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ABSTRACT

This handbook is part of a series of three, corresponding to the three grades (4, 6, and 8) at which Mastery Tests are administered. This publication was written as a resource for teachers developing mathematics programs for students in grades 7 and 8. The Mastery Test in Mathematics at Grade 8 assesses student performance for 36 instructional objectives. The following information is provided for each objective: (1) Appropriate Materials (manipulative materials to use for exploring the concept); (2) Enabling Skills and Activities (a description of prerequisite skills and activities); (3) Sample Lessons (student activities that build toward mastery); (4) Teacher Resource Materials; and (5) Mathematics Objectives and Sample Test Items. The instructional objectives are grouped into four categories: (1) conceptual understandings; (2) computational skills; (3) problem solving and applications; and (4) measurement and geometry. Two appendixes are included: Appendix A contains the sample test items and objectives; Appendix B is a list of commercial manipulatives and resource materials. (RH)

State of Connecticut

William A. O'Neill, Governor

Board of Education

Handbook for Teachers

GRADE 8 MASTERY TEST IN MATHEMATICS

This handbook was prepared by Betsy Y. Carter, mathematics consultant with the Connecticut State Department of Education.

CONTENTS

This *Handbook for Teachers, Grade 8 Mastery Test in Mathematics* represents another step toward our goal of insuring that all students master the mathematical concepts and skills required to become productive citizens. It relates to the achievement of three of the seven Board of Education goals--to improve the quality of instruction and curriculum, to improve effectiveness of teachers and teaching, and to insure equity for all children.

The 8th grade teachers' handbook is part of a series of three, corresponding to the three grades--4, 6, and 8--at which Mastery Tests are administered. The grade 4 handbook in mathematics was published in 1986; handbooks for teachers of grades 6 and 8 are being released in 1987.

The primary focus of this testing program is improvement in the achievement of Connecticut's students. Once teachers and administrators have reviewed class, school, and district test results, the task of using these results to improve student mastery of basic skills moves directly into the classroom. It is there--in thousands of Connecticut classrooms--where high quality teaching translates test results into meaningful instructional activities.

The *Handbook for Teachers, Grade 8 Mastery Test in Mathematics* includes specific instructional strategies and sample lessons that are keyed to each of the test objectives. You will also find sample test items and resource lists in the handbook.

I urge you to review carefully the Mastery Test results for your class and your school and to use this handbook as a resource for planning lessons and classroom activities that meet your students' needs for mathematics instruction.

Gerald N. Tirozzi
Commissioner of Education

The Handbook for Teachers, Grade 8 Mastery Test in Mathematics is intended as a resource for teachers as they develop a meaningful mathematics program for students in Grades 7 and 8. The Mastery Test in Mathematics at Grade 8 assesses student performance on 36 objectives. However, mathematics instruction should NOT be reduced to providing lessons planned to meet only those objectives. Rather, instruction aimed at mastery of the objectives should be integrated into a mathematics program that has as its primary goal the understanding of important mathematics concepts and the use of mathematical ideas to solve problems.

The Mastery Test objectives were chosen as significant benchmarks of growth. The limitations of a multiple-choice, paper-and-pencil test have inevitably influenced the final list of objectives. Test items most frequently assess skills at a symbolic, or abstract, level; a few items include pictorial information. Teachers are urged to continue to assess student development of concepts through classroom observation of students as they work with manipulative materials, collect and organize information, and solve problems.

The role of this handbook is to place the objectives in the perspective of a mathematics curriculum based on the way children develop mathematical skills and concepts. It is also designed to assist teachers in providing a mathematics program that continually moves through a sequence of concrete, pictorial, and abstract experiences as concepts are explored and objectives are mastered.

The ideas, activities, and sample lessons found in this handbook are illustrative of a mathematics curriculum that has problem solving as its central focus. Significant strands in the curriculum, which deserve equal time in the classroom, include:

- Organizing Information

  Classification
  Patterns
  Graphs and tables
  Probability
  Statistics
  Estimation

- Spatial Relationships

  Geometry
  Measurement
  Estimation

- Number Sense

  Order
  Place value
  Whole numbers, fractions, decimals, integers
  Operations with numbers
  Estimation
  Mental computation
  Computation with calculators

HANDBOOK COMPONENTS

In this handbook the mathematics Mastery Test objectives are organized in clusters. The objectives in each cluster share the same underlying mathematical concepts.

- Appropriate Materials. Exploration of concepts through the daily use of manipulative materials promotes understanding. A listing of useful manipulative materials is included for each cluster of test objectives. Appendix B is a list of suppliers of commercial math manipulatives.

- Enabling Skills and Activities. Students should NOT be asked to practice examples at the same level of difficulty as the test items without experiences at earlier stages. Therefore, this handbook provides a description of prerequisite skills and activities suitable earlier in the sequence of instruction. In general, each discussion of the developmental sequence spans cognitive abilities typical of students in grades 5 to 8.

- Sample Lessons. Included in the sample lessons are ideas that directly involve students in activities that build toward mastery. Each lesson has been designed to develop or reinforce several skills and concepts. This presents an opportunity for students to engage in the process of making connections and solving problems in the various strands of the curriculum. Rather than teach each skill in isolation, teachers are urged to explore ideas together so that students may begin to discover the structure of mathematics.

- Teacher Resource Materials. A list of teacher resource materials that can aid in the development of additional activities is provided for each cluster of objectives. The bibliography contains complete information about the resource materials, as well as citations for other professional research and reference materials.

- Mathematics Objectives and Sample Test Items. Appendix A contains a sample test item for each objective.

GUIOE TO INSTRUCTIONAL OBJECTIVES *

CONCEPTUAL UNDERSTANOINGS

* The test objectives are reorganized into clusters of related concepts. The numbering of objectives as listed on the Mastery Test Student Report has been maintained to serve as a cross-reference to the test. However, for instructional and mathematical clarity, a few objectives have been addressed out of numerical sequence.

COMPUTATIONAL SKILLS

PROBLEM SOLVING

MEASUREMENT AND GEOMETRY

I. COMPARING NUMBERS

Objective 1 Order fractions.

Objective 2 Order decimals.

Objective 3 Round whole numbers.

Objective 9 Identify points on number lines, scales and grids.

APPROPRIATE MATERIALS

Number lines
Numeral cards
Base Ten Blocks (Dienes or Powers of Ten)
Money

ENABLING SKILLS AND ACTIVITIES

Students develop a sense of number when they have many opportunities to estimate and count sets of discrete objects. A sense of the magnitude of numbers is developed by comparing sets of objects--for example, contrast 100 bottle caps with a collection of 1000. Number sense also is developed through estimation activities--have students guess how many bottle caps might fill a drinking glass, a large jar, and an aquarium.

Activities that require sorting, comparing, and ordering objects by length, area, volume, and weight all reinforce the idea of order. Such activities relate the size of objects to their corresponding numerical measure. The need for fractions becomes apparent when students measure objects which require the use of parts of a unit. The ruler is a number line with an application--to measure length. It also is a useful model for ordering fractions.

Place value materials provide another concrete model for comparing, estimating, and rounding whole numbers. Ask students to build a two-dimensional design or three-dimensional construction with place value blocks, estimate its value, and then organize and count the material to establish its value.

Activities with money reinforce place value and magnitude. Give a student some play money. Ask the student to select a certain number of pieces and make a money amount more or less than a specified amount, or between specified amounts. For example, with one-dollar bills, ten-dollar bills, twenty-dollar bills, and hundred-dollar bills available, make an amount between $300 and $350 using exactly five pieces of play money. Explore the different possible solutions.

Students may also play strategy games that require them to develop efficient ways to find a "Secret Number". Have one student write a secret number on a slip of paper. Other students in turn try to guess the number. The keeper of the secret number may only respond "It's larger," "It's smaller," or "You've guessed it." Begin the game with a "Secret Number" that is a whole number, and later, as students develop efficient strategies for finding the "Secret Number", use decimals, fractions, and mixed numbers.

Counting and skip counting forward and backward aid in sequencing numbers. Have students try skip counting with interesting rules, such as: count by 3 starting at 2; count by 7 starting at 2; count by 5 1/2 beginning at 8 1/4; or count by 20.2 starting at 11.04. Write the numbers down and look for a pattern. The number line is a useful way of organizing the numbers. Students discover patterns as they label positions on a number line.

SAMPLE LESSON Ordering Rational Numbers on a Number Line

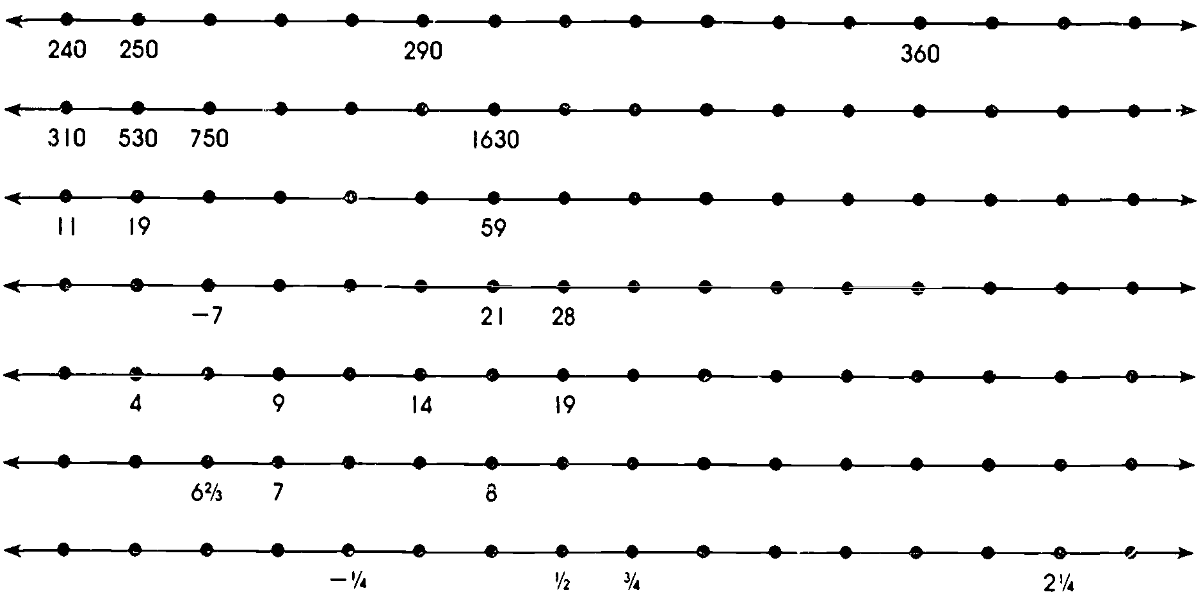
MATERIALS NEEDED

Number line sheets
Pencils

Make up a number line sheet, which consists of several blank number lines. Save this as a duplicating master. Make several copies. On each sheet label some of the points on each number line. Ask students to label the rest of the points. Begin with whole numbers.

