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ABSTRACT

An overview is first given of some main theoretical frameworks underlying current European research in mathematical thinking, learning, and teaching. Together with the information-processing and the (neo)-Piagetian approaches, both well known in the United States, two other research paradigms can be distinguished: the action-oriented approach to learning and instruction developed in the Soviet Union, and the so-called realistic approach. Each is described, with implications. In the next section, European research in the domain of elementary arithmetic word problems is discussed. The focus is on three topics: task characteristics, children's solution processes, and teaching experiments. It is demonstrated that this European research, which is largely unknown to American scientists, has provided insights and findings some of which replicate those of recent American studies, but others of which provide criticisms and completions on this work. A list of 57 references is appended. (MNS)

RESEARCH ON THE TEACHING AND LEARNING OF MATHEMATICS : SOME REMARKS FROM A EUROPEAN PERSPECTIVE

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SOME REMARKS FROM A EUROPEAN PERSPECTIVE

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1. Introduction

During the past twenty years there has been a major shift in American research on mathematics learning and thinking. During the first half of this century the American psychological landscape was dominated by behaviorism. At the end of the fifties, a new paradigm began to gain widespread support, namely the information-processing approach.

This emerging paradigm has provided a lot of promising new conceptualizations, findings, and research techniques with respect to human cognition in general and children's mathematical problem solving in particular (Frederiksen, 1984; Resnick, 1983). Romberg & Carpenter's (1985) chapter in the recently published Handbook of Research on Teaching illustrates this very well for the domain of elementary arithmetic word problem solving.

However, these authors also point to several difficulties experienced by curriculum builders and teachers who try to use the findings of recent research as a basis for improving instructional practice. One major problem is that until now most information-processing studies aimed at modeling the internal processes and knowledge structures underlying performance in a particular task domain; relatively little attention was paid to the issues of learning and instruction. Another important problem is that the mathematical content received little or no (appropriate) consideration. Finally, there was little concern for the realities of the normal classroom situation in which mathematics problem solving, learning and teaching take place.

Although behaviorism became rather well-known in Western Europe, this school has never been as dominant as in the United States. The typical European ideas of Wurzburg and Gestalt psychology remained more influential, especially with respect to instructional problems. Later on, these ideas were complemented by the developmental psychology of Piaget and by the action-oriented approach of learning and instruction coming from the Soviet Union. While these approaches still have a lot of adherents in Western Europe, the information-processing perspective that is now dominant in American instructional psychology, has also become more and more influential in Western Europe (De Corte, Lodewijks, Parmentier & Span, in press; Verschaffel, 1986).

In this contribution we first give an overview of some main theoretical frameworks underlying current European research in mathematics thinking, learning and teaching. It will be shown that - together with the information-processing and the (neo-)Piagetian approach (both well-known in the United States) - two other research paradigms can be distinguished : the action-oriented mentioned above, and the so-called realistic approach. Afterwards we discuss European work in one particular domain, namely elementary arithmetic word problems. It will exemplarily be demonstrated that this European research, which is largely unknown to many American scientists, has provided insights and findings some of which can be conceived as replications, but others as criticisms and completions on their own work.

2. Theoretical frameworks underlying European research on mathematics learning and thinking

As said before, besides the information-processing and the (neo-)Piagetian approach, two other theoretical frameworks take an important position in today's research on mathematics education in Europe, namely the action-oriented and the realistic approach. As the former two are rather well-known in the United States, we will only give a brief description of the latter two approaches.

The action-oriented approach

The action-oriented approach to psychological phenomena is largely influenced by the Soviet scientist Vygotsky (1896-1934). In a certain sense, his work parallels the American information-processing approach; indeed he also rejected the performance-oriented character of the then dominant associationistic Soviet psychology.

