*.

Q.E.D.

Now, as shown in the proof, the decomposition appearing in Theorem 6 takes the form

$$\mu_* = (\mu_* \circ \pi_1) + (\mu_* \circ \pi_2) + (\mu_* \circ \pi_3).$$

However, it follows from the definition of π_i and Propositions 5 and 6 that, for each i,

$$\begin{aligned} \mu_* \circ \pi_i &= \left[\mu_*\middle|S_* \cap S_{i_*}\right] \circ \pi_i \\ &= \left[\mu_*\middle|(S \cap S_i)_*\right] \circ \pi_i \\ &= \left[\mu\middle|S \cap S_i\right]_* \circ \pi_i. \end{aligned}$$

The sets $S \cap S_i$ ($i = 1,2,3$) are pairwise disjoint. Moreover, since the convex hull of the union of the sets $S \cap S_i$ is S, the values of μ on these sets--or, to put it differently, the functions $\mu|S \cap S_i$--determine μ on all of S. Thus, the additive decomposition of μ_* corresponds, in a sense, to a decomposition of μ into "building blocks" $\mu|S \cap S_i$ ($i = 1,2,3$). (Note that, if $S \cap S_i = \emptyset$--as would occur, for example, if i = 1 and all lotteries in S were continuous--then $\mu|S \cap S_i$ is merely the empty function.) Of course, the functions $\mu|S \cap S_i$ are measurable utility functions whenever μ is a measurable utility function; each merely represents the "restriction to $S \cap S_i$" of the preference ordering represented by μ.

We earlier alluded to the conjectures that individuals may have: (1) different frames of reference toward certainty and certain forms of uncertainty and (2) discontinuous risk preferences. To explore these points, we need:

Definition (18). A set S of lotteries is called *σ-convex* if, whenever $(L_i)_{i=1}^{\infty}$ is a sequence of elements of S and $(p_i)_{i=1}^{\infty}$ is a sequence of nonnegative numbers for which $\sum_{i=1}^{\infty} p_i = 1$, one has

$$\sum_{i=1}^{\infty} p_i L_i \in S.$$

If S is σ-convex, a function $\mu: S \to \mathbb{R}$ is called *σ-linear* if, for any sequences $(L_i)_{i=1}^{\infty}$, $(p_i)_{i=1}^{\infty}$ as just described, one has

$$\mu\left[\sum_{i=1}^{\infty} p_i L_i\right] = \sum_{i=1}^{\infty} p_i \mu(L_i).$$

Let μ and S be as in Theorem 6, and assume $\mathcal{D} \subset S$. Now, the "certain" lotteries are the degenerate lotteries, and these are discrete. Thus, all behavior of μ over "certainty" is reflected in its values at discrete lotteries. Furthermore, if either all discrete lotteries in S are simple or μ is σ-linear, then the behavior of μ over "certainty" *determines* its values at discrete lotteries. In either case, Theorem 6 shows that μ_* exhibits a natural split into a *discrete* part, ξ_1, that accounts for the behavior imposed (algebraically) on μ by its values at lotteries representing *certainties*, and a *continuous* part, $\xi_2 + \xi_3$, that accounts for the behavior of μ at lotteries complementary to the discrete ones—namely, the continuous lotteries. In this sense, μ may be viewed as being determined by two separate measurable utility functions, with one incorporating—and indeed being determined by—the behavior of μ over certainty.

This intrinsic split would appear to provide a type of theoretical plausibility for the conjecture that individuals may have a different frame of reference toward certainty than they have toward uncertainty in cases in which the nondegenerate discrete lotteries play only a formal role and the only pertinent uncertainty arises from continuous lotteries (as might occur, for example, if, through a central-limit-theorem-type process, many small, independent effects generated normal distributions that in turn generated the continuous lotteries in the individual's lottery space); for the split in μ_* that would be implied by such distinct frames of reference is already implicit in μ_*. Similarly, this intrinsic split provides a rationale for the conjecture that some individuals' risk preferences may be "discontinuous at certainty" (that is, may admit of a measurable utility function that is discontinuous at degenerate lotteries). This conjecture is supported by the observation that the computational ability of the human brain is limited; thus, there is no apparent *empirical* reason why an individual with preferences among certainties and preferences among continuous lotteries should be capable of so conforming these preferences that, when they are extended to mixtures, the limit properties necessary for continuity should hold. (Discontinuous responses to risk are discussed from another viewpoint in Kahneman and Tversky (*26*).)

Our examination of the decomposition of a measurable utility function suggests the following general method of *constructing* measurable utility functions from "pieces." Given a measurable utility function μ_i on a preference lottery space $(T_i, \succsim_i)$, where $T_i \subset S_j$ $(i = 1,2,3)$, let T be the convex hull of

$T_1 \cup T_2 \cup T_3$, and define a function $\mu: T \to R$ in the following manner: for any $L \in T$, by Proposition 3, there are unique s.d.f.'s $K_i \in S_{i_*}$ such that $L = K_1 + K_2 + K_3$. Since L is a convex combination of elements of T_1, T_2, and T_3, it follows that each $K_i \in T_{i_*}$. By appeal to Theorem 5, define

$$\mu(L) = \mu_{1_*}(K_1) + \mu_{2_*}(K_2) + \mu_{3_*}(K_3).$$

Clearly, μ is linear on T, since each μ_{i_*} is linear on T_{i_*}. Let $\succsim$ be the preference ordering $\succsim_\mu$ induced on T by μ (see p. 20). Then, μ is a measurable utility function for the preference space $(T,\succsim)$, and $\mu|T_i = \mu_i$. Our definitions of various "two-rule" measurable utility functions in sections 4 through 6 are based on this construction. This construction also establishes that distinct frames of reference toward certainty and "continuous" uncertainty, as previously discussed and qualified, *can* always be represented by some appropriate measurable utility function. In contrast, the existence of such distinct frames of reference would be *incompatible* with the traditional expected utility theory based on integrals, which implicitly requires that preferences be continuous.

By taking T_1 to be a set of simple lotteries and setting $T_3 = \emptyset$ in the above construction, we find that the elements of T are convex combinations of simple lotteries and lotteries having densities. Examples of such combinations arise in applied work when distributions having densities are truncated, as in the case of agricultural commodity price distributions that are truncated through the introduction of a support price, or in the case of income distributions that are truncated through the introduction of an income floor. In traditional expected utility theory, such distributions are evaluated as a whole. In contrast, our approach allows behavioral responses to the discrete and continuous constituents of such distributions to be treated independently. In fact, when, as in the examples cited, the only discrete distributions in the individual's formal lottery space that have any *practical* role are degenerate, our approach provides a method of modeling empirical behavior that is consistent with the axioms of expected utility theory, yet does not require that the individual's choices among certainties determine his/her choices among pertinent uncertainties (continuous probability distributions).

We will next examine the relationship between measurable utility functions defined on lottery spaces and functions defined on the real line that may serve as convenient representations of them. We will thereby clarify the use of so-called "utility functions" of income or wealth to represent behavior under risk.

We first describe how measurable utility functions give rise to "utility functions on $\mathbb{R}$." We also state conditions under which a measurable utility function can be represented as an "expected utility" functional in terms of a "von Neumann-Morgenstern utility function" defined on $\mathbb{R}$. We then adopt a somewhat different point of view. We take a function on $\mathbb{R}$ as given and investigate when there is a unique measurable utility function that gives rise to it. We obtain new representation theorems that provide a basis for interpreting discontinuous real functions as utility functions. We then consider implications of these results for the use of discontinuous utility functions in development theory (*30*).

4.1 Induced Utility Functions

Notation. We denote by "η" the natural one-to-one correspondence from $\mathbb{R}$ onto $\mathcal{D}$ defined by

$$\eta(r) = F_r \quad (r \in \mathbb{R}).$$

Definition. Let μ be an order-preserving function defined on a preference lottery space $(S,\succsim)$ for which $\mathcal{D} \subset S$. We call $\mu \circ \eta$ the *utility function induced* on $\mathbb{R}$ *by* μ and say $\mu \circ \eta$ is *induced by* $(S,\succsim)$. Moreover, if μ is linear (hence a measurable utility function on $(S,\succsim)$), we say $\mu \circ \eta$ is *measurably induced by* $(S,\succsim)$. (Thus, a function $f: \mathbb{R} \to \mathbb{R}$ is measurably induced by $(S,\succsim)$ if and only if there exists a measurable utility function μ on $(S,\succsim)$ such that $f = \mu \circ \eta$.)

Note that the meaning of "linearity" for μ as applied to degenerate lotteries is quite different from that for $\mu \circ \eta$ as applied to real numbers. For any $p \in [0,1]$, $r,s \in \mathbb{R}$, linearity of μ is expressed as

$$\mu(pF_r + (1-p)F_s) = p\mu(F_r) + (1-p)\mu(F_s).$$

In contrast, linearity of $\mu \circ \eta$ takes the form

$$[\mu \circ \eta](t) = at + b \quad (t \in \mathbb{R})$$

for some $a,b \in \mathbb{R}$. Failure to distinguish between a real number and its corresponding degenerate lottery could lead one mistakenly to assert that *every* measurable utility function μ on S, by definition, satisfied

$$\mu(pr + (1-p)s) = p\mu(r) + (1-p)\mu(s) \quad (p \in [0,1],\ r,s \in \mathbb{R}),$$

from which it would follow that

$$\begin{aligned}\mu(p) &= \mu(p\cdot 1 + (1-p)\cdot 0)\\ &= p\mu(1) + (1-p)\mu(0)\end{aligned}$$

$$= p[\mu(1) - \mu(0)] + \mu(0) \quad (p \in [0,1]),$$

so that μ would appear to be linear on (at least a subinterval of) $\mathbb{R}$. This apparent finding could tempt one to the unwarranted conclusion that μ is "risk neutral" (see p. 50).

4.2 "Expected Utility" Representations

As should by now be clear, a theory of measurable utility does not logically require the assumption that lotteries have expected values. However, the situation in which "the utility of a lottery equals its expected utility" (a notion that requires explication) is at the heart of traditional expected utility theory. Thus, we state a result (translated from Grandmont (*18*) into our language) giving conditions under which the situation cited obtains:

Theorem 7. Suppose $(S,\succsim)$ is a σ-convex preference lottery space such that $\mathcal{D} \subset S$. Then, the following conditions, taken together, are necessary and sufficient for the existence of a continuous bounded function $u\colon \mathbb{R} \to \mathbb{R}$ such that the function $\mu\colon S \to \mathbb{R}$ defined by

$$\mu(L) = \int_{-\infty}^{\infty} u(t)dL(t) \quad (L \in S)$$

is a measurable utility function for $(S,\succsim)$:

(1) For each $L_0 \in S$, the sets $\{L \in S \mid L \succsim L_0\}$ and $\{L \in S \mid L_0 \succsim L\}$ are closed in S (see Theorem 4 on p. 19).

(2) For all $L_1, L_2, L_3 \in S$ and all $p \in [0,1]$, if $L_1 \sim L_2$, then

$$pL_1 + (1-p)L_3 \sim pL_2 + (1-p)L_3.$$

For the proof, see Grandmont (*18*).

Definition. Let μ be a measurable utility function on a lottery space S. A function $u\colon \mathbb{R} \to \mathbb{R}$ for which

$$\mu(L) = \int_{-\infty}^{\infty} u(t)dL(t) \quad (L \in S)$$

is called a *von Neumann-Morgenstern utility function (for μ)*.

If u is a von Neumann-Morgenstern utility function for a measurable utility function μ defined on S, then the integral formula for μ may also be expressed as

$$\mu(L) = E(u \circ X)$$

whenever $L \in S$ and X is a random variable having L as its c.d.f. Moreover, if $\mathcal{D} \subset S$, then, for each $r \in \mathbb{R}$,

$$\mu(F_r) = \int_{-\infty}^{\infty} u(t)dF_r(t)$$

$$= u(r),$$

so that

$$\mu \circ \eta = u;$$

that is, u *is the utility function induced by* μ *on* $\mathbb{R}$, and $u(r)$ is "the utility of F_r" (as assigned by μ). Some treatments in the literature of expected utility theory obfuscate the distinction between the "nonrandom" function u on $\mathbb{R}$ and the random variable $u \circ X$, defined on a probability space.

