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ABSTRACT

Although perfectly scalable items rarely occur in practice, Guttman's concept of a scale has proved to be valuable to the development of measurement theory. If the score distribution is uniform and there is an equal number of items at each difficulty level, both the elements and the eigenvalues of the Pearson correlation matrix of dichotomous Guttman-scalable items can be expressed as simple functions of the number of items. Even when these special conditions do not hold, the values of the correlations can be computed easily by assuming a particular score distribution. These findings are useful in conducting research on the properties of scales. (Author)

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RESEARCH REPORT

SOME PROPERTIES OF THE PEARSON CORRELATION MATRIX OF GUTTMAN-SCALABLE ITEMS

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Some Properties of the Pearson Correlation Matrix of Guttman-Scalable Items

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Abstract

Although perfectly scalable items rarely occur in practice, Guttman's concept of a scale has proved to be valuable to the development of measurement theory. If the score distribution is uniform and there is an equal number of items at each difficulty level, both the elements and the eigenvalues of the Pearson correlation matrix of dichotomous Guttman-scalable items can be expressed as simple functions of the number of items. Even when these special conditions do not hold, the values of the correlations can be computed easily by assuming a particular score distribution. These findings are useful in conducting research on the properties of scales.

Guttman (1941, 1950a) developed the concept of an idealized type of attitude scale with the following property: "Persons who answer a given question favorably all have higher ranks on the scale than persons who answer the same question unfavorably. From a respondent's rank or scale score we know exactly which items he endorsed" (Suchman, 1950, p. 9). Although sets of items that follow this pattern rarely occur in practice, the concept of Guttman-scalable items has proved to be useful in the development of measurement theory. The properties described in this paper apply only to dichotomous Guttman items, though Guttman's theory comprises items with multiple score categories. To simplify the discussion, it will be assumed that these are cognitive items that are either correct or incorrect. For the case of cognitive rather than attitude items, the analogue of the scalability property described above is that items can be ordered according to difficulty such that individuals who answer a given item correctly also answer all previous items correctly.

One well-known property of dichotomous Guttman items is that, for n items, no two of which have the same marginals, the Pearson (phi) correlation matrix is of rank n (e.g. Torgerson, 1958, p. 312), despite the fact that the items can be ordered along a single dimension. It can be demonstrated that under certain uniformity conditions, both the elements and the eigenvalues of the Pearson correlation matrix, $\underset{\sim}{\phi}$ can be expressed as simple functions of the number of items. These results are closely related to Guttman's (1941, 1950c) findings on the principal components of scale analysis. Using a method that is now known as multiple correspondence analysis, Guttman obtained the latent structure of a transformation of the matrix of item responses. He did not, however, express the eigenvalues as simple functions of the number of items, nor did he point out the relation between the latent structure he derived and that of the phi matrix.

An understanding of the structure of the Pearson correlation matrix of Guttman items is of value in conducting research on the properties of scales. For example, in investigating methods of dimensionality assessment for dichotomous data, it is useful to determine the results of applying potential methods to Guttman items. It is advantageous to be able to generate the desired correlation matrices without generating the item responses themselves. A general form for the eigenvalues of the Pearson correlation matrix is also useful; these eigenvalues can be regarded as a standard to which the roots of other proximity matrices can be compared.

Notational Scheme

Table 1 gives a schematic representation of admissible response patterns, called a scalogram, for a set of n "types" of Guttman-scalable dichotomous items. In Guttman's terminology, items with the same marginal distribution (proportion correct) are said to be of the same type. The n + 1 rows of Table 1 correspond to the n + 1 permissible response patterns. The first n columns correspond

