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ABSTRACT

Problem solving that both requires and develops higher-order thinking skills is illustrated in the EQUATIONS Challenge Matches program, which consists of computer diskettes at the elementary, intermediate, and advanced levels plus a multi-level diskette. Samples of computer output are given, with the results of various types of experimentation with the program discussed and illustrated. Users are expected to design and perform experiments involving application of elementary mathematical ideas. The problem-solver can arrange for things to happen that will provide the information needed to deal with the problem. The series of exercises has been developed to provide users the opportunity to learn and use the fundamental reasoning and problem-solving skills of careful observation, logical deduction, mathematical analysis, asking good questions, scientific research by experimentation, and data gathering, organization, and analysis. (MNS)

THE LOGICAL, MATHEMATICAL, AND SCIENTIFIC REASONING INVOLVED IN THE PROBLEMS OF A SAMPLE RUN OF THE EQUATIONS CHALLENGE MATCHES RESEARCH MODE

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The Logical, Mathematical, and Scientific Reasoning Involved in the Problems of a Sample Run of the EQUATIONS Challenge Matches Research Mode

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Problem solving that both requires and develops higher-order thinking skills is illustrated in the EQUATIONS Challenge Matches program. [1] The fundamental reasoning skills involved in logic, mathematics, and science are useful for dealing with the challenging problems posed in the RESEARCH MODE of the EQUATIONS Challenge Matches program.

When the Multi-Level diskette of this program is used and Match R9 is selected, the following appears on the screen:

```
   Solution Research                   Resources   Solution    Solutions
Match R9    Number of Solutions: 12      used      entered     on list:

GOAL      : 6
RESOURCES: - - x / / 1 2 3 4
FORBIDDEN:
PERMITTED:
REQUIRED :

     SOLUTION CHECKING MODE
Please type a proposed Solution, or
CONTROL L to see the "next" page in the
List of Solutions or just press RETURN
to return to MOVE ENTRY MODE.                 Page number  1 of 1
?
```

What this says is that there are at least 12 different Solutions that can be built for the GOAL of 6 from the RESOURCES given. How many of them do you see?

The rules for building Solutions are:

1. A Solution expresses a number equal to the number expressed by the GOAL.
2. One Solution is different from a second Solution only if it uses different RAESOURCES from those used by the second Solution. Thus, 2x3 and 3x2 are not different Solutions, but 3x2 and (4-1)x2 are.
3. Only the RESOURCES available can be used in building a Solution. For example, neither 4+2 nor 2x3x1 are Solutions because there is no + available and there is only one x available.
4. The / indicates division, the x indicates multiplication, and the - indicates subtraction.
5. There are an unlimited number of parentheses available to indicate how to group parts of the expression. (However, multiplication can be expressed in a Solution only by using an x.)
6. Only single-digit numerals can be used in a Solution. Thus, for the GOAL of 3 with RESOURCES of 1 2 4 6 x /, the expression, 12/4, is not a Solution, but 6x2/4 is.

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[1] The EQUATIONS Challenge Matches program consists of four diskettes -- one each at elementary, intermediate, and advanced levels, and a multi-level diskette. The diskettes are available for Apple II, II+, IIc, and IIe microcomputers, and also for IBM PC and IBM PC-compatible machines. For details, write to WFF 'N PROOF, 1490 South Boulevard, Ann Arbor, MI 48104.

If you entered the Solution 3x2 at the ? prompt and then pressed RETURN, the screen would change to the following:

```
    Solution Research                             Resources   Solution    Solutions
Match R9    Number of Solutions: 12                 used      entered     on list:
                                                                             1
GOAL       : 6                                     23 x       3x2
RESOURCES: - - x / / 1 2 3 4
FORBIDDEN:
PERMITTED:
REQUIRED :

3x2
is a Solution.

      SOLUTION CHECKING MODE
Please type a proposed Solution, or
CONTROL L to see the "next" page in the
List of Solutions or just press RETURN
to return to MOVE ENTRY MODE.                          Page number  1 of 1
?
```

