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ABSTRACT

This framework gives direction for the development of mathematics programs in California schools. The introductory chapter discusses expectations and responsibilities: problem solving, calculator technology, computational skills, estimation and mental arithmetic, and computers in mathematics education; implementation of the recommended program, with the need for major changes in testing procedures, textbook development, and teacher preservice and in-service education are described. In chapter 2, the content and structure of the mathematics program are discussed, with sections on number, measurement, geometry, patterns and functions, statistics and probability, logic, and algebra. The third chapter focuses on the delivery of instruction in mathematics, considering teaching for understanding, reinforcement of concepts and skills, problem solving, situational lessons, use of concrete materials, flexibility of instruction, corrective instruction/remediation, cooperative learning groups, mathematical language, and questioning and responding. Standards for mathematics textbooks are presented in chapter 4. Chapters 5 through 8 discuss mathematics programs for kindergarten through grade 3, grade 3 through 6, grades 6 through 8, and grades 9 through 12, with suggested contents for each. An appendix describes two courses for preservice education. (MNS)

Mathematics Framework

for California Public Schools Kindergarten Through Grade Twelve

California State Department of Education
Bill Honig, Superintendent of Public Instruction

Publishing Information

When the *Mathematics Framework for California Public Schools* was adopted by the California State Board of Education on March 8, 1985, the members of the Board were Sandra J. Boese, President; David T. Romero, Vice-President; Henry Alder; Josie Gray Bain; Agnes Chan; Daniel M. Chernow; Gloria Hom; Angie Papadakis; Kenneth L. Peters; Mark Sedway; and John L. Ward.

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A list of other publications that are available from the Department can be found on page 46 of this publication

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Foreword

THIS *Mathematics Framework* follows in the spirit of *Raising Expectations,* which was adopted by the State Board of Education in 1983, and the *Science Framework Addendum,* published in 1984. As a statement of philosophy and vision, this framework holds that *every* student can enjoy and use mathematics to real advantage and that the power of mathematical thinking is not reserved for only an academic elite. It also holds that a more encompassing core curriculum is needed at all grade levels. The previously published *Model Curriculum Standards* gives reinforcement and elaboration to the high school curriculum described in this document. A curriculum guide is now under development to provide similar model guidelines for kindergarten through grade eight.

As described briefly in Section 1, the goals of this framework will require major initiatives in testing procedures, textbook development, and teacher education. We are, in fact, moving ahead in each area. Work has begun on a new California Assessment Program test in mathematics, and the first two subjects chosen for the Golden State examinations are algebra and geometry. Next year, basic instructional materials in mathematics will be adopted for kindergarten through grade eight. In addition, a review of secondary mathematics textbooks will be carried out pursuant to Senate Bill 813/1983. Most challenging, and ultimately most important, is a revitalized teacher recruitment and preparation process. I have been working with the leaders of our universities and colleges to ensure that the many thousands of new teachers entering our schools over the next decade will be prepared to teach the program characterized in this framework. This effort must be sustained.

Mathematics is more than ever a basic instructional area. The expectations we hold for future graduates must, however, far exceed the historical goal of proficiency in arithmetic. I commend this document to all readers, both for its philosophy and for its delineation of curriculum and instruction. It will take our best efforts to achieve these raised expectations. The active support of all participants in education is essential if California's momentum toward educational excellence is to be maintained.

Bill Honig

Superintendent of Public Instruction

State Board's Message

THE State Board of Education is pleased to present the *Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve* to educators. This framework represents a consensus on mathematics education among the curriculum specialists, teachers, and administrators who prepared the document, and we take this opportunity to express our appreciation to the writers of this framework, who carefully and thoughtfully addressed issues of concern to all of us.

We believe that this framework will be useful to those responsible for curriculum planning at the local level. Because the State Board of Education has a constitutional responsibility to adopt textbooks for students in kindergarten through grade eight, it is our intent that this framework influence the development of instructional materials at all levels.

We especially recognize Professor Henry Alder, who participated in the development of this framework and provided direction at critical points in the writing process. We join Dr. Alder in the hope that this framework will help students in this state appreciate the "power and beauty of mathematics."

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Preface

IN 1983 the State Board of Education appointed 15 leading mathematics educators to the Mathematics Curriculum Framework and Criteria Committee. The committee met 11 times between November, 1983, and January, 1985. During that time four complete drafts were prepared. Two of the drafts were sent to some 250 persons for review, and over half of the drafts sent were returned with comments. These critiques and the direct participation of eight additional major figures in mathematics education were vital to the preparation of the final document for examination by the State Board, which approved the framework in March, 1985.

We wish to pay special tribute to the committee members, who served without compensation. We are also grateful to the school districts that supported teacher participation on the committee by granting released time during the school year. Committee members met for three full weekends, and individual members spent many hours on their tasks between meetings. The framework was developed under the unusual circumstance of having a professor of mathematics sitting on the State Board. Professor Henry Alder of the University of California at Davis gave much of his time and close attention to the process and was instrumental in securing a broad consensus across the levels of mathematics education and, indeed, across the country. We in the State Department of Education appreciate both Professor Alder's continuing support and the dedication of the committee members.

The *Mathematics Framework* presents a vision of elementary and high school mathematics programs dependent not on expanded research but on expanded commitment. The developers of the framework know from their own experience that instruction in mathematics can be and has been delivered to match that vision. Nonetheless, while present knowledge is sufficient, major investments will be required to implement programs based on this framework.

The framework gives direction for the development of mathematics programs; it can and should serve as a major basis for teacher education and the development of instructional materials and tests. It is not, however, a curriculum guide, much less a completely defined program. Important companion documents are the *Model Curriculum Standards: Grades Nine Through Twelve* (First Edition), published by the State Department of Education in 1985; and the *Mathematics Curriculum Guide: Kindergarten Through Grade Eight*, to be published in 1986. Readers of this framework are encouraged to read other relevant documents and, most importantly, to seek the assistance of mat' ematics education specialists in school districts, colleges and universities, private consulting firms, and research and development laboratories.

Please send responses to the framework or other mathematics-related Department publications to the Mathematics Education Unit, California State Department of Education, 721 Capitol Mall, Sacramento, CA 95814-4785.

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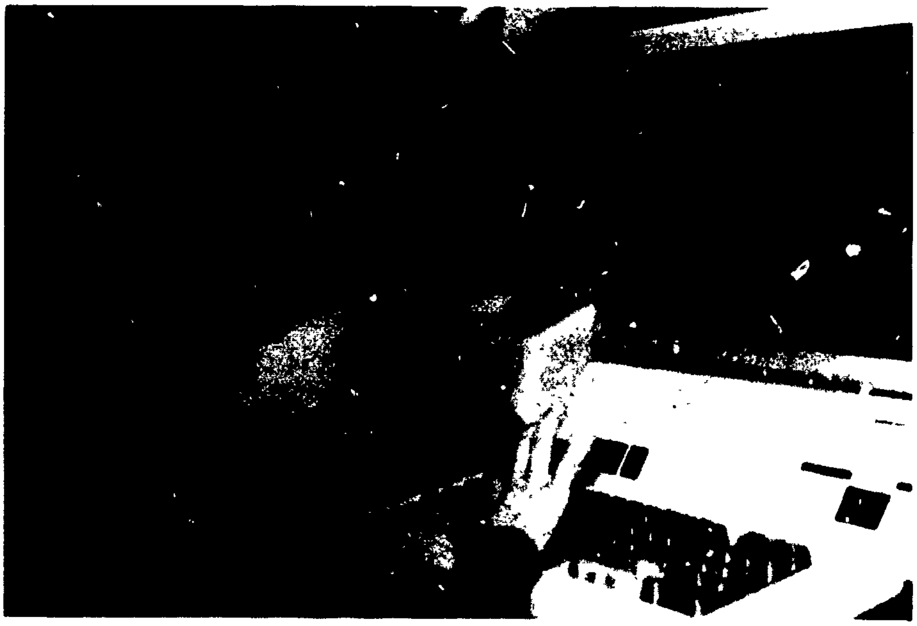

Courtesy of Shelley Gallagher

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Acknowledgments

THIS framework was developed by the Mathematics Curriculum Framework and Criteria Committee:

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Introduction to the Framework

THE inherent beauty and fascination of mathematics commend it as a subject that can be appreciated and enjoyed by all learners. The study of mathematics helps students to develop thinking skills, order their thoughts, develop logical arguments, and make valid inferences. Whether or not students will pursue a mathematics-based career, they should be encouraged throughout their years of schooling to develop a spirit of inquiry and an intellectual curiosity toward mathematics.

In addition to its intrinsic value, a knowledge of mathematics is essential to many other disciplines. As technology causes the role of mathematics in society to change, certain changes in emphasis and content must be made in mathematics education. Some long-established objectives, such as developing the number sense that comes from thorough mastery of the single-digit number facts and developing skill in estimation and mental arithmetic, must receive increased emphasis. Newer considerations, such as the effective use of calculators and computers, must be reflected in the mathematics program.

To enable all graduates to meet current and future demands, mathematics education must focus on students' capacity to make use of what they have learned in all settings. Mathematical power, which involves the ability to discern mathematical relationships, reason logically, and use mathematical techniques effectively, must be the central concern of mathematics education and must be the context in which skills are developed. Students who are mathematically powerful will:

- Have an attitude of curiosity and the willingness and ability to probe, explore, experiment, make conjectures, and persevere.
- Add, subtract, multiply, and divide whole numbers, decimals, and fractions with facility and accuracy. They will use:
    1. Estimation to aid in selecting a method for exact calculation and to establish the reasonableness of results
    2. Mental arithmetic for all single-digit operations and for simple manipulations, such as doubling, halving, and multiplying or dividing by powers of ten

3. A calculator correctly and confidently when mental calculation would be difficult or when pencil-and-paper calculation would be inefficient
4. Computer programs, as appropriate, to perform extensive or repetitive calculations, simulate real situations, and perform experiments that aid in the understanding of mathematical concepts

- Be able to extract information from data by:
  1. Processing data to create new information
  2. Using a variety of representational systems
  3. Using tables, graphs, and statistical measures selectively to organize and present information
  4. Judging the validity of statistics employed by others
- Deal successfully with problems by:
  1. Formulating key questions whose answers will provide useful information
  2. Selecting problem-solving strategies to fit given situations
  3. Seeking out appropriate data, experimenting, and finding patterns and similarities

Every student can and should develop mathematical power. A commitment to this goal will enable each student to develop his or her ability to enjoy and use mathematics.

Expectations and Responsibility

The mathematics program recommended in this framework reflects raised expectations for student achievement. The goal is for all students to be able to use mathematics with confidence; therefore, every student must be instructed in the fundamental concepts of each strand of mathematics, and no student should be limited to the computational aspects of the number strand. New concepts should be presented in such a way that all students can grasp the basic ideas. From a point of common understanding, the concepts and their interrelationships should be developed in increasing depth.

Most students will go beyond the fundamental concepts to achieve deeper and broader capability in mathematics; but even less capable students, by learning these concepts, will have appropriate experiences in all of the strands. They must not, for example, be deprived of work in geometry or probability in order to have more practice with narrow computational skills. Rather, they will continue to learn the new concepts of all of the strands and to integrate those concepts into their understanding throughout their school careers.

This expectation applies to all students, including students with special needs and those who come from groups that have historically been underrepresented in upper-level mathematics courses. The teaching methods advocated in Section 3, "Delivery of Instruction in Mathematics," are those that research has shown to be particularly beneficial to such students and those that should be employed so that all students can realize the expectations of this framework. Everyone who is interested in the quality of education must share these expectations. For each course or grade level, administrators and teachers must set specific, well-defined standards in terms of students' understanding of concepts and acquisition of skills at that level. Each grade that teachers give must be based on those standards and must accurately reflect the student's grasp of the essentials. Students and parents must know the standards, understand why they have been set, and take responsibility for meeting them.

Major Emphases of the Framework

The goal of this framework is to structure mathematics education so that students experience the enjoyment and fascination of mathematics as they gain mathematical power. The proposed program builds on the *Mathematics Framework and the 1980 Addendum*, published by the State Department of Education in 1982. Nevertheless, the 1985 framework recognizes changes in the use of mathematics. Although the underlying principles of mathematics are constant, the optimum structure for the presentation and use of mathematics has been shifting in response to the rapidly expanding importance of technology in solving problems. This section contains brief discussions of five major areas of emphasis that are reflected throughout the framework: (1) problem solving; (2) calculator technology; (3) computational skills; (4) estimation and mental arithmetic; and (5) computers in mathematics education.

Problem Solving

The special concern of this framework is the student's ability to use his or her knowledge and experience when encountering new and unexpected situations. In problem solving the student engages in a variety of purposeful but not necessarily sequential intellectual or physical activities. Problem solving involves applying one's knowledge, skills, and experience in efforts to resolve new or perplexing situations.

Although the student's textbook, as described in the textbook standards in Section 4, must include a substantial number of problem-solving situations, not all of the situations can be included in the textbook because some situations will be unexpected. The teacher must provide these problem-solving opportunities in class, recognizing that a problem-solving situation for one group of students may well be a routine application of familiar techniques for another. Teachers' editions of textbooks must supply additional situations from which appropriate choices can be made.

As students encounter more complex problems, they find that the skills required become intellectually more complex. Such situations will require the use of higher-level thinking skills by students before the situations can be understood. In working with more complex situations, students will formulate and model problems, screen relevant from irrelevant information, organize information, make conjectures and test their validity, analyze patterns and relationships, use inductive or deductive processes, identify or evaluate alternative mathematical approaches, find and test solutions, and interpret results.

All students should encounter new situations appropriate to their level of instruction. They must explore and experiment, ask appropriate questions, and bring forth the mathematical knowledge that will enable them to progress toward answering the questions. As students work with these complex situations, they will reinforce previously learned skills and extend their understanding of mathematical concepts.

Calculator Technology

Calculators are used pervasively because of their efficiency in computation. For this reason and because of their value in the teaching and learning of mathematics, calculators must be incorporated in the school's mathematics program.

From the primary grades on, calculators should be used for exploratory activities. As students understand basic concepts and learn the basic arithmetic operations, they should be taught how and when to use the calculator. And as more experience is gained, the student's effective use of calculators should increase greatly. With this development it is especially important to emphasize the students' understanding of place value and their ability to judge the reasonableness of the result of a calculation.

In the upper elementary grades, students should regularly be performing computations with calculators. At the same time they should learn that for some simple computations the use of the calculator is cumbersome or, worse, can obscure the understanding of the calculation being performed. As they gain experience, students should be expected to judge whether use of the calculator will be effective and efficient. Before the end of the sixth grade, students should have calculators continually available for use—in class, on homework assignments, and on tests. By the time students move into secondary schools, the use of calculators should be routine. Some examples of the advantages of calculators follow:

- Calculators decrease the time students must spend on computation and increase the time they can spend on the important aspects of problem solving: formulating questions, devising and evaluating strategies, and verifying and interpreting solutions.
- Calculators enable students to deal successfully with large, unwieldy numbers and allow slower students to complete assignments within time limits.
- At the secondary grade level, calculators allow students to explore solutions to algebraic equations in a way that is impractical with paper-and-pencil computation.

Some educators have expressed concern that the regular use of calculators will give students the mistaken impression that they will not have to learn the basic number facts. On the contrary, proper use of the calculator requires a knowledge of basic facts and strengthens number skills. Students must also develop skills in estimation and be able to recognize unreasonable answers. Making judgments about the results of calculation is more important to a student's mathematical power than performing numerous separate computations.

Calculators cannot serve as a substitute for good teaching nor for the use of concrete materials or manipulatives in the classroom. They should not be used for "busy work" drill and practice or to check answers to paper-and-pencil calculations. Perhaps most important, the use of calculators must not replace the development of the student's understanding of the meaning of arithmetic operations and the common algorithms used to perform those computations. With full and appropriate use of the calculator, all students, not just the most capable, can have the time and instructional support to learn to think through problems, knowing that lengthy computations will not be a barrier to success.