Table 1

Indicator Matrix for

Admissible Response Patterns

for n Dichotomous Guttman Items

		Incorrect Responses				Correct Responses					Frequency of
	Item	1	2	...	n	1	2	...	n	Row Total	Respondents
	1	1	1	...	1	0	0	...	0	n	f_1
	2	0	1	...	1	1	0	...	0	n	f_2
Response	3	0	0	...	1	1	1	...	.	n	f_3
Pattern	.	.	.	...	.	.	.	...	.	.	.
	.	.	.	...	.	.	.	...	.	.	.
	.	.	.	...	.	.	.	...	.	.	.
	n + 1	0	0	...	0	1	1	...	1	n	f_{n+1}
Column Total		1	2	...	n	n	n-1	...	1	n(n+1)	
Frequency of Responses		f_1	$\sum_{i=1}^{2} f_i$	...	$\sum_{i=1}^{n} f_i$	$\sum_{i=2}^{n+1} f_i$	$\sum_{i=3}^{n+1} f_i$	...	f_{n+1}		$F = \sum_{i=1}^{n+1} f_i$

to incorrect reponses to the n items; the next n columns correspond to correct reponses. In the body of the table, ones indicate cells in which observations occur, according to the definition of a Guttman scale; zeroes indicate cells in which it is impossible for observations to occur. The row and column totals shown on the inner margins of the table are the totals of the indicator variables. The outer margins of Table 1 give the number of subjects for each row and column of the table. The notation f_i represents the number of subjects giving response pattern i.

To simplify the presentation in this paper, the following assumptions are made:

1. There is a uniform number of items per type. Letting h_k denote the number of items of type k, k = 1, 2, ... n, this assumption can be expressed as $h_1 = h_2 = \ldots = h_n = h$. The results presented here concerning the elements and eigenvalues of Φ hold regardless of the value of h, provided that it is constant for all types. Therefore, it can be assumed without loss of generality that h = 1; that is, there is only one item per type. Because two Guttman-scalable items with the same proportion correct must have a Pearson correlation of 1, another way of stating this assumption is that no two items are perfectly correlated.

2. The frequencies are the same for each response pattern, i.e., $f_1 = f_2 = \ldots = f_{n+1} = f$. (For Guttman items, there is a one-to-one correspondence between response patterns and number-right scores. Therefore, another way of stating this condition is that the frequencies

Table 2

Response Frequencies for Two Dichotomous

Guttman Items, i and j ($i < j$)

Under Uniformity Conditions

		Item j: Correct (1)	Incorrect (0)	Total
Item i	Correct (1)	$n + 1 - j$	$j - i$	$n + 1 - i$
	Incorrect (0)	0	i	i
		$n + 1 - j$	j	$n + 1$

are the same for each number-right score.) This assumption and the assumption that all $h_k = h$ are referred to jointly as the uniformity conditions. Because the value of f does not affect the validity of the results presented here, we can assume $f = 1$. Based on the uniformity conditions and the further assumptions that $h = 1$, $f = 1$, the indicator variables and inner margins of Table 1 can then be treated as frequencies.

The Pearson Correlation between Guttman Items

The phi correlation between two items, i and j, can be expressed as

$$\phi_{ij} = \frac{f_{11}f_{22} - f_{12}f_{21}}{\sqrt{f_{+1}f_{+2}f_{1+}f_{2+}}} \qquad [1]$$

where f_{rc} represents the frequency in the r,c cell of the 2 x 2 table of responses to a pair of items, $r = 1, 2$; $c = 1, 2$; f_{+c} is the marginal frequency for column c and f_{r+} is the marginal frequency for row r. For a pair of Guttman items, the frequencies in the 2 x 2 table are as shown in Table 2 for two items, i and j, where $i < j$ (that is, item i is easier than item j). The number of subjects who get both items right is equal to the total number of subjects, $n + 1$, minus the index of the harder item, j. The number of subjects who get item i correct and item j wrong is simply the difference between their indices, j - i. Because i is the easier item, the number of subjects who answer both items incorrectly is equal to i. Finally, there can be no subjects who get item i wrong, but get item j correct. Using the computational formula in equation 1, the phi coefficient for items i and j can be computed as

$$\phi_{ij} = \frac{(n + 1 - j)(i)}{\sqrt{(n + 1 - j)(j)(n + 1 - i)(i)}} \qquad [2]$$

$$= \sqrt{\frac{i\ (n + 1 - j)}{j\ (n + 1 - i)}}\ , \ i < j\ .$$

An alternative derivation of equation 2 can be obtained by observing that, in a Guttman scale, the inter-item correlation is the maximum that can be achieved, given the marginal distributions for the items. The maximum phi coefficient that can be obtained from items with proportions correct p_i and p_j is

$$\text{Max}\ (\phi_{ij}) = \sqrt{\frac{p_j\ (1 - p_i)}{p_i\ (1 - p_j)}}\ , \ p_i > p_j\ . \qquad [3]$$

