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ABSTRACT

This case study examined an experienced secondary school mathematics teacher's knowledge and teaching of problem solving, using interviews, classroom observations, teaching documents, and experimental tasks. The informant revealed a broad interpretation of problem solving, integrated with mathematics but widely applicable. This interpretation appeared consistently in his knowledge of problem-solving content, knowledge of pedagogy, and instructional behavior. The informant's own background significantly influenced his knowledge of problem solving, which in turn shaped his teaching of problem solving. This study extends recent casework in mathematics teaching and has important implications for research, teaching, and educational policy. Appendices indicate data sources, interview schedules, task schedules, and a problem-solving hierarchy. (Author/MNS)

PROBLEM SOLVING WITH A SMALL "p":

A TEACHER'S VIEW

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Abstract

This case study examined an experienced mathematics teacher's knowledge and teaching of problem solving, using interviews, classroom observations, teaching documents, and experimental tasks. The informant revealed a broad interpretation of problem solving, integrated with mathematics but widely applicable. This interpretation appeared consistently in his knowledge of problem-solving content, knowledge of pedagogy, and instructional behavior. The informant's own background significantly influenced his knowledge of problem solving, which in turn shaped his teaching of problem solving. This study extends recent casework in mathematics teaching and has important implications for research, teaching, and educational policy.

PROBLEM SOLVING WITH A SMALL "p": A TEACHER'S VIEW

> (Math teacher, in interview, describing problem solving) When you have a situation that demands attention, either because of the positive things that the solution might bring or the negative situation that exists now that you want to alleviate, then that's a problem. And you seek to deal with that situation. At a certain point, you call what you have a solution, I guess. Sometimes it's a completely satisfying solution, such as balancing a checkbook; other times it's a temporary solution; and sometimes you give up, and you say this is no longer a problem. A very broad sense.
>
> I think students might not be making the split between problem solving and non-problem solving; the teachers are. For students, who don't divide up the curriculum like we do, I think it's all problem solving.

This teacher views problem solving in a distinctive way. It is not Problem Solving, the new find in math education, or Problem Solving, a topic to be squeezed in somewhere between Fractions and Quadratic Equations. It is problem solving with a small "p", applicable across disciplines and even outside of school, woven into the fabric of mathematics rather than stamped on top. But what does this approach to problem solving look like in detail? What alternative paradigms of problem solving might other math teachers hold? And how does the teacher's view of problem solving shape her teaching of it? These questions motivate the present study.

Problem solving as an issue in mathematics education is not newborn, but it has reached its adolescence in the 1980's: everyone knows it's around, but no one is quite sure how to handle it. Policy makers (California Department of Public Instruction, 1985) and educational researchers (Begle,

1979) agree with the National Council of Teachers of Mathematics (1980) that "problem solving [should] be the focus of school mathematics in the 1980s" (p. 1). Yet despite consensus on the importance of problem solving, it is an elusive concept to define and a complicated one to study. Briars (1982) and Kilpatrick (1985) describe various research programs in mathematical problem solving: analysis of problems; cognitive processes, whether through verbal protocols or computer simulations; expert-novice comparisons; and training sequences. Each of these approaches highlights a different facet of problem solving.

This extensive body of problem-solving research has examined the content, the student, and the instruction, but it has largely ignored the teacher (Grouws, 1985; Silver, 1985). Even when research has focused on teachers, it has typically concentrated on how they behave in the classroom rather than on how they think about the content, students, and instruction (Shulman & Elstein, 1975). Fortunately, a recent trend in mathematics education research has begun to correct this oversight by delving into teachers' knowledge and its relation to their teaching (Shulman, 1985). Thompson (1984) and Steinberg, Haymore, and Marks (1985) demonstrated in sets of case studies that math teachers' conceptions of mathematics strongly influence the way they teach. Cooney (1985), studying problem solving in Fred, a beginning math teacher, accounted for a good deal of his teaching behavior in terms of his conceptions of mathematics, problem solving, and teaching.

The present case study extends this line of research by investigating an experienced mathematics teacher's knowledge and teaching of problem solving. This study examines how the informant's background helped form his content knowledge of problem solving, how this in turn influenced his pedagogical knowledge, how these aspects of knowledge shaped his planning and teaching, and how features of the teaching context affected this process. These substantive outcomes also suggest a more formal model for the teaching of problem solving and similar topics.

Conceptual Framework

The categories and relationships of interest in this case study appear in Figure 1. Following Shulman's (1986) recent work, knowledge of content is considered separately from knowledge of teaching that content. Knowledge as used in this paper encompasses not only factual knowledge but also skills, beliefs, attitudes, and feelings. Problem solving will be defined by the teacher rather than by the researcher, since the chief purpose of this study is to discover the teacher's own conceptions of problem solving, together with their origins and effects. I did, however, construct a descriptive hierarchy of problem-solving behavior (see Appendix D, Figure 4) to assist in generating examples of various problem-solving features throughout the study. I also intended to use this hierarchy to analyze instances of problem solving in the data but found it difficult to apply reliably.

Knowledge of problem solving comprises three major categories: sources, content knowledge, and pedagogical knowledge. Sources are influences on the development of problem-solving knowledge and may include teaching experience, teacher education, other schooling, and non-academic factors. Content knowledge here connotes the teacher's knowledge of problem solving per se, apart from teaching. Three

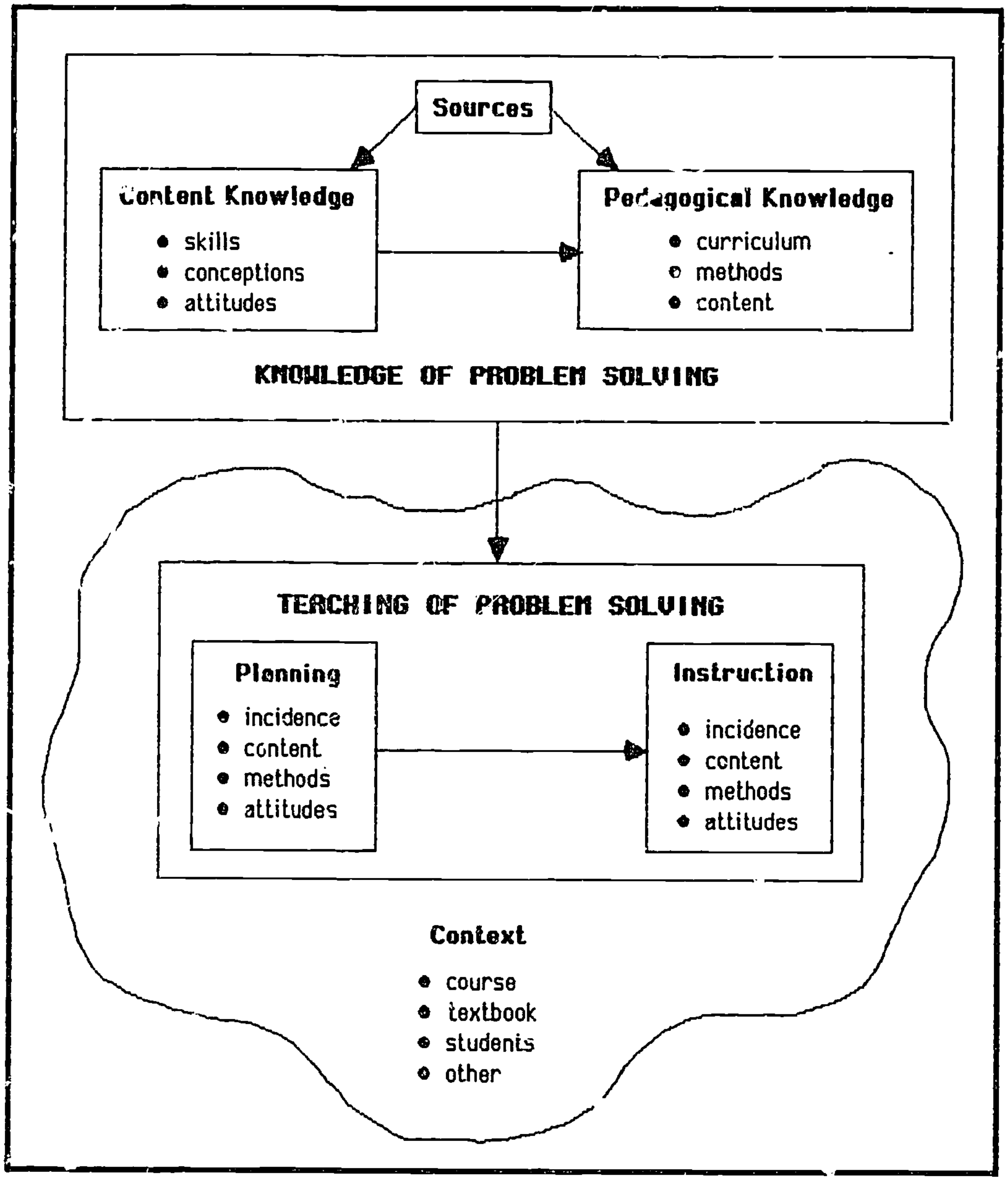

Figure 1. Influence of teacher's knowledge on teaching.

subcategories are useful for analysis. Skills represent the teacher's own problem-solving experience and competence, which many math educators argue are indispensable in an effective teacher of problem solving (e.g., House, 1980). Conceptions are knowledge and beliefs about the subject matter itself (e.g., "problem solving is central to mathematics"), while attitudes are affects and beliefs about the teacher's own relation to the subject matter ("I use problem solving all the time"). Pedagogical knowledge describes the teacher's knowledge of teaching problem solving in particular, not just of teaching in general. Again three types are important. Curriculum consists of general views of problem-solving instruction: its aims, significance, scope and structure, and its proper place in the total instructional program. Methods include specific teaching techniques and materials--for example, the use of cooperative small groups or of computer software to teach problem solving. Content as a subcategory of pedagogy refers to knowledge the teacher wants his students to acquire, such as a list of Polya's (1957) heuristics or a scheme for solving river-crossing problems.