Computational Skills

The availability of tools for performing basic numerical computations has increased significantly in the past decade. Students now have access to calculators and computers as well as to mental arithmetic, estimation, and paper and pencils for performing computations. It is important for students to develop computational skills through the use of *all* available methods and to understand when each method should be used.

Students should be taught procedures for adding, subtracting, multiplying, and dividing whole numbers, decimals, and fractions; but it is equally important for students to understand why computational algorithms are constructed in particular forms and sequences. This understanding provides a basis for future algebraic manipulations in higher-level mathematics courses. For this understanding to be achieved, a variety of teaching approaches, including extensive use of concrete materials, should be used.

As students progress through elementary school, they should increasingly be expected to exercise independent judgment in selecting the computational procedure and tool for a given problem. Before the end of the sixth grade, the student should have all procedures and tools continuously available and be responsible for making a choice among them based on the nature of the problem and the numbers involved. In grade seven and beyond, practice of previously taught computational skills should take place in the context of some application of these skills.

Estimation and Mental Arithmetic

Deliberate and thorough development of the ability to estimate and to do mental arithmetic should be a regular part of the instructional program at all levels. Calculators and computers have increased computing capacity beyond levels imaginable a generation ago. The more common use of calculators increases the need for mental computations and estimations. Although the development of these skills has always been implicit in mathematics instruction, they have not always been taught systematically and fully. Estimation has often been misconstrued as being equivalent to the approximation made in the rounding of numbers before calculations are performed. Mental arithmetic has often been used as a way to practice basic number facts rather than as a way to solve problems. Instead, these skills must be taught to improve number sense, logical reasoning skills, and the ability to solve problems.

Estimation. Estimation is the process by which the range of an answer is projected or the reasonableness of the result of a calculation is judged. It is an integral part of problem solving, not an alternate way to arrive at answers. Thus, estimation activities should be presented not as separate lessons but as a step to be used in all computational activities. Students should have enough experience with estimating that they discover and develop increasingly effective estimation techniques and learn to rely on their estimates as aids in computation and problem solving.

Another important skill allied to that of numerical estimation is the ability to determine whether a particular numerical solution to a problem is reasonable. Independent of any attempt to estimate or follow the calculation involved, the student should be taught always to ask, "Is my answer reasonable? Could it possibly be a solution? Within what range of numbers must my answer lie?" If the problem is to determine the average weight in a class of fifth grade students and the proposed answer is 12.5 pounds, then the student should realize that this answer is unreasonable. The creation of simple tests to indicate whether an answer is reasonable should be stressed. For example, an average value must lie between the smallest and largest values of the given data.

In the primary grades estimation activities should include instruction in vocabulary—the specialized use of words such as *around, about,* and *almost.* Of course, this instruction should be built on concrete experiences with estimation.

Some examples of estimation strategies that should be taught at the elementary level include the following:

1. *Comparison:* 46 + 38. Because both numbers are less than 50, the total is less than 100.
2. *Clustering:* 42 + 58 + 63 + 37. By observing that two of the numbers are close to 40 and two are close to 60, the sum can be estimated as (2 × 40) + (2 × 60), or 200.
3. *Rounding fractions:* 9/10 + 4/9. Since 9/10 is about 1 and 4/9 is about 1/2, the sum is approximately 1 ½.
4. *Reasonableness in context:* The height of a building cannot reasonably be 5 inches or 5 feet; the height of a person cannot reasonably be 5 inches or 50 feet.
5. *Compatible numbers:* 235 ÷ 12. Since 235 is close to 240, a multiple of 12, the quotient can be estimated as 240 ÷ 12 =20.

At the junior high level, estimation can be used in more sophisticated ways. As an example, consider the product 32.94 × 1.116. With two significant figures for each number, the product can be estimated by 33 × 1.1, or 33 plus one-tenth of 33, or approximately 36.

As students build their estimation skills, they will develop confidence in their ability to test the reasonableness of calculations. They will improve their understanding of number size, structure of the number system, and place value in the base-ten numeration system. In addition, they will increase their confidence and flexibility in solving problems.

Mental arithmetic. Mental arithmetic skills enable students to explore alternative computational approaches before using paper and pencil or calculator. As students develop these skills, they will begin to use simple algorithms to express calculations in forms compatible with mental arithmetic and develop nonstandard techniques for performing calculations. Some common strategies used in mental arithmetic include:

1. *Compensation:* 57 + 29. To eliminate the need to carry, the student can add 57 to 30 and subtract 1 to get 86.
2. *Computing left to right:* 426 + 567. By add the hundreds, tens, and ones separately an hen adding those sums (900, 80, and 13), the student gets a result of 993.
3. *Using the distributive property:* 35 × 18. Viewing this product as (35 × 20) – (35 × 2) or 700 – 70, the student gets a result of 630.

As students gain proficiency in men al arithmetic, they will improve their number sense, arithmetic skills, understanding of concepts, and ability to solve computational problems through a variety of approaches.

Computers in Mathematics Education

The use of computers, including microcomputers, is now pervasive in business and industry. In the schools, however, although microcomputers are common, they are used in mathematics education to only a limited degree. As hardware becomes both more sophisticated and less expensive and as software development continues to expand, computers will inevitably become major educational tools. In mathematics education, computers have special importance. Their value in creating geometric displays, organizing and graphing data, simulating real-life situations, and generating numerical sequences and patterns is already recognized. In the future, students will be able to interact with computer programs in highly individualized ways to explore and experiment with mathematical concepts.

The uses of computers in mathematics education are changing too rapidly for definitive statements about classroom applications to be practical at this time. Although large quantities of software are available, relatively little software has been designed to be an integral part of mathematics instruction; and it must be carefully evaluated to ensure that it will support and augment the school mathematics program. Program designers are strongly urged to seek software that meets instructional needs and that extends and enriches the learning experiences available to students.

This framework does not address courses in computer literacy, computer programming, or computer science. Current state direction for such courses can be found in *Computers in Education: Goals and Content*, adopted by the State Board of Education in July, 1984, and published by the State Department of Education in 1985.

Implementation of the Recommended Program

The mathematics program recommended in this framework can be implemented only when all facets of mathematics education work in concert. It is not enough that teachers and administrators agree on the goal of mathematical power for each student. Materials and methods that are designed specifically to meet this goal must be made available to the classroom teacher. Major changes must be made in testing procedures, textbook development, and preservice and in-service teacher education. These changes must be phased in gradually. *Note:* A suggested timetable is presented at the end of this section.

Testing

In support of the program called for by this framework, testing must be concentrated on students' understanding of mathematical concepts and their ability to use their knowledge in new situations. Teachers must also be able to assess the students' ability to carry out particular mathematical procedures. Assessment must enable the teacher to diagnose specific deficiencies that inhibit the development of broader skills.

Students must be tested on their ability to apply concepts and skills in situations that demonstrate their understanding of those concepts and skills. Testing for understanding should include items that require a sequence of processes. Examples would be recall of information, translation of problems into mathematical models, and the selection and execution of appropriate strategies. Because the objective of the tests is to measure the understanding of mathematical concepts and their application, the tests, including standardized tests, should be designed in most instances to allow for the use of a calculator. This recommendation does not, however, imply that calculators will be permitted in every testing situation.

Testing of problem solving , or the ability to apply concepts and skills in new and unexpected situations, should include items that require students to formulate mathematical problems, select alternate mathematical strategies for solving problems, make generalizations, and verify and interpret solutions. Testing of such processes is a challenge, as there is no readily determined correct answer nor quick and sure way to correct students' work. Staff development activities will be required for teachers to learn to develop, administer, and evaluate such tests.

For district or commercial test makers, the challenges are to prepare tests that allow students to demonstrate understanding and require students to apply their knowledge and skills in new situations and then to develop practical ways of scoring such tests. Experience in recent years with the scoring of writing samples may be applicable to mathematical problem solving. Although the effort will not be easy, future tests must be made to match the instructional strategies and textbook standards set forth in this framework.

Textbook Development

Textbooks must be restructured to include all strands and to integrate problem solving, mental arithmetic, estimation, use of calculators, and appropriate reinforcement in each strand. Major concepts from every strand—number, measurement, geometry, patterns and functions, statistics and probability, logic, and algebra—must be incorporated and interwoven throughout the text for each grade level. The new criteria for mathematics textbooks, other student materials, and teacher materials appear in Section 4. These criteria can be met only by the use of textbooks that foster understanding and the ability to use mathematics.

Teacher Preservice and In-service Education

The teacher will continue to be the decisive factor in student learning. But many teachers have learned mathematics in a way that leads them to view it as a collection of algorithms to practice until either mastery or exhaustion occurs. Mathematics should be viewed differently. The teacher must exhibit an attitude of exploration and invention, conveying the idea that all students can learn, enjoy, and use mathematics. To assist teachers in maintaining this attitude and implementing this framework, school districts, universities, and the state should provide major support for preservice and in-service programs.

In-service programs are needed to augment teachers' mathematical and pedagogical backgrounds. This framework calls for mathematics instruction that requires teachers to know more mathematics and to employ a greater variety of teaching techniques than most of their credential programs required. The content and delivery of instruction in mathematics are inseparable. Programs to enhance mathematics content must be taught according to the principles embodied in Section 3, "Delivery of Instruction in Mathematics," and programs to enhance teaching methodology must incorporate all of the mathematical strands embodied in Section 2, "Content and Structure of the Mathematics Program."

Teachers need the same opportunities to develop their understanding and their ability to apply their knowledge to new situations as students do, and such development does not occur in a one-time two-hour workshop on a single topic. Rather, well-planned, extended programs are needed in which teachers have the opportunity to see new techniques demonstrated in classrooms, try out new methods with their own students, and reflect on the changes in the curriculum. Further, teachers must receive coaching and support over a period of time to build their confidence and to see for themselves how content and methodology are related in their teaching. Most in-service programs will have to be substantially overhauled for these criteria to be met.

Preservice programs must be modified to prepare teachers adequately to carry out the goals of this framework in both mathematical content and methodology. Guidelines for the preparation of teachers, including content and methods, have been developed by the National Council of Teachers of Mathematics (Reston, Virginia, 1981). It is recommended that the levels of mathematical preparation found in the 1983 report of the Committee on the Undergraduate Program in Mathematics, Mathematical Association of America, be reflected in preservice programs (see the Appendix). These levels of preparation are also necessary for teachers who are reassigned to teach mathematics.

All of the courses listed as follows require two years of algebra and one year of geometry as prerequisites. A complete set of course descriptions can be found in the 1983 report. Course descriptions for Fundamental Mathematical Concepts I and II are included in the Appendix. Briefly, the levels and recommendations are as follows:

Level I
- Course 1: Fundamental Mathematical Concepts I
- Course 2: Fundamental Mathematical Concepts II
- Course 3: Geometry for Elementary and Middle School Teachers
- Course 4: Algebra and Computing for Elementary and Middle School Teachers

Level II
- Courses 1—4 from Level I
- Introduction to Calculus
- Four additional Level III courses (other than calculus)

Level III Calculus Sequence (three courses)
Discrete Mathematics
Introduction to Computing
Mathematics Appreciation
Linear Algebra
Probability and Statistics
Number Theory
Geometry
Abstract Algebra
History of Mathematics
Mathematical Modeling and Applications

Teaching Levels	*Recommended Minimum Preparation*
Early Childhood (Nursery, Kindergarten)	Courses 1, 3
Grades 1—6	Courses 1, 2, and 3
Elementary Mathematics Specialist	Level II
Middle School	Level II
High School	Level III
Calculus	Level III and advanced work in analysis

To effect such sweeping changes in teacher preparation will require a major investment. At the state level it will be necessary to review credential requirements and waiver programs, augment and reallocate state resources, and investigate ways to encourage cooperative efforts among school districts, institutions of higher education, and professional organizations. Some programs exist at the state, local, county, district, and university levels, but most are diffuse and small. A coordinated approach is essential.

Suggested Timetable

In addition to the changes in tests, textbooks, and teacher preservice and in-service education advocated in this section, it is essential that districts evaluate their entire mathematics programs with respect to the framework and take *at least* the following actions:

- *1985-86: Planning*
  1. Introduce the framework to administrators and teachers.
  2. Compare the current program with the framework and the model curriculum standards.
  3. Assess the changes needed to implement the framework and assign priorities.
  4. Begin to conduct an extensive in-service program on content and methods of the new framework.
  5. Consider the possibilities of departmentalizing the elementary program, establishing mathematics specialists at each school site, and providing other support to teachers.
- *1986-87: Continuing Implementation*
  1. Continue in-service programs on content and methods of the framework.
  2. Evaluate testing programs for consistency with the framework.
  3. Identify materials that might be adopted or adapted.
- *1987-88: Ongoing Evaluation*
  1. Adopt new materials that conform with the framework.
  2. Conduct in-service programs on new instructional materials while continuing in-service programs on the content and methods of the framework.
  3. Develop local curriculum guides and courses of study.
  4. Examine test results and modify testing programs.
  5. Plan development of additional materials.
- *1988 and Beyond:* Continuation of the above activities.

Content and Structure of the Mathematics Program

MATHEMATICS can be compared to the visible spectrum. By convention we say that there are several colors or color bands in the spectrum, although there are no defined boundaries between color bands. Similarly, seven areas or strands of mathematics are delineated in this section. The reader should keep in mind, however, that the division into strands is somewhat arbitrary, that more or fewer strands could be named, and that the strands frequently overlap.

In mathematics lessons, appropriate concepts and skills from previous lessons and all strands should be interwoven as much as possible. An elementary lesson in probability can provide experience with collecting and organizing data, reinforce concepts of graphing, and provide an introduction to the meaning of a function. Concepts and skills in ratios can be included in the study of trigonometry, indirect measurement, and functions in second-year algebra.

This section, which contains a listing of the major concepts that students will learn, is not designed to indicate the way in which the curriculum is presented and organized. The manner of presentation is described in Section 3, "Delivery of Instruction in Mathematics." Concerns and emphases at different grade levels are detailed in sections 5 through 8.

Number

The system of numbers should be presented as a system of expanding ideas, starting with the counting numbers and proceeding to the whole numbers, nonnegative rational numbers, the integers, all rational numbers, the real numbers, and the set of complex numbers Although development should, in general, be sequential, there must be considerable overlapping. Students should encounter all types of numbers long before they are ready to perform computations with them. Many activities should involve concrete experiences so that students develop a sense of what the numbers mean and how they are related before they are asked to add, subtract, multiply, or divide them. In the number strand the students in kindergarten through grade twelve should:

- Develop an understanding of the use of numbers to count, the order of numbers, absolute value, and the concepts of betweenness and density.
- Understand and use appropriate notation: place value, representation of fractions and decimals, and real and complex numbers.

- Understand the operations with real and complex numbers (addition, subtraction, multiplication, division, raising to a power, taking a root) and the relationships between operations.
- Understand the order of operations and use the properties of the real and complex number system to write and work with number sentences.
- Develop an appreciation of the nature of counting numbers, including such characteristics of numbers as whether they are prime or composite, even or odd; their factors or multiples; and their relationships to other numbers (greater than, equal to, less than, relatively prime to).
- Develop facility with a variety of methods of computation and be able to choose the most efficient and effective method of solving a given problem: mental arithmetic, paper-and-pencil algorithm, estimation, or calculator.
- Develop the concept of ratio and proportion and understand practical problems involving percent.