(Lord and Novick, 1968, p. 347). Guttman (1950b, p. 203) expresses the correlation between scalable items in a an equivalent form. This equation applies even if the uniformity conditions do not hold. Now, by noting that, under the uniformity conditions, the proportion correct for the i^{th} item in a Guttman scale can be expressed as $p_i = (n + 1 - i)/(n + 1)$, equation 2 can be obtained from equation 3.

It is useful to note that the expression for the correlation of the first with the n^{th} item can be simplified: Letting $i = 1$ and $j = n$ in equation 2,

$$\phi_{1,n} = \sqrt{\frac{1\ (n + 1 - n)}{n\ (n + 1 - 1)}} \qquad [4]$$

$$= \sqrt{\frac{1}{n^2}} = \frac{1}{n} .$$

Equation 2 can also be simplified slightly for the case of adjacent items. Letting $j = i + 1$, equation 2 becomes

$$\phi_{i,i+1} = \sqrt{\frac{i\ (n - i)}{(i + 1)\ (n + 1 - i)}} \qquad [5]$$

The correlation of the first with the second item and the second-last with the last item can both be simplified further. For $i = 1$, $j = 2$ or $i = n - 1$, $j = n$, equation 5 becomes

$$\phi_{1,2} = \phi_{n-1,n} = \sqrt{\frac{n - 1}{2n}} \qquad [6]$$

Thus, for a given number of items, these two "border" correlations are always equal. In fact, because the correlation matrix satisfies the definition of a simplex (Guttman, 1954, p. 274), it is symmetric with respect to its minor, as well as its major diagonal.

Eigenvalues of the Pearson Correlation Matrix

A scalogram for three Guttman items is given in Table 3. Because of the assumptions $h = 1$, $f = 1$, Table 3 is also a frequency table. Under the uniformity conditions, the correlations for $n = 3$ Guttman items can be found from equations 4 and 6 to be $\phi_{1,2} = \phi_{2,3} = \sqrt{1/3}$ and $\phi_{1,3} = 1/3$. This can be verified by computing the correlations directly from Table 3. The eigenvalues of this matrix are found to be 2, 2/3, and 1/3. To obtain an expression for the eigenvalues in terms of n, we can express the correlations as in equations 4 and 6 and then obtain a cubic equation for the eigenvalues, λ_i, in terms of n. We find that the cubic equation can be factored as follows:

$$[\lambda - (n + 1)/2]\ [\lambda - (n + 1)/6]\ [\lambda - (n + 1)/12] = 0$$

The roots can be expressed more generally as

$$\lambda_i = (n + 1)/[i(i + 1)], \qquad [7]$$

a result that holds for any n.[1] The smallest eigenvalue is thus $\lambda_n = (n + 1)/\ [n\ (n + 1)] = 1/n$; the largest is $\lambda_1 = (n + 1)/2$. Note that the proportion of variance attributable to the first principal component $\lambda_1/n = (n + 1)/2n$, approaches 1/2 as n approaches infinity, a somewhat surprising result.