Vygotsky's work is called action-oriented, because it takes as its basic notion the concept of activity (Bol, Haenen & Wolters, 1985; Cole, John-Steiner, Scribner & Souberman, 1978; De Corte, Span &

Carpay, 1980). According to Vygotsky, human activity can only be fully understood by investigating the nature and the causes of its development in the ontogenesis and the phylogenesis of man. In his opinion, the development of human functions, such as thinking and problem solving, is essentially socially and culturally determined. For Vygotsky "each function in the child's cultural development appears twice upon the scene, on two levels : first social, next psychological; first between people, as an interpsychic category, next within the child, as an intrapsychic category" (Bol, Haenen & Wolters, 1985, p. 7). Therefore, it is not at all surprising that the question how the development of human functions can be fostered through instruction, is a major problem in the action-oriented approach. Moreover, in their search for the answer to that question, activity psychologists do not only have to look for appropriate ways of teaching certain concepts, rules, techniques, etc.; but the content too is considered a crucial object of psychological analysis and investigation.

These characteristics of the action-oriented approach strongly influence the research methodology. Undoubtedly, the main research method is the teaching experiment, in which development-stimulating instructional materials and methods are being elaborated and tested. Frequently, these experiments are longitudinal, deal with relatively large parts of the school curriculum, and take place in school-like settings; consequently their results have a high ecological validity. A representative example of an experimental curriculum is Davydov's (1982) program for the introduction of the initial mathematical concepts and operations for the primary grades of elementary school.

As far as learning and thinking are studied without any systematic intervention - in so-called ascertaining experiments -, the data gathering is strongly process-oriented, using a rich set of diagnostic tasks and techniques (see e.g. Krutetskii, 1976).

In Western Europe the action-oriented approach entered in research on learning and instruction in the sixties, when publications from Vygotsky, his coworkers and disciples (e.g. Luria, Leont'ev, Gal'perin, Davydov) were translated into English, German, and Dutch. Their ideas gave rise to a lot of theoretical and empirical work in these countries. The Annual International Symposium on Activity Theory is one the major forums where European scientists exchange ideas, theoretical insights and empirical data with regard to that approach in psychology in general and in instructional psychology in particular (see e.g. Hedegaard, Hakkarainen, Engestrom, 1984; Bol, Haenen &

Wolters, 1985). Moreover, a number of these theoretical concepts and experimental teaching programs have found their way toward instructional practice in subject matters such as foreign language and mathematics teaching. For example, Gal'perin's theory of the stepwise formation of mental actions underlies several intervention programs for remedial mathematics teaching in the Netherlands and Belgium.

The realistic approach

Another approach, that becomes more and more influential in research on mathematics thinking, learning and teaching, is the so-called realistic approach, exemplified by the work of Bell (1978), Bishop (1983), Hilton (1984), Whitney (1985), and the Wiskobas group (Freudenthal, 1983; Streefland, 1984; Treffers & Goffree, 1985).

As in the action-oriented approach, research is focused on learning processes and how they can be optimally influenced by instruction. Again the main research methodology is the teaching experiment : one is not primarily interested in how children learn and think under given circumstances, but one tries to develop new instructional materials and teaching methods that are more in accordance with new theoretical insights.

However, there are also some important differences between both approaches of mathematics and mathematics learning. In the action-oriented approach mathematics is conceived as an academic discipline, consisting of a well-defined set of concepts, rules and techniques that have to be transmitted as efficiently as possible from one generation to the other. The realists, on the contrary, view mathematics mainly as human activity, leading at each level or age to genuine mathematical performances and products, some of which may be valuable and others not. Consequently, while in the activity theory mathematics instruction is strongly guided and systematized, realistic teaching tries to take explicitly into account children's proper contribution to the teaching-learning process; therefore pupils are stimulated to rely on their own constructions and productions while solving problems, and they are supported in gradually transforming these informal notions and strategies into more formal-mathematical concepts and solution methods (Treffers & Goffree, 1985).

The realistic approach too has produced a lot of experimental programs and materials, some of which have already been implemented in instructional practice. For example, more than one third of the textbooks nowadays used in Dutch primary schools, are provide more or less realistic mathematics instruction.

