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ABSTRACT

This paper proposes an hypothsis describing how some college students incorrectly view semantically laden letters and provides both clinical and written data that supports that hypothesis. Several word problems were given to nine college math students. These students were then interviewed as they attempted to solve these problems. In addition to the interviews, data were obtained from two related diagnostic mathematics tests. Test 1 was administered to 101 college students toward the end of the first semester of a calculus course and test 2 was administered to 153 different college students at the beginning of the same course. Overall, the study findings indicated that it is at the interface between the semantic content of a problem and its algebraic representation that students' skills appear to be most lacking. Implications for instruction, especially problem solving, are presented. Word problems given during the clinical interviews and the two diagnostic tests are appended. (RH)

Student Conceptions of Semantically Laden Letters in Algebra*

A Technical Report

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Central to most of the mathematics curriculum from algebra, to statistics, calculus and beyond is the concept of a variable. Most of modern mathematics rests upon symbolization processes that allow letters to represent unknown, varying quantities. Those symbolization processes take on even greater significance when viewed as a means for encoding world data into mathematical language. Letters in this context stand at the interface between meaning in language and meaning in mathematics. So, for example, in algebra, x may not only be thought of as a number, but also as a number or amount of some quantitative entity to which it refers such as number of books bought or number of dollars spent. One could say that the letter, in this context, carries a semantic load. It is these letters that will be referred to as semantically laden letters.

This paper proposes an hypothesis for describing how a large number of college students incorrectly view semantically laden letters and provides both clinical and written data that supports that hypothesis. Some implications of these findings will then be discussed. A description of previous and preliminary research is first in order.

Several authors have written about student conceptions of letters in algebra. Kuchemann (1978) for example attempts to classify algebra problems in an hie-archy according to the increasingly complex role that letters found in those problems play. He describes a cognitive developmental sequence that parallels the ascending complexity inherent in his classification scheme. Tonnessen (1980) attempts to define the concept of variable in its abstract form and to assess to what extent students' understanding of the concept matches his own. He does not, however, attempt to isolate the effects that the added semantic load of a letter has on students' understanding of that letter.

One such effect is the tendency on the part of students to view the letter as a nominal label for the referent. Clement (1982) and Rosnick (1981) both describe that tendency on the part of many college math students. These students tend to read an equation like 4C=5S as "four cheesecakes for every five strudels" rather than the appropriate, "four times the number of cheesecake is equal to five times the number of strudel". Kaput (1980) corroborated those findings and describes the tendency to associate a letter with a label for a concrete referent as a sympton of a "nominalist" tendency. Davis (1975), Galvin and Bell (1977), Matz (1979), and Kuchemann (1981) also discussed this tendency to associate letters with a concrete object rather than number or amount.

Preliminary Study

In search for further evidence for this "labels" or "nominalist" interpretation of semantically laden letters, the following problem was given to thirty statistics students, all of whom had had two semesters of calculus. The problem was developed to see whether, when faced with a blatant contradiction, students would recognize the fallacy of a labels reading of a letter. The problem reads as follows:

I went to the store and bought the same number of books as records. Books cost 2 dollars each and records cost 6 dollars each. I spent 40 dollars altogether. Assuming that the equation 2B + 6R = 40 is correct, what is wrong, if anything, with the following reasoning. Be as detailed as possible.

$$2B + 6R = 40$$

Since B = R, I can write:

$$2B + 6B = 40$$

$$8B = 40$$

This last equation says eight books is equal to 40 dollars. So one book costs 5 dollars.

It was thought that if students view letters as standing for labels, they would agree that 8B reads as "eight books" rather than "eight times B, the number of books bought " The results show that among those thirty students, 23 (77%) did not recognize the misinterpretation of the last line, giving some erroneous explanation of how the problem was done incorrectly. The fact that a large majority of the students did not see the error of interpreting 8B as "eight books", seems to support, at first glance, an hypothesis that says that students view letters as generic labels. However, closer inspection of student responses suggested that that hypothesis was not sufficient in describing the students' conceptions. By far, the most common incorrect answer to the above problem was that even though the number of books bought was the same as the number of records bought, B does not equal R because their prices are different. Thus, the letter B seemed to stand for more than just the word "books"; it also stood simultaneously for the price and number of the books.

In fact, in clinical interviews of college math students discussing the above problem, some students seemed to give several distinct interpretations of what the letters used had meant. What was intriguing about these interviews was that the students gave no indication whatsoever that these different interpretations were either inconsistent or contradictory. The meaning that these students seemed to impart to these letters was tenuous, unstable, and only sporadically quantitative.

An Hypothesis

This suggests the following hypothesis. For many students, the referent of a semantically laden letter appears to be an undifferentiated conglomerate. That is, the letter is identified with an entire, complex, overly generalized referent rather than a particular quantitative attribute of that referent. Thus, though the correct solution of a problem calls for creating a variable, x, that stands specifically for the number of books bought, students will use x to mean a general concept having to do with several aspects of books. Within this undifferentiated conception of what x means could be one or more quantitative attributes of books, or the name books, or the actual physical thing called a book, and/or etc. In a sense, students allow x to stand for "bookness"; i.e. it encompasses much of what, in the context of the problem, the word book or books implies.