Table 3

Frequency Table for

Three Dichotomous Guttman Items

		Incorrect Responses			Correct Responses			
		Item 1	2	3	1	2	3	Row Total
Response	1	1	1	1	0	0	0	3
Patterns	2	0	1	1	1	0	0	3
(Subjects)	3	0	0	1	1	1	0	3
	4	0	0	0	1	1	1	3
Column Total		1	2	3	3	2	1	12

In discussing the principal components of scale analysis, Guttman (1941, 1950c) does not analyze the correlation matrix. Instead, he derives the latent structure of related matrices that are transformations of the matrix of item responses. In the case of n dichotomous items, his $\underset{\sim}{G}$ matrix (1941, p. 331) is of dimensions 2n x 2n and can be expressed as

$$\underset{\sim}{G} = \underset{\sim}{D}_c^{-1/2}(S'S - \frac{1}{F}\underset{\sim}{c}\underset{\sim}{c}')\,\underset{\sim}{D}_c^{-1/2} \qquad [8]$$

where $\underset{\sim}{S}$ denotes the (n + 1) x 2n matrix of item responses (e.g., Table 3), $\underset{\sim}{c}$ is the 2n x 1 vector of column frequencies of $\underset{\sim}{S}$, F = n + 1 is the number of subjects, and $\underset{\sim}{D}_c^{-1/2}$ is the diagonal matrix of reciprocal square roots of these column frequencies. (For f > 1, the number of rows of $\underset{\sim}{S}$ would be expanded so that there were f rows for each of the n + 1 response patterns. The sample size F = f(n + 1) would be used in Equation 8. The dimensions of $\underset{\sim}{G}$ would be unchanged. For h > 1, the number of columns of $\underset{\sim}{G}$ would be expanded so that there were 2h, instead of 2, columns for each type of item. In this case, $\underset{\sim}{G}$ would be of dimensions 2hn x 2hn.) A general element of $\underset{\sim}{G}$ can be expressed as

$$g_{pq} = \frac{F_{pq} - \frac{F_p F_q}{F}}{\sqrt{F_p F_q}} \qquad [9]$$

where F_p and F_q denote the number of individuals in columns p and q, respectively, of $\underset{\sim}{S}$ (p, q = 1, 2, ...2n), and F_{pq} denotes the number of individuals who are represented in both columns p and q of

$\underset{\sim}{S}$. Guttman states that "this element is recognized to be precisely that used in the chi-square test of significance of association between two attributes" (Guttman, 1941, p. 332). More specifically, the Pearson chi-squared statistic for a pair of items is equal to the sum of squares of the four appropriate elements g_{pq} of $\underset{\sim}{G}$ (see Equation 11), multiplied by the sample size, F. Noting that, for any two items represented in the scalogram,

$$X_{ij}^2 = F\phi_{ij}^2 , \qquad [10]$$

we observe that the elements g_{pq} might be described more precisely as components of ϕ^2. In fact, the relation between the elements of the Pearson matrix and the elements of G can be expressed as

$$\begin{aligned} \phi_{ij} &= (g_{ij}^2 + g_{i,j+n}^2 + g_{i+n,j}^2 + g_{i+n,j+n}^2)^{1/2} \qquad [11] \\ &= [2(g_{ij}^2 + g_{i+n,j+n}^2)]^{1/2} \end{aligned}$$

For illustration, let us use Equation 11 to calculate $\phi_{1,2}$ from $\underset{\sim}{G}$ for the data of Table 2. The $\underset{\sim}{G}$ and $\underset{\sim}{\phi}$ matrices corresponding to Table 3 are given in Table 4. $\phi_{1,2}$ can be obtained from the elements of $\underset{\sim}{G}$ as follows:

$$\begin{aligned} \phi_{1,2} &= (g_{12}^2 + g_{15}^2 + g_{42}^2 + g_{45}^2)^{1/2} \\ &= [.35^2 + (-.35)^2 + (-.20)^2 + .20^2]^{1/2} \\ &= [2(.35^2 + .20^2)]^{1/2} \doteq .58 \end{aligned}$$

Table 4

Φ and G Matrices for

n = 3 Guttman Items under

Uniformity Conditions

Φ

1.00	.58	.33
.58	1.00	.58
.33	.58	1.00

G

.75	.35	.14	-.43	-.35	-.25
.35	.50	.20	-.20	-.5	-.35
.14	.20	.25	-.08	-.20	-.43
-.43	-.20	-.08	.25	.20	.14
-.35	-.50	-.20	.20	.5	.35
-.25	-.35	-.43	.14	.35	.75

$$\phi_{ij} = (g_{ij}^2 + g_{i,j+n}^2 + g_{i+n,j}^2 + g_{i+n,j+n}^2)^{1/2}$$
$$= [2(g_{ij}^2 + g_{i+n,j+n}^2)]^{1/2}$$