We also point out that, whenever a measurable utility function μ is defined on (a lottery space containing) $\mathcal{D}$, then, for any simple lottery

$$L = \sum_{i=1}^{n} p_i F_{r_i},$$

$\mu(L)$ too can be expressed as an "expected utility." In fact, if u is the utility function induced by μ on $\mathbb{R}$ and X is the obvious random variable taking the value r_i at i, then

$$\mu(L) = \sum_{i=1}^{n} p_i u(r_i)$$

$$= E(u \circ X).$$

A similar remark holds if μ is σ-linear and L is discrete. However, in these cases, if u is discontinuous at some r_i for which $p_i > 0$, then $E(u \circ X)$ cannot be expressed as a Stieltjes integral of u with respect to L, because that integral will not be defined (see pp. 47-48 for a further discussion of this point).

4.3 Measurable Utility Models for Real Functions

A common feature of economic studies involving risk is the adoption of a utility function on $\mathbb{R}$ (that is, a "utility of wealth" or "utility of income" function) to represent an economic agent's risk preferences. From the standpoint of measurable utility theory, this use of a function from $\mathbb{R}$ into $\mathbb{R}$ to represent a risk preference ordering raises several basic and important questions regarding the relationship between such functions and lottery preference spaces that may measurably induce them. For example, what functions from $\mathbb{R}$ into $\mathbb{R}$ are measurably induced by *some* lottery preference space? Can such a function be induced by more than one preference ordering defined on the same lottery space? An affirmative answer to the latter question would mean that some functions from $\mathbb{R}$ into $\mathbb{R}$ could serve simultaneously as utility functions induced by *incompatible* preference orderings on the *same* lottery space.

We begin by showing that *any* function $f: \mathbb{R} \to \mathbb{R}$ (even, for example, if discontinuous *everywhere*) is measurably induced by some lottery preference space. Specifically, we prove

Theorem 8. The relation "μ induces f" is a one-to-one correspondence from the set of all linear functions on H *onto* the set of all functions from $\mathbb{R}$ into $\mathbb{R}$.

(To understand the theorem, recall that H is the set of all simple lotteries and that a linear function μ on H is automatically a measurable utility function on the lottery preference space $(H, \succsim_\mu)$, where the preference relation $\succsim_\mu$ on H is defined by

$$L_1 \succsim_\mu L_2 \text{ if and only if } \mu(L_1) \geq \mu(L_2).)$$

Proof. We first prove that the correspondence is one-to-one. Suppose linear functions μ and ν on H induce the same utility function $f: \mathbb{R} \to \mathbb{R}$. Consider any simple lottery

$$L = \sum_{i=1}^{n} p_i F_{r_i} \quad (p_i \geq 0, \ \sum_{i=1}^{n} p_i = 1).$$

From the linearity property for μ and ν, it follows by induction that

$$\begin{aligned} \mu(L) &= \sum_{i=1}^{n} p_i \mu(F_{r_i}) \\ &= \sum_{i=1}^{n} p_i f(r_i) \\ &= \sum_{i=1}^{n} p_i \nu(F_{r_i}) \\ &= \nu(L), \end{aligned}$$

so that $\mu = \nu$. Thus, the relation "μ induces f" is one-to-one.

We next prove that this relation is onto. Let $f: \mathbb{R} \to \mathbb{R}$ be an arbitrary function. We will construct a linear function f^* on H that (viewed as a measurable utility function on $(H, \succsim_{f^*})$) induces f. In fact, for any $L \in H$ (where, say, $L = \sum_{i=1}^{m} p_i F_{s_i}$), put

$$f^*(L) = \sum_{i=1}^{m} p_i f(s_i).$$

Note that, for $f^*(L)$ to be well-defined, $f^*(L)$ must depend only on L and not on the particular representation of L chosen. But this condition is satisfied, for we have:

Lemma 2. Suppose m and n are positive integers, $s_1,\ldots,s_m$ and $t_1,\ldots,t_n$ (not necessarily distinct) numbers, and $p_1,\ldots,p_m$ and $q_1,\ldots,q_n$ (not necessarily distinct) nonnegative numbers such that

$$\sum_{i=1}^{m} p_i F_{s_i} = \sum_{j=1}^{n} q_j F_{t_j}.$$

Then, for any function $f: \mathbb{R} \to \mathbb{R}$,

$$\sum_{i=1}^{m} p_i f(s_i) = \sum_{j=1}^{n} q_j f(t_j).$$

Proof of Lemma. Without loss of generality, we may assume that each p_i and q_j is positive. Let S be the set of distinct s_i's; that is, let

$$S = \{s \in \mathbb{R} \mid \text{for some } i \in \{1,\ldots,m\},\ s = s_i\}.$$

For each $s \in S$, define

$$I_s = \{i \in \{1,\ldots,m\} \mid s_i = s\}.$$

Then, the sets I_s, $s \in S$, form a partition of $\{1,\ldots,m\}$, and we have

$$\sum_{i=1}^{m} p_i F_{s_i} = \sum_{s\in S} \sum_{i\in I_s} p_i F_{s_i}$$

$$= \sum_{s\in S} \left[\sum_{i\in I_s} p_i\right] F_s.$$

Similarly, defining

$$T = \{t \in \mathbb{R} \mid \text{for some } j \in \{1,\ldots,n\},\ t = t_j\}$$

and (for each $t \in T$)

$$J_t = \{j \in \{1,\ldots,n\} \mid t_j = t\},$$

we obtain

$$\sum_{j=1}^{n} q_j F_{t_j} = \sum_{t\in T} \left[\sum_{j\in J_t} q_j\right] F_t.$$

From our hypothesis, it follows that

$$\sum_{s\in S}\Big[\sum_{i\in I_s} p_i\Big]F_s = \sum_{t\in T}\Big[\sum_{j\in J_t} q_j\Big]F_t.$$

However, since the p_i's and q_j's are all positive, S and T are precisely the sets of points of discontinuity of the left and right sides, respectively. Thus, S = T, and we have

$$\sum_{s\in S}\Big[\sum_{i\in I_s} p_i - \sum_{j\in J_s} q_j\Big]F_s = 0.$$

Now, it is easily verified that, in the vector space over $\mathbb{R}$ of all functions from $\mathbb{R}$ into $\mathbb{R}$, the functions F_s, $s \in S$, form an independent set. It follows that, for each $s \in S$,

$$\sum_{i\in I_s} p_i = \sum_{j\in J_s} q_j.$$

Since

$$\sum_{i=1}^{m} p_i f(s_i) = \sum_{s\in S}\sum_{i\in I_s} p_i f(s_i)$$

$$= \sum_{s\in S}\Big[\sum_{i\in I_s} p_i\Big] f(s)$$

and (similarly)

$$\sum_{j=1}^{n} q_j f(t_j) = \sum_{t\in T}\Big[\sum_{j\in J_t} q_j\Big] f(t),$$

the lemma is proved.

Thus, $f^*: H \to \mathbb{R}$ is a well-defined function. It is clearly linear. Thus, it is a measurable utility function on $(H,\succsim_{f^*})$. Finally, observe that, since, for any $t \in \mathbb{R}$, $f^*(F_t) = f(t)$, f is the utility function on $\mathbb{R}$ induced by f^*.

Q.E.D.

Note that this result assures us that even arbitrary *unbounded* functions may serve as measurably induced utility functions on $\mathbb{R}$. *Thus, the use of H as a lottery space circumvents the "St. Petersburg Paradox"* (that is, the argument that the use of unbounded utility functions necessarily implies the existence of "infinite utilities"). Observations in this vein have already appeared in the literature (see 2, 41).

Our next theorem shows the connection between boundedness of real functions and σ-linearity of measurable utility functions. First, recall that H_σ is the set of all discrete lotteries. We shall need

Proposition 7. H_σ is σ-convex.

Proof. Suppose $\{L_i\}_{i=1}^{\infty}$ is a sequence of elements of H_σ and $\{p_i\}_{i=1}^{\infty}$ is a sequence of nonnegative numbers such that $\sum_{i=1}^{\infty} p_i = 1$. For each i, there exist a sequence $\{t_{i,j}\}_{j=1}^{\infty}$ of numbers and a sequence $\{q_{i,j}\}_{j=1}^{\infty}$ of nonnegative numbers such that

$$\sum_{j=1}^{\infty} q_{i,j} = 1$$

and

$$L_i = \sum_{j=1}^{\infty} q_{i,j} F_{t_{i,j}}$$

Now, it is a well-known fact from set theory that there exists a one-to-one correspondence, ϕ, from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. Thus,

$$\sum_{i=1}^{\infty} p_i L_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_i q_{i,j} F_{t_{i,j}}$$

$$= \sum_{n \in \mathbb{N}} p_{\phi(n)_1} q_{\phi(n)} F_{t_{\phi(n)}},$$

where $\phi(n)_1$ is the first coordinate of $\phi(n)$. Since each $p_{\phi(n)_1} q_{\phi(n)}$ is nonnegative and

$$\sum_{n \in \mathbb{N}} p_{\phi(n)_1} q_{\phi(n)} = \sum_{i=1}^{\infty} p_i \sum_{j=1}^{\infty} q_{i,j}$$

$$= \sum_{i=1}^{\infty} p_i \cdot 1$$

$$= 1,$$

we conclude that $\sum_{i=1}^{\infty} p_i L_i$ is discrete.

Q.E.D.

Proposition 7 permits us to consider σ-linear functions on H_σ. We can now prove

Theorem 9. The relation "μ induces f" is a one-to-one correspondence from the set of all σ-linear functions on H_σ *onto* the set of all bounded functions from $\mathbb{R}$ into $\mathbb{R}$.

(As in Theorem 8, a σ-linear function μ on H_σ is viewed as a measurable utility function on $(H_\sigma, \succsim_\mu)$.)

Proof. First, note that the correspondence is *into* the set of all bounded functions from $\mathbb{R}$ into $\mathbb{R}$. Indeed, suppose a σ-linear function μ on H_σ induced a utility function $f: \mathbb{R} \to \mathbb{R}$ that was *unbounded*—say, unbounded above. Then, for each $i \in \mathbb{N}$, there would exist an $r_i \in \mathbb{R}$ such that $f(r_i) > 2^i$. By the σ-convexity of H_σ and the σ-linearity of μ, we would have

$$\sum_{i=1}^{\infty} (1/2^i) F_{r_i} \in H_\sigma$$

and

$$\mu\left[\sum_{i=1}^{\infty} (1/2^i) F_{r_i}\right] = \sum_{i=1}^{\infty} (1/2^i) \mu(F_{r_i})$$

$$= \sum_{i=1}^{\infty} (1/2^i) f(r_i).$$

But this is impossible, since

$$\sum_{i=1}^{n} (1/2^i) f(r_i) > n$$

for each $n \in \mathbb{N}$. Thus, f must be bounded above. Similarly, f must be bounded below. We conclude that the correspondence is into (compare *18*, Lemma 2).

Next, suppose that σ-linear functions μ and ν on H_σ induce the same bounded utility function $f: \mathbb{R} \to \mathbb{R}$. Let

$$L = \sum_{i=1}^{\infty} p_i F_{s_i}$$

be any discrete lottery. Then

$$\mu(L) = \sum_{i=1}^{\infty} p_i \mu(F_{s_i})$$

$$= \sum_{i=1}^{\infty} p_i f(s_i)$$

$$= \sum_{i=1}^{\infty} p_i \nu(F_{s_i})$$

$$= \nu(L).$$

Thus, $\mu = \nu$. This proves that the correspondence is one-to-one.

Finally, to prove that the correspondence is onto, suppose $f: \mathbb{R} \to \mathbb{R}$ is an arbitrary bounded function. Define a function $f^*: H_\sigma \to \mathbb{R}$ as follows: for any $L \in H_\sigma$ (where, say,

$$L = \sum_{i=1}^{\infty} p_i F_{s_i}),$$

put

$$f^*(L) = \sum_{i=1}^{\infty} p_i f(s_i).$$

To prove that $f^*(L)$ is well-defined, we need

Lemma 3. Suppose $\{s_i\}_{i=1}^{\infty}$, $\{t_j\}_{j=1}^{\infty}$ are sequences of (not necessarily distinct) numbers and $\{p_i\}_{i=1}^{\infty}$, $\{q_j\}_{j=1}^{\infty}$ are sequences of (not necessarily distinct) nonnegative numbers such that

$$\sum_{i=1}^{\infty} p_i = \sum_{j=1}^{\infty} q_j = 1$$

and

$$\sum_{i=1}^{\infty} p_i F_{s_i} = \sum_{j=1}^{\infty} q_j F_{t_j}.$$

Then, for any bounded function $f: \mathbb{R} \to \mathbb{R}$,

$$\sum_{i=1}^{\infty} p_i f(s_i) = \sum_{j=1}^{\infty} q_j f(t_j).$$

Proof of Lemma. Without loss of generality, we may assume that each p_i and q_j is positive. We proceed as in the proof of Lemma 2, defining

$$S = \{s \in \mathbb{R} \mid \text{for some } i \in \mathbb{N},\ s = s_i\},$$

$$T = \{t \in \mathbb{R} \mid \text{for some } j \in \mathbb{N},\ t = t_j\},$$

and (for all $s \in S$ and $t \in T$)

$$I_s = \{i \in \mathbb{N} \mid r_i = s\},$$

$$J_t = \{j \in \mathbb{N} \mid t_j = t\}.$$

As before, we obtain

$$\sum_{s\in S}\left[\sum_{i\in I_s} p_i\right]F_s = \sum_{t\in T}\left[\sum_{j\in J_t} q_j\right]F_t$$

(here using the absolute convergence of the series to justify the rearrangement of terms).