Clinical Investigation

The above hypothesis is supported by evidence collected in the clinical investigation that is described in this section. Several word problems were given to nine college math students who were interviewed as they attempted to solve these problems. These interviews were recorded on audio tape and then transcribed and analyzed. Different problems were tried (see Appendix 1) but all had several things in common. They all could be solved by setting up two equations with two unknowns. The entities involved in the problems all had two attributes of consideration, e.g. price and quantity or weight and quantity. In each case, one of those attributes was given and the other was left unknown. All problems had integral solutions. It was believed that the problems in Appendix 1 would be useful in determining student conceptions of semantically laden letters and testing the above hypothesis because having two relevant attributes for each referent makes the differentiation between those attributes more crucial to the successful completion of the problem.

The subjects were chosen at random from among a group of volunteers who were enrolled in a statistics course designed for the social sciences at a large university. Most had had one or two semesters of calculus. The subjects were paid a nominal sum for their time. Subjects were told that the purpose of the interviews was to learn more about how students solve these types of problems and for that reason they should talk out loud as much as possible.

Interventions on the part of the interviewer were limited to questions that asked for clarification, amplification, or identification, such as "what did you mean by that?", "could you say a little more about that?", "could you read what this equation means in English", etc. Other questions aimed at students' understanding of letters, like "what does the x mean in this expression?". The interviewer was conscious of the pitfalls of asking leading questions and avoided doing so as much as possible. Each student was interviewed for approximately 45 minutes in which time they solved an average of just over three of the word problems of the type given in Appendix 1.

Due to the subjectivity inherent in any analysis of clinical interview data, it was felt that evidence used as support for the hypothesis should be strong and convincing. It was felt that simply witnessing students saying "B is books" or "x is turtles" was not strong enough. Though a student saying "B is books" might imply a lack of careful differentiation, it could be argued that the student understands the meaning and function of the letters but just has not learned (or has no reason) to articulate that. It could be further argued that if the student is capable of treating the letters quantitatively then his or her verbalizations are more or less irrelevant.

In a similar fashion, any one piece of evidence pointing to the existence of the conception that semantically laden letters refer to undifferentiated conglomerates could be circumstancially negated. However, if more than one piece of evidence were found, and if those pieces of evidence indicated in different ways that the conception is held by the student, the argument that is made becomes significantly more convincing.

The following four behaviors were each observed in several of the students' transcripts. Each one is considered by the author to be supporting evidence for the stated hypothesis. It was felt then that if any student, in a clinical interview, revealed three of these four behavioral criteria, it would be strong support for the hypothesis that that student tends to identify semantically laden letters with undifferentiated conglomerates.

The determination that three of four behavioral criteria validate the hypothesis is an arbitrary one. It is believed by the author that this standard for testing the hypothesis is sufficient but not necessary. That is, if less than three of the behaviors are observed it does not necessarily negate the existence of that conception in a student. For that reason, this standard is considered to be a conservative one. The four behaviors are as follows:

A. The student does not give clear, stable, quantitative definitions for the letters he or she uses.

B. The student uses a letter in contradictory ways, allowing the meaning of the letter to shift without recognizing or acknowledging that shift.

C. the student accepts the juxtaposition of two contradictory quantitative uses of the same letter. This differs from behavior B in that the dual usage of the letter seems to be occurring simultaneously.

D. The student is unable to drop part of the semantic load that a letter carries even when it is necessary and appropriate to do so.

To illustrate each of these behaviors, edited transcripts of several of the students will be presented and analyzed.

<u>Behavior A</u>. the omission of a clear, stable, quantitative

definition for the letters being used was evidenced by all nine students on at least some of the problems they tried and by eight of the students on all of the problems they tried. In the following transcript segment, note that Eileen manages to solve the problem without having a clear understanding of what the letters she uses mean. (Numbered notes and comments will follow the transcript segment and are considered an integral part of the text. They are separated from the transcript to allow the reader the option to read the transcript uninterrupted. The problem is given in full as example 3 in Appendix 1.)

S: (Reads) "A person went to the store and bought pecans and cashews and got a total of 100 nuts." So I'm going to put to start out with; P for pecans plus C for cashews is equal to 100. (Writes P + C = 100) [see note 1]. (Reads) "The number of pounds of pecans he bought was the same as the number of pounds of cashews. 8 pecans were 1 pound and 12 cashews weighed 1 pound. How many pounds of pecans did he buy?" So I'm looking at a total of 100 nuts and 8 pecans weigh 1 pound so 8 nuts, 8 of these weighs 1 pound, 12 of these weigh 1 pound and so I would just put that into proportion, I guess. Umm, I;m going to do the same thing I did with turtles and frogs [see note 2]. I'm going to say that P is equal to pecans [see note 3] and then, well actually, 8 pecans [see note 4]. So; yea, 8 is equal to pecans (writes 8P = pecans) and then 12P is equal to cashews (writes 12P = cashews) and then I'm going to say 8P plus 12P is equal to 100 (writes 8P + 12P = 100) [see note 5]. 20P is equal to 100. P is equal to 5 [see note 6]. Umm.