Teaching of problem solving likewise includes three major categories: planning, instruction, and context. Though planning precedes instruction, the two activities correspond closely, so each contains the same four subcategories. Incidence describes the frequency and emphasis of problem solving and the way it is incorporated into the mathematics class. Content here is the knowledge and techniques of problem solving that the teacher teaches, explicitly or implicitly. Methods include the procedures and materials which the teacher uses to teach problem solving but which are means and not ends with regard to student learning (e.g., the use of small groups). Attitudes are subjective beliefs and feelings about problem solving which the teacher communicates, again explicitly or implicitly. Context consists of those features of the particular teaching situation which call for adapting the teacher's general knowledge specifically to that situation. Various aspects of context may be important, but three are related closely to the content. Course refers to prescriptive factors such as curriculum outlines, advice from colleagues, or department policy. The textbook provides a particular treatment (or non-treatment) of problem solving, while the students bring a distinctive set of problem-solving skills, aptitudes, and attitudes to the class.

The relationships in Figure 1 depict the five main questions in this case study. What skills, conceptions, and attitudes does the teacher have regarding problem solving? How do these influence his ideas about teaching problem solving? How did the teacher acquire this knowledge? How does this knowledge shape his actual planning and teaching of problem solving? How does teaching context affect this process? Though this conceptual model was designed to explore these questions in the domain of problem solving, the same model could be applied to virtually any content area.

Methods and Procedures

Since the purpose of this study was not to determine the typicality of mathematical problem-solving instruction but to describe and explain it when it does occur, the informant had to be a math teacher who also taught problem solving in some form. One of my teacher education students recommended her master teacher, Sandy, who happened to be a casual friend of mine. After a brief interview which confirmed his suitability and willingness, Sandy became my informant. He was extremely cooperative and interested throughout the study and especially articulate in interviews, not surprising since he also does educational research at a nearby university.

Data for this study came from several different sources: nine structured interviews with the teacher, nine observations of classroom teaching, numerous documents, eight experimental tasks for the teacher, and one summary debriefing interview. (See Appendix A for a complete list of data sources, Appendix B for the interview schedules, and Appendix C for the task schedules.) At least two different types of data address each subcategory in the conceptual framework (see Appendix A, Table 1). The variety of sources serves both to increase the descriptive and explanatory power of the study and to increase construct validity through triangulation.

One or two 45-minute interviews each addressed background, content knowledge, pedagogical knowledge, and context, while shorter interviews conducted before and after each observation unit covered planning and instruction. The interviews included indirect as well as direct approaches: for example, "Suppose your math department organized a task force to recommend ways to include more problem solving in the curriculum, and they chose you as the chair of the task force. What would you do?" Each interview was audiotaped and later transcribed in full.

Three observation units of three days each explored instruction for one class on one topic, a different class on the same topic, and the same class on a different topic. Data in each class consisted of extensive handwritten notes and a short, open-ended debriefing interview after class. Documents included school course descriptions, unit and lesson plans, handouts and tests, reproductions of the textbook, and copies of other resources the teacher used. The experimental tasks probed Sandy's knowledge of problem solving in forms other than a direct verbal report, in an attempt to reduce the level of inference through multiple sources of evidence. Examples of the tasks are to solve a given math problem while thinking aloud, to classify a given set of math problems according to their perceived degree of problem solving, and to rank a set of 40 possible objectives for high school math according to their importance for the students. Five of the tasks were also audiotaped and transcribed.

Data analysis was a variation of Glaser and Strauss's (1967) constant comparative method. Analysis began during data collection. First I reduced each interview, observation, document, and task to a set of brief numbered notes, placing each instance relevant to one of the framework's conceptual categories into a single note. This stage involved selection but almost no commentary; the notes consisted of low-inference summaries, paraphrases, and quotations of the data. This technique not only reduced the mass of verbiage but also suggested themes to guide the remainder of the data collection. After reducing all the data to notes and studying them, I wrote a series of short memos delineating substantive themes, both descriptive and explanatory, and noted questions, contradictions, and conjectures. After organizing these themes I constructed coding categories, then returned to code some of the notes. I repeated this process twice, each time refining the memos, reducing the categories, and coding more of the notes for verification. Formal themes emerged eventually from the substantive memos. Finally, by the end of the third iteration, the themes formed a coherent structure and the coded data provided the supporting detail for the case study report.

Informant and Setting

> (Sandy, interview) Mostly I studied the social sciences and humanities, with a heavy emphasis of psych, sociology, anthropology. Some literature, history. Not a great amount of math. . . . I always liked math, yeah. I like it more since I'm teaching it. . . . And I think there are some complementary things about thinking in terms of mathematics that is a nice balance to other parts of my life. . . . I'm not one to believe that subject matter knowledge is the great decider of the effectiveness of the teacher. . . . My goal was to load up my credential as much as possible, sort of like a utility infielder in baseball, who can serve different roles in a school.

Sandy is an experienced and highly educated high school teacher with multi-disciplinary interests and abilities. He majored in psychology but also liked math and was good at it. In high school he went through the accelerated math program, including calculus, and in college he took courses in statistics, finite math, and computer science. Sandy holds an M.A. in education and a teaching credential in both math and social science. In addition, he has just completed a doctoral dissertation on teacher self-evaluation in math and social studies teachers. After eight years of full-time teaching, both math and social studies, he is still very happy with his career and plans to continue teaching high school, at least for the near future.

The quotation above illustrates a personal characteristic germane to the present study: Sandy's tendency to integrate and qualify, to reject extremes and dichotomies. This trait appears both in his simultaneous interest in social studies and math and in his minimizing of the importance of subject matter knowledge. He speaks of the varied disciplinary perspectives as balancing rather than competing. This characteristic manifests itself throughout the data and will enter the analysis later.

Sandy teaches Algebra 2, Algebra 1, and Algebra .5 in a medium-sized, integrated suburban high school. The courses are not tracked, and all of his classes span an ordinary range of student abilities. He describes the two higher-level courses as "college prep," "fast-paced," "challenging," and "difficult"; Algebra 2 in particular is "real solid." Algebra .5 is also "a serious math class" but moves more slowly. These terms are indicative of Sandy's serious, no-nonsense attitude toward his teaching. The present study focuses on Algebra 2 because Sandy's two sections of that course permitted more intensive observation and because preliminary data suggested that cross-course comparisons would add little insight.

Knowledge of Problem Solving

Content Knowledge

> (Sandy, interview) [If I'm driving home from work and I notice I'm low on cash] that's problem solving to me. Figuring out how am I going to get over to the bank, what's the best way to do it, is there any way I can put this off 'til tomorrow, do I have the money in the bank, how can I avoid traffic, can I do this and get where I have to be. That to me is a problem to solve.
>
> I would say most math problems would involve problem solving.

Conceptions. For Sandy, problem solving occurs whenever someone is motivated to perform a task which is not strictly mechanical. The actual carrying out of mechanical operations, such as stapling some papers or performing arithmetic operations, is usually not problem solving, but the task of having to get some papers stapled or of having to come up with a solution to an arithmetic problem does entail problem solving. However, what is mechanical for one person may not be for another, and so the act of stapling papers or performing arithmetic can also constitute problem solving. The designation depends on the performer as well as the task.

