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ABSTRACT

Focusing primarily on student cognitive processes, this second yearbook of the Finnish Association of Mathematics and Science Education Research contains five articles and a thesis summary on mathematics teaching and learning. Areas of investigation include: (1) learning styles and strategies; (2) processes and strategies in solving elementary verbal multiplication and division tasks; (3) use of a microcomputer as an educational tool and research instrument; (4) position of applications in junior secondary school mathematics teaching; (5) development of geometric thinking; and (6) development of a concept of number. Biographical notes of the contributing authors are also included. (ML)

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JYVÄSKYLÄN YLIOPISTO
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YEARBOOK 1984

Edited by Pekka Kupari

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PREFACE

The Finnish Association of Mathematics and Science Education Research published its first yearbook (Yearbook 1983) last year. The present volume, which covers research published in 1984, centers around papers presented by Finnish researchers at the Fifth International Congress on Mathematical Education (ICME 5) held in Adelaide, Australia, on August 24th - 30th, 1984. Although the articles have no common theme, most of them share an interest in students' cognitive processes as well as problem solving and application.

In the first article Leino puts forth the principles and general outlines of a project on learning styles and strategies, which he is directing. In the author's opinion to know and to understand the learner better is what the Finnish educational system needs most. The purpose of the project is to improve the possibilities of attaching more attention to the student's personality as a basis of instruction. Leino stresses that bringing teachers' and students' style profiles into the focus of educational research does not necessarily help teaching practice immediately. In order to be able to make use of the style the teacher has to know the theoretical basis of the styles and how they manifest themselves in the teaching-learning process.

Keranto's extensive article examines, from both a theoretical and an experimental viewpoint, the problems of the third stage of the author's longitudinal study begun in 1982 and discusses the following questions: a) What kind of solution strategies do children use in multiplication and division tasks whose mathematical structure corresponds to that of the types $q \cdot d = x$, $q \cdot x = a$ and $x \cdot d = a$, b) How are Piagetian abilities connected with the multiplication and division tasks mentioned above?, c) What role does the individual's memory capacity play in measurement and partitive division?, d) How and at what stage do pupils acquire the ideas of fractions and ratios involved in the concept of rational number and how does this connect with the contents investigated in the earlier stages of the study?, e) What kind of learning and teaching situations emphasize the interpretation and use of rational number as fractions and ratios? and f) How does the concept of rational number develop and what stages and associated solution strategies are involved in the process?

In the third article Björkqvist describes research of a kind which, as yet, has been rather limited in Finland. In this first attempt emphasis was put on developing the necessary routines for the simultaneous use of the microcomputer as an educational tool and a research instrument. The main reason was to develop a methodology for problem solving research in general, with an emphasis on school mathematics in as realistic situations as possible. Another set of goals were those connected with computer education - to know how to teach students how to use computers efficiently you need to know details about the way they think while they work with computers.

In the fourth article Kupari examines the position of applications in junior secondary school mathematics teaching. The author deliberates upon the problems of the present situation by concentrating on three questions: What role do applications play in subject matter? How are applications taught? and Of what significance is it to the pupil whether the applications are interesting or not? Then the author presents some empirical observations about the teaching of applications and learning results and sets forth some general outlines for improving the teaching of applications.

In the last article Silfverberg presents a research project in which an attempt was made to describe the development of pupils' geometrical thinking mainly at the three levels of the van Hiele theory. The main purpose of the study was to answer the following questions: First, can we perceive in the pupil's thinking a transition from a holistic way of perception to one analyzing and classifying properties and at what stage does such a transition take place? Secondly, in as far as a pupil recognizes, names, classifies and compares figures analytically, are these properties separate or connected with each other?

The yearbook ends with a summary of a doctoral thesis by Vornanen, approved in 1984. The thesis describes the development of a concept of number in first-graders by using a developmental psychology approach.

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COGNITIVE STYLES AND STRATEGIES

Jarkko Leino

RESEARCH PLANS

The purpose of education in school is to influence the personal development of the educand. The teachers at school and parents at home try to understand the child's thinking and acting, as well as guide them by means of encouragement, advice, and argument in the direction educationally valued by them. This guiding process often happens unconsciously and only in close connection with the actual situation like e.g. by means of remarks of unsuitable behavior etc. Through interaction with the immediate environment the child gradually adopts social rules and principles behind them, while the amount of his knowledge and experience keeps increasing and being structured. He learns proper ways of orientation to things, animals, people, situations etc. He adopts and develops the ways and their criteria which guide and control his thinking and acting.

Psychology has developed systems for describing personality and its components by means of which human behavior can be described and explained. These systems are important for educators because they make educators understand the educand, structure his ways of thinking and acting, find out difficulties in certain situations or tasks and reasons for those difficulties. Man's species-specific characteristics and ways of action serve as a foundation for descriptive systems. Individuals differ from each other in terms of each characteristic within certain limits.

There are naturally very different types of characteristics. Some physical ones can be directly seen and even measured (hight, age, slimness etc.) but intellectual characteristics can most clearly be seen only in actions (intelligence, impulsiveness, honesty etc.). Consequently describing many characteristics is at the same time describing anticipated, typical or potential activities (e.g. structure of abilities).

Human activities and performances have long been investigated in terms of the qualitative features as well as structure. In order to describe performances comprehensively, batteries of tasks have been developed which contain situations simulating human activities often in a comparatively simplified form. Attempts have been made to describe the so-called intelligent activities, verbal and problem solving activities in particular, as an important action characteristic of only the human being but, of course, other forms have also been investigated.

Action is characterized by similarities and differences. During the process of scientific development experimental psychology and differential psychology have become more differentiated from each other in spite of the same purpose of describing the regularities of psychological processes. Experimental psychology has concentrated on the effects of different experimental conditions in processes and to a great extent omitted individual differences including them in error term. Individual differences, on the other hand, form the starting point and explanatory basis for differential psychology. The differing starting-points of these two branches have understandably lead to different methodological solutions. Among the representatives of the two branches there have been those who have consciously tried to narrow the gap thus created. Some differential psychologists have tried to interpret traits considered as static to be dynamic and some representatives of experimental psychology have, in turn, tried to take individual differences into account in their formulations of theory. Inspite of that, fixed ideas often come to be seen consciously or unconsciously. Attention has been given to the fact even in Finland. For instance Wright (1984) considers the description of action strictly different from the description of traits based on individual differences and has reservations concerning the possibilities for combining these two description systems within the same research (Leino & Leino 1982).

It is the writer's opinion that cognitive psychology serves a possibility for combining the approaches mentioned above. Thus the starting-point would consist, on one hand, of conception of man as an information processor with all the different stages of the process as well as monitoring and controlling strategies, encoding and decoding processes and, on the other hand, the system theoretical description of the individual personality.
An excellent description of this approach is given by Royce and Powell (1983). A corresponding approach is also represented by the studies in which an attempt is made to investigate the traditional abilities by interpreting abilities as dynamic conceptions e.g. how a particular combination of abilities is seen at the level of action (see e.g. Leino 1981). We have the same question when cognitive styles and strategies are studied by asking what it means from the view point of the use of strategies that a person has a particular cognitive profile (combination of cognitive styles).

From the point of view of educational science the results of both experimental psychology and differential psychology create only conditions for investigating pupils' personal interactions in educational practice. In order to be able to guide the studying strategies of pupils who are very much different in terms of their styles the teacher needs plenty of information about styles and their manifestations, sensitivity to "read" the pupil, and skill to comparatively spontaneously apply this information in different teaching situations. This skill presupposes such a degree of internalization that he does not have to remain reflecting upon the activity; the teacher generally has to act spontaneously according to the demands of the situation.

There are no possibilities for reflecting upon consequences of different alternatives in the teaching situation. In a certain sense the question of how to guide a verbally gifted child and how to guide an impulsive child are analogical. Both styles and abilities are manifested in action and form a reality for personalizing instruction.

STYLES AND STRATEGIES

Man is an active, a purposeful, and preorientating creature, and not only responding to stimuli. Life with all its aspects has personal meaning to man which cannot be explained only by means of theories of conditioning. Even though many things during one's life span seem to take place randomly, purposefulness comes to be seen in the use made of situations. Friendship relations, career, and interests show conscious choice and goal-seeking behavior. Risks and strain are connected with many goals but man is ready to face them and even consciously takes risks.

In cognitive science man's interaction with his environment is considered as information processing. The flow of information in man's cognitive system can be described by means of different stages and speak about different functional units like sensory register, short-term memory, working memory, and long-term memory. From the point of view of learning the most important are awareness of information, monitoring, and control by means of which man selects, compares, manipulates, transforms, encodes, and decodes information in the memory. It is these mechanisms, monitoring and controlling processing, which essentially influence how things are learnt and how they can be used. Strategies and styles are connected with these questions of how.

To monitor and control mechanisms of the information of a certain situation are usually called strategies or cognitive styles according to how closely connected they are with the type of situation and the task. Bruner, Goodnow and Austin (1956) used the term strategy to describe the ways which the subjects used while looking for rules for categorizing the figure cards given to them. Later on the term has been particularly used by representatives of experimental psychology to denote ways of acting and thinking which are used in certain types of situations or tasks. In studies of school learning it has been used e.g. of the ways of retaining written texts (Wright et al. 1979) or the ways of solving certain types of elementary equations (Keranto 1984). Based on the preferences for different strategies the subjects have been categorized as e.g. holists or serialists, which is an approach typical of differential psychology. If the connection with the type of the task has simultaneously been given up, which is easily revealed by the language of the report, the transfer has already been made to the field of cognitive styles.

Cognitive style can be defined as a person's individual way of monitoring and controlling information processing. Styles are comparatively stable acquired habits of directing attention, abilities, and strategies in different tasks. Even though styles are comprehensive, several style dimensions are needed to cover e.g. ways of processing typical tasks in school. To change the styles generally presupposes long-lasting and systematic guidance.

Figure 1 clearly shows the position of styles in the comprehensive description of personality (Royce & Powell 1983, 13).

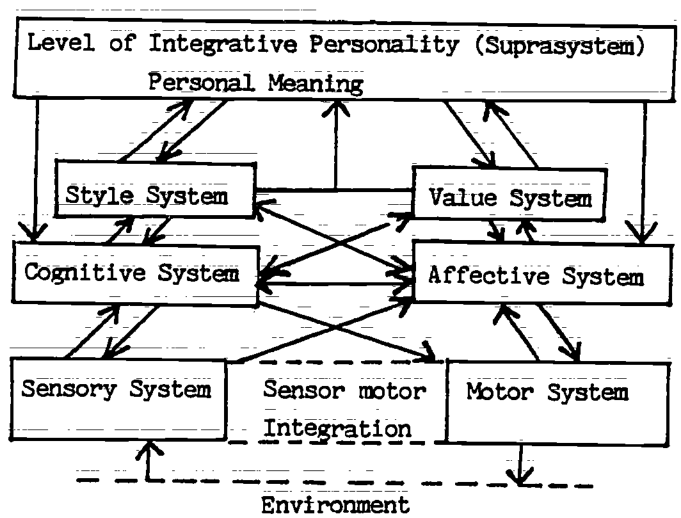

FIGURE 1. The Basic Systems and Interactive Relationship of Integrative Personality

As can be seen in the figure personality is considered as a suprasystem which consists of two lower-level systems, namely the systems of styles and values. Lower in the hierarchical description are the cognitive and affective systems which monitor the sensor motor system of senses and muscles. The higher the hierarchical level the more important the units are in the integration of personality, the stabler they are, and the greater priority they have in terms of action. The cognitive system contains the system of abilities as dynamic concepts which is to be understood as a factor description of potential skills (see also Gustafsson 1984). The affective system of the corresponding level can be considered as a factor description of emotions. The monitoring system of styles and values direct objectives of action of the lower systems so that styles are in charge of how processing goes on and values what goes on.

Cognitive styles have been investigated for decades in very many contexts and attempts have been made to extract ways of processes characteristic of individual in terms of which individuals are different from each other. The system of description of styles which has been used in our project is based on Letteri's research (1980). The system consists of seven dimensions which were originally discovered when an attempt was made to comprehensively explain the school achievements in academic subjects. The central stage descriptions necessary in information processing can be discovered even by means of logical analysis in the different dimensions of the system.

1. Focusing - Nonfocusing; the dimension is concerned with the way of selecting relevant details from the information offered by environment.
2. Field-Independent - Field-Dependent (Analytic - Global); the way of analyzing the field or perception and discovering the needed information in the complex situation.
3. Reflective - Impulsive; the dimension is concerned with how fast a person is in his decision-making process.
4. Tolerant - Intolerant; the dimension deals with the degree of a person's tolerance of ambiguous or unfamiliar information.
5. Leveling - Sharpening; the way of giving attention to items which seem familiar in the situation versus different compared with earlier items.
6. Broad versus Narrow Categorizing; the extent to which an individual uses many narrow versus few comprehensive categories in processing information.
7. Cognitive Complexity; the way of evaluating and dealing with the environment by means of either a complex or a relatively simple conceptual system.

The first two dimensions are connected with different ways of perceiving and they are to be seen in the perception strategies an individual deals with varying tasks while the two last mentioned dimensions describe constructing the conceptual system and its use in evaluating the information offered by the environment. The fifth dimension connects the perceiving and the conceptual system with one another and also as an influence on the development of cognitive constructs. The third and the fourth dimension describe how a person begins to perform a task or how he rejects it (the defence mechanism of self).

The number of style dimensions is altogether comparatively small which is quite natural in the beginning of the project. If the system proves inadequate it has to be complemented naturally. The relations between dimensions have not been analyzed more closely even though the correlations between the measures used have been low. So far the dimensions have been considered parallel even though e.g. Royce and Powell (1984, 135) consider Field-Independence - Field-Dependence more general than the other style dimensions mentioned above.

STYLES AND TEACHING

Relationships between cognitive styles and teaching and learning have not been much investigated. Styles are related to teaching methods preferred by teachers in such a way e.g. that Field-Dependent teachers prefer discussion method to lecturing which is teacher-centered (Messick et al.)(1976). Styles are also related to the subject that teacher represents (Rancourt & Dionne 1981). Styles explain as well subjects which students prefer. By investigating style profiles of students preferring each subject and by comparing them with the profiles of teachers of each subject a clear correspondence can be noticed at the senior secondary school level. It is to be seen that styles are connected with career preferences. The student whose style-profile resembles that of the teacher appreciates this kind of teacher more than the one with a different profile because it makes communication easier (Witkin et al. 1977). On the other hand the teacher can develop his repertoire so that he can better respond to the expectations of students of differing styles (Tinsman 1981).

The research results which we have received in our project show that styles are connected with teaching methods the students prefer and learning difficulties they have (Lapatto 1984). In studying mathematics Field-Independent students an solve problems better than Field-Dependent students who have difficulties in finding relevant information in a task, particularly if it contains surplus information. Reflective students like teacher-centered instruction better. Impulsive students, on the other hand, get easily tired of similar tasks and working so the tendency to make mistakes increases. These results correspond to those received in other countries where the relationship between Field-Dependency and problem solving, surplus information and teaching methods have, in particular, been investigated (Nummendal & Collea 1981, Roberge & Flexer 1983, Strawitz 1984).

The results we have received of the relationship between learning difficulties and styles compare well with those received by Letteri (Aimo & Viilo 1983, Letteri 1982). According to them if a pupil can be characterized by at least three of the following traits, namely Nonfocusing - Field-Dependent - Impulsive - Intolerant - Leveling - Broad Categorizing - Cognitively Simple, he has difficulties in school work.

Styles do not seem to change much during the school years if there is no systematic guidance to change them. Letteri's results (1982) compare well in this respect also with Finnish replication research (Viilo, in progress).

A study is in progress concerning the relationship between the styles and the strategies used in mathematical tasks and topics as well as attitudes towards mathematics. According to the theory styles should become manifest in the choice of strategies. In the remedy of learning difficulties the style profile is considered only as an indicator of inadequacies and weaknesses in processing. Argumentation has naturally to begin at the level of strategies but according to the basic idea the new strategies are gradually made more generalizable and their usefulness is really shown by means of their transferability through a variety of tasks.

The styles emphasize the student's characteristic ways of processing which makes our project different from psychological studies. Knowing the student's styles is a key to understanding him. Cognitively oriented psychological research on strategies can have a theoretically firm basis but educationally it remains quite superficial with few consequences of practical importance. The style system is very central for understanding the student's personality and gives a comprehensive basis for guiding the educational progress with the student as a startingpoint. To know and to understand the learner better is what our educational system needs most. The purpose of our project is not to change the school system but only to improve the possibilities of giving more attention to the student's personality as a basis of instruction. There are greater possibilities for these attempts now with the new school laws making it, in fact, easier to use flexible student grouping which in many cases means smaller groups.

In the so-called time resource quota system (tuntikehysjärjestelmä) the teacher of a certain subject gets more time for instructional purposes and each school is given a possibility to use flexible grouping. One experiment of our project concerns grouping on the basis of students' impulsivity which according to the pilot stydy was connected with the teaching methods students preferred but not with their school achievements (Lapatto 1984). By means of grouping teachers hopefully give more attention to student characteristics and hence make the choice of teaching methods easier.

Bringing teachers' and students' style profiles onto the focus of educational research does not necessarily help immediately teaching practice. In order to be able to make use of the style the teacher has to know the theoretical basis of the styles and how they manifest themselves in teaching-learning process. This helps him to become aware of his own styles and understand the students' different approaches to tasks and situations. Only in this way can he flexibly take these into account in his teaching and help the students who have learning difficulties.

It is characteristic of this project to employ teachers who are post-graduate students of education as researchers. Thus there is a close connection between research and real school learning situations. These teachers gather material concerning the diagnosis of styles and try to find out instructional means of using more effective strategies having their own students as subjects.

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PROCESSES AND STRATEGIES IN SOLVING ELEMENTARY VERBAL MULTIPLICATION AND DIVISION TASKS: THEIR RELATIONSHIP WITH PIAGETIAN EXPERIMENTS, MEMORY CAPACITY AND RATIONAL NUMBER

Tapio Keranto

INTRODUCTION

This report examines from both the theoretical and the experimental viewpoint the interrelationships between the mathematico-logical contents which were the objects of study in the third stage of a longitudinal study begun in 1982 (for reports of the first two stages of this study, see Keranto 1983a, 1983b, 1984). The mathematico-logical contents under investigation were the following:

A. Piagetian abilities (the understanding of transitive judgement, quantitative correspondence, and multiple correspondence) see e.g. Brainerd 1978, 1979; Copeland 1979; Flavell 1963; Keranto 1979, 1981, 1983a, 1984; Piaget 1952; Piaget et al. 1960)

B. Memory capacity (the retention of number words - in short term memory - throughout a given operation, e.g. the arranging of blocks into sub-sets of a given size (see Case 1972, 1980; Keranto 1983a, 1983b, 1984; Leino 1981, 1982).

C. Sequence skills (skills relating to the listing of number words, e.g. upward and downward listing from a given number by a given number of number words and listing in given intervals) (see Fuson & Hall 1982; Fuson et al 1982; Keranto 1981, 1983a, 1983b, 1984).

D. Multiplication skills and related counting strategies on elementary multiplication tasks of the type q·d=x.

E. Division skills and related counting strategies on elementary division tasks of the types x·d=a and q·x=a.

F. Rational number interpretations as rations, fractions and quotients (see e.g. Freudenthal 1973; Greeno 1976; Kieren 1976; Noelting 1980a, 1980,b; Payne 1976; Piaget & Inhelder 1952).

There is little knowledge based on both theoretical studies and experimental measurements as to how the contents outlined above interconnect in the mind of the individual at any given point of time. From the point of view of mathematical theory and of rational task analysis, these contents are related, and their interrelationships are examined briefly in the following section (for the idea of rational task analysis, see Resnick 1976).

THEORETICAL INTERRELATIONSHIPS

Of central importance in this study is the arithmetic of natural numbers and the way it relates to the concept of rational number. The basic operations performed with natural numbers are addition and multiplication. In Peano's axiomatic system these operations are explicitly related. The multiplication operation is defined as a recursive addition operation as follows: m (n+1)=m·n+m, where n+1 refers to n's successor n. Addition for its part is defined as a recursive operation using the idea of a natural number's successor: m+n = (m+n)+1 (for a more detailed exposition, see e.g. Landau 1960).

Peano's system gives prominence to the "holistic" nature of natural numbers and their connection with counting activity, of whose essential features it can be considered an abstraction (cf. Brainerd 1979; Keranto 1981). It is consistent with this to trace the process of learning to multiply natural numbers to the learning of number listing and addition. In this study number listing skills are represented by "listing from a certain number by a certain number and/or in certain intervals" (see Keranto 1983a, 1983b).In fact from the point of view of rational task analysis there is reason to assume that the most demanding listing skills concerned are on the level of elementary mental multiplication tasks. This point will be taken up later when strategies used in solving multiplication and division tasks are examined.