I: And what does P stand for?

S: P is a number again [see note 7]. So pecans would go 8 times 5 is equal to pecans; and that's 40. And then 12 times 5 is equal to cashews and that's 72? 60. And then to check it out, that's the number of nuts he bought and it should equal 100 because that was given in the beginning. And the question is how many pounds of pecans did he buy? [see note 8]. Well, he bought 40 pecans actually in number so I would take the 40 and, umm, I would take the 40 and divide it by 8 to get 5 because there are 8 nuts in every pound so, or if there is 8 nuts in 1 pound then how many

> pounds would there be for 40 nuts (writes 8 nuts/1 lb = 40/x lb) [see note 9]. So I'd go, 40 is equal to 8 nuts (writes 40 = 8x) [see note 10] and get x is equal to 5.

Later on the interviewer recapped this final segment.

> I: You multiplied [the 5] times 8 and got 40. Then you divided 40 by 8 to get 5 again.
>
> S: Yeah, So I probably could have stopped but I didn't recognize that that was the answer.
>
> I: Why do you think that... you didn't recognize the answer?
>
> S: Because probably not until I looked back did I realize that the P I was trying to find was pounds [see note 11]. If I realized it first then I would have said P is pounds [see note 12] and P is 5 then 5 pounds of pecans is what he bought. But not until you asked me what was P, what was P starding for then I realized that it had to be pounds.

1. Eileen verbally refers to the letter as meaning <u>pecans</u> but writes an equation that suggests that P represents the <u>number of pecans</u> bought.

2. Eileen had previously worked on the frogs and turtles problem (number 1 in Appendix 1) and despite numerous confused starts and a good deal of reversed reasoning (e.g., she said, "the number of turtles he bought was 3 times the number of frogs so T plus 3T is equal to the number of turtles and frogs he bought"), she eventually correctly solved the problem.

3. Again, note the broad, nonquantitative identification of P as "pecans".

4. P is associated with 8 pecans and in fact, if taken in isolation, this sentence says P = 8. Eight may be thought of here as an abstract quantitative attribute of pecans (8 pecans per pound).

5. The last verbalized definitions of P were that P is equal to

pecans and that P is equal to 8. But in order for this equation to be correct, P has to be the number of pounds of pecans sold. In any case, note the unacknowledged shift of meaning and utilization of P from P + C = 100 to 8P + 12P = 100. This is an example of behavior C.

6. This is the correct answer, but, as will be seen, Eileen doesn't yet recognize it as such.

7. She recognizes that P is quantitative but gives no clear definition of its meaning.

8. She does not know! This supports the claim that when she wrote 8P + 12P = 100, she was not thinking that P meant the number of pounds of pecans bought.

9. Note the introduction of the new variable x. This again supports the above claim.

10. Note the "labels" reading of the letter: 40 is equal to 8 nuts where nuts is said as she writes the x.

11. This seems to be an affirmation of the fact that as she solved this problem, Eileen did not have a clear, stable, quantitative definition of what P meant (behavior A).

12. Note now that "P is pounds" is consistent language with "P is pecans"; though she is closer to the correct idea, she once again refers to the letter in an overly general, undifferentiated manner.

Thus, though Eileen was able to solve this problem, she did so without a solid grasp of the meaning of the letters that she used. Furthermore, what descriptions of the meanings of the letters she had, shifted from a generalized label to a quantitative attribute to another quantitative attribute back to another generalized label.

This transcript therefore exemplifies behavior B as well as behaviors A and C.

Behavior B. The following transcript segment (of a student solving example 4 in Appendix 1) gives another example of a student performing behavior B. Note that though this problem is similar to the previously described Books and Records problems, it differs in two significant ways. First, in this problem the number of books and records bought is known and the prices are unknown, whereas in the original problem this was reversed. Secondly, this problem has not been done out for the student. Paul begins by reading the problem.

001 S: "A person went shopping for books and records. He spent a total of $72. The price of each book was the same as the price of each record. He bought 2 books and 6 records. What was the price of 1 record?"---He spent $72--He bought 2 books and 6 records. What was the price of one record? The price of each book was the same as the price of each reco[illegible]. So x is gonna be for... x equals x [see note 1] (Mumbles to self. Writes 72 = 2x + 6x; 72 = 8x; 9 = x) [see note 2]

002 I: What are you uh, remarking about?

003 S: Uh-- it just didn't look right for a minute.

004 I: Uh-huh.

005 S: What was the price of 1 record--(to self) The price of each book was the same as the price of each record. He bought 2 books and 6 records for $72--I'm getting confused here.

006 I: What's confusing?

007 S: --How do we write this answer? To-to find the price of the record. I suppose it's 6 records.

008 I: Ok. When you said "6 records" you underlined 6x (in 72 = 2x + 6x). Is that it?

009 S: Right [see note 3].

010 I: Ok--Can I ask you what x stands for, just so I can

be thinking with you?