Measurement

Measurement is a process of assigning numbers to represent certain quantitative attributes of an object or set. Because measurement may be viewed as a pairing of objects or sets with numbers, it can be considered a special example of a function. The study of measurement in kindergarten through grade twelve consists of:

- Becoming familiar with sets or objects to be measured: line segments, solids, time periods, weights, and angles
- Making informal comparisons, such as *taller/shorter, heavier/lighter,* and *greater than/less than/equal to,* and then refining these comparisons as more complex units are introduced or developed at each grade level
- Using nonstandard, arbitrary units of measure at first and then standard units, recognizing that standard units are needed for communication and simplified computation
- Recognizing that the best type of measurement for a given purpose depends on the physical aspects of the object to be measured
- Developing an understanding of the approximate nature of measurement
- Developing an understanding of accuracy and precision in numerical representations of measurements and recognizing the errors that result from using approximate measurements in calculations
- Understanding and using various systems of measurement, including the metric and U.S. customary systems, and developing the skill of making conversions within systems
- Using methods of indirect measurement (similar figures, the Pythagorean theorem, or trigonometric functions) to find quantities that cannot be measured directly
- Developing and being able to explain formulas as an efficient method of obtaining some measurements, such as area, volume, distance, rate, or time

Geometry

Geometry links students' perceptions of the real world with the mathematics that is used to solve many of the problems that arise in the students' lives. Number, measurement, and algebra provide tools for dealing with the quantitative features of the environment; geometry provides a visual approach to its organization. In geometry, students use visual and concrete experiences to gain insight into various branches of mathematics and to solve real-world problems. The understanding of geometric concepts should be developed gradually. Starting with the modeling of various aspects of the students' physical world, instruction should progress to more abstract concepts. Accordingly, students should study geometric objects of the two-dimensional plane and their corresponding analogues in three-dimensional space. However, the order is not specified because, for example, a ball might be more familiar to a child than a circle.

Instruction at the high school level should reach beyond the traditional treatment of geometry as a deductive system. Likewise, the deductive system should be used in areas of mathematics outside geometry (for example, elementary algebra). Numerical and algebraic techniques associated with coordinates, vectors, transformations, and trigonometry should be incorporated in the development of formal mathematics structures, such as the structure of synthetic geometry. In addition to the features cited previously, instruction in geometry in kindergarten through grade twelve should focus on the following:

- Developing the ability to recognize patterns through observation of sequences of geometric figures and/or the combinatorial data connected with them
- Developing a knowledge of geometric figures in the plane and in space through many experiences with concrete materials
- Introducing geometric concepts and terms informally, using vocabulary that is both correct and appropriate to the grade level

- Using transformations of the plane (reflections, translations, rotations, and dilations) to reinforce the development of geometric concepts, such as congruence, similarity, parallelism, symmetry, and perpendicularity
- Using coordinate geometry and mensuration theory to connect the major strands of geometry, measurement, numbers, and algebra

Patterns and Functions

The study of mathematical patterns and functions enables students to organize and understand most observations of the world around them. It involves discovery of patterns and relations, identification and use of functions, and representation of relations and functions in graphs, mathematical sentences or formulas, diagrams, and tables.

The search for patterns begins with concrete activities that focus on concepts such as symmetry, similarity, congruence, repetition, ordering, and equality. As students progress, they should learn more formal representations of those concepts and develop the ability to express the concepts in a variety of ways. Recognizing and working with numerical, geometric, and algebraic patterns will help students develop skill in inductive reasoning. This skill should be the precursor to a more disciplined use of inductive reasoning to devise strategies for solving problems. Attention to patterns and functions should be provided from kindergarten through grade twelve by:

- Encouraging the discovery of patterns and developing ways to describe or represent them
- Developing the understanding of a relationship between sets of numbers as a pairing of a member of one set with a member of another according to a rule that may be described verbally, visually, numerically, algebraically, or graphically
- Developing first informally, then with appropriate notation, the concept of a function as a special kind of relation in which each member of the first set is paired with just one member from the second set
- Using coordinate graphs at all levels to represent relationships

Statistics and Probability

Technological advances have created the ability to pursue almost limitless investigations with large quantities of real numerical data. Students must learn to interpret data with which they are confronted, make decisions based on the analysis of such data, and understand enough about these aspects of mathematics to assess derived information intelligently. The strand of statistics and probability must now be seen as including all stages of working with these data, from acquisition to application, and students must have the opportunity to begin developing their understanding in kindergarten.

The major themes that should be developed in the statistics and probability strand from kindergarten through grade eight are:

- Collecting and organizing data and presenting data in tables or graphs (bar, line, circle, and pictographs)
- Collecting empirical data and developing an intuitive notion of probability through performing probability experiments
- Counting possibilities by such means as organized lists and tree diagrams
- Developing theoretical probabilities based on methods of counting outcomes and comparing theory with experimental results
- Interpreting data using common measures of central tendency: mean, median, and mode
- Developing ways to measure dispersion through frequency distributions and graphs, starting with the range and preparing for the concepts of variance and standard deviation

The preceding themes should be continued at the high school level, with the following added:

- Developing methods for counting permutations and combinations
- Developing the concept of a probability function
- Applying binomial distributions in sampling and testing hypotheses
- Using random samples from a finite population to make an estimate of the population mean
- Relating two variables—curve fitting and rank correlation
- Exploring the laws of uncertainty: probability distributions of a chance variable, mean, and variance of probability distributions

Logic

The ability to reason logically is both a prerequisite for learning mathematics and a desired outcome of mathematics instruction. Mathematics provides an excellent context in which to make students aware of the logical structures they need to function successfully in any setting.

At the elementary level, logical thinking should be developed by helping students to:

- Recognize patterns not only in mathematics but also in such diverse disciplines as history, art, music, and economics.
- Organize their ideas and develop an understanding of the thought processes involved in logical reasoning.
- Understand and use terms such as *all, some, and, or, if . . . then,* and *not* in a mathematical context.
- Decide whether a particular mathematical construct fits a given set of conditions.
- Recognize a specific application of a general principle.
- Make valid inferences drawn from everyday experiences.

At the high school level, students should develop the ability to:

- Understand the deductive method as a way of thinking and use the basic rules for forming valid inferences in solving problems.
- Understand the relationship between assumptions and conclusions and thus test the implications of ideas.
- Judge the validity of reasoning that claims to establish proof.
- Recognize the difference between inductive and deductive reasoning and identify the process that is being used at a given time.
- Use several modes of inductive reasoning, such as enumeration, analogies, extensions of a pattern of thought, and the formulation of hypotheses.
- Use standard deductive methods to prove theorems, not only in geometry but in all other areas, including proof by contradiction and mathematical induction.
- Apply formal and informal logical reasoning processes in all mathematics courses.

Algebra

Algebra is often referred to as the "language of mathematics." It is simultaneously a tool and a product. Algebraic symbols make it easy to think about and represent mathematical ideas clearly and to show that certain ideas or relationships are true for all or a certain set of numbers. Algebra is concerned not only with the symbols for the elements of a system and their manipulation but also with the structure of a system defined by the basic properties of its operations. It links the other strands and is thus a prerequisite for the study of higher mathematics.

The major themes that should be developed in algebra in kindergarten through grade eight include:

- The notion of a variable
- The order of operations
- Solution of some equations and inequalities
- Evaluation of algebraic expressions given particular numbers to replace the variables
- Writing of statements, including variables, to represent given problems or number patterns

The content to be covered at the high school level in the algebra strand should include:

- Operations with and simplification of algebraic expressions
- Solution of equations and inequalities (linear, quadratic, rational, polynomial, and other algebraic) and their applications
- Graphing of linear, quadratic, polynomial, rational, algebraic, exponential, logarithmic, and circular functions and relations
- Solution of systems of mathematical sentences algebraically and graphically and their applications
- Use of vectors, matrices, and determinants
- Writing of mathematical equations or inequalities to represent a given problem

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Delivery of Instruction in Mathematics

TO ISOLATE the acquisition of mathematical knowledge from its uses and its relationships is to limit the depth of understanding achieved. Mathematical concepts and skills must be learned as part of a dynamic process, with active engagement on the part of the student. In other words, delivery of instruction is inseparable from curricular content. In this section the characteristics of instruction that will promote students' acquisition of mathematical power are described. The characteristics are more interlocking than distinct; all are directed at developing the students' abilities to enjoy and use mathematics.

Teaching for Understanding

Those persons responsible for the mathematics program must assign primary importance to a student's undertanding of fundamental concepts rather than to the student's ability to memorize algorithms or computational procedures. Too many students have come to view mathematics as a series of recipes to be memorized, with the goal of calculating the one right answer to each problem. The overall structure of mathematics and its relationship to the real world are not apparent to them.

The assumption is sometimes made that students who can perform an arithmetic computation understand the operation and know when to apply it. Teachers know and test results indicate that students are fairly competent at performing computations but have difficulty applying their skills to problem-solving situations.

Teaching for understanding emphasizes the relationships among mathematical skills and concepts and leads students to approach mathematics with a commonsense attitude, understanding not only how but also why skills are applied. Mathematical rules, formulas, and procedures are not powerful tools in isolation, and students who are taught them out of any context are burdened by a growing list of separate items that have narrow application. Students who are taught to understand the structure and logic of mathematics have more flexibility and are able to recall, adapt, or even recreate rules because they see the larger pattern. Finally, these students can apply rules, formulas, and procedures to solve problems, a major goal of this framework.

Teaching for understanding may be contrasted with teaching rules and procedures for their own sake in the following ways:

Teaching for understanding	Teaching rules and procedures
Emphasizes understanding	Emphasizes recall
Teaches a few generalizations	Teaches many rules
Develops conceptual schemas or interrelated concepts	Develops fixed or specific processes or skills
Identifies global relationships	Identifies sequential steps
Is adaptable to new tasks or situations (broad application)	Is used for specific tasks or situations (limited context)
Takes longer to learn but is retained more easily	Is learned more quickly but is quickly forgotten
Is difficult to teach	Is easy to teach
Is difficult to test	Is easy to test

Teaching for understanding does not mean that students should not learn mathematical rules and procedures. It does mean that students learn and practice these rules and procedures in contexts that make the range of usefulness apparent.

Reinforcement of Concepts and Skills

Many of the basic mathematical concepts are learned in the primary grades. It is a mistake, however, to assume that the initial learning will be retained without reinforcement. Concepts and skills from all of the strands of mathematics must be continually reinforced and extended. Major ideas must be learned by focusing initially on the basic concept and then repeating experiences with the concept in a variety of new settings. With each new experience a student's understanding of and ability to use the concept is expanded. In the mathematics program, opportunities must be created for individual students to achieve an increasingly more abstract or generalized understanding.

As new or extended concepts are developed, they must be connected to what students already know. Students must be able to combine single or simple skills to solve practical problems, such as those involving ratio, proportion, or percent. Relationships among concepts and skills should be continually stressed through activities designed or selected to demonstrate these relationships. Students should experience mathematics as a cumulative subject made up of interrelated strands rather than as a series of disjointed topics made up mostly of memorized operations to be applied separately on command.

Homework assignments are often used in mathematics programs to provide students with the additional exposure they need to internalize a concept. This is an appropriate and worthwhile practice and should be used as an opportunity to give students meaningful and creative experience that supplements the activities of the classroom. The involvement of parents and family members in a student's mathematical development can be accomplished by designing out-of-school assignments that cause students to work on problems with others, including their families. Homework should not be used for repetitive drill on material already learned. Isolated practice is no more appropriate at home than at school.

Problem Solving

Instruction in mathematics must be planned to maximize each student's experience in solving challenging real-world and abstract problems. To gain confidence and proficiency in problem solving, students must experience the reward of arriving at solutions through their own efforts. Therefore, students must take an active role in this activity and not simply observe the teacher or other students leading the way. Students must understand that problem solving is a process, with solutions coming often as the result of exploring situations, stating and restating questions, and devising and testing strategies over a period of time. The thinking and reasoning skills that go into problem solving are essential.

In problem solving the teacher should serve as a group facilitator rather than as a directive group leader. In this role the teacher should encourage the class to consider a variety of alternative and even unpromising approaches in solving problems associated with given situations. In addition, the teacher needs to create an atmosphere in which students understand that being temporarily perplexed is a natural state in problem solving. Adequate time for problem solving must be provided because the students, not the teacher, must do the thinking, make decisions, and find successful means to solve the problems.

To help students develop the attitudes and strategies useful in problem solving, teachers should:

- Model problem-solving behavior whenever possible, exploring and experimenting along with students.
- Create a classroom atmosphere in which all students feel comfortable trying out ideas.
- Invite students to explain their thinking at all stages of problem solving.
- Allow for the fact that more than one strategy may be needed to solve a given problem and that problems may require original approaches.
- Present problem situations that closely resemble real situations in their richness and complexity so that the experience that students gain in the classroom will be transferable.

Procedures in Problem Solving

The procedures used in problem solving may be divided into components to guide instruction. One such characterization is:

- Formulating problems
- Analyzing problem and selecting strategies
- Finding solutions
- Verifying and interpreting solutions

In the course of solving problems, students should experience these different aspects so they they can identify steps to take when they are confronted with a new situation. The teacher is a facilitator in the classroom, assisting students to use these processes for solving problems. The teacher should not, however, present the components as something to be followed in a lock-step sequence and should not reject alternative processes. For example, in analyzing a problem, a student might make a conjecture about the nature of the solution that in turn suggests a strategy to be tried. At any stage a creative solution is more valuable than a burdensome routine. When a student has an insight that leads quickly to a problem solution, the whole class can benefit from a discussion of the knowledge and thinking skills that were used.

Formulating problems. Real problems, mathematical or otherwise, do not usually exist in simple, easy-to-identify forms. More often, they are embedded in descriptions of puzzling or complex situations. The ability to analyze situations for potential mathematical relationships and to pose problems whose solutions might clarify those relationships or provide new information is a skill to be developed and nurtured throughout the mathematics program. Some critical skills involved in problem formulation are:

1. Verbalizing a question or questions that could be answered from the information given
2. Identifying mathematical questions that arise from given mathematical models (graphs, systems of equations, tables, and diagrams)
3. Formulating reasonable mathematical hypotheses or conjectures based on given information or a general description of a situation
4. Identifying missing or extraneous information

Students must have extensive experience with a variety of situations that they can explore and analyze to formulate mathematical questions. In the primary grades and throughout the other grades, students must be given frequent opportunities to discuss problem situations in their own words. Through these discussions teachers can assess students' understanding of concepts and mathematical vocabulary and their ability to apply their knowledge to a situation. In addition to discussions, students must have experience in writing problems.

Analyzing problems and selecting strategies. Students should have a collection of strategies to use in analyzing problems. Some strategies that students can use include:

1. Making a model
2. Drawing a picture
3. Organizing information in a table
4. Finding a simpier related problem
5. Acting out the situation
6. Restating the problem
7. Looking for patterns
8. Guessing and checking the result
9. Working backwards

The major role of the teacher in problem analysis is to encourage the students to think about possible strategies. A response of "That's right" can shut off the thinking process by suggesting that there is only one correct strategy. Questions such as, "Would that work? How could you find out?" or "What other approaches might work?" indicate to students that a variety of strategies might be effective and invite them to analyze and discuss their thinking.

Finding solutions. Students should carry out the plan or strategies that have been chosen. If the strategies selected do not work, the students should go back to the formulation and analysis stages and try again. Students also need to have a positive attitude toward risk taking if they are to have success in attacking new problems. Teachers must, therefore, create an atmosphere in which students are rewarded for devising and trying a likely approach, even if a solution is not achieved. Through discussing their ideas about approaches and results, students will become comfortable with the fact that some problems have more than one solution and others have none.

Verifying and interpreting solutions. A critical component of problem solving is the verification and interpretation of the solution that is obtained. Before any result is accepted, students should review the validity of the model and the accuracy of the mathematical procedures that were used to find the solutions. Major concerns may include the following:

Is the problem solved? Is the answer reasonable? Does the answer match the estimate?

What simplifications were made to solve the problem? Were they too broad or too restrictive? Did they ignore significant features of the situation?

How was the analysis done? Were the mathematical procedures performed correctly? If some of the numbers in the problem are changed, how does that change the solution?

How can the model be improved? Would another model work? Can the model be generalized or extended to other problems?

Summary of Problem Solving

Students must be actively involved in the processes of problem solving. The teacher should encourage students to think through these processes, foster discussion of ideas and approaches, and guide students to consider the reasonableness of their procedures. The processes of problem solving are neither a recipe nor magic but rather a collection of strategies that students can use to solve problems.