Repeat the activity with other sheets. Vary the value of the interval and the size and type of the numbers used. For example, start with whole numbers between 0 and 500 and have the interval worth 10. Later, include larger numbers and try intervals worth 20, 25, 50, 100, 150, 200, 220, and so on. Then use integers and intervals such as 7, 8, 11, 12, and so on. Vary the starting points. Introduce intervals worth 1/2, 1/4, 1 1/2, and once again vary the kinds of numbers used as starting points.

Encourage students to make up similar number line problems and exchange them with each other. Explore the patterns that emerge.



TEACHER RESOURCE MATERIAL

The Mathworks
The Good Time Math Event Book

II. PLACE VALUE

Objective 4 Round decimals to the nearest whole number, tenth and hundredth.

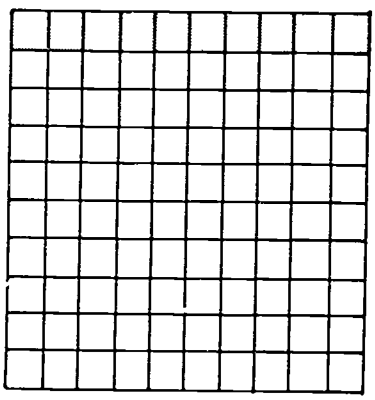
Objective 5 Multiply and divide whole numbers and decimals by 10, 100 and 1000.

APPROPRIATE MATERIALS

Decimal Squares
Fraction Bars
Money
Number lines
Grid paper

ENABLING SKILLS AND ACTIVITIES

As students develop a sense of numerical order and magnitude they are prepared to "cluster" numbers based on their comparative sizes. Before students round numbers on the symbolic level, they should develop some visual models. Such models include building decimal values with Decimal Squares and fraction values with Fraction Bars. Grid paper may be used to draw pictures of decimal amounts--a 10 x 10 grid may represent the unit, ten squares are a tenth (0.1) and one square is a hundredth (0.01). Have students build or draw models of three numbers and compare them to see which two are closest in value; for example, build 0.10 and 0.20. Then build 0.17 and compare it to the other two decimal values. Which one is closest to 0.17?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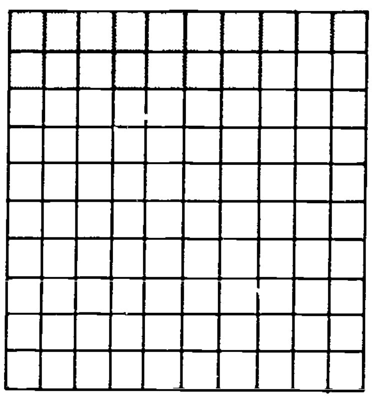
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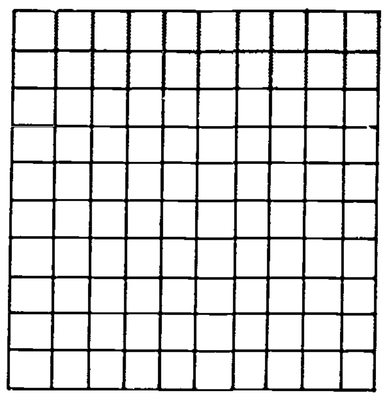
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Money also may serve as a model--is \$5.43 closer to \$5.00 or \$6.00? What is the difference in pennies?

Compare \$5.50 to \$5.00 and \$6.00 and note that the difference in pennies is the same. Establish a common rule for rounding \$5.50 and handling similar situations.

The number line is a pictorial model that requires the student to relate the distance between points to the relative value of a specified point. Ask students to describe the numbers (to the nearest tenth) greater than 3.5 that are closer to 3.5 than to 4.0 (3.6 and 3.7). Repeat the problem but describe the numbers to the nearest hundredth (the numbers from 3.51 to 3.74). Find the numbers on the number line. Discuss the number 3.75 as an example of rounding up.

As students work on rounding with decimal fractions they also should be rounding common fractions. This will help them see the relationship in size between decimals and fractions.

Once students develop skill rounding decimals to whole numbers, tenths, and hundredths, they should use rounded forms to explore alternate strategies for computation.

For example:

$$\begin{array}{r} 2.99 \\ \underline{+\ 4.36} \end{array}$$

may be found by rounding and adjusting

$$\begin{array}{r} 3.00 \\ \underline{+\ 4.35} \end{array}$$

Rounded forms are clearly useful for estimation and mental computation Once students have developed rounding skills, these skills allow the students to develop many strategies for solving computation problems. The strategies are not limited to standard procedures--students will invent interesting algorithms.

Students see patterns for multiplication and division when they multiply and divide whole numbers and decimals by 10, 100, and 1000. Explore multiplication and division by multiples of 10 with place value materials and pictorial representations. Begin by multiplying one-digit numbers by 10, then two-digit numbers by 10, and so on. Ask students to record answers to examples and arrange them in a chart. Explore alternate symbolic ways to record the operations and results--such as by using expanded forms or exponential notation.

Build models for decimal computation problems with decimal squares. Record the results pictorially on centimeter grid paper. Complete a chart. Compare the results.

SAMPLE LESSON Exploring Multiplication Patterns

MATERIALS NEEDED

- Pencils
- Record sheet (see illustration)
- Calculators

Make up some Multiplication Pattern Record Sheets. Allow students to use calculators as they fill in the blanks. You may find that as students observe the pattern for placement of the decimal point and the use of zero as a placeholder they will not use the calculator but quickly enter the missing numbers.

Multiplication Pattern Record Sheet

Multiply each number by:

Number	10	100	1,000	10,000	0.1	0.01
5						
6						
7						
8						
9						
10						
11						
12						
13						
⋮						

Have students make up similar sheets with different numerals in the number column, and then complete the sheets, thereby continuing to investigate multiplying by tens, hundreds, thousands, tenths, hundredths, and thousandths. Create similar sheets for division and have students try them.

On later worksheets include decimal fractions and mixed numbers in the number column.

TEACHER RESOURCE MATERIAL

The Mathworks
Base Ten Mathematics
Estimation and Mental Computation, The NCTM 1986 Yearbook
Decimal Squares

III. FRACTIONS, DECIMALS AND RATIOS

Objective 6 Identify fractions, decimals and percents from pictorial representations.

Objective 7 Convert fractions to decimals and vice versa.

Objective 8 Convert fractions and decimals to percents and vice versa.

Objective 10 Identify ratios and fractional parts from given data.

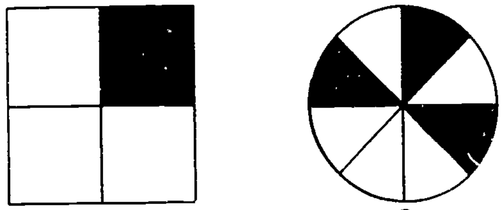
APPROPRIATE MATERIALS

Pattern Blocks
Fraction Action Game
Construction paper
Grid paper
Fraction strips
Fraction Bars
Proportional Fraction Blocks
Decimal Squares

ENABLING SKILLS AND ACTIVITIES

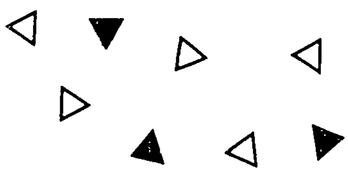
Textbook activities with fractions, decimals, and percent move much too quickly to paper-and-pencil work at the symbolic level. Students attempt to learn a seemingly endless set of mysterious rules that are unconnected to the meaning of rational numbers. They ignore picture representations of equivalents and try to manipulate symbols according to prescribed rules. As a result, performance with rational numbers, and common fractions in particular, is poor.

Fractions involve the idea of separating something into two or more equal parts. Each part is a unit fraction. The fraction symbol indicates the number of parts the unit is separated into (denominator) and those parts under consideration (numerator).



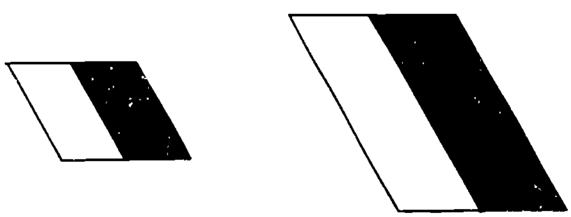
$\frac{1}{4}$ $\frac{3}{8}$

Another fraction relationship involves the fractional part of a whole number set. The fraction symbol represents one or more equal subsets of a larger set of discrete objects. For example, the whole number set below has 8 objects. Three of the 8 objects are shaded--this represents 3/8.



$\frac{3}{8}$

However, in either case, the size of the whole piece, or the amount of objects in a set, may differ and yet have the same fraction label. Equivalent fraction labels may represent the same ratio.

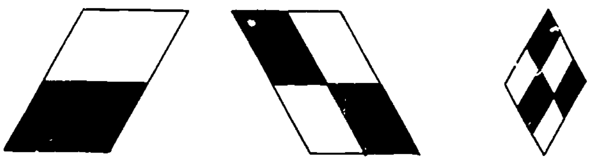


Here 1/2 is used to label one out of two parts in each case. The ratio of part to whole is the same for each picture, but the halves are not the same size (area). Similarly, half a grape is not the same size (volume) as half a basketball.

The same applies to fractions that describe parts of whole number sets. Consider 1/4 of each set as illustrated below. The ratio is the same, but the number of objects in the sets is different.

$\frac{1}{4}$

Furthermore, a rational number may be represented in more than one way--not only as equivalent common fractions, but as equivalent decimal fractions.



...and so on.

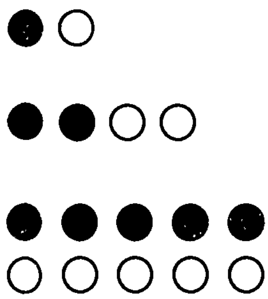
1/2 = 2/4 = 3/6 = 50/100 = ...5 = ... .50 and so on.

Not only do we use fraction notation to indicate parts of a whole, and parts of a set, but we also indicate correspondences between sets with ratio notation.

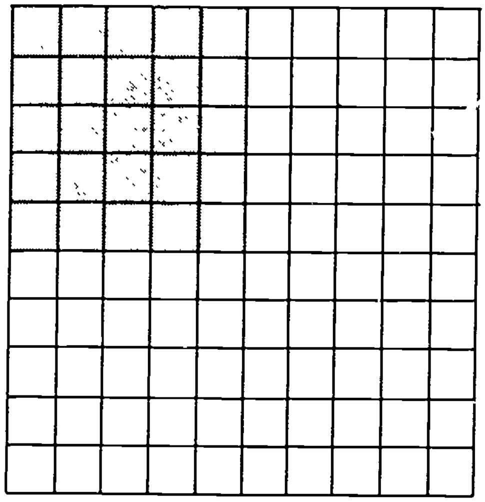
1:2

2:4

5:10



The use of a special ratio, parts per hundred, is yet another way to compare part to whole.