011 S: Um,--(Pauses) [see note 4].

012 I: What were you thinking?

013 S: Items purchased [see note 5].

014 I: Items purchased? Uh-huh. So, read this first line to me again. [$72 = 2x + 6x$]

015 S: $72 equals 2 items plus 6 items [see note 6].

016 I: Ok--And what is it that you're troubled by now?

017 S: It doesn't seem to be answering the question.

018 I: How's that?

019 S: Uh---

020 I: When you say 9 is equal to x, what do you conclude from that?

021 S: I don't know [see note 7].

022 I: Er, x means? What does x mean?

023 S: A minute ago I said x means items purchased.

024 I: Uh-huh.

025 S: 9 items purchased.

026 I: Uh-huh. Does that seem reasonable to you?

027 S: Yeah.

028 I: Ok.

029 S: ---(30 seconds)---No.

030 I: What was that?

031 S: I said no, it doesn't seem reasonable.

032 I: Oh.

033 S: 8 items purchased.

034 I: Uh-huh. You pointed to the 8x when you said that.

035 S: Yeah. X is items purchased. This is all very

confusing [see note 8].

036 I: Uh-huh. It's a confusing problem.

037 S: So I guess uh,--it's $9 for 1 record.

038 I: How did you decide that? Now what you just did was, you put dollar signs on the 9 and the 72 and you just said $9 for each record. How did you come to that conclusion?

039 S: There was 8 items purchased.

040 I: Uh-huh.

041 S: For a total of $72. And the price of each book was the same as the price of each record. So everything costs $9.

042 I: I see. Um, so you say 8 items were purchased when you look at which equation?

043 S: $72 = 8x; 8 items.

044 I: Uh-huh. So x means what?

045 S: Items purchased.

046 I: Items purchased. And then you have x is equal to $9.

047 S: Right. Items purchased equals $9; those were the prices. $9 per item purchased [see note 9].

048 I: It sounds like you have a question in your voice when you're saying that.

049 S: Uh, each item costs $9.

050 I: Uh-huh; So say again now specifically what x means.

051 S: The cost for an item [see note 10].

052 I: The cost for an item. Did you, er, were you thinking that when you first wrote the equation down? You just said x is the cost of 1 item. Were you thinking that when you first wrote it down?

053 S: I don't think all that consciously, but maybe a little bit subconsciously. I think that's what it had to be; yeah [see note 11].

1. It is unclear what x equals at this point. When he says "x equals x" it is possible that that is parallelling the sentence "the price of each book was the same as the price of each record", but that is not clear.

2. This is the correct answer but it seems that he doesn't recognize it as such yet.

3. The implication is that x is a label meaning records.

4. The long pause might indicate that he doesn't clearly know what x means (behavior A).

5. X seems to stand for both records and books so it is generalized further to stand for items purchased.

6. X is used as a label for the word "items".

7. This confirms that he doesn't recognize that he has the right answer and therefore seems to confirm that he doesn't recognize that x stands for the price of records (or books).

8. Again, this is a labels interpretation of x.

9. X is simultaneously "items purchased" and the quantitative value $9, clearly implying behavior B.

10. Finally, Paul is able to give a good definition of the letter

11. If in fact Paul was thinking correctly subconsciously, it must have been pretty far back in his consciousness because he didn't recognize the right answer when he saw it. A better explanation might be that because the qualitative concept "items purchased" was mixed with the quantitative attribute, price of an item, Paul became confused.

<u>Behavior C</u>. As was the case with Eileen's transcript, the

preceeding transcript not only exemplifies in particular one of the behaviors (in this case behavior B) but also gives strong hints of other behaviors (A and C). The following transcript segment focuses on behavior C, but also includes some of the other behaviors. Behavior C is distinguished by the juxtaposition of two quantitative interpretations of one letter without any remorse or discomfort on the part of the problem solver.

Ray is working on the same pecans and cashews problem that Eileen worked on (example 3 in Appendix 1). After he reads the problem, he writes down the information using a labels approach. Thus, he writes 8P = 1 lb and 12C = 1 lb to mean that there are eight pecans and, respectively, twelve cashews to a pound. He quickly follows this by writing P = C, saying "the number of pounds of pecans he bought was the same as the number of pounds of cashews". Note that this is an example of behavior B. P first means the word "pecans" and then means the number of pecans bought. Ray becomes stuck in trying to solve the problem algebraically and resorts to a trial and error, arithmetic solution. He is successful after several attempts. We pick up the transcript at that point.

047 I: So you solved it by trial and error.

048 S: Right. I just plugged in a couple of numbers. I'm sure I could have worked it out in a formula.

049 I: Could you do that for me? I'd be interested in seeing the formula.

050 S: You want a formula. Well, you know P equals C, so...So it's got to be, umm 12C + 8P = 100 and let me see. P equals 3/4C because 8 is...No, 2/3, I'm sorry (writes P = 2/3C). 8 is two thirds of 12, right [see note 1].