Situational Lessons

Mathematical concepts and skills often should be developed and reinforced through situational lessons. Such lessons begin with a description of an interesting, challenging situation from which a number of activities can emerge. To be effective, the situation should have immediate significance for students, perhaps be drawn from their lives, and involve families or community. The situation need not, however, be true to life. Students are usually interested in unfamiliar or imaginary places; and, because they lack firsthand knowledge, they are open to the use of reasoning. Good situational lessons exemplify mathematical problems as they occur, helping students to see the broad range of applications of mathematics. They contrast sharply with narrow exercises that provide practice with specific procedures.

Whatever their origin, situations should be complex enough so that several problems can be identified and pursued, a variety of approaches can be used, and the lesson can be studied over several class periods. Teachers should take the opportunity to demonstrate the connections among several concepts and to note the progress of individual students as they employ the various methods of solving problems that are required. Follow-up lessons can be planned to provide assistance with certain areas of difficulty. Teachers' editions of textbooks should provide examples of situational lessons that include the basic situation, the possible problems for students to pursue, the indicators of success that teachers should look for, and the types of follow-up that would be appropriate for certain areas of difficulty.

Use of Concrete Materials

Most learners, including those adults who usually function at the abstract level, profit from using concrete materials when they encounter a new concept or a difficult problem. In the classroom the use of concrete materials is frequently valuable for developing an understanding of new concepts, relating new concepts to what has already been learned, and representing elements of a problem. Concrete materials provide a way for students to connect their understandings about real objects and their own experiences to mathematical concepts. They gain direct experience with the underlying principles of each concept. For example, the use of place-value materials permits students literally to see when regrouping is appropriate in addition and subtraction. Ideas of probability are difficult for most students without extensive experience with spinners, dice, or other objects.

Manipulative materials can be used profitably to introduce concepts, even at the high school level. When students have internalized a concept through the use of concrete materials, they can move toward abstract representations of the concept. The use of manipulative materials does not, however, guarantee that students will transfer their learning to the symbolic representation of a concept. The role of the teacher is to assist students in making the connection between the concrete and the abstract. In addition to using concrete materials, students may work with pictures, drawings, diagrams, and other representations of objects. When students have explored concepts through concrete and representational experience, they will be ready to use symbols. In some cases at the secondary level, an abstract introduction of a procedure can be enhanced by the use of physical models. In algebra, for example, factoring can be taught first through the use of numerals and variables and then through work with base-ten or algebraic blocks. The teacher must determine the appropriate combination of concrete and symbolic models so that students gain a thorough understanding of concepts.

To be effective, teachers must do more than just provide concrete materials for their demonstrations. Students need to interact with the materials directly, continue to use materials as they develop understanding, and refer to the materials as they solve problems.

Flexibility of Instruction

Daily mathematics lessons should be structured to address the diverse needs of students through a program of ongoing diagnosis and assessment of each student. To make provisions for students who grasp a concept quickly, are working at an average pace, or need more time to learn concepts, the teacher should vary assignments. Although the assignments will vary in complexity, it is critical that the essential concepts be presented to all students. This can be accomplished through flexible classroom grouping, including whole-group instruction, cooperative learning groups, and individualized instruction; teacher-directed remediation; and horizontal enrichment.

Selection of textbooks and classroom management must be compatible with a variety of learning styles and rates. For each lesson, gradations of difficulty must be incorporated in the regular materials and must not be limited to supplementary materials. Alternative approaches must be delineated for those students who have failed to master a concept so that students do not continue to meet the same approach and the same failure repeatedly. From explaining ideas in their own words, using concrete materials and visual aids, and working on problems that are related to their own experience, students will have more opportunities to succeed. With such flexibility all students will be able to benefit from textbooks written for their grade levels and to learn the concepts and skills designated for those levels.

Parents should be apprised of their children's progress in mathematics. At the beginning of the school year, they should be informed of their children's status in mathematics instruction, expectations concerning homework assignments, and activities that can be done at home to aid student progress. When changes in student assignment and performance take place, parents should be notified.

Corrective Instruction/Remediation

Regular diagnosis and assessment of the students' work should continue throughout each year as an integral part of the mathematics program so that specific areas of difficulty can be identified and corrective instruction provided as soon as it is needed. Assessments should indicate the depth of comprehension as well as the pattern of errors. In analyzing students' errors, the teacher should listen to students' reasoning before deciding on a plan for corrective instruction. Corrective instruction should be based on diagnostic information and should be focused on the development of understanding in each of the strands, not on drill with rote procedures. This instruction should use interesting and fresh alternative approaches for students who have failed to master a concept.

It must never be assumed that once a student is placed in one type of group or provided with a particular method of instruction, he or she will remain in that mode indefinitely. Regular assessments must occur to ensure continuous progress.

Students who have been identified as needing the services provided by state and federal compensatory funding must be provided with a comprehensive, articulated mathematics program that goes beyond emphasis on computational skills development. In mathematics these services have frequently focused on practicing algorithms rather than on achieving understanding and have reinforced the pattern of the less capable students not learning the basic content of each strand of mathematics. For all students to learn and understand the material presented in this framework, regular and supplementary instruction must support the aims of understanding and using mathematics from all of the strands.

Parents of students in need of corrective instruction must be informed of their children's progress and of specific ways in which they can assist their children with activities and assignments that may be done at home.

Cooperative Learning Groups

To internalize concepts and apply them to new situations, students must interact with materials, express their thoughts, and discuss alternative approaches or explanations. Often, these activities can be accomplished well in groups of four students. Working in small groups increases each student's opportunity to interact with materials and with other students while learning. Students have more chances to speak in a small group than in a class discussion; and in that setting some students are more comfortable speculating, questioning, and explaining concepts in order to clarify their thinking. By brainstorming, exploring various approaches, and solving problems cooperatively, students can gain confidence in their individual abilities.

When a cooperative climate has been established, small groups that are heterogeneous in their composition have the added value of promoting positive attitudes toward others, regardless of individual differences. As a group works together, the differences will be less important than the task at hand. More capable students can assist others. The questions asked by students who do not yet understand an idea will help the entire group to bring its thinking into focus. Individual strengths will be highlighted so that students who have difficulty in some areas will have a chance to contribute their special skills in other areas.

Mathematical Language

In their mathematics classes students will encounter a distinct mathematics vocabulary. They will also find that commonly used words, such as *average*, have special meanings in mathematics. Students can grasp the meaning of these technical words only when the words are defined in terms that students understand and that are associated with familiar ideas. When teachers introduce new concepts and related vocabulary in a context that is familiar to students and continue to use the vocabulary in a clear and consistent way, students can assimilate the new words in their own vocabularies.

As new vocabulary is introduced, students' developmental levels must be taken into account. The reading of mathematics is especially challenging. Current textbooks do not enable a student to review material alone, and the repetition which helps to build understanding through the use of new words in context is typically lacking. Close attention must be paid to the students' points of reference and experiences. Often, verbal messages alone are not sufficient to define terms and concepts for students. Teachers must be ready to use concrete materials and illustrations as well as a variety of verbal cues, such as comparisons and examples. At every level, students should be required to use mathematical terminology appropriate to that level in their writing and discussions so that they will develop the ability to express themselves clearly and precisely. Expanded glossaries in textbooks should be used to support this goal.

Special care must be taken with the significant proportion of limited-English-proficient students. Although the learning styles and mathematical abilities of these students match those of the general student population, the students' limited proficiency in English must be considered when the instruction they are to be given in mathematics is being designed. Demonstrations and use of concrete materials and visual aids will be even more important for these students than for the general student population in building understanding of terms and concepts. Careful planning and use of resources will help limited-English-proficient students meet the general expectation of achieving understanding and developing skills in all of the strands.

Questioning and Responding

The questions asked by teachers or presented in textbooks can have a major impact on the way students think and reason. Teachers can use stimulating questions to get students involved in a lesson, improve students' understanding of the concepts under study, and assess the level of that understanding. Some characteristics of good questions follow:

- Good questions call on students to analyze and synthesize as well as to recall facts.
- They prod students to pose their own questions and to explore ways of answering them.
- Questions such as the following encourage students to explain, experiment, explore, and suggest strategies:
  1. How did you solve the problem?
  2. Why did that approach work (or not work)?
  3. What is another way to solve that problem?
  4. What different kinds of problems could be solved in the same way?

The way in which a teacher responds to students' answers can influence the answers as much as the questions do. If a teacher reacts to an answer in a way that signals conformity (through praise, criticism, or other value judgment), students will perceive that their thinking process is not valued as much as guesses to the answers that the teacher has in mind. The most effective teachers' responses are those that promote further thinking. Some examples follow:

1. *Allowing time after students' responses.* By pausing after a student answers a question, the teacher indicates that the students, not the teacher, are responsible for the thinking in the classroom. During the pause all the students have time to consider the response that was

ven. Further, the teacher can ask several students for their opinions about the response.

2. *Accepting students' responses.* If the teacher accepts a student's response without labeling it right or wrong, the student can examine and compare his or her ideas with those of other students without risk of rejection.
3. *Asking for clarification.* By asking a student to be more specific, to elaborate, or to rephrase a statement, the teacher indicates that the student's ideas are worth exploring and that the ability to explain his or her thinking is more important than the ability to use the same wording that the teacher or the textbook might have used. This approach requires students to use mathematical terminology correctly to clarify ideas in their own minds so that they can explain them to others.
4. *Suggesting sources of further information.* By answering a question with a reference to materials or information sources that may have the answer, the teacher encourages students to explore various sources and to become independent learners.

Standards for Mathematics Textbooks

THE standards in this section will serve as criteria for the adoption of mathematics textbooks for kindergarten through grade eight. For grades nine through twelve, these standards should be used as guidelines for the formal review of textbooks.

The *overall standard* is as follows:

1. Mathematics textbooks and other student materials will be consistent with the content and presentation of the mathematics curriculum as outlined in this framework.

The specific standards deal with content, organization and presentation of lessons, students' assignments, assessment, teachers' materials, and auxiliary materials.

The *content* of students' textbooks satisfies the following:

2. Major concepts or precursors of concepts from every strand—Number, Measurement, Geometry, Patterns and Functions, Statistics and Probability, Logic, and Algebra—are incorporated and are interwoven throughout the text for each grade level.
3. Problems in the text require the student to apply concepts and skills from all the strands in a variety of practical situations.
4. Multiple models and strategies for problem solving are developed and illustrated throughout the text, with emphasis on:
    a. Developing strategies
    b. Guessing, estimating, and exploring
    c. Selecting appropriate data
    d. Interpreting results and evaluating the approach used
    e. Analyzing errors
5. Mental arithmetic is emphasized to build students' confidence in relying on estimation before, during, and after calculation.
6. Calculators are presumed to be available for students to use in performing computations during the process of problem solving and in exploratory and discovery activities.
7. Examples and exercises show how mathematics is applied in other disciplines, such as natural science, social science, art, music, business, medicine, and law, and in everyday life.

8. The style of the text promotes student comprehension through such aids as:
   a. Subheads to assist in previewing material
   b. Paraphrasing, explanations, and comparisons to clarify important points
   c. Italic, boldface, large, or colored type to set off key words and concepts
   d. Illustrations to clarify points that are made in the text
9. There is a glossary or index of mathematical terms, with accurate definitions, matched to the understanding expected at each grade (grades three and above).

The organization and presentation of *lessons* in the textbook satisfy the following:

10. Presentation of mathematical concepts and skills demonstrates the beauty of mathematics and allows students to experience the fascination and excitement that mathematics provides.
11. The sequence of lessons provides for a steady, complete progression in concept and skill acquisition.
12. Lessons for every student, below as well as above average, include the major concepts and skills of every strand. No student is excluded from studying some areas because of difficulty with other areas.
13. For students who grasp the major concepts quickly, the textbook provides more complex lessons or extensions.
14. Lessons at all grade levels are designed to include the use of concrete materials so that students have ample opportunity to investigate concepts through firsthand experience with physical objects.
15. New or extended concepts are presented explicitly in relation to previously learned material.
16. Concepts from two or more strands frequently occur in the same lesson
17. Lessons often begin with problem situations that:
    a. Are interesting and challenging for the student.
    b. Require students to formulate mathematical problems.
    c. Stimulate creative approaches to problem solution.
    d. Anticipate a number of lessons emerging from or contributing to the problem-solving process.
18. Some lessons are designed to have students work together in small groups throughout the problem-solving process.

Assignments based on the textbook satisfy the following:

19. Each set of problems requires a variety of operations or solution techniques or both.
20. Problem sets are classified into subsets of varying degrees of difficulty and are properly identified as such.
21. Previously learned skills are reinforced through problems that require their use in new situations.
22. In calculating, students are consistently required to choose among estimation, mental arithmetic, paper and pencil, and the calculator according to the nature of the problem to be solved.
23. Problem sets include ample practice of skills necessary for later application, such as calculation or use of ratio, proportion, and percent.
24. The student is often directed to activities outside the textbook, such as:
    a. Obtaining data from real situations
    b. Developing computer programs from information in the textbook
    c. Utilizing computer software for extensions or applications not specifically identified in the textbook

Assessment and evaluation materials satisfy the following:

25. Materials are included that provide a means for regular assessment of students':
    a. Knowledge of mathematical concepts and skills
    b. Ability to apply these concepts and skills appropriately and correctly in given situations
    c. Ability to identify appropriate procedures, explain reasoning, and demonstrate techniques for problems solving

Teachers' materials satisfy the following:

2·. Teachers' materials:
    a. Provide suggestions keyed to specified learner outcomes for activities that will reinforce, extend, and enrich the learning of all students.
    b. Include procedures for teachers to use in analyzing students' errors and diagnosing gaps in students' problem-solving approaches.
    c. Demonstrate how teachers can integrate other content areas in mathematics lessons.
    d. Describe models or depict strategies for the teacher to use in questioning, responding, clarifying, or extending students' learning activities.

e. Indicate how concrete materials can be employed to present concepts, pose problems, and serve as tools for solving problems.

f. Provide a variety of ways to direct students to extend learning beyond the textbook.

g. Give direction on appropriate student groupings for activities, including successful strategies for small-group work.

h. Provide alternative strategies for evaluating students' learning, as demonstrated in class discussion, written and oral explanations, and visual presentations.

i. Provide for corrective instruction that uses fresh alternative approaches for students who have failed to master a concept.

j. Contain several schedules showing alternate ways to incorporate long-term (lasting several days) problem-solving projects, shorter-term lessons, and review of skills.

k. Contain references to materials outside the textbook that provide additional information, activities, and resource materials for teachers and students.

l. Suggest ways in which teachers might design lessons that match the developmental characteristics and learning modalities of students.

m. Provide information to use with parents in explaining various teaching strategies (for example, emphasis on problem solving, use of mental arithmetic, and use of the calculator).

Auxiliary/supplementary materials satisfy the following:

27. Instructional materials that accompany the student's textbook deepen or extend textbook material rather than provide for repetitive practice with narrow skills.
28. Computer software accompanying the text:
    a. Is consistent with the recommendations of this framework
    b. Provides extensions that reinforce textbook material
    c. Enhances the learning process beyond simple computational results through appropriate use of tutorials, simulations, and problem-solving activities
    d. Has been evaluated for technical features

Mathematics Programs:
Kindergarten Through Grade Three

Special Concerns

Expectations for Mathematics

By the time children begin kindergarten, they have counted and sorted objects, shared toys, and explored two- and three-dimensional shapes. They bring with them the natural curiosity that is essential for problem solving. In addition, the increasing use of electronic devices in the home has brought many children into contact with sophisticated technology at an earlier age than ever before. The mathematics program that children encounter in school must build on familiar experiences to extend the children's understanding and appreciation of mathematics. The instructional program in kindergarten through grade three should be structured so that:

- Classroom atmosphere fosters the development of logical thinking and problem solving.
- Students use concrete materials often in varied activities
- Students have many opportunities to explore, investigate, and discover.
- Students are continually encouraged to interact with each other to enhance understanding through verbalizing and visualizing.
- Relationships among mathematical skills and concepts are emphasized.