Because $\underset{\sim}{G}$ has two rows and columns corresponding to each item, one for the correct response and one for the incorrect response, it may be regarded as a redundant means of expressing the Pearson matrix, $\underset{\sim}{\phi}$. $\underset{\sim}{G}$ has n non-zero roots, which are identical to those of $\underset{\sim}{\phi}$. Guttman (1950c) finds the latent structure of two transformations, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ (pp. 338-339), of $\underset{\sim}{S}$ that differ slightly from $\underset{\sim}{G}$. Because $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are the minor product moment and major product moment matrices, respectively, of a rescaled version of $\underset{\sim}{S}$, their non-zero roots are identical. There are n roots that are 1/n times the roots of $\underset{\sim}{\phi}$ or $\underset{\sim}{G}$, as well as an extraneous root of 1. The non-trivial roots of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are interpretable as squared correlation ratios. The largest non-trivial root of $\underset{\sim}{A}$ is equal to the maximum value of the ratio of variance between categories (between columns of $\underset{\sim}{S}$) to total score variance, obtained by assigning scores to subjects (rows of $\underset{\sim}{S}$) in an optimal fashion. Similarly, the largest non-trivial root of $\underset{\sim}{B}$ is the maximum value of the ratio of variance between subjects (rows of $\underset{\sim}{S}$) to total variance, obtained by assigning weights to categories (columns of $\underset{\sim}{S}$) in an optimal way. The succeeding roots are the maximum squared correlation ratios for the residualized matrices. Analysis of multiway contingency tables through derivation of the eigenstructures of transformed response matrices such as $\underset{\sim}{G}$, $\underset{\sim}{A}$, and $\underset{\sim}{B}$ is now commonly referred to as multiple correspondence analysis (see Tenenhaus and Young, 1985, for an extensive review and Zwick and Cramer, in press, for an illustration of the relation between this approach and other multivariate techniques in the case of a two-way contingency table).

Discussion

Under certain uniformity conditions, the elements and eigenvalues of the Pearson correlation matrix for dichotomous Guttman items can be expressed as simple functions of the number of items. These relations may prove to have applications in the investigation of the properties of scales. For example, in conducting research on the dimensionality of dichotomous data, it is of interest to determine the results of applying potential methods of dimensionality assessment to perfect Guttman scales. A method cannot be considered acceptable if it is known to produce the wrong answer for dichotomous items. Equations 2, 4, 5, and 6 allow the generation of the desired correlation matrices without generation of the item responses themselves. Equation 7 may also be useful; eigenvalues of possible transformations of ϕ (e.g., see the section on image analysis in Zwick, 1986) or of other proximity matrices can be compared to "baseline" values obtained from Equation 7.

It is important, however, to recognize the effect of relaxing the assumption that all $f_k = f$. (The assumption that all $h_k = h = 1$ is not implausible; furthermore, if two items are perfectly correlated, one can be discarded without loss of information.) The results of allowing the f_k to be unequal can best be demonstrated by example. Suppose once again that $n = 3$, producing $n + 1 = 4$ response patterns. It is likely that the intermediate response patterns, 2 and 3, will be more common than 1 and 4, which represent all-incorrect and all-correct patterns, respectively (see Table 3). Let us assume a simple model in which