Some of the types of problems Sandy describes are especially relevant to his teaching. Learning to differentiate concepts and apply them is a form of problem solving: for example, "Identify which of these equations are quadratic equations," and "I don't have to think what a red light means any more, but that, when I was learning, that was a problem to be solved." Another form of problem solving closely related to this is learning to select and apply procedures, in a sort of heuristic rather than mechanical fashion: "If I give them a set of quadratic equations not written in standard form, and asked them to solve them using a variety of methods. . . . Some that can be factored, some that aren't quadratic, some that they get imaginary roots using the quadratic formula." Interactional and organizational tasks, such as working together in groups or dealing with the teacher or taking notes, are another important type of problem solving. One of the most significant forms of problem solving for Sandy himself is his teaching, "this great intellectual challenge, to teach well." He distinguishes interpersonal and professional problems from academic problems in that, with the former, "as an individual you can't necessarily control the outcome; one's own actions are not enough to arrive at a solution that might be desired." Most "important life tasks" are of this sort.

In Sandy's view virtually all math problems entail problem solving, but some more so than others. In one of the experimental tasks (see Appendix C, Task 2) Sandy read a set of 23 problems appropriate to high school algebra, varying in content, conceptual difficulty, and conventionality. Examples are "Graph the equation $x^2 + 4y^2 - 16y - 20 = 0$" and "Would you and your family be comfortable living in a 440 square foot house? Why or why not?" He decided that every one would involve problem solving for his students and for himself. In classifying the problems according to their probable degree of problem solving, however, he rated all the more conceptual and less standard problems (such as the house example above) as excellent or good and virtually all the more standard problems (such as the graphing example) as fair or poor. In another task (see Appendix C, Task 4) Sandy chose a topic, then generated various problems which he thought represented either low or high degrees of problem solving within that topic. Again he selected problems with less standard forms as involving more problem solving, because "there may be more tasks layered in, there's options that they may have to explore, there may be more alternatives. It may take more sticking-to-it power for them . . . so it may take a higher degree of motivation." As for word problems, Sandy refers to them as "typical" or "traditional" problem solving. He believes they involve maximal problem solving and are important, but they don't define problem solving for him as they do for some teachers and in some textbooks.

Problem solving refers to solution methods as well as problem types. Although Sandy never used the word "heuristics," several times he listed various Polya-style heuristics as typical problem-solving skills: for instance, "Figure out what skills you need, what is the problem asking, lay out a plan of attack, evaluate your answer at the end, work cooperatively, . . . what resources are necessary, do you have the ability to even begin this problem, . . . is this problem solvable?" These "broader, overarching skills" apply across content areas: "I don't see mathematical problem solving as being much different than other types of problem solving." What does vary across disciplines or applications is the underlying content skills, which are different from but necessary for successful problem solving: "There's a distinction between the mathematics involved and the mathematics problem solving." "In order to solve a problem you need the background content." At the human level, "There is a generalized problem-solving ability that people have," though "I wouldn't generalize it to all parts of that person's life." "People may get hung up on the skills involved." For example, Sandy considers himself good at solving many kinds of problems, but he lacks skills in plumbing. Consequently he could not solve a plumbing problem directly but would do so via a more general heuristic, seeking help--that is, calling a plumber.

Skills and attitudes.

> (Sandy, interview) I'd say I'm a good problem solver. . . . If I don't understand something, I will either stop and ask for help, or find a way to go through it. . . . For example, if you gave me a calculus problem to solve now, I think I could solve it, even though I haven't seen calculus for fifteen years. I'm not saying I could do it on my own, but if you said, okay, you know, come back in X amount of time with the solution, if it was a reasonable amount of time, I think I could find the solution. Because I would go backwards to the point, I'd be willing to put in the effort and time to figure out what background I needed, and then I would . . . rebuild my understanding from there.

In arithmetic, where he has the skills, Sandy seems justified in calling himself a good problem solver. He solved two non-standard problems competently, using a variety of heuristics in a deliberate and organized fashion. For instance, in a "how many ways can you make change for a dollar given certain coins" problem (see Appendix C, Task 1F), he stated his simplifying assumption that "all the pennies are the same"; made a simplifying substitution, reducing eight pennies to five; chose a search-tree strategy based on size of coin, and carried it out systematically; noted an alternative strategy along the way; worked backward as well as forward; checked his work when finished, discovering a missed combination and an unnoticed relationship; and restated his assumption with the final solution, which was correct.

Sandy considers himself skilled at academic and educational problems but not so good with physical problems. Not surprisingly, his attitudes run in the same direction. He enjoys his dissertation and the challenge of how to teach well but doesn't like problems involving his car or income tax. As for mathematics, Sandy seems to like problems more because they relate to his teaching than for their own sake. One of his tasks (see Appendix C, Task 3) was to rate 21 math problems, varying widely in topic and difficulty, according to how much he would enjoy solving them. He consistently rated teaching-related and conceptual problems high (e.g., "How are complex numbers like real numbers, and how are they different?"), number theory problems low (e.g., "Find MAID if O^3 = DAD and $(IM)^2$ = MOM"), and logic problems and strategy games intermediate; algebraic problems ranked low if simple and higher if harder. Sandy's own use of problem solving parallels his attitudes for the most part. Problem solving is "a crucial ability" in Sandy's life, one he uses "all the time": in planning lessons, responding to students, writing his university dissertation, figuring his income tax, juggling his schedule. Mathematical problem solving, however, he uses a great deal in his teaching but little if at all outside of teaching.

Pedagogical Knowledge

Curriculum and content.

> (Experimental task--see Appendix C, Task 6) Given a list of 40 things that students might learn in high school math, Sandy rated each on a five-point scale according to its importance. The items included skills, concepts, real-world applications, problem solving, and affects; they also covered arithmetic, algebra, geometry, and unspecified topic areas. Examples include "Use a calculator efficiently to perform arithmetic"; "Explain the relationship between area and linear measure"; "Use functions to represent real-life relationships"; "Detect unreasonable results"; "Show enthusiasm for doing mathematics."
>
> Sandy spontaneously created two columns, for college- and non-college-bound students. He consistently rated concept, application, skill, and problem-solving items high for the college group, but reduced the rankings of concepts, applications, and advanced skills for the non-college group, leaving problem solving and basic skills at the top. He ranked affective items consistently low for both groups.

Sandy sees problem solving as a critical component of mathematics teaching; once he even described the content as a vehicle for problem solving. He is aware of the current educational climate: "I think there is a public emphasis on problem solving; it's somehow seen as a better thing to be teaching." Characteristically, however, he qualifies this view: "I don't think it's quite that neat. . . . I think it's much more linked than the popular conception of problem solving." That is, he also values math content skills, without which mathematical problem solving is impossible, and believes that problem solving must be fully integrated with the math curriculum. "In order to solve a problem you need the background content; even to understand content. . . hopefully it would be good to put in the context of a problem where the content arose." Better understanding of the content is one of Sandy's goals for teaching problem solving.

Another of his goals for problem-solving instruction is the development of generalized skills and affective qualities such as confidence and persistence. This is one of Sandy's rare references to affect and at first glance appears to contradict his low evaluation of affective objectives in the task described above. However, four of the five affects in the task were specifically related to mathematics--for example, "express appreciation for the beauty of mathematics," or "plan to continue studying mathematics"--whereas the affective attributes Sandy values are generic across subject areas. This is another instance of his cross-disciplinary propensity and is perfectly consistent with his views of problem solving in general.

The problem-solving content that Sandy wants his students to learn consists of general heuristics: defining the problem, determining the goal, developing a strategy, using previously learned skills, getting help if stuck, working through the problem, interpreting the answer, and using alternative methods. The vehicle he cites for teaching these heuristics is all the mathematical problems encountered in the standard curriculum, including but by no means limited to word problems. Sandy sometimes supplements the textbook's problems with algebra problems from other sources but never with problems unrelated to the current mathematical topic. This is consonant with his own relative preference for problems related to standard algebra content over problems of number theory, logic, and strategy games which are sometimes used to teach problem solving.

Sandy believes that, besides using heuristics to solve standard math problems, his students are perpetually practicing two other forms of problem solving in class. Learning concepts and procedures--that is, the process of learning mathematics--is an important kind of problem solving, one related to the subject matter but equally applicable in any discipline. Interacting with classmates and the teacher is another form of problem solving, this one essentially unconnected to the subject matter. These views of the problem-solving content he teaches exemplify Sandy's broad conceptions of what constitutes problem solving and where it resides.

Methods.

> (Sandy, interview) I think that problem solving can be taught using different methods. . . . It's not a question of whether, which method but the question would be if they're done well.

Sandy recognizes several different methods for teaching problem solving: modeling, questioning, simulations, media--the last two especially in social studies. He thinks that varying the teaching method is helpful because of individual differences, temporal variations, and the need for a range of representations. "As a teacher one hopes that, doing as many methods as possible, that you sort of magically hit people along the way." He also thinks that cooperative small group work might be useful, although it requires a delicate mix of conditions to work and even then it may be overrated.