Next, we prove that $S = T$ by showing that S and T are precisely the sets of points of discontinuity of the left and right sides, respectively. It will be enough to prove this for S. (Note that S need not here be finite; indeed, it could even be dense in $\mathbb{R}$.)

Toward this end, for each $s \in S$, put

$$P_s = \sum_{i\in I_s} p_i.$$

Note that each P_s is positive. Put

$$F = \sum_{s\in S} P_s F_s,$$

and consider any $r \in \mathbb{R}$. For any $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 < r < r_2$, we have

$$F(r_2) - F(r_1) = \sum_{s\in S} P_s[F_s(r_2) - F_s(r_1)].$$

Since this series is uniformly convergent on $\mathbb{R}$, we obtain

$$F(r+) - F(r-) = \sum_{s\in S} P_s[F_s(r+) - F_s(r-)]$$

(where the "+" and "-" denote right- and left-hand limits). However, for each $s \in S$, $F_s(r+) - F_s(r-)$ is either 1 or 0, according to whether r is, or is not, equal to s. Thus, $F(r+) - F(r-)$ is positive if and only if $r \in S$, which proves that S is precisely the set of points of discontinuity of F. Arguing similarly for T, we conclude that $S = T$.

Thus, putting

$$Q_t = \sum_{j\in J_t} q_j$$

for each $t \in T$, we have

$$\sum_{s\in S}(P_s - Q_s)F_s = 0.$$

Note that this series is uniformly convergent on $\mathbb{R}$, and consider any $s_0 \in S$. Reasoning as before, we obtain

$$\sum_{s\in S}(P_s - Q_s)[F_s(s_0+) - F_s(s_0-)] = 0,$$

which reduces to

$$(P_{s_0} - Q_{s_0})[F_{s_0}(s_0+) - F_{s_0}(s_0-)] = 0.$$

It follows that $P_{s_0} = Q_{s_0}$. But

$$\sum_{i=1}^{\infty} p_i f(s_i) = \sum_{s\in S} P_s f(s)$$

and

$$\sum_{j=1}^{\infty} q_j f(t_j) = \sum_{t\in T} Q_t f(t)$$

(see the proof of Lemma 2). Thus, the lemma is proved.

Thus, the function $f^*: H_\sigma \to \mathbb{R}$ is well-defined. Furthermore, it is σ-linear. (To establish this point, let $\{L_i\}_{i=1}^{\infty}$ be a sequence of elements of H_σ, and let $\{p_i\}_{i=1}^{\infty}$ be a sequence of nonnegative numbers such that $\sum_{i=1}^{\infty} p_i = 1$. Then, drawing on the proof of Proposition 7 (and using the same notation), we can write

$$\begin{aligned} f^*\left[\sum_{i=1}^{\infty} p_i L_i\right] &= f^*\left[\sum_{n\in\mathbb{N}} p_{\phi(n)_1} q_{\phi(n)} F_{t_{\phi(n)}}\right] \\ &= \sum_{n\in\mathbb{N}} p_{\phi(n)_1} q_{\phi(n)} f(t_{\phi(n)}) \\ &= \sum_{i=1}^{\infty} p_i \sum_{j=1}^{\infty} q_{i,j} f(t_{i,j}) \\ &= \sum_{i=1}^{\infty} p_i f^*(L_i).) \end{aligned}$$

Since $f^*(F_t) = f(t)$ for each $t \in \mathbb{R}$, we conclude that f^* (viewed as a measurable utility function on $(H_\sigma, \succsim_{f^*})$) induces f. It follows that the correspondence under consideration is onto.

Q.E.D.

Theorems 8 and 9 obviously imply that (necessarily distinct) preference relations on distinct lottery spaces may measurably induce the same utility function on $\mathbb{R}$. More important, however, is the question of whether distinct preference relations on the *same* lottery space can measurably induce the same utility function on $\mathbb{R}$. We will now demonstrate that they can when these preference relations are not required to be continuous. In fact, we will construct uncountably many distinct preference relations, all of which are defined on the same lottery space and measurably induce the same utility function on $\mathbb{R}$. Both in form and notation, our construction will parallel that given during our discussion of the decomposition of measurable utility functions (pp. 29-30). In particular, for each $t \in \mathbb{R}$, we will specify measurable utility functions μ_{it} defined on lottery spaces $T_{it} \subset S_i$ ($i = 1,2,3$) and will use these to define a measurable utility function μ_t on the convex hull, T_t, of $T_{1t} \cup T_{2t} \cup T_{3t}$.

To proceed, consider an arbitrary function $f\colon \mathbb{R} \to \mathbb{R}$. By Theorem 8, there exists a (unique) measurable utility function, μ_1, on H that induces f. For each $t \in \mathbb{R}$, put $T_{1t} = H$, $\mu_{1t} = \mu_1$, $T_{3t} = \emptyset$, and (correspondingly) $\mu_{3t} = \emptyset$. Let T_{2t} be the lottery space consisting of all elements of S_2 with finite mean. Define a preference ordering, $\succsim_{2t}$, on T_{2t} by the rule:

$$L \succsim_{2t} M \text{ if and only if } L(t) \leq M(t) \quad (L,M \in T_{2t}).$$

(Thus, for example, if L and M were the c.d.f.'s of profit random variables X and Y, respectively, "$L \succsim_{2t} M$" would mean that the probability of realizing a profit exceeding t would be at least as great for X as for Y.) Choose any $a_t > 0$ for which $a_t > 2f(t)$. The function $\mu_{2t}\colon T_{2t} \to \mathbb{R}$ defined by

$$\mu_{2t}(L) = a_t\left[1-L(t)\right] \quad (L \in T_{2t})$$

is clearly a measurable utility function that represents $\succsim_{2t}$. Let T_t be the convex hull of $T_{1t} \cup T_{2t} \cup T_{3t}$. (Of course, all the T_t, $t \in \mathbb{R}$, are identical; let T be this common set.) Then, in accordance with our earlier construction (pp. 29-30), the rule

$$\mu_t(L) = \mu_{1t_*}(L_1) + \mu_{2t_*}(L_2) + \mu_{3t_*}(L_3)$$

(where $L = L_1 + L_2 + L_3 \in T$, $L_i \in S_{i_*}$ ($i = 1,2,3$)) defines a measurable utility function μ_t, and a corresponding preference relation $\succsim_t$, on T. By the definition of μ_{1t} (see the proof of Theorem 8) and our construction, $\succsim_t$ measurably induces f.

Next, we show that $\succsim_s$ and $\succsim_t$ are distinct whenever $s \neq t$. Suppose $s \neq t$, and observe that there clearly exist lotteries $L_2, M_2 \in T_{2s} = T_{2t}$ such that $L_2(s) < M_2(s)$ and $L_2(t) > M_2(t)$. Then, $L_2, M_2 \in T$,

$$\mu_s(L_2) = 0 + a_s\left[1-L_2(s)\right],$$

and, similarly,

$$\mu_t(L_2) = a_t\left[1-L_2(t)\right],$$

$$\mu_s(M_2) = a_s\left[1-M_2(s)\right],$$

and

$$\mu_t(M_2) = a_t\left[1-M_2(t)\right],$$

so that $\mu_s(L_2) > \mu_s(M_2)$ and $\mu_t(L_2) < \mu_t(M_2)$. We conclude that $\succsim_s$ and $\succsim_t$ are distinct orderings.

We have thus demonstrated that incompatible preference orderings defined on the same lottery space can measurably induce the same utility function on $\mathbb{R}$. Since f was arbitrary, it follows that, *in problems involving risk, no assumptions concerning a utility function on $\mathbb{R}$ are sufficient to characterize the underlying risk preferences. Rather, the risk preferences can only be characterized by assumptions at the more abstract level of the preference ordering itself.*

As we shall see below, the key to the preceding construction lies in the fact that each μ_t, and thus (by the remarks following Theorem 4) each $\succsim_t$, is *discontinuous*. To establish discontinuity, suppose $t \in \mathbb{R}$, and choose $b_t \in [0,1]$ such that

$$b_t \neq 1 - \left[f(t)/a_t\right].$$

For each $n \in \mathbb{N}$, let L_n be any element of T_{2t} such that

$$L_n(t - n^{-1}) = 0,$$

$$L_n(t) = b_t,$$

and

$$L_n(t + n^{-1}) = 1$$

(there is an obvious piecewise-linear lottery that will do). Then, $L_n \to F_t$ weakly as $n \to \infty$. However, for each n,

$$\begin{aligned}\mu_t(L_n) &= a_t\left[1 - L_n(t)\right]\\ &= a_t(1 - b_t)\\ &\neq f(t)\\ &= \mu_t(F_t).\end{aligned}$$

It follows that μ_t is not continuous at F_t. Thus, $\succsim_t$ is discontinuous.

The preference relations $\succsim_t$ ($t \in \mathbb{R}$) constitute our first "concrete" example of the new type of discontinuous risk preference ordering described in the introduction (p. 2). Since each μ_t is discontinuous, and since every measurable utility function satisfies the Herstein-Milnor axioms for measurable utility theory (*23*), this example establishes that the Herstein-Milnor axioms do not imply that measurable utility functions are continuous. (There seems to have been a lack of clarity on this point in the literature. One source of confusion may have been an overly liberal interpretation of Herstein and Milnor's careful assertion that one of their axioms "approximately states that an individual's preference ordering is continuous with regard to probability distributions" (*23*, p. 293). Nor has the situation been helped by the fact that the Archimedean axiom of expected utility theory (*7*, p. 292) is even more widely known (sometimes in slightly changed form) as the "continuity axiom" (see, for example, *22*, p. 53). Our example also shows that the continuity axiom does not imply continuity for measurable utility functions.)

The preceding results naturally raise the question of whether distinct *continuous* preference relations on the same lottery space can measurably induce the same utility function on $\mathbb{R}$. The answer is given by

Theorem 10. Let S be a lottery space containing $\mathcal{D}$. Suppose $\succsim_1$ and $\succsim_2$ are continuous preference relations on S that measurably induce the same utility function, f, on $\mathbb{R}$. Then, $\succsim_1$ and $\succsim_2$ are identical.

Proof. It follows from our assumptions that there exist measurable utility functions μ and ν that represent $\succsim_1$ and $\succsim_2$, respectively, and that induce the same utility function, f, on $\mathbb{R}$. Furthermore, by the remarks following Theorem 4, μ and ν must be continuous.

It clearly suffices to prove $\mu = \nu$. For this, we will use the following well-known result:

Lemma 4. For any lottery L there exists a sequence of simple lotteries that converges weakly (indeed, uniformly on $\mathbb{R}$) to L.

Proof of Lemma. For any integer $n \geq 3$, define intervals

$$I_k = [(k-1)/n, k/n) \quad (k = 1, \ldots, n-1)$$

and

$$I_n = [(n-1)/n), 1].$$

Let L be any lottery. Since $I_1, \ldots, I_n$ are disjoint and exhaust [0,1], the sets

$$J_k = L^{-1}(I_k) \quad (k = 1, \ldots, n)$$

are disjoint and exhaust $\mathbb{R}$. Moreover, since L is a lottery, the sets J_k have the following properties: $J_k < J_\ell$ whenever $k < \ell$ (that is, each element—if any—of J_k is less than each element—if any—of J_ℓ whenever $k < \ell$). Each J_k is an interval (possibly empty). J_1 and J_n are nonempty. $J_2, \ldots, J_n$ are bounded below. Each of $J_2, \ldots, J_n$ is closed on the left; that is, each contains its left-hand endpoint.