Problem Solving

From kindergarten on, skill in solving problems should be continually developed and should be based on situations appropriate to the students' level of understanding. Within this grade span particular emphasis should be placed on students' talking and listening to each other; making decisions; and collecting, organizing, and interpreting information in a variety of settings.

Concrete Materials

Concrete materials must be used extensively in the primary grades. Children in these grades are not abstract thinkers. Yet, these children are often asked to perform at an abstract level when they are given pages of exercises to complete before they have had sufficient experience to understand the concepts at a concrete level. Most of their experiences with mathematics should center on teacher-directed activities that build the understanding of concepts through the use of concrete materials and models. Every child should be involved in hands-on activities, and discussion among students must be an integral part of these activities. Through careful questioning, teachers must help students see relationships and make observations, generalizations, predictions, and interpretations. Teachers must assist students in making the connection between concrete experiences and the symbolic representation of a concept.

Classification

Classifying and sorting are especially valuable activities in the primary grades because they help students develop the ability to think analytically. In the past these activities have been used primarily in kindergarten and first grade and have required students to work with only one attribute to sort objects. All students in primary grades should have many opportunities to carry out varied classifying and sorting activities with one or more attributes.

Students should discuss their observations, conjectures, and conclusions, deciding whether they are reasonable and logical. The teacher should facilitate the discussions, encouraging students to use words such as *all, some, none, if . . . then,* and *not.*

Computation

Skill in computation is important at all levels in mathematics and is necessary in solving many but not all problems that arise in practical situations. However, the practice of a skill in isolation is seldom effective in developing the understanding required to make the skill useful. Instruction in computational algorithms should emphasize understanding the procedures that are being used.

For example, it is now common to teach two-place addition and subtraction without regrouping before introducing regrouping. This approach leads students to focus on separate procedures and hinders their understanding of the basic operations. Most students view a problem such as $\begin{array}{r}43\\ \underline{+25}\end{array}$ as two one-place problems pushed together ($\begin{array}{r}4\\ \underline{+2}\end{array}$ and $\begin{array}{r}3\\ \underline{+5}\end{array}$) and do not think of adding 40 and 20. Because students get a great deal of practice in this kind of problem, they assume that $\begin{array}{r}43\\ \underline{+29}\end{array}$ can be calculated in the same way and see nothing wrong with the answer of 612. After learning a new procedure in which 1 is carried or put on top of the tens' column every time, students are confused when presented with a mixture of problems that require regrouping and problems that do not. The child who asks, "Do I regroup on this problem?" has almost no understanding of the concept of two-place addition or subtraction. This dilemma can be avoided by teaching multidigit addition and subtraction with and without regrouping simultaneously, using manipulative place value materials, and relating the process to realistic situations. When manipulative materials are used, it is easy to demonstrate when regrouping is needed; and there is no need to teach regrouping as a separate algorithm.

Place Value

A firm understanding of place value is a prerequisite for all work in arithmetic. Students who do not understand the concept of place value cannot progress through the four basic operations without difficulty. They learn the operations by memorizing an increasing number of seemingly unrelated facts and procedures. Even the most capable students become confused when the load becomes too great, usually when they must learn multiplication and division. Mathematics becomes increasingly mysterious for these students, who have little hope of understanding the content of high school-level mathematics courses. It is essential that place value be given major emphasis in the primary grades and that students have frequent experience with manipulative materials that demonstrate place value.

Estimation

From kindergarten on, students should form the habit of estimating before calculating any answers. Activities on estimation should be focused on the range of a reasonable answer rather than on formal procedures. For example, in an attempt to find out how many second grade students there are in two classrooms and setting up a computation such as 34 + 29, estimation questions might be, "Will there be fewer or more than 100 students? How do you know?"

Program Content

The primary program in mathematics should enable the student to do the following:

Number

1. Count by ones, twos, fives, and tens.
2. Use cardinal and ordinal numbers to compare and order quantities.
3. Demonstrate an understanding of the meaning of the four basic operations.
4. Have facility with basic addition, subtraction, and multiplication facts.
5. Estimate answers to computational problems. For a given problem, be able to decide whether a proposed numerical answer is reasonable.
6. Add or subtract two three-digit whole numbers, multiply a two-digit number and a one-digit number, and divide a two-digit number by a one-digit number.

7. Choose the appropriate operation in a given situation.
8. Interpret word problems by using role playing, pictures, and models.
9. Write mathematical expressions and sentences, using operational symbols and symbols of equality and inequality to represent a situation. (The inequality symbols should be used only when students fully understand the concepts they represent.)
10. Use informally the concepts of commutativity, associativity (grouping), and the identity properties of zero and one.
11. Use concrete materials to recognize, represent, and compare halves, thirds, and fourths.
12. Use money to recognize, represent, and compare decimal values.
13. Recognize decimal and fractional equivalents for halves, fourths, and tenths.

Measurement

1. Use nonstandard, metric, and customary units of measure to estimate and measure length, volume, and weight.
2. Use digital and traditional clocks to tell time.
3. Read and interpret Celsius and Fahrenheit temperatures on thermometers.
4. Choose an appropriate unit of measure and use a variety of measurement instruments.
5. Recognize and count money.

Geometry

1. Use visual attributes and concrete materials to identify, classify, and describe common geometric figures and models, such as rectangles, squares, triangles, circles, cubes, and spheres. Use correct vocabulary.
2. Use several geometric shapes to make other geometric shapes.
3. Decide whether figures are congruent and whether they are similar.
4. Explore the tiling of the plane and the filling of space.

Patterns and Functions

1. Identify, verbalize, and extend a pattern in a sequence of objects.
2. Use a concrete model to create a pattern and represent that pattern symbolically in a table.
3. Describe the relationship given in a table or by a sequence of objects.
4. Determine a location by using ordered pairs of numbers on a rectangular grid.

Statistics and Probability

1. Collect, organize, represent, and interpret data derived from surveys and experiments conducted by the students.
2. Create and interpret concrete, pictorial, and symbolic graphs.
3. Predict outcomes and carry out simple activities involving probability.

Logic

1. Classify and sort objects, using one or more attributes by observing relationships and making generalizations.
2. Make reasonable or logical conjectures and conclusions about situations with concrete materials, using such words as *and, or, if . . . then, all, some, none, not,* and *out of.*

Mathematics Programs:
Grades Three Through Six

Special Concerns

Expectations for Mathematics

In the third through sixth grades, many students establish their basic attitude toward learning mathematics and their perception of what mathematics is and how it can be used. It is essential that the mathematics program for these grades promote the understanding and enjoyment of mathematics and provide a high degree of motivation. Students must have many experiences applying what they learn in interesting, useful ways. They must also have many opportunities to explore and experiment with new ideas. The program for this grade span must break away from its traditional teaching of rote, isolated computational skills and instead must strive for the development of higher-level thinking skills.

Problem Solving

Continual emphasis should be placed on development of the problem-solving skills of formulation, analysis, selection of strategies, solution techniques, and verification and interpretation of solutions. Students at this level should be able to suggest and use several strategies for solving problems, such as guessing and checking, making a model or drawing, making an organized list, working backwards, or considering a simpler related problem. Within the problem-solving context, students should organize collected information or data in a variety of ways and begin to develop means of analyzing and interpreting that information. In addition, students should be provided with opportunities to work with concrete materials and to discuss problems with each other. Teachers should model problem-solving approaches themselves and should continue to encourage students to ask questions and to make statements. devise their own solutions, think of related problems from their own experience, and use what they have learned as they meet new situations.

Understanding of Numbers and Arithmetic Operations

Whole numbers and decimals should be taught together, with an emphasis on place value; and students should have many experiences reading and writing large numbers and decimals. Concepts of percent should be introduced at an intuitive, concrete level. In addition, students should learn equivalencies of simple fractions, decimals, and percents (for example, that one-half is the same as .5, which is the same as 50 percent). Students must develop a firm understanding of what a fraction represents, including the interpretation of a fraction as an indicated division. Manipulative materials and pictures should be used to enhance the understanding of fractions and to develop methods of adding, subtracting, and multiplying fractions.

Students must understand the algorithms for operations with whole numbers, decimals, and fractions; but proficiency with paper-and-pencil calculation is not the primary goal. For example, students can understand the division process by using divisors of one and two digits. The use of divisors with more than two digits is time-consuming and nonproductive. The calculator must emerge as a tool to enhance the understanding of computation and as a device that will enable students to enlarge the content of mathematics they can study.

Incorporation of the calculator must go hand in hand with emphasis on mastery of single-digit number facts and thorough knowledge of place value and decimals so that students will confidently and accurately estimate the results of any calculations they are going to perform. Instruction in mental computational skills should place particular emphasis on short division and on multiplication and division by powers of ten, by two, and by five. With this focus it should be assumed that before the end of the sixth grade, the student will at all times, on tests as well as on in-class and out-of-class assignments, have the choice of estimation, mental arithmetic, paper and pencil, or a calculator to obtain a computational result.

Measurement and Geometry

Students at each of these grade levels must have hands-on experiences to provide a basis for concepts of measurement and geometry. They should extend their work beyond linear measurement to volume and weight and be abie to make reasonable estimates of these quantities for common objects. Concepts of perimeter, area, and volume should be explored through experimentation with materials such as tiles, grid paper, and blocks before abstract formulas are developed. Students should investigate ways to compare sizes of nonsimilar figures. They should use straightedges, protractors, and compasses to draw, measure, compare, and investigate relationships among angles. Students should develop and extend their perception of spatial relationships by investigating congruence, similarity, transformation, symmetry, and tesselation (tiling) by using concrete materials and recognizing examples as they occur in the environment.

Graphing, Data Analysis, and Interpretation

To be able to understand the data presentations of others, students must have experiences in collecting and presenting information. They should design surveys to collect data, organize and tally results, and present their findings in bar, line, or circle graphs. With data to analyze, students should develop an understanding of the mean, median, mode, and range and be able to use each in describing and comparing sets of data.

Estimation and Mental Arithmetic

In grades three through six, students should be taught a variety of strategies, such as clustering, comparing, using compatible numbers, and rounding as they continue to build their estimation skills and gain confidence in their ability to test the reasonableness of an answer. (See Section 1 for specific examples.) Decimals and fractions as well as whole numbers should be included. In addition, students should develop facility with several strategies of mental arithmetic, including compensating, computing left to right, and using the distributive property.

Program Content

e upper elementary program in mathematics should enable the ent to do the following:

Number

1. Recall basic addition, subtraction, multiplication, and division facts without hesitation.
2. Estimate answers to computational problems and be able to decide whether a proposed numerical answer is reasonable.
3. Find the sum, difference, product, or quotient of whole numbers and decimals in problem situations.
4. Use various techniques of mental arithmetic in a variety of situations.
5. Select and use an appropriate method of calculation—estimation, mental arithmetic, paper and pencil, or a calculator—according to the nature of the problem and explain the choice.
6. Use the concept of place value to identify the value represented by digits within numbers, compare whole numbers and decimals, estimate, and round off.
7. Interpret problems by using pictures and models and role playing and, when appropriate, translate into mathematical expressions and explain the process.
8. Use the concepts of commutativity and associativity, the identity properties of zero and one, and the distributive property informally.
9. Identify the prime factors of a number, the least common multiple, and the greatest common factor of two numbers.

10. Use concrete materials to represent fractions, their equivalences, and the operations of addition, subtraction, and multiplication.
11. Understand and use ratio and proportion to solve problems.
12. Identify common equivalent expressions (decimals, percents, and fractions) and explain why they are equivalent.
13. Understand the concept of fractions and their order and, on the basis of this understanding, find their sums, differences, and products.

Measurement

1. Choose an appropriate unit of measure (metric and U.S. customary) to measure length, volume, and weight.
2. Continue to develop skill in using measurement instruments.
3. Convert units within a system of measurement.
4. Measure various figures and derive the appropriate formulas to find perimeter, area, and volume.
5. Use estimation to approximate measurement.

Geometry

1. Identify two-dimensional geometric figures, such as circles and polygons, and three-dimensional geometric figures, such as cylinders, cones, and rectangular prisms.
2. Apply the basic terminology (such as point, line, plane, angle, parallel, perpendicular, and diameter) to geometric figures.
3. Use a protractor, compass, and straightedge to draw and measure angles and for other constructions.
4. Construct models of three-dimensional figures.
5. Determine whether two or more geometric figures are similar or congruent and explain the reasoning that is used.
6. Demonstrate whether a figure has one or more lines of symmetry and explain the reasoning that is used.
7. Explore transformations of geometric figures, including translations (slides), rotations (turns), reflections (flips), and dilations (stretching and shrinking).
8. Explore the tiling of the plane and the filling of space.

Patterns and Functions

1. Find a pattern in a sequence of whole numbers and extend the sequence.
2. Extend patterns represented in tables or as ordered pairs and propose a rule to describe the relationship.
3. Identify and graph points in the coordinate plane and describe the result.

Statistics and Probability

1. Collect, organize, represent, and interpret data, using lists, tables, and graphs (bar, line, and circle).
2. Interpret data, using the mean, median, mode, and range.
3. Predict, perform, and record results of simple probability experiments.
4. Use simple tables or tree diagrams to represent possible outcomes of an experiment and assign probabilities to these outcomes.

Logic

1. Classify and sort objects, using one or more attributes by observing similarities and differences, describing and recording relationships, and making generalizations.
2. Make reasonable or logical conjectures and conclusions about situations represented by concrete materials or by sentences, using the words *and, or, if . . . then, all, some, none, not,* and *out of* when appropriate.
3. Act out, model, and use charts to organize information and solve simple logic problems.

Algebra

1. Recognize that each number from a given set can be used as a replacement for a variable in an expression or equation.
2. Find replacements for variables that make simple number sentences true.
3. Use variables in algebraic expressions to represent arithmetic relationships and represent basic properties of numbers.
4. Evaluate simple formulas.

Mathematics Programs:
Grades Six Through Eight

Special Concerns

Expectations for Mathematics

The middle or junior high school grades are important transition years in the mathematics curriculum. Most students in these grades are developing the ability to think more abstractly. They have had many experiences applying skills and strategies for solving problems. The mathematics program for these years should provide:

- Strengthening of previously taught skills, with a focus on greater depth of understanding in all strands
- A variety of application assignments that incorporate more than one strand
- More opportunities for thinking independently, solving complex problems, and working in small groups
- Preparation for the more rigorous content and expectations of the high school program

The middle or junior high school mathematics program must ensure that all students who successfully complete the entire program are prepared for Algebra 1 or Math A. Mathematics programs for students who experience difficulty must provide instruction in all strands and provide an avenue that will allow these students to enroll in the normal high school mathematics program.

By the end of the eighth grade, students will have completed the study of the rational number system, including the basic operations on the set of positive and negative numbers and zero, common fractions, and terminating and repeating decimals; the ordering of the rational numbers; ratio, proportion, and percent; and an introduction to the irrational numbers. In addition, students will have completed a substantial study of measurement and its applications; relationships among geometric elements, including the Pythagorean theorem; and introductory concepts of formal algebra and statistics.

Concrete Materials

Although students in grades six through eight are developing increasing competence in working with the abstractions of mathematics, they should have frequent opportunities to use concrete materials so that they can develop new concepts and relate those concepts to what was taught previously.

Arithmetic Operations

Before the end of the sixth grade, students should know how and when to use a calculator and should use it on a regular basis. In grades six through eight, students should be using basic arithmetic operations in a variety of situations involving measurement, geometry, functions, and statistics and probability. If a student cannot determine the operation or sequence of operations necessary to solve applied problems, the student should be provided remediation based on individual diagnosis.