Of the various teaching methods, Sandy definitely prefers modeling and asking questions, combined into a single composite technique: "lecture-student-questioning, where I sort of lead them through the stages through my questioning." This choice is partly a function of Sandy's instructional goals; he wants his students to learn a sort of internal question-and-answer dialogue, and so he models this dialogue in his teaching. He believes that this process demonstrates successful problem solving for the students and at the same time stimulates them to think. Sandy's choice of teaching method is also very much a consequence of his management style; he strongly prefers to retain the locus of control in his classroom. This avoids discipline problems, ensures that the right questions arise and that the lesson takes the desired course, and makes the lesson easier to teach. For instance, in two planning tasks Sandy described "have them discover the formula" as a method involving maximal problem solving. However, he would implement that method by guiding the whole class through the discovery with lecture-questions-discussion, thus controlling the direction and pace of the exercise at an aggregate level.

Another, more subtle factor in Sandy's choice of method may be his uncertainty about how students actually learn problem solving. He thinks the learning process is long and complicated, that it may entail accumulating experience and confidence through practice, and that at some point it may culminate in a leap to intuitive understanding, but he is very tentative about all this. He is "not sure that a good lecture-teacher-questioning has any less value than a group problem-solving situation for students" but offers little rationale for this view. Sandy, lacking a clear conception of the role of the student in learning, seems to base his teaching method largely on instructional goals and management considerations, whereas a teacher who held strong conceptions of the learning process might be expected to adapt her methods accordingly, taking more account of means in relation to goals.

Sources

Sandy credits his problem-solving skills to three sources: "Blessing of genetics, and blessings of a good education, and being blessed with good models along the line." In his schooling, including mathematics and math education classes, his teachers modeled problem solving but never taught it explicitly. Outside of school, his problem-solving ability developed through his own experience and through the modeling of friends. He cites these same sources, plus his experience in teaching and planning, as major influences on his pedagogical knowledge of problem solving, whereas his teacher education program and informal conversations with colleagues have had little impact. In other words, whatever Sandy learned about problem solving, he learned implicitly rather than didactically, and mainly in non-mathematical contexts. From his experience he seems to have abstracted a collection of general heuristics, which he applies to life situations and to specialized fields where he has content skills. This history accounts largely for Sandy's conceptions, described earlier, of what problem solving is and how to teach it.

Another aspect of Sandy's background seems to have helped shape his knowledge of problem solving: his tendency to qualify and integrate things in general. This trait appears in his very relative definition of problem solving: a problem doesn't always have a clear-cut presentation or solution, it is a problem only in its relation to the solver, virtually any non-trivial task constitutes a problem, problems form a continuum of complexity, problem solving is essentially the same across content areas, and the solution process may include such steps as studying, researching, or seeking help. This relativistic tendency is manifest throughout the data--for instance, in Sandy's view that students do not fall intellectually into "haves" and "have-nots" but that all students can learn problem solving, though perhaps at different rates.

Sandy's background in social studies is also an obvious candidate for explaining his views of problem solving, inasmuch as the social sciences are perceived as promoting more relativistic thought than is mathematics. This hypothesis is contradicted, however, by Fred, Cooney's (1985) math teacher, who was also trained in social studies but holds very different views of problem solving from Sandy. It seems likely that undergraduate major is a poor predictor of a teacher's basic conceptions of subject matter, both because of the greater salience of other, more personal attributes and experiences and because of the diversity of depth and perspective possible within the study of a given discipline.

In summary, Sandy's ideas about the curriculum and teaching content of problem solving reflect in detail his knowledge of problem solving. Both interpret the concept itself broadly, view its substance as a set of general heuristics which depend for their application on specific subject-domain skills, consider it as an integral part of mathematics, and see it as an ability of major importance. The methods he describes for teaching problem solving are somewhat coupled to his views of the problem-solving content but are just as much a consequence of his own history and of generic pedagogical features such as conceptions of the learning process and style of classroom management. Sandy's knowledge of problem solving in general, both content and pedagogy, unarguably reflects his characteristic integrative tendency and his personal learning experiences. Why these particular characteristics are especially salient for Sandy is unclear. Further analysis of the significance of the teacher's personality and learning history calls for intensive case studies and for psychological studies extending the notion of concept learning to broad domains like problem solving. More generally, the relative influence of personal, general pedagogical, and content-specific factors on teachers' knowledge bears further investigation.

Teaching of Problem Solving

Planning

> (Document summary) Sandy's lesson plan for Monday is two handwritten pages. The top line reads, "Last of conic sections --hyperbola. Similar to ellipse." The rest of the lesson plan consists of comparative lists of characteristics of an ellipse and a hyperbola, followed by four worked-out hyperbolic graphing problems and a textbook homework assignment. The only other note is next to the first graph: "to help draw graph, draw in 2 lines, asymptotes--curve approaches more + more closely.

Sandy's explicit planning for the teaching of problem solving is very limited. His written plans at both the weekly unit and daily lesson level deal almost exclusively with mathematical content; they include no references to pedagogy in general or to problem solving in particular. When he talks about lessons he has prepared to teach, however, he describes several important ways those lessons contain problem solving: in learning concepts, procedures, and heuristics, and sometimes in interactional or organizational tasks. In other words, Sandy does plan to teach problem solving, all the time, but the problem solving is embedded in the mathematics teaching and so does not merit writing down. In Sandy's case the influence of planning on instruction, hypothesized in the conceptual framework, is negligible.

Instruction

Methods.

> (Classroom observation; dialogue is slightly condensed) The teacher has explained how to recognize the type of conic section from its equation, and the students have identified several simple examples (neither translated nor rotated).
>
> *Teacher:* It gets trickier; with extra terms, you need to shift things around. (Indicates $4x^2 + y^2 + 24x - 4y + 36 = 0$.) Does it have one squared term or two?
>
> *Students:* Two.
>
> *T:* So that eliminates what?
>
> *Ss:* A parabola.
>
> *T:* Are the coefficients equal or unequal?
>
> *Ss:* Unequal.
>
> *T:* That eliminates what?
>
> *Ss:* A circle.
>
> *T:* Are they added or subtracted?

Ss: Added.
T: So what is it?
Ss: An ellipse.
T: Right. The extra terms just translate the center.

Sandy's actual teaching corresponds quite closely to his descriptions of how he teaches, relying mostly on lecture interspersed with questions. He often asks several questions in a row to lead the class through an entire problem. He usually directs his questions to the class as a whole, and several students answer in chorus. Most of the time the students can and do answer Sandy's questions in a brief phrase or a single word. When they answer incorrectly, he usually supplies the correct answer, often without further discussion. When a student asks a question, Sandy almost alwa[illegible] either answers it directly, refers it to the class for an immediate a[illegible]ef answer, or defers it indefinitely; a question may lead to an extended exp[illegible]tion by the teacher but rarely to any sort of discussion by the students. In other words, although teacher and students exchange a lot of questions and answers during each class, student participation is for the most part highly constrained; classroom activity is strongly teacher-centered and teacher-controlled. Sandy describes his students as cognitively active, but their behavioral role is fairly passive. Their overt tasks in class are to listen, to respond with brief answers to the teacher's questions, to practice solving problems like those just demonstrated, occasionally to ask questions or make comments, and rarely to participate in small group work.

The occasional exceptions to the above patterns deserve mention. Not all of Sandy's questions are procedural and rhetorical; some require deeper thought. Examples are, "Why does the graph of $xy = k$ lie in only the first and third quadrants," and "Suppose we had an ellipse and a hyperbola; what are the options, how many points could they have in common?" Furthermore, when a student answered "six" to this last question, Sandy asked the same student to draw his solution, after which the student ventured that "the most there can be is four." In a class on the hyperbola, a student asked, "What if the B piece is not a perfect square?" Sandy first asked the class for ideas, then gave two examples, whereupon a student responded correctly, "Use the square root."

The same pattern-with-exception appears in the students' assignments. Their nightly homework consists of practicing standard problems of the sort covered in class. The one review sheet and two chapter tests that Sandy gave during my observations consisted of exactly the same types of problems. However, the comprehensive take-home quiz on the conic sections differed drastically. All nine questions focused on concepts, all but one asked for verbal responses, several called for justifications, and most entailed more complex forms of exploration and reasoning than usual: for example, after graphing four similar hyperbolas on the same set of axes, "How does the value of B in [the intercept equation of a hyperbola] affect the shape of a graph?" Furthermore, Sandy allowed the students to work together on the quiz. This quiz grade introduced some measure of problem-solving performance into Sandy's evaluations of the students.

> (Classroom observation) Instructions written on blackboard for a small group exercise on systems of quadratic equations:
> A. Break into groups of 3 or 4.
> B. Check each other's graph of pg. 336.
> C. Work together on pp. 339-342.
> D. Refer to "Key" if all are stuck. Check answers.