$f_4 = f_1$ and $f_2 = f_3 = kf_1$, where k is a positive integer. The values of the correlation coefficients and the largest eigenvalue, λ_1, of ϕ are given in Table 5 for selected values of k. The correlation coefficients can be computed by first noting (e.g., from Table 1) that under the hypothesized model, the proportions correct for the three items are $p_1 = (1 + 2k)/(2 + 2k)$, $p_2 = (1 + k)/(2 + 2k) = 1/2$, and $p_3 = 1/(2 + 2k)$. Then by application of equation 3, the correlation coefficients are found to be $\phi_{12} = \phi_{23} = \sqrt{1/(1 + 2k)}$ and $\phi_{13} = 1/(1 + 2k)$ (The values of λ_1 in Table 5 were obtained numerically.) A value of k = 1 corresponds to the case discussed above, in which all $f_k = f$. It is clear that as k increases and the distribution of subjects becomes more peaked, the inter-item correlations become smaller. This is not surprising when we consider that, for a fixed scale score, Guttman items are independent of one other. In fact, it is easily verified that, for any pair of adjacent score patterns, any two Guttman items are independent. By making k larger, we are increasing the proportion of subjects whose scale scores are 1 or 2. By the time we reach k = 100, ϕ begins to resemble the identity matrix. In short, the properties of the correlation matrix of dichotomous Guttman items can be affected substantially by the distribution of subjects.

Table 5

Values of Correlation Coefficients (ϕ_{ij})

and Largest Eigenvalue (λ_1) for

Selected Values of k[1]

k	$\phi_{1,2} = \phi_{2,3}$	$\phi_{1,3}$	λ_1
1	.58	.33	2.0
2	.45	.20	1.7
10	.22	.05	1.3
100	.07	.01	1.1

[1]It is assumed that $n = 3$, $f_4 = f_1$, and $f_2 = f_3 = kf_1$.

Of particular interest is the case in which scores follow a normal distribution. For n = 8 items, Table 6 shows the proportions correct for each item and the eigenvalues of ϕ under two conditions: (a) a uniform distribution of number-right scores, as above, and (b) an approximately normal distribution of number-right scores, created by setting $f_1 = f_9 = 4$, $f_2 = f_8 = 7$, $f_3 = f_7 = 12$, $f_4 = f_6 = 17$, and $f_5 = 20$. Under the normal conditions, the proportions correct for the items are no longer equally spaced, as they are under the uniform conditions. The largest eigenvalue is smaller than in the uniform case; the remaining roots are larger. The results for the normal case, as well as the results for k = 2 in Table 5, show that substituting a more realistic score distribution for the uniform distribution does not result in an increase in the size of the first eigenvalue. In fact, the size of the first root is largest for U-shaped distribution; that is, when there are more subjects in the extreme score patterns and fewer in the intermediate patterns.

If the researcher wishes to know the size of the correlations and the roots of the correlation matrix for n Guttman items with a symmetric score distribution, the values obtained from Equations 2, 4, 5, 6, and 7 may be adequate estimates. An alternative is to assume a specific score distribution and apply procedures analogous to those used to obtain the values in Table 5.

Table 6

Proportions Correct and Eigenvalues of $\underset{\sim}{\phi}$ for

Eight Guttman Items Under Two Conditions

Proportions Correct		Eigenvalues	
Uniform	Approx. Normal	Uniform	Approx. Normal
.89	.96	4.5	3.5
.78	.89	1.5	1.7
.67	.77	.8	1.0
.56	.60	.5	.6
.44	.40	.3	.4
.33	.23	.2	.3
.22	.11	.2	.3
.11	.04	.1	.2

Footnote

[1]In an appendix to an article on factor analysis of Guttman-scalable items, Burt (1953, p. 21) gives the same expression for the "factor-variances" obtained by analyzing a matrix of "product moment coefficient[s] applied to the data after they have been transformed to standard measure" (1953, p. 11). The scores he describes are obtained from the items by subjects matrix of 0-1 item responses by multiplying by n and then centering each row. Burt makes a clear distinction between a correlation based on standardized scores and a "product-moment correlation for a twofold point distribution (ϕ)" (Burt, 1950, p. 169; 1953, p. 20). In fact, if correct responses to an item, k, are assigned a score of a_k and incorrect responses a score of b_k, then ϕ_{ij} is invariant across all possible values of a_k and b_k, $k = i, j$. The correlations described by Burt are therefore identical to phi coefficients. Because his factor-variances are obtained through principal component analysis of the correlation matrix, they are identical to the eigenvalues of $\underset{\sim}{\phi}$.

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