Eighth Grade Course

The mathematics course offered in the eighth grade must prepare students for the high school mathematics program outlined in this framework. This course must not be merely a rehash of the mathematics taught in grades four through seven, nor should it be limited to traditional prealgebra topics. It must prepare students for their first high school mathematics course, Algebra I or Math A; therefore, it must contain topics from geometry, logic, probability, statistics, and algebra. Students must extend their experience in every strand and have the opportunity to use the mathematics learned in grades four through seven in new ways and in new contexts. This approach will ensure coverage of the entire content for grades six through eight (see "Program Content," the second major part of this section). Students must be prepared to solve problems for which the method of solution is not clear at the outset and which may require mathematical background from several strands. They must be prepared to pose, argue, prove, or demonstrate mathematical statements.

Sample Problems in the Eighth Grade Course

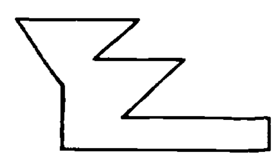
The purpose of the following set of examples enumerated by strand, is to give a general idea of the level and type of questions and problems that students should encounter in the eighth grade course. The list is mixed in its content to include, in some cases, specific sample problems and, in others, general descriptions of problems. The list is meant to provide a few representative examples of the general level and approach of the course and is by no means complete.

Number

1. Investigate square roots. Construct line segments of length $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, and so on. Estimate square roots of larger numbers and check. Use the divide-and-average method with a calculator to get closer and closer to the square root of a given number
2. Demonstrate that a terminating or repeating decimal can be represented as a fraction and vice versa. Use a calculator to compare decimal expansions of some rational and irrational numbers, with attention to the limitations of the calculator display.
3. Use a calculator to determine estimates of other (third, fourth, fifth) roots of numbers.
4. Use ratio and proportion to find missing parts of similar figures and to solve problems involving percent.
5. Devise a demonstration to show how many one million represents.
6. Use a scientific calculator or computer to solve problems involving very large or very small numbers.
7. Write a set of directions for a younger student, explaining how to add 2/5 and 1/3. Then use a picture and write an explanation as to why you add fractions the way you do.

Measurement

1. Demonstrate experimentally for a younger student that the volume of a cone is 1/3 the volume of a cylinder with the same base and height.
2. Use and explain three different methods for approximating the area of an irregular figure, such as:



3. Obtain areas of regular polygonal figures and use figures inscribed in a circle to approximate π.
4. Use similar triangles and the Pythagorean theorem to make indirect measurements.

40

Geometry

1. Given two sides and the included angle, draw a triangle. Do the same when given three sides or two angles and a side. Draw two different triangles having an angle of 20°, an adjacent side of five units, and the opposite side of four units.
2. Use a straightedge and a compass to inscribe a triangle, square, hexagon, octagon, decagon, or pentagon within a circle.
3. Given a figure involving the basic constructions, construct it in a different size.
4. Given the coordinates of a figure in the plane, rotate it through 180°, 90°, or 45°. Translate it. Reflect it.
5. Construct models of the five platonic solids and investigate relationships among the number of edges, faces, and vertices.
6. Draw two-dimensional top, front, and side views to represent a three-dimensional object. Reverse the process to draw a three-dimensional representation based on top, side, and front views.
7. Investigate the number of diagonals of various regular polygons and develop a rule.

Patterns and Functions

1. Given a table of values, such as

-2	-1	0	1	2	3	4	5	10	100
-8	-5	-2	1	4	7				

find the missing values, sketch a graph, and find a formula by guessing and checking.

2. Make a table, and sketch a graph of each function:

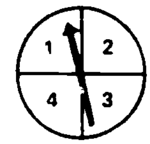
$y = \frac{1}{2}x - 3 \qquad y = |x - 4| \qquad y = x^2 - 5$

$y = x(x - 2)(x - 4) \qquad y = 1/x$

3. Given the sequence 0, 7, 26, 63, 124, . . . , find the next two terms and find an expression to represent the nth term.
4. Draw a graph that represents the length of each day in two-week intervals in one year.

Statistics and Probability

1. If your favorite baseball star is batting .297, what is the probability he will not get a hit in his next time at bat?
2. In a game with a spinner like the one shown here that tells you how far to move your marker on a board, what is the probability you will move exactly 12 spaces in your first three turns? What is the probability you will move fewer than 11 spaces?



3. What are the possible results of rolling two dice and subtracting the number showing on the face of one from that showing on the face of the other in such a way that a non-negative result is obtained? Guess which result is most likely. Try the experiment 50 to 100 times and record your results. Make a chart to find the theoretical probability of each outcome.
4. Conduct class or school surveys of interest to students. Analyze the data by calculating the mean, median, mode, and range, using a frequency distribution when appropriate. Make graphs to represent the information, including pie charts based on percentages and drawn with a protractor.
5. Gather data such as heights of students, head sizes, shoe sizes, or test scores. Have students make a bar graph based on a frequency distribution. Use the graph to explore the idea of measures of central tendency and dispersion.

Logic

1. Explore informally the distinction between a statement and its converse in a variety of contexts.
2. Given the equation $5x - 7 = -3$, prove, by justifying each step in the solution of the equation, that 4/5 is the only solution.

Algebra

1. Use a guess-and-check approach to develop an equation or inequality to represent a word problem.
2. Use formulas to solve specific problems, given the appropriate data. Students should work with formulas from the physical and biological sciences and social science as well as standard formulas, such as those for distance and area.

Eighth Grade Algebra Course

It is strongly recommended that an algebra course be offered to well-prepared eighth grade students who have already completed the eighth grade mathematics course previously described.

This course should be a full and complete algebra course so that on finishing it, students will be prepared for the next course in the college preparatory sequence (see Section 8). For this course offering to be established, the following conditions must be met:

- Each eighth grade algebra teacher must be fully qualified to teach the course and must be capable of adding depth and breadth to the concepts and skills under study. A significant increase in the quality of preservice and in-service education of mathematics teachers in grades six through eight will, therefore, be required.
- Before entering an eighth grade algebra course, students must have completed the study of all the mathematics recommended for kindergarten through eighth grade and must have demonstrated a high level of understanding and proficiency in the concepts and skills of the program.
- The selection of students for an eighth grade algebra class must be based on demonstrated achievement, the satisfaction of clearly stated prerequisites, and the recommendation of teachers.
- Students must have the study habits, maturity, and motivation necessary for success in an algebra program.

Program Content

The middle school or junior high program in mathematics should enable students to do the following:

Number

1. Estimate answers to all computational problems. For a given problem, the student should be able to decide whether a proposed numerical answer could be a possible solution.
2. Find the sum, difference, product, or quotient of whole numbers, fractions, and decimals, choosing an appropriate method of calculation—estimation, mental arithmetic, paper and pencil, or the calculator—according to the nature of the problem.
3. Use the order relations to compare whole numbers, fractions, and decimals; and locate these numbers on the number line.
4. Understand and perform basic operations on the full set of rational numbers—positive, negative, and zero; develop an awareness of the underlying structure on which these operations are based; and recognize the various forms for representing rational numbers.
5. Use the associative and commutative laws of addition and multiplication and the distributive law to simplify numerical calculations.
6. Convert among percent, fractional, and decimal equivalents; and solve for the unknown in a percent problem.
7. Use ratios and proportions to solve a variety of problems, including those involving percent.
8. Find the square roots of whole numbers, fractions, and decimals whose square roots are rational; and estimate irrational square roots as a check of results obtained with calculators.
9. Find the prime factors of whole numbers; and use prime factorization to find factors, multiples, greatest common factors, and least common multiples of a set of whole numbers.
10. Evaluate numerical expressions involving positive integral exponents; represent the prime factorization of a whole number in exponential notation; and convert whole numbers from base-ten notation to scientific notation and vice versa.
11. Use the additive and multiplicative laws of exponents to simplify arithmetic expressions involving positive integral exponents and to multiply and raise numbers represented in scientific notation to powers.

Measurement

1. Choose an appropriate unit of measure and use the appropriate formulas to find the perimeter, circumference, and area of polygons and circles, the volume and surface area of selected solids, and the measures of angles.
2. Convert units of measure of length, area, and volume within a system of measurement.
3. Find perimeter, area, surface area, and volume of irregular geometric figures.
4. Explore the approximate nature of measurement and the degree of error, using the concept of rounding.
5. Use a ruler and protractor with proficiency.

Geometry

1. Use and give examples to represent geometric terms, such as acute, right, obtuse, complementary, supplementary, and vertical angles; paral-

lel, perpendicular, and intersecting lines and planes; and measures of angles. Identify two- and three-dimensional geometric figures.

2. Describe relationships between figures (congruent, similar) and perform transformations (rotations, reflections, translations, and dilations).
3. Demonstrate relationships involving geometric elements (for example, the sum of the measures of the angles of a triangle—180°; two points needed to determine a line; three noncollinear points needed to determine a plane; the Pythagorean theorem; symmetry about a point and a line; supplementary and complementary angles; scalene, isosceles, equilateral, and right triangles).
4. Use knowledge of geometric terms, figures, or relationships to solve problems.
5. Construct geometric figures, using compass and straightedge.

Patterns and Functions

1. Determine the function rule that represents a relationship and find the value of the function for a given value of the variable.
2. Graph simple relations and functions in all quadrants of the coordinate plane.
3. Represent functions in several ways: with a graph, as ordered pairs of numbers, in a verbal statement, or as an algebraic rule.

Statistics and Probability

1. Represent the probability of an event as a fraction.
2. Find the empirical probability of an event from a sample of observed outcomes.
3. Find the probability of complementary events and of mutually exclusive events.
4. Generate a frequency distribution for a given list of data; and compute the mean, median, mode, and range.
5. Extract valid information from graphs, tables, and schedules.
6. Use a list or tree diagram to count permutations or combinations.

Logic

1. Explore the notion of mathematical implication.
2. Explore the meaning of the logical connectives *and, or, if . . . then,* and *not.*
3. Determine the validity of simple arguments.
4. Determine the equivalence of logical expressions.
5. Perform simple deductive and inductive reasoning exercises.

Algebra

1. Translate English phrases and sentences into algebraic expressions and vice versa.
2. Simplify algebraic expressions.
3. Substitute numerical and algebraic terms for variables in algebraic expressions; and use the standard order of operations to evaluate these expressions.
4. Solve linear equations of the form $ax + b = c$, using integers, fractions, and decimals.
5. Solve simple inequalities; and display solution sets on the number line
6. Represent mathematical patterns, using variables.

Mathematics Programs:
Grades Nine Through Twelve

Special Concerns

Orientation

A successful high school mathematics program must have teachers who are willing to experiment with new ideas and to dedicate themselves to high standards for all students. It must provide as well a commitment to help students perceive the beauty and fascination of mathematics and develop confidence in their ability to use mathematics.

Student Performance

Teachers of mathematics must ensure that

- Students satisfactorily complete the study of *all* of the recommended content in each course in such depth that subsequent related concepts and skills can be developed with some assurance of understanding and success.
- Standards of student performance are such that later courses can be completed without extensive reteaching or review of content previously taught.

Alternative High School Programs

Every high school mathematics program must offer alternative sequences of courses. A thorough, ongoing diagnosis for each student is essential for recommending student placement. Equally important, advice on placement as well as the course offerings themselves must be planned to keep options open for the student. Students who lack either preparation or motivation to begin a college preparatory sequence in the ninth grade should have the opportunity, if ready, to enter the sequence in the tenth or even eleventh grade. At the same time some students who take Algebra 1 in the ninth grade solely to keep the four-year college preparatory option open should have worthwhile opportunities in the tenth or eleventh grade if they choose not to complete the college preparatory sequence. Finally, there must be well-developed, intensive mathematics courses for students who are not functioning at the expected level on completion of the eighth grade. The high school mathematics program should make it possible for students to take mathematics during all four years and should provide counseling to encourage students to do so.

Figure 8-1 presents an outline of the recommended alternative mathematics programs for high school students. The solid lines represent the most likely paths; the dotted lines represent alternatives that should be viewed as not only possible but highly desirable for some students. Students should take at least two years of mathematics past the level of the eighth grade course.

The major recommendations in terms of Figure 8-1 follow:

- There should be four options for ninth grade students:

1. The college preparatory sequence, beginning with Algebra 1, is described in detail under "Suggested Content for High School Courses," the last part of this section. The structure of these courses should be determined according to the recommendations contained in this framework and the deliberations of local curriculum groups charged with implementing those recommendations. The depth and breadth of the courses should be based on the outlines included under "Suggested Content for High School Courses" and should not be determined by the capability or interest of the majority of students enrolled or considering enrollment at a particular time.

   Students who are not fully prepared should complete appropriate courses before attempting to enroll in the regular college preparatory sequence. Students capable of attending college should be enrolled in mathematics courses during all four years of high school. If feasible, the school should provide an honors

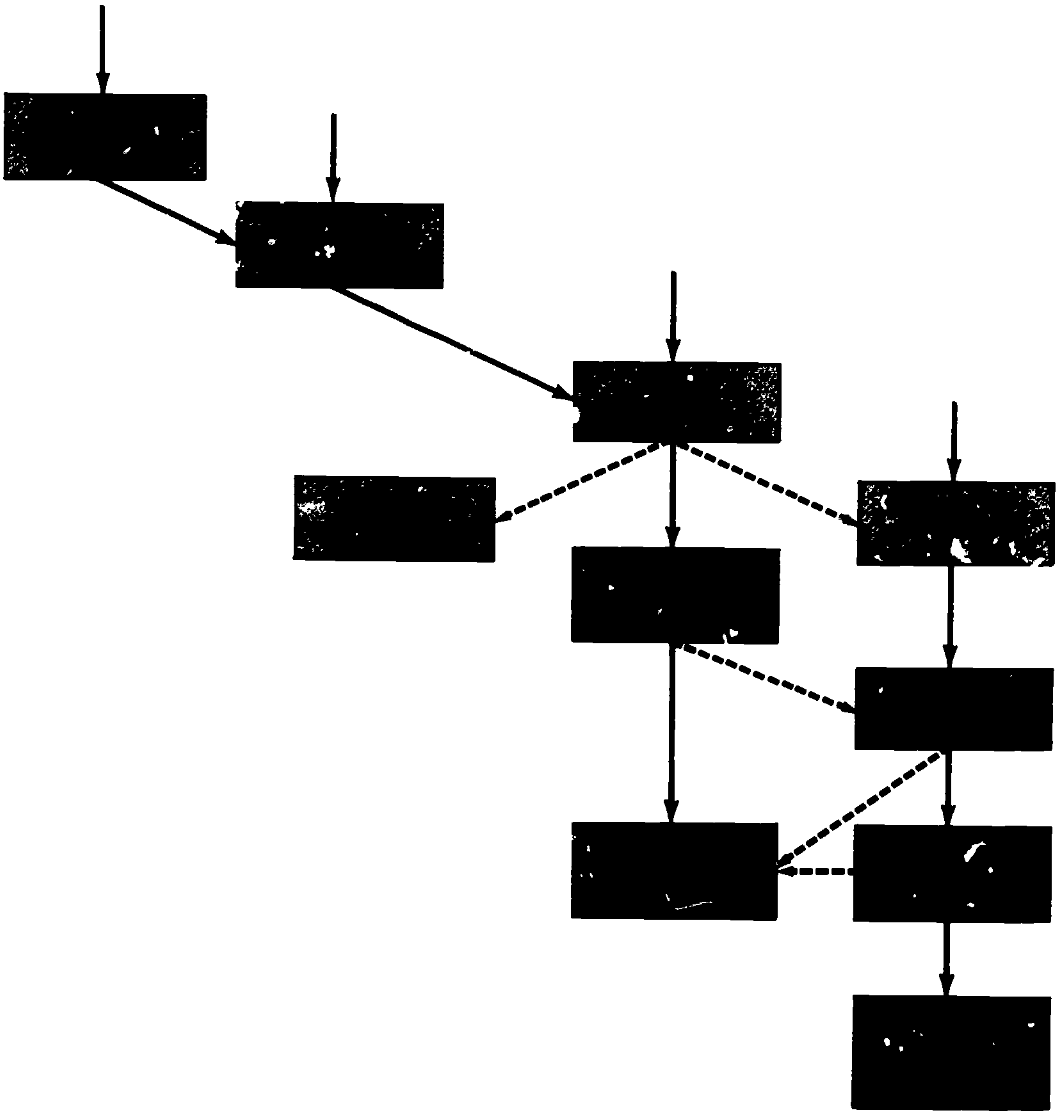


Fig. 8-1. Recommended alternative mathematics programs for high school students

sequence for academically exceptional students beginning at the Algebra 2 level or earlier. The courses in this sequence should meet honor-level criteria established by the University of California and should include all of the mathematical content required to enter an advanced placement calculus course.