051 I: So P is equal to two thirds C.

052 S: So when you multiply...No, it doesn't work either.

053 I: what were you about to do?

054 S: I was going to multiply 8 times two thirds but it comes out to 16, it comes out to 16 thirds and that's not a very nice figure to work with [see note 2].

055 I: What does this equation mean, 12C plus 8P equals 100. Could you read this in English?

056 S: In English? Umm, well, you know there is 100 nuts and there are 12 cashews to a pound and there is 8 pecans to a pound so you know from the formula there is going to be...the proportion of nuts is going to be um, the total number of nuts is going to be in this proportion [see note 3].

057 I: What does the letter C stand for?

058 S: Cashews. And P stands for pecans [see note 4]. But they're worried about weight, how many pounds of pecans did he buy. You know it's equal to the number of pounds of cashews he bought so, 8 pecans weigh 1 pound and 12 cashews weigh 1 pound. So, you know there is 100 nuts. Let me check this over again. I want to make sure this proportion is right. 12 cashews in a pound and 8 pecans in a pound so 8 over 12 it's got to be two thirds [see note 5].

1. Ray has written two equations, 12C + 8P = 100 and P = 2/3C. Both equations are correct but only if you allow the letters to have different quantitative meanings. In the former, P for example, has to stand for the <u>number of pounds of pecans</u> bought. In the latter, P has to stand for either the number 8 or the <u>number of pecans</u> bought. This exemplifies behavior C.

2. Ray tries to solve the above two equations simultaneously but rejects that, not because the letters mean different things in each case, but merely because the numbers come out to be non integral.

3. the equation here is very loosely defined. It is not clear that

for Ray, P and C stand for the number of pounds of pecans and cashews, respectively, nor is it clear that, in Ray's mind, multiplication is occurring between the coefficients and the letters.

4. This hints of a generalized labels definition.

5. Despite acknowledging that the problem is "worried" about weight, Ray does not let go of trying to relate P and C in terms of the numbers of individual pecans and cashews, respectively, in a pound.

Eventually, Ray gives up on the equation, P = 2/3C, only to try the semantically identical equation, 3/2P = C. When that also produces a non integral result, he goes back to rereading the problem. He rediscovers that he can write P = C and solves the problem accordingly. What is important to note is that he never completely rejects the idea of relating P with C via the equation P = 2/3C. The interviewer asks:

> 067 I: What do you think was wrong with what you were doing over here when you said P was equal to two thirds C?
>
> 068 S: Well, I wasn't, I wasn't, umm...I was trying to solve..I wasn't...
>
> 070 S: ...What I was doing over here, I was trying to make C and P equal in a proportional sense, you know, by using a fraction, by dividing the 8 into 12 or the 12 over the 8, when...
>
> 071 I: What's wrong with that?
>
> 072 S: Well, it's not necessary because it's given, it's C equals P...

Thus it seems that for Ray, there is nothing wrong or inconsistent with writing P = 2/3C, it is just that it is not necessary in solving the problem. Ray does not seem to recognize that the meaning of the P is different in P = 2/3C from its meaning in 8P + 12C = 100. This apparent acceptance of the juxtaposition of two contradictory quantitative uses of the same letter is an example of behavior C.

Behavior D. Perhaps the most vivid evidence of a student's conception of semantically laden letters as undifferentiated conglomerates can be found in behavior D. Behavior D occurs when a student cannot suspend the association of a letter with its complex referent, even though it is expedient or essential to do so. That is, the fact that the letter is semantically laden interferes with the student's ability to apply standard algebraic rules to that letter.

For example, Beth is working, in the following transcript segment, on the Books and Records problem given on page 2. The solution is done out for the student with the "trick" conclusion. She is very troubled by the equation 2B + 6B = 40 which appears in the text of the problem because, as she says, "why would you miltiply the cost of each record [$6] times the amount of books [B]." Even though the amount of books equals the amount of records "B is not equal to R because um, you cannot substitute B for R or R for B...and have everything else be right." It appears that she cannot let go of the fact that the referent of B is "books" and thus B should in no way be related to any attribute of records, specifically their 6 dollar cost. B has more meaning than just its quantitative value. Eventually Beth correctly recognizes that 5 books and 5 records were bought. But she is still convinced that the algebraic solution that was given to her is incorrect. She is still troubled by the equation, B = R.

196 S: ...B does not equal R because B is a book and R is a record so a book doesn't equal a record [see note 1].

197 I: Okay. B is a book and R is a record?

198 S: Right. A-an amount...an unspecified amount...it could be zero, it could be 50 million, but in this case, 5 happens to be an answer [see note 2].

199 I: Okay. But just because both of them are 5, they're still not equal?

200 S: Right. Because books aren't equal to...Not if-it's just like saying up..to-you know, tomato is to an orange.

201 I: Uh-huh, uh-huh...uh-huh.

202 S: You know, the amounts can happen to be the same but the just the amount of-of the book [see note 3]...I don't know. I'm getting myself in deeper with this...(9 secs)..the amounts here are the same but they're not the amounts of the same thing [see note 4].