.25 or 25%

Students should have many opportunities to work with manipulative materials that demonstrate the meaning of fractions, decimals, and percents. They should use a variety of materials, and record results pictorially and symbolically. This will enable them to construct visual images of rational numbers. Then they can recall the images and use them to connect meaning to the symbols. It is the coordination among the various concrete models that develops fraction, decimal, ratio, and percent concepts. The further matching of fraction models and images to the mathematical symbols will enable students to construct methods for comparing and computing with rational numbers.

We must be reasonable about the kinds of rational numbers that students investigate. Work with common fractions whose denominators are limited to 2, 3, 4, 5, 6, 8, 10, 12, 16, and 18 is sufficient for students to develop understanding of fraction concepts. Similarly, tenths, hundredths and, later, thousandths are reasonable to explore for development of place value concepts with decimal fractions and for computational facility with decimal notation. The focus of work with percents should include 100%, 50%, 25%, 10%, 5%, 1%, and "multiples" such as 200%, 150%, and so on.

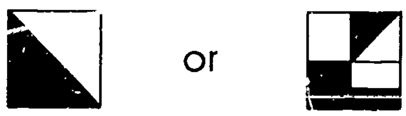
SAMPLE LESSON Exploring Equivalence of Rational Number Forms

MATERIALS NEEDED

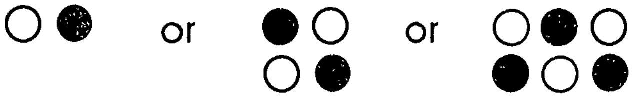
Colored pencils

A recognized method for involving students in the development of skills and concepts is to require them to write about their discoveries. An ongoing lesson in the classroom may consist of students working together cooperatively to produce their own book about rational numbers. The production of a book that illustrates the equivalence of rational number forms will help students gain insight into the various notations used to symbolically record rational numbers.

Production of the book may be organized into a small group project, with each student responsible for specific pages. The first chapter in the book should illustrate the various ways to show pictorially, and symbolically, a specific rational number. For example, a few pages might be devoted to exploring one-half on grid paper. The pages would contain illustrations of fractional equivalents for 1/2 as part of a whole, such as:



Other pages would include equivalent forms of 1/2 as a subset of a larger set of discrete objects. Make drawings to depict equivalent ratios.



24

One-half as a decimal, as a ratio, and as a percent would also be illustrated. Mathematical sentences that state the equivalence relationships among certain fractions should be written. Students should create problems that involve the use of the fraction in interesting applications.

Other chapters, written by other groups of students, would illustrate other commonly occurring rational numbers such as one-third, one-fourth, one-tenth, three-fourths, and so on.

The completed book may be shared with other classes and each student should have a copy to bring home.

TEACHER RESOURCE MATERIAL

Fractions with Pattern Blocks
Mathematics...A Way of Thinking
Fraction Bars
Decimal Squares

IV. ADDITION AND SUBTRACTION--WHOLE NUMBERS

Objective 12 Add and subtract whole numbers less than 10,000.

APPROPRIATE MATERIALS

Place value material (Dienes Blocks and Powers of Ten)
Place value boards
Place value cards
Money

ENABLING SKILLS AND ACTIVITIES

The written procedures for addition and subtraction involve the manipulation of symbols. Traditionally we have relied on the student's ability to memorize algorithms that direct the steps involved in computation. However, many of the procedures quickly become separated from the reality of the numbers being operated on. The procedure of borrowing and carrying, for example, is so divorced from the quantities involved that it makes no difference whether one is dealing with units, tens, or hundreds, or even decimals--a number to the left is crossed off, a value one less than the number is written above it, and a little numeral one is placed in front of the number on the right.

$$\begin{array}{r} 3\,\overset{1}{\cancel{2}}\,{}^{1}2 \\ \underline{-1\,1\,3} \end{array} \qquad \begin{array}{r} \overset{4}{\cancel{5}}\,{}^{1}0\,6 \\ \underline{-1\,3\,2} \end{array} \qquad \begin{array}{r} \overset{5}{\cancel{6}}\,{}^{1}\overset{9}{\cancel{0}}\,{}^{1}\overset{9}{\cancel{0}}\,{}^{1}0 \\ \underline{-3\,5\,4\,7} \end{array}$$

When students apply such procedures prematurely, they arrive at an unquestioned "answer". Errors that creep into the algorithm are undetected and "senseless" mistakes, such as subtracting and obtaining a value that is larger than the minuend, are made.

Research findings indicate that students need to spend more time dealing with quantities through the manipulation of concrete materials that preserve the meaning of the numbers and operations. Research also has shown that students can construct various methods for dealing with computation at the symbolic level if they are allowed to monitor their work through the use of concrete models and pictures. The development of nonstandard computational strategies provides more flexibility, not only with paper-and-pencil calculations, but also with estimation and mental computations.

Arithmetic should not be seen as a collection of meaningless rules, but as an opportunity to think. Inventing computational strategies and alternate algorithms is problem solving.

Students should be encouraged to experiment and develop several strategies for addition and subtraction of whole numbers. They also should be encouraged to select and use the strategy that is most efficient for the circumstances: some strategies are simpler to use for mental computation, others for paper-and-pencil computation, and still others for estimation of answers and for checking the reasonableness of calculator displays.

SAMPLE LESSON Computation as Problem Solving

MATERIALS NEEDED

Paper and pencil
Place value material

Begin with the whole class. Ask them to write down all the different ways to find the sum of 159 and 472.

In the beginning many students will indicate the use of the standard algorithm

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 159 \\ +\underline{472} \\ 631 \end{array}$$

Ask students if they can think of other ways to find the sum. Discuss each suggestion as it is made. Students will begin to put variations together and may make suggestions such as:

$$\begin{array}{r} 159 \\ +\underline{472} \end{array} \Rightarrow \begin{array}{r} 160 \\ \underline{471} \end{array} \quad \text{or} \quad \begin{array}{r} 150 \\ +\underline{475} \end{array} \; \begin{array}{l} \text{Adjust answer} \\ \text{add } 9 \\ \text{subt } 3 \end{array} \quad \text{or add 6 to answer}$$

or $475 + 125 = 600$ adjust
add 31

or $500 + 120 + 11$, and so on.

Repeat the procedure with other addition problems. Then allow the students to work in pairs or small groups. Give them similar problems to do together. Have groups compare their strategies.

Repeat the procedure with larger numbers. Look for patterns. Is it easier to add 18 by adding 20 and then subtracting 2?

$$\begin{array}{r} 137 \\ +\underline{\ 18} \end{array} \Rightarrow \begin{array}{l} 137 + 20 - 2 \Rightarrow \\ \quad 157 \quad - 2 = 155 \end{array}$$

Then ask students to investigate different ways to subtract. A few strategies they might suggest include:

$$\begin{array}{r} 47 \\ -\underline{19} \end{array}$$

$$47 - 19 \Rightarrow 48 - 20 = 28$$

or

$$47 - 19 \Rightarrow 47 - 20 + 1 = 27 + 1 = 28$$

or

$$47 - 19 \Rightarrow 50 - 20 - 2 = 30 - 2 = 28$$

Continue the activity as an ongoing way to practice number work and improve computational skill. Ask students to try some mental computations. When paper and pencil is not used, do they find nonstandard strategies more useful?

TEACHER RESOURCE MATERIALS

Learning from Children
Developing Computational Skills: The 1978 NCTM Handbook
Journal for Research in Mathematics Education, January 1985

V. MULTIPLICATION AND DIVISION--WHOLE NUMBERS

Objective 13 Multiply and divide 2- and 3-digit whole numbers by 1- and 2-digit numbers.

APPROPRIATE MATERIALS

Place value materials (Dienes Blocks or Powers of Ten)
Counters
Grid paper
Money

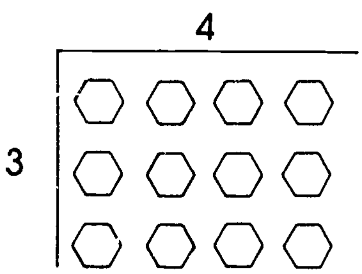
ENABLING SKILLS AND ACTIVITIES

Multiplication on the counting level is demonstrated as the combining of several subsets, each subset containing the same number of objects. This is the repeated addition model (4 x 3 is 3 + 3 + 3 + 3). Division is demonstrated as the reverse operation--the repeated subtraction of subsets that are the same size.

Students often have rather undeveloped ways of "doing" division. If you give a student a pile of objects and ask him to "divide them up among five students," you may observe the student deal the objects out one at a time. Sometimes, a more sophisticated counting strategy--dealing the objects out in groups of two, three, five, or ten--is used. Rarely does a student count all the objects and then apply the appropriate division by five fact. Students do not readily connect the strategies for doing division with counting objects to the procedures for doing division with paper and pencil. These connections can be encouraged by using concrete and pictorial models as the symbolic work with division is developed.

The mastery of basic multiplication and division facts is simplified when students understand that the relationship among two factors and their product can be used to find the basic facts.

Counters arranged to display the repeated addition model may be rearranged in an array.



$$3 \overline{)\,12}^{\;4} \qquad 3 \times 4 = 12$$

$$f \overline{)\,p}^{\;f} \qquad f \times f = p$$

When the array is labeled, a clear relationship between multiplication and division is established. The two factors and product displayed also are consistent with the standard symbol used for division.

It is helpful to use the factor-factor-product terminology rather than to introduce more terms, like dividend and quotient. Division is the process of finding a missing factor. When students see the reversibility of the procedure, they find that there are fewer facts to remember. The definition of division as the search for a missing factor also is consistent with its algebraic treatment.

Once students have mastered the basic facts and concepts, the development of strategies for multiplication and division is appropriate. Students should have some opportunity to work computation problems in expanded form so the implications of place value can be explored.

There are several different algorithms for doing multiplication and division with paper and pencil. All of them must account for the value of the digits in the factors and product. Counters and place value blocks may be used to illustrate the role place value plays in any algorithm. Time must be spent working at this level before moving to work totally at the symbolic level. With an understanding of the role of place value in multiplication and division, students will be able to make good computational estimates and develop strategies for mental computation. For example, a student was observed transforming division by a two-digit divisor into a simpler problem--division by one digit.

$$16\overline{)\;6480} \Rightarrow 8\overline{)\;3240}$$

SAMPLE LESSON Exploring the Role of Place Value in Multiplication and Division Through Expanded Forms and the Distributive Property

MATERIALS NEEDED

- Place value blocks
- Grid paper
- Pencils

Have place value blocks available so students may explore various ways to decompose and rearrange the numbers.

Begin with the whole class. Ask them to write down all the different ways to find the product of 58 and 4.