This feedback would verify that 3x2 is a Solution, list the RESOURCES used in it in an orderly way, indicate that you had achieved 1 of the 12 different Solutions possible, and prompt you with a ? for your next entry. If you then entered 2x3 as a Solution at the ? prompt, what would appear of the screen is the following:

```
    Solution Research                             Resources   Solution    Solutions
Match R9    Number of Solutions: 12                 used      entered     on list:
                                                                             1
GOAL       : 6                                     23 x       3x2
RESOURCES: - - x / / 1 2 3 4
FORBIDDEN:
PERMITTED:
REQUIRED :

2x3
is a Solution, but the set of RESOURCES
used in it does not differ from those
used in 3x2.

      SOLUTION CHECKING MODE
Please type a proposed Solution, or
CONTROL L to see the "next" page in the
List of Solutions or just press RETURN
to return to MOVE ENTRY MODE.                          Page number  1 of 1
?
```

How many additional Solutions do you see? Do you see all 12 of them? In this sample run of R9, suppose that the person saw and entered, one at a time as the ? prompt appeared: 3x2/1, 3x(4-2), 3x(4-2)/1, 3x(4/2), and 3x(4/2)/1. After the final entry, the following would be on the screen:

```
Solution Research                         Resources   Solution     Solutions
Match R9    Number of Solutions: 12          used      entered      on list:
                                                                       6
GOAL      : 6                               23  x      3x2
RESOURCES: - - x / / 1 2 3 4               123  x/     3x2/1
FORBIDDEN:                                 234  -x     3x(4-2)
PERMITTED:                                 234  x/     3x(4/2)
REQUIRED :                                1234  -x/    3x(4-2)/1
                                          1234  x//    3x(4/2)/1

3x(4/2)/1
is a Solution.

      SOLUTION CHECKING MODE
Please type a proposed Solution, or
CONTROL L to see the "next" page in the
List of Solutions or just press RETURN
to return to MOVE ENTRY MODE.                    Page number  1 of 1
?
```

How about it; have you run out of gas about here the way that our hypothetical person did? Or do you see another Solution to enter at the ? prompt? Do you want your students to learn to think deeply about such elementary math problems (and some more advanced ones, too)? There are effective ways to get into the habit of developing these deeper thinking skills in mathematics by more self consciously getting into the logical and scientific reasoning involved.. You can do it (among a host of other ways) by playing EQUATIONS, by matching wits with the computer in going through the DIG Math program, and by both cooperating and competing with other players while battling the computer in EQUATIONS Challenge Matches.

If you do not see any more Solutions, then the thing to do next in this RESEARCH MODE is to simply press RETURN to go to the MOVE ENTRY MODE, where the following will appear on the screen:

```
Solution Research                         Resources   Solution     Solutions
Match R9    Number of Solutions: 12          used      entered      on list:
                                                                       6
GOAL      : 6                               23  x      3x2
RESOURCES: - - x / / 1 2 3 4               123  x/     3x2/1
FORBIDDEN:                                 234  -x     3x(4-2)
PERMITTED:                                 234  x/     3x(4/2)
REQUIRED :                                1234  -x/    3x(4-2)/1
                                          1234  x//    3x(4/2)/1

SOLUTION     NUMBER OF SOLUTIONS
 LENGTH    Before Moves   After Moves
   11          0
    9          0
    7 .......  4   ........
    5          7
    3          1
Solutions Eliminated .....
```

```
      MOVE  ENTRY  MODE
Enter moves, CS to Check a Solution,
Q to Quit or CONTROL L to see the "next"
page in the List.                      Page number  1 of 1
?
```

In the MOVE ENTRY MODE you can conduct experiments and gain valuable information by making moves of the Resources that Forbid or Require their use. To illustrate how to do such experiments, suppose that you decide to Forbid the 1 that is in the Resources. You can do so by entering F1 at the ? prompt. What will appear on the screen if you enter F1 is the following:

```
    Solution Research                   Resources  Solution    Solutions
Match R9    Number of Solutions: 12       used     entered     on list:
                                                                  6
GOAL      : 6                           23 x       3x2
RESOURCES: - - x / /   2 3 4           123 x/      3x2/1
FORBIDDEN: 1                           234 -x      3x(4-2)
PERMITTED:                             234 x/      3x(4/2)
REQUIRED :                            1234 -x/     3x(4-2)/1
                                      1234 x//     3x(4/2)/1

SOLUTION     NUMBER OF SOLUTIONS
 LENGTH   Before Moves   After Moves
   11          0             0
    9          0             0
    7 .......  4 ........    0
    5          7             3

    3          1             1
Solutions Eliminated .....   7

      MOVE  ENTRY  MODE
Enter moves, CS to Check a Solution,
Q to Quit or CONTROL L to see the "next"
page in the List.                      Page number  1 of 1
?
```