2. There should be a sequence of mathematics courses, labeled here as Math A and Math B and described more completely under "Suggested Content for High School Courses," whose content requires the student to learn mathematics beyond the kindergarten through eighth grade program described in this framework but does not require the student to begin a traditional college preparatory sequence. These courses should cover much of the content of Algebra 1 and geometry. After completing one or both of these courses successfully, students could transfer to the college preparatory sequence.

3. A course should be offered for students who have facility with concepts and skills of the elementary school program but have not mastered the additional or more complex content of the middle school or junior high program. The junior high remediation course should include the content of the mathematics program for grades six through eight (see the sample for the eighth grade course in Section 7). The content must be presented in a manner appropriate for high school students.

4. Finally, a course should be offered for students lacking the basic concepts and skills taught in the elementary program. This course should be highly individualized and should be taught by someone who has both deep understanding of the fundamentals of all the strands and special expertise in diagnosing learning difficulties.

- Courses should be designed specifically to allow students to shift from one sequence to another.

  1. The junior high school remediation course should have enough flexibility in its design to allow some students from the elementary remediation course to transfer in during the year.

  2. Although Math A and Math B contain much of the content of Algebra 1 and geometry (as well as significant content from the Number, Measurement, Statistics and Probability, and Logic strands), they do not together incorporate all of Algebra 1. Thus, some additional instruction in specific concepts or skills of algebra must be provided for students who, while taking Math B, decide to enter the college preparatory sequence the following school year. One possibility is for these students to receive the additional instruction through extra assignments during the second semester of Math B. If there are several Math B students opting for geometry in the following year, a more manageable approach might be to provide a three- to four-week summer class covering the remaining material from Algebra 1. A third alternative is to allow the student to transfer to the second semester Algebra 1 course at the midpoint of Math B.

- There should be a varied set of electives. Each elective should be offered with the expectation that it will be useful to some students who had taken several different sequences of courses.

  1. There should be at least one course (labeled *Elective* on Figure 8-1) that would be an alternative to Math B. Such a course would be oriented toward the use of mathematics in practical problem situations.

  2. There should be an elective (labeled *Math C* on Figure 8-1) that would be of interest to two sets of students: (1) those from Math B or geometry who do not wish to complete the college preparatory sequence but want an additional year of mathematics; and (2) those who have completed Algebra 2 and have a definite need for additional mathematics but do not expect to pursue a mathematically based college program. Suggested topics for such a course are included under "Suggested Content for High School Courses" in this section.

  3. The size and composition of the student population, the qualifications of the staff, and the availability of eighth grade algebra are all factors in determining a school's college preparatory sequence. The content of the Algebra 1, geometry, Algebra 2, and math analysis courses, as described under "Suggested Content for High School Courses," encompasses all of the mathematics that must be covered prior to enrollment in a college-level calculus course. Since college-bound students should be enrolled in a mathematics course every year of high school, there should be one or more electives in the senior year of high school, such as linear algebra, statistics and

probability, or discrete mathematics. For many students any of these courses may be a preferred alternative to an advanced placement course in calculus.

4. Calculus offered in high school should be taken by those students who are strongly prepared in algebra, geometry, trigonometry, and coordinate geometry and who can demonstrate mastery of these subjects. The calculus course should be a full-year course and should prepare the enrolled students to take one of the College Board's advanced placement calculus examinations.

Qualifications of Teachers

It is essential that mathematics teachers be qualified to teach assigned courses. Preparation of secondary mathematics teachers should follow the guidelines outlined in Section 1 under "Teacher Preservice and In-service Education."

Uses of Technology

All high school mathematics courses should make full use of calculators and computers as tools to facilitate the learning and application of mathematical concepts:

- Scientific calculators should be used to expedite the development of mathematical concepts and skills at every possible opportunity. Scientific calculators should replace the use of all common tables, such as those for roots and powers, logarithms, and trigonometric functions.
- Computers can be used for simulating complex situations, performing mathematical experiments to clarify concepts, constructing graphs, finding zeros of polynomials, and performing complex matrix operations.
- Students should be instructed in the development of algorithms and programming techniques that can be applied in specific mathematical situations.

Delivery of Instruction

Instruction in the high school mathematics program should:

- Involve students in active learning situations for, on the average, at least half of the period. Individual seatwork should take up not more than one-third to one-half of the period.
- Require that students describe in writing and in speech the process used in solving a given problem.
- Involve questioning techniques that create student interaction, encourage exploration, diagnose processes that students have used, enhance transfer, motivate, and challenge students to think.
- Stimulate curiosity on the part of the students and set the stage so that students ask questions and contribute comments in a discussion.
- Enable students to see the teacher engaging in problem-solving activities as a regular part of the instructional program.
- Enhance students' understanding of mathematics by identifying and stressing difficult parts of a process and explaining why as well as how a process works.
- Demonstrate the continuity of mathematics by explaining the relationship of the current work to concepts learned in the past and to be learned in the future.
- Model and require careful, correct use of mathematical vocabulary.
- Use a variety of instructional formats (small groups, interactive groups, teacher-led discussions, student presentations, laboratory learning experiences, and so on).
- Involve high but appropriate expectations that are visible to students and are at the same time consistently expressed. For example, students are expected to complete all assignments and to do well on examinations.
- Involve the use of calculators wherever possible to (1) remove the tedium of arithmetic computation; (2) enhance the study of mathematical relationships such as inverse functions; and (3) speed up the development of mathematical concepts by reducing the time spent on intermediate computations.
- Encourage students to solve problems in a variety of ways and to accept solutions in many different forms.
- Provide opportunities for students to become involved in in-depth problems for periods of several days. As they work toward solutions, students should investigate and experiment, develop new mathematical skills and concepts, and apply previously acquired knowledge.

Program Content

The senior high school program in mathematics should enable students to do the following:

Number

1. Use operations on positive and negative rational numbers, including fractions and decimals, to

determine sums, differences, products, and quotients quickly and accurately.

2. Select and use the symbols of equality and inequality, operational symbols, and properties of the number system to write mathematical expressions and sentences that satisfy given conditions.
3. Evaluate formulas and numerical expressions involving powers and roots, grouping symbols, and the rules for the order of operations to solve a variety of applied problems.
4. Locate integers on a number line and approximate the location of rational and irrational numbers to demonstrate the concept of ordering.

Measurement

1. Understand and apply the relationships between the precision of measurements and the accuracy of a calculation based on the measurements.
2. Make conversions within measurement systems, using conversion tables and equivalence of units.
3. Select and use appropriate formulas and procedures to determine a measure when a direct measurement tool is not available.

Geometry

1. Use the terminology of geometry, including terms that describe angles, lines, polygons, circles, and three-dimensional figures: cubes, cones, cylinders, prisms, pyramids, and spheres.
2. Understand and apply the basic postulates, definitions, and theorems of Euclidean geometry.
3. Employ the concepts of reflection, rotation, and translation to demonstrate symmetry and congruence of figures. Use ratio and proportion to demonstrate similarity.
4. Perform standard geometric constructions with a compass and a straightedge.
5. Apply the Pythagorean theorem and right-triangle trigonometry in practical problems.
6. Solve simple algebraic problems involving properties of polygons, including special quadrilaterals—square, rhombus, rectangle, parallelogram, and trapezoid—and special triangles—isosceles, equilateral, and right (including 30° - 60° - 90°, 45° - 45° - 90°).
7. Apply the principles of coordinate geometry to graph lines, determine the slope and intercept of a line and the midpoint of a line segment, and determine the areas of special geometric figures.

Patterns and Functions

1. Identify functions and inverse functions by inspection of their graphs.
2. Translate graphs vertically and horizontally.
3. Graph linear inequalities in two variables.
4. Graph nonlinear functions that represent practical situations and interpret the graphs.

Statistics and Probability

1. Use counting procedures to solve combinatorial problems.
2. Develop an understanding of the common mathematical properties of the mean of a set of data.
3. Use mathematical expectation to make judgments about the possible outcomes of random events.
4. Distinguish between independent and dependent events and use conditional probabilities.
5. Explain the significance of varying values of statistical measures.
6. Choose appropriate statistical measures to describe data.
7. Identify and explain misuses of statistical measures.
8. Estimate probabilities of events based on empirical data and use these results to make inferences.

ogic

1. Distinguish between inductive and deductive reasoning and explain when each is appropriate.
2. Use inductive reasoning to generate hypotheses.
3. Use deductive reasoning to reach conclusions.
4. Recognize when the conditions of a definition are met.
5. Identify the distinction between a necessary condition and a sufficient condition. Explain what is meant by a necessary and sufficient condition.
6. Recognize and explain flaws in invalid arguments.

Algebra

1. Solve a linear equation or inequality in one variable and explain the steps used.
2. Formulate and solve systems of linear equations or inequalities algebraically or graphically.
3. Solve practical problems involving direct and inverse variation.
4. Solve practical problems involving ratio, proportion, and percent.
5. Simplify and evaluate algebraic expressions involving positive and negative integral exponents and square roots.
6. Perform the operations of addition, subtraction, and multiplication on binomials.
7. Simplify rational algebraic expressions with monomial denominators.
8. Factor polynomials by removing the greatest common monomial factor.

Suggested Content for High School Courses

Description

This is the first year of a two-year sequence that presents the fundamental concepts and skills required in a technological society. The course should emphasize applications and use an instructional approach appropriate for students with a concrete rather than an abstract learning style. Students' competencies in each of the areas will range from minimal understanding to full appreciation of major implications and applications. All students are expected to achieve minimal understanding in each area. Most will achieve beyond this level.

Students who demonstrate a thorough understanding of the course content may, with the recommendation of the instructor, enroll in Algebra 1.

Content

A. Measurement

- Choose appropriate units for measuring objects, rates, and energy.
- Find measures of length, area, volume, and weight to predetermined accuracy.
- Use conversion tables and equations to approximate measurements between measurement systems

B. Geometry

- Identify and apply the basic postulates of Euclidean geometry.
- Apply the terminology of geometric elements, including angles, lines, polygons, circles, and three-dimensional shapes: spheres, cubes, cones, pyramids, and cylinders.
- Use transformations of the plane to demonstrate similarity, symmetry, and congruence of figures.
- Perform standard geometric constructions with a compass and a straightedge.
- Apply the Pythagorean theorem and right-triangle trigonometry in practical problems.

C. Statistics and Probability

- Construct and use frequency tables to find the mean and dispersion of a set of data.
- Determine the sample space for the outcomes of an experiment.
- Estimate the probability of an event, using empirical data.

D. Logic

- Distinguish between inductive and deductive reasoning and explain when each is appropriate.
- Use inductive reasoning to generate hypotheses.
- Recognize when the conditions of a definition are met.

E. Algebra

- Apply the rules for the order of operations on positive and negative rational numbers.
- Evaluate algebraic expressions involving the basic operations, the real numbers, and the concept of absolute value.
- Solve linear equations and inequalities algebraically and by successive substitutions.
- Graph linear equations and inequalities in two variables.
- Translate graphs horizontally and vertically.
- Formulate and solve problems involving ratio and proportion.
- Add, subtract, and multiply binomials. Add and subtract polynomials.
- Simplify algebraic expressions involving integral (positive, negative, zero) exponents.
- Simplify and evaluate numerical expressions involving square root.
- Use the function concept to solve practical problems.

Course Title: Math B

Description

This is the second year of a two-year sequence that presents the fundamental concepts and skills required in a technological society. The course should emphasize applications and use an instructional approach appropriate for students with a concrete rather than abstract learning style. Students' competencies in each of the areas will range from minimal understanding to full appreciation of major implications and applications. All students are expected to achieve minimal understanding in each area. Most will achieve beyond this level.

Content

A. Measurement

- Solve geometric problems (perimeter, area, and volume) in various measurement systems.
- Select and use appropriate formulas and procedures to determine a measure when a direct measurement is not available.

- Determine the precision of measurements necessary to produce a calculation of desired accuracy based on the measurements.

B. Geometry

- Recognize and apply the basic postulates, definitions, and theorems of Euclidean geometry, including parallel and perpendicular lines, congruent triangles, arcs and angles of circles, and similar triangles.
- Solve problems involving properties of polygons, including special quadrilaterals—square, rhombus, rectangle, parallelogram, and trapezoid—and special triangles—isosceles, equilateral, and right (30° - 60° - 90°, 45° - 45° - 90°).
- Apply the principles of coordinate geometry to graph lines and circles and determine the slope and intercept of a line and the midpoint of a line segment, the center and radius of a circle, and the areas of simple geometric figures.
- Use the three basic trigonometric ratios to find angles and sides of right triangles.

C. Statistics and Probability

- Distinguish between inclusive and exclusive events. Distinguish between independent and dependent events and use conditional probabilities.
- Explain the significance of varying values of statistical measures.
- Choose appropriate statistical measures to describe data.
- Identify and explain misuses of statistical measures.
- Graph data to study the relationship between two variables.

D. Logic

- Identify what is necessary and sufficient in an argument.
- Recognize and explain flaws in invalid arguments.

E. Algebra

- Extend knowledge of the fundamentals of algebra expected of all students in Math A.
- Formulate and solve systems of linear equations and inequalities algebraically and graphically.
- Formulate and solve linear equations.
- Graph nonlinear functions and interpret the graphs.
- Simplify rational algebraic expressions with monomial denominators.
- Factor polynomials by removing the greatest common monomial factor.
- Factor polynomials of the form $a^2 - b^2$ and $a^2 \pm 2ab + b^2$.
- Simplify and evaluate algebraic expressions involving square roots.

Description

This is the third course in the continuum for students who are not interested in pursuing the college preparatory sequence. Although some time is devoted to consumer/business application skills, the primary concern of Math C is preparing young adults for living in a technological society that requires a high level of mathematical understanding.

The course should be activity-oriented and should concentrate on the concepts of mathematics not emphasized in previous courses.