At first many students will indicate the use of the standard algorithm.

$$\begin{array}{r} \scriptstyle 3 \\ 58 \\ \times\ 4 \\ \hline 232 \end{array}$$

Ask students if they can think of other ways to work with the numbers. Discuss each suggestion as it is made. Students will begin to invent strategies and may make suggestions such as:

$$58 \times 4 \Rightarrow 50 \times 4 + 8 \times 4 \Rightarrow$$

$$200 + 32 = 232$$

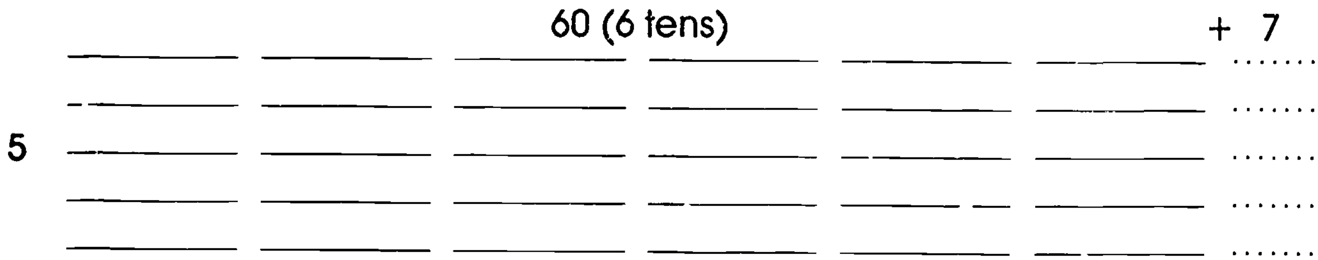
$$\begin{array}{r} 58 \\ \times\ 4 \\ \hline 32 \\ 200 \\ \hline 232 \end{array}$$

$$58 \times 4 = (50 + 8) \times 4$$

32

Students may wish to build concrete models to see the effect of place value on the partial products.

________ represents a 10-rod.



	60	+ 7
5	300	35

$$\begin{array}{r} 67 \\ \times\ 5 \\ \hline 35 \\ 300 \\ \hline 335 \end{array}$$

Repeat the procedure with other multiplication problems. Allow students to work in pairs or groups. Give each group the same problem. Have groups compare their models and their written strategies.

Repeat the procedure with larger numbers. Look for patterns.

$$\begin{array}{r} 27 \\ \times 38 \\ \hline \end{array}$$

	30	+ 8
20	600	160
+ 7	210	56

$$\begin{array}{r} 27 \\ \times\ 38 \\ \hline 56 \\ 160 \\ 210 \\ 600 \\ \hline 1026 \end{array}$$

In other lessons explore division. Students may come up with strategies such as:

$$3\,\overline{)\,45} \quad\Rightarrow\quad \begin{array}{l} \qquad 10+5=15 \\ 3\,\overline{)\,30+15} \end{array}$$

$$4\,\overline{)\,128} \quad\Rightarrow\quad \begin{array}{l} \qquad 30+2=32 \\ 4\,\overline{)\,120+8} \end{array}$$

$$9\,\overline{)\,108} \quad\Rightarrow\quad \begin{array}{l} \qquad 10+2=12 \\ 9\,\overline{)\,90+18} \end{array}$$

Continue the activity as an ongoing way to practice number work and improve computational skills. Ask students to try some mental computations. When paper and pencil are not used, do they find some of their nonstandard strategies useful?

TEACHER RESOURCE MATERIALS

The Mathworks
The I Hate Mathematics Book
Math for Smarty Pants
Base Ten Mathematics
Those Amazing Tables
Middle Grades Mathematics Project--Factors and Multiples
Arithmetic Teacher--September, 1986

VI. COMPUTATION WITH DECIMALS

Objective 14 Add and subtract decimals (to hundredths) in horizontal form.

Objective 15 Identify the correct placement of the decimal point in multiplication and division of decimals.

Objective 22 Compute sums, differences, products and quotients using a calculator.

APPROPRIATE MATERIALS

Decimal Squares
Grid paper
Construction paper
Money
Calculators

ENABLING SKILLS AND ACTIVITIES

Decimals are an important mathematical topic. The increasing availability of calculators and computers and the increasing use of the metric system demand that decimals be given a more prominent place in the mathematics curriculum. However, as with other concepts, sufficient time must be allocated to the development of decimal concepts through the use of manipulative materials and pictorial models.

National- and state-level assessments of upper elementary and middle school student performance with decimals indicates that students do not understand decimals. They have difficulty identifying the place value of a digit in a two-place decimal and often ignore the decimal point and treat computation with decimals as computation with whole numbers. They do not perceive decimals as a special case of fractions. Decimals are fractions with denominators that are multiples of ten, and decimal notation uses place value to indicate the denominators.

The study of decimals should come after students have developed basic fraction concepts and have worked with fractions whose denominators are 10 and 100. Students must have opportunities to use fraction symbols and decimal notation interchangeably as they label concrete and pictorial models. As with fractions, students need time to construct strong visual images of decimals. The images will allow them to connect meaning to the decimal notation.

The use of rounded forms and estimation of computational results with decimals is most important. Students must have ways of estimating reasonable answers to problems that involve decimals and fractions in order to establish the magnitude of the answer and to check for the position of the decimal point in both paper-and-pencil and calculator computations.

As with whole numbers, students should explore alternate strategies for computing with decimals. For example:

$$2.99 + 3.74 \Rightarrow 3.00 + 3.73 = 6.73$$

SAMPLE LESSON Exploring Decimal Fractions and Computation with the Calculator

MATERIALS NEEDED

Paper and pencil
Calculators

Each student will need a calculator. Paper and pencil will be used to record the strategy used. Once students understand the activity, problems may be made up on ditto sheets.

Enter a decimal fraction, such as .073 into the calculator. Ask the students if they can make the display read .73 by using the multiplication key only once in combination with some digits. Students should discover that they need to enter " X 10 = " to obtain the desired number.

Repeat the procedure--enter 2.3 and try to change the display to 230 again using the multiplication key only once. Continue the activity and explore the effect of multiplication by multiples of ten.

Later, pose the problem of entering .87 and trying to obtain the display .0087. What operation key should we use?

In subsequent lessons students may explore addition or subtraction. For example, try to change the display from 30.587 to 32.586 using one operation key. Later, try more challenging problems, such as changing 45.73 to 166.73 with one or more operations. Compare and discuss strategies.

TEACHER RESOURCE MATERIAL

The Mathworks
Decimal Squares
Focus on Decimals
Calculator Explorations and Problems

VII. COMPUTATION WITH FRACTIONS

Objective 16 Add and subtract fractions and mixed numbers.

Objective 17 Multiply fractions and mixed numbers.

APPROPRIATE MATERIALS

Fraction strips
Fraction Circles
Fraction Bars
Proportional Fraction Blocks
Pattern Blocks
Cuisenaire Metric Blocks
Fraction books (made by students)

ENABLING SKILLS AND ACTIVITIES

It may appear simpler, both to teacher and student, to just put in place a few computational rules for dealing with fractions. We ask students to memorize rules because we have given up the idea that students can understand fractions, or that there is any advantage to understanding them. In reality, an understanding of concepts for fraction computations will make the process simpler. Estimation and mental computation with fractions are possible only if fraction concepts are grasped. Even paper-and-pencil computation is easier if concepts are understood well enough to allow for the use of simplified strategies. For example, if only the rule for multiplying two fractions is used, students do the computation as follows:

$$3\frac{3}{4} \times \frac{5}{5} = \frac{\overset{3}{\cancel{15}}}{4} \times \frac{5}{\underset{1}{\cancel{5}}} = \frac{15}{4} = 3\frac{3}{4}$$

However, with an understanding that $\frac{5}{5} = 1$

the answer more readily follows $3\frac{3}{4} \times \frac{5}{5} = 3\frac{3}{4}$

Repeated addition, or multiplication, of fractions is a natural place to begin computational work with fractions. The models involved are easy to visualize. To find a fractional part of a whole number set, i.e., 3/4 of 24, students need to understand the part-to-whole relationship and that the parts are all the same size. On the other hand, addition and subtraction of fractions not only require the part-to-whole relationship but also the idea of equivalence.

SAMPLE LESSON Developing Computational Strategies for Fractions

MATERIALS NEEDED

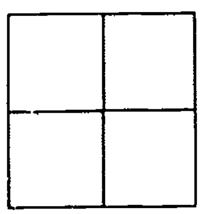
- Fraction books (made by students)
- Poster or construction paper
- Scissors
- Glue
- Pencils

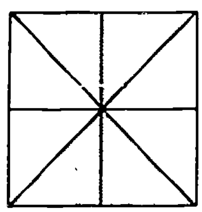
Let's begin with multiplication and follow a sequence with several models. To find 1/3 of 45, three equal subsets must be built from 45 counters. The quantity in one subset is the answer--1/3 of 45 is 15. The sentence is written as 1/3 x 45 = 15.

To find 2/3 of 45, the amount is again allocated to three equal subsets. The quantity in two of the subsets is the answer. Therefore, 2/3 of 45 is 30. Three-thirds would be all three subsets, or 45.

Students should practice partitioning sets and labeling the results. They will begin to connect the multiplication/division facts with the operation and will not need to physically construct the set and subsets.

Before going on with other operations with fractions, students must develop the idea of equivalence. Student-made fraction books is one model for developing equivalence. The books consist of pages of pictorial models and symbolic sentences that show the part-to-whole relationship and equivalence.



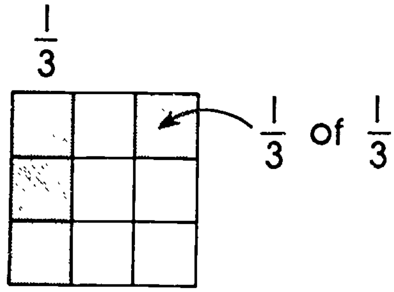


$$1 = \frac{2}{4} + \frac{2}{4}$$

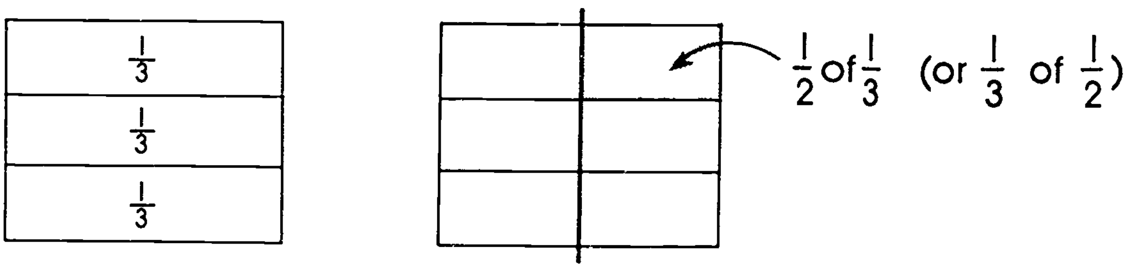
$$1 = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}$$

$$1 = \frac{2}{4} + \frac{4}{8}$$

Often, as students work with the books, they spontaneously volunteer another way to label a section such as 1/9.