The other way to seek to define natural numbers is based on the idea of bijective function. Natural numbers are defined as finite cardinal numbers which are equivalence classes of sets of equal power. In this way the operation of adding and multiplying natural numbers comes to be defined thus: let a=card(A) and b=card(B); $a+b:=card(A \cup B)$ where $A \cap B = \emptyset$ and $a \cdot b := card(A \times B)$.

Thus in the cardinal approach addition and multiplication are not explicitly bound to each other as in Peano's ordinal system. For the multiplication operation is defined as the power of the product set. Moreover the cardinal approach views numbers as individual entities: in other words it is an "atomistic" approach in contrast to the "holistic" character of Peano's system (cf. Brainerd 1979, Keranto 1981).

Piagetian Abilities and Arithmetical Skills

Piaget's theory of developmental psychology gives no direct expression of how such things as conservation, classification and certain relational judgments are connected to the usual contents of school mathematics teaching and the strategies applied to them. It has been one task of the longitudinal study to seek to explain how success/failure on tasks such as those involving conservation, classification and transitive reasoning relate to and influence success in school arithmetic and measurement tasks and the kinds of strategies used.

On a general level positive correlations have been observed between Piagetian ability sum variables and arithmetical sum variables. This does not, however, necessarily show anything more than that Piagetian tasks may be "good" predictors of arithmetic performance at a certain school level in the same way as intelligence tests are "good" predictors of success in certain subjects (cf. Hiebert & Carpenter 1982). This kind of "global" approach does not give any clear indications as to how Piagetian abilities relate to school mathematics and whether training of these abilities has any influence on performance in arithmetical tasks.

On the specific level certain types of arithmetical addition and subtraction tasks logically require a grasp of conservation and class inclusion. This is particularly the case with problems where the addend or minuend/subtrahend is the unknown quantity, and with tasks relating to the comparison schema. In the case of tasks relating to combine-and-change schemata, in which the part or the start set & change are given, demands on logical inference ability are not so great. This picture is supported by correlations obtained in experimental measurements (see Hiebert & Carpenter 1982; Keranto 1983a, 1983b). On the other hand, even high correlations between Piagetian abilities and performance on addition and subtraction tasks do not indicate whether Piagetian abilities are necessary prerequisites of arithmetic skills and of the acquisition and training of their associated strategies. "All" they show is that the more demanding arithmetic tasks are more closely connected with logical inference abilities than mathematical tasks that can be solved by direct counting algorithms or routine counting. In fact, results suggest that pupils may use considerably developed counting strategies, e.g.

counting-on strategies, though they are non-conservers (Hiebert et al. 1982; Keranto 1983a, 1983b, 1984).

For the purpose of the present study, it would be important to know how Piagetian abilities are connected with oridinary school multiplication and division tasks. This is here examined from the theoretical point of view. As I have observed in my previous studies, Piaget and the Geneva school in general have focussed their attention from the outset on prenumerical functions and their internalisation as "mental" operations. Multiplication and division operations in the development and training of the concept of number form no exception. The research sample is dominated by the cardinal view of number based on one-one correspondence. Counting and the search for the significance of number words are of secondary importance. Consequently, for Piaget multiplication is bound up with the understanding of multiple correspondences (see Keranto 1979; Inhelder & Piaget 1958; Piaget 1952). On the operational level the child should be able to generalise N+N as the multiplication 2xN, N+N+N as 3xN etc. In addition, if it is known that a b=c, then the child should be able to understand on the operational level that a=c:b and its inverse. In other words, as in the case of addition operations, Piaget requires the understanding of inverse relation for multiplying operations, too. A typically Piagetian experiment designed to define the developmental level of the multiplication operation reveals one further notable feature. Piaget seeks to link the understanding of multiplication with partitive division and moreover to a specific one-one correspondence strategy. In other words the strategy hinted at is "one for me, one for you, one for her etc." This experiment is examined below (cf. also Keranto 1979; Piaget 1952, 203-220).

There are bunches of 10 flowers (e.g. 5 bunches) and 10 flower vases. The child places the flowers from one bunch into the vases (one in each) then transfers them to a jug, then does the same with the second bunch. At this point the question is posed: if the flowers in the jug are put back into the flower vases, how many flowers will there be in each? The same procedures and questions are repeated with the remaining flower bunches. If the child is able to infer the multiple correspondences 2 and 1, 3 and 1, 4 and 1, 5 and 1, then according to Piaget this is an indication of the understanding of multiple relation on the operational level. It should be noted that logically the above process is connected with conservation and transitive judgement. For looking at it logically the child must think of the number of flowers in the group as the same regardless of its spatial form. In addition, by means of putting the flowers into the vases (observed/concrete one-one correspondence) the quantitative equivalence of the flower bunches is established by transitive reasoning. It is not surprising, therefore, that Piaget links the operational developmental level of multiplication and division ideas with the developmental level of conservation and transitive reasoning.

For the purposes of the experimental section of this study the above experimental design was developed in the direction of a standard measure in such a way that when difficulties arose during the performance process, the conservation and transitive judgments involved in the process could be checked by means of standard questions (see Appendix 2).

Counting Strategies Relating to Multiplication and Division Skills

The multiplication and division tasks in the experimental section of this study are in mathematical structure of the types $q \cdot d = x$, $q \cdot x = a$ and $x \cdot d = a$. Further, the study does not restrict itself to partitive division situations for division tasks, but seeks also to investigate measurement divisions carried out with corresponding numbers. In a partitive division situation, the number of the part sets into which the basic set is to be divided is given. In measurement division, it is the number of objects in the part set which is given. The task is to determine the number of sub-sets (see Appendix 1: partitive division tasks PARD and measurement division tasks MEAD). There is no experimental evidence as to what kinds of solution strategies children use on the multiplication and division tasks concerned. The investigation of this is one of the main goals of this study (see Section 2: Theoretical interrelationships). Here the question is considered from the theoretical point of view.

On the basis of the previous studies concerning addition and subtraction strategies it can be assumed that children may use the following strategies in internalising and abbreviating operations on the multiplication type m•n=x (cf. Keranto 1983a, 1983b, 1984):

1. Long processes: The child takes the required number of part groups (times) containing the number of members indicated by the multiplicand and counts the members of the whole group one by one; e.g. 3•5=?: 1,...5,1,...5,1,...5; 1,...15—> 15

2. Abbreviated processes: the child makes use of now better developed number-listing skills and is able to do the task mentally; e.g. 3•5=?: 5,10,15 or 5,10,11,12,13,14,15—> 15

3. Knowledge or derived knowledge: the answer comes off the shelf" (t<2s); it emerges from the child´s explanations whether answers to previous tasks have been made use of (derived knowledge)

What then are the possible counting strategies relating to division skills? Since counting strategies relating to measurement and partitive division situations are different to some extent, they are here treated separately to start with. First, the measurement division strategies:

1. Long processes: the child takes the number of objects to be divided and groups them into sub-sets of the size indicated by the divisor and finally counts the number of sub-sets; e.g. x•5=15: 1,...,5;1,...,5;1,...,5;1,2,3 -->3

2. Abbreviated processes: the child makes use of now better developed number-listing skills and is able to do the task mentally; e.g. x•5=15; 5 is 1., 10 is 2., 15 is 3. -->3

3. Knowledge or derived knowledge: the answer is taken "off the shelf" (t<2s); it emerges from the child's explanations whether use has been made of the inverse relation of multiplication and division either in connection with previous tasks or multiplication tables; e.g. the answer to the task x•5=15 is 3, because 3•5=15, or because "I just did the 'same' sum.".

Finally, the partitive division strategies are as follows:

1. Long processes: the child uses systematic "one-one systems" (cf. Piaget) or a "try and see" strategy; e.g. 5•x=15; one for the first,...one for the fifth, a second for the first... a second for the fifth, three for the first... three for the fifth -->3; or using the "try and see" strategy starts with the 5-member groups, in which case the whole set divides out equally but there is not the appropriate number of sub-sets; continues with 4-member groups, leading to unequal division; then continues with 3-member groups, which yields 5 equal part-sets-->3

2. Abbreviated processes: the child uses now more developed number-listing skills linked to a "try and see" strategy; e.g. 5•x=15; 5, 10, 15 , or the reverse, and notices that there is not the appropriate number of parts; then tries with 4, 8, 12, 16 and notices it does not work; then tries in intervals of 3 and so finds that 3 is the answer.

3. Knowledge or derived knowledge (see the corresponding level for measurement division).

As can be seen from the above, the strategies used on measurement and partitive division situations at levels 1 and 2 differ to some extent. In fact with measurement division what the number of sub-sets should be is given. Presumably as a result of this, the tasks in question can be solved on the concrete operational level without any strain on the memory. In addition, the trial and error procedure is presumably easy to eliminate. For it is known at the outset how to begin grouping or from which number to begin listing and by how many. In the case of partitive division the situation is different when the subject does not know or cannot use a systematic one-one strategy. While using the "try and see" strategy, he has to retain in memory the number of sub-sets. This constitutes an additional load on memory as compared with the measurement division situation. At present there is no knowledge of what role the individual's memory capacity plays in such division tasks. This question is investigated in this study through the development of a memory capacity measure IPC related specifically to division situations (see Appendix 4). In the test in question the subject has to group a set of 15 objects into sub-sets of 3 while at the same time retaining in memory a certain number of orally given number words. From the previous studies there are grounds to assume that memory capacity measured in this way should correlate particularly with performance on multiplication and division tasks (on the content-specific nature of memory capacity, see Keranto 1983a, 1983b).

The foregoing analysis suggests that number-listing skills in particular are closely connected with multiplication and division skills and strategies. It is to be assumed that this will be observable in the experimental section of this study in the form of a high correlation between number-listing skills and multiplication and division skills. In addition it can be assumed that Piagetian abilities will not show a marked correlation with multiplication and division skills. As pointed out above, Piaget's experimental designs are suggestive of a certain specific strategy (the "one-one" structure) and of partitive division.

From Natural Numbers to Rational Numbers

Addition and multiplication with natural numbers are central counting operations. In other words the results of adding or multiplying two natural numbers is always a natural number. The situation is different as regards the inverse of these operations, subtraction and division. This means that using the set of known natural numbers it is not always possible to solve the equations $m+x=n$ and $m\ n=x$ where $m,n \in N$.
For "uninhibited" performance of mathematical operations, therefore, the range of numbers needs to be expanded. The expansion of set N to a set of whole numbers Z is initially carried out in such a way that the solution of the equation $m+x=n$ is also possible in cases where $n<m$. For the set of ordered pairs $(m,n) \in N\times N$ an equivalence relation is defined which divides the product set into the equivalence classes we refer to as whole numbers. Thereafter multiplication and division and order are determined for the set of "numbers" in question in such a way that the basic properties of natural numbers are "preserved" in set Z, too. In fact it is easy to demonstrate that the set of non-negative whole numbers and the set of natural numbers are identical. In other words the set of natural numbers and the set of non-negative whole numbers are describable as unequivocal inversions of each other such that addition and multiplication and ordering are preserved (for further details see e.g. Pehkonen 1978,137-175).

The set of whole numbers Z is still deficient, however, in that it does not always allow a solution of the other equation $m\ x=n$ where $m,n \in Z$. The set Z is therefore enlarged into the set Q of rational numbers, where the equation is always soluble provided that $m \neq 0$. The expansion is carried out here too by means of an equivalence relation. An equivalence relation is defined for the set ZxZ which divides the members of this set into equivalence classes known in mathematics as rational numbers. For the set of rational numbers obtained, addition and multiplication and ordering are then defined in such a way that the properties of natural numbers are preserved. In fact it can be shown that rational numbers of the form (m,1) are isomorphous with whole numbers, in other words (m,1) and m are equivalent where $m \in Z$. Finally this leads to the familiar result that rational numbers can be expressed as a quotient of two whole numbers, i.e. $(m,n)=m:n$, where $n \neq 0$.

In the elementary school syllabus, an algebraic approach to the expansion of the number range described above is to be observed. As early as the fourth grade, the concept, reading and writing of whole numbers, as well as their ordering by size and comparison, is dealt with (see Kouluhallitus 1982). But it is also to be noticed that the teaching of rational numbers is begun on the third grade with the teaching of the concept being based on the idea of fractions. Also introduced is the expression of rational numbers as decimals. On these bases it is then sought to deepen and broaden the concepts of whole numbers and positive rational numbers in parallel and separately from each other. It may well be asked, as Leino with good reason has done, how desirable it is to adopt whole numbers in elementary school mathematics teaching from the point of view of both learning and of needs (see Leino 1977,75-78). Unfortunately the situation at the moment is such that there is no knowledge based on detailed empirical research of how pupils understand whole numbers and of what conscious processes and other factors the learning of finite numbers demands. This would be an important and interesting area for research, but is not taken up any further in this study. Instead the focus of attention in the present study is on how and at what stage pupils acquire the ideas of fractions and ratios involved in the concept of rational number and how this connects with the contents investigated in the earlier stages of the longitudinal study.

Rational Numbers as Rations, Fractions, and Quotients

The learning and teaching of rational numbers involves several interpretations in which different conscious and pedagogical structures and strategies are emphasized (for the different interpretations of rational number, see Kieren 1976). Of central importance in this study are the kinds of learning and teaching situations which emphasize the interpretation and use of rational numbers as fractions and ratios. Also of interest are the "precomparative" judgements involved in division situations with discrete object sets. These may be related via partitive division processes to the idea of rational number most closely connected to continuous models. By comparative judgements here is meant the comparison between the relative sizes of two ratios in quantitative contexts. More precisely, the focus of interest here are those situations where the pupil has to be able to decide whether the relational parts of certain fraction, ratio or partitive division contents are equivalent or not. The intention is thus to investigate how the concept of rational number develops and what stages and associated solution strategies are involved in the process.

Rational Numbers as Ratios

The mental development of the idea of ratio and proportion has been investigated in various mental contexts. Piaget's studies in particular have been pioneering (probability and chance / Piaget & Inhelder 1951; geometrical uniformity / Piaget et al 1960; certain physical laws, speed and time / Inhelder & Piaget 1958, Piaget 1970, 1971). Karplus and Peterson, and later Noelting, have tried to develop Piaget's line of research in such a way that it might be possible to present the levels of operational thought by means of executive strategies (Karplus & Peterson 1970; Noelting 1980a, 1980b). Given the line of research and the experimental aims of the present longitudinal study, Noelting's experimental designs and research angle are of particular note. Indeed, Noelting seems to have developed a measure of the concept of ratio number by means of which the mental level of the concept of ratio in situations involving comparison can be reliably measured. The following table gives a condensed picture of Noelting's experimental design, with a typical task and description for each level.

TABLE 1. Developmental Levels of the Concept of Ratio According to Noelting (cf. Noelting 1980a;231)

Stage	Level	Age1)	Typical task2)	Description
IA	Preoper-ational	3;6	■■■■□ ■□□□□ 4:1 vs. 1:4	Comparison of first term
IB	Intuitive opera-tions level	6;4	■□□ ■□□□□□ 1:2 vs. 1:5	Same first terms comparison of second terms
IC		7;0	■■■□□□□ ■■□ 3:4 vs. 2:1	Inverse proportion
IIA	Concrete operations	8;1	■□ ■■□□ 1:1 vs. 2:2	Ratio 1:1 equivalence class
IIB		10;5	■■□□□ ■■■■□□□□□□ 2:3 vs. 4:6	"Whatever ratio" equivalence class
IIIA	Formal opera-tions level	12;2	■□□□ ■■□□□□□ 1:3 vs. 2:5	Ratios where one of the corres-ponding terms is a multiple of the other
IIIB		15;10	■■■□□□□□ ■■■■■□□□□□□□□ 3:5 vs. 5:8	"Whatever ratio"

1 Age = age where 50 % of the age group su.-eed
2 ■ = orange
□ = water

The basic question posed in this experiment is: which mixture tastes sweeter (or more of orange), or do both mixtures taste the same (cf. Appendices 3 and 5)

Noelting distinguishes two main types of solution strategy to be used on this task. Where the subject operates on the "inner state" of each mixture (juice vs. water) and bases his judgement on the products he obtains (in certain cases on the results of division), Noelting talks of a "within strategy". For instance, on the IC stage task 3:4 vs. 2:1 the subject may say that "this one is weak and this one is juicy, so this one (2:1) is sweeter"; on the stage IIA task 1:1 vs. 2:2 the subject may say that "this one is just right, and so is this one, so they are just as sweet" (these examples are taken from interviews conducted during the experiments in the present study). On the stage IIIA task 1:3 vs. 2:5 the subject may work out how much water there is to one part juice and come to the figures 3 and 2 1/2; therefore the 1:3 mixture is weaker and the 2:5 mixture sweeter. On this basis it can be observed that the within strategy is related to and leads to the understanding of percentage numbers. The other main type of strategy is the between strategy and relates and leads to common denominator algorithms, as the following example will show. Using the between strategy on the stage IIIA task 1:3 vs. 2:5, the starting point or focal point is the "lower" inner state 1:3. This is worked on (e.g. converted to higher terms) so that the quantities of juice are equal: "There should now be six glasses of water to two of juice. Since there are five glasses of water in this one, this (2:5) is the sweeter mixture."

Noelting's theoretical and empirical results strongly suggest that the system described above forms a hierarchy (Table 1.). Indeed the basic idea is that the strategies at each stage contain the essential features of the strategies of the previous stage. For the pupil to advance from one stage to another a qualitative change has to happen in the strategies used. According to Noelting, the remark "Now I get it!" signals that the pupil has made the appropriate modifications to his executive strategies (cf. learning with understanding / Ausubel 1963; equilibrium / Beth & Piaget 1966; adaptive restructuring / Noelting 1980b). As can be seen in Table 1, stage IA tasks are solved via comparison between the first terms (between strategy). This operation is included in the stage IB strategy, in which by means of the between strategy the first terms are found to be the same and the second terms different. To solve stage IC tasks it is not sufficient to concentrate on one "between" relationship at a time; there also has to be an examination of the "within" relationships. To solve stage IIA tasks the child should understand that these tasks are solved either by means of the within strategy or by means of between strategy. At stage IIB, multiplication and/or division operations become necessary. The solution of stage IIIA tasks requires for the first time a synthesis of two operational systems. The pupil has actually to be able to combine multiplication and addition operations with each other. Since at stages IIIA and IIIB the products of certain operations are subject to further operations, it is now a question of operations on operations or formal operations (cf. e.g. Beth & Piaget 1966).

The above analysis leads to the logical conclusion that multiplication and division skills become really necessary on tasks at the IIB stage. The experimental measurements in the present study were carried out on second graders ranging in age between 8 and 9. According to the age estimates presented by Noelting, most of the pupils in question might be expected to be at stages IC and IIA. Thus it is to be expected that the stages measured on the ratio test will not be particularly related to the multiplication and division skills and their associated counting strategies examined in the present study.

Rational Numbers as Fractions and Quotients

When we seek to answer the question "How great a part?" or "What part of the whole?" we are dealing with the fraction interpretation of rational numbers. This interpretation centres on the division of a certain whole into parts of equal size and the simultaneous observation of the quantitative relationship of the parts thus formed and the original whole. We also speak of fractions of a whole and write these as a/b. Here b is known as the denominator and indicates the number of equal parts into which the whole has been divided, and a as the numerator, showing how many equal parts are observed. As is apparent from what was said in the previous section, on the more demanding ratio number tasks pupils may make use of the fraction symbolics they have learned. In other words the ratio may be expressed as a/b, but this is fundamentally a question of the quantitative comparison of two parts of the whole. If for instance we have 4 parts juice and 6 parts water, then the quantity of juice is 4/10 or 2/5 of the mixture, and the quantity of water correspondingly 6/10 or 3/5. But the ratio of juice to water can also be expressed using the fraction symbol system as 4/6 or 2/3, but this must be read as "four to six". This suggests, then, that the teaching of the symbolic expression involved in the concept of rational number should begin with fraction interpretations and the symbolics learnt can then be employed in the teaching and learning of more demanding ratio number tasks. (This proposal is in keeping with the elementary school mathematics syllabus; Kouluhallitus 1982).

The problem is at what stage to teach fractions and to what observational models to use from the point of view of fraction interpretations. Both surface models and set models have been used in school mathematics. Instructional research carried out has shown surface models to be clearly superior in the teaching of tasks involving the addition or subtraction of fractions of a different denominator and verbal ratio problems. Surface models also led to better results on tasks involving part-part and part-whole comparisons (Greeno 1976; Payne 1976). These results suggest that it is more difficult for pupils to perceive part-whole relationships when models involving discrete sets of objects are used. It would appear that set models are closely related to the idea of ratio number examined above where comparison focuses on the relationship between parts. In addition, the results indicate that the idea of equivalent ´fraction numbers´ is learnt more easily with the help of surface models. The equivalence of the ´fraction numbers´ 1/2 and 2/4 can be observed in a very concrete way with the help of ´rectangular´ or ´round´ models, where the size of the units to be compared remains constant (cf. Greeno; Leino 1977,77; App. 3 and 6). With the set theory approach the situation is different. For the change in the denominator entails a change in the number of members in the set. The quantitative invariance of the units to be compared is thus lost. The consequences are evident in the results obtained in instructional research.