Both the action-oriented and the realistic approach have some interesting characteristics lacking in much current American cognitive-psychological research, namely the focus on the teaching of mathematics, the concern for the contents of mathematics instruction, the broad-spectrum and longitudinal character of the teaching experiments, and the ecological validity of their results.

However, the action-oriented and the realistic approach seem also to share some negative aspects. First, the analyses of the activities and processes in the teaching-learning situations are - generally speaking - much less precise, fine-grained and formalized than in the current American cognitive-psychological approach. Second, both approaches are less concerned with methodological rigor (e.g. by testing the significance of a result, or dealing with the problems relating to the use of verbal protocols as data).

3. European research on elementary arithmetic word problems

In their chapter Romberg & Carpenter (1985) give an overview of current research on elementary arithmetic word problem solving, as one example to illustrate that the information-processing approach has provided a lot of insights, findings and techniques, that are not only of great theoretical value, but also of practical importance.

In their overview two related categories of investigations can be distinguished. Some researchers have collected empirical data concerning the level of difficulty of different types of word problems, the strategies children use to solve those problems, and the nature of their errors (Carpenter, 1985; Carpenter & Moser, 1982; Fuson, 1982; Steffe, Von Glaserfeld, Richards & Cobb, 1983). Others have constructed computer-implemented models in an attempt to provide a unified account of the internal processes and cognitive structures underlying children's performance on verbal problems (Briars & Larkin, 1982; Riley, Greeno & Heller, 1983). Remarkably, in Romberg & Carpenter's (1985) overview not one reference to European work on this topic can be found. Of course, the authors do not claim their review to be comprehensive. But as they consider this work as representative for the current (American) research on mathematics learning and thinking, and its implications for instructional practice , it might be interesting to confront their picture with some European work in the same domain. The section is organized around three topics : (1)

task characteristics, (2) children's solution processes; (3) and teaching experiments.

Task characteristics

A main characteristic of current American work on elementary arithmetic word problems is its focus on the semantic structure of the problem. In most publications four different classes of problems based on that dimension, are distinguished : Change, Combine, Compare and Equalize problems. Examples of each of these four problem types are presented in Table 1. Each of these categories is further subdivided in distinct problem types depending on the identity of the unknown quantity; for the Change, Compare and Equalize problems further distinctions can be made depending on the direction of the event (increase or decrease) or the relationship (more or less).

Table 1

During the last few years several European researchers too have focused on the semantic structure of elementary addition and subtraction word problems too. While some of them have adopted the above-mentioned classification schema, others have developed a somewhat different categorization. For example, Vergnaud and Durand's (1976; see also Vergnaud, 1982) classification scheme of the basic categories of relationships in simple addition and subtraction word problems, consists of six types. Three of them - composition of two measures, transformation linking two measures, and a static relationship linking two measures - parallel the first, the second and the third problem type in Table 1 respectively. However, it is difficult, if not impossible to classify the other three into that classification schema. Table 2 gives the names of these three problem types together with an example.

Table 2

Differences in the level of difficulty of these categories, but also between distinct problem types within one category, have been investigated intensively, especially in France (see e.g. Vergnaud, 1982; Marthe, 1979; Escarabajal, 1985). Because these studies deal with types of problems that are absent in the American classification schema, their results cannot be accounted for in the current theoretical models in the U.S. In this respect we mention that Kintsch & Greeno (1985) made this criticism on their own model, acknowledging that there are elementary addition and subtraction word problems, that are related to those in Table 1, but not similar enough to be handled by the comprehension and the problem-solving structures implemented in their model.

Furthermore, several European researchers refer to another task characteristic which - together with the semantic structure - seems to have an important impact on the relative difficulty of arithmetic word problems, as well as on the processes children use to solve them. Indeed, word problems can also be situated along a "reality" dimension, ranging from traditional, stereotype, lean word problems on the one hand, to rich so-called context problems on the other (Aebli, 1985; De Corte & Verschaffel, in press a; Nesher, 1981; Treffers & Goffree, 1985).