It follows that there exist numbers

$$a_1 \leq a_2 \leq \ldots \leq a_{n-1}$$

such that

$$J_1 = (-\infty, a_1),$$

$$J_n = [a_{n-1}, \infty),$$

and

$$J_k = [a_{k-1}, a_k)$$

for $k = 2, \ldots, n-1$. Define $L_n: \mathbb{R} \to [0,1]$ by

$$L_n(t) = \begin{cases} 0 & \text{if } t \in J_1 \\ 1 & \text{if } t \in J_n \\ k/n & \text{if } t \in J_k \quad (1 < k < n). \end{cases}$$

Then, L_n is a simple lottery (since, as is easily proved, the simple lotteries are precisely those having finite range). Moreover, since $L(t) \in I_k$ whenever $t \in J_k$, we have $|L_n(t) - L(t)| \leq 1/n$. Thus, the lemma is proved.

Continuing with the proof of the theorem, let L be any element of S. By the lemma, there exists a sequence $(L_n)_{n=1}^{\infty}$ of simple lotteries converging weakly to L. Since μ and ν agree at each degenerate lottery, they must agree at each L_n. But then $\mu(L) = \nu(L)$, by the continuity of μ and ν. We conclude that $\mu = \nu$, and the theorem is proved.

Q.E.D.

4.4 Implications for Discontinuous Utility in Peasant Agriculture

In his study of discontinuous utility (30), Masson argued that an expected utility model derived from a utility function of income with a jump discontinuity may be an appropriate representation of farmer behavior when a disaster-avoidance motive is present. He based his conclusions on the findings of O'Mara's study (32) of the diffusion of technical change in and around a farm project in Mexico. Masson pointed to directly estimated utility functions arising out of O'Mara's study as in fact providing empirical evidence of discontinuous utility. The use of discontinuous utility functions had earlier been suggested by Roy in the context of his "safety-first" theory (39). A "jump point" of such a utility function might be, for example, the level of income at which bankruptcy occurs.

Neither Masson nor Roy, however, attempted to establish the legitimacy of discontinuous utility functions within a theory of behavior under risk. We will now address this issue with particular attention to the use of discontinuous utility functions when a riskless asset is available. The reader may find it of interest to contrast our methods with the graphical methods used by Pyle and Turnovsky (35) (whose work also considers implications of riskless versus risky assets, though in a different context).

The assumption that an economic agent has the choice of holding a riskless asset is formally the assumption that his/her lottery space contains degenerate (hence simple) lotteries. The consideration of lottery spaces containing simple lotteries is also important for another reason: in many empirical studies of behavior under risk, it is the subjects' expressed preferences among various putative simple lotteries that are used to construct utility functions of income. Thus, any realistic model intended to reflect the behavior observed in such studies must assume that the subjects' lottery space contains simple lotteries.

Now, the common presumption is that functions on $\mathbb{R}$ that are put forth as utility functions for risky choices are to be interpreted as von Neumann-Morgenstern utility functions (see p. 32). In the traditional case in which the utility function of income is continuous, the inclusion of simple lotteries in the lottery space poses no difficulty. However, when the (proposed) utility function is *discontinuous*, it cannot always be interpreted as a von Neumann-Morgenstern utility function, for its integral with respect to a simple (and hence discontinuous) lottery may be undefined (see below). How, then, is such a function to be interpreted, and on what basis can it characterize behavior under risk? Specifically, how can it be related to a preference ordering of risky prospects? To place these questions in sharper focus, let us review the theoretical justification for using *continuous* functions as "utility functions" of income and then contrast this case with that of discontinuous functions.

We begin by pointing out that, within the traditional theory of behavior under risk, the fundamental economic datum is a preference ordering defined on a lottery space. When, for analytical convenience, one pursues a risk-related study by selecting a continuous function $u: \mathbb{R} \to \mathbb{R}$ and designating it a "utility function" of income, one is merely defining a preference ordering of lotteries implicitly rather than explicitly. Indeed, in the usual case in which expected utility maximization is intended, the rule

$$\mu(L) = \int_{-\infty}^{\infty} u dL$$

defines a linear function μ on the (necessarily convex) set of all lotteries L for which the integral is finite. Then, in the usual manner (see p. 20), μ determines a preference ordering $\succsim_\mu$ for which μ is a measurable utility function. In this way, u determines $\succsim_\mu$. Yet, $\succsim_\mu$ also measurably induces u by means of μ. One can summarize these relationships among u, μ, and $\succsim_\mu$ by the statement that any continuous function $u: \mathbb{R} \to \mathbb{R}$ determines a preference lottery space that, in turn, measurably induces u. It is this correspondence between continuous real functions and preference lottery spaces that justifies interpreting the former as "utility functions."

For *discontinuous* functions $u: \mathbb{R} \to \mathbb{R}$, however, the relationship to preference lottery spaces is less apparent. As already noted, the Stieltjes integral

$$\int_{-\infty}^{\infty} u dL$$

may not even be *defined* when u is discontinuous and L is simple. Thus, the arguments used in the continuous case are not applicable here, and we are confronted with the task of proving that the use of discontinuous real functions as utility functions is not merely spurious, but, rather, can be justified by the existence of an appropriate correspondence between such functions and preference lottery spaces. We now proceed toward a resolution of this issue.

First, suppose an individual has available no riskless asset—that is, that there are no degenerate lotteries in his/her lottery space. In fact, suppose that he/she chooses only from among continuous lotteries representing bounded random variables. Assume $u: \mathbb{R} \to \mathbb{R}$ is continuous except perhaps at finitely many points. Then, for any one of his/her lotteries, L, the Stieltjes integral

$$\int_{-\infty}^{\infty} u dL$$

is well-defined, so that u, though discontinuous, may serve as a von Neumann-Morgenstern utility function, defining a measurable utility function (and associated preference lottery space) as previously described. In this situation, u is legitimized as a utility function by the same relationship to a preference lottery space as would be enjoyed by a continuous real function.

Alternatively, suppose now that the individual *does* have a choice of holding a riskless asset; that is, suppose that his/her lottery space contains the degenerate lotteries. Then, if u has a discontinuity at t_0, both u and F_{t_0} are discontinuous at t_0, and u will not be Stieltjes integrable with respect to F_{t_0} (that is,

$$\int_{-\infty}^{\infty} u dF_{t_0}$$

will not exist). This observation follows directly from the definition of the integral. (In fact, from among the various Stieltjes sums

$$\sum_{i=1}^{n} u(s_i')[F_{t_0}(s_{i+1}) - F_{t_0}(s_i)],$$

one could always select two that differed by nearly $u(t_0+) - u(t_0-)$, yet whose subdivisions

$$\ldots\ s_{i-1},\ s_i,\ s_{i+1}\ \ldots$$

were arbitrarily fine.) Thus, u cannot fulfill the role that the traditional expected utility framework requires of a von Neumann-Morgenstern utility function—namely, of serving as the integrand of a Stieltjes-integral functional that assigns to every lottery in the choice space its "expected utility."

How, then, when a riskless asset is available, can discontinuous utility on the real line be rationalized within an acceptable theory of preference behavior under risk? A simple answer is provided by Theorems 8 and 9, for they establish that *any* function $u: \mathbb{R} \to \mathbb{R}$, even if discontinuous at *every* point, determines a unique measurable utility function on the lottery space of all simple lotteries (or, if u is bounded, even a unique σ-linear measurable utility function on the "larger" lottery space of all discrete lotteries) that, in turn, induces u. Thus, discontinuous utility functions on the real line, while not necessarily consistent with the maximization of expected von Neumann-Morgenstern utility, *are* consistent with the maximization of *measurable* utility. Our theorems legitimize the use of such functions as representations of behavior under risk (as in Masson's study (*30*)) by showing that each may be associated with a measurable utility function, and thus with a preference ordering, defined on a lottery space.

In establishing one-to-one correspondences between real functions and measurable utility functions, Theorems 8 and 9 assume that preference comparisons are to be made only among simple lotteries or among discrete lotteries. It is natural to ask whether similar correspondences hold when continuous lotteries (in conjunction with simple or discrete lotteries) are allowed in the lottery space. We will now show that such correspondances do not hold. In fact, although any discontinuous function on $\mathbb{R}$ is still measurably induced by some preference ordering, such an ordering is not unique. Thus, if a discontinuous utility function of income were used to model behavior under risk for both simple (or discrete) and continuous lotteries, no underlying risk preference ordering could be uniquely identified (unless, of course, additional restrictions were introduced).

To formalize and prove this assertion, let $f: \mathbb{R} \to \mathbb{R}$ be any function, and let μ be any measurable utility function defined on the space C of all continuous lotteries. We will construct a measurable utility function that induces f, yet agrees with μ on C. (We consider here only the conjunction of simple lotteries and continuous lotteries; when f is bounded, the proof for the discrete lottery case is similar.) Now, by Theorem 8, there is a (unique) measurable utility function, f*, defined on H that measurably induces f. Let L be the convex hull of $H \cup C$. Then, every lottery L in L can be written as a convex combination

$$L = p_L S_L + (1-p_L)C_L,$$

where $0 \leq p_L \leq 1$, S_L is a simple lottery, and C_L is a continuous lottery. Furthermore, as can be demonstrated by a short argument based on the lottery decomposition results described previously (pp. 20-21), p_L, $p_L S_L$, and $(1-p_L)C_L$ are unique. We are thus assured that the measurable utility function, ν, defined on L by

$$\nu(L) = p_L f^*(S_L) + (1-p_L)\mu(C_L) \quad (L \in L)$$

is well-defined. Clearly, ν measurably induces f. Moreover, $\nu|C$—the restriction of ν to C—equals μ. Thus, we have constructed a measurable utility function (and, therefore, a preference ordering—namely, the ordering $\succsim_\nu$ determined by ν) that encompasses both f and μ in the senses desired. Consequently, since μ was arbitrary, f cannot identify a unique preference ordering, and our assertion is proved. It follows that the empirical use of discontinuous utility functions may be spurious unless the lottery space is properly restricted, as in Theorems 8 and 9.

A final issue requiring clarification is the interpretation of "jump size" at points of discontinuity of utility functions of income. Masson (30) interprets a "larger" drop in utility values at points of discontinuity as signifying a "more serious" economic disaster. From the standpoint of measurable utility theory, however, no such interpretation is warranted. For an induced utility function contains no more information on behavior under risk than its underlying preference ordering, and the latter is concerned solely with *order* of preference, not *strength* of preference.

An individual is "risk averse" if, whenever confronted with a risky prospect, he/she prefers a guaranteed payment equal to the expected value of the prospect, to the prospect itself.

The concept of risk aversion is usually considered within the framework of the traditional expected utility theory based on integrals. Within that framework, risk aversion is known to be equivalent to the concavity of a von Neumann-Morgenstern utility function. In this section, we investigate the relationship of risk aversion to concavity within the more general setting of measurable utility theory.

We first present the basic definitions. We then show that the equivalence between risk aversion and concavity holds (in a weakened form) for measurable utility functions when preferences are continuous, but may fail in one direction, and "appear" to fail in the other, when preferences are discontinuous. Finally, we consider the implications of the "apparent" failure for the empirical identification of risk aversion.

5.1 Definitions

Recall that the *mean* of a lottery L (here denoted E(L)) is defined as the Stieltjes integral $\int_{-\infty}^{\infty} t\,dL(t)$ whenever the integral exists. Recall also the function η: $R \to D$ defined earlier (p. 31).

Definition. Let $(S,\succsim)$ be a preference lottery space in which each lottery has a finite mean. Assume $\eta\left[E(S)\right] \subset S$, where E(S) is the set of means of elements of S. Then, $(S,\succsim)$ (or $\succsim$) is called:

(1) *weakly risk averse* if, for each $L \in S$, $F_{E(L)} \succsim L$;

(2) *risk neutral* if, for each $L \in S$, $F_{E(L)} \sim L$; and

(3) *weakly risk loving* if, for each $L \in S$, $L \succsim F_{E(L)}$.

If $\succsim$ is replaced by $\succ$ in (1) or (3), then $(S,\succsim)$ (or $\succsim$) is called *strongly risk averse* or *strongly risk loving*, respectively.

Most treatments of risk aversion leave the impression that the notion requires a utility framework, or even an expected-utility framework, for its definition (for example, see (*22, 27, 44*)). In fact, however, risk aversion is a purely ordinal concept.