Sandy did devote one of the nine lessons I observed to small group work. He had referred to cooperative small groups as a potentially useful means of teaching problem solving, especially for group interactive problems per se, and had occasionally asked this class to work in small groups. However, he had not suggested any structure or operating rules for the groups, nor did he on this day. Most of the students formed groups larger or smaller than the recommended three or four, most students worked individually rather than cooperatively within the groups, most of the inter-student talk was not related to mathematics, and most math-related talk dealt only with comparing answers. Roughly half a dozen students appeared to experience some sort of cooperative mathematical problem solving, while the rest did something else. Perhaps Sandy's casual handling of the group exercise reflects his ambivalence about the real value of group work, which in turn may be a function of his hazy understanding of students' learning processes.

Content.

> (Classroom observation) At the beginning of class one day a student asks to see how to graph the homework problem $4x^2 - 4y^2 = 64$.
>
> *Teacher:* Okay. I can't see A or B and it's not equal to one. What can we do?
>
> *Student:* Divide by four.
>
> *T:* I'd divide by 64. (He does so.) Then A is four, because the first piece is A.
>
> *Ss:* Huh?
>
> *T:* If I can't remember about A and B, forget about them; instead, I want to know which pair of points works in the equation: $(\pm 4,0)$ or $(0,\pm 4)$. [He demonstrates that the first works and the second doesn't.] So I can always check the points.
>
> *S:* Can't you just say X goes first, so put it on the X axis?
>
> *T:* Yes, that's the easy way.

Many of Sandy's statements and questions in class are heuristic in nature. In the vignette above, for instance, he stated relevant given conditions, asked for possible approaches, suggested an alternative route if stuck, and acknowledged and evaluated a third alternative. The statements and the questions together constitute modeling of how he himself might solve the problem and of how he hopes the students will eventually learn to solve similar problems. In other words, in his teaching Sandy employs extensively the heuristics he says he wants the students to learn as problem-solving content. Furthermore, the value of the heuristics lies in their generality, and he illustrates this fact by applying them to many diverse situations. Nevertheless, Sandy did not once refer explicitly to the heuristics themselves during the nine classes I observed. Researchers studying the role of metacognition in mathematical problem solving believe that strategy selection and monitoring need to become conscious processes for students. "Bringing these decisions out of the unconscious into the realm of conscious planning seems to be an important part of problem-solving instruction (McLeod, 1985, p. 271). So despite Sandy's constant modeling of heuristics in class, his neglecting to talk about them may exclude them from students' awareness and thus severely limit their utility for the students' own problem solving.

Sandy uses this heuristic modeling and questioning to teach both concepts and procedures. In the given vignette, for instance, he reinforced the concept of intercepts and the procedure of graphing a conic section not in intercept form. Other examples of concept learning abound: "How do I know the long axis of an ellipse?" "Number 41 is a little different; how?"

"What does B represent in a hyperbola, in whose graph it doesn't appear?" Instances of procedures are equally numerous: "The first thing I want to know is what A and B are; how do I find them?" "The next thing is to find the center; what is it?" "Now we know all we need to graph it." When describing the problem-solving features of his lessons, Sandy explicitly mentions instances of concept learning, procedure selection and use, and social interaction as well as heuristics. These processes clearly constitute problem-solving instruction for him.

<u>Incidence and attitudes</u>.

> (Sandy, interview) In good teaching [problem solving] should be emphasized and inherent, an inherent part of everything. In bad teaching I think problem solving is still going on, but it may be not being presented as the greatest usefulness.

Problem solving is incorporated into Sandy's teaching in a distinctive way. In his view it is inherent in just about everything: the teacher's planning, explanations, examples, and questions; the students' answers in class, homework problems, and learning of mathematics; and the interactions among students and teacher. The problem solving remains inherent, however, and never becomes the object of discussion. The mathematical forms of problem solving are closely associated with the mathematics content of the course, though in an unvarying way. That is, Sandy uses a relatively fixed set of heuristic devices to teach virtually all mathematical ideas and solve all problems, regardless of the particular math content. In addition, Sandy contends that he and his students continually practice more generic forms of problem solving.

How much problem solving Sandy's Algebra 2 students actually do is difficult to measure. By his estimation problem solving is a major and pervasive feature of the class, whereas an observer with a different definition of problem solving might say that it is minimally represented. The point here is not to adjudge one view or another correct, but to recognize that thoughtful people can hold very different interpretations with some conviction. The central image of the present case study is one teacher's vision of problem solving; other teachers no doubt harbor their own visions. One of the practical tasks of problem-solving research is to discover the ways that teachers envision problem solving. The parallel theoretical task, often cited in the literature (e.g., Grouws, 1985; Shumway, 1982), is trying to define problem solving in a way that permits reliable identification of instances and non-instances. These two tasks should proceed hand in hand. The collective wisdom of teachers can help researchers devise a definition with significance for the education of children, and such a definition can help researchers interpret teachers' views. The combined results should assist math educators in developing goals and means to improve problem-solving instruction.

Since Sandy never talks to his students explicitly about problem solving or its components, the attitudes he conveys are difficult to tie to problem solving per se. The way he treats it in his teaching carries an implicit message, however: problem solving is an inherent and unvarying part of mathematics that doesn't merit scrutiny in its own right, it's just one of those things that you try to get better at doing. As for Sandy's goals of teaching such attitudes as confidence and persistence, he displays these attributes in his teaching, but the students' limited problem-solving activity may hinder their own development of these traits.

To sum up Sandy's instruction, he actually teaches problem solving in the way he thinks it should be taught. His predominant instructional methods are modeling and lecture-questioning, and the problem-solving content consists of heuristics used implicitly to teach mathematical concepts and procedures expressly. That is to say, his problem-solving instruction follows directly from his pedagogical knowledge of problem solving. Much more problematic is the relationship between Sandy's teaching and his students' learning. Polya (1957) described succinctly what many educators would agree is the appropriate role of the teacher of problem solving: "The student should acquire as much experience of independent work as possible. . . . The teacher should help, but not too much and not too little, so that the student shall have a *reasonable share of the work* [italics in original]" (p. 1). In his directiveness Sandy assumes the lion's share of the work. Also, with few exceptions the problems he asks his students to do are routine, though not necessarily trivial, and the concepts he questions them about seldom address a very high level of generalization or a novel application. If the students are acquiring strategic knowledge at all, it may be limited to particular forms in certain narrow domains rather than the flexible and powerful body of heuristics Sandy himself seems to possess. Questions about what conceptions and skills students develop represent important follow-up research for conceptual studies of teachers and teaching.

Context

Textbook, course, and students. Although Sandy recognizes that the textbook can have a powerful effect on what is taught, its effect on his teaching of problem solving in Algebra 2 is minimal. He uses the book mainly as a source for standard math problems, supplementing as he feels necessary. Inasmuch as he teaches problem solving through the medium of these standard problems, virtually any Algebra 2 text would serve as well. The course content, determined by a one-page department outline and by the textbook, guides the mathematical topics Sandy teaches but has no effect on problem solving. As for differences in students' knowledge or ability, these influence the kinds of problems Sandy gives them. Nevertheless, he would use essentially the same heuristic content and teaching methods for a class full of good students or poor students as for average ones. Any differences would be a matter of shading: a faster or slower speed, an emphasis on sophisticated or simple skills.

<u>Other</u>.

> (Sandy, interview) Their behavior is either going to limit or broaden the alternatives of types of instruction that I can use. . . . If I have to worry about discipline problems or students being able to sit next to each other or talk with each other, then I can't do the group work, or I can't do other techniques I might want to use.
>
> So let's say I decided to do it that way, the more non-teacher-centered. . . . It would probably be a two-day lesson. . . . But in 50 minutes it's more difficult, with other housekeeping tasks and things to do. So that's a concern.

Classroom discipline is a very important issue for Sandy. Although he seems to have no trouble maintaining order, he dislikes discipline and takes some pains to ensure that it doesn't become a problem. At the beginning of the year, for instance, he distributed to all his students a one-page course introduction listing, not course content, but the mutual responsibilities of students and teacher in his classroom. Sandy's heavy reliance on teacher-centered instruction stems from his concern with discipline. Small wonder, then, that management considerations impose constraints on problem solving. "Say I see a set of problems that I'd like the students to do. . . . The first thing I think of is behavior. . . . That's the first line I consider in everything I do. Is this going to create potential explosive situations?"

Time is an important constraint on Sandy's teaching of problem solving. The yearly calendar sets the overall pace, the weekly calendar disturbs natural content units, and the clock forces him to run too rapidly through individual lessons. After some lessons Sandy remarked that they were too rushed, with no time for student questions; the lessons he felt better about were the slower ones with adequate time for questions. This time pressure hinders more student-centered, time-intensive teaching methods.

The other significant contextual influence on Sandy's teaching of problem solving is his own welfare. Less teacher direction often requires making up extra worksheets, the use of manipulatives involves going to the store for marbles, not handing out answer keys during group work means answering the same questions over and over, and doing group work at all calls for reteaching the same things several times. Sometimes Sandy simply chooses not to put his energy into these kinds of activities, given the demands of teaching five classes of kids each day, performing all the ancillary tasks, and still maintaining a personal life. "My energy and health. . . is a major factor in the equation."