According to Treffers & Goffree (1985) context problems can be distinguished from traditional school word problems with respect to their form, content and function. The presentation of context problems is not restricted to short piece of text followed by a question, but can also be in the form of a game, a play, a story, newspaper-cuttings, graphs, or a combination of them. They are built around meaningful, attractive and realistic situations : in a context problem it does matter whether the task - e.g. an area problem - deals with building a parking lot for cars or with stacking boxes in a store. Consequently, a context problem invites and sometimes even forces children to bring in and use their real-world knowledge and personal experiences in solving the problem. It is important to note that this task characteristic is not only utilized for its emotional or motivational value; it is also assumed that it will significantly influence the knowledge structures and the solution processes children apply to understand and solve a particular problem. Finally, it is expected that appropriately choosen context problems can fulfill a number of functions in elementary mathematics instruction (such as concept formation, model building, application...) much better than traditional word problems.

It must be acknowledged that until now we know of little or no studies in which the effects of this task characteristic on children's thinking and learning has been systematically analyzed. There is an obvious need for such research. However, it certainly will have to take into account Nesher's (1981; Nesher & Katriel, 1977) theoretical analysis of the presuppositions that underly traditional school word problems. It can also take the advantage of the empirical work on related task characteristics such as "personalizing" word problems (Zweng, 1979), and the difference between problem-solving in school and real-world settings (Carrahar, 1985).

Finally we point to the fact that while recent American research on elementary arithmetic word problem solving alomst exclusively focuses on the operations of addition and subtraction, the European investigators have been doing a lot of work on multiplication and division problems too (see e.g. Bell, Swan & Taylor, 1981; Greer, 1985; Fischbein, Deri, Nello & Marino, 1985; Vergnaud, 1983).

Analysis of children's solution processes

Recent European research has also provided a lot of empirical data and theoretical analyses with respect to the appropriate as well as the inappropriate knowledge structures and thinking processes underlying children's solutions of elementary addition and subtraction word problems. While many of the results replicate those of American research, there are also important dissimilarities and additions. We will briefly illustrate this statement referring to one of our own investigations in which thirty first graders were individually interviewed three times during the school year : at the very beginning in September, in January, and at the end in June (Verschaffel, 1984; see also De Corte & Verschaffel, 1985a, 1985b, in press a, in press b). Each time they were administered eight elementary addition and subtraction word problems : four Change, two Combine and two Compare problems. Each problem was read aloud by the interviewer who, then, asked the child to perform the following tasks : (1) to retell the problem, (2) to solve it, (3) to explain and justify their solution strategy, (4) to build a material representation of the story with puppets and blocks, and (5) to write a matching number sentence. The individual interviews were videotaped and the data were submitted to a quantitative as well as a qualitative analysis.

One of the most significant contributions to recent American research on children's solution strategies for elementary arithmetic word problems is certainly the work of Carpenter & Moser (1982; Carpenter, 1985). First, they found that young children dispose of a rich and varied set of strategies to solve different types of elementary addition and subtraction word problems. Second, they observed an obvious development in the level of internalization of these solution strategies : initially problems were mainly solved using material and verbal counting strategies; subsequently children shifted to mental solution strategies based on known number facts. Finally, their data yield evidence that children's material and verbal solution strategies for subtraction problems are significantly influenced by the semantic structure underlying these problems. More specifically, it was found that children tend to solve each subtraction problem with that kind of strategy that corresponds most closely to its semantic structure, as illustrated in Table 3.

Table 3

Our longitudinal study yielded several findings concerning children's solution strategies that are, generally speaking, consistent with those of Carpenter & Moser. However, using a more elaborated classification schema for solution strategies and a more varied set of problem types than in their investigation, we also obtained findings allowing us to complement their analysis in two respects : (1) the relationship between the semantic structure of subtraction problems and children's solution strategies does not only hold for the lowest two levels of internalization (i.e. material and verbal counting strategies), but also for the mental level, where strategies are based on known number facts; (2) the solution strategies for addition problems are also strongly influenced by the semantic structure of the problem (De Corte & Verschaffel, in press a).