The above observations indicate that the learning of the concept of rational number is at first tied to certain models and interpretations. It is apparently only after a long process of teaching and learning that the pupil is able to switch freely between models and interpretations. This is an area which the present study also seeks to survey empirically. In connection with the fraction interpretation a "cake" model is used. The pupil has to compare certain fractions (portions) of equal-sized cakes and decide if there is more to eat in one than the other or the same amount. The fractions in question are chosen to correspond with the number ratios in Noelting´s experiment (see App. 3). For instance, Noelting´s task ratios 1:2 vs. 2:4 were taken as corresponding to the fractions 1/3 and 2/6 in the cake test (see App. 5 and 6). On the basis of the theoretical observations above, it can be assumed that there will not be a marked correlation between the ratio and fraction tests at the second-graders´ level.

The above does not say anything directly about how the fraction concept develops and at what stage its fundamentals could be taught to pupils. Both Piaget's research and studies carried out in the U.S. suggest that second-graders ought generally to be ready to learn the basics of fractions and the associated symbolics (Payne 1976; Piaget et al. 1960). This essentially concerns surface models, where division is into two, three or four equal parts. It is a different matter at what stage the pupil understands the idea of equivalent fractions and can apply this knowledge in the addition and subtraction of fractions with different denominators. Empirical research is needed to throw light on this. Similarly research is needed into the question of what mental schemes and strategies are required for the pupil to be able to understand the fraction a/b as a quotient a:b of the numbers a and b. Instructional research is also needed to find out how best to teach this relationship between the fraction a/b and the quotient a:b.

Touching on this question, the present study investigates experimentally the proportional reasoning of second-graders in partitive division situations. As in the "juice" and "cake" tests, so in the "chocolate" test the pupil has to divide it to different proportions. The pupil has actually to be able to decide which of two groups gets most pieces of chocolate or if they get the same amount (see App. 4 and 7). The difference between this and the previous tests is that in this case the tasks can be solved fairly easily using multiplication and division strategies. Thus it can be assumed that the chocolate test will correlate more significantly with multiplication and division skills than will the juice and cake tests. It is the task of the next stage of the research programme to investigate how multiplication and division skills are connected with the ability to extract out a part of a number or quantity. In this way it will be possible to seek an answer to the question of how the pupil discovers the connection between the fraction a/b and the quotient a:b.

EMPIRICAL INTERRELATIONSHIPS

The main tasks of the experimental part of this study are presented in the form of questions as follows:

1. What kind of counting strategies are used by second-graders in solving elementary multiplication and division tasks and what are the frequencies of such strategies?

2. How and to what extent are Piagetian abilities, memory capacity and number-listing skills related to multiplication and division skills and to each other?

3. How and to what extent do multiplication and division skills, and the number-listing skills (listing by certain intervals) closely associated with them, develop during the second year of school?

4. How and to what extent do tests measuring proportional reasoning involving ratio, fraction and partitive division contents form hierarchies and correspond to each other?

5. How and to what extent are Piagetian abilities, memory capacity, number-listing skills and multiplication and division skills related to proportional reasoning with ratio, fraction and partitive division contents?

The questions posed above are examined with the help of frequency tables and correlation and regression analyses. T-tests calculated are not reported here, since they did not provide any additional information on these questions.

Method

Measurements for the third stage of the longitudinal study were taken out in individual interviews held during the period Sept. 9 - Nov. 18, 1983 at the training school of the University of Tampere, Teacher Training College of Hämeenlinna. The subjects were pupils beginning their second school year. Each pupil (N=36) took part in four interviews, conducted by the author himself. The first test studied number-listing skills and solution processes on multiplication and division tasks. This took the pupil 30-40 minutes depending on the level of skills and processes used, and included a short break (see App. 1) The second test measured Piagetian abilities, and lasted 15-20 minutes (App. 2). The third interview was concerned with charting the understanding of ratios and fractions, and took 20-30 minutes (App. 3). The fourth interview measured memory capacity and the idea of partitive division, again lasting 20-30 minutes depending on the pupil´s ability (App.4). Measurements for the development of the more demanding number-listing skills and the solution processes on the multiplication and division tasks were perfomed during the April 1984.

The Measures

The types of tasks used and the abbreviations together with their explanations are outlined below. The test type abbreviations and number of items per test type are given in parentheses. The individual tasks are set out in detail in Appendices 1-4.

Piagetian abilities: "Length Transitive inference" (LTR/4). "Cardinal Transitive inference" (CTR/2), "Equivalence Conservation" (EC/3). "Identity Length Conservation" (ILC/2). "Multiplicative Correspondence" (MSP/3).

Of the above, CTR and MCP are new variables developed further from Piaget´s multiple correspondence tasks (cf. Piaget 1952, 203-220). The other variables were also used in the earlier studies (see Keranto 1981, 1983a, 1984). A brief examination of the mental development of these variables is given in connection with the presentation of results on the first and third question.

Memory Capacity: "Information Processing Capacity" (IPC/15). The task presentation and combinations of numbers used in this IPC measure were as for the ESP(S) measure in the earlier studies (cf. Keranto 1983a, 1983b, 1984) except that now the pupil worked with wooden blocks. The pupil was told the numbers to be remembered, which he had to repeat straight away. Then the blocks (15 in random array) had to be grouped into threes. When the grouping was complete, the pupil had to announce the numbers remembered. It was the intention that the IPC tasks should correspond in content as far as possible with the functional processing used on content division tasks.

Number-listing skills: "Count UP from x BY 'N'" (UPBY/4), "Count DOWN from y BY 'N'" (DOBY/4). "Count UP from x BY the SAME number m" (SABY/6). The task types were the same as the more demanding of those used in the two earlier stages of the longitudinal study (cf. Keranto 1983a, 1983b, 1984). One new task was "continue in intervals of six: 6, 12..." (SABY/6).

Multiplication skills: "Multiplication" (MULTI/5). The test involved elementary verbal tasks of the type m n=x, selected to fall in the 1-20 number range and using the school text book as an aid ("ecological validity").

Division skills: "Measurement division" (MEAD/6), "Partitive division" (PARD/6). Content and partitive division tasks were chosen in such a way as to correspond as well as possible with the multiplication tasks and with each other both contextually and numerically.

Understanding of basics involved in the rational number concept: "Proportional reasoning in RATIO-content" (RATIO/19), "Proportional reasoning in FRACTION-content" (FRACT/19), and "Proportional reasoning in PARTITION-content" (PART/15).

Proportional reasoning in connection with the concept of ratio was tested by means of Noelting's "juice" test (Noelting 1980a, 1980b). The test involves the pupil comparing the sweetness of mixtures poured into two jugs. For each task there was a pictorial representation showing the quantities of juice and water to be poured into the jugs (see App. 5). Proportional reasoning involved in the fraction interpretation was investigated by means of the "cake" test. The fractions involved were chosen to correspond with the ratios in the juice test. Thus the numbers used were made "constants". Proportional reasoning in partitive division was investigated by means of the "chocolate" test. As with the juice and cake tests, here too the pupil was able to use pictorial aids (see App. 6 and 7).

Scoring

For Piagetian abilities and the process involved in performance the scoring was as follows: LTR and CTR: 0 for failure, 1 for success on task but inadequate reasoning, 2 for success and correct reasoning; EC: 0 for correspondence not "conserved", 1 for "conservation" but unsatisfactory reasoning, 2 for an explanation based on reversibility, compensation, numbers or non-addition/non-subtraction; ILC and ELC: 0 for an immediate "wrong" answer of "shorter" but inability to explain why, 1 for the right answer without sufficient explanation or for an eventual wrong answer but evident understanding of compensation, 2 for a correct and well-reasoned answer "it's not cut", "you can straighten it again" etc.; MCP: 0 for lack of understanding, 1 for the right answer but no or insufficient explanation, 2 for a well-reasoned answer.

Memory capacity scoring was dichotomous as follows: 0 for performance inadequate in some respect (incorrect grouping, numbers given needed repeating, numbers remembered in the wrong order, one of the numbers forgotten during grouping), 1 for completely satisfactory performance.

Number-listing skills were scored on a trichotomous scale: UPBY and DOBY: 0 for failure, 1 for success using finger counting, 2 for correct solution arrived at mentally (for more detail on memory strategies, see Keranto 1983a,1983b, 1984); SABY: 0 for inability to manage two "steps", 1 for success in handling two or more "steps" by listing the intervening numbers, 2 for successfully completing two or more "steps" without number-listing.

Multiplication skills scoring again involved the performance process and the strategies observed: 0 for failure even with the help of blocks or fingers; 1 for a solution process based on concrete aids, where the pupil takes from the box the required number of sub-groups of a certain size and finally counts the number of objects in the whole group (long processing); 2 for mental strategies based on the more demanding number listing skills mentioned above, e.g. on the task 5 2=? counting 2,4,6,8,10, or 2,4,6,8,9,10 giving the answer 10; 3 for a promptly given answer based on knowledge or derived knowledge (e.g. noticing the relationship between the different tasks; solution time < 2s).

Division skills scoring corresponded to that used on multiplication skills: MEAD and PARD: 0 for failure even using blocks or fingers; 1 for success using concrete aids: on MEAD tasks the pupil groups directly into sub-groups of the number indicated by the divisor and finally counts the number of sub-groups; on PARD tasks the situation is more problematic. If the pupil uses a "try and check" strategy, he has to bear in mind the whole time that the division has to work out evenly and that the number of groups must be that indicated by the divisor; if he uses the "one at a time for each" strategy (cf. MCP tasks) the situation is easier as regards memory capacity; 2 for performance based on mental processing using addition/subtraction/number-listing skills; 3 for a prompt answer based on knowledge or derived knowledge (e.g. noticing the connection with the multiplication tasks).

Understanding of basics involved in the rational number concept the scoring here was: RATIO: 0 for failure or an answer based on guessing; 1 for competent performance, which was based mainly on "within" or "between" strategies. Using the "within" strategy the pupil examines the "internal" proportions of the water and juice to be mixed in the jug and then compares these proportions; e.g. 2:2 and 3:3 are both "just right", in other words equally sweet. With the "between" strategy the relation between the parts of water or juice are compared with either the water or the juice as constant; e.g. in the case of 1:3 and 2:5 the pupil may make the juice the constant such that 2:6 = 1:3 and then compare the ratios 2:6 and 2:5. The right and wrong strategies used by the pupils will not be examined in any greater detail in this study, but will be the concern of a later stage of the study.

FRACT: 0 for inability to do the task mentally or for an answer based on guessing or, where pictures were used, on a direct visual comparison; 1 for the right answer and adequate reasoning. As can be seen from Appendix 3, a figure was always shown after a mental attempt or performance. It so happened, however, that the pictorial material used invited a direct visual comparison (the required fractions of the cakes were ready shaded/cf. school textbooks). For our purposes it would probably have been better to use pictures without the shading, in other words such as to require the pupil to seek and compare the portions to be observed for himself (cf. Nurmi, Reinikka & Tiira 1984). It is for that reason that the scoring used was dichotomous and based on correct/incorrect mental performance.

PART: 0 for an unsatisfactory guess or answer based on direct visual observation; 1 for competent mental performance or performance based on distribution of the figures ("chocolate bars"). PART tasks could have been scored trichotomously, but this analysis was left for the further stages of the study.

Sum Variables

The following sum variables were formed in connection with the above task types and scoring:

$$LTR = \sum_{1}^{4} LTR_i \;,\quad CTR = \sum_{1}^{2} CTR_i \;,\quad TR = LTR + CTR \;,\quad EC = \sum_{1}^{3} EC_i \;,\quad LC = \sum_{1}^{2} ILC + \sum_{1}^{2} ELC$$

$$\text{ja } MCP = \sum_{1}^{3} MCP_i \;,\quad IPC = \sum_{1}^{15} IPC_i \;,\quad UPBY = \sum_{1}^{4} UPBY_i \;,\quad DOBY = \sum_{1}^{4} DOBY_i \text{ ja } SABY = \sum_{1}^{6} SABY_i$$

$$MULTI = \sum_{1}^{5} MULTI_i \;;\quad MEAD = \sum_{1}^{6} MEAD_i \text{ ja } PARD = \sum_{\neq 2}^{6} PARD_i \;;\quad RATIO = \sum_{1}^{19} RATIO_i \;,$$

$$FRACT = \sum_{\neq 1,3,5,8}^{10} FRACT_i \quad \text{ja } PART = \sum_{2}^{15} PART_i$$

The formation of the sum variables is supported not only by criteria of content but also by the correlation and hierarchy analyses of the items relating to the RATIO, FRACT and PART variables shown in Appendix 9. The hierarchy analysis of the IPC test is given in Appendix 10.

Results

Results Pertaining to the First Question and the Frequencies and Development of Piagetian Abilities and Number-Listing Skills

An attempt to answer the first experimental question is made using the following table, which shows the frequencies of strategies used on multiplication and division tasks.

TABLE 2. Frequencies of Strategies Involved in Multiplication and Division Skills (N=36). (In parentheses below, figures where technical errors are taken into account.)

	MULTI1 (4:2)	MULTI2 (3:4)	MULTI3 (3:5)	MULTI4 (3:6)	MULTI5 (5:2)	MEAD1 (8:4)	MEAD2 (12:4)	MEAD3 (10:2)	MEAD4 (12:3)	MEAD5 (15:5)	MEAD6 (18:6)	PARD1 (8:4)	PARD2 (10:2)	PARD3 (12:4)	PARD4 (12:3)	PARD5 (15:5)	PARD6 (18:3)
failure	0 (3)	0	0	4 (6)	0 (5)	0	0	0	0 (1)	0	0	0 (1)	0	4 (5)	1 (4)	2 (6)	2 (3)
aids	16 (13)	11	5	17 (15)	21 (17)	21	28	26	27 (26)	14	21	20 (19)	9	25 (24)	27 (24)	27 (23)	24 (23)
abbrev. process	18	17	23	14	12 (11)	7	7	5	5	10	4	8	0	6	4	5	3
knowledge	2	8	8	1	3	8	1	5	4	12	11	8	27	1	4	2	7

On the basis of this table the following observations can be made:

1. The number of unsuccessful performances on multiplication and division tasks was very small if we exclude technical errors due to carelessness. The instances of failure on content division

tasks PARD3-6 were mainly due to the use of an inclusion strategy in the content division situation. In addition there were two failures on task PARD3 due to the unsuccessful use of the "try and check" strategy.

2. On the multiplication tasks MULTI1-3 the main strategy was "abbreviated mental processes", i.e. the pupil used the more demanding number-listing skills in arriving at the solution. On tasks MULTI4-5 a slight tendency becomes apparent to "shift" to the use of concrete aids. In the case of division tasks, long processes relying on concrete aids formed the main strategy. An exception to this was the content division task PARD2, which from the point of view of the performance process proved in many cases to be an addition task 5+5=10. The table does not directly show how many of the children used a "one-one" strategy on the long processes level on content division tasks; the number was in fact between three and five. This result is parallel to those in the earlier studies, in which the formation of numerical equivalence was one area of investigation; there, too, the one-one strategy was of a very low frequency (cf. Keranto 1981, 1983a).

3. Use of factual knowledge was, as expected, low in frequency. Derived knowledge was used mainly on tasks MULTI2, MEAD6 and PARD6, i.e. on those tasks where the pupils were able to make conscious use of the relationship and similarities between the tasks. In other cases of factual knowledge, the answer was "simply" known.

4. On the basis of the frequency distributions it can be tentatively observed that only some multiplication and division tasks involving a certain mathematical equation are closely connected in the mind of the child. More detail of the highest intercorrelations between items can be found in Appendix 8, but mention can be made here of the following: MEAD1, PARD1 (8:4=?) (0.68***); MEAD2, PARD3 (12:4=?) (0.72***); MULTI1 (4 2), PARD1 (0.65***); MULTI5 (5 2), MEAD3 (10:2) (0.65***). Other

correlations relating to tasks involving a certain mathematical equation were below the 0.60 level (App. 8).

Before proceeding to results in the second main area of the investigation, it will be helpful to examine the table below showing the frequencies of scores obtained on Piagetian tasks and number-listing tasks.

TABLE 3. Frequencies of Scores Based on the Performance Process on Piagetian Tasks and Number-Listing Tasks

LTR1	LTR2	LTR3	LTR4	CTR1	CTR2	EC1	EC2	EC3	ILC1	ILC2	MCP1	MCP2	MCP3			
0	0	0	0	0	1	0	0	0	6	7	7	8	1	1	1	failure
2	2	2	2	4	4	3	4	4	14	12	14	14	16	14	14	inadequate reasons
34	34	34	34	32	31	33	32	32	16	17	15	14	19	21	21	valid reasons

SABY1	SABY2	SABY3	SABY4	SABY5	SABY6	UPBY1	UPBY2	UPBY3	UPBY4	DOBY1	DOBY2	DOBY3	DOBY4	
0	0	5	9	1	18	0	0	1	6	2	4	2	8	failure
0	14	23	21	7	16	11	13	12	13	10	12	13	13	aids/intervening numbers used
36	22	8	5	28	2	25	23	23	17	24	20	21	15	done mentally

On the basis of Table 3, the following can be observed:

1. In the case of transitive reasoning and conservation of observed one-one correspondence nearly all the pupils had attained the level of valid explanation. A binomial test indicated that a significant ($p<.01$) and highly significant

(p<.001) development had occurred on tasks LTR1 and LTR3 within the span of a year (cf. Keranto 1983a; also see Keranto 1981, 1983b; Siegel 1958).

2. Conservation judgements relating to length were still clearly at a developmental stage. A binomial test showed a significant and fairly significant development on tasks ILC1 and ELC1 within the span of a year (cf. Keranto 1983a).

3. Piagetian multiple correspondence tasks were performed successfully from the point of view of accuracy by almost 100 % of the pupils. This result is consistent with that obtained in a doctoral dissertation directed by the author (Toivonen & Tuomi 1984).

4. Accuracy of performance on number-listing by certain intervals revealed the following scale of difficulty: SABY1--SABY2--SABY5--SABY3--SABY4--SABY6. This corresponds to the order observed in the fall term of 1982, but differs from the spring 1983 order with respect of SABY3 and SABY4 (cf. Keranto 1983a,1983b). UPBY and DOBY frequencies are similar to those obtained in spring 1983 (Keranto 1983b). In other words these skills showed no noticeable improvement in the interval between the spring and fall measurements 1983. This was also observed in binomial tests. In the case of SABY skills, however, a statistically significant advance was seen to have occurred during the same period in the ability to list numbers in intervals of three and five (tasks SABY3 and SABY5).

Results Pertaining to the Second Question

The second question is elucidated with the help of the regression models presented below which were derived from stepwise selective regression analyses (the level of significance at least 5 % with both t-values and F values).

The correlation matrix used in the regression analysis appears in Appendix 10.

TABLE 4. Regression Models of the MULTI,MEAD and PARD Variables, where the Predictors are Piagetian, IPC and Number-Listing Skill Variables (N=36). Coefficients are Values.

object		predictors TR,EC,LC,MCP = PIA	R (%)
MULTI	=	0.38 LC	14
MEAD	=	0.38 MCP	14
PARD	=	0.35 LC	12
		predictors PIA,IPC,SABY,UPBY,DOBY	
MULTI	=	0.60 SABY + 0.38 UPBY	72
MEAD	=	0.49 SABY + 0.43 UPBY	62
PARD	=	0.37 SABY + 0.34 DOBY	40
		(SABY,UPBY,DOBY)=SEQ predictors PIA,IPC,SEQ,MEAD,PARD	
MULTI	=	0.37 MEAD + 0.38 SABY + 0.25 DOBY	77
		predictors PIA,IPC,SEQ,MULTI,PARD	
MEAD	=	0.53 MULTI + 0.42 PARD	74
		predictors PIA,IPC,SEQ,MULTI,MEAD	
PARD	=	0.76 MEAD	58

The main points to be observed from the above are:

1. The predictive power of Piagetian variables with regard to multiplication and division variables is relatively low; this is consistent with logical analysis.

2. Number-listing skills represented a highly significant predictor of the deviation of multiplication and division variables; again this is consistent with logical analysis.

3. IPC could be omitted from the models: most of the pupils were at the three-bit level.

4. The above results were in keeping with the explanatory models presented in the two earlier stages of the longitudinal study concerning the relative explanatory power of Piagetian variables and number-listing skill variables as predictors of the variance found with addition and subtraction variables, the best predictors being number-listing skill variables (cf. Keranto 1983a, 1983b, 1984).