For precision, we will need:

Definition. A function f: $R \to R$ is *weakly* (respectively, *strongly*) *concave* if

$$f(ta + (1-t)b) \geqslant tf(a) + (1-t)f(b)$$

(respectively, $f(ta + (1-t)b) > tf(a) + (1-t)f(b)$)

whenever $t \in (0,1)$, $a,b \in \mathbb{R}$, and is *weakly* (respectively, *strongly*) *convex* if $-f$ is weakly (respectively, strongly) concave.

5.2 Relation to Concavity

We can now prove:

Theorem 11. Let $(S,\succsim)$ be a preference lottery space for which each lottery has a finite mean and $D \subset S$. Assume $(S,\succsim)$ measurably induces on $\mathbb{R}$ a utility function, f. Then:

(1) If $\succsim$ is weakly risk averse (respectively, weakly risk loving; risk neutral), then f is weakly concave (respectively, weakly convex; affine). The analogous statement holds if "weakly" is replaced by "strongly."

(2) If f is weakly concave (respectively, weakly convex; affine) *and* $\succsim$ *is continuous*, then $\succsim$ is weakly risk averse (respectively, weakly risk loving; risk neutral).

Proof. Let μ be a measurable utility function for $(S,\succsim)$ that induces f. To prove (1), suppose $\succsim$ is weakly risk averse, and consider any $a,b \in \mathbb{R}$, $p \in (0,1)$. Put

$$L = pF_a + (1-p)F_b;$$

then $L \in S$ and

$$E(L) = pa + (1-p)b.$$

Since

$$F_{E(L)} \succsim L,$$

it follows that

$$\mu(F_{E(L)}) \geq \mu(L).$$

Thus,

$$\mu\left[\eta(pa + (1-p)b)\right] \geq p\mu\left[\eta(a)\right] + (1-p)\mu\left[\eta(b)\right];$$

that is,

$$f\left[pa + (1-p)b\right] \geq pf(a) + (1-p)f(b).$$

Thus, f is weakly concave.

If $\succsim$ is weakly risk loving, the proof is similar. Finally, if $\succsim$ is risk neutral, then it is both weakly risk averse and weakly risk loving, so that f is both weakly concave and weakly convex, hence (by Lemma 1, with $S = \mathbb{R}$) affine.

The second half of (1) is proved similarly.

To prove (2), suppose f is weakly concave and $\succsim$ is continuous. Then, by the remarks following Theorem 4, μ must be continuous. We will need the following two lemmas (see *11*, pp. 150-51, *33*, pp. 211-12):

Lemma 5. If L is any lottery with a finite mean, then

$$\lim_{t \to -\infty} tL(t) = \lim_{t \to \infty} t(1-L(t)) = 0.$$

Proof of Lemma. It is known that any lottery L with a finite mean satisfies

$$\int_0^\infty t\,dL(t) = \int_0^\infty 1-L(t)dt$$

and

$$\int_{-\infty}^0 t\,dL(t) = -\int_{-\infty}^0 L(t)dt.$$

(*Note:* We adopt the convention here that Stieltjes integrals are taken over *closed* intervals. In particular, $\int_{-\infty}^0 t\,dL(t)$ refers to $(-\infty,0]$, not $(-\infty,0)$. Such specificity is not necessary, of course, for Riemann integrals.) Now, for any $r > 0$, we have (by integration by parts for Stieltjes integrals)

$$\int_0^r t\,dL(t) = rL(r) - \int_0^r L(t)dt$$

$$= r\left[L(r)-1\right] + \int_0^r 1-L(t)dt$$

and

$$\int_{-r}^0 t\,dL(t) = rL(-r) - \int_{-r}^0 L(t)dt.$$

If we now take limits as $r \to \infty$ and apply the results just cited, we obtain at once the desired conclusions.

Lemma 6. For any lottery L with a finite mean, there exists a sequence $\{L_n\}_{n=1}^{\infty}$ of simple lotteries such that

$$L_n \to L \text{ weakly}$$

(indeed, uniformly on $\mathbb{R}$) and

$$E(L_n) \to E(L) \text{ as } n \to \infty.$$

Proof of Lemma. Consider any integer $n \geq 3$. We will construct a simple lottery L_n such that

$$|E(L_n) - E(L)| < 1/n$$

and, for each $t \in \mathbb{R}$,

$$|L_n(t) - L(t)| \leq 1/n.$$

This will clearly be sufficient to establish the lemma.

Since $\lim_{t\to-\infty} L(t) = 0$, $\lim_{t\to\infty} L(t) = 1$, and $\int_{-\infty}^{\infty} t\,dL(t)$ exists—and, by Lemma 5—there exists a $B > 0$ such that

$$\left| \int_{-B}^{B} t\,dL(t) - \int_{-\infty}^{\infty} t\,dL(t) \right| < 1/4n, \tag{1}$$

$$BL(-B) < 1/4n, \tag{2}$$

$$B\left[1-L(B)\right] < 1/4n, \tag{3}$$

$$L(t) < 1/n \text{ whenever } t < -B, \tag{4}$$

and

$$L(t) > 1 - 1/n \text{ whenever } t \geq B. \tag{5}$$

Define $L_n(t) = 0$ for all $t < -B$ and $L_n(t) = 1$ for all $t \geq B$. Then $|L_n(t) - L(t)| < 1/n$ whenever $t \in (-\infty,-B) \cup [B,\infty)$. Our definition of L_n on $[-B,B)$ must await our construction of an appropriate partition of $[-B,B]$ into subintervals.

Toward this end, note that, by a standard theorem on Stieltjes integrals (*40*, p. 108), there exists a $\delta > 0$ such that, for any finite sequence of numbers $b_1 < b_2 < \ldots < b_m$ for which

$$b_1 = -B,$$

$$b_m = B,$$

and

$$\text{each } b_{i+1} - b_i \text{ is less than } \delta,$$

and for any choices $t_1,\ldots,t_{m-1}$ of points $t_i \in [b_i,b_{i+1}]$, one has

$$\left|\sum_{i=1}^{m-1} t_i\left[L(b_{i+1}) - L(b_i)\right] - \int_{-B}^{B} t\,dL(t)\right| < 1/4n.$$

We define a particular such finite sequence as follows: as in the proof of Lemma 4, we partition [0,1] into intervals

$$I_k = [(k-1)/n, k/n) \quad (k = 1,\ldots,n-1)$$

and

$$I_n = [(n-1)/n, 1],$$

obtaining thereby intervals

$$J_k = L^{-1}(I_k) \quad (k = 1,\ldots,n)$$

and numbers

$$a_1 \leq a_2 \leq \ldots \leq a_{n-1}$$

such that

$$J_1 = (-\infty, a_1),$$

$$J_n = [a_{n-1}, \infty),$$

and

$$J_k = [a_{k-1}, a_k) \quad (k = 2,\ldots,n-1).$$

We next (in effect) subdivide those J_k that lie (at least partly) within $[-B,B]$ into subintervals of length less than δ. More precisely, we define the set S to be the union of: (1) the set of all a_k (if any) for which $-B < a_k < B$ and (2) any one finite subset $\{c_1,\ldots,c_\ell\}$ of $[-B,B]$ such that

$$-B = c_1 < \ldots < c_i < c_{i+1} < \ldots < c_\ell = B$$

and each $c_{i+1} - c_i$ is less than δ. We may write S as

$$S = \{b_1, \ldots, b_m\},$$

where $b_i < b_j$ whenever $i < j$. Necessarily, $b_1 = -B$ and $b_m = B$. Moreover, $b_{i+1} - b_i < \delta$ for each $i = 1, \ldots, m-1$. Thus, the finite sequence $b_1, \ldots, b_m$ satisfies the conditions of the theorem cited.

Now, define L_n on $[-B, B)$ by

$$L_n(t) = L(b_i) \text{ if } t \in [b_i, b_{i+1}).$$

Clearly, L_n (having now been defined on all of $\mathbb{R}$) is a simple lottery.

Observe that each subinterval $[b_i, b_{i+1})$ is contained in some $J_k = L^{-1}(I_k)$. Thus, if $t \in [b_i, b_{i+1})$, then $L(t), L(b_i) \in I_k$, that is, $L(t), L_n(t) \in I_k$. Since I_k has length $1/n$, it follows that

$$|L_n(t) - L(t)| \leq 1/n.$$

Since, for all $t \in (-\infty, -B) \cup [B, \infty)$, a similar inequality was established earlier, we conclude that

$$|L_n(t) - L(t)| \leq 1/n$$

for all $t \in \mathbb{R}$, as was to be shown.

Finally, we prove that $|E(L_n) - E(L)| < 1/n$. Applying the cited integration theorem to our $b_1, \ldots, b_m$, choosing (in the language of that theorem)

$$t_i = b_{i+1} \in [b_i, b_{i+1}] \quad (i = 1, \ldots, m-1),$$

and (for brevity) putting

$$W = \sum_{i=1}^{m-1} b_{i+1}\left[L(b_{i+1}) - L(b_i)\right],$$

we obtain

$$\left|W - \int_{-B}^{B} t\,dL(t)\right| < 1/4n.$$

However,

$$\int_{-\infty}^{\infty} t dL_n(t) = b_1 L_n(b_1) + \sum_{i=1}^{m-2} b_{i+1}\left[L_n(b_{i+1})-L_n(b_i)\right] + b_m\left[1-L_n(b_{m-1})\right]$$

$$= -BL(-B) + W - B\left[L(B)-L(b_{m-1})\right] + B\left[1-L(b_{m-1})\right]$$

$$= W - BL(-B) + B\left[1-L(B)\right].$$

Thus,

$$|E(L_n) - E(L)| \leq \left|W - \int_{-B}^{B} tdL(t)\right| + \left|\int_{-B}^{B} tdL(t) - \int_{-\infty}^{\infty} tdL(t)\right|$$

$$+ BL(-B) + B\left[1-L(B)\right]$$

$$< 4(1/4n)$$

$$= 1/n,$$

which completes the proof of the lemma.

Returning now to the proof of the second half of the theorem, suppose f is weakly concave and $\succsim$ (and thus μ) is continuous, and consider any $L \in S$. By Lemma 6, there is a sequence $\{L_i\}_{i=1}^{\infty}$ of simple lotteries (hence, elements of S) such that $L_i \rightarrow L$ weakly and $E(L_i) \rightarrow E(L)$ as $i \rightarrow \infty$. We may write

$$L_i = \sum_{j=1}^{n_i} p_{ij} F_{t_{ij}}$$

for each i. Since f is weakly concave, we have

$$\mu\left[F_{E(L_i)}\right] = f\left[E(L_i)\right]$$

$$= f\left[\sum_{j=1}^{n_i} p_{ij} t_{ij}\right]$$

$$\geq \sum_{j=1}^{n_i} p_{ij} f(t_{ij})$$

$$= \mu\left[\sum_{j=1}^{n_i} p_{ij} F_{t_{ij}}\right]$$

$$= \mu(L_i)$$

for each i. However, as $i \to \infty$, we have $L_i \to L$ weakly and (since $E(L_i) \to E(L)$)

$$F_{E(L_i)} \to F_{E(L)}$$

weakly. Thus, by the continuity of μ,

$$\mu\left[F_{E(L)}\right] \geq \mu(L),$$

so that

$$F_{E(L)} \succsim L.$$

Since $L \in S$ was arbitrary, it follows that $\succsim$ is weakly risk averse.

For the case in which f is weakly convex, the proof of the corresponding result is similar. Finally, if f is affine, it is both weakly concave and weakly convex, and the corresponding result follows from the two previous cases.

Q.E.D.

In the usual expected utility theory based on integrals, part (2) of Theorem 11 follows immediately from Jensen's Inequality (*8*, p. 47). However, in the more general measurable utility theory presented here, we do not know that μ can be expressed as an integral. In particular, Theorem 7 (p. 32) does not apply, since we are not assuming that S is σ-convex. Our proof applies to such non-σ-convex lottery spaces as the space of all lotteries having a finite mean (which is, in a natural sense, the "largest" lottery space over which risk aversion can be considered).

Theorem 11 raises the question of whether weak concavity of f implies weak risk aversion for $\succsim$ when $\succsim$ is *not* continuous. We now show that it does not; in fact, not even *strong* concavity of f would suffice.