We also obtained interesting data on children's typical errors and misconceptions with respect to elementary arithmetic word problems. While some of these errors and misconceptions have again been described and formally modeled by American researchers (Briars & Larkin, 1984; Riley, Greeno & Heller, 1983), we discovered other deficiencies that are not mentioned in their work. We illustrate this with respect to one typical error on Combine problems with one of the subsets unknown ("Pete has 3 apples; Ann has also some apples; Pete and Ann have 9 apples altogether; how many apples does Ann have ?").

In our study this problem was frequently answered with the largest given number, namely 9. The computer models mentioned above attribute children's failure on such problems to a lack of understanding of the part-whole relation. This relation is not explicitly stated in the verbal text; consequently children who do not yet master the appropriate knowledge structure, interpret each sentence separately and cannot infer the part-whole relation between the two given sets in the problem. Although we admit that in a number of cases this explanation accounts for children's largest-given-number error, we hypothesized that a lot of these errors were not due to the absence of the part-whole schema, but to a misinterpretation of the word "altogether", namely as referring to each person's property, and, therefore, leading to a faulty interpretation of the third sentence ("Pete and Ann have 9 apples altogether") as "Pete and Ann each possess 9 apples". This hypothesis was confirmed by the retelling data of a significant number of children and also by their materializations of the problem situation.

More examples of errors and misconceptions for which the computer models mentioned above cannot account, are described by De Corte & Verschaffel (1985a, 1985b), by Fischer (1979) and by Escarabajal (1985). Several American researchers, e.g. Kintsch & Greeno (1985) and Carpenter (1985), also have stressed the limitations of the current models. A main criticism is that the text-processing component, i.e. the variables and processes contributing to the construction of an appropriate representation of the problem text, is not sufficiently elaborated. Therefore it is not surprising that these models cannot account for several findings concerning the appropriate as well as the inapproriate understanding processes, neither for the task and subject variables affecting them. Kintsch & Greeno (1985) recently made a first step towards meeting this criticism. Similar work is going on in France by Escarabajal, Kayser, Nguyen-Xuan, Poitrenaud and Richard (1983).

Teaching experiments

As said before, the teaching experiment is a major method in both the action-oriented and the realistic approach. Consequently this methodology is used intensively in European research on mathematics learning and teaching. Meanwhile the application of teaching experiments has more and more become recognized as a valid research strategy in the information-processing (e.g. Carpenter, Moser & Bebout, 1985; Fuson & Hall, 1984; Lindvall, Tamburino & Robinson, 1982; Willis & Fuson, 1985) and the (neo-)Piagetian (Case, 1983) approaches too. Two examples of the use of the teaching experiment in

European research on elementary arithmetic word problem solving will now briefly be discussed.

Based on the results of the above-mentioned longitudinal investigation of children's problem-solving skills and processes with respect to elementary arithmetic word problems on the one hand, and on a critical analysis of the use of that kind of problems in six current Flemish elementary arithmetic textbooks on the other (De Corte, Verschaffel, Janssens & Joillet, 1985), we developed an experimental program for the teaching of word problems in the first grade.

Some main characteristics of this program can be summarized as follows (1) The teaching of word problems is not postponed until children have learned the formal-arithmetical operations of addition and subtraction; on the contrary, word problems are presented before introducing these arithmetic operations and the related number sentences. (2) In order to fulfil their concept formation and application function optimally, a whole range of types of elementary arithmetic word problems is presented to the children. (3) Contrary to prior experimental programs (Wolters, 1983) and current instructional practice, children are taught distinct types of schematic representations to understand and solve different kinds of verbal problems. Table 4 gives the diagrams for the three main categories of addition and subtraction word problems used in our investigations.