The experiment conducted by moving the 1 to Forbidden means that the Forbidden 1 can no longer be used in constructing Solutions. This has the effect of eliminating all 4 of the possible Solutions that use 7 of the Resources and eliminating 4 of the 7 possible Solutions that use 5 of the Resources, but not eliminating the 1 Solution that uses 3 of the Resources. This, in turn, means that with the 1 Forbidden, there are still 3 possible Solutions that use 5 Resources (in other words, that are of length 5), and 1 possible Solution of length 3.

By examining the six Solutions on the list of those that have already been written, we can see that there is already one there of length 3 that does not use the Forbidden 1; so we already have that Solution.

However, with respect to the Solutions of length 5, it is quite a different story. Since there are only four numerals in the original set of Resources, when the 1 is Forbidden, all three of the remaining Resources must appear in each of the 3 Solutions of length 5. This means that each of these 3 Solutions of length 5 must contain the 2, the 3, and the 4. But when we look at the Solutions of length 5 on the list of those that have already been discovered, only two of them contain the three Required numerals. This means that there is another Solution of length 5 that contains the numerals, 2, 3, and 4, that we have not yet discovered. But what is that other length 5 Solution?

What operation signs does it use? How can we find out?

We can find out by conducting further experiments -- by making further moves.

If we examine the Resources available, we see that the only operation signs there are: - - x / and /. A Solution of length 5 will contain exactly two operation signs, and there are just five possible pairs of operation signs from those that are available, namely: - -, - x, - /, x /, and / /.

In the list of Solutions already discovered, the two of length 5 that contain the 2, the 3 and the 4, use the pairs of operations: - x, and x /. So, the unknown Solution of length 5 that uses the numerals, 2, 3, and 4, must use one of the three following pairs of Solutions: - -, - /, or / /. We can find out just which pair by conducting the three experiments of making the moves of Requiring 2 3 4 - -, Requiring 2 3 4 - /, and Requiring 2 3 4 / /.

When 2 3 4 - - are Required, the feedback is:
1 Solution possible of length 7
0 Solutions possible of length 5
0 Solutions possible of length 3
This indicates that there is no Solution of length 5 that uses the Resources, 2 3 4 - -.

When 2 3 4 - / are Required, the feedback is:
2 Solutions possible of length 7
0 Solutions possible of length 5
0 Solutions possible of length 3
This indicates that there is no Solution of length 5 that uses the Resources, 2 3 4 - -.

When 2 3 4 / / are Required, the feedback is:
2 Solutions possible of length 7
1 Solution possible of length 5
0 Solutions possible of length 3
This indicates that we have found what we are looking for; there is one Solution of length 5 that uses the Resources, 2 3 4 / /.

Once we know what Resources are used in a Solution, it is not too difficult to discover how to order and group them to construct a Solution.

We observe, first, that Solutions of length 5 are either of the form, No(NoN), or of the form, (NoN)oN, where N's indicate numerals and o's indicate operations.

We observe, second, that there are no expressions of the second form from the available Resources that equal 6, that is, there is no (N/N)/N = 6, where the N numerals are 2, 3, and 4.

We, then, ask ourselves such questions as:
3 divided by what (constructed from 24 /) = 6?
2 divided by what (constructed from 34 /) = 6?
4 divided by what (constructed from 23 /) = 6?
and we conclude that in two of the three cases, we get the results that we are looking for, namely:

3/(2/4) = 6 and 4/(2/3) = 6

So, we have found another Solution, which we can easily check by going back into SOLUTION CHECKING MODE and entering, for example, 3/(2/4).

After this new one has been added, our list of seven discovered Solutions includes:

3x2   3x2/1   3x(4-2)   3x(4/2)   3/(2/4)   3x(4-2)/1   3x(4/2)/1

So, we now have: the only Solution of length 3, 4 of the 7 Solutions of length 5, and 2 of the 4 Solutions of length 7.