To prove our assertion, we consider once again the discontinuous preference orderings $\succsim_t$, and the corresponding discontinuous measurable utility functions μ_t ($t \in \mathbb{R}$), constructed earlier (pp. 42–44). Recall that each μ_t was constructed partly from—and, in turn, measurably induced—an arbitrarily given function, f, on $\mathbb{R}$. In particular, we may suppose f to be strongly concave. Resuming the assumptions and notation of that construction, we fix $t \in \mathbb{R}$ and choose any $L \in T_2$ such that $L(t) = 1/2$ and $E(L) = t$ (the c.d.f. of some uniform density centered at t will obviously do). Then,

$$\mu_t\left[F_{E(L)}\right] = f\left[E(L)\right]$$

$$= f(t).$$

However,

$$\mu_t(L) = a_t\left[1-L(t)\right]$$

$$= a_t/2$$

$$> f(t)$$

by the definition of a_t. Thus,

$$L >_t F_{E(L)},$$

from which it follows that $\succsim_t$ is not weakly risk averse.

Theorem 11 establishes that, in measurable utility theory, as in the traditional expected utility theory based on integrals, risk aversion implies concavity for measurably induced utility functions on $\mathbb{R}$. In the latter theory, a von Neumann-Morgenstern utility function that is measurably induced by a weakly risk averse preference ordering must be weakly concave. Thus, in this case, the utility of a lottery is expressed as a Stieltjes integral with a weakly concave integrand. However, the situation in measurable utility theory is more subtle: notwithstanding part (1) of Theorem 11, we will now construct a weakly risk averse preference ordering that, over all continuous lotteries corresponding to bounded random variables, is represented by an expected utility functional whose integrand is *not* concave. In fact, if, for some interval $[a,b]$, we restrict the set of lotteries considered to those arising from random variables taking values only in $[a,b]$, then we can even specify the integrand to be *strongly convex*.

To accomplish the construction, put $T_1 = H$, let T_2 be the set of all continuous lotteries that arise from bounded random variables (note that T_2 is convex), and let T be the convex hull of $T_1 \cup T_2$. Given any weakly concave function $u: \mathbb{R} \to \mathbb{R}$, define a measurable utility function

$$\mu_1: T_1 \to \mathbb{R}$$

by

$$\mu_1(L) = \int_{-\infty}^{\infty} u dL \quad (L \in T_1).$$

Let μ_2 be any measurable utility function on T_2 satisfying the following property ("Property P"):

For each $M \in T_2$, $u\left[E(M)\right] \geq \mu_2(M)$.

(We give examples in the next paragraph.) Finally, define a measurable utility function, μ, on T by

$$\mu\left[\lambda L + (1-\lambda)M\right] = \lambda\mu_1(L) + (1-\lambda)\mu_2(M) \quad (\lambda \in [0,1],\ L \in T_1,\ M \in T_2);$$

this definition is justified by a previous result on the uniqueness of lottery decompositions (see p. 49). Then, μ is weakly risk averse. In fact, suppose $H \in T$, and note that there exist $\lambda \in [0,1]$, $L \in T_1$, and $M \in T_2$ such that

$$H = \lambda L + (1-\lambda)M.$$

Thus, by the weak concavity of u and Property P of μ_2, and letting X be any random variable whose c.d.f. is L, we have

$$\begin{aligned}
\mu(F_{E(H)}) &= \mu_1(F_{E(H)}) \\
&= u\left[\lambda E(L) + (1-\lambda)E(M)\right] \\
&\geq \lambda u\left[E(L)\right] + (1-\lambda)u\left[E(M)\right] \\
&\geq \lambda u\left[E(X)\right] + (1-\lambda)\mu_2(M) \\
&\geq \lambda E(u \circ X) + (1-\lambda)\mu_2(M) \\
&= \lambda\mu_1(L) + (1-\lambda)\mu_2(M) \\
&= \mu(H).
\end{aligned}$$

We now exhibit a measurable utility function μ_2 satisfying Property P and allowing μ on T_2 to be expressed as an integral of a nonconcave function. For this, let $v: \mathbb{R} \to \mathbb{R}$ be any continuous nonconcave function dominated by u—that is, for which

$$v(x) \leq u(x)$$

for all $x \in \mathbb{R}$. Define $\mu_2: T_2 \to \mathbb{R}$ by

$$\mu_2(M) = \int_{-\infty}^{\infty} v\,dM \quad (M \in T_2).$$

Then, for each $M \in T_2$, letting Y be any random variable whose c.d.f. is M and applying Jensen's Inequality, we obtain

$$u\left[E(M)\right] = u\left[E(Y)\right]$$

$$> E(u \circ Y)$$

$$> \Sigma(v \circ Y)$$

$$= \mu_2(M),$$

so that μ_2 satisfies Property P. Since $\mu(M) = \mu_2(M)$ for all $M \in T_2$, μ has the desired representation as an integral over T_2. Alternatively, it is clear that, if we had defined T_2 as the set of all continuous lotteries that arise from random variables whose values all lie in some specified interval [a,b], and if we had required that u dominate v only over [a,b], then we could have chosen v to be strongly convex.

5.3 Implications for Identifying and Modeling Risk Aversion

To understand how the previous examples relate to the separate problems of identifying and modeling behavior in the presence of risk, consider first an investigator who wishes to construct an economic model involving risk averse behavior. Within the *traditional* interpretation of expected utility theory, the investigator would, in effect, assume that measurable utility functions take the form of expected utility integrals, and he/she would adopt a concave von Neumann-Morgenstern utility function as a "generator" of the risk behavior. This approach, however, carries the implicit assumption that the individuals under study have the same frames of reference toward certainty (degenerate lotteries) and "continuous uncertainty" (continuous lotteries), since the measurable utility of a lottery is determined, through the integral formula, by the utility values assigned to certainties.

If, on the contrary, the researcher does not wish to rule out *a priori* the possibility of different frames of reference--the possibility that an individual's risk preferences for certainties may differ from his/her risk preferences for continuous lotteries, so that his/her risk preferences are discontinuous--then the assumption of concavity for v would not guarantee risk averse behavior. For, as our first example demonstrated, even a strongly concave utility function on $\mathbb{R}$ can be measurably induced by a preference ordering that is *not* risk averse.

Our second example, however, suggests that some risk averse preference orderings might manifest the *illusion* of being determined by a nonconcave--or even strongly convex--utility function on $\mathbb{R}$. For, as we established in that example, preferences among certain continuous lotteries can be determined by such an "apparent" utility function even when the full preference ordering is risk averse. In a work developed within the classical expected utility framework, Hildreth and Knowles cite several empirical studies (including their own) of individuals' risk preferences that produced apparently risk neutral or risk loving responses (in effect, "nonconcavities" in the utility functions on $\mathbb{R}$) in cases where one might expect the decisionmakers' true preferences to be risk averse (*24*, p. 33). They characterize these nonconcavities as inaccurate and suggest various possible explanations for their occurrence (see also *37*).

The theory that we have delineated here, however, provides an alternative hypothesis: the nonconcavities are legitimate; interpreted within the context of measurable utility theory, they do not contradict risk aversion. Rather, the respondents in the studies, while being risk averse, may have had discontinuous, "two-rule" type measurable utility functions (reflecting dichotomous behavior toward certainty and "continuous uncertainty"), and they may have responded to some queries in the studies as though using their "continuous uncertainty" rule (μ_2, in the notation of our second example, with v nonconcave) rather than their "certainty" rule (μ_1, with u necessarily concave). Of course, we can only assert here that this hypothesis is a logical possibility; the determination of its empirical applicability is beyond the scope of this report.

We now reconsider and generalize two economic models of behavior under risk presented by Rothschild and Stiglitz (38). The first of these models concerns production with an uncertain output price, while the second involves saving with an uncertain interest rate. Rothschild and Stiglitz assume that an economic agent compares lotteries on the basis of their expected utilities; in our approach, an economic agent compares lotteries according to their *measurable* utilities. We will find that, if the marginal utility of money is greater under certainty than under "continuous uncertainty" (the meaning of this distinction will become clear shortly), then (under additional routine assumptions) the optimal production level and the optimal saving rate will be lower than the respective solutions of the expected utility models. No assumption of risk aversion is required.

6.1 Optimal Production Levels Under Price Uncertainty

We now consider a model of production with uncertain output price in which the producer is assumed to be endowed with a "two-rule" measurable utility function that allows, but does not require, him/her to use one preference rule for choosing among certainties and another for choosing among continuous lotteries. As we have seen, this assumption is compatible with measurable utility theory. (In this connection, recall our observation (p. 29) that the values of a measurable utility function at simple lotteries are determined by its values at certainties.) We will examine first- and second-order conditions and obtain an explicit solution of the former that expresses output in terms of the two preference rules and certain features of the price variable. We will find that, in the two-rule case, the producer will generally choose a utility-maximizing output level different from the one he/she would choose if following a more restrictive expected utility-maximizing approach. In particular, if the producer's marginal utility of money is greater under certainty than under "continuous uncertainty" and both the expected utility and measurable utility models have interior solutions, then the latter model will generate a lower optimal production level than the former. Furthermore, it is possible to have a zero production level and a positive production level that are simultaneously optimal. A particular pricing situation to which the analysis applies is that of a random price that is truncated below through the introduction of a "support price," as with agricultural commodities.

To set the stage for the analysis, let p (output price) be a bounded random variable on a probability space $(\Omega, \mathcal{B}, P)$ and $C: [0,\infty) \to \mathbb{R}$ (the cost function) a function such that $C''(Q) > 0$ whenever $Q \geq 0$. For each $Q \geq 0$, define a random variable $\pi(p,Q)$ (profit) by

$$\pi(p,Q) = pQ - C(Q).$$

We will use the convention that, whenever X is a random variable, its c.d.f. is denoted F_X. (Note that, in the special case in which X is constant, F_X is merely a degenerate lottery as defined earlier.) We assume the producer has a measurable utility function, μ, defined at each profit lottery $F_{\pi(p,Q)}$ $(Q \geq 0)$ and at each degenerate lottery F_t $(t \in \mathbb{R})$; of course, F_t represents "money amount t with certainty." More specifically, and in the spirit of endowing μ

with a two-rule form that generalizes the expected utility approach, we suppose that, for each lottery L for which the following integrals exist, we have

$$\mu(L) = \int_{-\infty}^{\infty} u_1 d\left[\pi_1(L)\right] + \int_{-\infty}^{\infty} u_2 d\left[[\pi_2+\pi_3](L)\right],$$

where the functions

$$\pi_i : S \to S \quad (i = 1,2,3; \ S \text{ the set of all s.d.f.'s})$$

(not to be confused with profits) are projection operators of the type described on pp. 24–25 (so that $\pi_1(L)$ and $[\pi_1+\pi_2](L)$ are the discrete and continuous "parts" of L, respectively) and $u_1, u_2 : \mathbb{R} \to \mathbb{R}$ are functions for which $u_1'(t) > 0$, $u_2'(t) > 0$, $u_1''(t) < 0$, and $u_2''(t) < 0$ for all $t \in \mathbb{R}$. As indicated, the domain of μ is taken to be the set of all lotteries L for which the integrals are finite. This set is clearly convex, and it contains each degenerate lottery and, since p is bounded, each $\pi(p,Q)$ $(Q \geq 0)$. Finally, we assume that p, u_1, u_2, and C are sufficiently "nicely behaved" to permit repeated differentiation under the Stieltjes integral sign (see 33, p. 409).

We wish to determine maxima $Q \geq 0$ of $\mu(F_{\pi(p,Q)})$ by examining first- and second-order conditions. However, obtaining these conditions is not quite straightforward, as the Stieltjes integrals in the definition of $\mu(F_{\pi(p,Q)})$ do not display Q (with respect to which we are to differentiate) very explicitly. We need to rewrite these integrals as Lebesgue integrals with random-variable integrands that display Q as a parameter. This change of form will allow differentiation under the integral signs. Moreover, in conformity with the "two-rule" form of μ, we wish these random variables to correspond, respectively, to the discrete and continuous lotteries that appear in the canonical convex decomposition of $F_{\pi(p,Q)}$ (see Proposition 3, p. 22). Thus, we need to determine this decomposition of $F_{\pi(p,Q)}$. We will do so in a way that explicitly relates the constituents of the decomposition to p, C, and Q.