Table 4

During the school year 1983-84 this program was implemented in one first-grade class. The experimental group was administered a pretest and a posttest consisting of eight addition and subtraction word problems, representing eight different types of problems in terms of the classification schema mentioned above. The results were compared with a control group taught according to the regular arithmetic program. The performance scores revealed that the experimental group made nearly twice as much progress from the pretest to the posttest as the control group. A qualitative analysis of children's solution processes showed that the observed difference in performance was mainly due to differences in the first stage of the solution process. For example, while most control children used only one kind of diagram to represent all the problem types - namely venn-diagrams -, the children of the experimental group spontaneously applied different diagrams to represent the distinct problem types (De Corte & Verschaffel, 1985c).

Although it is not related to elementary addition and subtraction, we also mention a representative example of a realistic program for teaching "long division", developed by the Wiskobas team (Treffers & Goffree, 1985). The course is based on the principle of "gradual progressive mathematisation" : in the beginning of the program children are stimulated to construct their own strategies to solve division problems; gradually these informal strategies are transformed into more efficient and formal solution methods, until the algorithm for long division is reached. Table 5 shows a few examples of strategies that children applied during the course for problems such as "342 stickers are fairly distributed among 5 children; how many does each of them get ?".

Table 5

Other important characteristics of the teaching program are the following : (1) the major role of word and context problems, serving both a concept building and an application function; (2) the very interactive character of the learning process; (3) the non-directive nature of the teaching process in the sense that a child is never forced to apply a solution method that is more abbreviated and/or formalized, than it feels comfortable with.

The value of the Wiskobas program on long division has recently been demonstrated by Rengerink (1983). After 25 sessions children taught according to the program solved significantly more division problems correctly than a control group who got traditional instruction (although at that time not all the experimental children used the most abbreviated version of the division algorithm). Moreover, the children who followed the Wiskobas program were significantly better at recognizing the appropriate arithmetic operation in practical problem situations ("Do I have do add, subtract, multiply or divide the two given numbers in the problem ?").

4. Conclusions

In their most interesting chapter, Romberg & Carpenter (1985) have shown that recent American research on mathematics learning and teaching has produced a rich set of findings. These results not only provide us with more insight in the processes underlying the acquisition and instruction of mathematical knowledge and skills, but they also call for important changes in instructional practice. However, the authors also criticize current research and suggest several areas that scholars should concentrate their efforts on, such as (1) investigating how mathematics learning proceeds, (2) studying effective ways for teaching mathematics, and (3) analyzing the content of mathematics education.

In this contribution we first discussed two main theoretical frameworks underlying current European research in mathematics thinking, learning and teaching, that, besides the familiar information-processing and the (neo-)Piagetian approaches, are less well-known in the United States, namely the action-oriented and the realistic approach. Afterwards, it was shown exemplarily mainly with respect to elementary addition and subtraction word problems, that European research has provided a lot of findings that, on the one hand, replicate those of recent American studies, but on the other, complement this work or call for revisions. To some degree, these European investigations, can be conceived as examples of the above-mentioned directions Romberg & Carpenter (1985) want American scholars to move into.

Notes

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Table 1 Four problem types distinguished in recent American research

Combine :	Pete has 3 apples; Ann has 6 apples; how many apples do they have altogether ?
Change :	Pete had 3 apples; Ann gave him 6 more apples; how many apples does Pete have now ?
Compare :	Pete has 3 apples; Ann has 6 apples more than Pete; how many apples does Ann have ?
Equalize:	Pete has 3 apples; Ann has 9 apples; what could Pete do to have as many apples as Ann ?

Table 2 Three additional problem types distinguished by Vergnaud (1982)

A composition of two transformations : Pete won 6 marbles in the morning; he lost 9 marbles in the afternoon; in total he lost 3 marbles.

A transformation linking two static relations : Pete owed Ann 6 marbles; he gave her already 4; he still owes Ann 2 marbles.

A composition of two static relationships : Pete has 3 apples more than Ann; Ann has 6 apples more than Joe; Pete has 9 apples more than Joe.

Table 3. Material solution strategies for three types of subtraction word problems (Carpenter & Moser, 1982)

Change 2 : Pete had 6 apples; he gave 2 apples to Ann; how many apples does Pete have now ?