1. The letter B seems to be associated with a single prototypical book.

2. Now the letter is quantitative in value. Note that she says 5 is the answer. She does not identify what it answers; she does not indicate what it is the amount of.

3. In the phrase "the amount of the book", she still seems to be referring to a single book.

4. This last sentence is a key indicator of behavior D.

Results of Clinical Interviews

The transcripts that have been used thus far in the paper are noteworthy in the degree to which they illustrate the various behavioral criteria. They are not, however, exceptional. Each of the above students demonstrated similar conceptions in other problems they attempted. Those students not included among the above examples also exhibited variations on the same kinds of behaviors. In fact, eight of the nine students who were interviewed exhibited at least three of the four behavioral criteria. Thus it could be conservatively stated that eight out of the nine students demonstrated an ill-defined, nebulous view of the letters they were working with. The letters, for them, were associated with both qualitative and quantitative attributes of the complex referent. Furthermore, there is a sense that these various attributes were not recognized as being distinct from each other but rather as parts of some undifferentiated conglomerate.

Table 1 shows the distribution of observed behaviors.

	Behaviors			
	A	B	C	D
Beth	x	x		x
Maria	x	x	x	x
Ray	x	x	x	
Janet	x	x	x	x
Paul	x	x		x
Eileen	x	x	x	
Margaret	x	x	x	
Victoria	x	x	x	
Liz	x			x

Note that, as mentioned above, all nine students exhibited behavior A, a lack of a clear stable, quantitative definition of variables being used. Note also that all students except Liz exhibited behavior B.

Thus according to the standard set up by this author, the clinical interview data strongly support the hypothesis that many students (in this case, at least eight out of nine college statistics students) view semantically laden letters as undifferentiated conglomerates.

Written Diagnostic Tests

In addition to the clinical interviews, the written results from two related diagnostic tests were scored and analyzed. The first test (Appendix 2) was administered to 101 college students enrolled in the first semester of a calculus course designed for students in the social and biological sciences. The test was given towards the end of the semester. The second test (Appendix 3) was given to 153 different students enrolled in the same calculus course. These students were just beginning their semesters.

The diagnostic tests were originally designed with more goals in mind than are relevant to this study. Those goals that were relevant to the current discussion are the following:

1. To obtain raw data on the overall success rate for the types of problems that were given in the clinical interviews.

2. To see to what extent the four behavioral criteria given on page 7 are observable in student responses and to determine whether a correlation exists between the exhibiting of some of those four behaviors and success or failure on the problems.

In each set of tests, each student was given two of four possible problems. The format of the tests is given in Appendices 2 and 3. Students were randomly given tests containing either problems 1 and 4 or tests with problems 2 and 3. Each problem was given on a separate sheet of paper and the order of the problems was randomly mixed.

1. Raw data. Data indicating overall student ability on these problems are given in Tables 2 and 3. Table 2 is the overall success rate on these tests. Table 3 gives data on individual performances. It tallies how many people got both, one, or no problems correct. In the column labeled "score" in Table 3 are the average number of

problems done correctly.

Table 2: Overall Success Rate on the Written Diagnostic Tests.

	No. of Problems 2/person)	No. Correct	% Correct
Test 1	202	102	50%
Test 2	306	144	47%
Total	508	246	48%

Table 3: Individual Student Performances on the Written Diagnostic Tests

	No. of People	No. of people getting: both problems correct		one problem correct		no problem correct		score
Test 1	101	34	34%	34	34%	33	33%	1.01
Test 2	153	44	29%	56	37%	53	35%	0.94
Total	254	78	31%	90	35%	86	34%	0.97

The ≥ data can be briefly summarized by saying that overall, there was less than a 50% success rate on these algebra problems. Students, on an average, got slightly fewer than one out of two problems correct with 34% of the students getting no problems correct.

2. Evidence for the four behaviors in the written data. The second goal of this phase of the research was to answer questions like the following. Are the above results in any way related to student conceptions of semantically laden letters? Is there evidence in the written data for any of the four behavioral criteria indicating the conception of letters as undifferentiated conglomerates?

In attempting to answer these questions, some issues pertaining to research methods became self evident. Difficulties were encountered in attempting to analyze written data in the same manner in which the clinical interview data was analyzed. It was found that written data alone tended to be more ambiguous than clinical interview data vis-a-vis revealing what the student may have been thinking. Identical written responses may have been derived from very different conceptual backgrounds.

Thus, the written work alone was not as reliable an indicator of student conceptions. The differences between written and clinical interview data are underscored further when one attempts to use the behavioral criteria that had been developed for the clinical interviews in analyzing written work. Many of these behaviors have a verbal component, making it difficult to observe the behavior in the written work. With the exception of behavior A, the amount of evidence in written work for the other behaviors is greatly deflated from what it had been in the clinical interviews.