To begin, observe that, for any $Q > 0$, $t \in \mathbb{R}$, we have

$$F_{\pi(p,Q)}(t) = P(pQ-C(Q) \leq t)$$

$$= F_p\left[(t+C(Q))/Q\right].$$

Thus,

$$F_{\pi(p,Q)} = F_p \circ \left[(I+C(Q))/Q\right],$$

where I is the identity function on $\mathbb{R}$ and, as usual, "$\circ$" denotes composition of functions. By means of this formula, we may use the canonical convex decomposition of F_p to obtain that for $F_{\pi(p,Q)}$. Now, put

$$A = \{r \in \mathbb{R} \mid P(p^{-1}\{r\}) > 0\},$$

$$\Omega_1 = \bigcup_{r \in A} p^{-1}\{r\},$$

and

$$\Omega_2 = \Omega/\Omega_1$$

(where "/" denotes the set-theoretic complement). Then, A is at most countable, and $\Omega_1, \Omega_2 \in \mathcal{B}$. Consider first the case in which p is a "proper mixture" of a discrete lottery and a continuous lottery, that is, the case in which $P(\Omega_1) > 0$ and $P(\Omega_2) > 0$. Let $(\Omega_1, \mathcal{B}_1, P_1)$, $(\Omega_2, \mathcal{B}_2, P_2)$ be the conditional probability spaces induced by $(\Omega, \mathcal{B}, P)$ on the events Ω_1, Ω_2, respectively, and define restricted mappings

$$p_1 = p \big| \Omega_1$$

and

$$p_2 = p \big| \Omega_2 .$$

Clearly, p_1 and p_2 are (bounded) random variables on $(\Omega_1, \mathcal{B}_1, P_1)$ and $(\Omega_2, \mathcal{B}_2, P_2)$, respectively. Furthermore, it can be shown without much difficulty that p_1 is discrete, p_2 is continuous, and

$$F_p = P(\Omega_1)F_{p_1} + P(\Omega_2)F_{p_2} .$$

Consider next the case in which p is itself discrete. Then $P(\Omega_1) = 1$ and $P(\Omega_2) = 0$, and by choosing $p_1 = p$ and p_2 to be any (bounded) continuous random variable, we can still assert

$$F_p = P(\Omega_1)F_{p_1} + P(\Omega_2)F_{p_2} .$$

This formula likewise holds if p is continuous, for then $P(\Omega_1) = 0$ and $P(\Omega_2) = 1$, and we need only choose $p_2 = p$ and let p_1 be any (bounded) discrete random variable. Thus, in all cases, if $Q > 0$, we can assert that

$$F_{\pi(p,Q)} = P(\Omega_1)F_{p_1} \circ \left[(I+C(Q))/Q\right] + P(\Omega_2)F_{p_2} \circ \left[(I+C(Q))/Q\right]$$

$$= P(\Omega_1)F_{p_1Q-C(Q)} + P(\Omega_2)F_{p_2Q-C(Q)},$$

so that

$$\mu(F_{\pi(p,Q)}) = \int_{-\infty}^{\infty} u_1 d\Big[P(\Omega_1)F_{p_1Q-C(Q)}\Big] + \int_{-\infty}^{\infty} u_2 d\Big[P(\Omega_2)F_{p_2Q-C(Q)}\Big]$$

$$= P(\Omega_1)E\Big[u_1\circ(p_1Q-C(Q))\Big] + P(\Omega_2)E\Big[u_2\circ(p_2Q-C(Q))\Big].$$

It follows (see *33*, p. 409) that, for any $Q > 0$,

$$[\mu(F_{\pi(p,\cdot)})]'(Q) = P(\Omega_1)E\Big[\Big[u_1'\circ(p_1Q-C(Q))\Big]\Big[p_1-C'(Q)\Big]\Big]$$

$$+ P(\Omega_2)E\Big[\Big[u_2'\circ(p_2Q-C(Q))\Big]\Big[p_2-C'(Q)\Big]\Big],$$

while

$$[\mu(F_{\pi(p,\cdot)})]''(Q) = P(\Omega_1)E\Big[\Big[u_1''\circ(p_1Q-C(Q))\Big]\Big[p_1-C'(Q)\Big]^2$$

$$+ \Big[u_1'\circ(p_1Q-C(Q))\Big]\Big[-C''(Q)\Big]\Big]$$

$$+ P(\Omega_2)E\Big[\Big[u_2''\circ(p_2Q-C(Q))\Big]\Big[p_2-C'(Q)\Big]^2$$

$$+ \Big[u_2'\circ(p_2Q-C(Q))\Big]\Big[-C''(Q)\Big]\Big]$$

$$< 0$$

by virtue of our assumptions about the signs of the various derivatives. Thus, $\mu(F_{\pi(p,\cdot)})\big|(0,\infty)$ can have at most one maximum and $\mu(F_{\pi(p,\cdot)})$ can have at most two. Any maximum, Q^*, of $\mu(F_{\pi(p,\cdot)})$ must satisfy either

$$Q^* = 0 \text{ ("nonproduction")} \tag{1}$$

or

$$Q^* > 0 \text{ and } [\mu(F_{\pi(p,\cdot)})]'(Q^*) = 0. \tag{2}$$

However, although condition (2) is sufficient to ensure that Q^* is a maximum of $\mu(F_{\pi(p,\cdot)})\big|(0,\infty)$, it is not generally sufficient to ensure that Q^* is a maximum of $\mu(F_{\pi(p,\cdot)})$. To find a maximum of $\mu(F_{\pi(p,\cdot)})$ when there is a $Q^* \in \mathbb{R}$ satisfying condition (2), one needs to compare $\mu(F_{\pi(p,Q^*)})$ with

$$\mu(F_{\pi(p,0)}) = \mu(F_{-C(0)})$$

$$= u_1\left[-C(0)\right].$$

The latter, of course, is the μ-utility of fixed costs. Thus, in the two-rule model, even if p is continuous (so that all profit random variables associated with positive production levels are continuous), there is always at least one profit lottery, representing fixed costs, whose utility must be determined from u_1 rather than from u_2.

Now, condition (2) holds for $Q > 0$ if and only if

$$\begin{aligned}
0 &= [\mu(F_{\pi(p,\cdot)})]'(Q) \\
&= P(\Omega_1)E\left[\left[u_1'\circ(p_1Q-C(Q))\right]p_1\right] - P(\Omega_1)C'(Q)E[u_1'\circ(p_1Q-C(Q))] \\
&\quad + P(\Omega_2)E\left[\left[u_2'\circ(p_2Q-C(Q))\right]p_2\right] - P(\Omega_2)C'(Q)E[u_2'\circ(p_2Q-C(Q))] \\
&= P(\Omega_1)A_1(Q) - P(\Omega_1)C'(Q)B_1(Q) + P(\Omega_2)A_2(Q) - P(\Omega_2)C'(Q)B_2(Q) \\
&= [P(\Omega_1)A_1 - P(\Omega_1)C'B_1 + P(\Omega_2)A_2 - P(\Omega_2)C'B_2](Q),
\end{aligned}$$

where, for given u_1, u_2, p_1, p_2, and C, the functions $A_1, A_2, B_1, B_2: (0,\infty) \to \mathbb{R}$ are defined in the obvious manner. Observe that, since $[\mu(F_{\pi(p,\cdot)})]''$ is negative throughout $(0,\infty)$, $[\mu(F_{\pi(p,\cdot)})]'$ is decreasing and thus invertible. Accordingly, when condition (2) is satisfied, its unique solution is

$$Q = [P(\Omega_1)A_1-P(\Omega_1)C'B_1+P(\Omega_2)A_2-P(\Omega_2)C'B_2]^{-1}(0).$$

Let us compare this solution with the one that arises under classical expected utility theory (38). We assume $(\Omega,\mathcal{B},P)$, p, C, $\pi(p,Q)$ (for any $Q \geq 0$), and u_1 are as in the preceding discussion. Define a measurable utility function μ by the "expected utility" rule

$$\mu(L) = \int_{-\infty}^{\infty} u_1 dL$$

for each lottery L for which the right-hand side exists (the set of all such L is convex). Then, for all $Q \geq 0$,

$$\mu(F_{\pi(p,Q)}) = E\Big[u_1 \circ (pQ - C(Q))\Big],$$

$$[\mu(F_{\pi(p,\cdot)})]'(Q) = E\Big[\Big[u_1' \circ (pQ - C(Q))\Big]\Big[p - C'(Q)\Big]\Big],$$

and

$$[\mu(F_{\pi(p,\cdot)})]''(Q) = E\Big[\Big[u_1'' \circ (pQ - C(Q))\Big]\Big[p - C'(Q)\Big]^2$$

$$+ \Big[u_1' \circ (pQ - C(Q))\Big]\Big[-C''(Q)\Big]\Big]$$

$$< 0.$$

Thus, $\mu(F_{\pi(p,\cdot)})$ has at most one maximum, and $Q^* \in [0,\infty)$ is a maximum if and only if either

$$Q^* = 0 \text{ and } [\mu(F_{\pi(p,\cdot)})]'(Q^*) \leq 0 \qquad (1')$$

or

$$Q^* > 0 \text{ and } [\mu(F_{\pi(p,\cdot)})]'(Q^*) = 0. \qquad (2)$$

Of course, (1') amounts to the condition

$$E\Big[u_1'(-C(0))\Big[p - C'(0)\Big]\Big] \leq 0,$$

which reduces to

$$E(p) \leq C'(0).$$

This is a sufficient conditio: for 0 to be a maximum (that is, for nonproduction to be optimal) in the present case, but not in the two-rule case. Likewise, condition (2) guarantees a maximum in the present case, but not in the two-rule case.

Now, in the expected utility model, we have, for each $Q > 0$ (using the same decomposition of $F_{\pi(p,Q)}$ as before),

$$\mu(F_{\pi(p,Q)}) = \int_{-\infty}^{\infty} u_1 dF_{\pi(p,Q)}$$

$$= P(\Omega_1) \int_{-\infty}^{\infty} u_1 dF_{p_1 Q - C(Q)} + P(\Omega_2) \int_{-\infty}^{\infty} u_1 dF_{p_2 Q - C(Q)}$$

$$= P(\Omega_1)E\left[u_1\circ(p_1Q-C(Q))\right] + P(\Omega_2)E\left[u_1\circ(p_2Q-C(Q))\right],$$

so that

$$\begin{aligned}[\mu(F_{\pi(p,\cdot)})]'(Q) &= P(\Omega_1)E\left[\left[u_1'\circ(p_1Q-C(Q))\right]p_1\right] \\ &\quad - P(\Omega_1)C'(Q)E\left[u_1'\circ(p_1Q-C(Q))\right] \\ &\quad + P(\Omega_2)E\left[\left[u_1'\circ(p_2Q-C(Q))\right]p_2\right] \\ &\quad - P(\Omega_2)C'(Q)E\left[u_1'\circ(p_2Q-C(Q))\right] \\ &= P(\Omega_1)A_1(Q) - P(\Omega_1)C'(Q)B_1(Q) \\ &\quad + P(\Omega_2)A_3(Q) - P(\Omega_2)C'(Q)B_3(Q),\end{aligned}$$

where A_1, B_1 are defined as before and (for given u_1, p_1, p_2, and C) the functions A_3, B_3: $(0,\infty) \to \mathbb{R}$ are defined in the obvious manner. Thus, when condition (2) is satisfied in this model, its unique solution is

$$Q = [P(\Omega_1)A_1 - P(\Omega_1)C'B_1 + P(\Omega_2)A_3 - P(\Omega_2)C'B_3]^{-1}(0).$$

It follows that the expected utility model and the two-rule model generally give rise to different optimal production decisions. In particular, if p is continuous, the solution of condition (2) is

$$Q = [A_3 - C'B_3]^{-1}(0)$$

in the expected utility model and

$$Q = [A_2 - C'B_2]^{-1}(0)$$

in the two-rule model. In this case, the latter solution is obtained by substituting u_2 for the von Neumann-Morgenstern utility function, u_1, in the former.

To compare the models further, suppose that each has an interior solution (that is, a solution of condition (2) that is optimal over $[0,\infty)$) and that $u_1' > u_2'$ on $\mathbb{R}$. (This inequality may be described heuristically as the assumption that the marginal measurable utility of money is greater under certainty than under "continuous uncertainty.") Then, examining the expansions of $[\mu(F_{\pi(p,\cdot)})]'$ (which, we recall, are comprised of decreasing functions) in both models jointly with condition (2) reveals that the solution of the expected utility model must exceed that of the two-rule model. Thus, in brief, when the marginal utility of money is greater under certainty than under continuous uncertainty in a two-rule model, maximizing measurable utility leads to a lower optimal production level than maximizing expected utility.