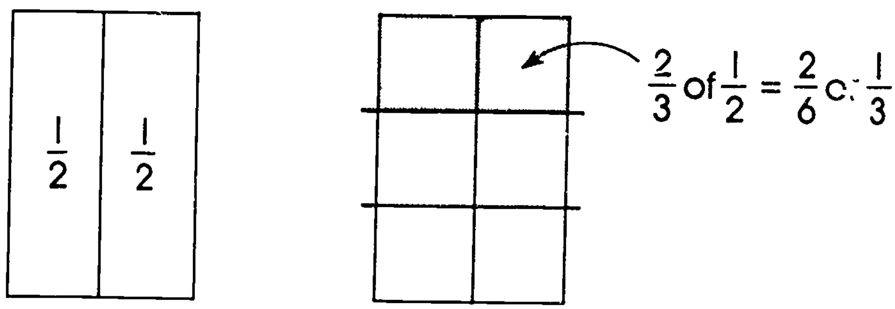


This is a good model to use for extending the operation to the multiplication of two fractions. Students should make more pages in their fraction books. Have them label the sections to name what part one fraction is of another-- 1/2 of 1/3 is 1/6.

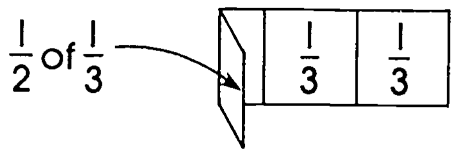


Now there are 6 parts. Each part is 1/6. This is written as: 1/2 x 1/3 = 1/6.

Similarly,



Folding fraction strips is another model. Fold the unit length in thirds. Fold one-third in half.



Students should continue this process and then look for patterns. For example, make a list like this:

1/2 x 1 = _____

1/2 x 1/2 = _____

1/2 x 1/3 = _____

1/2 x 1/4 = _____

1/2 x 1/5 = _____

1/2 x 1/6 = _____

What is the "rule"?

What is 1/2 x 1/12?

What is 1/2 of 2/12; or 1/2 of 4/12; and so on? Explore the patterns visually and symbolically.

Make tables and record the patterns that appear when other denominators--3, 4, 5, 6, 8, 9, 10, 12, 16, and 18--are used. These are realistic in terms of practical use and the set of them is small enough to allow students to explore them all at concrete and pictorial levels.

Repeat the procedure for mixed numbers:

1/2 x 1/2 = _____

1/2 x 1 1/2 = _____

1/2 x 2 = _____

1/2 x 2 1/2 = _____

1/2 x 3 = _____

and so on.

Try

1/2 x 1/4 = _____

1/2 x 2/4 = _____

1/2 x 3/4 = _____

1/2 x 4/4 = _____

1/2 x 5/4 = _____

and so on.

With the same limited set of 11 denominators, explore strategies for addition and subtraction. Students who have spent time writing fraction sentences for the pages in their fraction books will already have determined a way to add with the model that utilizes equivalent forms. Have them write and compare addition sentences. Write more sentences that describe parts of a fraction picture. For example, 1/3 + 1/3 = 2/3; or 1/4 + 2/4 = 3/4. During addition and subtraction with like denominators students will count the parts. Ask them to write many sentences with like denominators.

The work with like denominators will naturally evolve into writing sentences with unlike denominators. If students have had sufficient time with the fraction books, they will not suggest that 1/2 + 1/4 is 2/6. The visual model that they have developed will help them think through the situation. They will see that 1/2 + 1/4 covers 3/4. They will also know that 1/2 is the same as 2/4. The problem can be rewritten as 2/4 + 1/4. Students who have developed these insights will make better estimates of sums and differences that require the use of mixed numbers.

There is no need to work with least common multiples and greatest common factors. Students should rely on visual models and estimating strategies. The LCM and GCF are interesting topics for an exploration of number theory. Nor do we need to find the common denominator through the application of rules at the symbolic level. Allow the students to think their way through using fraction concepts.

Once students have automatically developed their own strategies, extend the work with fractions to include mixed numbers. With models, as well as the distributive property already used in work with whole numbers, students will extend their strategies to organize problems such as

1/2 of 1 1/2 as 1/2 of 1 and 1/2 of 1/2

$$1/2 \times 1\ 1/2 = 1/2 \times 1 + 1/2 \times 1/2 =$$

$$1/2 + 1/4 \qquad = 3/4$$

TEACHER RESOURCE MATERIALS

Fractions with Pattern Blocks
Solid Sense in Mathematics 4-6

VIII. PERCENT

Objective 18 Determine the percent of a number.

APPROPRIATE MATERIALS

Grid paper
Money
Calculators

ENABLING SKILLS AND ACTIVITIES

The term "percent" implies parts per hundred. Students should recognize percentage as a special ratio involving parts per hundred: Y percent is the ratio Y to 100 or Y parts of 100 parts. When the set under discussion is not equal to 100 parts, "X" percent of the set is equal to the ratio of the parts being considered to the parts in the set. For example, if a set has 12 objects and 3 of them are considered, "X" percent (X parts per hundred or X : 100) is equal to the ratio 3 parts per 12 -- 3 : 12. This may be expressed as the proportion

$$\frac{X}{100} = \frac{3}{12}$$

Using simple equivalent fractions we know that 3/12 is equivalent to 1/4 and 25/100 is also equivalent to 1/4.

Therefore, X is 25 and 3 is 25% of 12.

The more general statement of the proportion is

$$\frac{X}{100} = \frac{A}{B}$$

Given the value of any two of the variables, the value of the third variable in the proportion may be found. The use of the proportion does not require students to memorize three algorithms for the three different "cases" of percent problems.

The basic nature of percentage is the special ratio of parts per hundred. Consistent use of the general proportion reduces the symbolism and algorithmic development and establishes a method that is based on the nature of percentage.

To determine the percent of a number, such as 12% of 50, we can write the proportion

$$\frac{12}{100} = \frac{A}{50}$$

and, through identifying an equivalent fraction, we find

$$\frac{12}{100} = \frac{6}{50}$$

so 12% of 50 is 6.

However, many students in grades 6 and 7 need to see the special ratio in concrete and pictorial terms. Using 100 counters, money or 10 x 10 grids, they should find various percents.

Count out 10% of the counters.

What is 25% of one dollar?

Shade in 50% of the hundred grid.

Repeat activities with these models and various percents. Then try these models with other bases. For example, shade in 50% of a 4 x 4 grid.

Shade in 25% of a 4 x 4 grid.

Shade in 75% of a 3 x 4 grid, and so on.

What fractional parts have been shaded?

Record the results. Look for patterns.

25% is

$$\frac{25}{100} = \frac{1}{4}$$

On the 4 x 4 grid have you shaded one out of every four squares?

Shade in 25% of a 2 x 2 grid.

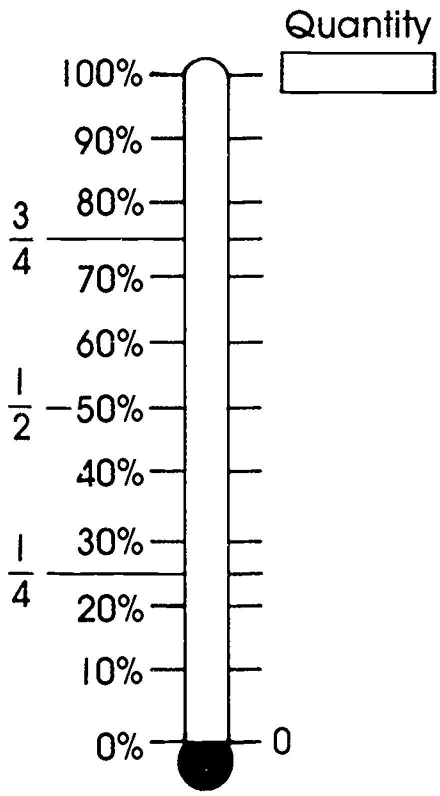
Students should work with percents such as 100%, 50%, 25%, 10%, 1%, and multiples of 10%. They should find percentages through estimation, the use of patterns and their own strategies. Students also should use the hand-held calculator. Have them estimate or mentally compute the percent and then try to discover how the percent key may be used to make the calculator print the expected value on its display.

To increase their understanding of the special ratio underlying percent problems, students should work with pictorial models. The sample lesson illustrates such a model.

SAMPLE LESSON The Percent Thermometer

MATERIALS NEEDED

Paper and pencil
Ditto master copies of the Percent Thermometer



The Percent Thermometer

The Percent Thermometer permits students to estimate and match the corresponding parts of a quantity with the percent (or fraction) on the opposite scale. The use of this pictorial model provides a visual image of the equivalence relationship that defines percent.

To introduce the Percent Thermometer begin by placing a number, such as 600, in the Quantity Box. Have students suggest the numbers that may be filled in on the scale across from the corresponding percent. For example, what is 50% (or half way up the scale to 600)? What is 25%? What is 10%? What is 20%? What do you think 15% will be?

On another copy of the Percent Thermometer ask students to enter 300 in the Quantity Box and complete the scale. Estimate reasonable answers for 17%, 23% and so on.

Repeat the procedure with other quantities. Through the use of the model, students will discover reasonable answers and useful strategies for finding percentages. For example, 30% is three times the value at 10%. 45% is halfway between the values at 40% and 50%. 35% is equal to the value at 30% plus the value at 5%. Provide plenty of time for students to make these and similar discoveries. Then provide opportunities to share and discuss the strategies.

The Percent Thermometer may be used to solve percent problems.

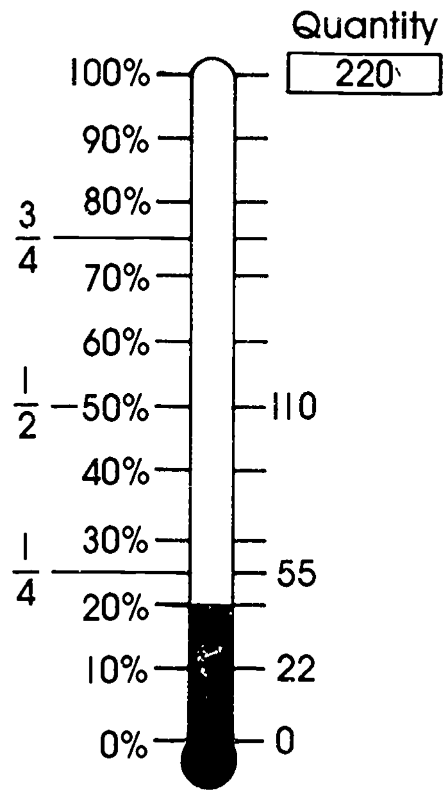
What is 20% of 220?

Place 220 in the quantity box.

Shade in the thermometer to show 20%.

What value goes opposite 20%?

In the beginning, students may find it helpful to close in on the value by finding 50%, 25%, and 10%.



TEACHER RESOURCE MATERIALS

The Mathworks

IX. ESTIMATION

Objective 19 Estimate sums and differences of whole numbers and decimals including making change.

Objective 20 Estimate products and quotients of whole numbers and decimals.

Objective 21 Estimate fractional parts and percents of whole numbers and money amounts.

Objective 11 Identify an appropriate procedure for making estimates with decimals and fractions.

APPROPRIATE MATERIALS

Money
Number lines
Newspapers, magazines, and store fliers
Calculators

ENABLING SKILLS AND ACTIVITIES

The ability to estimate the results of computations is an essential skill. With the increasing use of calculators and computers, estimation is more commonly used than paper-and-pencil computations to judge the reasonableness of an answer displayed on a machine.