The figure below illustrates on the "macro-level" how Piagetian variables relate to multiplication and division variables:

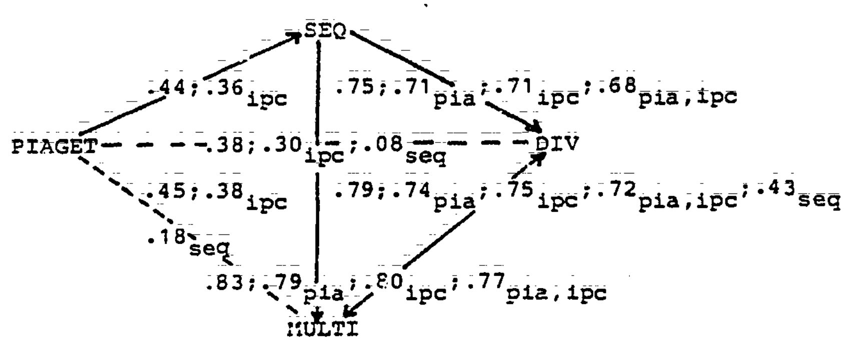

FIGURE 1. A Model of Dependency Relationships of the Sum Variables PIAGET, SEQ, MULTI and DIV, where .pia, .ipc, .seq Indicate that the Variables PIAGET, IPC and SEQ are Constants. PIAGET=TR+EC+EC+LC+MCO; SEQ=SABY+UPBY+DOBY; DIV=MEAD+PARD (N=36).

The figure shows that the connection of Piagetian abilities to multiplication and division skills comes via number-listing skills. Another point is that making "IPC abilities" constant is of little significance as regards the relationship of number-listing skills with multiplication and division skills. It is also noticeable that multiplication and division skills are significantly correlated when the SEQ sum variable is the constant.

The results suggest that the use of Piagetian tasks logically relating to multiplication and division skills in the training of these skills is highly problematic. Although positive statistically significant correlations are observed between Piagetian variables and multiplication and division skills variables, this does not necessarily mean that Piagetian abilities are prerequisites for the solving of the elementary multiplication and division tasks concerned. A similar indication is given by the results of a study on this particular question directed by the author (Toivonen & Tuomi 1984).

As regards the relationship between IPC abilities and multiplication and division skills, there is at least a fairly significant correlation between the IPC variable and the MULTI and MEAD variables. The reason that correlations did not turn out to be any higher than this may well be that most children were already on at least the three-bit leavel; only four were on the two-bit level. In other words most of the children were able to retain three numbers in memory while at the same time grouping 15 blocks randomly arranged into groups of three. Logically this amount of memory capacity ought to be sufficient for the information required by MULTI, MEAD and PARD tasks to be retained in memory during processing. This and the content-dependent nature of the IPC measure are reflected in the results (cf. Keranto 1983a,1983b, 1984).

Results Pertaining to the Third Question

An answer to the question of the extent to which multiplication and division skills develop during the second year of school can be presented with the help of the following tables:

TABLE 5. The Development of Multiplication During the Second School Year (N=36)

		spring 1984			
	level	a)	b)	c)	
	a)	5	12	***2	Binomial test
fall 1983	b)	0	5	10	p<.001***
	c)	0	0	2	

TABLE 6. The Development of Division During the Second School Year (N=36)

		spring 1984						
	level	a)	b)	c)	a)	b)	c)	
	a)	16	6	***7	6	1	***13	Binomial
fall 1984	b)	0	0	2	0	1	11	test
	c)	0	0	5	0	1	2	p<.001***
		measurement division			partitive division			

The tables show that during the second school year (pupils aged 8-9) there is a highly significant development in the direction of mental processing in multiplication and division skills (p<0.001). In the case of multiplication, there appears to be a clear development from level a) (long processes based on external aids) via b) (abbreviated processes based on the more demanding number-listing skills) to level c) (knowledge or derived knowledge). As regards division tasks, however, the typical development of processes is more problematic and requires further investigation.

"Listing in certain intervals" skills develop as a rule through the following stages:

a) the child is unable to continue the number sequence;

b) the child lists the intervening numbers in his mind, silently;

c) the child is able to list in certain intervals without listing the intervening numbers.

With reference to these levels of skill and to performance on the different listing tasks it was possible to chart the levels of ability among the pupils as shown in the following figure:

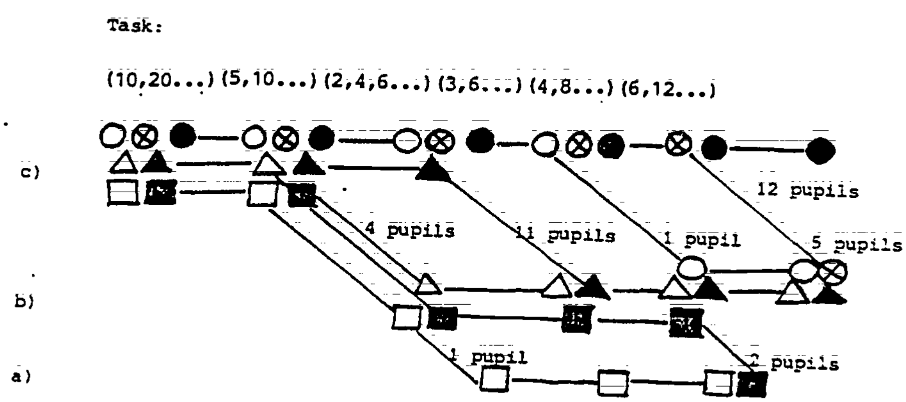

FIGURE 2. The Solution Profiles on "Listing in Certain Intervals" Tasks, Measured in Spring 1984 (N=36)

This cross-sectional examination indicates that "listing in certain intervals" skills develop in a certain order; and indeed in the longitudinal study as a whole it is evident that individual developing processes do follow this line of development.

Results Pertaining to the Fourth Question

The question of the hierarchical nature and intercorrespondences of ratio, fraction and partitive division tests was examined with the help of the scales used by Guttman (Guttman 1944; Keranto 1983a, 68-71). Table 7 shows the extent of hierarchy in the RATIO test, lists the solution frequencies and gives a description of the performance at each "level".

In the table the RATIO test can be seen to possess a complete hierarchy in the full sense of the word. The recommended criterion values would "memely" be CR = 0.90, MMR < 0.80 and PPR > 0.70 (cf. Keranto 1983a; Noelting 1980a; White & Saltz 1957). In addition there is a clear decrease performance as we move from the "intuitive level" tasks to tasks on the "low concrete operations level", and from there to the "high concrete operations level" tasks. Tasks on the "formal operations level" were as expected, too difficult for the second- grade pupils. The result obtained is consistent with those in Noelting's experiments, and the order of difficulty corresponds largely with the order Noelting has presented. Slight differences were observed within each sub-level (cf. Noelting 1980a,228). Similar results were obtained in another study carried out under the supervision of the author (Nurmi, Reinikka & Tiira 1984).

TABLE 7. Order of Difficulty of Items in the RATIO Test and Classification of Level (N=36); Hierarchy figures CR, MMR and PPR

"Level"	Item	Combin.		Freq.	Description
low	2	4:1	1:4	36	succeeds by
intuitive	4	3:1	2:2	36	comparing first
average intuitive	3	1:1	1:2	36	or second terms
					difference vs.
high	5	2:1	3:3	35	similarity
intuitive	8	2:2	3:4	32	
low concr.	6	1:1	2:2	20	Equivalence class
operations	7	2:2	3:3	20	1:1 "right"
high concr.	9	1:2	2:4	4	"whatever" equiv-
operations	10	4:2	2:1	3	alence class
formal	11	2:3	1:2	1	one or other set
operations	12	2:1	4:3	0	of terms multiple
	13	2:1	3:2	0	
	14	2:3	1:2	0	conversions
	15	6:3	5:2	0	necessary -
	16	3:2	4:3	0	operations on
	17	5:2	7:3	0	operations
	18	3:5	5:8	0	
	19	5:7	3:5	0	

CR 1.00, MMR = 0.70 and PPR = 1.00

Combin. = combination; Freq. = solution frequeny

The following table presents the mental solution frequencies on the FRACT test, together with description of performance processes and hierarchy values.

TABLE8. Order of Difficulty of Items in the FRACT Test, "Level" and Description of Process. Hierarchy Values CR, MMR and PR (N=36)

"Level"	Item	Combin.		Freq.	Description
intuitive	2	4/5	1/5	35	succeeds by comparing
	4	3/4	2/4	34	number of "eatables"
	5	2/3	3/6	10?	understands inverse
??	3	1/2	1/3	8	rel.of size & number
concrete	6	1/2	2/4	8	comparison to half
operations	7	2/4	3/7	5	with mental picture
					of parts
	8	2/4	3/7	5	
	9	1/3	2/6	5	comparison to third,
	10	4/6	2/3	5	mental picture
formal	11	2/5	1/3	0	conversion skills
operations	...	...			needed
	19	5/12	3/8	0	

Items 2-10: CR = 0.91, MMR = 0.88, PPR = 0.40
Items 2,4,6,7,9,10: CR = 0.96, MMR = 0.88, PPR = 0.67

Table 8 shows that there was clearly a less well-defined hierarchy on the FRACT test than on the RATIO test. The extent of hierarchy and intercorrespondence among performances increases if only items 2,4,6,7,9 and 10 are considered. The FRACT tasks on the "formal operations level" were, as expected, altogether too difficult, as on the RATIO test. In contrast with the RATIO test, the FRACT test "hierarchy" was made somewhat problematic by items 3, 5 and 8. Logically items 5 and 8 should be solved via comparison with a half. Thus item 5 should have occupied a place after, not before, items 6 and 7 on the scale of difficulty. Otherwise the scale of difficulty observed seems to be what would logically be expected.

From the point of view of rational task analysis, the RATIO and FRACT tests have only a partial hierarchical correspondence where the number correspondences used are a:b & c:d vs. a/a + b & c/c + d. The correlation between the RATIO and FRACT variables of 0.45** suggests that in the mind of the second-grade pupil the schemes involved in the understanding of ratio and fractions are not yet well co-ordinated. In other words the suggestion is that these contents are quite independent of each other without sharing any "umbrella scheme" which would enable free movement between models. Similar results were obtained in connection with the FRACT test in the study already referred to (Nurmi et al. 1984).

The results in connection with the PART test are similarly presented below with the help of Table 9:

TABLE 9. Order of Difficulty on the PART Test and Description of Performance Levels. Hierarchy Values CR, MMR and PR (N=36)

Level	Item	Combin.		Freq.	Description
intuitive	2	15:3	12:3	35	succeeds without
1-20	3	12:6	12:4	35	multiplication
					and division
	5	36:4	32:4	34	calculations
20-100	6	42:7	49:7	34	
	4	25:5	20:5	33	
concrete	8	9:3	8:3	24	mental
operations	7	4:2	6:3	23	calculations
					needed
	10	24:6	28:7	21	or use of
20-100	12	27:3	28:4	21	"chocolate bar"
	9	16:4	18:6	20	models
	13	48:6	40:5	20	
	14	35:7	32:8	19	
	15	45:9	42:7	18	
	11	30:5	36:6	17	

CR=0.96, MMR = 0.76 and PPR = 0.86

Again the table shows high values for degree of hierarchy. The order of difficulty for PART items can be considered "perfect". The correlations of the PART sum variables with the RATIO and FRACT variables were 0.37* and 0.47* respectively. This again indicates their "independence" in the pupil´s mind at this stage. Results in the other study referred to (Nurmi et al 1984) were similar.

Results Pertaining to the Fifth Question

On the question of how Piagetian abilities, memory capacity, number-listing skills and multiplication and division skills interrelate with proportional reasoning, the following seeks to provide an answer with the help of regression models and partial correlation models (see also App. 11).

TABLE 10. The Regression Models of RATIO, FRACT and PARD. Predictors PIAGET, IPC, SEQ Variables and MULTI, MEAD and PARD Variables (N=36)

object		predictors TR, EC, LC, MCP = PIA	R (%)
RATIO	=	no significant predictors	-
FRACT	=	0.37 MCP	14
PART	=	0.47 LC + 0.38 MCP	53
object	=	predictors SABY, UPBY, DOBY = SEQ	
RATIO	=	0.34 SABY	12
FRACT	=	0.50 SABY	25
PARD	=	0.51 SABY	26
object		predictors MULTI, MEAD, PARD = MUL	
RATIO	=	0.47 MULTI	22
FRACT	=	0.51 MULTI	26
PART	=	0.60 MULTI	36
object	=	predictors PIA, IPC, SEQ, MUL	
RATIO	=	0.47 MULTI	22
FRACT	=	0.51 MULTI	26
PART	=	0.36 LC + 0.36 MULTI + 0.31 MCP	64

The best single predictor proved to be the MULTI variable relating multiplication skills. It is logical that number-listing variables and multiplication and division variables best predict proportional reasoning in the case of partitive division contents (the PART variable); their predictive power in respect of proportional reasoning is weakest in the case of ratio contents. Piaget variables as predictors "behave" similarly. With the PART variable in particular, Piagetian variables were high predictors. It is worth noting that the Piagetian tasks in this study relate logically to multiplication and division skills and not directly to ratio contents. This is empirically shown in the fact that there were no significant predictors of the RATIO variable among the Piagetian variables. It is a matter for further research to show how and to what extent Piaget's tasks measuring the understanding of probability relate to the RATIO test. Logically these may be expected to show a close empirical relationship (see e.g. Chapman 1975, Falk et al. 1980, Piaget & Inhelder 1951).

The above results are naturally enough reflected in and complemented by the partial correlation model which follows.

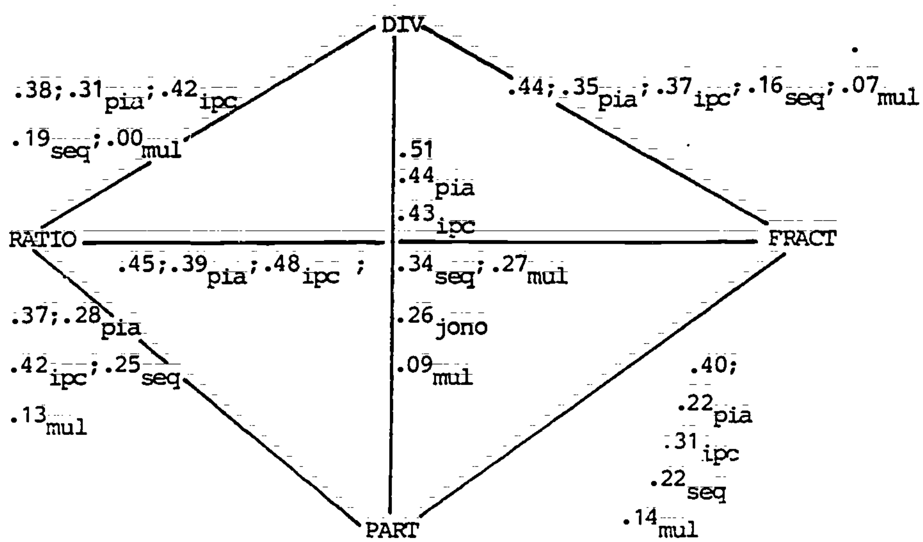

FIGURE 3. A Model of the Dependency Relationships of the Sum Variables RATIO, FRACT, PART and DIV = MEAD + PARD; Subscripts Indicate the Variables PIA, IPC, SEQ and MULTI Taken as Constant.

The first thing that is striking in the above is that the relationship of division skills with proportional reasoning in ratio, fraction and partitive division contents are explained via the MULTI sum variable. Secondly, it is notable that some of the relationships between RATIO variables and FRACT and PART variables are explained via the PIAGET = TR + EC + LC + MCP sum variable. Thirdly, it is noticeable that the intercorrelations between the variables in the model are relatively little affected by the IPC variable (cf. App. 11).

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APPENDIX 1. Tests Used to Measure Number Listing Skills and Multiplication and Division Skills.

SABY1 Continue counting in tens: 10,20,...
SABY2 Continue counting in twos: 2,4,6,...
SABY3 Continue counting in threes: 3,6,...
SABY4 Continue counting in fours: 4,8,...
SABY5 Continue coumbting in fives: 5,10,...
SABY6 Continue counting in sixes: 6,12,...

UPBY
Practice 1 Count up from 5 by 2 numbers: 6,7.
UPBY 1 and 2 Count up from 6 by 5 numbers; by 8 numbers
UPBY 3 and 4 Count up from 14 by 5 numbers; by 8 numbers

DOBY
Practice 2 Count down from 7 by 2 numbers: 6,5
DOBY 1 and 2 Count down from 9 by 5 numbers; by 8
DOBY 3 and 4 Count down from 14 by 5 numbers; by 8

On the following tasks (UNIFIX) counting blocks are available for use; pupilsare urged to make use of these if they cannot solve the task mentally.
On the multiplication tasks MULTI1-5 the blocks (20) are randomly arranged in a box.
On the division tasks the blocks are laid out on the table in the number indicated by the dividend.

MULTI1 There are 4 children. Each child picks 2 apples. How many apples did they pick altogether?

MEAD1 You use 4 nuts to fix a wheel to a car. How many wheels can you fix to a car with 8 nuts?

MEAD2 There are 12 winter tyres in the garage. How many cars can be fitted with winter tyres? 4 tyres go on one car.

PARD1 There are 8 strawberries. These are shared out equally to 4 children. How many does each child get?

MULTI2 New tyres are bought for 3 cars. How many tyres are needed altogether?

PARD2 There are 10 sweets. These are shared out equally between 2 children. How many does each child get?

PARD3 There are 12 apples. These are shared out equally between 4 children. How many does each child get?

MEAD3 There are some hens in a hen-pen. Altogether they have 10 feet. How many hens are there altogether? Hens have two feet.

MULTI3 There are 3 children. Each child collects 5 mushrooms. How many do they collect altogether?

MEAD 4 A hook is screwed to the wall with 3 screws. How many hooks can be screwed to the wall with 12 screws?

PARD4 There are 12 sweets. These are shared out equally between 3 children. How many does each child get?

MEAD5 A radio needs 5 batteries. How many radios is 15 batteries enough for?

PARD5 There are 15 pictures. These are shared out equally between 5 children. How many does each child get?

MULTI4 There are 3 children. Each child buys 6 stamps. How many do they buy altogether?

MEAD6 A torch takes 6 batteries. How many torches is 18 batteries enough for?

PARD6 There are 18 sweets. These are shared out equally between 3 children. How many does each child get?

MULTI5 There are 5 children. Each child picks 2 apples. How many apples do they pick altogether?

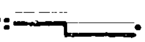
APPENDIX 2. The Piagetian Abilities Measure.

LTR1 Three (Cuisenaire) rods of equal length are used. The child sees that A=B and B=C; is then asked: Is A longer, shorter, or the same length as C? The child does not see A and C at the same time.

LTR2 As for LTR1 except with (10-block) strings of UNIFIX blocks.

A piece of wire is needed, about 20 cm in length.

ILC1 If you bend this wire, will it be shorter?

ILC2 The wire is bent thus: . Is the wire now shorter, longer, or the same length?

In addition to the wire, a piece of string of the same length is needed. The pupil may compare and see they are the same length.

ELC1 If one of these is bent, will it be shorter than the other?

ELC2 Analogous with ILC2

10 paper cups are randomly arranged on the table, together with one large glass jar and blue, white, yellow and red UNIFIX blocks in random groups of 15. The child is instructed to take from the blue group one block for each cup. After this the superfluous blacks are removed from the table.

E1 Are there now as many blue blocks as cups?

EC1 Now empty the cups into the glass jar. Are there still as many blue blocks as there are cups?

The same procedures are repeated with white blocks for tasks E2 and EC2.

CTR1 Now compare the blue and white blocks in the glass jar.

Are there more blue blocks than white, less, or the same number?

MCP2 Let's imagine that you put the blocks in the jar back into the paper cups so that there's the same number of blocks in each cup. How many blocks would there then be in each cup?

The above procedures are repeated with yellow blocks for tasks E3 and EC3.

CTR2 Now compare the blue and yellow blocks in the glass jar. Question as in CTR1.

MCP2 Now imagine that you put the blocks in the jar back into the paper cups so that there's the same numberin each cup. Question as in MCP1.

The procedures are again repeated with the red blocks.

This time only taks E4 and MCP3 are given.

LTR3 As for LTR1 except that now A<B and B<C

LTR4 As for LTR2 except that now A<B and B<C

Nb. After each task the child is asked Why?

APPENDIX 3. The Measure of Proportional Reasoning in Ratio and Fraction Contents.

The "juice" test (understanding of the idea of ratio) or RATIO test:

The basic question always presented is: which of the jugs contains the sweeter juice (or the juice tasting more of orange), or does the juice in each jug taste the same? The visuals are shown in Appendix 5. The juice:water ratios are as follows:

RATIO1	2:0 - 1:1				
RATIO2	4:1 - 1:4	RATIO3	1:1 - 1:2	RATIO4	3:1 - 2:2
RATIO5	2:1 - 3:3	RATIO6	1:1 - 2:2	RATIO7	2:2 - 3:3
RATIO8	2:2 - 3:4	RATIO9	1:2 - 2:4	RATIO10	4:2 - 2:1
RATIO11	2:3 - 1:2	RATIO12	2:1 - 4:3	RATIO13	2:1 - 3:2
RATIO14	2:3 - 3:4	RATIO15	6:3 - 5:2	RATIO16	3:2 - 4:3

RATIO17 5:2 - 7:3 RATIO18 3:5 - 5:8 RATIO19 5:7 - 3:5

The "cake" test (understanding of the fraction idea) or FRACT test:

The basic question presented throughout is: who has got more to eat, or have they both the same? A practice question is given comparing a whole cake and half a cake. After each mental attempt at a question the pupil tries the task with the help of pictures.