Suggested Topics

- Programming concepts and algorithms using BASIC, Logo, or other languages
- Consumer applications (taxes; credit card purchasing, insurance, installment buying, and loans; comparison shopping)
- Data collection; organization and analysis of data
- Statistical measures of central tendency and dispersion
- Misuses of statistics
- Probability (fair and unfair games; mathematical expectation related to life insurance rates; quality control)
- Logical reasoning (valid arguments; inferences; reasoning patterns used in advertising and political speeches)
- Measurement and measurement instruments, such as calipers and transits
- Geometry (ratio; proportion; similarity; trigonometric ratios; indirect measurement)
- Laboratory experiments in which mathematics is used to describe or explain results

Description

This course is a formal development of the algebraic skills and concepts necessary for students who will take a geometry course and other advanced college preparatory courses. In particular, the instructional program in this course should provide for:

- Use of a full range of problem-solving skills and processes in the context of the development of algebraic skills and concepts
- Consistent exposure to and use of logical reasoning processes in the development of algebraic skills and concepts
- Systematic but not pedantic use of the function concept throughout the course
- Understanding of the structure of the real number system as a basis for the skills and concepts that are developed
- Substantial exposure to the development and application of right triangle trigonometry as a precursor to the development of circular function trigonometry in later courses
- Consistent interweaving of all the strands throughout the course

Content

- Variables; algebraic symbols and expressions; evaluation of expressions and formulas; translation from words to symbols
- Integers, rational numbers, real numbers, and the properties of each; absolute value
- Solutions and applications of linear equations and linear inequalities
- Polynomial expressions and operations; factoring techniques
- Rational expressions and operations; rational equations (including proportions) and applications
- Radicals, operations, and simplifications; algebraic expressions and equations involving radicals
- Solutions and graphs of linear equations and inequalities: concepts of slope, y-intercept, parallelism, and perpendicularity
- Linear systems in two variables—solution by graphical and algebraic techniques
- Solutions (by factoring, completing the square, and using a formula) and applications of quadratic equations
- Application of quadratic solutions to maximum, minimum problems
- Operations with expressions involving integral exponents
- Applications of right-triangle trigonometry
- Development of the concept of function and graphs of functions
- Solution of word problems, including geometric as well as algebraic applications
- Introduction to probability; statistical measures of central tendency and dispersion

Course Title: Geometry

Description

This course is a formal development of the geometric skills and concepts necessary for students who will take Algebra 2 and/or other advanced college preparatory courses. In particular, the instructional program should provide for the:

- Consistent use of algebra throughout the course to reinforce the skills and concepts developed in Algebra 1
- Exploratory development of the formal representation of logical arguments
- Application of logical principles to geometric proofs
- Use of a full range of problem-solving skills in the development of geometric concepts
- Extension of trigonometry to angles greater than 90° as a precursor to the development of circular function trigonometry in later courses
- Consistent interweaving of all the strands throughout the course

Content

- An adequate set of postulates to support the proof of geometric theorems
- Perpendicularity, parallelism, congruence, and similarity relationships in two and three dimensions
- Mensuration theory, as applied to developing the formulas for measuring lengths, perimeters, areas, and volumes of common geometric figures
- Coordinate geometry, including slope of a line, midpoint of a segment, distance formula, point-slope-intercept forms of the equation of a line, and equation of a circle
- Relationships in circles and polygons—algebraic applications
- Introduction to transformational geometry, including translations, reflections, rotations, and similarity transformations
- Original proofs and proofs of theorems, including direct, indirect, coordinate, and transformational techniques
- Introduction to symbolic logic
- Locus, geometric constructions, and concurrence theorems
- Use of algebraic concepts and skills in geometric situations, including solution of problems involving the Pythagorean theorem
- Introduction to the trigonometric functions of angles greater than 90° and special angle relationships

Description

Algebra 2 is a full-year course. It should expand on the mathematical content of Algebra 1 and geometry. Review of those concepts should be integrated throughout the course. Emphasis should be placed on abstract thinking skills, the function concept, and the algebraic solution of problems in various content areas. Content areas that should be emphasized include the solution of systems of quadratic equations, logarithmic and exponential functions, the binomial theorem, progressions and series, the complex number system, and right and oblique triangle trigonometry. Calculators and computers should be used throughout the course to aid in the solution of problems and in making estimates and approximations to determine whether the solutions that are obtained are realistic.

Content

- Simplification of algebraic expressions, including fractional exponents and radicals
- Solution of linear equations and inequalities, including those involving absolute value and rational expressions
- Operations on polynomials
- Solution of quadratic equations by factoring, completing the square, and using a formula; properties of roots
- Solution of quadratic inequalities
- Solution of polynomial equations; rational roots; Descartes' Rule of Signs
- Solution of systems of linear equations with two and three variables: homogeneous, dependent, and inconsistent systems; use of determinants and matrices
- Linear, quadratic, and polynomial functions—their graphs and properties
- Introduction to the graphing of quadratic equations in two variables—conic sections
- Permutations and combinations; the binomial theorem
- Arithmetic and geometric sequences and series
- Function concept, including notation, composition, inverse, and arithmetic operations on functions
- Graphs of exponential and logarithmic functions; solution of exponential and logarithmic equations
- Solution of word problems, including estimation and approximation
- Complex numbers—notation, graphical representation, and applications
- Solution of systems of quadratic equations in two unknowns
- Mathematical probability; finite sample spaces; conditional probability
- Trigonometric applications—solution of oblique triangles through the use of the laws of sines and cosines

Description

Math Analysis is a full-year course that blends together all of the precalculus concepts and skills that must be mastered prior to enrollment in a college-level calculus course. A functional approach integrating as many concepts as possible should be used throughout the course.

Content

- Development of the trigonometric functions through the use of the concept of circular functions
- Graphical characteristics of the trigonometric functions—including translations, amplitude, change of period, domain, range, and sums and differences of functions
- Inverse trigonometric functions—notations and graphs
- Trigonometric identities, including addition and double-angle and half-angle formulas
- Use of degree and radian measures
- Solution of trigonometric equations
- Polar coordinates and vectors; graphical representations
- Solution of problems related to force and navigation
- Trigonometric form of complex numbers and De Moivre's theorem
- Mathematical induction
- Analytic treatment of the conic sections, including translations and rotations
- Rational functions and their graphs
- Parametric equations and their graphs
- Lines and planes in space—three-dimensional coordinate geometry
- Introduction to vectors in the plane and space
- Characteristics of graphs of functions
- Concept of a limit: definition, application, convergence/divergence

The content of these courses has been established by the College Board. Copies of these course outlines may be obtained directly from the College Board.

Suggested Advanced Elective Courses

[illegible]

Suggested Topics

- Matrix operations: order, equality, addition, multiplication by a scalar, matrix multiplication
- The algebra of 2 x 2 matrices: determinant function, invertible matrices, matrix representation of complex numbers
- Matrices applied to systems of linear equations
- Column matrices as geometric vectors: algebra of vectors, vectors and their geometric representation, inner product of two vectors, vector spaces and subspaces
- Transformations of the plane: rotations and reflections, linear transformations, matrix representations of transformations, characteristic vectors and values

[illegible]

Suggested Topics

- Permutations
- Combinations
- Empirical determination of probability
- Inclusive and exclusive events
- Dependent and independent events
- Bayes' theorem
- Measures of central tendency and dispersion
- Binomial and normal distributions
- Sampling techniques
- Testing simple statistical hypotheses
- Regression, correlation, and the significance of a correlation coefficient

APPENDIX

Descriptions of Courses I and II for Preservice Education

Source: Taken from *Recommendations on the Mathematical Preparation of Teachers.* Prepared by the Committee on the Undergraduate Program in Mathematics. Washington, D C Mathematical Association of America, 1983, pp. 9—15. Used with permission.

Note: As stated in the document cited, "the course descriptions in these recommendations are not intended to be complete course descriptions but rather to identify topics that are particularly important for teachers."

Fundamental Mathematical Concepts I

This course provides prospective teachers with part of the background needed for teaching the content of contemporary elementary mathematics programs, including the development of the whole number system, geometry, and measurement. The prerequisites are three years of college preparatory mathematics, including two years of algebra and one year of geometry, and demonstrated mastery of the basic algorithmic skills of the K-8 mathematics curriculum.

Objectives

The objectives of Fundamental Mathematical Concepts I should be to provide teachers with the ability to:

1. Identify, develop, and solve problems that are related to the child's environment and involve the mathematical concepts and principles usually taught in the elementary grades.
2. Identify and use problem-solving strategies appropriate to these grades.
3. Illustrate prenumeration concepts (attributes, classification, ordering, patterns, and sets).
4. Illustrate and explain number and numeration concepts (cardinal and ordinal numbers, place value).
5. Explain and develop the usual algorithms for the four basic operations with whole numbers and illustrate these operations, using models and thinking strategies appropriate to the elementary grades.
6. Relate the properties of the whole number system to the basic algorithms and to their use in problem solving.
7. Use estimation in numerical calculation and problem-solving situations.
8. Identify examples in the child's environment of simple geometric shapes and their properties.

9. Develop basic planar relationships (parallelism, perpendicularity, . . .) and model them with examples from the child's environment.
10. Model spatial relations and illustrate their properties, using classroom objects.
11. Use standard and nonstandard (paper clips, erasers, body measures, . . .) units in measuring length, perimeter, area, capacity, volume, mass, weight, angle, time, and temperature.
12. Design classroom experiences illustrating the geometric and measurement concepts appropriate to the various grade levels.
13. Use mathematical terminology and symbolism appropriately while working with elementary school children.
14. Describe the historical and cultural development of some of the main mathematical concepts and principles usually taught in the elementary grades.

Topics

While there may be considerable variation in the topics selected to achieve these objectives, the following are particularly suitable for inclusion in such a course:

Problem Solving. Problem-solving strategies (including guess and test, pattern and searches, models, related problems, formulas, algorithms, and simulations); applications of strategies in traditional and nontraditional settings; preparation for use of strategies throughout the remainder of the course and following courses.

Prenumber Concepts. Properties of joining, separating, and comparing sets; set equivalence; set inclusion; Cartesian products; and the use of set concepts in problem solving (Venn diagrams, exhaustive listings, . . .).

Whole Numbers and Their Operations. Historical role of number systems; base ten numeration system; place value and its relation to grouping in operations; models for each of the four basic operations; properties of the basic operations; common error patterns in student computation with basic algorithms.

Number Theory. Divisibility; prime and composite numbers; sieve of Eratosthenes; infinitude of primes; divisibility rules for whole numbers through 11 (excluding seven); lcm, gcd, and relative primeness.

Geometry. Basic concepts and properties of two- and three-dimensional spaces (including point, line, plane, space, segment, ray, betweenness, parallelism, perpendicularity, congruence, similarity, symmetry, and basic shapes and solids); use of geoboards, paper folding, and other models in illustrating these concepts and relationships; basic concepts and properties of geometric transformations (slides, flips, and turns).

Measurement. Process of measurement (selection of a unit, covering with the unit, "counting" the number of units used); application of measuring, using standard and nonstandard units of length, area, volume, capacity, mass, and their relationships; estimation of measures; perimeter, area, and volume of standard geometric figures; indirect measurement (similar figures, Pythagorean theorem, . . .).

Fundamental Mathematical Concepts II

This course focuses on the development of the real number system and its subsystems, probability, statistics, and basic computer concepts. The prerequisite is Fundamental Mathematical Concepts I.

Objectives

The objectives of Fundamental Mathematical Concepts II are to provide teachers with the ability to:

1. Identify, develop, and solve problems that are related to the child's environment and involve the mathematical concepts and principles for the appropriate grade levels.
2. Explain the concepts of fractions (including decimals), integers, ratio, proportion, and percentage, using models appropriate for the elementary grade levels.
3. Explain and develop the standard algorithms for the four basic operations for integers, positive and negative rational numbers (including decimal notation), and real numbers.
4. Illustrate the standard algorithms, using properties of the number of systems involved and appropriate models.
5. Recognize other algorithms for the basic operations and explain them, using appropriate models or properties of the number systems.
6. Use estimation wherever appropriate, in particular to pose and select alternatives concerning reasonable responses.
7. Solve simple problems involving probability, inference, and the testing of hypotheses.
8. Solve simple problems involving the reporting of data (measures of central tendency, dispersion, expectation, and prediction).
9. Make appropriate use of calculators and computers in problem solving and in exploring and developing mathematical concepts.
10. Explain the role and the uses of computers in our society.
11. Use mathematical terms and symbolism appropriate to different elementary grade levels.
12. Describe the historical and cultural significance of some of the major mathematical concepts and principles contained in the elementary school mathematics curriculum.

Topics

While there may be considerable variation in the topics selected to achieve these objectives, the following topics seem to be particularly suitable for inclusion in such a course:

The Real Number System and Its Subsystems. Extension of whole numbers to integers, models for integers, operations with integers, properties of integers (order, absolute value, . . .); extension of integers to the rational numbers (fractional form); operations with rational numbers; properties of rational numbers; decimal expansions of rational numbers; irrationality of $\sqrt{2}$; irrational numbers; Pythagorean theorem; extension of the rational numbers to the real numbers; relations of types of real numbers to forms of decimal expansions; properties of real numbers; applications of real numbers; relationship of real numbers to classical geometry problems; rounding and truncating real numbers on a calculator and a computer, ratio, proportion, percentage, applications of proportions in algebra and geometry.

Probability. Relative frequency experiments; methods of counting (tree diagrams, exhaustive listings, permutations, combinations); sample spaces; joint events, independent events, dependent events, complementary events; Monte Carlo methods.

Statistics. Organization and presentation of data (line, circle, and bar graphs; stem and leaf plots); role of scales and possible bias in graphs; analysis of the observed distributions (mean, median, mode, range, variance, standard deviation); box-and-whisker plots; sampling as it applies to consumers of sample research literature.

Computer Appreciation. History of computing (Pascal to PASCAL); impact of computers on contemporary school mathematics curricula; nature of an algorithm; elementary programming; writing simple programs correlated to the elementary school curriculum; role of computing in mathematics.

Publications Available from the Department of Education

This publication is one of over 600 that are available from the California State Department of Education. Some of the more recent publications or those most widely used are the following:

Bilingual-Crosscultural Teacher Aides: A Resource Guide (1984)	$3 50
Boating the Right Way (1985)	4 00
California Private School Directory	9 00
California Public School Directory	14.00
Career/Vocational Assessment of Secondary Students with Exceptional Needs (1983)	4 00
College Core Curriculum: University and College Opportunities Program Guide (1983)	2 25
Computer Applications Planning (1985)	5.00
Computers in Education: Goals and Content (1985)	2 50
Curriculum Design for Parenthood Education (1982)	4 00
Guide for Vision Screening in California Public Schools (1984)	2.50
Handbook for Planning an Effective Foreign Language Program (1985)	3.50
Handbook for Planning an Effective Mathematics Program (1982)	2.00
Handbook for Planning an Effective Reading Program (1983)	1.50
Handbook for Planning an Effective Writing Program (1983)	2.50
History—Social Science Framework for California Public Schools (1981)	2.25
Improving the Attractiveness of the K—12 Teaching Profession in California (1983)	3.25
Improving the Human Environment of Schools: Facilitation (1984)	5 50
Improving Writing in California Schools: Problems and Solutions (1983)	2.00
Individual Learning Programs for Limited-English-Proficient Students (1984)	3.50
Instructional Patterns: Curriculum for Parenthood Education (1985)	12.00
Literature and Story Writing: A Guide for Teaching Gifted and Talented Children (1981)	2.75
Manual of First-Aid Practices for School Bus Drivers (1983)	1 75
Martin Luther King, Jr., 1929—1968 (1983)	3.25
Mathematics Framework for California Public Schools (1985)	3.00
Model Curriculum Standards: Grades Nine Through Twelve (1985)	5.50
Raising Expectations: Model Graduation Requirements (1983)	2.75
Reading Framework for California Public Schools (1980)	1.75
Resources in Health Career Programs for Teachers of Disadvantaged Students (1983)	6.00
School Attendance Improvement: A Blueprint for Action (1983)	2.75
Science Education for the 1980s (1982)	2.50
Science Framework for California Public Schools (1978)	3.00
Science Framework Addendum (1984)	3 00
Selected Financial and Related Data for California Public Schools (1985)	3.00
Standards for Scoliosis Screening in California Public Schools (1985)	2 50
Studies on Immersion Education: A Collection for U.S. Educators (1984)	5 00
Trash Monster Environmental Education Kit (for grade six)	23.00
University and College Opportunities Handbook (1984)	3.25
Visual and Performing Arts Framework for California Public Schools (1982)	3.25
Wet 'n' Safe: Water and Boating Safety, Grades 4—6 (1983)	2.50
Wizard of Waste Environmental Education Kit (for grade three)	20.00
Work Permit Handbook (1985)	6.00

Orders should be directed to:

California State Department of Education
P.O. Box 271
Sacramento, CA 95802-0271

Remittance or purchase order must accompany order. Purchase orders without checks are accepted only from government agencies in California. Sales tax should be added to all orders from California purchasers.

A complete list of publications available from the Department, including apprenticeship instructional materials, may be obtained by writing to the address listed above.

A list of approximately 140 diskettes and accompanying manuals, available to members of the California Computing Consortium, may also be obtained by writing to the same address.

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