Students should develop estimation skills as they develop computational strategies. The teaching of estimation should not be limited to rounding, but should involve the exploration of many strategies including compensation, clustering, finding compatible numbers, front-ending, and recognizing computational patterns with special numbers. Have students discuss their different approaches to the same situation.

Estimation activities can be integrated into other curriculum topics, such as measurement. The relationship between problem solving and estimation should be emphasized--looking back at a solution includes judging the reasonableness of the answer obtained.

Some prerequisite skills for successful computational estimation include the ability to do 1- and 2-digit computations, to work with rounded numbers, and to compare and order numbers. Students must have an understanding of place value concepts and the nature of the decimal number system and its basic operations.

In addition to developing skills in rounding, place value, and operating with rounded numbers, it is essential that students develop a sense that multiplication and division in general have much greater effects than do addition or subtraction on the order of magnitude of the result. Students should practice operating with tens--adding and subtracting tens and multiples of ten. Multiplying and dividing by multiples of ten should be carefully explored to further develop an understanding of the size of a number.

Computational estimation activities should be accompanied by work with pictures or models such as number lines or place value materials. Students also need to estimate capacity, length, area, and volume with real materials to enhance their sense of number. Such activities lead them to discover that estimates are easier to use. It is important for students to develop a tolerance for guessing and accept that more than one answer may be reasonable.

Estimation problems may be presented in a variety of formats. Some problems require the establishment of a reference number; for example, "Will $30 be enough to pay the grocery bill?" Some require a decision about the reasonableness of an answer; for example, "The calculator shows 324 multiplied by 8 as 6592. Is that right?" Some are open-ended; for example, "About how much is 3478 x 42 1/2?"

Pilot test results at the eighth grade level indicate that students have the most difficulty not only with addition, subtraction, and multiplication of common fractions, but also with estimating fractional parts and percents of whole numbers and money amounts, and identifying appropriate procedures for making estimates with decimals and fractions. The sample lesson that follows offers some strategies for using rounded forms to judge the reasonableness of answers to computational problems that involve fractions and decimals.

SAMPLE LESSON Estimation with Fractions and Decimals

MATERIALS NEEDED

Paper and pencils
Calculators

The fractions we use are bound on either side by whole numbers that are slightly less and slightly more in value. Begin by asking students to find the range inhabited by each number.

For example, 4 2/3 lies between 4 and 5. It is closer to 5. 3 1/4 lies between 3 and 4, and it is closer to 3.

Therefore, to estimate the value of 4 2/3 + 3 1/4 we round 4 2/3 UP to 5 and we round 3 1/4 DOWN to 3.

4 2/3 + 3 1/4 is about 8.

Now, reverse the process. Ask students to list pairs of numbers that together have a sum of about 8. The list should be lengthy.

Repeat the procedure with other fractions and other operations. For instance, find 3 1/4 X 12 7/8. Establish the range for each number. Find a reasonable product--39. Find other pairs of fractions and mixed numbers that result in a product of about 39.

Similarly, try the procedure with decimal fractions. Find 35.12 divided by 4.89. Establish the ranges and rounding strategies. Find a reasonable estimate--7. To further improve estimation skills, be sure to reverse the process and list other pairs of numbers that will result in a quotient close to 7.

TEACHER RESOURCE MATERIALS

Developing Computational Skills: The 1978 NCTM Handbook
Estimation and Mental Computation: The 1986 NCTM Handbook
Krypto
Developing Skills in Estimation
GUESS: Guide to Using Estimation Skills and Strategies
Calculator Explorations and Problems

X. PROCESS PROBLEM SOLVING

Objective 31 Solve process problems involving the organization of data.

Objective 23 Interpret graphs, tables and charts.

Objective 27 Solve problems involving elementary probability.

APPROPRIATE MATERIALS

Attribute Blocks
Tangrams
Geoboards
Pattern Blocks
Place value material
Money
Grid paper
Calculators
Computers

ENABLING SKILLS AND ACTIVITIES

Two areas of problem solving are developed in a curriculum that focuses on mathematical thinking--process problems and translation problems. Translation problems (also called word or story problems) require the problem solver to operate at a symbolic level. Words are translated into mathematical symbols, mathematical operations are done with the symbols, and the resulting mathematical sentence is translated back into words. The problem solver may not have seen a particular word problem before (the combination of words is new), but the problem may be translated into a mathematical statement which uses familiar symbols, operations, and concepts.

Although words and symbols are involved in process problems, the context is different. Process problem solving involves working with novel situations. The context of the problem may require spatial skills. For example, the problem may entail finding all the different rectangles that can be constructed on a geoboard. The problem solver must find a systematic way to find all possible solutions.

A variety of strategies may be utilized during the process of solving the problem. Guess-test-revise is a simple approach. Other strategies include building a model, drawing a picture, or acting out the problem. Information from the problem may need to be organized. A list, table or graph may be employed. Once the information is organized, the problem solver may search for patterns or reorganize the problem into smaller, simpler parts. The problem solver may use the components to work backward or analyze the pieces logically.

For students to develop these strategies, the curriculum must emphasize classifying, patterning, graphing, estimating, and making predictions. Students must have many opportunities to analyze and structure situations.

Students must be:

- faced with problems in which the approach is not apparent and encouraged to generate and test many alternative approaches;
- allowed sufficient time for discussion, practice, and reflection on problems and problem-solving strategies;
- encouraged to solve different problems with the same strategy and apply different strategies to the same problem;
- provided many experiences solving problems in small groups; and
- involved in the collection and organization of data.

SAMPLE LESSON How Many Ways Can You Arrange These?

MATERIALS NEEDED

Paper and pencil
Numeral cards (0 - 9)

Ask students to work in pairs or small groups. Give each student a blank piece of paper. Have available a set of numeral cards for each group.

Ask students to find all the numbers that can be made using exactly two numerals. Observe how the students approach the problem. Some students may need to physically rearrange the cards to help them find all the permutations. They may begin in a random fashion and slowly begin to see a possible pattern Other students may organize the task by exchanging the cards two at a time. Others may organize the information into a table which they quickly complete. Allow the students to use their own strategies.

Next, ask the students to find all the three-digit numbers that can be made with the three numerals 1, 2, and 3. Again observe the strategies used. Then ask students to make all the three-digit numbers that can be formed using the numerals 5, 6, and 7. Did some students generalize and use the previous strategy, or even improve upon it?

Continue with similar problems, such as:

Find all the five-digit numbers possible using the numerals 1, 2, 3, 4, and 5.

Find all the four-digit numbers possible using the five digits 1, 2, 3, 4, and 5.

Find all the three-digit numbers possible using the same five numerals.

Is there a way to predict the number of possible permutations of five-digit numbers made from six digits?

TEACHER RESOURCE MATERIALS

Middle Grades Mathematics Project: Probability
Attribute Acrobatics
SPACES: Solving Problems of Access to Careers in Engineering and Science
Visual Thinking
Spatial Problem Solving with Paper Folding and Cutting
What's My Logic?
Make It Simpler
Building Thinking Skills
Problem Solving in School Mathematics: The 1980 NCTM Yearbook

XI. TRANSLATION PROBLEM SOLVING

Objective 24 Solve 1-step problems involving whole numbers and decimals, including averaging.

Objective 25 Solve 1- and 2-step problems involving fractions.

Objective 26 Solve problems involving measurement.

Objective 28 Estimate a reasonable answer to a given problem.

Objective 29 Solve problems with extraneous information.

Objective 30 Identify needed information in problem situations.

APPROPRIATE MATERIALS

Calculators
Computers

ENABLING SKILLS AND ACTIVITIES

Translation problems require the problem solver to change a story problem from words to mathematical symbols. The symbols are manipulated through the use of the appropriate mathematical operations. The resulting mathematical statement is translated back into words and the answer is checked for reasonableness in terms of numerical magnitude and its fit within the context of the problem. Solving translation problems requires the student to operate at an abstract level.

To successfully solve translation problems, students must have good reading comprehension and good skills applying mathematical operations (addition, subtraction, multiplication, and division) with various kinds of numbers (whole numbers, fractions, decimals, percents, and integers). Students should be able to use numbers as quantifiers, scales, measures, and rates.

Students must understand that number sentences provide specific information, that they are general statements that may be used to describe similar situations, and that the components of word problems provide pieces of information that help develop the number sentences.

The following sequence of experiences is helpful in developing translation problem skills:

Match a number sentence to a picture or diagram, and match a diagram to the appropriate number sentence.

Draw a diagram to illustrate a word problem.

Write several stories that fit a general number sentence.

Write questions that may be answered from given pieces of information.

Match number sentences to word problems.

Write number sentences to fit word problems.

Some classroom teaching strategies for developing translation problem-solving skills include activities in which students practice reading skills such as selecting main ideas. Encourage students to put problems into their own words. Have students note the important information given in a problem, and have them identify useless or confusing information. Ask students to determine the question to be answered, select specific information necessary for the solution, and choose the appropriate operation(s).

Provide opportunities for students to work in pairs or small groups. Have them estimate answers and discuss strategies for making the estimates.

After completing the problem have students look to see if they might have solved it differently. Ask students to test the reasonableness of their answers.

SAMPLE LESSON Creating and Publishing Word Problems

MATERIALS NEEDED

Paper and pencil
Magazines and/or computer graphics

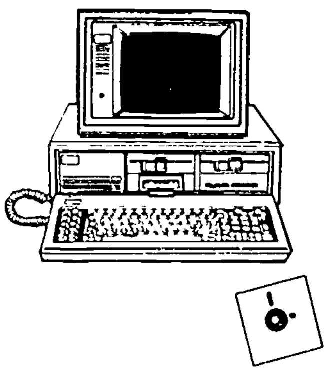
When students write their own word problems, they develop the organizational skills needed to solve problems. They also analyze information and generalize ideas.

You can organize word problem writing as a class project. Have students work in small groups. Their task is to write, illustrate and publish word problem books.

Each student should create some pieces of information. They are to exchange the information scenarios with other students in the group. Working together, students are to write questions that can be answered from the information, illustrate the situations, and compile them into a book.

Be sure to have students use fractions, decimals, percents, and integers. Groups of students may be assigned various kinds of problems--measurement, averages, consumer problems, and so on.

SOME PROBLEMS BY TED



Computer hardware is on sale at 30% off. Software is marked down 15%. June wants to buy a computer, screen, disk drive and software package.

How much will it cost her, including a sales tax of 6%?

	Regular Price
Computer	$965.00
Screen	$189.00
Disk drive	$459.00
Software	$349.00

The problems may be illustrated by any combination of student drawings, pictures cut from old magazines, or computer graphics generated with software such as Newsroom or Printshop. The group should produce an answer page.

Place the word problem books at an activity center and have students read them and do each others' problems. Share the books with other classrooms. Have each student select a favorite problem, then collect and reproduce them as a word problem newsletter distributed to parents.