The content-free issues of time and management are important features of context for Sandy, as for most teachers, and they affect all of his teaching. In contrast, content-related features such as textbook, course description, and students' ability have relatively little effect on his content or methods for teaching problem solving. As noted earlier, Sandy views problem solving as central to mathematics in general but essentially independent of mathematical topics, and this conception carries over into his instruction. Consequently, problem solving is a fairly constant element in his teaching regardless of variations in content.

The Lower-Case Curriculum

The descriptive portion of this study has answered the five guiding research questions in the case of this one teacher. Often Sandy's particular traits or choices suggested the kinds of alternatives other teachers might display. But how do these results generalize? And what do they signify?

Sandy's knowledge of problem solving seems to hold a great deal of explanatory power for his teaching of problem solving. Similar outcomes derived from the other studies of mathematics teachers mentioned earlier. Consequently, a model for classifying teachers' views of problem solving would be useful. Cooney (1985) suggested such a model, which I have borrowed and extended (see Figure 2). Two dimensions of teachers' views emerged from the few available cases: the visibility of the problem-solving process (that is, whether it is taught implicitly or explicitly) and the connection to the accompanying mathematical content (whether problem solving is isolated from or integrated with the mathematics). Though there is some middle ground, these variables are essentially bimodal. When they

		CONNECTION TO MATHEMATICAL CONTENT	
		Isolated	**Integrated**
VISIBILITY OF PROBLEM-SOLVING PROCESS	**Implicit**	vitamins	sugar
	Explicit	layer	nuts

Figure 2. Cake model for teachers' views of problem solving.

are crossed as in Figure 2, each of the four cells can be represented as a part of a walnut layer cake. Imagine the entire cake as the mathematics content of a course, the specified ingredient as the problem-solving content, and a slice as a typical lesson or unit. If problem solving is a layer, either thick or thin, then it can be seen and tasted for itself but is distinct from the rest of the mathematics. The walnuts are not only tasty nuggets in their own right, they also permeate the entire cake and give it its flavor. The sugar is likewise inseparable from the cake but is unrecognizable per se. As for the vitamins, they may be present and may even do some good, but who knows and who cares.

Both Sandy and Fred, Cooney's novice teacher, expressed their opinions that problem solving is a crucial part of mathematics and that problem-solving heuristics are very important for students to learn, which seemed to put them in the Explicit-Integrated category (nuts). Fred, however, was ambivalent, repeatedly providing indirect evidence that he saw problem solving as a dispensable supplement to mathematics. Cooney described Fred's conceptions of problem solving as like a layer of cake, and this more isolated stance actually governed his teaching. Interestingly, in their instruction both teachers shifted toward the implicit extreme by not teaching problem solving in any explicit way, though Sandy's approach remained integrated and Fred's isolated. One use for this classificatory scheme, then, is to examine the consistency between what teachers believe and how they teach.

Sandy views problem solving like sugar in a cake. This exemplifies problem solving with a small "p"--not, "Okay, kids, today we're going to do Problem Solving!" but an integrated, low-profile approach to the topic, spread throughout the math curriculum and useful in other areas as well. This approach may be more common than we suspect because it is not ostentatious. The concept applies not only to problem solving in mathematics but to other content areas which are not part of the standard academic curriculum, such as critical thinking in history, inductive thinking in science, self-expression in English, organizational skills in accounting, and group interactional skills in physical education. I call these areas the "lower-case curriculum."

Recently these areas have received a lot of attention in education; clearly they represent crucial abilities in their respective fields and beyond. Yet despite their importance, they are rarely part of the official curriculum, seldom one of the dozens of things that have to be covered in a given course. As a result, the teaching of these areas--what, how, and how much--varies wildly. Moreover, these areas are by nature relative to the primary subject matter fields and so are sometimes hard to identify. At first glance a classroom observer could conclude that one of the lower-case areas is not being taught at all, whereas closer scrutiny might reveal that it is woven into the primary content in subtle ways.

The cake model for problem solving provides a way to analyze the teaching of any area in the lower-case curriculum. This model only describes teachers' views of the content, however, and doesn't discriminate finely enough among teaching styles. It needs to be expanded to encompass learners' experiences as well (see Figure 3). In addition to visibility of the process and connection to the primary content, the extended model includes engagement of the students (i.e., whether they are passive or active with respect to the lower-case content) and prevalence of their experience (whether they encounter this content rarely or frequently).

<u>TEACHERS' VIEWS</u>

VISIBILITY: implicit vs. explicit

CONNECTION: isolated vs. integrated

<u>LEARNERS' EXPERIENCES</u>

ENGAGEMENT: passive vs. active

PREVALENCE: rare vs. frequent

Figure 3. Dimensions of teaching the lower-case curriculum.

The new model provides niches for several prototypical ways of treating problem solving, and these are probably mirrored in other lower-case areas. For example, a teacher who takes problem solving seriously may very well have her students memorize a litany of Polya-type heuristics, practice them on a few examples, and continue on to the next topic (high visibility, low elsewhere--problem solving with a capital "P"). Fred often gave his students unusual or recreational problems, puzzles, and games to make math fun, but he ignored the kinds of solution strategies these exercises might have provoked (low visibility and connection, high engagement and prevalence). Some math educators advocate simply giving students a lot of interesting, challenging problems in each mathematics topic and letting them develop their own strategies (low visibility, high elsewhere). These are very distinct ways of teaching problem solving, and they are likely to have very different consequences for the students. This model could be used to study the effects on learning of various teaching styles.

The four dimensions in this model capture much of what is important about Sandy's teaching of problem solving: implicit treatment of the process of problem solving, integration with the mathematics, passive involvement by the students, and frequent opportunities to use problem solving (though this is arguable, as discussed earlier). Even this characterization is inadequate, however. For instance, it suppresses Sandy's more or less constant use of the same heuristics, which qualifies the extent of his integration of the problem solving with the mathematics. In other words, we still need case studies to mine the riches of teachers' knowledge and its effects on their teaching. As these studies accumulate in problem solving and other lower-case domains, perhaps the model developed here can be useful in characterizing teachers' approaches to these non-standard subjects and in facilitating cross-case comparisons.

Conclusion

This study of Sandy demonstrates a particular view of problem solving in mathematics teaching. It also bolsters the conclusions of earlier studies that a teacher's conceptions of her subject matter have a profound impact on what and how she teaches. The findings carry important implications for several areas. Teacher education programs may be more effective if they concentrate on ways of thinking relative to the subject matter rather than on general pedagogical techniques. Criteria for selection of teacher candidates may grow in importance as we learn more about the role of personal attributes in teaching. Competency testing, which looms larger every day, may be fairer and more productive if it takes account of teachers' treatment of non- standard but important topics like problem solving. Classroom observation, whether for the purpose of research, supervision, or evaluation, should be extensive and sensitive enough to detect subtle ways of incorporating the lower-case curriculum into teaching. Textbooks, commonly supposed to dictate the mathematics curriculum, may in their present forms hold little significance for the teaching of problem solving.

The results of this study also bear on classroom teaching. The lack of customary forms and materials for teaching problem solving means that teachers must talk with one another, read, attend in-service programs, experiment, or find other ways to discover the range of alternatives and to choose intelligently among them. Since personal factors seem to strongly affect teaching style, teachers should try to become aware of these effects, both negative and positive. On the other hand, teachers' knowledge doesn't always translate accurately into their instruction, so teachers ought to look for such inconsistencies and try to reconcile them. Finally, teachers and administrators need to recognize the severe constraints that course curricula, time, and classroom organization and management place on the teaching of problem solving, and they should work together to alleviate these constraints and facilitate the attainment of important instructional objectives.

Starting from the handful of recent case studies of mathematics teachers, research could extend in three different directions. One road leads toward students, trying to link up with teacher effectiveness and instructional studies; we would like to know what effect Sandy's teaching of problem solving had on his students. Another route leads toward cognitive science, exploring how teachers acquire knowledge of broad topics and then transform these into instructional choices. Perhaps the most important avenue lies straight ahead, toward teachers themselves. We need additional case studies to illuminate the range of conceptualization and instruction, and cross-case comparisons to demonstrate sources of variation. We must flesh out our skeletal case knowledge with more studies of real teachers in real classrooms, thinking as teachers think and doing what teachers do. That's where we'll learn about teaching.

References

Begle, E.G. *Critical variables in mathematics education.* Washington, D.C.: Mathematical Association of America & National Council of Mathematics, 1979.

Briars, D.J. "Implications from information-processing psychology for research on mathematics learning and problem solving." In F.K. Lester, Jr., & J. Garofalo (Eds.), *Mathematical problem solving: Issues in research.* Philadelphia: The Franklin Institute Press, 1982.

California State Department of Public Instruction. "Mathematics framework for California public schools, kindergarten through grade twelve." Sacramento, 1985.