As an example of a situation where the foregoing analysis can be applied and within which a mixed-price lottery arises naturally, consider the introduction of a price support for an agricultural commodity. Assume (for simplicity) that the presupport price is a (bounded) continuous random variable, p_0. Suppose that a support price, $s \in \mathbb{R}$, is introduced, so that, in effect, p_0 is truncated below at s, forming a new random price variable, p. We assume that $P(p_0 < s) > 0$. Then,

$$p = \begin{cases} s & \text{if } p_0 < s \\ p_0 & \text{if } p_0 \geq s, \end{cases}$$

so that

$$F_p(t) = \begin{cases} 0 & \text{if } t < s \\ F_{p_0}(t) & \text{if } t \geq s \end{cases}$$

and (as is easy to show)

$$F_p = F_{p_0}(s)F_s + \left[1-F_{p_0}(s)\right]G,$$

where G is a continuous lottery defined by

$$G(t) = \begin{cases} 0 & \text{if } t < s \\ [F_{p_0}(t)-F_{p_0}(s)]/[1-F_{p_0}(s)] & \text{if } t \geq s. \end{cases}$$

In this case, we have

$$P(\Omega_1) = P(p_0 \leq s),$$

$$P(\Omega_2) = P(p_0 > s),$$

$$F_{p_1} = F_s,$$

and

$$F_{p_2} = G.$$

Furthermore, $F_{p_0}(s)$ is merely the probability that the presupport price will be equaled or exceeded by the support price, while $G(t)$ is the conditional probability that the presupport price will lie at or below t if it exceeds the support price. These relationships help us interpret the expression for the optimal production level Q arrived at earlier.

6.2 Optimal Saving Rates Under Uncertainty

We next investigate a model in which an individual who has different preference rules for certainty (degenerate lotteries) and "continuous uncertainty" (continuous lotteries) must decide how much of his/her initial wealth $W_0 > 0$ to consume in the current period and consequently how much to invest for consumption in the next period. We assume that the rate of return on saving is represented by a random variable. We will examine first- and second-order conditions and show how a solution may be obtained. We will find that the possibility an individual may be indifferent between (a) consuming W_0 entirely in period 1 and (b) investing a certain part of W_0 for later consumption cannot generally be ruled out. Finally, we will compare our model with the expected utility version of (*38*), which ours generalizes, and will find that the two models determine different solution values. In particular, when the marginal utility of money is greater under certainty than under continuous uncertainty, the optimal saving rate will be lower than the rate obtained through traditional von Neumann-Morgenstern expected utility maximization.

To begin, let r (the rate of return on investment) be a bounded, continuous random variable, and suppose that a saving rate of $s \in [0,1]$ generates a "certain" first-period consumption of $(1-s)W_0$ and an "uncertain" second-period return of sW_0r. The choice space over which the individual seeks to maximize his/her intertemporal utility is

$$\{(F_{(1-s)W_0}, F_{sW_0r}) \mid s \in [0,1]\}$$

(where, as earlier, "F_X" denotes the c.d.f. of the random variable X). Note that this choice space consists not of lotteries but of *ordered pairs* of lotteries. However, the definitions presented in our discussion of the existence of measurable utility functions (pp. 11-13) are sufficiently general to encompass "multidimensional lotteries" and "multidimensional utility": for if S and T are any lottery spaces such that S contains each $F_{(1-s)W_0}$ ($s \in [0,1]$) and T each F_{sW_0r} ($s \in [0,1]$), then the choice space we are considering is contained in the Cartesian product

$$S \times T,$$

which is a mixture space. (In fact, $S \times T \subset \Phi \times \Phi$, where Φ is the set of all functions from $\mathbb{R}$ into $\mathbb{R}$. Since Φ is a vector space over $\mathbb{R}$, $\Phi \times \Phi$ is a vector space over $\mathbb{R}$ under the inherited operations of coordinate-wise addition and scalar multiplication. Since $S \times T$ is a convex subset of $\Phi \times \Phi$, it has a mixture-space structure.) We will specify S and T, define a measurable utility function, μ, on $S \times T$, and assume that the individual seeks to find an element of the choice space that maximizes μ over the choice space. Furthermore, following Rothschild and Stiglitz (*38*), we desire that μ: (1) be additively separable between periods, (2) discount period 2 utility, and (3) allow for different behavior toward certainty and "continuous uncertainty." (A full treatment of the economic implications of additive separability for measurable utility functions would take us beyond the bounds of our present purpose. Additive separability in expected utility theory is discussed in Pollak (*34*).)

To guarantee these properties, let $u_1, u_2 \colon \mathbb{R} \to \mathbb{R}$ be functions for which $u_1''(t) < 0$ and $u_2''(t) < 0$ for all $t \in \mathbb{R}$. (We also assume that u_1, u_2, and r are sufficiently "nicely behaved" to allow repeated differentiation under the Stieltjes integral sign (*33*, p. 409).) Define a measurable utility function

$$\eta_1 \colon H \to \mathbb{R}$$

by

$$\eta_1(L) = \int_{-\infty}^{\infty} u_1 dL \quad (L \in H)$$

and a measurable utility function

$$\eta_2 \colon J \to \mathbb{R}$$

by

$$\eta_2(M) = (1-\delta)\left[\int_{-\infty}^{\infty} u_1 d\pi_1(M) + \int_{-\infty}^{\infty} u_2 d[\pi_2+\pi_3](M)\right] \quad (M \in J),$$

where J is defined as the (convex) set of all lotteries M for which the right-hand side of the equation above is finite, the π_i are the usual projection operators, and $\delta \in (0,1)$ is interpreted as the "pure rate of time discount of utility." Finally, define a measurable utility function, μ, on $H \times J$ by

$$\mu\left[(L,M)\right] = \eta_1(L) + \eta_2(M) \quad \left[(L,M) \in H \times J\right].$$

Note that, since r is bounded, each $(F_{(1-s)W_0}, F_{sW_0 r})$ $(s \in [0,1])$ is in Domain (μ). Then, when $s \in (0,1]$, we have

$$\mu\left[(F_{(1-s)W_0}, F_{sW_0 r})\right] = \int_{-\infty}^{\infty} u_1 dF_{(1-s)W_0} + (1-\delta)\left[0 + \int_{-\infty}^{\infty} u_2 dF_{sW_0 r}\right]$$

$$= u_1\left[(1-s)W_0\right] + (1-\delta)E\left[u_2 \circ (sW_0 r)\right],$$

while $s = 0$ implies

$$\mu\left[(F_{(1-s)W_0}, F_{sW_0 r})\right] = \mu\left[(F_{W_0}, F_0)\right]$$

$$= u_1(W_0) + (1-\delta)u_1(0).$$

Thus, μ exhibits the desired properties (1), (2), (3).

For brevity, with μ, W_0, and r understood, define ξ: $[0,1] \to \mathbb{R}$ by

$$\xi(s) = \mu\left[(F_{(1-s)W_0}, F_{sW_0 r})\right] \quad (s \in [0,1]).$$

Then, if $s \in (0,1]$, we have

$$\xi'(s) = u_1'\left[(1-s)W_0\right](-W_0) + (1-\delta)E\left[[u_2' \circ (sW_0 r)]W_0 r\right]$$

and

$$\xi''(s) = u_1''\left[(1-s)W_0\right](-W_0)^2 + (1-\delta)E\left[[u_2'' \circ (sW_0 r)]W_0^2 r^2\right]$$

$$< 0.$$

Thus, $\xi\big|(0,1]$ can have at most one maximum, and ξ can have at most two. Any maximum, s^*, of ξ must satisfy either

$s^* = 0$ (no wealth saved), (1)

$s^* = 1$ (all wealth saved), (2)

or

$0 < s^* < 1$ and $\xi'(s^*) = 0$. (3)

However, although condition (3) is sufficient to ensure that s^* is a maximum of $\xi\big|(0,1]$, it is not generally sufficient to ensure that s^* is a maximum of ξ. To find a maximum of ξ when there is an $s^* \in \mathbb{R}$ satisfying condition (3), one would need to compare $\xi(s^*)$ with $\xi(0)$. Note that $\xi(0)$ is the μ-utility associated with "no wealth saved;" it depends on u_1, whereas $\xi(s^*)$ depends on u_1 and u_2.

Now, condition (3) holds for $s \in (0,1)$ if and only if

$$\begin{aligned} 0 &= -u_1'\left[(1-s)W_0\right] + (1-\delta)E\left[\left[u_2'\circ(sW_0 r)\right]r\right] \\ &= A(s) + (1-\delta)B_2(s) \\ &= [A + (1-\delta)B_2](s), \end{aligned}$$

where, for given u_1, u_2, W_0, r, and δ, the functions $A, B_2\colon [0,1] \to \mathbb{R}$ are defined in the obvious manner. However, since A' and B_2' are everywhere negative, $A + (1-\delta)B_2$ is decreasing and thus invertible. Accordingly, if condition (3) holds, its unique solution is

$$s = [A + (1-\delta)B_2]^{-1}(0).$$

Let us compare these results with those of the classical model that portrays the individual as operating by means of a single, expected utility-type preference rule. We assume W_0, r, u_1, δ, and η_1 are as before. Define measurable utility functions

$$\eta_2\colon \mathcal{J} \to \mathbb{R}$$

and

$$\mu\colon \mathcal{H} \times \mathcal{J} \to \mathbb{R}$$

by

$$\eta_2(M) = (1-\delta)\int_{-\infty}^{\infty} u_1\,dM \quad (M \in \mathcal{J})$$

and

$$\mu\left[(L,M)\right] = \eta_1(L) + \eta_2(M) \quad ((L,M) \in \mathcal{H} \times \mathcal{J}),$$

where now $\mathcal{J}$ is defined as the set of all lotteries M for which the integral appearing in the definition of η_2 is finite. Put

$$\xi(s) = \mu\left[(F_{(1-s)W_0}, F_{sW_0 r})\right]$$

for each $s \in [0,1]$. Then, for each $s \in [0,1]$,

$$\xi(s) = u_1\left[(1-s)W_0\right] + (1-\delta)E\left[u_1 \circ (sW_0 r)\right],$$

$$\xi'(s) = u_1'\left[(1-s)W_0\right](-W_0) + (1-\delta)E\left[\left[u_1' \circ (sW_0 r)\right]W_0 r\right],$$

and

$$\xi''(s) = u_1''\left[(1-s)W_0\right]W_0^2 + (1-\delta)E\left[\left[u_1'' \circ (sW_0 r)\right]W_0^2 r^2\right]$$

$$< 0.$$

Thus, ξ has at most one maximum, hence exactly one (since it is continuous on $[0,1]$). (Note that, in the two-rule model, by contrast, ξ was not, in general, continuous at 0.) Moreover, $s^* \in [0,1]$ is the maximum if and only if either

$$s^* = 0 \text{ and } \xi'(s^*) \leq 0, \tag{1'}$$

$$s^* = 1 \text{ and } \xi'(s^*) \geq 0, \tag{2'}$$

or

$$0 < s^* < 1 \text{ and } \xi'(s^*) = 0. \tag{3}$$

Observe that condition (1') reduces to

$$-u_1'(W_0) + (1-\delta)u_1'(0)E(r) \leq 0,$$

that is (assuming $u_1'(0) \neq 0$),

$$E(r) \leq [1/(1-\delta)][u_1'(W_0)/u_1'(0)].$$

This inequality is a sufficient condition for 0 to be the maximum of ξ in the present model but not in the two-rule model. Likewise, condition (2') and condition (3) each guarantees a maximum in the present model, but not in the two-rule model. Now, condition (3) holds for $s \in (0,1)$ if and only if

$$-u_1'\left[(1-s)W_0\right] + (1-\delta)E\left[\left[u_1' \circ (sW_0 r)\right]r\right] = 0,$$

which can be expressed as

$$A(s) + (1-\delta)B_1(s) = 0,$$

where $A\colon [0,1] \to \mathbb{R}$ was defined previously and $B_1\colon [0,1] \to \mathbb{R}$ is defined in the obvious manner. However, since $A' + (1-\delta)B_1'$ is everywhere negative,

$A + (1-\delta)B_1$ is invertible. It follows that, if condition (3) holds, then its unique solution must be

$$s = [A + (1-\delta)B_1]^{-1}(0).$$

We conclude that condition (3) generally has different solutions in the classical and two-rule models. The solution formulas differ in that the two-rule model replaces the classical model's von Neumann-Morgenstern utility function, u_1, in B_1, with u_2, reflecting the different treatment of uncertainty in period 2.

In particular, suppose each model has an interior solution (that is, a solution of condition (3) that is optimal over [0,1]) and $u_1' > u_2'$ on $\mathbb{R}$ (that is, "the marginal measurable utility of money is greater under certainty than under continuous uncertainty"). Then, examination of the expansions of ξ in both models jointly with condition (3) reveals that the expected utility model yields a higher optimal saving rate than the two-rule measurable utility model.

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