TEACHER RESOURCE MATERIALS

Mind Benders
Wollygoggles and Other Creatures
Arithmetic Teacher, February 1982 (Focus Issue on Teaching Problem Solving)
Problem Solving in School Mathematics: The 1980 NCTM Yearbook
Techniques for Problem Solving

XII. GEOMETRIC FIGURES

Objective 32 Identify figures using geometric terms.

APPROPRIATE MATERIALS

Geoboards
Pattern Blocks
Tangrams
Pentominoes
Mira
Wooden or plastic geometric solids
Cuisenaire Metric Blocks
Cuisenaire Rods
Cuisenaire Polydrons
Centimeter cubes
Orbit kit
Protractors
Compasses
Grid paper

ENABLING SKILLS AND ACTIVITIES

Geometry topics provide a rich and enjoyable strand in the mathematics curriculum. The development of spatial skills is an important part of a mathematics program that focuses on thinking skills.

The study of geometry should not be restricted to the memorization of a vocabulary. Students must work with manipulative materials in order to explore spatial relationships. Students should use materials such as tangrams, Pattern Blocks and geoboards to construct two-dimensional shapes. Manipulatives such as centimeter cubes, metric blocks, rods, polydrons, and the Orbit kit may be used to explore three-dimensional solids. At the pictorial level students should use protractors, compasses, straightedges, and Miras to reproduce shapes and create representations of the solids.

Geometrical ideas such as shape, size, length, area, volume, congruence, similarity, and symmetry can all be discovered during exploration activities with manipulatives. Spatial relationships also can be investigated by considering the transformation of shape pieces as they are rotated or flipped.

Shapes and solids can be compared and classified by size, number of sides, number of angles, types of angles, or number of faces. Vocabulary should be built as a natural extension of classification activities.

SAMPLE LESSON Classifying Geometric Shapes

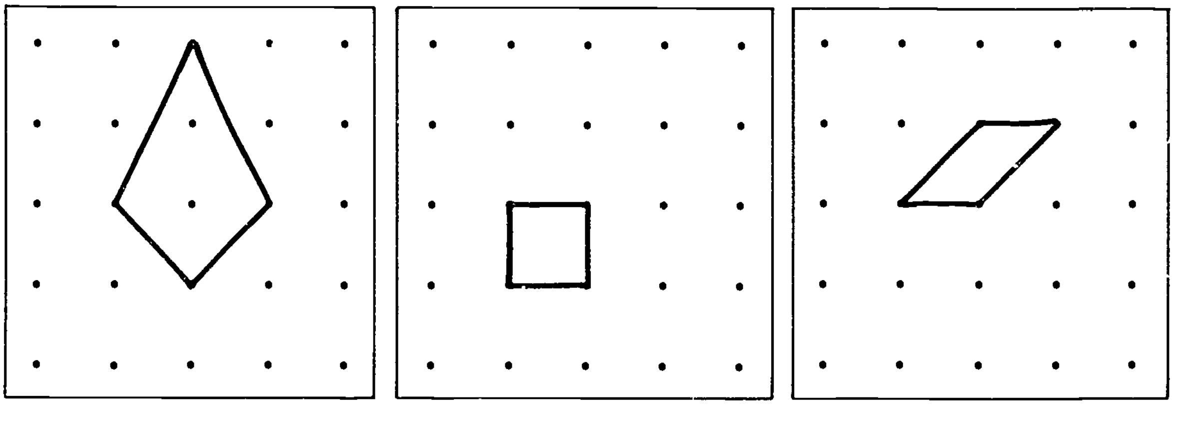
MATERIALS NEEDED

Geoboards and geobands
Geodot paper
Pencils

Have students work in small groups. Give each student a geoboard, a geoband and several sheets of geodot paper.

Tell the students to construct a shape on the geoboard with one band. The only requirement is that the band must touch exactly four pegs. Then have each student use pencil to reproduce the shape on one section of the dot paper.

Ask students to clear the geoboard and then build another, different shape on the geoboard that follows the same rule--one band that touches exactly four pegs. Students should reproduce the shape on another section of the geodot paper.



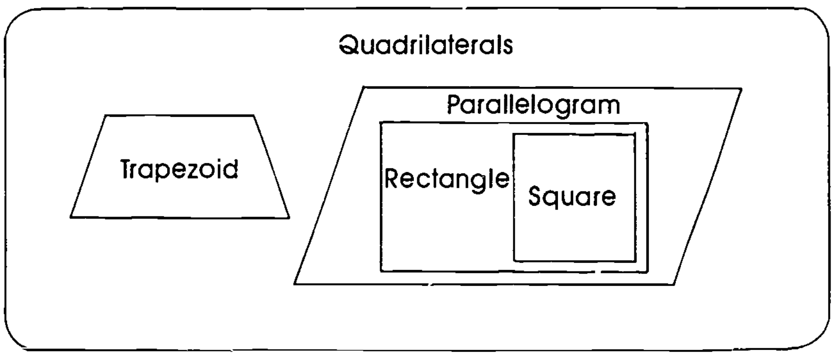
It is possible to produce many shapes while working with the same rule. Have students repeat the procedure to make several designs each.

Then ask them to cut apart the geodot designs and place them in the center of the group's workspace. Each small group of students should then work together to sort the designs. The students should decide how to sort them.

Once the designs have been sorted, ask students to try to find the special names for the designs in each category. Students should use textbooks and other reference materials such as dictionaries and mathematics resource books.

The students should compare the results of each small group. Did they find names like quadrilateral, trapezoid, parallelogram, square?

Which shapes can be classified with more than one name? Make a chart or graph to show the relationships between the classification categories.



Repeat the process to investigate the classification of other shapes. Use another rule, such as one band that touches exactly three pegs, to explore the different kinds of triangles.

TEACHER RESOURCE MATERIALS

Middle Grades Mathematics Project: Spatial Visualization
Mira Math for Junior High School
The Mirror Puzzle Book
Visual Thinking Cards
Spatial Problem Solving with Cuisenaire Rods
Spatial Problem Solving with Paper Folding and Cutting
Let's Pattern Block It

XIII. MEASUREMENT

Objective 33 Measure and determine perimeters and areas.

Objective 34 Estimate lengths, areas, volumes and angle measurement.

Objective 35 Select appropriate metric and customary units and measures.

Objective 36 Make measurement conversions within systems.

APPROPRIATE MATERIALS

Color Tiles
Geoboards
Geodot paper
Centimeter cubes and base ten rods (Powers of Ten)
Tangrams
Pattern Blocks
Construction paper squares
Grid paper (inch and centimeter)
Transparent grids
Ceramic tiles
Cuisenaire Metric Blocks
Rulers and meter sticks

ENABLING SKILLS AND ACTIVITIES

Students should continue to explore the measurement of length, area, and volume with standard and nonstandard units. Often, real-life situations which require students to estimate capacity, area or length do not require them to use standard units. They may need to estimate how many tiles will cover a floor, how many pieces of border trim will outline a bulletin board or how many rocks will fill a dry well. Estimation practice with nonstandard units helps develop spatial skills.

Use objects such as paper clips, toothpicks, and straws for measuring length. Estimate and measure with nonstandard units such as handspans, footspans, and strides. Use large dry lima beans, ceramic tiles, and construction paper squares for measuring area. Estimate the capacity of jars, aquariums, and irregular containers with Unifix cubes, pebbles, and bottle caps.

Estimation practice with standard units helps students develop a frame of reference for the sizes of units of measure. Don't just use a ruler or meter stick; other materials that yield standard measures include Cuisenaire Powers of Ten rods, Cuisenaire Metric Blocks, and Color Tiles. The advantage with these materials is that they are physically countable; this reinforces the idea that measurement is done by repeatedly using a uniform unit. Be sure to measure the same lengths or areas with different nonstandard and standard units so students can compare units. Organize the results into a table. Identify patterns--find a common ratio between measures when two different objects are used.

When students are provided with many opportunities to estimate and measure they understand the physical attributes of length, area and perimeter, volume and surface area. They also construct a mental image of the unit of measure that helps them decide whether it is an appropriate unit; for example, should we use a centimeter or a meter as the unit when we measure the length of the gymnasium?

As students deal with area, provide them with opportunities to organize and compare direct measures of length and area. Do not tell students to use a certain formula--allow them to explore the relationships between lengths of sides and the area of shapes. They will discover some patterns and generalize the formulas.

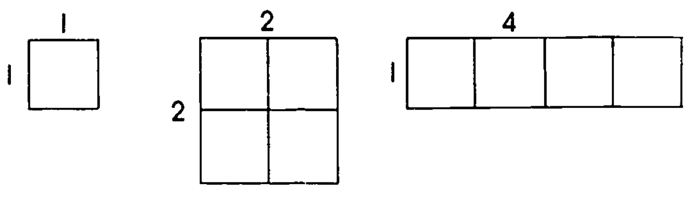
SAMPLE LESSON Organizing Information About Sides and Area

MATERIALS NEEDED

- Color Tiles
- Pencils
- One-inch grid paper
- A record sheet (see illustration)

Each student will need 30 Color Tiles, several sheets of grid paper, a pencil and a copy of the record sheet (if you do not have Color Tiles, you may use ceramic tiles or construction paper squares). Tell students to make the smallest possible square. Ask them to draw it on the grid paper. Discuss the characteristics of the square--each side spans the distance between adjacent lines on the grid--there are no lines in between. We will call that length one unit--the square has a length of one and a width of one. We will call the area of the square one square unit. Label the drawing and write the measures on the record sheet.

Next ask the students to build a rectangle with two tiles. Ask students to describe its sides (length and width) and area. Draw the rectangle on grid paper and label its sides. Record the measures on the record sheet.



Record Sheet

# of tiles	L	W	A
1	1	1	1
4	2	2	4
4	1	4	4

Continue the procedure of making rectangles with three tiles, then four and so on to 30. With certain numbers of tiles, more than one rectangle is possible. Record all the variations that are found; some will be the special case rectangle--the square. Draw and label the rectangles and complete the record sheet.

Discuss the results. Is there a pattern? Can anyone guess what the *sides* might measure for a rectangle made with 31 tiles, or 32? What would be the area of a rectangle built with a length of 8 tiles and a height of 5 tiles? Make more predictions. What is the rule?