FRACT2	4/5 - 1/5	FRACT3	1/2 - 1/3	FRACT$	3/4 - 2/4
FRACT5	2/3 - 3/6	FRACT6	1/2 - 2/4	FRACT7	2/4 - 3/6
FRACT8	2/4 - 3/7	FRACT9	1/3 - 2/6	FRACT10	4/6 - 2/3
FRACT11	2/5 - 1/3	FRACT12	2/3 - 4/7	FRACT13	2/3 - 3/5
FRACT14	2/5 - 3/7	FRACT15	6/9 - 5/7	FRACT16	3/5 - 4/7
FRACT17	[illegible]/7 - 7/10	FRACT18	3/8 - 5/13	FRACT19	5/12 - 3/8

In connection with the mental attempts on the tasks FRACT2-19 the numerical information was presented as follows: e.g. FRACT2: take 4 of the 5 parts and take 1 of the 5 parts. In addition the pupil is asked at the beginning of each task to say by himself how many equal parts the cake is divided into and how many parts are to be taken.The visuals are of the type shown inAppendix 6.

NB. After each RATIO and FRACT task the child is asked Why?, in other words has to explain and justify his answer.

APPENDIX 4.

The Memory Capacity Measure.

For the following tasks you get all the numbers to be remembered in advance. Always repeat the numbers I give you at the beginning. After that group these blocks (15 in random array) into groups of three. When you have finished say the numbers I gave to you at the start in the same order. Let's practise. To begin with group these blocks into threes. (After this the numbers 5 and 3 are given.) Repeat then numbers. Try and remember them now whilst you do the grouping you have learnt. (When the pupil has finished grouping, the pupil is asked to say the numbers again - if he does not already volunteer them.)

IPC1	9,16	IPC2	15,8	IPC3	14,17	IPC4	12,17
IPC5	18,11	IPC6	9,14,5	IPC7	8,10,12	IPC8	8,17,13
IPC9	11,9,7	IPC10	12,18,13	IPC11	7,5,18,14		
IPC12	6,9,12,14	IPC13	10,16,14,8	IPC14	11,16,13,18		
IPC15	12,6,9,17						

The Measure of Proportional Reasoning in Partitive Division Contents.

The "chocolate" test (understanding the idea of sharing) or PART test:

The basic question presented throughout is: who gets more (the boys or the girls) or do they get the same? It is emphasized and explained that it is a question of whether there are more/less pieces per boy or per girl.

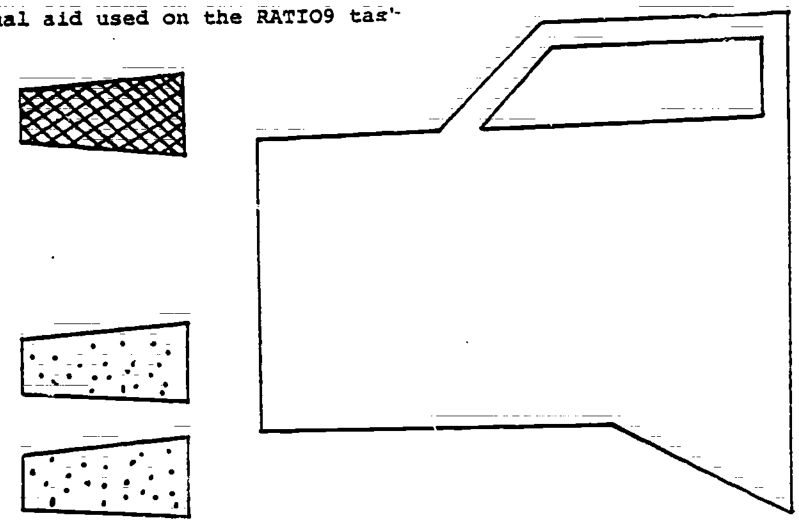
PARTI is a practise item: 2 girls get 2 pieces and 1 boy 2 pieces.

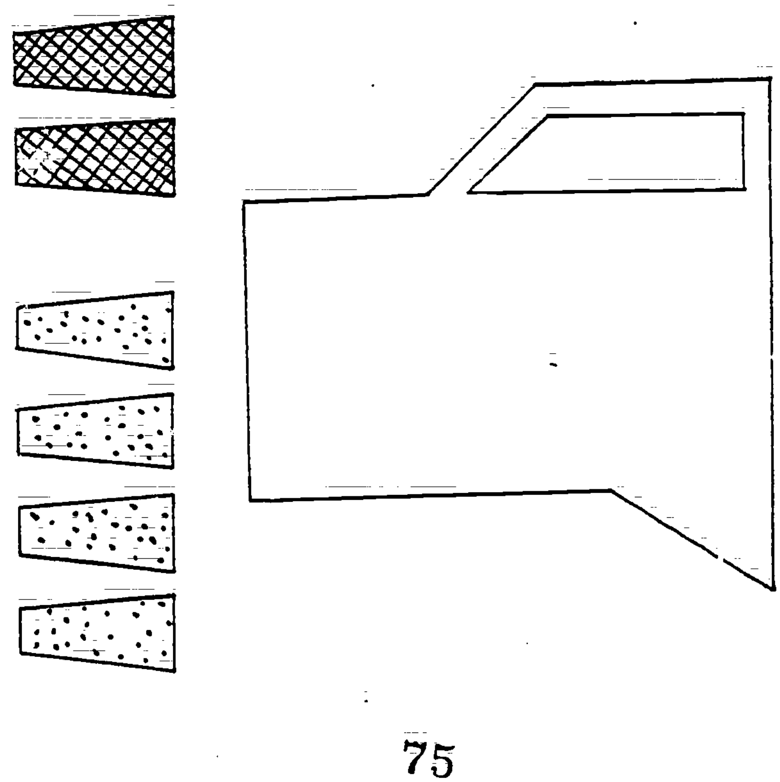
PART2	15:3 - 12:3	PART3	12:6 - 12:4
PART4	25:5 - 20:5	PART5	36:4 - 32:4
PART6	42:7 - 49:7	PART7	4:2 - 6:3
PART8	9:3 - 8:2	PART9	16:4 - 18:6
PART10	24:6 - 28:7	PART11	30:5 - 36:6
PART12	27:3 - 28:4	PART13	48:6 - 40:5
PART14	35:7 - 32:8	PART15	45:9 - 42:7

On tasks PART2-8 the picture was shown only after a mental attempt. On the remaining tasks the picture was present throughout.

APPENDIX 5

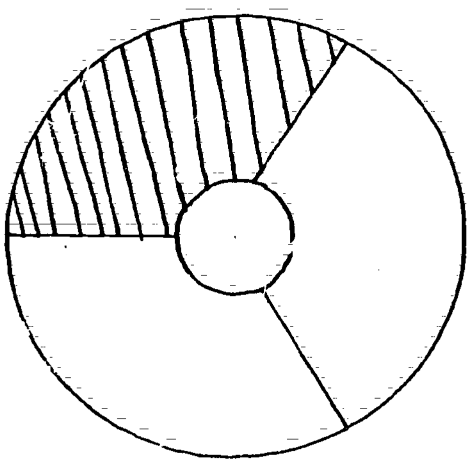
The visual aid used on the RATIO9 tas'-





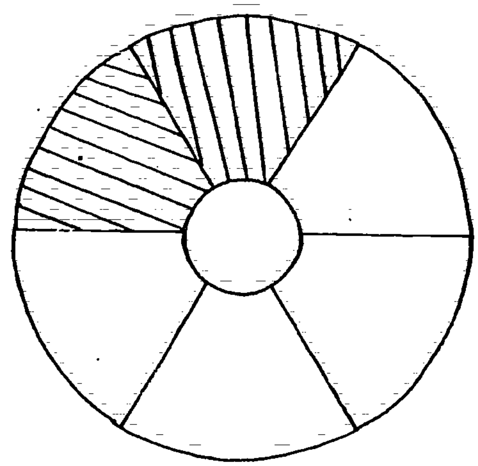
APPENDIX 6

The visual aid used on the FRACT9 task



3 OSAAN, MUKAAN 1

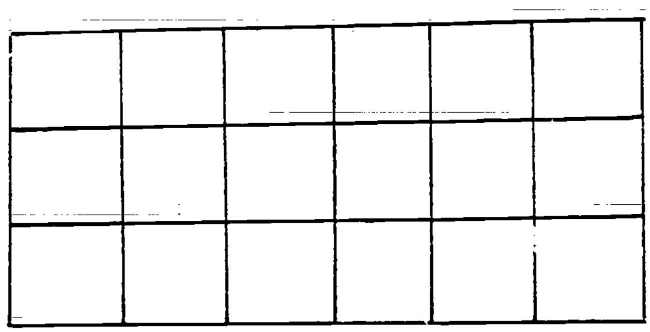
= Divided into 3 parts; take 1



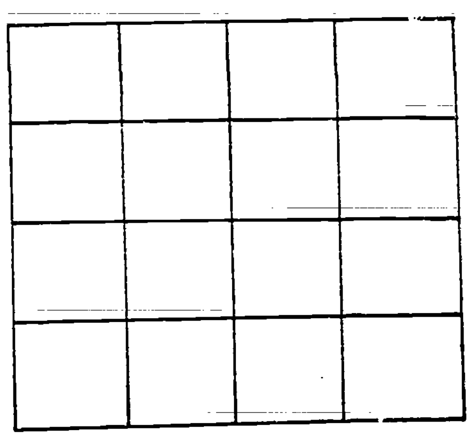
6 OSAAN, MUKAAN 2

= Divided into 6 parts; take 2

18 PIECES
6 BOYS



16 PIECES
4 GIRLS



APPENDIX 8. Correlations Between the Variables Relating to Multiplication and Division Skills MULTI1-5, MEAD1-6 and PARD1-6 (N=36).

	MEAD1	MEAD2	MEAD3	MEAD4	MEAD5	MEAD6	PARD1	PARD2	PARD3	PARD4	PARD5	PARD6
MULTI1	42	42	46	58	51	36	65	37	57	30	19	47
MULTI2	52	38	39	41	67	44	44	38	41	22	23	29
MULTI3	58	52	44	55	51	48	48	41	45	27	58	39
MULTI4	67	29	63	47	55	34	46	54	27	03	27	24
MULTI5	52	47	65	50	56	41	46	34	48	48	30	51
MEAD1							68	39	40	23	46	27
MEAD2							59	28	72	54	55	50
MEAD3							42	35	24	35	41	30
MEAD4							63	32	55	41	54	46
MEAD5							55	42	40	41	29	47
MEAD6							52	31	42	48	21	47

APPENDIX 9. The "Inner" Correlations of the Component Variables of the Sum Variables TR, EC, LC, MCP, SABY, UPBY, DOBY, MULTI, MEAD and PARD.

	CTR2	LTR1	LTR2	LTR3	LTR4
CTR1	87	69	69	69	69
CTR2		46	46	46	46
LTR1			100	100	100
LTR2				100	100
LTR3					100

	ILC2	ELC1	ELC2
ILC1	66	85	71
ILC2		76	87
ELC1			86

	MCP2	MCP3
MCP1	72	72
MCP2		100

	EC2	EC3
EC1	85	85
EC2		100

	SABY3	SABY4	SABY5	SABY6
SABY2	50	60	41	46
SABY3		79	26	57
SABY4			37	55
SABY5				38

	UPBY2	UPBY3	UPBY4
UPBY1	88	64	36
UPBY2		74	39
UPBY3			51

	MULTI2	MULTI3	MULTI4	MULTI5
MULTI1	58	49	51	66
MULTI2		47	54	51
MULTI3			40	47
MULTI4				67

	DOBY2	DOBY3	DOBY4
DOBY1	50	66	56
DOBY2		38	61
DOBY3			66

	MEAD2	MEAD3	MEAD4	MEAD5	MEAD6
MEAD1	50	58	64	45	32
MEAD2		41	65	50	53
MEAD3			66	54	26
MEAD4				52	35
MEAD5					74

appendix continues

APPENDIX 9. (cont.)

	PARD2	PARD3	PARD4	PARD5	PARD6
PARD1	39	57	23	30	43
PARD2		21	07	18	13
PARD3			44	46	49
PARD4				25	43
PARD5					53

APPENDIX 10. Intercorrelations of the Sum Variables SABY, URBY, DOBY, MULTI, MEAD, PARD, MCP, CTR, LTR, TR, LC, EC, IPC and a Chronological Age Variable AGE (N=36).

	UPBY	DOBY	MULTI	MEAD	PARD	MCP	CTR	LTR	TR	LC	EC	IPC	AGE
SABY	48	54	78	70	56	35	16	33	28	27	26	51	05
UPBY		86	67	67	51	20	37	30	37	25	42	23	-05
DOBY			70	65	55	21	41	31	40	29	39	21	-03
MULTI				80	66	31	32	32	36	38	34	38	07
MEAD					76	38	23	24	26	27	27	43	10
PARD						23	13	25	22	35	15	24	04
MCP							27	02	15	46	36	48	-06
CTR								57	86	54	78	02	23
LTR									91	44	75	04	26
TR										54	86	02	28
LC											53	23	17
EC												17	17
IPC													01

APPENDIX 11. The Correlations of the Sum Variables SABY, UPBY, DOBY, MULTI, MEAD, PARD, MCP, TR, LC, EC, IPC and the Variable AGE with the Sum Variables RATIO, FRACT and PART (N=36).

	RATIO	FRACT	PART
SABY	34	50	51
UPBY	30	34	35
DOBY	25	34	39
MULTI	48	51	60
MEAD	30	45	48
PARD	42	36	48
MCP	22	37	59
TR	24	23	45
LC	17	29	65
EC	25	17	53
IPC	-03	31	40
AGE	-11	04	-01
RATIO		45	37
FRACT			40

appendix continues

APPENDIX 11. (cont.)

Hierarchy Values, Order of Difficulty and Solution Frequencies on the IPC Test

Item	Combination	Solution Srequency
1	9,16	36
2	15,8	36
3	14,17	36
4	12,17	36
5	18,11	36
9	11,9,7	36
7	8,10,12	35
6	9,14,5	33
12	6,9,12,15	30
10	12,18,13	29
8	8,17,13	28
13	10,16,14,8	22
11	7,5,18,14	21
14	11,16,13,18	15
15	12,6,9,17	14
CR= 0.90, MMR = 0.74 ja PPR = o.62		

COMPUTER ANALYSIS OF COGNITIVE PROCESSES IN PROBLEM SOLVING

Ole Björkqvist

INTRODUCTION

During the last 15 years theory and research about thinking have acquired a richness in details that previously simply was not there. This development towards sophistication coincides with the emergence of the computer as a general research tool. Indeed, in the information processing parad ... the computer analogy of the brain plays a prominent part.

The interest in the cognitive processes of students in schools is connected with the principle of individualization of education. With detailed knowledge of thought processes it is assumed that a matching of teaching methods and learning characteristics is possible.

In mathematics, the traditional analysis of cognitive processes is an indirect one. The object is the product - the calculations written down on a paper, or something equivalent. More direct methods involve the use of interviews or "thinking loud" while solving problems. The tape recorder and the video tape recorder provide means of repeating the sequence for later detailed analysis.

The microcomputer now takes this evolution of methodology one step further. Besides being a widely used instructional tool, it can also be used as a powerful research instrument. The requirements include program routines that store each depression of a key into a protocol memory of sufficient size and preferably a possibility of recording the time elapsed between

successive key depressions. The later is used if the time variable is of interest, for instance if the various stages of solution are of unequal difficulty to a student.
There are limitations, of course. Only certain kinds of problem-solving episodes can be investigated - primarily those in which the presentation of the problem and all the work is done on the computer. If the problem involves paper work, the products are of a traditional type and cannot be immediately included in the computer analysis.

A great advantage of the computer is the fact that many of the responses can be classified automatically, and if statistics or calculations based on the classifications are needed, the results can be printed out immediately after the problem session. In other cases a classification cannot be made in advance. Then the printout can be scanned for patterns and the research itself of exploratory type.

Since the problem session is virtually soundless, the microcomputer allows the research on cognitive processes to be taken from the clinical laboratories into the classrooms. This is a welcome development, as the value of clinical results has always been at least somewhat in question. It can be argued that the use of the microcomputer makes the problem solving research artificial in a different way, but this is not necessarily so. The computers are here to stay in mathematical education, and whether we like it or not, they are changing the type of problems that will be dealt with in schools. The changes in the classrooms and the changes in research methods converge on the same medium, the computer, and thus there is hope for research relevant to actual practice.

A DESCRIPTION OF THE TASK

In the research reported here, emphasis was put on developing the necessary routines for the simultaneous use of the microcomputer as an educational tool and a research instrument.

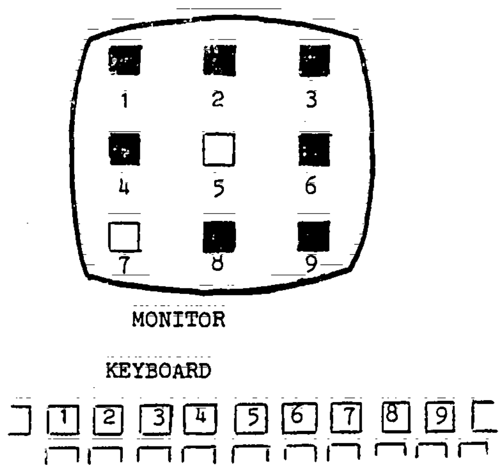


FIGURE 1. The Keyboard and the Monitor with a Pattern of Dots.

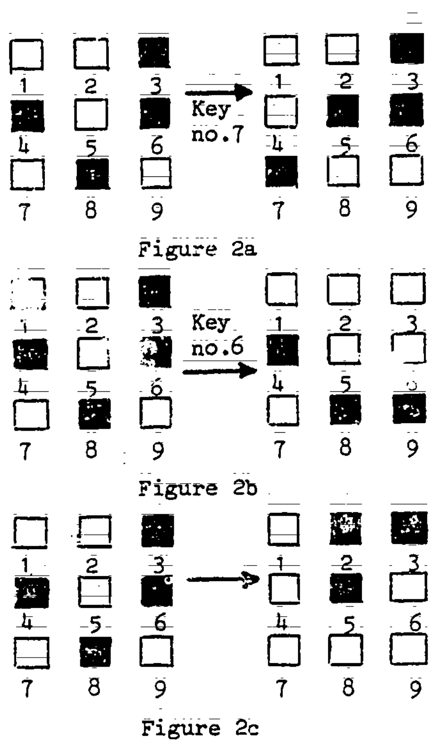


FIGURE 2. The Effects of some Key Depressions.

Being a first attempt, the work was not performed in a classroom, but in a research laboratory. The subjects were upper secondary school students.

The subjects were given a task in the form of a problem presented on a microcomputer screen. To motivate the students, the problem selected was an abstract game based on a commercial but not too well known electronic game, MERLIN (reg. trade mark). When adapted to a microcomputer, it has the advantage that it requires the use of very few keys, and thus the responses are easily classified.

The problem solver works with a 3 x 3 pattern of dots on the monitor (Figure 1), each dot identified by one of the digits 1 - 9, and some of them initially lighted, while others are dark. By pressing the numerical keys on the keyboard, also numbered 1 - 9, the solver can change the parity of a portion of the dots, so that some initially doark dots turn light, while some originally light dots darken. However, the action of each key is unknown at the beginning of the game, and the preliminary stages involve detection of their effects. There is a symmetry to be discovered, but it is rather well disguised by the simultaneous lightening and darkening going on.

The keys corresponding to the corner dots, those numbered 1, 3, 7, and 9, change the parity of exactly the corresponding corner dot, but also change the parity of the three dots next to it; for instance, pressing key number 7 changes the parity of the dots numbered 4, 5, 7, and 8 (Figure 2a).

A key corresponding to a side dot, those numbered 2, 4, 6, or 8, changes the parity of exactly the same side dot, but also changes the parity of the two other dots along the same side; for instance, pressing key number 6 changes the parity of the dots numbered 3, 6, and 9 (Figure 2b).

The key numbered 5 has the special effect of changing the parity of the dots numbered 2, 4, 5, 6, and 8 (Figure 2c).

The object of the adapted version of the game is to find a sequence of key depressions that lights all the dots. A mathematical analysis of the game shows this always to be possible (Gibbs 1982). For each starting configuration there is a unique optimum solution which never involves more than 9 key depressions to be performed in no specific order. A later depression of a key that has been used just nullifies the effect of the previous depression. Thus, each key either has to be touched exactly once or must not be touched at all. However, to find that out, or even to develop a strategy, the solver generally goes through long repetitive sequences. Games of more than 200 key depressions are not unusual for a beginner.