Behavior C, for example, requires that two quantitative uses for the letter be juxtaposed without the tudents' recognition of any contradiction. This does seem to occur often in the written work. However, the students' thoughts and views are left for conjecture. One student wrote the following series of equations:

$$r + g = 91$$

$$1r + 3g = 91$$

$$1r + 4(3g) = 91$$

Does each successive equation imply a refinement and rejection of the one before or is there a belief that all three equations are correct but only the last one is needed to solve the problem? Without the verbal input, this remains unclear.

Behavior D is even more closely linked to verbal data. It is evidenced when a student verbally reports something akin to "I cannot replace B with R because even though the numbers are the same, books are different from records". In written work, it is often the case that students do not appropriately replace one variable with another. However, without the verbal input, it cannot be claimed that behavior D is occurring.

Nevertheless, even though the extrapolation of the categories and behavioral criteria to written work is problematic, some significant results are still quite evident. One such result is a very simple one but one which has strong pedagogical implications. It is the fact that of the 245 students who used algebra in at least one of the problems they attempted, 212 of them (87%) exhibited behavior A in at least one problem solution. That is, 212 students either gave no definition for the variables they used or the definitions given were not quantitative. It is interesting to note that those students who consistently gave clear, quantitative definitions for the letters they used scored significantly better on these problems than did students who exhibited behavior A. The former students had a mean score of 1.54 whereas the latter students had a mean score of 0.88. (The reader will recall that a student was given a score of 2 if both problems were done correctly, 1 if one problem was correct, and zero if no problems were correct.) A two-tailed t-test establishes that the difference was significant at the $p < .001$ level.

In addition to looking for behavior A in the written responses, the written data were scrutinized for evidence of the other behaviors as well. It was found that 125 of the 254 students (49%) exhibited at least 2 of the 4 behaviors on at least one of their two problems. The mean score for these students was 0.70 whereas tha mean score for the remaining students was 1.22, a difference which was significant at $P < .001$.

The reader should be reminded, however, that because the three behaviors B, C, and D are evidenced much less frequently in written work than in clinical intervie s, these latter statistics are probably very different than they would be had the data been collected via clinical interviews.

One other finding in the written data should be noted: that it is clear that students' conception of semantically laden letters goes beyond a "labels approach". They do evidence the use of .etters as labels quite often. But they also treat the letters quantitatively and demonstrate other qualitative applications for which a "labels" description is inadequate.

Because of the inadequacy of the written data, one cannot assert with the same degree of confidence as was the case with the clinical interview data, that students have improper conceptions of semantically laden letters. Specifically, it cannot be established from the written data alone whether the conception of semantically laden letters as undifferentiated conglomerates is prevalent among non physical science undergraduates. However, there is nothing in the data to contradict this. Certainly, the poor showing in general and the even poorer algebraic showing in particular could be predicted by a hypothesis that says that students conceive of letters as having amorphous and undifferentiated meaning. These students' overwhelming tendency to neglect to define the letters they are using as values of quantitative attributes of the referents could also have been predicted by that hypothesis. That the written data does not contradict, but, if anything, lends support to the clinical interview data gives this latter data added importance.

Implications

The implications of these findings are important, not only in algebra, but in geometry, statistics, calculus and, in fact any subject, mathematical or not, that uses symbols to stand for unknown, variable quantities. The role that letters as variables play is often a fundamental, crucial, yet tacit one in these subjects. Some of the difficulties that students encounter may be directly connected with their misconceptions concerning the role of the letters.

Consider, for example, calculus. Surely, students have difficulty grasping concepts like limit, continuity and the derivative. But many of their difficulties might very well relate to their understanding (or lack thereof) of variables. Semantically laden letters abound in applied calculus (maximum/minimum problems are notable examples). It might well be the case that students' legendary difficulties with these problems might be as much connected with their improper conception of the variables that they use as it is an inability to apply the principles of calculus.

Consider, also, the concept of a random variable in probability and statistics. Part of the difficulty in teaching that concept may be connected with students' confusion about distributions and probability. But much of their confusion may also be related to their unawareness of the need to carefully define their variables.

As a further example, consider that in Chemistry, letters are often used to stand for things. H2O signifies a molecule with two atoms of hydrogen and one atom of oxygen. However, there are certainly plenty of places in chemistry where letters take on more traditional roles as variables (in an algebraic sense). One might ask whether the legitimacy of using letters to stand for "things" adds to students' confusion concerning the use of letters as quantitative variables, contributing to difficulties the students face in learning the subject.

Implications of the data reported in this paper also stretch outside of the realm of symbolization processes and into the realm of cognitive science. Briefly, the difficulties students have in clearly discerning and differentiating between various attributes of a complex referent might be related to difficulties in concept differentiation in general.

A more general pedagogical implication of these data is one that argues for a focusing of the entire mathematics curriculum away from the basic skills of manipulation of expressions and equations and towards an emphasis on semantically laden problems and the development of general problem solving skills. Most of the students who were interviewed in the studies reported above showed little difficulty in manipulating and solving equations and performing other skills of basic algebra. Where their difficulties became evident was in the creation and or interpretation of those equations. It is thus at the interface between the semantic content of a problem and its algebraic representation where students' skills appear to be most lacking. Yet it is the ability to interface semantic content with an algebraic representation that is crucial to the application of mathematics to practical situations.