Cooney, T.J. "A beginning teacher's view of problem solving." *Journal for Research in Mathematics Education*, November 1985, *16*, 324-336.

Glaser, B.G. & Strauss, A.L. *The discovery of grounded theory: Strategies for qualitative research.* New York: Aldine Publishing Co., 1967.

Grouws, D.A. "The teacher and classroom instruction: Neglected themes in problem-solving research." In E.A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives.* Hillsdale, NJ: Lawrence Erlbaum Associates, Inc., 1985.

House, P.A. "Risking the journey into problem solving." In S. Krulik & R.E. Reys (Eds.), *Problem solving in school mathematics* (1980 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: NCTM, 1980.

Kilpatrick, J. "A retrospective account of the past 25 years of research on teaching mathematical problem solving." In E.A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives.* Hillsdale, NJ: Lawrence Erlbaum Associates, Inc., 1985.

McLeod, D.B. "Affective issues in research on teaching mathematical problem solving." In E.A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives.* Hillsdale, NJ: Lawrence Erlbaum Associates, Inc., 1985.

National Council of Teachers of Mathematics. "An agenda for action: Recommendations for school mathematics of the 1980s." Reston, VA: NCTM, 1980.

Polya, G. *How to solve it* (2nd ed.). Princeton, NJ: Princeton University Press, 1957.

Shulman, L.S. "On teaching problem solving and solving the problems of teaching." In E.A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives*. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc., 1985.

Shulman, L.S. "Those who understand: A conception of teacher knowledge." *American Educator*, Spring 1986, *10*, 8-15, 43-44.

Shulman, L.S. & Elstein, A.S. "Studies of problem solving, judgment, and decision making: Implications for educational research." In F.N. Kerlinger (Ed.), *Review of Research in Education* (Vol. 3). Itasca, IL: F.E. Peacock Publishers, 1975.

Shumway, R.J. "Problem-solving research: A concept learning perspective." In F.K. Lester, Jr., & J. Garofalo (Eds.), *Mathematical problem solving: Issues in research*. Philadelphia: The Franklin Institute Press, 1982.

Silver, E.A. "Research on teaching mathematical problem solving: Some underrepresented themes and needed directions." In E.A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives*. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc., 1985.

Steinberg, R., Haymore, J., & Marks, R. "Teachers' knowledge and structuring content in mathematics." Paper presented at the annual meeting of the American Educational Research Association, Chicago, April 1985.

Thompson, A.G. "The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice." *Educational Studies in Mathematics*, May 1984, *15*, 105-127.

APPENDIX A. DATA SOURCES

Interviews

I1	Background
I2	Personal knowledge
I3F	Instructional knowledge
I3G	Instructional knowledge
I4	Contextual knowledge and adaptation
I5	Pre-unit (2) (I5A, I5B)
I6	Post-unit (2) (I6A, I6B)
I7	Debriefing (respond to first draft of writeup)

Tasks

T1	Solve math problem, think aloud (2) (T1F, T1G)
T2	Classify math problems as Problem Solving, not PS, or not sure
T3	Rank math problems according to personal enjoyment
T4	Write examples of PS and non-PS problems in a given topic
T5	Write methods for teaching a given topic with and without PS
T6	Rank importance of various student learnings in math curriculum
T8	Write (or rewrite) a lesson plan to emphasize PS

Observations

OA	Reference cycle (3) (OA1, OA2, OA3)
OB	Topic variation (3) (OB1, OB2, OB3)
OC	Class variation (3) (OC1, OC2, OC3)

Documents

D1	Course descriptions/syllabi: teacher, school, department, district
D2	Text sections taught during observations (2) (D2A, D2B)
D3	PS resources used by teacher
D4	Unit plans for observations (2) (D4A, D4B)
D5	Daily lesson plans for observations (6) (D5A1, D5A2, ... , D5B3)
D6	Handouts or other materials during observations

DATA SOURCES / CONCEPTUAL CATEGORIES	Interviews	Observations	Tasks	Documents
SOURCES	I 1, I 2, I 3			
KNOWLEDGE				
Personal				
skills	I 2		T 1	
conceptions	I 2		T 2	
attitudes	I 2		T 3	
Pedagogical				
curriculum	I 3, I 4		T 6	D 3
methods	I 3, I 4		T 5	D 3
content	I 3, I 4		T 4	D 3
TEACHING				
Planning				
incidence	I 5		T 8	D 4, D 5
content	I 5		T 8	D 4, D 5
methods	I 5		T 8	D 4, D 5
attitudes	I 5		T 8	
Instruction				
incidence	I 6	OA,OB,OC		
content	I 6	OA,OB,OC		
methods	I 6	OA,OB,OC		D 6
attitudes	I 6	OA,OB,OC		
CONTEXT				
course	I 4			D 1
textbook	I 4			D 2
students	I 4	OA,OB,OC		
other	I 4	OA,OB,OC		

Table 1. Data sources by conceptual category.

APPENDIX B. INTERVIEW SCHEDULES

INTERVIEW #1: BACKGROUND

Introduction

I'm pleased that you're interested in participating in this research project. Let me explain it further, then you can tell me whether you want to go ahead with it.

Pre-interview

Explain study:
-- recent emphasis on PS in math ed, we expect teachers to teach it, but we don't know much about how this happens
-- I'm interested in the teacher: what he thinks of PS and how he teaches it
-- this entails interviews, a few little tasks, some classroom observation, and some document collection

Your commitment:
-- about 5 hours of interviews
-- four observation cycles of 2 to 3 days each; no intervention, just brief questions before and after class

Your benefits: talking about some of these ideas tends to stimulate teachers' thinking in interesting and productive ways

Questions?

Are you willing to participate?
-- (If no, "thanks" and leave: otherwise continue)

Discuss access: Whom should I speak with, if anyone? How approach him/her? Will kids have any problem with my observing quietly from back of room?

OK if I audiotape our interviews?
-- (If yes, start tape)

Reference info

1. Name
   Address
   Phone #: home; hours
   Phone #: work; hours
   Pseudonym

2. School name, location
   Pseudonym

3. Courses you are now teaching; describe briefly
   Teaching schedule, incl. rooms
   Times and places usually available for interviews

Professional background

4. Tell me a little about your own education....
   (Undergraduate career? Graduate career? Teaching credential?)
   (What turned you on academically? What did you like about it? ... off? ... dislike?)
   (What were you especially good at? Bad at?)
   (You must have taken some math in college.... You probably did pretty well in it.... You must have liked it.... Did you plan to use the math later?)

5. And then somehow you became a teacher....
   (You've taught high school math.... What else?)
   (How did you get into teaching? When?)
   (There must be some things about teaching that you like and some you dislike....)
   (So you're teaching now. What will you be doing next year? In 5 years? In 15?)

Post-interview

Access: make necessary arrangements

Discuss observation plan:
-- composition of the four units
-- criteria for topics: significant PS, coherent, convenient

Select topic and dates for first 1 or 2 units
-- schedule observation dates
-- schedule short pre-observation interview
-- schedule short post-observation interview
-- ask to have copies of text sections, unit & lesson plans, and handouts; offer to copy these when convenient

Schedule Interview #2 if convenient (allow an hour maximum)

Ask to have copies of course descriptions and syllabi; offer to copy these

Conclusion

I'm delighted you've agreed to participate in this study. It's going to be a very interesting experience for me; I hope it is for you too. Thank you very much for the time you've donated. I'll see you on ....

END OF INTERVIEW

INTERVIEW #2: PERSONAL KNOWLEDGE

Introduction

As you know, this research is concerned with problem solving in mathematics teaching. In today's interview I hope to learn about your ideas about problem solving in general, without regard to teaching; in the next interview we will discuss problem solving in the context of teaching.

Definition 1

1. If I said "problem solving" to ten different people, they'd probably think of ten different things. Suppose I said "problem solving" to you.
   (When I say "problem solving," what comes to mind?)

2. I have a friend who is an excellent problem solver. What do you think my friend is like?
   (What characteristics do you think my friend has that make him an excellent problem solver?)
   (Try to think of someone you know who is a good problem solver; describe.)
   (How would that person be different from a poor problem solver?)

3. a. Can you give me examples of two math problems, one of which entails what you consider "problem solving" and one of which doesn't?
   (Separate)
   (Think of some problems you've used recently in your teaching.)

   b. (What makes ___ an example of problem solving while ___ isn't?)
   (Separate)

Acquisition

4. You're a math teacher; you must be a pretty good problem solver.
   (How would you describe yourself as a problem solver?)
   (How good a problem solver are you compared to other math teachers? other teachers in general? other ___ majors?)
   (There must be some kinds of problems that are easy for you.)
   (There must be some kinds of problems that are hard for you.)
   (Do you have a particular style of problem solving? Can you describe it?)