Separating from : using objects or fingers, a set of 6 objects is constructed; 2 objects are removed; the answer is the number of remaining objects.

Change 3 : Pete had 2 apples; Ann gave him some more apples; now Pete has 6 apples; how many apples did Ann gave him ?

Adding on : a set of 2 objects is constructed; elements are added until there is a total of 6 objects; the answer is found by counting the number of elements added.

Compare 1 : Pete has 6 apples; Ann has 2 apples; how many apples does Pete have more than Ann ?

Matching : a set of 6 and a set of 2 objects are constructed and matched one to one until one set is exhausted; the answer is the number of objects remaining in the unmatched set.

Table 4. Diagrams for the three main categories of addition and subtraction word problems (De Corte & Verschaffel, 1985c)

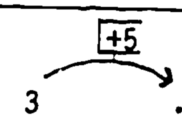
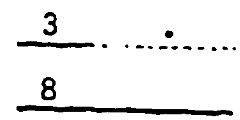
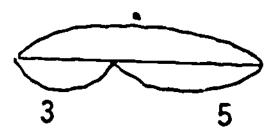
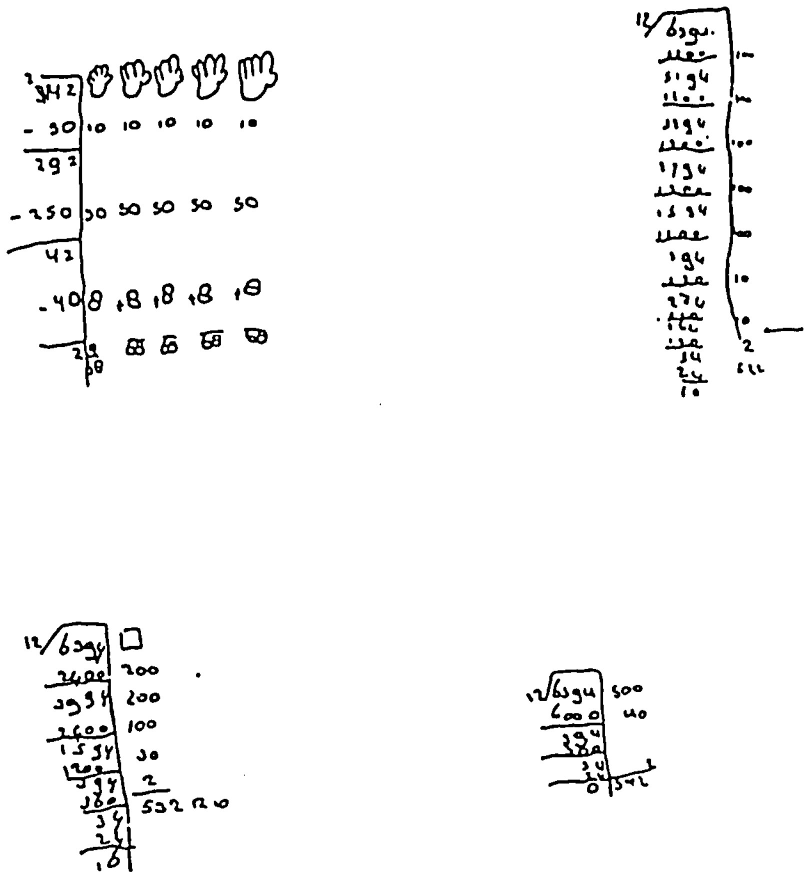
Type	Example	Diagram
Change	Pete had 3 apples; Ann gave him 5 more apples; howe many apples does Pete have now ?	
Compare	Pete has 3 apples; Ann has 8 apples; how many apples does Ann have more than Pete ?	
Combine	Pete has 3 apples; Ann has 5 apples; how many apples do they have altogether ?	

Table 5. Some examples of strategies from children working at different levels of abbrevation (Treffers & Goffree, 1985).



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