The game is not very sensitive to the specific starting configuration. In fact, using a completely random strategy, the expected length of a game varies between 511 and 607 key depressions. In that sense all the configurations that arise during a game are comparable to the starting position, and the number of games played is less important for research purposes than the total number of situations a specific subject has faced.

PROCEDURE

After several tryouts, the game was played by 10 high-ability senior secondary boys, all motivated enough to play it for up to two hours. Protocol materials from 81 games were accumulated, the total number of positions to be analyzed numbering 3411. This is typical of the capacity of a computer- the enormous amount of information that can be gathered using just a small number of subjects.

	configuration 0 = dark 1 = light	optimal keys to press in given situation	key chosen	R right W wrong	E dark F light	corner/side/5-key	seconds to contemplate	cumulative time	average time per key depression
0	000010100	2356	1	W/5	E/7	C	7	7	7
1	110100100	23561	3	R/5	E/5	C	3	10	5
2	101111100	2561	2	R/4	E/3	S	1	11	3.66667
3	010111100	561	1	R/3	E/4	C	2	13	3.25
4	100001110	56	9	W/7	E/6	C	6	19	3.8
5	100010111	569	2	W/6	E/4	S	2	21	3.5
6	011010111	5692	5	R/4	F/6	S	6	27	3.85714
7	001101101	692	1	W/6	E/4	C	3	30	3.75
8	111011101	6921	4	W/5	E/2	S	3	33	3.66667
9	011111001	69214	4	R/5	F/6	S	1	34	3.4
10	111011101	6921	8	W/5	E/2	S	7	41	3.72727
11	111011010	69213	4	W/4	E/3	S	3	44	3.66667
12	011111110	692134	2	R/6	F/7	S	8	52	4
13	100111110	69134	6	R/5	F/6	S	2	54	3.85714
14	101110111	9134	3	W/5	F/7	C	5	59	3.93333
15	110101111	91343	5	W/4	E/2	S	7	66	4.125
16	100010101	913435	5	R/6	F/4	S	3	69	4.05882
17	110101111	91343	3	R/5	E/2	C	3	72	4
18	101110111	9134	1	R/4	F/7	C	5	77	4.05263
19	011000111	934	7	W/6	E/5	C	5	82	4.1
20	011110001	9847	9	R/4	F/[illegible]	C	2	84	4
21	011101010	847	5	W/6	E/[illegible]	S	4	88	4
22	001010000	8475	7	R/4	E/7	C	9	97	4.21739
23	001100110	845	1	W/6	E/5	C	7	104	4.33333
24	111010110	8451	3	W/5	F/3	C	2	106	4.24
25	100001110	84513	3	R/5	E/5	C	1	107	4.11538
26	111010110	8451	9	W/5	E/3	C	2	109	4.03704
27	111001101	84519	9	R/5	F/6	C	1	110	3.92857
28	111010110	8451	5	R/4	F/6	S	2	112	3.86207
29	101101100	841	9	W/6	E/4	[illegible]	2	114	3.8
30	101110111	8419	3	W/5	F/7	[illegible]	[illegible]	120	3.87097
31	1101[illegible]1111	84193	1	R/5	F/7	[illegible]	[illegible]	130	4.0625
32	0000[illegible]1111	8493	9	R/4	F/5	[illegible]	[illegible]	134	4.06061
33	000000100	843	7	W/6	F/1	C	4	138	4.05882
34	000110010	8437	7	R/4	E/6	C	2	140	4
35	000000100	843	1	W/6	E/3	C	2	142	3.94444
36	110110100	8431	9	W/5	E/4	C	3	145	3.91892
37	110101111	84319	3	R/5	E/2	C	7	152	4
38	101110111	8419	7	W/5	F/7	C	18	170	4.35897
39	10100[illegible]101	84197	5	W/4	E/6	S	3	173	4.325
40	111111011	841975	7	R/6	E/1	C	24	197	4.80488
41	111001101	84195	3	W/4	F/6	C	7	204	4.85714
42	100010101	841953	6	W/3	E/5	S	3	207	4.81395
43	101011100	8419536	6	R/7	F/5	S	1	208	4.72727
44	100010101	841953	4	R/6	E/5	S	3	211	4.68889
45	000110001	81953	8	R/5	E/6	S	4	215	4.67391
46	000110110	1953	9	R/4	E/5	C	4	219	4.65957
47.	000101101	153	3	R/3	E/5	C	5	224	4.66667
48	011110101	15	2	W/7	F/6	S	3	227	4.63265
49	100110101	152	2	R/3	E/4	S	1	228	4.56
50	011110101	15	1	R/2	E/3	C	1	229	4.4902
51	101000101	5	5	R/1	E/5	S	4	233	4.48077

FIGURE 3. Sample Printout.

The following data were recorded by the computer (an example of a printout is given in Figure 3):

1. Identifying number of position in game.
2. Configuration of lighted dots.
3. Set of optimal (right) keys to press in the given situation.
4. Key actually chosen.
5. Correctness of key.
6. Parity of the dot corresponding to the key.
7. Position of the corresponding dot (corner/side/middle).
8. Time taken to contemplate situation.
9. Cumulative time.
10. Cumulative average time per situation.

Included were some of the variables characterizing the situation and assumed to influence the decision of the problem solver, as well as variables describing the response. Based on the output, new variables were defined such as "reflectivity of thought", calculated as the ratio of the time taken to contemplate a given situation to the average time per situation. Another secondary variable of interest to the analyses was the frequency with which a subject nullified a previous attempt by repeating it. This would most often be a repetition of a single key depression, but there occurred instances where up to four key depressions were nullified through repetition, showing that the subject [illegible]ed to return to a specific configuration and restart from it. The proportion of such repetitive key depressions defined the variable "repetition".

Some variables were expected to show a correlation with learning, i.e., with time on task. To investigate such dependence, which conceivably might show within the course of one game, games of more than 60 key depressions were subdivided into equal parts of not more than 60 (and not less than 30) key depressions. The number of such periods of work provided a progressive measure of the interaction between the student and the computer. The periods of work are termed intervals in the following.

SELECTED RESULTS

With only 10 subjects, any group measures are, of course, less illuminative than individual profiles and results showing intraindividual change. The large number of positions encountered by each subject makes the latter reasonably reliable. The group measures, such as mean time taken to contemplate a situation and preferences for certain keys, are of interest to show the general properties of the problem and to indicate psychological consequences of its structure.

The total effective playing time for the subjects varied between 30 minutes and 139 minutes, as calculated directly from the computer output. However, as the mean time taken to contemplate a situation also showed large variation, from 3.8 seconds to 29.5 seconds (not particularly highly correlated with the total playing time), the number of situatior [illegible] ered by each subject varied from only 98 to a hig [illegible] n terms of intervals of work, the range was [illegible] 15, with the exception of two extremes at 2 and 24. Th [illegible] ater the number of situations, the more information was gathered about the subject, of course. On the other hand, extremes in the form of short total playing time or only a small number of situations per game may indicate fatigue or mastery of the problem, respectively, and they are thus particularly interesting to the study of individual differences in working with a microcomputer.

In Figure 4 it is shown why the mean time taken to contemplate a situation, as calculated from the total output for one subject, is less satisfactory than a graph showing the change of that same variable when the subject proceeds to ne intervals of work. For the three subjects, the overall means were approximately equal, but the curves are far from flat and the shape of each one may be interpreted differently.

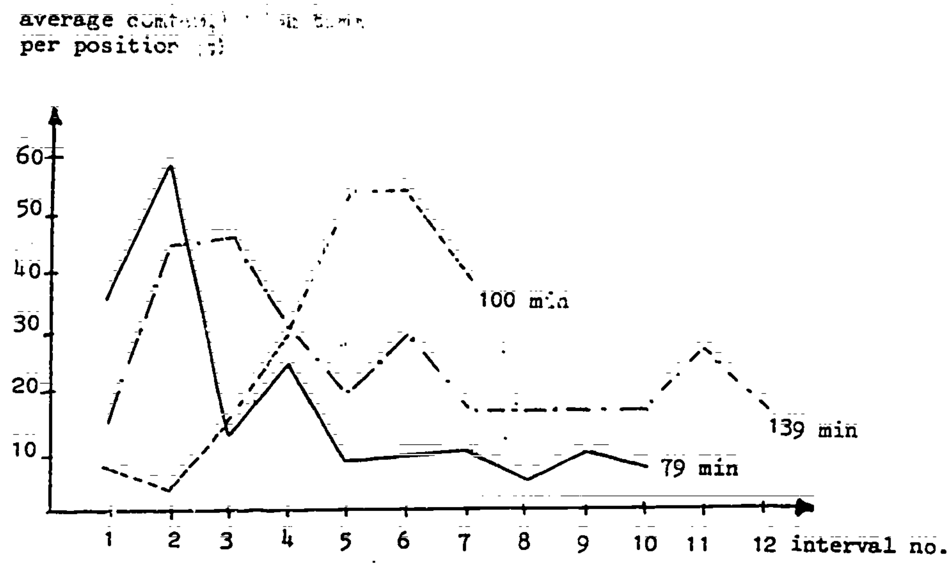


FIGURE 4. Average Contemplation Time per Position during Progressive Intervals of Work for Three Subjects. Total Time on Task also Indicated.

The 3 x 3 pattern of dots itself included three types of dots, with possibilities of preference for certain kinds, either as a psychological bias or as a conscious strategy. Table 1 gives the distribution of the keys pressed, according to their classification as corner, side, or 5 keys. For each game, the "optimal" keys in the starting configuration have been subtracted, since they were forced rather than open choices. With an unbiased selection of keys, the percentages would be expected to be 44.4, 44.4, and 11.1, respectively. Key number 5, however, was used significantly more often than one ninth of the time, as might be anticipated from the central position of the number 5 dot and the special effect of the number 5 key. The differences between individual subjects in the use of the corner and side keys, e.g. between subjects 3 and 4, point toward the use of strategies that involve the symmetries of the pattern of dots.

In the overall preference order of the keys, no. 5 was followed by nos. 6 (12. 4 %), 2 (11.6 %), 3 (10.9 %), 4 (10.4 %), 1 (10.2 %), 8 (10.2 %), 9 (9.3 %), and 7 (8.9 %). The less frequent use of the keys corresponding to the third row of dots reflects a tendency to use the normal direction of reading, starting from the top, in the systematic attempts to find out the effect of each key. Interestingly enough, there was a significant preference for the rightmost column (39.5 %) over the leftmost column (32.5 %).

TABLE 1. Distribution of Key Depressions (Percent) according to the Classification of the Keys.

Subject	Corner	Side	5-key
1	38.8	41.1	20.1
2	41.2	44.3	14.4
3	45.7	35.8	13.5
4	31.2	58.4	10.4
5	41.7	40.0	18.3
6	43.6	46.9	9.5
7	42.2	42.2	15.6
8	32.3	55.4	12.3
9	46.5	38.6	14.9
10	30.5	43.5	26.0
Total	39.4	44.6	16.0

Turning now to variables that are more closely connected with actual strategies of solution, the repetition of key depressions showed an interval dependence typical of learning. In Figure 5, the proportion of key repetitions for the same three students as in Figure 4 is plotted as a function of the interval of work. All the students had at least a short period of random attempts, which typically included a greater proportion of repetitions, during the initial play.

As is evident from the list of primary variables (Figure 3), the parity of the dot corresponding to a key was hypothesized to be an important ... its selection. This would be true during the period of random attempts, and the effect would be expected to remain, since the touch of a certain key always does have an effect on the corresponding dot, in addition to various others.

In fact, the structure of the problem made it possible to judge the influence of the parity on the choice of key. This was so because there is, on the average, no difference in correctness between the two parities, even though for a specific configuration keys corresponding to dark dots may be predominantly right and keys corresponding to light dots

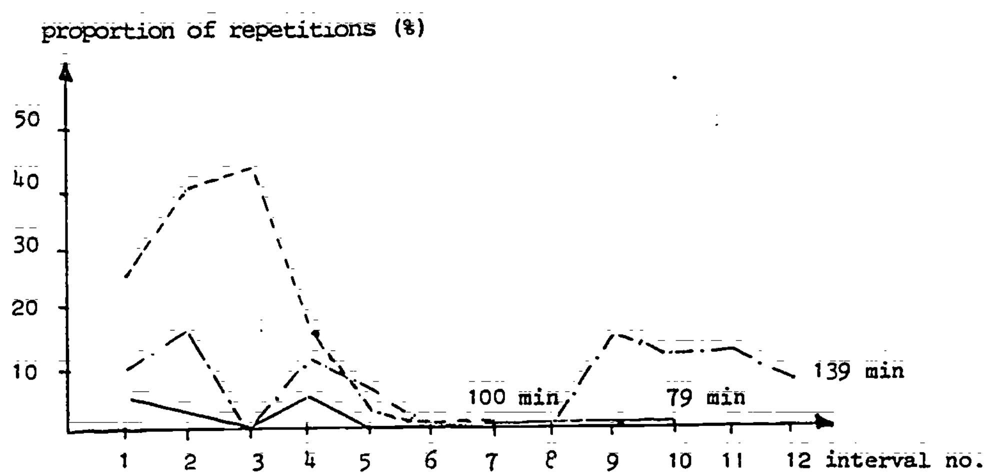


FIGURE 5. Proportion of Key Repetitions for Three Subjects.

predominantly wrong, or vice versa. This can be mathematically demonstrated via symmetry arguments. Considering all the 512 possible configurations of dots, and the nine possible choices of key for each configuration, one quarter of the key depressions are right choices corresponding to dark dots, one quarter right choices corresponding to light dots, one quarter wrong choices corresponding to dark dots, and one quarter right choices corresponding to light dots.

However, the subjects deviated from this straight-forward distribution of key depressions. As expected, there was an overall preference for keys corresponding to dard dots, the percentages being 58 to 42 in favor of dark dots (Table 2). The key depressions corresponding to light dots were evenly divided between right and wrong choices, in accordance with probability expectation. The key depressions corresponding to dark dots, on the other hand, were clearly more often right than wrong. This seems to reflect a characteristic of some conscious strategies employed - pressing sequences of keys corresponding to dark dots so that the total number of dark dots is decreased. In a number of cases the configuration itself gives a clue to likely correct choices. The high incidence of right choices corresponding to dark dots thus is partly an artifact of the game, notably the last stage of it, and partly a reflection of the characteristics of the successful strategies preferred by the players.

Facing a configuration which is dominated by light dots, the a priori probability of choosing a key corresponding to a light dot may be great enough to overcome the tendency to choose keys of the opposite kind. Thus there would be expected to exist a level where the two tendencies balance each other. This was analyzed by calculating the average number of dark dots in the situations met by a subject, as a function of the kind of key chosen. The results for the different subjects are given in Table 3.

TABLE 2. Distribution of Keys Pressed (Percent) according to Parity (E = dark, F = light) and Correctness (R = right, W = wrong).

Subject	RE	RW	WE	WF	E	F
1	27.7	27.5	19.7	27.2	47.4	52.6
2	30.6	25.1	23.3	21.0	53.9	46.1
3	37.5	19.0	24.7	18.8	62.2	37.8
4	44.3	17.7	19.2	18.7	63.5	36.5
5	34.1	19.7	22.1	24.1	56.2	43.8
6	31.0	24.2	22.[illegible]	22.7	53.1	46.9
7	40.8	13.3	25.5	20.4	66.3	33.7
8	41.9	21.8	17.[illegible]	18.4	59.8	40.2
9	33.2	19.9	26.3	18.9	61.3	38.7
10	35.0	21.8	[illegible] 5	19.6	58.5	41.5
Total	35.6	21.0	[illegible].6	20.8	58.2	41.8

The grand average of 4.07 (statistical expectation 4.50) shows the magnitude of the tendency towards play with as many light dots as possible, i.e., to eliminate the dark dots. When the key chosen was associated with a dark dot, the average number of dark dots in the configuration was 4.41 (statistical expectation 5.00). The same average when the chosen key was associated with a light dot was 3.59 (sstatistical expectation 4.00). The last number shows that an average of 5.41 dots had to be lit on the screen for the a priori probability of choosing one of the corres[illegible]ding keys to balance the opposite tendency. In Table 3 averages are also given for the number of dark dots when the correctnesss of the chosen key is considered in conjunction with its parity.

Again, it must be emphasized that the averages for individual subjects in certain cases reflect strategies adhered to. For quite a while subject number 1 tried to make all the dots dark. His notion (which was expressed when he was interviewed afterwards) was that it should be easy to light all the dots from that configuration. Of course this involves an understanding that symmetry can be used. Table 2 shows that to fulfil his plan he had to press keys corresponding to light dots more often than keys corresponding to dark dots, being the only subject to do so. In Table 3, the value 4.29 (for the average number of dark dots in situations where a key corresponding to a light dot was chosen) is a deviating number which is specifically relevant to that strategy.

TABLE 3. Average Number of Dark Dots in Configuration according to Parity (E = dark, F = light) and Correctness (R = right, W = wrong) of Key Pressed.

Subj.	RE	RF	WE	WF	R	W	E	F	Tot
1	4.82	4.58	4.65	3.98	4.70	4.28	4.75	[illegible]	4.51
2	4.73	3.82	4.31	3.52	4.31	3.94	4.55	[illegible].58	4.15
3	3.98	3.27	4.11	2.89	3.74	3.58	4.03	3.08	3.67
4	4.63	4.17	4.59	3.95	4.50	4.27	4.62	4.05	4.41
5	4.35	3.90	4.20	3.22	4.19	3.69	4.29	3.52	3.96
6	4.32	3.46	4.55	3.18	3.94	3.85	4.41	3.33	3.90
7	4.48	3.69	4.68	3.70	4.28	4.24	4.55	3.70	4.27
8	4.44	3.46	4.91	3.39	4.11	4.14	4.58	3.43	4.12
9	4.08	3.19	4.04	3.40	3.75	3.78	4.06	3.29	3.76
10	4.23	3.88	4.26	3.17	4.10	3.77	4.24	3.55	3.96
Tot	4.41	3.74	4.43	3.44	4.16	3.96	[illegible]	3.59	4.07

The differences between the strategies of the subjects were great. The same subject also would vary his attempts, e.g., selecting keys corresponding to side dots for a long while and then turning to the corner dots. The futility of some efforts

19	101010111	123	6	W/6	E/3	S	57	1183	59.15
20	100011110	1236	4	W/5	E/4	S	2	1185	56.4286
21	000111010	12364	6	R/5	F/4	S	55	1240	56.3636
22	001110011	1234	9	W/5	F/5	C	119	1359	59.087
23	001101000	12349	3	R/5	F/3	C	11	1370	57.0833
24	010110000	1249	9	R/4	E/6	C	185	1555	62.2
25	010101011	124	6	W/6	F/5	S	210	1765	67.8846
26	011100010	1245	3	W/5	F/4	S	17	1782	66
27	011100101	12463	1	R/5	E/4	C	12	1794	64.0714
28	101010101	2463	6	R/4	E/4	S	38	1832	63.1724
29	100011100	248	4	R/3	E/5	S	4	1836	61.2
30	000111000	28	8	R/2	E/6	S	4	1840	59.3548
31	000111111	2	2	R/1	E/3	S	2	1842	57.5625

0	110010101	1235678	1	R/7	F/5	C	34	34	34
1	000100101	235678	6	R/6	E/6	S	12	46	23
2	001101100	23578	5	R/5	E/5	S	26	72	24
3	011010110	2378	6	W/5	E/4	S	14	86	21.5
4	010011111	23786	7	R/5	F/6	C	14	100	20
5	010101001	2386	4	W/5	F/4	S	18	118	19.6667
6	110001101	23864	3	R/5	E/4	C	3	121	17.2857
7	101010101	2864	6	R/4	E/4	S	15	136	17
8	100011100	284	4	R/3	E/5	S	7	143	15.8889
9	000111000	28	8	R/2	E/6	S	2	145	14.5
10	000111111	2	2	R/1	E/3	S	1	146	13.2727

0	000101011	1679	9	R/4	E/4	C	19	19	19
1	000110000	167	4	W/6	F/2	S	22	41	20.5
2	100010100	1674	6	R/4	E/6	S	7	48	16
3	101011101	174	4	R/3	E/3	S	34	82	20.5
4	001111001	17	5	W/7	F/5	S	97	179	35.8
5	011000011	175	6	W/3	E/5	S	7	186	31
6	010001010	1756	5	R/4	E/6	S	80	266	38
7	000110000	176	4	W/6	F/2	S	2	268	33.5
8	100010100	1784	3	W/5	E/6	C	76	344	38.2222
9	111001100	17643	9	W/4	E/4	C	20	364	36.4
10	111010111	176439	8	W/3	F/7	S	14	378	34.3636
11	111010000	1764398	2	W/2	F/4	S	3	381	31.75
12	000010000	17643982	5	W/1	F/1	S	28	409	31.4615
13	010101010	176439825	8	R/9	F/4	S	5	414	29.5714
14	010101101	17643925	2	R/8	F/5	S	4	418	27.8667
15	101101101	1764395	8	W/2	E/3	S	30	448	28
16	101101010	17643958	2	W/1	E/4	S	1	449	26.4118
17	010101010	176439582	4	R/9	F/4	S	4	453	25.1667
18	110001110	17639582	6	R/8	F/5	S	1	454	23.8947
19	111000111	1739582	7	R/7	F/6	C	47	501	25.05
20	111110001	139582	6	W/3	E/3	S	2	503	23.9524
21	110111000	1395826	9	R/7	E/4	C	123	626	28.4545
22	110100011	135826	2	R/6	F/5	S	5	631	27.4348
23	001100011	13586	7	W/4	E/5	C	6	637	26.5417
24	001010101	135867	1	R/6	E/5	C	28	665	26.6
25	111100101	35867	4	W/4	F/6	S	58	723	27.8077
26	011000001	358674	2	W/3	F/3	S	7	730	27.037
27	100000001	3586742	7	R/7	E/7	C	37	767	27.3929
28	100110111	358642	3	R/6	E/3	C	4	771	26.5862
29	111101111	58642	5	R/5	E/1	S	39	810	27
30	101010101	8642	2	R/4	E/4	S	9	819	26.4194
31	010010101	864	8	R/3	E/5	S	2	821	25.6563
32	010010010	64	6	R/2	E/6	S	2	823	24.9394
33	011011011	4	4	R/1	E/3	S	1	824	24.2353

FIGURE 6. Documentation of Three Consecutive End-games by One of the Subjects.

would be quite obvious, and the strategy consequently abandoned by the student. In this respect a similarity with traditional problem solving in mathematics may be pointed out - when students do not know what to do they may try one strategy after the other until they find a solution, either by way of sudden insight or by accident. When they have found a strategy that works, it is tempting to use it all the time without looking for another, perhaps better, strategy.