5. It must have been an interesting process, your learning about problem solving.
   (How did you learn what you know about problem solving?)
   (Through explicit teaching, or implicitly?)
   (In math classes? In other classes? Elsewhere? From particular people?)

6. Imagine a course entitled "Mathematical Problem Solving" at your (undergraduate) college.

   a. Can you describe this course for me?
   (What topics would it cover? What would class sessions be like?)

   b. Suppose this course had been available when you were a student.
   (Would you have taken it? Would you have enjoyed it?)
   (Should it be an option for math majors? Math teachers? A requirement?)

Definition 2

7. Some people see problem solving as a skill, some as an art, and there must be many other opinions. How do you see problem solving?
   (In what ways?)

8. Some people say that mathematical problem solving is different from problem solving in general, and some say they're the same. What would you say?
   (In what ways are they different? Alike?)

Significance

9. You must do a fair amount of problem solving.

   a. (Is problem solving important in your life?)
      (Do you use it often? Does it make a difference in your life?)

   b. (Where and when do you use problem solving?)
      (In mathematics? In other academic areas? In non-academic areas?)

   c. (Is problem solving important in most people's lives?)
      (For what people is it important?)

   d. (Do you enjoy problem solving?)
      (What kinds?)
      (What do you like about it?)

Definition 3

10. We've talked about problem solving from various perspectives. To sum up, there are lots of different ways to define "problem solving." How would you define it?

Closing

That's all for this interview; thank you very much. In our next interview I'll ask you about problem solving in the context of teaching (high school, junior high school, middle school) mathematics; maybe you could think about that in the meantime.

END OF INTERVIEW

INTERVIEW #3F: INSTRUCTIONAL KNOWLEDGE

Introduction

As you recall, this research is concerned with problem solving in mathematics teaching. In our last interview I explored your thinking about problem solving itself; today I hope to learn about your ideas about the teaching of problem solving.

General

1. Can problem solving be learned? (Elaborate)
   Can problem solving be taught? (Elaborate)

Curriculum

2. Different people hold various views of the role of problem solving in the high school curriculum. Can you describe some of these views to me? I'm less interested here in your preferences than in your knowledge of alternatives.
   (What are some views of the importance of problem solving? Of its place in the curriculum? Of its purpose in the curriculum?)
   (Could problem solving be fully integrated with other mathematics teaching? For example? Could it be taught separately?)

3. Now I'd like to hear about your own preferences. What do you think should be the role of problem solving in the high school curriculum?
   (What is the purpose of teaching problem solving? Is it important?)
   (Should problem solving be taught in mathematics courses? In other subjects? As a separate course?)

Methods

4. For the next series of questions I'd like you to differentiate teaching method from teaching content. For instance, a teacher may lecture on the quadratic formula; the students are supposed to learn the quadratic formula as content, but the teacher uses lecture as a method to teach it. Can you describe some different methods that teachers might use to teach problem solving in mathematics?
   (What are some of the advantages and disadvantages of these various methods?)

5. You must prefer some of these methods to others.
   (Which methods do you use? How often?)
   (Are all of these methods that you use equally effective?)

Content

6. Now let's turn to the content of problem solving instruction. What do you actually try to teach the students about problem solving?
   (What problem solving techniques do you teach?)
   (What kinds of problems do you use to teach problem solving? Include specific examples.)

7. What other problem solving content might other teachers teach their students?
   (What are some of the advantages and disadvantages of these alternatives?)

Sources

8. You seem to have a wide knowledge of the teaching of problem solving; this knowledge must have come from various sources.
   (Are you familiar with any writings about problem solving? If so, could you tell me about some of those ideas and their sources?)
   (Do you recall any ideas about teaching problem solving which came from:
   your teacher education program?
   professional contact with other teachers: in school, at conferences, etc.?
   your independent reading and study?
   your textbook or teacher's manual or other curriculum materials?
   your own teaching experience?
   any other sources?)

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NOTE: For lack of time, remainder of this interview is included in I3G.

Summary

This last series of questions pertains explicitly to your own views.

9. Suppose your math department organized a task force to recommend ways to include more problem solving in the curriculum, and they chose you as the chair of the task force. What would you do?

10. Can you give me an example of a lesson you have taught, [illegible] taught, or would like to teach, which emphasizes problem solving?
    (In what ways is problem solving a part of this lesson?)

11. What do you expect your Algebra 2 students to know or be able to do by the end of this year with regard to problem solving?
    (What learning experiences will they have had during this year that will help them achieve that goal?)

12. Is there anything you'd like to add, anything I should have asked you in this interview but didn't?

Closing

Thank you very much. In our next interview I'll ask about the influences of context -- for example, course descriptions, textbooks, and students -- on the teaching of problem solving in high school mathematics; you might want to think about that in the meantime.

END OF INTERVIEW

INTERVIEW #3G: INSTRUCTIONAL KNOWLEDGE

Introduction

As you may recall, we ran short of time in our last interview; I'd like to complete that now. Then I'll ask you to clarify some ideas that arose in earlier interviews. Finally, I have a couple of brief tasks for you to do. Okay?

Curriculum

This series of questions completes last week's interview on curriculum. These questions pertain explicitly to your own views.

1. Suppose your math department organized a task force to recommend ways to include more problem solving in the curriculum, and they chose you as the chair of the task force. What would you do?

2. Can you give me an example of a lesson you have taught, seen taught, or would like to teach, which emphasizes problem solving?
   (In what ways is problem solving a part of this lesson?)

3. What do you expect your Algebra 2 students to know or be able to do by the end of this year with regard to problem solving?
   (What learning experiences will they have had during this year that will help them achieve that goal?)

Clarification

The next few questions are designed to help me understand better some of the things you've said in previous interviews.

4. When I asked (I2) whether mathematical problem solving is different from other kinds of problem solving, you indicated that mathematical skills are necessary as prerequisites; I'd like to explore that further.

   a. Is problem solving in mathematics distinguishable from math itself?

   b. Are there identifiable problem solving skills per se, apart from subject matter (e.g., mathematical) skills and knowledge?

   c. You said "I generally think that there is a generalized problem solving ability that people have." Can you tell me more about this ability?
      (Is it innate or learned?)
      (Does it consist of a collection of techniques and/or skills, or is it more diffuse?)

5. When I asked you for a definition of problem solving (I2), you said "a situation you have to do something about and you're motivated to solve it."

   a. Does <u>every</u> task which requires your attention qualify as a problem, and doing it as problem solving?
      (e.g., noticing you're low on cash as you drive home from work?)
      (e.g., getting hungry about noon on Saturday?)
      (e.g., responding to my questions in this interview?)

   b. Do you differentiate "problem" from "task", or "problem solving" from "task performance"?
      (How? What is the relationship between them?)

6. In discussing problem solving, you also mentioned (I5B) the students' describing models -- e.g., a laser beaming to the moon, two spaceships meeting.

   a. Are these examples of problem solving?

   b. Do you differentiate "mathematical problem solving" from "applications of mathematics"?
      (How? What is the relationship between them?)

7. Once when I asked you about problem solving (I6B) you emphasized the students' understanding what a system of equations is.

   a. Is this an example of problem solving?

   b. Do you differentiate "problem solving" from "concept learning"?
      (How? What is the relationship between them?)

<u>Closing</u>

Thank you very much. In our next interview I'll ask about the influences of context -- for example, course descriptions, textbooks, and students -- on the teaching of problem solving in high school mathematics; you might want to think about that in the meantime.

END OF INTERVIEW

INTERVIEW #4: CONTEXTUAL KNOWLEDGE AND ADAPTATION

Introduction

In our last interview we discussed your ideas about the teaching of problem solving in general. Today I'd like to explore your thinking about the teaching of problem solving in particular contexts, how you might adapt your teaching to fit specific situations.

General

1. When you start planning a course at the beginning of the year, how do you go about deciding what to teach and how to teach it?

   How do you make adjustments to that initial plan as the school year progresses?

Course

2. The school's course outlines list only the mathematical topics to be covered; they say nothing about problem solving. What significance does this have for you?

   Do you talk with other teachers of the same courses or your department chair about problem solving?
   (What is said? What are the effects of this communication?)

3. You're currently teaching Algebra 2, Algebra 1, and Algebra .5 . Does problem solving feature equally in the way you teach all three?
   (How does it vary? For what reasons?)

   Are there other math courses in this school in which you would teach problem solving more, or less, or differently?
   (In what ways? Why?)

Text

4. In general, what do you think of your current Algebra 2 textbook?

   How do you use the text?

   How do you expect your students to use the text?

5. What do you think of the Algebra 2 text from a problem solving point ot view?
   (Does its presentation encourage or discourage problem solving?).
   (What about the problems themselves?)

   Do you use the text to teach problem solving?
   (To what extent? In what ways?)
   (Do you supplement the text for problem solving instruction?)

   If you were to rewrite the text to emphasize problem solving, what would you do?