If we now focus attention upon discovering the remaining pair of Solutions of length 7, we observe that all Solutions of this length will contain the numerals 1 2 3 4, and three operation signs. There are just five sets of triples of operation signs possible from the available Resources, namely: - - x, - - /, - x /, - / /, and x / /. We observe that the - x / and x / / triples have already been used in the three Solutions of length 7 that have already been discovered. So, the only three triples that remain to be tested are - - x, - - /, and - / /.

We can test these three triples by conducting experiments that make the moves of Requiring 1 2 3 4 - - x, Requiring 1 2 3 4 - - /, and Requiring 1 2 3 4 - / /.

When 1 2 3 4 - - x are Required, the feedback is:
1 Solution possible of length 7
0 Solutions possible of length 5
0 Solutions possible of length 3
This indicates that there is one Solution of length 7 that uses the Resources, 1 2 3 4 - - x.

Notice that an alternative way of acquiring the same information (and more) with the given Resources is to move to Forbidden the other operation signs, namely: / /.

When / / are Forbidden, the feedback is:
1 Solution possible of length 7
3 Solutions possible of length 5
1 Solution possible of length 3
This also indicates that there is one Solution of length 7 that uses the Resources, 1 2 3 4 - - x. In addition, it indicates that there are 3 Solutions of length 5 that use pairs of operations from the set consisting of - - x. This will turn out to be useful information in determining the other Solutions of length 5, but for the moment, we should focus on the next Solution of length 7. Since it uses two subtraction signs and a multiplication sign, one possible strategy to construct an expression equal to 6 is to use the x to express a number larger than 6 and the use the pair of -'s to reduce it to 6. The only pairs of numerals available, that when multiplied express a number larger than 6, are the pairs 2, 4 and 3, 4. When we explore the first pair, we quickly see that 4x2 = 8, and when we ask: 8 - ? = 6, we are quickly led to the Solution, 4x2-(3-1).

When 1 2 3 4 - - / are Required, the feedback is:
0 Solutions possible of length 7
0 Solutions possible of length 5
0 Solutions possible of length 3
This indicates that there is no Solution of length 7 that uses the Resources, 1 2 3 4 - - /.

When 1 2 3 4 - / / are Required, the feedback is:
1 Solution possible of length 7
0 Solutions possible of length 5
0 Solutions possible of length 3
This indicates that there is one Solution of length 7 that uses the Resources, 1 2 3 4 - / /. We could have inferred that there was a Solution with these Resources from what we already knew. We knew that there were only three remaining triples of operations to be tested for whether they were used in the two unknown Solutions of length 7. Since - - x are used in one of the unknown length 7 Solutions and - - / are not used in one, the remaining - / / must be used in the other unknown one. Hence, the experiment in which 1 2 3 4 - / / are Required can be regarded as a replication to confirm the theorizing that has been done. It is somewhat trickier to detect the parallel that exists between a Solution that is possible by means of this set of Resources and one that we have already discovered. But there is such a connection, and once we notice it we are home. The parallel Solution is 3/(2/4), and the connection is that this and the new Solution both

involve the idea of dividing by a fraction; they both contain a pair of /'s. The connection is expressed in the following equalities:

3/(2/4) = 3/(1/2) = 3/(1/(4-2)).

This analysis, in turn, reveals one of the 3 unknown Solutions of length 5, namely, 3/(1/2).

After these three new ones have been added, our list of ten discovered Solutions includes: 3x2 3x2/1 3x(4-2) 3x(4/2) 3/(1/2) 3/(2/4) 3x(4-2)/1 3x(4/2)/1 4x2-(3-1) and 3/(1/(4-2)). So, we now have the only Solution of length 3, a total of 5 of the 7 Solutions of length 5, and all 4 of the Solutions of length 7. All we need are the 2 remaining Solutions of length 5.

What do we already know about the 2 unknown solutions of length 5 from the experiments already performed? We know from the experiment in which the / / are Forbidden that there are 3 Solutions of length 5 in which the pairs of operations - - and - x are used. But among the Solutions of length 5 already discovered there is only 1 that uses either of these pairs, namely, the 3x(4-2) that uses the - x. So, there are 2 more Solutions of length 5 that use either another - x or a - -. What experiments will indicate which pairs are in the unknown length 5 Solutions, and possibly provide other useful information as well?