In the printout there were to be found instances where intermediate goal configurations could be clearly identified. Figure 6 shows the way one of the subjects ended three consecutive games. He obviously had learned that from a situation with the number 2, 4, 6, and 8 dots dark you can obtain the final goal by pressing exactly those same dots. The symmetry of the configuration is certain to make it easy to memorize.

The variable "reflectivity of thought" revealed many interesting details with regard to specific configurations of dots. To screen out the situations that corresponded to the longest relative contemplation times, an arbitrary value of 2 was taken as an operative criterion. Only those situations with values exceeding 2 were considered interesting enough to be included in the analysis. Their number was about 11 % of all the positions in the original analysis. The highest value found was 10, i.e., the students never contemplated a particular situation longer than 10 times the average contemplation time during a certain interval of work.

The configurations that most often were associated with high reflectivity of thought were characterized by either a small number of dark dots (= apparent closeness to the final goal), some obvious symmetry, or both. The six most frequent configurations are depicted in Figures 7a-f. For the purposes of this classification, any configurations that may be obtained by rotation or reflection were considered equivalent.

Figure 7a shows a configuration that is intrinsically difficult. The correct keys to press are the keys numbered 2, 4, 6, 7, and 9. However, out if the 21 occurrences associated with careful consideration (high reflectivity of thought) only 4 led to depression of a correct key. Figure 7b, on the other hand, is intrinsically easy. There are 6 correct keys to press, and 12 occurrences out of 13 actually led to a correct key depression. The difficulties of the other configurations fell between those two extremes.

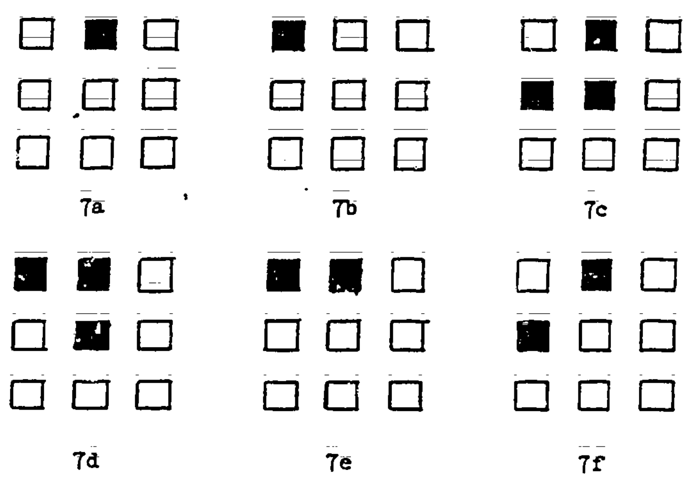


FIGURE 7. The Six Configurations of Dots Most Frequently Associated with High Reflectivity of Thought.

The configuration in Figure 7a, being the most frequent high-reflectivity configuration and at the same time an interestingly difficult one, was also investigated using the full documentation. Altogether it occurred 81 times, 30 of which led to a correct choice of key. The distribution of the attempts is given in Figure 8 along with the high-reflectivity attempts, using the pattern of the dots. The number 2 dot, the only dark one, evidently acted as the center of attention. However, the longer the students contemplated the situation, the more apt were they to choose keys corresponding to the neighboring dots rather than the number 2 dot itself. Since the number 1, 3, and 5 keys are all wrong whereas the number 2 key is correct, long contemplation in this singular case does not seem to benefit the process of solution.

17	19	12		5	2	6
5	16	4		2	3	0
1	6	1		0	3	0

FIGURE 8. The Distribution of the Key Selections in Response to Configuration 7a. On the Right, the Distribution of the Hight-Reflectivity Responses.

SUMMARY

The main reason for this piece of research was not to find individual similarities or dissimilarities in the solution of the specific problem, but rather to develop a methodology for problem solving research in general, with an emphasis on school mathematics in as realistic situations as possible. Gradually, technically more difficult analyses should be possible as microcomputers become more versatile.

Even in this simple problem, where the responses of the students were strictly limited, a number of interesting features of the solution process were identified and quantified. The structure of the problem was such as to admit a variety of strategies, from random attempts to the use of sophisticated symmetry arguments. At the same time, the number of dark dots on the monitor acted as a distractor rather than a real measure of advance towards the final goal. The switches between strategies, the reflectivity of thought at particular stages, and the return to previous configurations via repetition of key depressions were some elements of the solution process that could not have been studied equally closely without the microcomputer functioning as a research tool.

Another, slightly different set of goals in mind for the research were those connected with computer education - to know how to teach students how to use computers efficiently you need to know details about the way they think while they work with computers.

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APPLICATIONS IN THE JUNIOR SECONDARY SCHOOL LEVEL MATHEMATICS

Pekka Kupari

INTRODUCTION

In the teaching of school mathematics applications have an essential role. Applications help to prove the usefulness of mathematics, by means of applications mathematical techniques can be usefully practiced, and their use promotes conceptual understanding. The value of applications has always been understood and recognized, but it is generally known that at the present time, the position of applications in school mathematics leaves a lot to be desired. The effect of this can be seen also in learning results. Several recent studies of school achievement at this age level (e.g. Foxman et al. 1980, Hart 1981, NAEP 1979) have indicated that the application skills of pupils are deficient and that the application process has become mechanized. These defects were clearly disclosed also by the extensive national assessment of mathematics teaching in the comprehensive school in Finland which was carried out in 1979. (Korhonen & Kupari 1983). As a result, applications have been given special emphasis in the effected curricular reforms and development programs for the improvement of teaching. This was the case also in Finland when the comprehensive school mathematics syllabus was reformed in autumn 1982.

What, then, are the reasons for this deplorable state of applications and how could we bring about a new and more inspiring phase - these are questions to which we should try to find some answers here. In the following, I shall try to provide a few themes for the deliberation of these issues. I am going to focus mainly on the single question: "How are applications used in teaching at this age level, and what problems emerge from this, in view of the learner's development?" Finally, I shall also present a few ideas on what could be done to help the teachers.

HOW ARE APPLICATIONS USED IN TEACHING?

The word "applications" is used to describe a wide range of different mathematical activities. In his excellent review on applications Burkhardt (1983) gives a good definition of applications: "An application involves the use of mathematics in describing a situation from outside mathematics, usually involving a mathematical model reflecting some aspects of that situation".

However, an application is always a relative concept in many ways. Applications can be viewed in relation to the different sub-factors of the teaching entity from the standpoint of: 1) learning material, 2) teaching practice, and 3) the pupil. It may be asked, then, to what degree the ambiguity of applications is a consequence of this relativity. Next, I shall discuss these factors in more detail.

What Is the Role of Applications in the Subject Matter?

We may begin by asking how applications are used in the subject matter. In principle, the position of applications in the learning context is one of these two: in the teaching situation, applications come last, or they are the carrying idea in teaching new stuff.

a) When applications follow the teaching of a new thing, they are used as a means to illustrate the new mathematical technique. In this case, we speak of illustrations, and the teaching method which is used is the traditional text-book centered exposition and exercise approach. Several illustrations of a given technique are usual. For example, linear growth might be illustrated by different shopping and travelling problems and, correspondingly, exponential growth by compound interest and radioactive decay.

b) When applications provide the central theme of instruction they deal with real situations from outside mathematics and the aim is to discuss these situations by means of different models. By way of example, Burkhardt mentions a study of personal finance which might look at income and expenditure or the use of a mathematical model of capital and interest. The many aspects and viewpoints of such realistic situations make the discussion strategically more demanding for the student, so that the technical demand of the mathematics has to be lower.

Among the major objectives in teaching pupils of this age is to accustom them to use mathematics in forming an organized perception of their environment and to develop the thinking skills of pupils. In the pursuit of these aims applications of real situations play a central role. On the other hand, the significance of illustrations is obvious in the acquisition of firm arithmetic skills, which is one of the main objectives of teaching at the primary level.

In dealing with "real situations" the student must outline and analyse the problem, he has to choose the strategy for solving the problem and he must form a model and decide on the techniques to reach the solution. In doing so, the student is compelled to rely on his own thinking a' ' he should be encouraged to do so by all means. The student should be encouraged to explain, argue and defend his own thoughts and ideas on solving the problem. Faulty solution efforts should be analysed together with the pupil, because they often help in leading the way to the correct solution. If pupils' thinking processes are maintained in this way, it will be sure to give them confidence in their own ability to cope with situations.

Why does the subject matter contain so few realistic applications? There are several reasons for this. First, teachers feel that these applications are messy and uncertain involving a wide range of open demands which makes them unpleasant. In dealing with these problems it is not certain that there is any single right answer. Therefore teachers think that these application problems are not decent mathematics which, in their opinion, is characterized by clarity and precision.

Secondly, teachers believe that realistic applications are too difficult for pupils. It is true that realistic problems do require a broader view and wider experience and maturity than imitative exercises. But since, according to Burkhardt, no signs of the development of these qualities can be seen in the mathematics classrooms it is unreasonable to set the pupils tasks for which they have no readinesses. It should be remembered, however, that during this transition phase when pupils change from children to young adults, these qualities should be allowed to develop.

Thirdly, applications of realistic situations demand a lot of time. At least as far as Finnish mathematics teachers are concerned, lack of time is regarded as a central problem. When, on the one hand, the increase of applications in an approved development trend, but the other hand, people want to retain the main emphasis of teaching on the practice of arithmetic skills, we drift to a disastrous situation. In order to include applications, the teaching process must be speeded up and this happens at the expense of teaching the fundamentals which are necessary for the applications. Thus the instruction deteriorates even more.

How to Teach Applications?

Secondly, we will look at applications in relation to teaching practices. Also in this respect two approaches are possible.

a) First, we may speak of standard models of situations in teaching of applications. These are important in building up a "tool kit" of useful models and techniques (a term used by Burkhardt). They also provide practice in mathematical techniques and reinforce understanding by providing concrete illustrations of abstract concepts. Standard models are taught didactically by exposition and imitative exercise. It is assumed, after a sufficient number of illustrations, that the student will be able to recognize the characteristics of the standard situation when he meets a similar situation (pattern recognition). An example of this might be calculating the two basic cases of percentages: how much is a given percent of a number and what percentage is a number of another?

However, learning outcomes indicate clearly that some points are easily misremembered or different cases get mixed up. Thus it would seem, that by using only standard models it is not possible to adopt new methods and to use them in practice. The reason for this is, that the repetition of similar situations cannot result in the formation of higher level concepts and generalizations which is the prerequisite for assimilation of things and which should be aimed at in this age period, according to Piaget and Ausubel (e.g. Bell et al. 1983, Resnick & Ford 1981).

b) On the basis of what has been said above, it is obvious that pupils are able to remember only a small fraction of the models which they employ in the standard applications. Evidence and experience have shown that even small deviations from the standard models of situations confuse students. Thus it is of the highest importance for every pupil to acquire skill and experience in tackling new situations. When a student is faced by a new situation he or she must resort to his/her own store of knowledge in choosing and, if necessary, in adapting models and techniques. As new situations help the pupil to distinguish the essential features of solution methods he/she also learns to make generalizations. The reason why new situations have not been very popular in classrooms follows to a great extent from the tradition of exposition and exercise in teaching where the tasks children are asked to perform involve close imitation of processes demonstrated by the teacher. The role of the teacher is that of a manager, explainer and corrector. It goes without saying, that after the instruction pupils succeed in these imitative exercises much better than in tasks where they have to select mathematics from their own "tool kit" of techniques.

In approaching new situations the didactic method based on exposition and exercise is not appropriate. The teaching methods have to include more open styles of teaching where the pupil leads at his or her own level with the teacher acting more as an adviser and fellow pupil. In practising the tackling and perception of situations the methods of general problem solving could be applied, for example following the model of Polya (Polya 1957, Resnick & Ford 1981).

What Is the Role of Applications from the Student's Viewpoint?

In the foregoing, applications have been discussed in relation to student's characteristics at this age level. Next, we will deal with one final dimension, in other words, the various ways in which pupils may experience application problems. According to Burkhardt (see classification on the next page) applications differ as far as the interest level of the problems to the students is concerned. According to the rating there are five kinds of applications: 1) action problems, 2) believable problems, 3) curious problems, 4) dubious problems and 5) educational problems. This is of course only one way of rating applications from the pupil's standpoint. However, it

Classification of Applications according to the Interest Level of Problems (Burkhardt 1983)

1) Action problems concerning decisions which will affect the student's own life.
   Example: Organizing one's time to balance the various things one needs or should like to do.

2) Believable problems are action problems related to the student's plans for the future or for someone you care about.
   Example: "How can I borrow some money most cheaply?" or "Is it worth my studying for two more years, or should I try to get a job?"

3) Curious problems are simply fascinating, intellectually, aesthetically, or in some other way.
   Example: "Why are total eclipses of the sun rarer than those of the moon?" or "How can sap be 'drawn' up a tree 100 meters high?"

4) Dubious problems are intended just to make one practise mathematics.
   Example: "Calculate the area of a right-angled triangle when its base is 6 cm and the adjacent acute angle is 45 ."

5) Educational problems belong to the category of dubious problems which illuminate some mathematical content or concept so beautifully and enticingly that students want to solve them.
   Example: "One drachma was invested at 5 % compound interest in the year 759 BC. To what sum has it grown?"

illustrates very well the point how many different kinds of problems there are and, at the same time, how few types of applications are used in classrooms. According to Burkhardt, applied mathematics has rarely aimed higher than curious problems and for many students it has been almost entirely discussion of dubious problems. Utilization of this whole range of applications would certainly be significant for the differentiation of teaching, because at this stage of education pupils' readinesses and motivation vary dramatically. We should remember however, that even educational problems always seem curious to some pupils and, correspondingly, some pupils experience believable problems as highly dubious. Therefore, the choice of application problems is of vital importance in mathematics teaching.

EMPIRICAL OBSERVATIONS

In Finnish comprehensive school mathematics teaching the role of applications is largely as described above. Teachers consider lack of time the main reason for limited use of realistic application situations in teaching. This in turn results from the fact that teaching is too strictly tied to the textbook. Since no change has occured in the amount of teaching contents, at least not in the direction of a cut down, although teaching time has been clearly reduced (and a decision made in spring 1985 will further reduce the number of weekly lessons in grade 9 from 4 to 3), it follows without saying, that time-consuming material will be eliminated or efforts made to speed up the teaching process. According to Leino (1984) the elimination has been directed to problems related to concrete contexts and to the number of applications. On the other hand, the emphasizing of applications and the increase in their number has been a recognized and desired development trend during the last ten years. This is illustrated well by the fact that the number of applications in one 8th-grade textbook almost trebled during 1974-79. Below, I will examine in greater detail learning results in the area of applications, as compared to other achievements, and the relationships between learning results and teaching. The scrutiny will be based on research material collected in 1979 during the national assessment of mathematics teaching in the comprehensive school.

The population of this extensive situational survey consisted of 2251 9th-graders and 116 teachers from 40 junior secondary schools. Data was collected both on cognitive learning results in mathematics and on structural, attitudinal and process features of the student, the teacher, the home and the school. The assessment of cognitive achievements focussed on measuring objectives defined as most essential (so-called basic objectives and core subject matter, Kouluhallitus 1975). Each student was presented with one out of nine test versions containing together a total of 180 items. Each version had been so constructed that, by using a one-parameter logistic model (Rasch's model), we could produce comparable scores for all students. The comparable student scores and item parameters also allowed estimations of various subscores, for example, regarding different behaviour categories.

The test items were classified according to Wilson's (1971) behaviour classification as follows: computation, comprehension and application and analysis. The number of items at each level was:

- computation	95 items	(53 %)
- comprehension	29 "	(16 %)
- application and analysis	56 "	(31 %)

In view of teaching, the weighting of the areas was correct in that the dominating role of knowledge and computation was clearly emphasized. On the other hand, the proportions of comprehension and application items did not reflect the true position of these areas in mathematics teaching.

TABLE 1. The Achievements of 9th-graders at Different Levels of Behaviour (Estimated Means of Relative Scores, Standard Deviations and Reliabilities, N=2184).

Level of behaviour	$\bar{z}$	s	rel.
Computation	0.49	0.22	.965
Comprehension	0.53	0.24	.897
Application and analysis	0.48	0.22	.943

The results in Table 1. are quite surprising. When analysing learning results, we had observed that a majority of students had very inadequate application and problem solving skills. The perception and analysis of items seemed to be especially difficult and therefore students' attempts to solve problems were totally mechanical. So, we were led to expect mastery of application items to be clearly inferior to mastery of computation items. This was, however, not the case, as achievements in all areas were very similar. On an average, students mastered 50 % of the items in different categories of behaviour.

Reasons for the uniformity of achievements in different areas may be sought in the items presented and in the nature of teaching. Firstly, many of the application and analysing items were illustrative i.e. verbally expressed arithmetic operations. Generally they contained only one operation (so-called one-step problems, NAEP 1979), the only additional difficulty compared with "mechanical" items thus being reading comprehension.

The items were now reclassified so that illustrations were placed either in the computation or comprehension category. An attempt was also made at utilizing factor analysis, but the factorizations of individual test versions did not produce viable solutions.

TABLE 2. The Achievements of 9th-graders at Different Levels of Behaviour after Reclassification of Items

Level of behaviour	$\bar{z}$	s	rel.
Computation	0.51	0.22	.969
Comprehension	0.55	0.23	.914
Application and analysis	0.40	0.23	.918

Now, the outcome based on reclassification is more along the predicted lines. Achievements in the application area were about 0.6 standard deviation weaker than those in the area of computation and this difference can already be considered comparatively great. In other words, the mastery of more realistic application items, which are important from the standpoint of students' application skills, was significantly weaker than could be assumed at first (Table 1.). Teachers considered these items a definite part of the core subject matter, although this kind of items had not, on an average, received very much attention in teaching.

After this, it was interesting to see what impact the weighting of teaching contents had on students' achievements.

TABLE 3. Teachers' Answers to the Question "What aspects do you emphasize in your mathematics teaching?" (N=114)

	Practical computation	Thinking and deduction skills	Application to everyday situations	Application to other subjects
Hardly ever	0.0	1.8	1.8	16.7
Sometimes	20.2	43.9	36.0	64.9
Frequently	79.8	54.3	62.2	18.4

TABLE 4. Teachers' Answers to the Question "How often do you give your students homework to the following effect?" (N=114)

	Items similar to those solved in the class	Applications of subject matter discussed in the class	Solutions not discussed in the class	Individual problems
Hardly ever	1.8	3.5	41.6	66.4
Sometimes	2.7	46.9	56.6	26.5
Frequently	95.5	49.6	1.8	7.1

According to results presented in Tables 3. and 4. teachers put strong emphasis on practical computation skill, although a good half of the teachers often underlined the importance of applying mathematics to everyday situations and of thinking and deduction skills. As for homework, the traditional exposition and exersice came out strongly, as almost all teachers gave homework based on problem types solved in the classroom. Again, about 50 % of teachers often used items requiring application of subject matter discussed in the class. Homework containing new situations was very incidental.