TEACHER RESOURCE MATERIALS

Middle Grades Mathematics Project: Mouse and Elephant--Measuring Growth
Measuring in Metric
Middle Grades Mathematics Project: Similarity and Equivalent Fractions
From Here to There with Cuisenaire Rods--Area, Perimeter and Volume
Dot Paper Geometry

APPENDIX A
MATHEMATICS OBJECTIVES AND SAMPLE TEST ITEMS

OBJECTIVE	SAMPLE TEST ITEM
CONCEPTUAL UNDERSTANDINGS	
1. Order fractions.	Which group of fractions is in order from LEAST TO GREATEST? a. $\frac{1}{16}$, $\frac{3}{8}$, $\frac{1}{2}$ b. $\frac{1}{2}$, $\frac{1}{16}$, $\frac{3}{8}$ c. $\frac{1}{16}$, $\frac{1}{2}$, $\frac{3}{8}$ d. $\frac{3}{8}$, $\frac{1}{2}$, $\frac{1}{16}$
2. Order decimals.	Which of these decimals has a value between 0.62 and 0.75? a. 0.613 b. 0.065 c. 0.760 d. 0.732
3. Round whole numbers.	Which of these numbers when rounded to the nearest thousand is 20,000? a. 19,449 b. 20,680 c. 19,725 d. 19,250
4. Round decimals to the nearest whole number, tenth, and hundredth.	What is 14.367 rounded to the nearest hundredth? a. 15.00 b. 14.37 c. 14.40 d. 14.368

OBJECTIVE	SAMPLE TEST ITEM

CONCEPTUAL UNDERSTANDINGS, continued

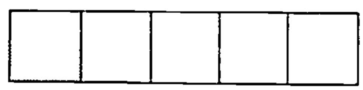
5. Multiply and divide whole numbers and decimals by 10, 100, and 1000.

$32.8 \times 100 =$

a. 32.800
b. 32.80
c. 3.280
d. 3280

6. Identify fractions, decimals, and percents from pictorial representations.

What part of this block is shaded?



a. 0.2
b. 0.4
c. 0.5
d. 0.1

7. Convert fractions to decimals and vice versa.

What is another way of writing 0.922?

a. $\frac{922}{10}$

b. $\frac{92.2}{1000}$

c. $\frac{922}{1000}$

d. $\frac{922}{100}$

OBJECTIVE	SAMPLE TEST ITEM

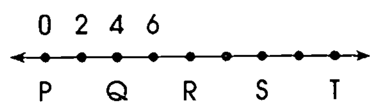
CONCEPTUAL UNDERSTANDINGS, continued

8. Convert fractions and decimals to percents and vice versa.

What is another way of writing 90%?

a. 0.90
b. 0.09
c. 9.0
d. 90

9. Identify points on number lines, scales, and grids.



The value 14 would be located between points --

a. P and Q
b. R and S
c. Q and R
d. S and T

10. Identify ratios and fractional parts from given data.

Jane's flower garden has 15 petunias and 10 geraniums.

What fractional part of her garden is petunias?

a. $\frac{15}{10}$

b. $\frac{3}{5}$

c. $\frac{3}{25}$

d. $\frac{1}{5}$

OBJECTIVE	SAMPLE TEST ITEM
COMPUTATION	
11. Identify an appropriate procedure for making estimates with decimals and fractions.	2 + 3 is a good way to estimate which of the following? a. 1.9 + 3.2 b. 2.7 ÷ 1.3 c. 2.5 + 4.1 d. 1.4 + 3.9
12. Add and subtract whole numbers less than 10,000.	1922 + 392 = a. 2314 b. 2324 c. 2414 d. 2224
13. Multiply and divide 2- and 3-digit whole numbers by 1- and 2-digit numbers.	323 ÷ 15 = a. $21\frac{7}{15}$ b. 25 c. $21\frac{8}{15}$ d. $21\frac{3}{5}$
14. Add and subtract decimals (to hundredths) in horizontal form.	62.32 - 24.1 = a. 59.91 b. 38.22 c. 61.91 d. 599.1

OBJECTIVE	SAMPLE TEST ITEM

COMPUTATION, continued

15. Identify the correct placement of the decimal point in multiplication and division of decimals.

$1.8\overline{)59.67}$

a. 33.15
b. 3.315
c. 331.5
d. 0.3315

16. Add and subtract fractions and mixed numbers.

$$\begin{array}{r} 16\frac{3}{4} \\ +\ 9\frac{3}{5} \\ \hline \end{array}$$

a. $27\frac{1}{5}$

b. $27\frac{7}{20}$

c. $26\frac{1}{10}$

d. $26\frac{7}{20}$

17. Multiply fractions and mixed numbers.

$3\frac{1}{4} \times \frac{3}{5} =$

a. $1\frac{4}{5}$

b. $3\frac{3}{10}$

c. $1\frac{19}{20}$

d. $4\frac{13}{20}$

OBJECTIVE	SAMPLE TEST ITEM
<u>COMPUTATION</u>, continued	
18. Determine the percent of a number.	What is 20% of 225? a. 450 b. 11.25 c. 45 d. 25
19. Estimate sums and differences of whole numbers and decimals including making change.	ABOUT how much is $79.29 - $21.56? a. $40.00 b. $90.00 c. $100.00 d. $60.00
20. Estimate products and quotients of whole numbers and decimals.	ABOUT how much is $5.2\overline{)80.36}$? a. 16 b. 160 c. 40 d. 1.6
21. Estimate fractional parts and percents of whole numbers and money amounts.	ABOUT how much is $\frac{2}{5}$ of 407? a. 80 b. 160 c. 800 d. 200

OBJECTIVE	SAMPLE TEST ITEM

PROBLEM SOLVING AND APPLICATIONS
(with a calculator available)

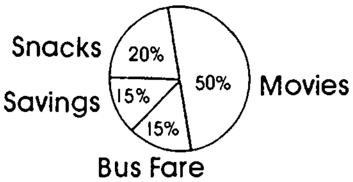
22. Compute sums, differences, products, and quotients using a calculator.

64.9 x 487.5 =

a. 31,638.75
b. 31,606.3
c. 3,160.63
d. 31,537.05

23. Interpret graphs, tables, and charts.

Joe's Allowance Budget



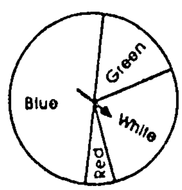
If Joe's allowance is $20.00 each week, how much does he save in a week?

a. $3.50
b. $3.00
c. $10.00
d. $1.50

24. Solve 1- and 2-step problems involving whole numbers and decimals including averaging.

At the swimming meet, Emilio had scores of 7.5, 8.0, 7.3, 8.0, and 9.2. What was his average score at the meet?

a. 9.2
b. 7.3
c. 8.0
d. 40.0

OBJECTIVE	SAMPLE TEST ITEM
PROBLEM SOLVING AND APPLICATIONS (with a calculator available), continued	
25. Solve 1- and 2-step problems involving fractions.	Judy had 16 3/4 feet of tubing. She gave 6 1/2 feet of it to Bob. How much tubing does Judy have left? a. $10\frac{1}{2}$ feet b. $9\frac{1}{2}$ feet c. $10\frac{1}{4}$ feet d. $9\frac{3}{4}$ feet
26. Solve problems involving measurement.	Teresa bought a 2-pound bag of peanuts. She divided it evenly among 4 people. How many ounces of peanuts did each person get? a. 8 b. $\frac{1}{2}$ c. 32 d. 4
27. Solve problems involving elementary probability.	If you spin the spinner, which color is MOST likely to come up?  a. Red b. Blue c. Green d. White

OBJECTIVE	SAMPLE TEST ITEM
PROBLEM SOLVING AND APPLICATIONS (with a calculator available), continued	
28. Estimate a reasonable answer to a given problem.	Over a three-month period, a balloon company sold 8847 red balloons, 2767 black balloons, 9064 yellow balloons, and 6247 blue balloons. A *reasonable estimate* of the total number of balloons sold in this three-month period is -- a. 2500 b. 10,000 c. 35,000 d. 25,000
29. Solve problems with extraneous information.	The Santo family took a 12-day trip. They drove 8 hours a day and traveled 3840 miles. How many miles per day did they average on the trip? a. 96 b. 480 c. 320 d. 960
30. Identify needed information in problem situations.	Karen's mother sells shoes. This month she sold $725.00 worth of shoes. In order to find out how much of a commission she earned, what else do you need to know? a. How many shoes she usually sells in a month b. The rate of commission c. Whether or not the shoes were on sale d. How many days she worked

OBJECTIVE	SAMPLE TEST ITEM

PROBLEM SOLVING AND APPLICATIONS
(with a calculator available), continued

31. Solve process problems involving the organization of data.

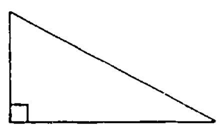
Jane is shorter than Jill. Jill is taller than Mary. Mary is shorter than Jane. Select the choice where the girls are listed from shortest to tallest.

a. Mary, Jane, Jill
b. Jane, Jill, Mary
c. Mary, Jill, Jane
d. Jane, Mary, Jill

MEASUREMENT AND GEOMETRY
(with a calculator available)

32. Identify figures using geometric terms.

This figure is called --



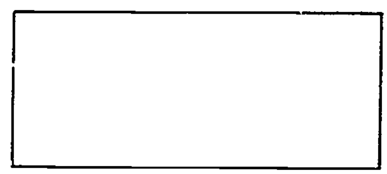
a. an isosceles triangle
b. a scalene triangle
c. an equilateral triangle
d. a right triangle

OBJECTIVE	SAMPLE TEST ITEM

MEASUREMENT AND GEOMETRY
(with a calculator available), continued

33. Measure and determine perimeters and areas.

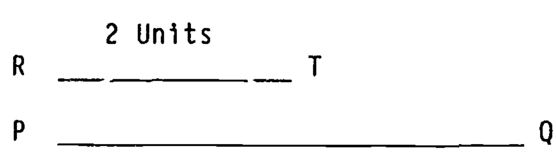
Use your ruler to help you find the perimeter of this figure.



a. 10 cm
b. 14 cm
c. 25 cm
d. 32 cm

34. Estimate lengths, areas, volumes, and angle measures.

If line segment RT is 2 units, ABOUT how long is line segment PQ?



a. 2 units
b. 4 units
c. 7 units
d. 8 units

OBJECTIVE	SAMPLE TEST ITEM
<u>MEASUREMENT AND GEOMETRY</u> (with a calculator available), continued	
35. Select appropriate metric or customary units and measures.	Which of the following is best for measuring the length of a crayon? a. Milliliters b. Kilometers c. Centimeters d. Grams
36. Make measurement conversions within systems.	A truck weighs 12,000 pounds. How many *tons* is that? a. 10 b. 2 c. 4 d. 6

APPENDIX B
SUPPLIERS OF COMMERCIAL MATERIALS

Supplier	Materials
Burt Harrison & Co. P.O. Box 732 Weston, MA 02193-0732 (617) 647-0674	Manipulatives
Creative Publications 5005 West 110th Street Oak Lawn, IL 60453 (800) 624-0822	Manipulatives Resource materials
Cuisenaire Co. 12 Church Street New Rochelle, NY 10805 (914) 235-0900	Manipulatives Resource materials
Dale Seymour Publications P.O. Box 10888 Palo Alto, CA 94303 (800) 872-1100	Resource materials
Didax 6 Doulton Place Peabody, MA 01960 (617) 535-4757	Manipulatives Resource materials
National Council of Teachers of Mathematics 1906 Association Drive Reston, VA 22091 (703) 620-9840	Professional publications Resource materials

Note: The above list includes suppliers of materials mentioned in this handbook. The list is NOT exhaustive, but does include many regional suppliers. Your school office may have their catalogs, as well as those of other suppliers. The addresses provided here are for your convenience in acquiring information.

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