When - x is Required, the feedback is:

2 Solutions possible of length 7
2 Solutions possible of length 5
0 Solutions possible of length 3

This indicates that 1 of the 2 unknown length 5 Solutions uses - x and that the other one uses - -. But what numerals are used with these? The experiment in which the 1 was Forbidden indicates that there are 3 Solutions of length 5 that use 2 3 4. All 3 of these have already been discovered. So, the 2 remaining unknown length 5 Solutions must use 1 2 3 or 1 2 4 or 1 3 4.

When the 4 is Forbidden, the feedback is:

0 Solutions possible of length 7
2 Solutions possible of length 5
1 Solution possible of length 3

Both of the length 5 Solutions that use the trio of digits 1 2 3 have already been discovered.

When the 3 is Forbidden, the feedback is:

0 Solutions possible of length 7
1 Solution possible of length 5
0 Solutions possible of length 3

No Solutions of length 5 that use 1 2 4 have been discovered; so, 1 of the 2 unknown Solutions of length 5 uses 1 2 4 and the other unknown Solution must use 1 3 4, the only remaining possibility?

So, what is now known? The results of the experiments indicate that the numerals in the 2 unknown length 5 Solutions are 1 2 4 and 1 3 4, and that the operations in them are - x and - -. Armed with this information some problem-solvers will see how to put the sets of numerals and sets of operations together appropriately to construct the Solutions, (4-1)x2 and 3-(1-4).

That completes the entire list of 12 different Solutions, and the problem posed to find them is solved. The final 6 of these Solutions were discovered (in this account) by conducting 11 experiments. These experiments provided data that helped to focus attention on concepts that were used in the Solutions, but were not immediately apparent to our hypothetical problem-solver by his or her self observation of his or her knowledge of mathematics. By this account our hypothetical problem-solver was unable to solve the problem by observational science alone, that is, by merely observing his or her state of mathematical knowledge. But when mere observational science is extended to include experimental science, the solving of the problem is achieved. When a problem-solver can experiment, he or she has available much more powerful tools for solving problems than when all that it is possible to do is to observe. Seemingly simple problems involving only elementary arithmetic drawn from Challenge Matches of the EQUATIONS Game illustrate dramatically this powerful difference. Mathematical knowledge and reasoning alone just are not enough (for most persons) to do the job. But when such knowledge is supplemented by logical reasoning and scientific experimentation, the problem-solver's capacity is magnified many-fold.

Those few readers whose depth of understanding of arithmetic permitted them to easily solve the problem above without resorting to experiment may wish to test their intuitions further by the pair of problems below. These should persuade most readers that it is relatively easy to generate problems in this fashion that will go to the edge of any learner's current understanding.

SECOND PROBLEM.

GOAL: 2
RESOURCES: + x L L 3 4 6 6 9
Number of different Solutions: 8

L indicates the logarithm operation; thus b L a indicates the Log of a to the base b. For example, 10 L 100 = 2.

THIRD PROBLEM.

GOAL: 18/25
RESOURCES: + x x / / A 3 5 5 6 where A = 3/5
Number of different Solutions: 7

Logic, mathematics, and scientific experimentation tend to be taught separately in specialized courses rather effectively isolated from each other. The Research Mode of the Challenge Matches of EQUATIONS provides (1) problems that require for solution an integrative use of logical, mathematical, and scientific reasoning and (2) an efficient software laboratory for learning and practicing such skills.

ABSTRACT

The Research Mode of the EQUATIONS Challenge Matches can be viewed as a software laboratory. Users of the program design and perform experiments to obtain information that they need to solve problems involving application of elementary mathematical ideas. Use of the program provides striking illustration of how the illuminative power of observation science (like astromony) is enhanced when it is extended to become experimental science (like physics). Instead of merely observing carefully the current state of affairs (her own mathematical knowledge) and the unfolding stream of events that are occurring (what she is noticing about the problem as stated), the problem-solver can arrange for things to happen that will provide exactly the information that is needed to deal with the problem. The series of exercises on the EQUATIONS Challenge Matches diskette have been developed to provide users the opportunity to learn and use the fundamental reasoning and problem-solving skills of careful observation, logical deduction, mathematical analysis, asking good questions (which in this context is equivalent to designing good experiments), scientific research by experimentation, and data gathering, organization, and analysis. Instead of being presented in an isolated fashion in different courses, all of these powerful techniques are brought together in dealing with a single problem.