Next, the study was focussed on students (N=702) whose teachers had in their teaching often emphasized the application of mathematics to everyday situations and had often given these students homework requiring application of subject matter dealt with in the class (=appliers). The teaching in this group was regarded as having been most application-centred. This student group's achievements in different behaviour categories were then compared with the achievements of other students (N=1482). It was assumed that the emphasizing of applications would also manifest itself in the form of better performance.

TABLE 5. The Estimated Means (z) and Standard Deviations (s) of Student Groups in Different Behaviour Categories.

Student group / Behaviour category	Appliers (N=702)		Others (N=1482)		
	$\bar{z}$	s	$\bar{z}$	s	t
Computation (107 items)	0.48	0.23	0.52	0.22	3.91 +++
Comprehension (36 items)	0.53	0.24	0.56	0.23	2.81 ++
Application and analysis (37 items)	0.38	0.24	0.41	0.23	2.81 ++

+++ = $p < 0.01$, ++ = $p < 0.1$

However, the hypothesis was not correct. On the contary, other students had significantly better means both in application and analysis and in computation and comprehension. The outcome was unexpected, and it was considered a result of the way students from different sets were divided into these groups. As a matter of fact, the group of appliers did include distinctly more students from the lowest set, i.e. the general course, than other groups. The impact of this factor was eliminated by examining the results by set.

TABLE 6. The Estimated Means (z) and Standard Deviations (s) of Student Groups in Different Behaviour Categories by Set

Student group / Behaviour category	Extensive course				Intermediate course				General course			
	Appliers N=249		Others N=621		Appliers N=223		Others N=621		Appliers N=230		Others N=240	
	$\bar{z}$	s	$\bar{z}$	s	$\bar{z}$	s	$\bar{z}$	s	$\bar{z}$	s	$\bar{z}$	s
Computation	0.69	0.15	0.69	0.15	0.48	0.17	0.45	0.17	0.26	0.13	0.27	0.13
	n.s.				t=2.26 *				t=0.83 n.s.			
Comprehension	0.74	0.15	0.73	0.16	0.52	0.18	0.49	0.17	0.30	0.15	0.31	0.14
	t= 0.85 n.s.				t= 2.22 *				t= 0.75 n.s.			
Application and analysis	0.59	0.19	0.58	0.19	0.36	0.17	0.33	0.16	0.17	0.11	0.18	0.11
	t= 0.70 n.s.				t= 2.36 *				t= 0.98 n.s.			

* = $p < 0.5$

We now noticed that there were no differences in the achievements of the groups either in the extensive course or in the general course. In the intermediate course the achievements of appliers were better to an extent ($p<0.5$) which only just reached the level of significance. However, we have to bear in mind that the means of appliers were higher in all behaviour categories and not only in application and analysis.

The results we obtained are interesting in several respects. They confirm clearly that application items are very much seen in an illustrative sense, which favours computation-centred "mechanistic" teaching. Only a limited number of more realistic application tasks were used in teaching because of lack of time, and therefore, their mastery was considerably weaker. Items which are more important in view of learning had received less attention.

Results regarding relationships between the weighting of teaching and learning results reinforce further the picture we have obtained from the teaching of applications. According to teachers' own replies, they emphasize practical computation as well as thinking skills and application of mathematics to everyday situations, but obviously largely on the terms of computation skill. The illustrative and imitative approach in the teaching of applications was manifested e.g. in that when (strong) emphasis was put on applications the results in all behaviour categories improved in the intermediate course. This was not the case in the extensive and in the general course, because there the teaching went either below or above students' "reception level". On the other hand, this outcome seems to suggest that increased utilization of carefully chosen realistic and also new application situations in teaching would improve the mastery of all behaviour categories for all students, but especially the application skills of mathematics.

WHAT COULD BE DONE TO HELP THE TEACHERS?

On the basis of what has been said above, there can be no question of that clear changes in the teaching of applications at this age level are necessary in order to make sure that pupils' characteristics and changes which occur in them will be taken into consideration. The present-day teaching of applications is characterized by a clear dominance of illustrations, by teaching practice based on the imitation of standard models of situations and by one-sided application tasks. It is obvious, that this kind of education cannot promote pupils' development in the direction of objectives. The responsibility for the change of teaching style lies mainly with the teachers, and therefore we should find means to help them. This is a challenge to which there is no clear-cut answer, but many ideas are sure to be found. The following table presents a proposal in principle on the development areas.

TABLE 7. A Model for the Development of Teaching Applications

Problem	Development area	
What is the role of applications in the learning content?	Development of teaching materials	Changing external presures
How are applications taught?	Development of teaching styles	
What role do applications play from the pupil's point of view?	Development of teaching arrangements	

This table presents each development area linked with a certain problem, although it must be understood that they are in close interaction with each other. In teaching materials there is a special need for multi-level, interesting application material. According to Burkhardt, there is a shortage of material particularly in the area of easy, interesting application problems relevant to everyday life. Possibilities should also be investigated of utilizing the fund of application materials around the world.

We know from experience that it is very difficult to change teaching practices and styles. A large number of different development projects have been carried out all over the world, but no major changes have been produced. For example, it has been found, that printed material has very little influence on teacher's teaching style. (Burkhardt 1983). On the other hand, the success of some projects in the USA, for example, suggests that pupils have positive attitudes to application-centered teaching and can also cope well with it.
The development of teaching styles has direct impacts on teaching arrangements and some of the emerging questions might be: "What is the optimum size of teaching groups suitable for various open styles of teaching?" or "What arrangements are necessary for the use of microcomputers in the teaching of applications?"

Also various kinds of external pressures have an immense impact on education. Up till now, the social pressures, for example, have been for better technical skills rather than the ability to cope with practical situations. We should try to change these pressures coming from varous sectors (including the society, parents, school administration, teachers' associations, examinations, media) to make them more favourable to application skills.

Finally, we must remember, that implementation of changes in the educational system is a very slow process. Therefore we must be prepared for longterm work and failures as well. All efforts must be aimed in the same direction. This implies the necessary support of extensive research and development activities as well as close communication between the various influential parties.

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STUDY OF LEARNING CONCEPTS RELATED TO TRIANGLES AND QUADRANGLES ON THE BASIS OF VAN HIELE'S THEORY

Harry Silfverberg

INTRODUCTION

The place of geometry in the curriculum of the comprehensive school is felt to be problematic, and with reason. Placing the subject matter of geometry in between other matters in fairly short periods may make the perception of wholes more difficult and lead into misunderstandings as to what degree of mastery is to be expected at different stages (Pehkonen 1982).

The content and method of teaching at any given stage are influenced not only by the inner structure of the subject matter taught but also by the level of maturity of the learners. Yet very little is known about the geometrical thinking of pupils at different levels of maturity. One way of describing the development of geometrical thinking is offered by the van Hiele theory of phased change in geometrical thinking, already published in 1957. According to this theory, the development of geometrical thinking is characterized by the lessening of holistic thinking, by the shifting of the focus of perception from the visual shape of the figure into its properties, the organizing of these properties and the assumption of ever more complicated mathematical structures.

It is obvious that a wider knowledge of geometry as well as separate concepts are acquired by degrees. Since many geometrical concepts are used long before their exact mathematical content has been clarified, it follows that at the preliminary phases of learning such concepts may be regarded as natural concepts, of whose inner structure and assumption cognitive psychology has been produced a lot of information (Rosch 1973, Rosch et al. 1975, 1976).

In the present study (Silfverberg 1984) we made an attempt to describe the development of pupils' geometrical thinking mainly at the three lowest levels of the van Hiele theory, which were considered the most essential as far as school mathematics was concerned. In particular we tried to answer the following questions: first, can we perceive in the pupils' thinking a transition from a holistic way of perception into one analyzing and classifying properties, and if yes, at what stage does such a transition take place? Secondly, in as far as a pupil recognizes, names, classifies and compares figures analytically, making use of the properties of the figure in an explicit way, are, then, these properties separate or connected with one another?

THEORETICAL BASIS

Dutch scholars Pierre van Hiele and Dina van Hiele-Geldof put forth in their doctoral theses a theory of the levels of mental development in the learning of geometry, a transition from one level to another, and classroom strategies assisting the transition. The theory can be considered to be of didactic interest at least for two reasons: first, the theory presents through what kind of phases geometrical learning progresses and how these phases follow one another. Secondly, the transition from one level to another presupposes the mastery of activities pertaining to the previous level in essence. Especially if the language of instruction is above the level of the pupils' thinking, the pupil cannot grasp the instruction given (van

Hiele 1959).

The levels of development, later referred to simply as the van Hiele levels, can be described as follows (Burger 1981, Geddes 1981, Hoffer 1981, 1982, Mayberry 1983, Pehkonen 1982, van Hiele & van Hiele-Geldof 1958, Wirszup 1976).

Van Hiele Levels of Development in Geometry

Level 0. Visualization

At this basic level figures are percieved as total entities. Recognition, naming, classifying, comparison, description etc. are carried out on the basis of the visual shape of the figure, not on the basis of its properties. The pupil is able to recognize and name ordinary geometrical figures. He does not pay attention to the connection between the whole and its components. His thinking is concrete, and geometrical concepts are mainly names of objects and figures, not so much mental constructions.

Level 1. Description

According to van Hiele, the qualitative change of mental development while moving on from level 0 to level 1 presupposes the changing of visual structures into geometrical structures. The abstract level of thinking rises, as operating with concrete objects changes into operating with geometrical symbols. At this level the pupil consciously focuses his attention also on the properties of the figure, but the properties remain disconnected, since the relations between them are not discerned. The pupil can compare figures with the help of the relations between the components. Also, the pupil can classify figures with the help of their properties and not only by relying on their similarity. The pupil discovers properties belonging to all triangles, squares etc. and he can draw comparisons between groups of figures on the basis of their properties. At this level, however, it is not possible for the pupil to explain how the properties of a certain group of figures are related to one another.

Level 2. Abstraction

At this level the pupil can formulate and use definitions and follow deductive conclusion. He identifies necessary and sufficient properties for characterizing a class of figures. He can make use of the properties of a figure when examining whether a class of figures is included in another. He understands some of the relations between the properties. Greater connections between theorem groups are not grasped, neither is geometry understood as an axiomatic theory.

Level 3. Deduction

At the level of deduction the pupil can conclude what follows from a given fact and discerns relations between theorem groups. Differences between definitions, axioms and statements are understood. A pupil operating at this level can recognize what has been given in a problem and what is required.

Level 4. Mathematical rigour

At the highest van Hiele level, the pupil can compare different axiomatic systems, for example different geometries. He understands the limitations and possibilities of hypotheses and axioms. He can use mathematical models to represent abstract systems and to describe various phenomena through such models.

While looking for the distinguishing signs of holistic perception at the level of visualization, we relied particularly on E. Rosch's observations on the assumption of natural concepts (Rosch 1973, Rosch et al. 1975, 1976). The visual structures of level 0 cannot be constructed on explicitly perceived common properties. Instead, the formation of classes is explained to have resulted from a sufficient similarity between objects belonging to the same class, a feature which Rosch calls "family resemblance", adapting Wittgenstein. Objects belonging to the same class need not have properties common to all. Thus class divisions are not necessarily rigid.

At the level of description the pupil adopts the means needed for analyzing similarity and difference. The pupil learns to make use of the properties as an instrument of recognizing, naming and classifying. As stated by the van Hieles, a single property of a figure may become the signal of a concept, i.e. the distinguishing sign for this concept. Since the properties at the level of description are not organized, hierarchic structures cannot be presented in the knowledge structure.

At the level of abastraction, according to the van Hiele theory, the properties of geometrical figures form a partly organized system. Generally it is not until this level that an exact and economic definition of concepts becomes possible and the class divisions become rigid.

Even though the van Hiele theory was originally meant to be used to explain the development of geometrical thinking, it was later applied to chemistry and economics as well, at least in Holland. In addition, A. Hoffer has drawn the outlines for the application of the model to examining the learning of geometrical transformations and real numbers (Hoffer 1982).

CONDUCTING THE STUDY

The main body of the tests consisted of drawing, naming and defining triangles and quadrangles as well as tasks related to discovering the common properties of the figures. Triangles and quadrangles were supposed to be familiar enough to all comprehensive school pupils tested, the youngest of whom were in the fifth grade (aged 10-11) and the oldest in the ninth grade (aged 14-15). The first test (triangles) was given to 127 pupils and the second (quadrangles) to 136 pupils. Both the tests were taken by 121 pupils.

RESULTS

As for the first main question, the results were in accordance with the prediction of the van Hiele theory. In connection with the drawing, recongnizing and comparing of figures, geometrical concepts were used in a holistic way.

As indicators for holistic conceptual perception in connection with drawing we regarded the pupil´s limited capacity to modify when producing a series of four different triangles/quadrangles and giving a finite estimate of the possible number of different triangles and quadrangles. A limited capacity of modification in the whole series produced (triangles) was apparent especially among the fifth-graders. With regard to quadrangles it was apparent among pupils in higher grades as well. About three out of four fifth-graders thought there was a finite number of different triangles and quadrangles. Even in the higher grades only one out of three was sure that in principle it would be possible to draw an infinite number of different figures. The results may have been distorted by the fact that the pupils were inclined to regard figures belonging to a different type only as different figures, in which case there would naturally be a finite number of such classes.

In the test of recognizing triangles the pupils had to pick out the triangles out of a group of 14 given plane figures. The four true triangles and they only were recognized as triangles by about a fourth of the fifth-graders and about a half of the pupils in the upper grades. The most common mistake was to accept triangular figures having partly or totally curved parts in them as genuine triangles. Out of the 15 quadrangles presented, squares were recognized almost flawlessly, but the rectangles and the parallelograms were correctly recognized by no more than a fourth of the pupils at any grade level. There seemed to be two main reasons for the difficulties with the recognition of rectangles and parallelograms, the first of which was the fairly constant tendency to try and avoid overlapping in classification. For example only 19.0 % of the ninth-graders accepted the square as a rectangle and 9.5 % the rectangle as a parallelogram. The second factor causing faults in recognition was the fact that the names of even the most common basic figures were not familiar to all pupils. Of the basic fugures presented the square, the rectangle and the parallelogram (all three) were correctly named by about a half of the pupils in the upper grades. The square and the rectangle (both two) were correctly named by 43.5 % of the fifth-graders and by 72.5 % of the other testees. Such a classification was interpreted as a sign of a recognition which takes place in an holistic way on the basis of resemblance rather than on the basis of defining properties.

The comparison of figures proved a hard task at every grade level. Properties used as a support for comparison were classified as either holistic or analytical. The higher the grade, the smaller the ratio of the holistic concepts was. Owing to the crude method of estimation used, no more than a half of even the eighth and the ninth-graders were found to have used a clearly analytical way of examination.

As for the pupils who did use an analytical way of examining the properties, we studied whether the properties formed a partly organized system. As a characteristic of an order of properties we regarded the pupil's capacity to satisfactorily define a triangle/quadrangle and his ability to understand that a class of figures may be included in another. As stated before, class inclusion with regard to quadrangles rarely occured. An acceptable definition of a triangle and a quadrangle was only given by a handful of pupils. A certain property becoming a signal, a feature mentioned by van Hiele, was quite apparent. The signal of a triangle was its shape being triangular and that of a rectangle its being oblong.

Even though the test battery used was not specifically designed to place pupils at the various van Hiele levels, we can with fair certainty say that nearly all pupils tested would have placed on the two lowest van Hiele levels. By way of comparison we might mention that, according to Pyskalo (1968), with the help of classroom strategies based on the van Hiele theory it is possible for all pupils to attain the level of description

DISCUSSION

The holistic way of recording and manipulating information which was discovered in the test results of especially the youngest pupils seems to be a natural starting point in the process of learning a new thing. Learning the concepts of elementary geometry probably starts by forming the appropriate visual images. Such a holistic recording of information seems to be a typical and effective way of storing visual information. Yet from the point of view of learning geometry it is disadvantegous if information can only be manipulated in this form. The exact definition of concepts and understanding the relations between the concepts cannot be mastered by visual means alone; for that we need knowledge of the properties of the figures. A sign of relying too heavily on perception was indicated by most pupils' inclination to classify quadrangles disjunctively. We should be able to make the best possible use of the spontaneously born or purposefully made visual structures, but we should also be able to depart from them when need be. We need, however, to know more about what these visual structures are like and how geometrical structures proper can be constructed on the basis given by them. In as far as the van Hiele theory is to be used as a framework for planning the teaching of geometry, we need to further clarify the content of each level and their relation to one another. One problem is that the levels are so concept-bound. There is evidence of the fact that pupils are at different levels as to understanding different concepts (Mayberry 1983). The mere placement of pupils at the different van Hiele levels is not of much use as such. We also need to know through what kind of classroom strategies the transition from one level onto another can be assisted. This aspect of the van Hiele theory falls out of the scope of the present preliminary study.

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THE DEVELOPMENT OF A CONCEPT OF NUMBER IN FIRST-GRADERS (A DEVELOPMENTAL PSYCHOLOGY APPROACH)

SUMMARY OF A THESIS

Irma Vornanen

This study's primary goal is to apply Piaget's principles of developmental psychology to the development of numerical concept in first-graders. The study has to be conducted in three parts. The first section, discussed below was carried out between 1981 - 84, the second partis to be carried out 1983 - 86, and the third part in 1985 - 88. The first part deals with Piaget's understanding of how the child's concept of number develops, the principles of developmental psychology that Piaget believes applicable to the teaching of mathematics, and other thoughts that Piaget has on the subject of developing teaching techniques for mathematics. With these as a foundation a program for developing the numerical concepts of first-graders will be developed: a didactic solution and the training of teachers, experimentation and research fundamentally connected with it. The first part of the study stresses the theoretical aspect of the subject, the second empirical, and the third concentrates on practical application. The research is being carried out in conjunction with the continuing education of teachers. The role of the teachers in the development of teaching procedures has been significant.

The didactic solution, based on Piaget's numerial concept theory, was developed to be appropriate to planned situations intergrated into the normal teaching of mathematics in the first school year. The educational and instructional influences characteristic of school and also the orientation of teaching towards the creation of the abilities presupposed by school instruction must be considered. Thus the theoretical basis was expanded to include Vygotsky's and Ausubel's concepts of the development of teaching together with learning. In the didactic solution the question was one of increasing sensitivity to developmental readiness; a prerequisite for the start of meaningful mathematical learning at school. With respect to the training of teachers the goal was to develop the pedagogical readiness that is required if significant development in first-graders' understanding of numbers is to take place. The testing of the didactic solution in practice is based upon cooperation between the researcher and the teachers. The study is one which presupposes cooperation; a stage-by-stage R & D (Research & Development)-type of developmental study.

The research problems of the first part of the study deal with: 1) the level of first-graders' numerical development at the beginning of school and certain developmental factors, the type of day-care received the previous year and the significance of age in relation to it 2) the significance of a developmental program on the development of the concept of numbers, success in a mathematics course and the prevention of learning difficulties and 3) the teachers' opinions and experiences concerning the developmental program. In solving the first problem emphasis is placed on explaining the thought processes behind the answers. The notes taken by the teachers in the test situation play a significant role in the analysis of results. In solving the second problem the attempt is to draw conclusions about the situation prevailing at the moment in respect to the problem proposed by Piaget as being mathematics' central didactic problem, that is the fusion of the child's natural numerical development process and a teaching curriculum. In solving the second problem the experimental set-up is that typical of field studies: pre-test, retest, control groups. In solving the third problem the attempt was to gain knowledge of the program's relevance to the goal set by the study as a whole. The information was gathered from a questionnaire presented to the teachers. There are 262 subjects in the study and 19 teachers.

According to the results obtained, there were various levels of development among the 6 - 7-year-olds starting school. On the basis of how the children react and respond in the experimental situation and from the kind of explanations they give for their answers, conclusions can be drawn about the thought processes involved. The attempt to estimate the general level of development of childrens' numerical concept was unsuccessful due to inexactness in both the instrument of measure and measurement procedures. The measure proved, however, to be quite informative from a developmental point of view. The child's age appeared to be significant to the level of numerical development at the start of school. Those born in the beginning of the year had a headstart compared to those born at the end of the year. On the other hand, whether or not the children attended nursery school or received any other form of daycare the preceding year did not appear to have any effect on the level of numerical development. The effect of a developmental program based upon Piaget's theories, on cautious appraisal, appeared to be positive. On the other hand the program had only little positive effect on the prevention of learning difficulties, and there were even indications of effect in a negative direction. The teachers' opinions and experiences concerning the relevance of the developmental program were for the most part positive. However the need for further developing the program became evident.

In accordance with the nature of a developmental study, the study at hand was assembled from the study data available at the present moment. On this basis a program for developing the development of numerical concept in first-graders was designed. The program was tested in practice. According to the feedback received, the program proved to be functional. Need for correction, supplementation, and specification was observed in the study's various parts: the didactic solution, the training of teachers, research. The program must therefore be further developed. Feedback on the trial use was also obtained, suggesting in which direction further research is needed.

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