If one accepts that the ability to apply mathematical skills is itself an essential skill, then the data reported on above are extremely important. They indicate that the lack of clarity and stability in the definition and use of letters to represent semantic referents is likely to be a significant factor in students' difficulties with word problems. The association between some of the behavioral indicators of a student's view of letters as undifferentiated conglomerates and poorer scores on the written diagnostic tests is one of the indicators of that. If one views an inability to solve word problems as symbolic of a general inability to apply the basic skills, then students' misuse of semantically laden letters is itself symbolic of a significant shortcoming of the mathematics curriculum. The very premise for teaching these students mathematics, that they will be able to apply some of what they learn, is called to question.

It is this that suggest the need for the reorienting of the mathematics curriculum away from a focus on basic skills and towards a goal of "understanding" and the ability to apply mathematical tools to real world situations. Educators must question, for example, the value of requiring business and social science majors to learn calculus when the data show that these students' understanding of algebra as it models the world is greatly lacking. Educators must emphasize general problem solving skills at all levels, especially those skills that are required for the solution of semantically laden problems.

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APPENDIX 1

Some of the Problems Given to Students During the Clinical Interviews.
(Not all students received all of the problems)

1. A boy bought a collection of frogs and turtles. The number of frogs that he bought was three times the number of turtles that he bought. Frogs cost 3 dollars each and turtles cost 6 dollars each. He spent $60.00 altogether. How many frogs did he buy?

(This problem was given to only two students, Ann and Beth).

2. I went to the store and bought the same number of books as records. Books cost two dollars each and records cost dix dollars each. I spent $40 altogether. Assuming that the equation 2B + 6R = 40 is correct, what is wrong, if anything, with the following reasoning. Be as detailed as possible.

$$2B + 6R = 40 \text{ Since } B = R, \text{ I can write}$$
$$2B + 6B = 40$$
$$8B = 40$$

This last equation says 8 books is equal to $40. So one book costs $5

3. A person went to the store and bought pecans and cashews. He bought a total of 100 nuts. The number of pourds of pecans he bought was the same as the number of pounds of cashews. 8 pecans weight 1 pound and 12 cashews weigh 1 pound. How many pounds of pecans did he buy?

4. A person went shopping for books and records. He spent a total of $72. The price of each book was the same as the price of each record. He bought two books and six records. What is the price of one record?

5. A person went shopping for books and records. He spent a total of $72. The number of books bought was the same as the number of records bought. Books cost two dollars each and records cost six dollars each. How many records were bought?

APPENDIX 2

Written Diagnostic Test 1

(Students who took the written diagnostic test were given the following instructions)

Please answer the following two problems. SHOW ALL OF YOUR WORK. DO NOT ERASE. Just lightly cross out your mistakes.
Your answers will be kept completely confidential and will in no way affect your grade in this course.

(Half of the students were given problems 1 and 4; the other half were given problems 2 and 3. On the original test, each problem was given on separate sheets.)

1. A Biology teacher bought a collection of frogs and turtles. The number of frogs that he bought was three times the number of turtles that he bought. Frogs cost 3 dollars each and turtles cost 6 dollars each. He spent $60.00 altogether. How many frogs did he buy?

2. A woman had a container of red and green blocks that weighed a total of 91 ounces. Red blocks weigh one ounce each. Green blocks weigh three ounces each. The number of red blocks was four times the number of green blocks. How many red blocks did she have?

3. A Biology teacher bought a collection of frogs and turtles. The price of one frog was three times the price of one turtle. He bought three frogs and six turtles. He spent $60.00 altogether. What is the cost of one frog?

4. A woman had a container of red and green blocks that weighed a total of 91 ounces. She had one red block and three green blocks. The weight of a red block is four times the weight of a green block. How much does one red block weigh?

APPENDIX 3

Written Diagnostic Test 2

(Students who took the written diagnostic test were given the same instructions given in Appendix 2.)

(Half of the students were given problems 1 and 4; the other half were given problems two and three. On the original test, each problem was given on separate sheets.)

1. A biology teacher bought a collection of frogs and turtles. The number of frogs that he bought was four times the number of turtles that he bought. Frogs cost one dollar each and turtles cost three dollars each. He spent $91.00 altogether. How many frogs did he buy?

2. A woman had a container of red and green blocks that weighed a total of 60 ounces. Red blocks weigh three ounces each. Green blocks weigh six ounces each. The number of red blocks was three times the number of green blocks. How many red blocks did she have?

3. A biology teacher bought a collection of frogs and turtles. The price of one frog was four times the price of one turtle. He bought one frog and three turtles. He spent $91.00 altogether. What is the cost of one frog?

4. A woman had a container of red and green blocks that weighed a total of 60 ounces. She had three red blocks and six green blocks. The weight of a red block is three times the weight of a green block. How much does one red block weigh?